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Report Information

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Abstract

Advanced Mobility is a critical research area under the Naval S&T Focus Area of *Platform Design and Survivability*. Enhanced mobility is a key objective in the design of new platforms, but one may also enhance mobility of existing platforms through advanced modeling and control, enabling current assets to operate within larger environmental envelopes. This final report concerns accomplishments by a Virginia Tech research team toward – in collaboration with the Australian Defence Science & Technology (DS&T) Group, organized under the auspices of The Technical Cooperation Program (TTCP) – concerning submarine maneuvering in elevated sea states. In this scenario, advanced modeling and control can improve disturbance rejection and maneuvering performance, enabling new mission capabilities for operations at or near the surface.

The goal of this effort is to critically assess the effectiveness of energy-based nonlinear control approaches for motion control of a submarine in high sea states. More specifically, the team will work to develop a parametric model structure that faithfully represents the motion of a submarine maneuvering in a seaway but is amenable to nonlinear control design, to design a model-based predictive motion control law, and to assess the performance of this autopilot design by comparing its performance to conventional designs. The specific objectives of the collaboration are:

- To develop theory and methods to obtain energy-based parametric models for submarine maneuvering in high sea states for use in motion prediction and control.
- To validate the proposed models with experimental data.
- To demonstrate the use of these parametric models for motion prediction.
- To develop robust, predictive, energy-based motion control algorithms.
- To compare the performance of these nonlinear controllers with the performance of controllers developed using linear methods.

The Virginia Tech team is focusing development of a suitable parametric motion model and on the development of motion prediction strategies to support control design and implementation.

Note: This grant restored funding that had been cut from an earlier grant (N00014-14-1-0651), enabling the investigators to restore the originally proposed computational flow modeling effort. The abstract and technical section of this report are therefore identical to the final report for the earlier grant. The final report statement, however, includes contributions made during the period of performance for this follow-on grant (and omits earlier contributions under the preceding grant).

Technical Section

Following is a summary of the originally proposed research tasks and the technical approach.

Task 1: Control-Oriented Modeling

- Review of prior efforts to unify maneuvering and seakeeping models for model-based motion control of a vessel in a seaway.
- Development of a reduced order model (ROM) amenable to energy-based nonlinear control design.

Task 2: Hydrodynamic Analysis

- Development of a strip-theory model of the vessel using widely available computational tools and comparison to experimental data to identify gaps in modeling.
- RANS/URANS modeling of the vessel with validation to experimental data. Emphasis shall be placed on conditions and physics not captured by reduced order, strip-theory modeling.

Task 3: Motion Prediction

- Development of a motion prediction scheme to aid feedforward disturbance rejection.

As proposed, the effort requires the use of CFD to identify parameter values in a low-order mathematical model that can be used to predict vessel motion in waves. To enable validation, the effort focuses on a well-studied geometry: The Joubert (BB2 variant), shown below. Captive model experiments at the Australian Maritime College will be used to assess the adequacy and improve the quality of the model.

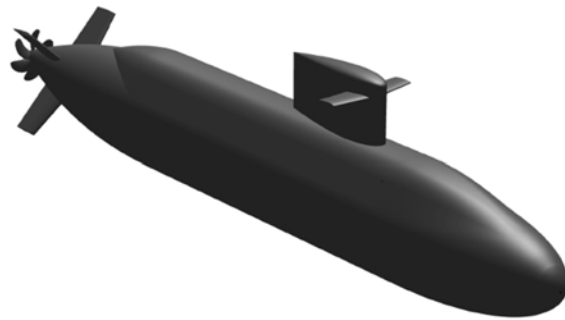


Figure 1. The Joubert (BB2 Variant), fully appended

Final Report Statement

During the project period of performance (7/1/2016 through 6/30/2020, including a one-year no-cost extension), the team:

- Completed a Lagrangian mechanics (potential flow) model for maneuvering and seakeeping by a rigid body moving near a free surface. The maneuvering component includes memory effects, modeled using a nonlinear extension of the Cummins equation. The seakeeping component includes Froude-Krylov forcing due to ambient wave effects. Approximating memory effects with a finite-dimensional filter, the model is amenable to the design of nonlinear estimation and control methods that make use of the Lagrangian mechanical system structure.
- Developed a parameter computation approach for this control-oriented motion model using the time-domain panel method Aegir. That effort involved reformulating elements of the original Lagrangian nonlinear maneuvering and seakeeping (LNMS) model developed by Battista and modifying aspects of the Aegir source code.
- Supported efforts by Dr. F. Valentinis, with Australia's Defence Science and Technology (DST) Group to adapt the Lagrangian nonlinear maneuvering (LNM) model to the Joubert BB2 and to obtain experimental validation of the Lagrangian nonlinear maneuvering and seakeeping (LNMS) model.
- Presented the results in 6 international conference presentations and 2 doctoral dissertations. **Note:** Several of these publications appeared during the overlapping periods of performance for Grant Nos. N00014-16-1-2749 and N00014-14-1-0651. Both grants were acknowledged in the publications.

Refereed Journal Articles

- "A maneuvering model for an underwater vehicle near a free surface - Part 2: Incorporation of the free surface memory," T. Battista*, F. Valentinis, and C. Woolsey (In review.)
- "Computing hydrodynamic forces for a unified maneuvering and seakeeping model of submerged vessel motion," S. Jung*, S. Brizzolara, and C. A. Woolsey (In review.)
- "Parameter computation for a Lagrangian mechanical system model of a submerged vessel moving near a free surface," S. Jung*, S. Brizzolara, and C. A. Woolsey (In review.)

- "A maneuvering model for an underwater vehicle near a free surface - Part 1: Motion without memory effects," T. Battista*, F. Valentinis, and C. Woolsey. *IEEE J. Oceanic Engineering* 45(1), pp. 212-226, January 2020.
- "Nonlinear control of a subscale submarine in emergency ascent," F. Valentinis and C. A. Woolsey. *Ocean Engineering* 171, pp. 646-662, January 2019.

Books And Chapters

- "Model-Based Path Planning," A. Wolek and C. A. Woolsey. In *Sensing and Control for Autonomous Vehicles: Applications to Land, Water and Air Vehicles*, T. I. Fossen, K. Y. Pettersen, and H. Nijmeijer, Editors. [In press.]

Technical Reports

- *Determining Parameters for a Lagrangian Mechanical System Model of a Submerged Vessel*, S. Jung, Doctoral Dissertation, Virginia Tech, January 2020.

Note: T. Battista also completed his doctoral dissertation during this period of performance, but that dissertation is excluded here. Also, T. Battista is not tallied among the project personnel in the ONR Statistics section.

Contributed Presentations

- "Longitudinal, near-surface maneuvering of a prolate spheroid," Y.-C. Chen*, S. Jung*, and C. A. Woolsey, *Proc. 12th IFAC Conf. on Control Applications in Marine Systems, Robotics, and Vehicles*, Daejeon, South Korea, September 2019.
- "An immersed spheroidal pendulum oscillating near a free surface," B. Andersen*, C. Beardsley*, M. Gates*, J. Schlafman*, T. Battista*, S. Jung*, and C. A. Woolsey, *Proc. MTS/IEEE OCEANS*, Charleston, SC, October 2018.
- "An analytical approximation of a force prediction model for a prolate spheroid moving at a constant forward speed parallel to a calm free surface," S. Jung*, T. Battista*, F. Valentinis, S. Brizzolara, E. Paterson, and C. Woolsey, *Proc. 13th Int. Conf. on Hydrodynamics*, Incheon, Korea, September 2018.
- "Effects of depth on added mass obtained from virtual PMM tests of a submerged prolate spheroid," S. Jung*, T. Battista*, E. Paterson, and C. Woolsey, *Proc. MTS/IEEE OCEANS*, September 2017, Anchorage, AK.
- "An energy-Casimir approach to underwater vehicle depth and heading regulation in short crested waves," T. Battista*, S. Jung*, C. Woolsey, and E. Paterson, *Proc. Conference on Control Technology and Applications*, Kohala Coast, HI, August 2017. (Invited, Peer-reviewed)
- "A dynamic model for underwater vehicle maneuvering near a free surface," T. Battista, C. A. Woolsey, T. Perez, and F. Valentinis, *Proc. 10th IFAC Conf. on Control Applications in Marine Systems*, Trondheim, Norway, September 2016. [Best Student Paper Award]

* denotes a graduate or undergraduate student researcher.

Patents

- None

Honors

- Best Student Paper Award: Battista*, Woolsey, McCue-Weil, Paterson, and Valentinis, *IFAC CAMS*, 2016.

Related Sponsored Work

- NSF Grant No. CMMI-1435484: Bioloocomotion Analysis and Design Using Geometric Control and Averaging Methods (Woolsey, PI) focuses on the use of nonlinear control design and analysis methods for simultaneous design optimization of the morphology and gait of biomimetic systems. The dynamic model used to incorporate unsteady, nonuniform flow effects on submarine motion in waves may also be applied to biomimetic systems.
- NSF Grant No. CMMI-1635143: Collaborative Research: Unsteady Hydrodynamics and Geometric Control of Pisciform Locomotion (Woolsey, PI) builds on efforts begun under the preceding grant, but with a focus on experimental validation of low-order motion models for aquatic (e.g., pisciform) locomotion systems.

ONR Statistics

Grad Students(total):	1
PI/Co-PI Women:	0
PI/Co-PI Minority:	0
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Grad Students Minority:	0
Post Docs Students:	0
Post Doc Women:	0
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Degrees Granted:	1
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Determining Parameters for a Lagrangian Mechanical System Model of a Submerged Vessel Maneuvering in Waves

Seyong Jung

Dissertation submitted to the Faculty of the
Virginia Polytechnic Institute and State University
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy
in
Aerospace Engineering

Craig A. Woolsey, Chair
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Blacksburg, Virginia

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surface, maneuvering and seakeeping

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Determining Parameters for a Lagrangian Mechanical System Model of a Submerged Vessel Maneuvering in Waves

Seyong Jung

(ABSTRACT)

In this dissertation, an approach for determining parameters for a nonlinear Lagrangian mechanical system model of a submerged vessel maneuvering near waves is presented. The nonlinear model with determined parameters is capable of capturing nonlinear effects neglected by other linear models, and therefore can be applied to improve maneuvering performance and expand the operating envelope for submerged vessels operating in elevated sea states.

To begin, a first principles Lagrangian nonlinear maneuvering (LNM) model for a surface-affected submerged vessel derived by using Lagrangian mechanics [1] is reformulated to allow the application of data from a medium fidelity potential flow code. In the reformulation process, the order of integration and differentiation in the integro-differential parameters are switched and partial derivatives of the Lagrangian function are computed with readily available data from the panel code solution. As a result, all model parameters can be computed individually using the panel code, wherein the need for additional numerical discretization is circumvented in the computation process through use of solutions already performed by the basic panel code, enabling higher accuracy and lower computational cost. Furthermore, incident wave effects are incorporated into the reformulated LNM model to yield a Lagrangian nonlinear maneuvering and seakeeping (LNMS) model. The LNMS model is numerically validated by confirming the proposed methods and by comparing steady and unsteady hydrodynamic force calculations from the LNMS model against panel code computations for various vessel motions in calm water and in plane progressive waves. Finally, methods for computing physically intuitive components of the model parameters, as well as methods for

making approximations of the terms accounting for memory effects are presented, leading to a model formulation amenable to control design.

By applying the methods proposed in this dissertation, each and every parameter of the Lagrangian mechanical system model of a submerged vessel maneuvering in waves can be obtained accurately and with computational efficiency by using a potential flow panel code. The resulting nonlinear motion model provides higher model fidelity than existing unified maneuvering and seakeeping models, especially in applications such as nonlinear control design and simulation.

Determining Parameters for a Lagrangian Mechanical System Model of a Submerged Vessel Maneuvering in Waves

Seyong Jung

(GENERAL AUDIENCE ABSTRACT)

A unified maneuvering and seakeeping model for a submerged vessel maneuvering near waves describes mathematically the relationship between input values to the dynamical system, such as thrust from the propulsors, and output values from the system, such as the position and orientation of the vessel. This unified model has a wide range of applications, ranging from vessel hull form optimization in the early design phase to motion controller tuning after the vessel has been constructed. In order for a unified model to make accurate predictions, for instance, for a submerged vessel making a rapid turn near large waves, nonlinear effects have to be included in the model formulation. To that end, a nonlinear motion model for a marine craft affected by a free surface has been developed [1] using Lagrangian mechanics. This dissertation describes an approach for determining the parameters of the nonlinear motion model using a potential flow panel code, which is originally designed to determine flow velocity of the fluid and pressure distribution over marine vessels. The nonlinear motion model is reformulated and the software implementation is modified to support parameter computations. In addition, the methods are numerically validated by comparing computations using the model against solutions output by the panel code. Compared to traditional parameter estimation approaches, the proposed methods allow for a more accurate and efficient determination of parameters of the nonlinear potential flow model for a submerged vessel operating near waves. The resulting Lagrangian nonlinear maneuvering and seakeeping (LNMS) model with determined parameters is able to capture critical nonlinear effects and has applications such as nonlinear control design, rapid design optimization and training simulator development.

Dedication

Dedicated to my father.

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I wish to express my deepest gratitude to everyone who has supported me during my often challenging yet gratifying doctoral journey. I am especially grateful to my advisor Dr. Craig Woolsey for all his wisdom and guidance, and his timely encouragements. I would also like to thank my co-advisor Dr. Stefano Brizzolara for helping me overcome obstacles and offering a new point of view on hard problems. I am also indebted to my committee members, Dr. Eric Paterson and Dr. Francis Valentinis, for helping me with their extensive knowledge in marine craft hydrodynamics and motion control throughout my research. I would also like to thank Dr. Thomas Battista for laying out the groundwork and always willing to lend a helping hand.

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Chapter 1

Introduction

1.1 Background and Motivation

The word model comes from the Latin *modulus*, meaning “measure” or “standard”. As suggested by its root, a model is a pattern or guide used to represent a system. Characteristics of the real system are incorporated into the model according to its purpose. Therefore, the degree to which a model resembles the real system depends on the model’s intended use. A model should describe the real system well enough to produce accurate results, e.g., predict values of interest within error bounds. On the other hand, the model cannot be so complex that the costs, such as computational or experimental costs, involved in employing the model become too expensive. A model which adequately resembles the real system can help us in situations where using or experimenting on the actual system is impractical or inefficient.

Ljung [2, 3] categorizes models into mind, verbal, graphical, physical, mathematical and software models. A mind model is based on intuition and experiences, and it is how we answer questions in our minds. As for a verbal model, it describes a system in words. An example of a verbal model is “if the bank interest rate goes up, then unemployment rate will rise” [2]. Similarly, a graphical model describes the properties of a system using numerical tables and/or plots, e.g., a properties table for a compressor used for parts selection. In contrast, a physical model is usually a smaller or larger copy of the actual system, which can be used to investigate various physical phenomena, as well as to evaluate aesthetic qualities.

On the other hand, a mathematical model describes mathematically the relationship between quantities that can be observed in the system [2]. A mathematical model can further be classified into subcategories depending on its structure, such as linear vs nonlinear, static vs dynamic, etc. [4]. And finally, a software model is one that is used in a computer simulation and can be a combination of lookup tables and mathematical models, usually to represent complicated systems.

For a marine craft, depending on the application, a graphical, physical, mathematical, software, or a combination of these models can be applied to the early stages of conceptual design to well beyond the ship launching phase. Mathematical models of marine craft, such as the ones described in [5, 6, 7, 8, 9], are especially useful for vessel design, motion prediction, stability analysis and model-based controller design. Moreover, mathematical models have historically been developed independently as maneuvering models or seakeeping models, but the body of theory which unifies the two concepts is limited in its extent and its utility because of restrictive assumptions. Maneuvering models were developed to study course keeping and maneuvering of marine craft in otherwise undisturbed waters [10], a condition that prevails in confined regions, such as harbors, but is uncommon in the open ocean. Many of the maneuvering models for subsurface vessels were built on classical models for vessels submerged in unbounded fluid [11] by incorporating the effects of a calm free surface. Well-known maneuvering models include the ship steering equations of motion described in [12], the model for maneuvering in deep and confined waters in [13], the nonlinear rolling coupled steering model for high speed container ships in [14, 15] and various others presented in, for example, [6]. More recently, a first principles Lagrangian nonlinear maneuvering (LNM) model has been developed using Lagrangian mechanics [9, 16] for use in the design and tuning of control and estimation schemes and for efficient simulation. On the other hand, seakeeping models have been formulated for marine craft traveling in a seaway under restrictions such

as zero or constant forward speed. Early interest in the seakeeping problems were triggered by a paper by Weinblum and St.Denis [17], and extensive studies have been carried out since then by employing analytical [5, 18] as well as numerical methods [19, 20]. Seakeeping models can be used, for example, to assess the dynamic stability of steady, forward motion in various environmental conditions, but their use is typically restricted to simple motions such as this. In order to overcome some of the limitations of maneuvering models and seakeeping models, efforts have been made to unify the concepts for maneuvering in waves. However, these efforts have typically imposed strict conditions. For example, the conditions for slender-body theory have been assumed [18, 21], or that the vessel's nominal motion is purely in surge and in a following sea [22], or that the ship moves only in the horizontal plane and does not deviate significantly from a straight course [23]. More recent and more advanced models allow small perturbations from a nominal speed and heading [6, 8, 24, 25]. To relax many of these assumptions, a nonlinear Lagrangian mechanical system model for a submerged vessel operating near the surface in waves was developed by Battista [1]. This potential flow model can be effective in capturing nonlinear hydrodynamic forces due to aggressive maneuvers or significant wave forcing, and can improve performance and expand the safe operating envelope of prior unified models.

The basic structure of the nonlinear Lagrangian mechanical system model developed by Battista [1] is used in this dissertation. This model has been constructed based on first principles into a parametric ordinary differential equation (ODE) form by basically following a procedure such as the one outlined in [2]. Extensive studies have been carried out in the past for parameter estimation of various forms of marine craft ODE models by using experimental data, high-fidelity computational fluid dynamics (CFD) data, or both. Captive model experiments, such as planar motion mechanism (PMM) tests have been performed [26, 27] to obtain experimental data. By performing PMM tests, one may obtain parameters

for widely used motion models, such as the Abkowitz model [28]. Alternatively, PMM tests [29, 30] or free running tests [31] have been simulated using CFD to derive hydrodynamic coefficients for marine craft models. In some cases, both experimental data and CFD data have been used [32]. Although these methods may provide accurate parameter values for existing motion models, they can be costly and time consuming. Moreover, these methods are not necessarily well suited to parameter determination for alternative models, such as the potential flow Lagrangian mechanical system model presented in this dissertation.

1.2 Collaborative Research Effort

This dissertation contributes to an effort supported in part by the U.S. Office of Naval Research, of developing a model for a free surface-affected submerged vessel. This is part of a larger international collaborative effort that started with the hypothesis that the application of nonlinear model-based control can expand the operating envelope and provide robust control for a submerged vessel maneuvering in a seaway at periscope depth. The following objectives were set in order to confirm the hypothesis:

1. **Parametric model design** - Construct a nonlinear control-oriented parametric model which incorporates the vessel dynamics as well as near-surface hydrodynamic effects.
2. **Parameter determination** - Develop a method to compute or approximate the parameters of the model using numerical methods.
3. **Model validation** - Determine the degree to which the mathematical model is accurate by comparing motion and force predictions using the model to predictions from higher fidelity simulations and from experimental measurements.
4. **Control design** - Develop robust nonlinear control algorithms for the parametric model.
5. **Performance evaluation** - Evaluate controller performance through simulation and

comparison to existing linear control schemes.

This dissertation describes an efficient method of determining the parameters for the non-linear control-oriented model developed by Battista [1], for a submerged vessel maneuvering in waves. The parametric model developed by Battista is described in detail in Chapter 2. The modeling process is built on the abstraction of the problem of interest [33], where the relevant factors (Figure 1.1) that affect the motion of a marine craft maneuvering in waves are identified.

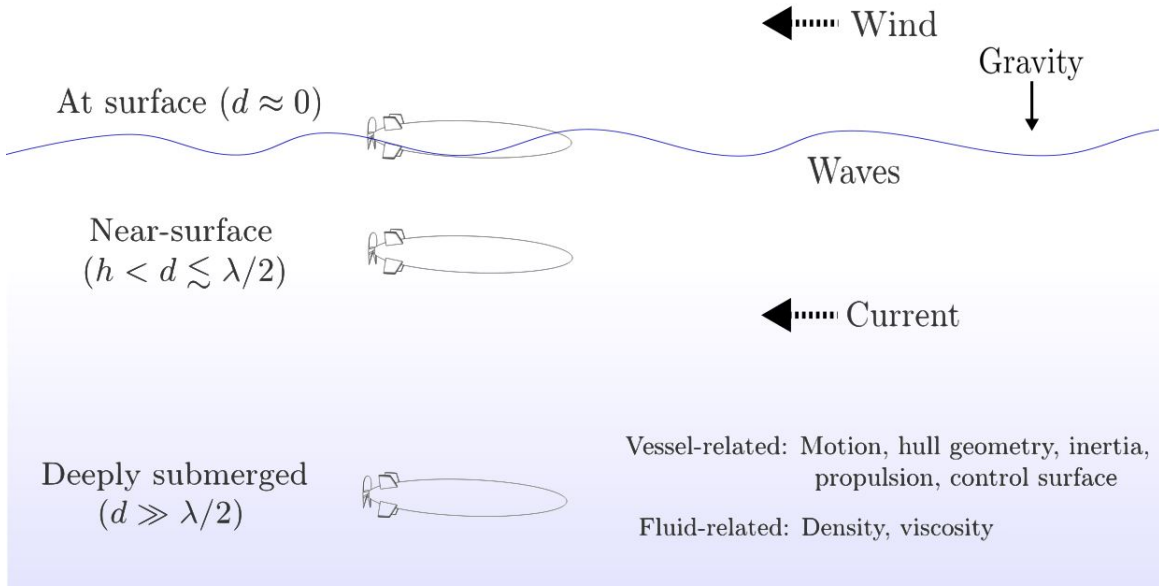


Figure 1.1: Factors that affect dynamics of marine craft.

As can be seen in the Figure 1.1, depending on the operating depth of the vessel, different groups of external factors affect the dynamics of the vessel. The depth d of the vessel is measured from the still water level to the geometric center of the vessel, and the distance h is measured from the geometric center to the top of the vessel. At the surface, forces due to wind, waves, currents and gravity affect the vessel, while at near-surface depths, only forces due to waves, currents and gravity affect the vessel. The half-wavelength depth ($\lambda/2$) is

the wave base, the depth at which the movement of the fluid due to surface waves becomes negligible [34]. Therefore, when a vessel is deeply submerged at depths much greater than $\lambda/2$, the vessel is considered to be in unbounded fluid, and only forces due to currents and gravity affect the vessel. In all cases, there are additional vessel-related factors, such as hull geometry, inertia of the body, and forces due to propulsion and control surfaces that affect the dynamics, while fluid-related factors such as fluid density and viscosity are also factors.

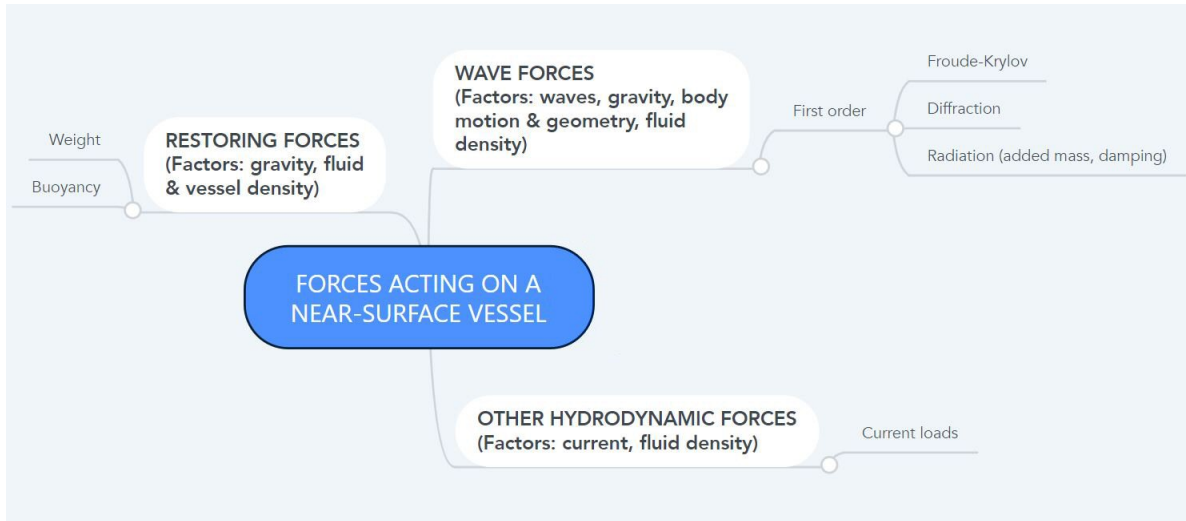


Figure 1.2: Mindmap of forces acting on a marine craft in the model developed by Battista [1].

The model developed by Battista [1] considers the near-surface case, while propulsion and appendages are not included in the model. A first principles model of a marine craft was constructed by applying a Lagrangian mechanics approach that incorporates near-surface hydrodynamic effects on the vessel. The model was developed under two assumptions, the linearized free surface boundary condition and potential flow. Due to the chosen relevant factors and assumptions, the forces and moments shown in Figure 1.2 are considered as the ones acting on the marine craft, and therefore incorporated into the model. First order

wave forces, including the Froude-Krylov, diffraction and radiation forces generated due to external factors such as waves, gravity, body motion, body geometry and fluid density are considered. In addition, current loads on the body due to the currents and fluid density are considered. Finally, restoring forces, including the body weight and buoyancy force due to factors such as gravity and the fluid and vessel density are considered. While viscosity is neglected due to the potential flow assumption, viscous effects can be incorporated into the model in future work using semi-empirical methods such as in [35].

This dissertation proposes a method of determining the parameters of the model developed by Battista [1] by first, reformulating the model for parameter computation using data from a modified potential flow code, then, directly computing the parameters of the reformulated model by using the modified panel code, and finally, presenting a method for approximating the integro-differential terms and replacing them with a form more amenable to control design. The steps taken for parameter determination are outlined in the next section.

1.3 Overview and Contributions

In Chapter 2, a control-oriented Lagrangian nonlinear maneuvering and seakeeping (LNMS) model for a marine craft, developed by Battista [1], is presented. First, the rigid body and fluid mechanics related to the model is described. Then, the Lagrangian mechanics for a body moving in an ideal fluid is introduced, and the resulting LNM and LNMS models are shown. In addition, a simple application of the LNM model [36] is presented. In Chapter 3, the LNM model from the previous chapter is reformulated for parameter computation using data from a modified potential flow code. Novel reformulation methods are introduced and the resulting hydrodynamic force model, LNM model, and LNMS model are presented. In Chapter 4, the parameters of the reformulated model from the previous chapter are computed

by applying data from a time-domain seakeeping panel code that has been modified to output the necessary numerically derived values. Then, force predictions using the reformulated model are numerically validated by comparison to computations from the basic panel code solution. The computed data can be used to determine the state-dependent parameter functions. In addition, the convolution integral terms related to memory effects may be approximated and replaced in the model to produce a model form more amenable to control design. An example is given where the parameters of the LNM model are determined by fitting state-dependent parameter components using computed data and by approximating the memory term of the model [37]. Finally, in Chapter 5, conclusions and future work are presented.

The main contributions of this dissertation are as follows.

- **A novel method for determining parameters of the Lagrangian nonlinear maneuvering (LNM) model:** Computation and approximation of parameters using a potential flow panel code drastically decreases computational cost compared to parameter estimation using high fidelity CFD or experimental data.
- **A method for reformulating the LNM model:** Reformulating the LNM model developed by Battista [1, 9] results in the reformulated hydrodynamic force model (3.30) and the reformulated LNM model (3.34). The proposed reformulation method allows computation of the model parameters while avoiding additional numerical discretization, thus increasing accuracy and decreasing computational cost.
- **A method for incorporating wave forces into the reformulated LNM model:** Introducing wave forces into the reformulated LNM model (3.34) yields the reformulated LNMS model (3.37), which can be used to predict motion and compute forces on a submerged vessel maneuvering in waves.
- **Numerical validation of the reformulated models:** Section 3.5 presents numerical

validation results by computing hydrodynamic forces using the reformulated hydrodynamic force model (3.30) and the reformulated LNMS model (3.37), and by comparing them to force calculations from the basic panel code.

Chapter 2

Parametric Model Design

A first principles Lagrangian nonlinear maneuvering and seakeeping (LNMS) model for a surface-affected marine craft was developed by Battista [1] using Lagrangian mechanics. This model accounts for nonlinear hydrodynamic forces neglected by existing linear unified maneuvering and seakeeping models, which may be important for aggressive maneuvers in waves. The model assumes potential flow with the expectation that **real fluid effects, such as viscosity, can be incorporated by a semi-empirical method or otherwise** after the fact. In addition, the linearized free surface boundary condition is applied, and therefore only first order wave forces are considered. The final form of the Lagrangian mechanical system model is a set of ordinary differential equations (ODEs) that represent the potential flow hydrodynamic effects for nonlinear vessel motion. The model is amenable to nonlinear control design, rapid design optimization, application to training simulators, etc. This chapter summarizes the derivation presented in Battista's dissertation [1], establishing the notation adopted in the current work and providing the basis for later discussion in which the model is reformulated to a more adequate form for parameter computations.

2.1 Rigid Body Dynamics

2.1.1 Kinematics

The reference frames and notation used by Battista [1] are shown in Figure 2.1. The free surface is denoted by $\mathcal{S}$. The earth-fixed reference frame is located at the undisturbed free surface $\mathcal{S}_0$. The z -axis points down in the direction of gravity. The origin O' of the body-fixed reference frame is fixed at the geometric center of the body $\mathcal{B}$. The generalized coordinates $\mathbf{q} = (\mathbf{x}_b^T, \boldsymbol{\theta}_b^T)^T$ describe the configuration of the body, where the vector $\mathbf{x}_b = [x_b, y_b, z_b]^T$ denotes the position of O' in the earth-fixed frame and $\boldsymbol{\theta}_b = [\phi_b, \theta_b, \psi_b]^T$ represents roll, pitch and yaw Euler angles. The quasivelocity vector $\boldsymbol{\nu} = (\mathbf{v}^T, \boldsymbol{\omega}^T)^T$ is composed of the translational velocity $\mathbf{v} = [u, v, w]^T$ and rotational velocity $\boldsymbol{\omega} = [p, q, r]^T$ of the body-fixed frame with respect to the earth-fixed frame, expressed in the body-fixed frame.

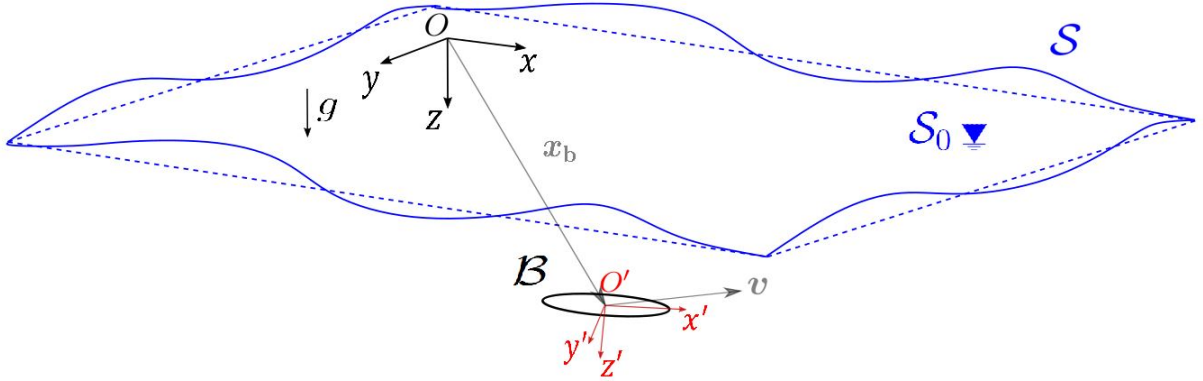


Figure 2.1: Submerged vessel near waves.

A proper rotation matrix $\mathbf{R}(\boldsymbol{\theta}_b)$ which maps free vectors from the body-fixed frame to the earth-fixed frame by a 3-2-1 sequence using Euler angles is defined. Then, $\mathbf{x}_b$ is related to

the translational velocities $\mathbf{v}$ as

$$\dot{\mathbf{x}}_b = \mathbf{R}(\boldsymbol{\theta}_b)\mathbf{v}$$

where the rotation matrix can be explicitly expressed as

$$\mathbf{R} = \begin{pmatrix} \cos \theta_b \cos \psi_b & \sin \phi_b \sin \theta_b \cos \psi_b - \cos \phi_b \sin \psi_b & \cos \phi_b \sin \theta_b \cos \psi_b + \sin \phi_b \sin \psi_b \\ \cos \theta_b \sin \psi_b & \cos \phi_b \cos \psi_b + \sin \phi_b \sin \theta_b \sin \psi_b & -\sin \phi_b \cos \psi_b + \cos \phi_b \sin \theta_b \sin \psi_b \\ -\sin \theta_b & \sin \phi_b \cos \theta_b & \cos \phi_b \cos \theta_b \end{pmatrix}$$

Also, a transformation matrix $\mathbf{L}(\boldsymbol{\theta}_b)$ which relates $\dot{\boldsymbol{\theta}}_b$ to $\boldsymbol{\omega}$ as follows

$$\dot{\boldsymbol{\theta}}_b = \mathbf{L}(\boldsymbol{\theta}_b)\boldsymbol{\omega}$$

can be defined as

$$\mathbf{L}(\boldsymbol{\theta}_b) = \begin{pmatrix} 1 & \sin \phi_b \tan \theta_b & \cos \phi_b \tan \theta_b \\ 0 & \cos \phi_b & -\sin \phi_b \\ 0 & \sin \phi_b \sec \theta_b & \cos \phi_b \sec \theta_b \end{pmatrix}$$

By combining $\mathbf{R}$ and $\mathbf{L}$, the kinematic transformation matrix can be defined as

$$\mathbf{J} = \begin{pmatrix} \mathbf{R}(\boldsymbol{\theta}_b) & \mathbf{0} \\ \mathbf{0} & \mathbf{L}(\boldsymbol{\theta}_b) \end{pmatrix} \quad (2.1)$$

which can be used to express the following relation

$$\dot{\mathbf{q}} = \mathbf{J}(\boldsymbol{\theta}_b)\boldsymbol{\nu} \quad (2.2)$$

2.1.2 Kinetics

As opposed to Newtonian mechanics, where equations of motion are derived by relating external forces and moments to the time rate of change of the translational and angular momenta, Lagrangian mechanics generates equations of motion based on principles of work and energy through the application of Hamilton's principle to the time integral of the system Lagrangian [38]. By defining the Lagrangian as $\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t) = T - V$, where $\mathbf{q}$ is a set of n independent generalized coordinates, $T(\mathbf{q}, \dot{\mathbf{q}}, t)$ is the system kinetic energy and $V(\mathbf{q}, t)$ is the system potential energy, Hamilton's principle is satisfied by the set of n equations known as the Euler-Lagrange equations

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} \right) - \frac{\partial \mathcal{L}}{\partial \mathbf{q}} = 0 \quad (2.3)$$

Nonconservative forces, such as forces due to viscosity, can be accounted for by incorporating the generalized forces $\mathbf{Q}$, leading to the equations of motion

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} \right) - \frac{\partial \mathcal{L}}{\partial \mathbf{q}} = \mathbf{Q} \quad (2.4)$$

The Euler-Lagrange equations (2.4) can be modified to accomodate quasivelocities such as the body velocities $\boldsymbol{\nu}$ [39], yielding the Boltzmann-Hamel equations, presented here in matrix-vector form [40],

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \boldsymbol{\nu}} + \mathbf{G}(\boldsymbol{\nu}) \frac{\partial \mathcal{L}}{\partial \boldsymbol{\nu}} - \mathbf{J}(\mathbf{q})^T \frac{\partial \mathcal{L}}{\partial \mathbf{q}} = \mathbf{J}(\mathbf{q})^T \mathbf{Q} \quad (2.5)$$

where

$$G_{ij} = \sum_{k=1}^n \sum_{l=1}^n \sum_{m=1}^n J_{ki} J_{lm} \nu_m \left(\frac{\partial J_{kj}^{-T}}{\partial q_l} - \frac{\partial J_{lj}^{-T}}{\partial q_k} \right)$$

Here, J_{ij} is the i, j element of the transformation matrix $\mathbf{J}$. For the chosen kinematic transformation matrix $\mathbf{J}$ in (2.1),

$$\mathbf{G}(\boldsymbol{\nu}) = \begin{pmatrix} \hat{\boldsymbol{\omega}} & \mathbf{0} \\ \hat{\mathbf{v}} & \hat{\boldsymbol{\omega}} \end{pmatrix} \quad (2.6)$$

where the hat operator $\hat{\cdot}$ represents the cross product equivalent matrix of a vector satisfying $\hat{\mathbf{a}}\mathbf{b} = \mathbf{a} \times \mathbf{b}$ for vectors $\mathbf{a}$ and $\mathbf{b}$.

2.1.3 Rigid Body Equations of Motion

The equations of motion for a rigid body can be derived by applying the process in the previous section. If we define the rigid body mass matrix $\mathbf{M}_b$ with respect to the body-fixed frame, the rigid body kinetic energy is then

$$T_b = \frac{1}{2} \boldsymbol{\nu}^T \mathbf{M}_b \boldsymbol{\nu} \quad (2.7)$$

and the potential energy, when only gravitational forces are considered, is

$$V_b = -mg \bar{\mathbf{x}}_{\text{cm}} \cdot (\mathbf{R}^T \mathbf{i}_3) \quad (2.8)$$

Here, m is the rigid body mass, $\bar{\mathbf{x}}_{\text{cm}}$ is the location of the center of mass in the body-fixed frame and $\mathbf{i}_3$ is the unit vector aligned with the z -axis in the earth-fixed frame. Therefore, the quantity $\mathbf{R}^T \mathbf{i}_3$ is the unit vector defining the direction of gravity in the body-fixed frame.

Therefore, the rigid body Lagrangian is

$$\mathcal{L}_b = \frac{1}{2} \boldsymbol{\nu}^T \mathbf{M}_b \boldsymbol{\nu} + mg \bar{\mathbf{x}}_{\text{cm}} \cdot (\mathbf{R}^T \mathbf{i}_3) \quad (2.9)$$

Now, the equations of motion for a rigid body can be derived using the rigid body Lagrangian (2.9) and the Boltzmann-Hamel equations (2.5). The terms on the left-hand side of (2.5) are

$$\begin{aligned}\frac{d}{dt} \frac{\partial \mathcal{L}_b}{\partial \boldsymbol{\nu}} &= \mathbf{M}_b \dot{\boldsymbol{\nu}} \\ \mathbf{G}(\boldsymbol{\nu}) \frac{\partial \mathcal{L}_b}{\partial \boldsymbol{\nu}} &= \mathbf{G} \mathbf{M}_b \boldsymbol{\nu} \\ \mathbf{J}(\mathbf{q})^T \frac{\partial \mathcal{L}_b}{\partial \mathbf{q}} &= \begin{pmatrix} 0 \\ mg \bar{\mathbf{x}}_{\text{cm}} \times (\mathbf{R}^T \mathbf{i}_3) \end{pmatrix}\end{aligned}$$

By substituting the right-hand side of (2.5) as $\mathbf{F}$, the vector of exogenous forces and moments, the equations of motion for a rigid body can be written as

$$\mathbf{M}_b \dot{\boldsymbol{\nu}} + \mathbf{G} \mathbf{M}_b \boldsymbol{\nu} - \begin{pmatrix} 0 \\ mg \bar{\mathbf{x}}_{\text{cm}} \times (\mathbf{R}^T \mathbf{i}_3) \end{pmatrix} = \mathbf{F} \quad (2.10)$$

2.2 Fluid Mechanics

A fluid system can be modeled by studying fluid statics, which considers fluids at rest, and fluid dynamics, which considers fluids in motion. Fluid dynamics is a mathematically complex field, and numerical methods are often required to solve the governing equations. Three fundamental laws govern the motion of a fluid: conservation of mass, conservation of momentum and conservation of energy. These laws are applied to potential flow theory to derive the model in [1], with the expectation that some of the the simplifying assumptions due to potential flow theory can be accounted for separately, as an addition to the original potential flow model.

2.2.1 Potential Flow Theory

Potential flow is an idealization of fluid dynamics where the flow is assumed to be inviscid, incompressible and irrotational. We can define a space and time-dependent potential function $\phi(\mathbf{x}, t)$, as a continuous function that satisfies the laws of conservation of mass, conservation of momentum and conservation of energy. The Eulerian flow velocity $\mathbf{V}_f(\mathbf{x})$ can be expressed as the gradient of the potential function.

$$\mathbf{V}_f(\mathbf{x}) = \nabla \phi(\mathbf{x}) \quad (2.11)$$

This expression of the flow velocity and the assumptions for potential flow are applied to the laws of conservation to yield the governing equations, described in detail in [41, 42], and are used in the model derivation by Battista [1].

Conservation of Mass

For an incompressible fluid, the change in mass in a differential volume of fluid is zero. That is,

$$\frac{D}{Dt}(\rho dV) = 0$$

where $\frac{D}{Dt}$ is the material derivative and dV is an infinitesimal volume of the fluid. Evaluating the material derivative and substituting the fluid velocity with the velocity potential expression in (2.11),

$$-\rho \nabla \cdot (\nabla \phi) = 0$$

Therefore, conservation of mass in potential flow reduces to Laplace's equation:

$$\nabla^2 \phi = 0 \quad (2.12)$$

Conservation of Momentum

Conservation of momentum can be expressed mathematically by applying Newton's second law to a differential element of fluid so that the material change in momentum is equal to the sum of internal and external forces $\mathbf{F}$:

$$\frac{D}{Dt}(\rho dV \mathbf{V}_f) = \mathbf{F}$$

This relation yields the Navier-Stokes equations. Applying the assumptions of the fluid being inviscid, incompressible and irrotational leads to the following momentum equations:

$$\rho \left(\frac{\partial \mathbf{V}_f}{\partial t} + \frac{1}{2} \nabla (\mathbf{V}_f \cdot \mathbf{V}_f) \right) = \nabla(\rho g z) - \nabla p$$

where p is fluid pressure. Now, substituting the fluid velocity with the velocity potential expression in (2.11),

$$\nabla \left(\rho \frac{\partial \phi}{\partial t} + \frac{1}{2} \rho \nabla \phi \cdot \nabla \phi - \rho g z + \nabla p \right) = 0$$

This expression can be integrated to yield the unsteady Bernoulli equation,

$$\rho \frac{\partial \phi}{\partial t} + \frac{1}{2} \rho \nabla \phi \cdot \nabla \phi - \rho g z + p = C(t) \quad (2.13)$$

The constant can be absorbed into the velocity potential [43] to yield

$$\rho \frac{\partial \phi}{\partial t} + \frac{\rho}{2} \nabla \phi \cdot \nabla \phi - \rho g z + p = 0 \quad (2.14)$$

Conservation of Energy

The total energy of an infinitesimal fluid volume can be expressed as

$$E = \rho dV \left(\frac{1}{2} \mathbf{V}_f \cdot \mathbf{V}_f + e \right) \quad (2.15)$$

where e is the specific internal fluid energy, which accounts for thermal energy within the fluid. Conservation of energy can be expressed as the material change in fluid energy being equal to the rate of work done P on the fluid:

$$\frac{DE}{Dt} = P \quad (2.16)$$

By applying equation (2.15) to the left-hand side of (2.20), and substituting the fluid velocity with the velocity potential expression in (2.11),

$$\frac{DE}{Dt} = \rho dV \left(\nabla \left(\frac{\partial \phi}{\partial t} + \frac{1}{2} \nabla \phi \cdot \nabla \phi \right) \cdot \mathbf{V}_f + \rho \frac{\partial e}{\partial t} + \mathbf{V}_f \cdot \nabla e \right) \quad (2.17)$$

On the other hand, the rate of work done on the fluid in equation (2.20) can be expressed using the fundamental definition of rate of work due to body and pressure forces [42]:

$$P = \rho dV \left(\nabla(\rho g z) \cdot \mathbf{V}_f - \nabla p \cdot \mathbf{V}_f - p \nabla \cdot \mathbf{V}_f \right) \quad (2.18)$$

Combining equations (2.17) and (2.18) and applying equation (2.13) yields a much simpler expression:

$$\rho \left(\frac{\partial e}{\partial t} + \mathbf{V}_f \cdot \nabla e \right) = -p \nabla \cdot \mathbf{V}_f \quad (2.19)$$

Substituting the fluid velocity with the velocity potential expression in (2.11) and applying

equation (2.12) yields

$$\frac{De}{Dt} = 0 \quad (2.20)$$

This equation indicates that internal energy is constant in potential flow. The conservation laws described thus far indicate that a set of initial values and boundary conditions need to be specified to solve for an unsteady potential flow problem described by the partial differential equations (2.12), (2.14) and (2.20).

Boundary Conditions

When considering a rigid body moving in a semi-infinite volume of fluid, where the sides and bottom of the volume of fluid are infinitely far away, conditions are set at the boundaries of the body surface and the fluid free surface. The zero-penetration condition must be satisfied at the body surface, while the freely deforming free surface must also satisfy a dynamic boundary condition.

First looking at the rigid body surface, the impermeable boundary must satisfy the zero-penetration condition, and therefore the fluid velocity normal to the body must be equal to the normal velocity at each point on the body surface. Let $\mathbf{r}$ represent the position of a point on the body surface relative to the origin of the body-fixed frame, and let $\mathbf{n}$ represent the corresponding unit normal vector directed outward from the fluid. Then, the rigid body boundary condition can be expressed as

$$(\nabla\phi \cdot \mathbf{n})|_{\mathcal{B}} = (\mathbf{v} + \boldsymbol{\omega} \times \mathbf{r}) \cdot \mathbf{n}$$

By defining a body geometry vector

$$\mathbf{b} = \begin{pmatrix} \mathbf{n} \\ \mathbf{r} \times \mathbf{n} \end{pmatrix} \quad (2.21)$$

the rigid body boundary condition can be rewritten as

$$(\nabla\phi \cdot \mathbf{n})|_{\mathcal{B}} = \mathbf{b} \cdot \boldsymbol{\nu} \quad (2.22)$$

Moving on to the kinematic and dynamic free surface boundary conditions, the kinematic free surface boundary condition states that a fluid particle on the free surface remains there always. In addition, the vertical velocity of the fluid particles on the free surface and the vertical velocity of the surface itself are equal. This statement can be written as

$$\left[\frac{D}{Dt}(z - \eta) \right] \Big|_{z=\eta} = \left[\frac{\partial\phi}{\partial z} - \frac{\partial\eta}{\partial t} - \frac{\partial\phi}{\partial x} \frac{\partial\eta}{\partial x} - \frac{\partial\phi}{\partial y} \frac{\partial\eta}{\partial y} \right] \Big|_{z=\eta} = 0$$

where η is the free surface elevation.

The dynamic free surface boundary condition states that the surface pressure of the fluid is constant and equal to the atmospheric pressure p_a . This can be expressed as

$$\left[-\frac{p - p_a}{\rho} \right] \Big|_{z=\eta} = \left[\frac{\partial\phi}{\partial t} + \frac{1}{2} \nabla\phi \cdot \nabla\phi - gz \right] \Big|_{z=\eta} = 0$$

Details on derivation of the free surface boundary condition can be found in [41].

To simplify the problem, small wave heights, and consequently small wave slopes, at the surface $\mathcal{S}$ can be assumed in order to linearize the free surface boundary conditions. Taking the first order approximation of $\mathcal{S}$ as the undisturbed free surface $\mathcal{S}_0$ leads to adequate

predictions in deep water, even in the presence of waves with slopes far from being vanishingly small [44]. The resulting linearized kinematic and dynamic free surface boundary conditions can be written as

$$\left. \frac{\partial \phi}{\partial z} \right|_{S_0} = \left. \frac{\partial \eta}{\partial t} \right|_{S_0} \quad (2.23)$$

$$\left. \frac{\partial \phi}{\partial t} \right|_{S_0} = g \eta|_{S_0} \quad (2.24)$$

These boundary conditions can be combined by taking the time derivative of equation (2.24) and substituting it into equation (2.23).

$$\left. \frac{\partial^2 \phi}{\partial t^2} \right|_{S_0} - g \left. \frac{\partial \phi}{\partial z} \right|_{S_0} = 0 \quad (2.25)$$

2.3 Lagrangian Mechanics for a Body Moving in an Ideal Fluid

As described in Subsection 2.1.2, the system Lagrangian $\mathcal{L} = T - V$ can be determined by defining the system kinetic energy T and the system potential energy V . Now, if we consider a system composed of a rigid body and the ambient fluid, then the total system Lagrangian $\mathcal{L}$ can be defined as the sum of the rigid body Lagrangian $\mathcal{L}_b$ and the fluid Lagrangian $\mathcal{L}_f$,

$$\mathcal{L} = \mathcal{L}_b + \mathcal{L}_f \quad (2.26)$$

and these two Lagrangians can be written as

$$\mathcal{L}_b = T_b - V_b \quad (2.27)$$

$$\mathcal{L}_{\mathcal{F}} = T_{\mathcal{F}} - V_{\mathcal{F}} \quad (2.28)$$

where T_b and $T_{\mathcal{F}}$ are the rigid body and fluid kinetic energies, respectively, and V_b and $V_{\mathcal{F}}$ are the rigid body and fluid potential energies, respectively. The equations of motion for the rigid body can be obtained by substituting the decomposed Lagrangian (2.26) into the Boltzmann-Hamel equations (2.5). Now, the equations of motion can be derived by correctly identifying the energy functions T and V for the rigid body and the fluid.

Kinetic Energy of Fluid with a Free Surface

The kinetic energy within a fluid volume $\mathcal{F}$ is

$$T_{\mathcal{F}} = \iiint_{\mathcal{F}} \frac{1}{2} \rho \mathbf{V}_f \cdot \mathbf{V}_f dV$$

where we can replace $\mathbf{V}_f$ with $\nabla\phi$ (2.11) and rewrite the kinetic energy as

$$T_{\mathcal{F}} = \iiint_{\mathcal{F}} \frac{1}{2} \rho \nabla\phi \cdot \nabla\phi dV \quad (2.29)$$

The following vector calculus identity can be applied to equation (2.29):

$$\begin{aligned} \nabla \cdot (\phi \nabla \phi) &= \nabla\phi \cdot \nabla\phi + \phi \nabla^2 \phi \\ &= \nabla\phi \cdot \nabla\phi \end{aligned} \quad (2.30)$$

where the Laplacian $\nabla^2\phi$ vanishes due to conservation of mass (2.12).

Now, the divergence theorem can be applied to the resulting equation to yield the following

form of the kinetic energy of the fluid.

$$\begin{aligned}
 T_{\mathcal{F}} &= \iiint_{\mathcal{F}} \frac{1}{2} \rho \nabla \cdot (\phi \nabla \phi) dV \\
 &= \frac{1}{2} \rho \iint_{\mathcal{B}} (\phi \nabla \phi) \cdot \mathbf{n} dS + \frac{1}{2} \rho \iint_{\mathcal{S}} (\phi \nabla \phi) \cdot \mathbf{n} dS
 \end{aligned} \tag{2.31}$$

This form indicates that the kinetic energy within the fluid can be expressed as the flux of the quantity $\phi \nabla \phi$ normal to the body surface $\mathcal{B}$ and the free surface $\mathcal{S}$.

Potential Energy of Fluid with a Free Surface

As described in Section 2.2.1, only body forces due to gravity are considered, and in addition, thermal energy contributions may be neglected. Therefore, only gravitational potential energy is considered, and may be expressed as

$$V_{\mathcal{F}} = - \iiint_{\mathcal{F}} \rho g z dV \tag{2.32}$$

The potential energy within the fluid can then be expressed in terms related to the rigid body and the free surface [1],

$$V_{\mathcal{F}} = \rho g \mathcal{V} z_b + \iint_{\mathcal{S}} \frac{1}{2} \rho g z^2 dS \tag{2.33}$$

where $\mathcal{V}$ is the volume of fluid displaced by the rigid body.

General Lagrangian Expressions

Using the integral forms of the fluid kinetic energy (2.29) and the fluid potential energy (2.32), the fluid Lagrangian can be written as

$$\mathcal{L}_{\mathcal{F}} = \iiint_{\mathcal{F}} \left[\frac{1}{2} \rho \nabla \phi \cdot \nabla \phi + \rho g z \right] dV \quad (2.34)$$

The fluid Lagrangian can also be expressed in terms of surface integrals over the rigid body and the free surface:

$$\mathcal{L}_{\mathcal{F}} = \frac{1}{2} \rho \iint_{\mathcal{B}} (\phi \nabla \phi) \cdot \mathbf{n} dS + \frac{1}{2} \rho \iint_{\mathcal{S}} (\phi \nabla \phi) \cdot \mathbf{n} dS - \frac{1}{2} \rho \iint_{\mathcal{S}} g z^2 dS - \rho g V z_b \quad (2.35)$$

2.4 A Lagrangian Nonlinear Maneuvering (LNM) Model

In this section, the nonlinear model for a surface-affected marine craft is derived. Hydrodynamic effects due to incident waves are incorporated in the subsequent section. The following assumptions are applied to the Lagrangian nonlinear maneuvering (LNM) model derivation:

1. The conditions of potential flow theory (incompressible, inviscid and irrotational flow) hold.
2. The linearized free surface boundary conditions (2.23 and 2.24) hold.

2.4.1 Composition of the Total System Lagrangian

The fluid Lagrangian in the total system Lagrangian shown in (2.26) can be decomposed into the fluid body Lagrangian $\mathcal{L}_{\mathcal{B}}$ and the free surface Lagrangian $\mathcal{L}_{\mathcal{S}}$, and the total system

Lagrangian can be rewritten as

$$\mathcal{L} = \mathcal{L}_b + \mathcal{L}_B + \mathcal{L}_S \quad (2.36)$$

The following sections describe each of these components of the total Lagrangian, which are expressed as functions of the velocity potential.

Rigid Body Lagrangian

As described in (2.27), the rigid body Lagrangian is defined as the rigid body kinetic energy T_b minus the rigid body potential energy V_b . As shown in (2.7), the rigid body kinetic energy can be written as

$$T_b = \frac{1}{2} \boldsymbol{\nu}^T \mathbf{M}_b \boldsymbol{\nu} \quad (2.37)$$

As for the rigid body potential energy, a term accounting for the torque induced by the misaligned center of mass and center of buoyancy is added to the expression in (2.8) and is written as

$$V_b = -mgz_b - mgr_{cm} \cdot (\mathbf{R}^T \mathbf{i}_3) \quad (2.38)$$

We now have an expression for the rigid body Lagrangian

$$\mathcal{L}_b = \frac{1}{2} \boldsymbol{\nu}^T \mathbf{M}_b \boldsymbol{\nu} + mgz_b + mgr_{cm} \cdot (\mathbf{R}^T \mathbf{i}_3) \quad (2.39)$$

Fluid Body Lagrangian

The fluid Lagrangian in (2.35) can be reorganized in terms of surface integrals over the rigid body $\mathcal{B}$ and the free surface $\mathcal{S}$:

$$\mathcal{L}_F = \iint_{\mathcal{B}} \left(\frac{1}{2} \rho \phi \nabla \phi + \frac{1}{2} \rho g z^2 \mathbf{i}_3 \right) \cdot \mathbf{n} \, dS + \iint_{\mathcal{S}} \left(\frac{1}{2} \rho \phi \nabla \phi + \frac{1}{2} \rho g z^2 \mathbf{i}_3 \right) \cdot \mathbf{n} \, dS \quad (2.40)$$

From this expression of the fluid Lagrangian, the fluid body Lagrangian can be defined as the first term in the equation, i.e., the surface integral over the rigid body $\mathcal{B}$:

$$\mathcal{L}_{\mathcal{B}} = \iint_{\mathcal{B}} \left(\frac{1}{2} \rho \phi \nabla \phi + \frac{1}{2} \rho g z^2 \mathbf{i}_3 \right) \cdot \mathbf{n} dS \quad (2.41)$$

In the fluid body Lagrangian expressed above, the first term of the integrand can be simplified by applying the rigid body boundary condition (2.22). In addition, the second term of the integrand can be replaced with the first term in the fluid potential energy in (2.33). Therefore, the fluid body Lagrangian can be expressed as

$$\mathcal{L}_B = \frac{1}{2} \rho \iint_{\mathcal{B}} \phi \mathbf{b}^T dS \boldsymbol{\nu} - \rho g V z_b \quad (2.42)$$

The velocity potential in the fluid body Lagrangian expression can be decomposed into the following form:

$$\phi(\mathbf{q}, \boldsymbol{\nu}, t) = \phi_K(\boldsymbol{\nu}) + \phi_s(\mathbf{q}, \boldsymbol{\nu}) + \phi_m(\mathbf{q}, \boldsymbol{\nu}, t) \quad (2.43)$$

which combines approaches suggested by Kirchhoff [11] and Cummins [45]. Here, Kirchhoff's potential ϕ_K is augmented by a state-dependent free-surface potential ϕ_s and a state- and time-dependent memory effect potential ϕ_m . The potentials can be expressed in terms of normalized potential vectors $\boldsymbol{\varphi}$, where the elements of $\boldsymbol{\varphi}$ represent the velocity potential due to a body motion with unit velocity in the respective mode. Now, the decomposed velocity potential can be expressed as

$$\phi(\mathbf{q}(t), \boldsymbol{\nu}(t), t) = (\boldsymbol{\varphi}_K + \boldsymbol{\varphi}_s(\mathbf{q}(t))) \cdot \boldsymbol{\nu}(t) + \int_0^t \boldsymbol{\varphi}_m(\mathbf{q}(\tau), t - \tau) \cdot \boldsymbol{\nu}(\tau) d\tau \quad (2.44)$$

Substituting equation (2.44) into the fluid body Lagrangian in (2.42) yields

$$\mathcal{L}_{\mathcal{B}} = \frac{1}{2}\rho \iint_{\mathcal{B}} \left[(\boldsymbol{\varphi}_{\mathcal{K}} + \boldsymbol{\varphi}_{\mathcal{s}}(\mathbf{q})) \cdot \boldsymbol{\nu} + \int_0^t \boldsymbol{\varphi}_{\mathcal{m}}(\mathbf{q}(\tau), t - \tau) \cdot \boldsymbol{\nu}(\tau) d\tau \right] \mathbf{b}^T dS \boldsymbol{\nu} - \rho g V z_{\mathbf{b}} \quad (2.45)$$

Now, we can define the added mass matrix as

$$\mathbf{M}_{\mathcal{B}} = \rho \iint_{\mathcal{B}} \boldsymbol{\varphi}_{\mathcal{K}} \mathbf{b}^T dS \quad (2.46)$$

and also define the surface-perturbed added mass matrix as

$$\Delta \mathbf{M}_{\mathcal{B}}(\mathbf{q}) = \frac{1}{2} (\mathbf{M}_2(\mathbf{q}) + \mathbf{M}_2^T(\mathbf{q})) \quad (2.47)$$

where

$$\mathbf{M}_2(\mathbf{q}) = \rho \iint_{\mathcal{B}} \boldsymbol{\varphi}(\mathbf{q})_{\mathcal{s}} \mathbf{b}^T dS \quad (2.48)$$

We can also define the fluid kernel matrix

$$\mathbf{K}_{\mathcal{B}}(\mathbf{q}(\tau), t - \tau) = \frac{\rho}{2} \iint_{\mathcal{B}} \mathbf{b} \boldsymbol{\varphi}_{\mathcal{m}}^T(\mathbf{q}(\tau), t - \tau) dS \quad (2.49)$$

While the deeply submerged added mass matrix $\mathbf{M}_{\mathcal{B}}$ accounts for the effects of an unbounded fluid volume, the surface-perturbed added mass matrix $\Delta \mathbf{M}_{\mathcal{B}}$ accounts for the asymmetry in the fluid volume due to the free surface boundary and the fluid kernel matrix $\mathbf{K}_{\mathcal{B}}$ accounts for the past motions of the vessel. Substituting the newly defined matrices into the fluid body Lagrangian expression in (2.45) yields

$$\mathcal{L}_{\mathcal{B}} = \frac{1}{2} \boldsymbol{\nu}^T (\mathbf{M}_{\mathcal{B}} + \Delta \mathbf{M}_{\mathcal{B}}(\mathbf{q})) \boldsymbol{\nu} + \boldsymbol{\nu}^T \int_0^t \mathbf{K}_{\mathcal{B}}(\mathbf{q}(\tau), t - \tau) \boldsymbol{\nu}(\tau) d\tau - \rho g V z_{\mathbf{b}} \quad (2.50)$$

Free Surface Lagrangian

The second term in the fluid Lagrangian expression in (2.4.1), which is the surface integral over the free surface, can be defined as the free surface Lagrangian:

$$\mathcal{L}_S = \iint_S \left(\frac{1}{2} \rho \phi \nabla \phi + \frac{1}{2} \rho g z^2 \mathbf{i}_3 \right) \cdot \mathbf{n} dS \quad (2.51)$$

The integral from the free surface Lagrangian can be evaluated over the undisturbed free surface $\mathcal{S}_0$ when applying the linearization assumption. The resulting linearized free surface Lagrangian can be written as

$$\mathcal{L}_{S_0} = \iint_{S_0} \left(\frac{1}{2} \rho \phi \nabla \phi + \frac{1}{2} \rho g \eta^2 \mathbf{i}_3 \right) \cdot \mathbf{n} dS \quad (2.52)$$

For brevity, hereon, the linearized free surface Lagrangian will be referred to as the free surface Lagrangian. The linearized dynamic free surface boundary condition in (2.24) can be applied to the free surface Lagrangian in (2.52) to yield the following expression:

$$\mathcal{L}_{S_0} = -\frac{\rho}{2} \iint_{S_0} \left[\phi \frac{\partial \phi}{\partial z} + \frac{1}{g} \left(\frac{\partial \phi}{\partial t} \right)^2 \right] dS$$

Using the following identity,

$$\frac{1}{2} \frac{\partial^2 \phi^2}{\partial t^2} = \left(\frac{\partial \phi}{\partial t} \right)^2 + \phi \frac{\partial^2 \phi}{\partial t^2}$$

the free surface Lagrangian becomes

$$\mathcal{L}_{S_0} = -\frac{\rho}{2} \iint_{S_0} \left[\frac{-\phi}{g} \left(\frac{\partial^2 \phi}{\partial t^2} - g \frac{\partial \phi}{\partial z} \right) + \frac{1}{2g} \frac{\partial^2 \phi^2}{\partial t^2} \right] dS$$

According to the combined linearized free surface boundary condition (2.25), the quantity

in the parenthesis is equal to zero. Therefore, the free surface Lagrangian can simply be written as

$$\mathcal{L}_{S_0} = -\frac{\rho}{4g} \iint_{S_0} \frac{\partial^2 \phi^2}{\partial t^2} dS \quad (2.53)$$

The decomposed velocity potential in (2.44) can be substituted into the above free surface Lagrangian to yield the following form:

$$\begin{aligned} \mathcal{L}_{S_0} = & -\frac{\rho}{4g} \iint_{S_0} \left[\boldsymbol{\nu}^T (\boldsymbol{\varphi}_K \boldsymbol{\varphi}_K^T + \boldsymbol{\varphi}_K \boldsymbol{\varphi}_s^T(\mathbf{q}) + \boldsymbol{\varphi}_s \boldsymbol{\varphi}_K^T + \boldsymbol{\varphi}_s(\mathbf{q}) \boldsymbol{\varphi}_s^T(\mathbf{q})) \boldsymbol{\nu} \right. \\ & + 2\boldsymbol{\nu}^T (\boldsymbol{\varphi}_K + \boldsymbol{\varphi}_s(\mathbf{q})) \int_0^t \boldsymbol{\varphi}_m^T(\mathbf{q}(\tau), t - \tau) \boldsymbol{\nu}(\tau) d\tau \\ & \left. + \left(\int_0^t \boldsymbol{\varphi}_m^T(\mathbf{q}(\tau), t - \tau) \boldsymbol{\nu}(\tau) d\tau \right)^2 \right] dS \end{aligned} \quad (2.54)$$

We can now define the following elements:

$$\begin{aligned} \mathbf{A}_{S_0} &= -\frac{\rho}{2g} \iint_{S_0} \boldsymbol{\varphi}_K \boldsymbol{\varphi}_K^T dS \\ \mathbf{B}_{S_0}(\mathbf{q}) &= -\frac{\rho}{2g} \iint_{S_0} (\boldsymbol{\varphi}_s(\mathbf{q}) \boldsymbol{\varphi}_K^T + \boldsymbol{\varphi}_K \boldsymbol{\varphi}_s^T(\mathbf{q})) dS \\ \mathbf{C}_{S_0}(\mathbf{q}) &= -\frac{\rho}{2g} \iint_{S_0} \boldsymbol{\varphi}_s(\mathbf{q}) \boldsymbol{\varphi}_s^T(\mathbf{q}) dS \\ \mathbf{D}_{S_0}(\mathbf{q}, \mathbf{q}(\tau), t - \tau) &= -\frac{\rho}{2g} \iint_{S_0} [\boldsymbol{\varphi}_K + \boldsymbol{\varphi}_s(\mathbf{q})] \boldsymbol{\varphi}_m^T(\mathbf{q}(\tau), t - \tau) dS \\ \varepsilon_{S_0}(t) &= \iint_{S_0} \phi_m(t)^2 dS \end{aligned}$$

These terms can be substituted into the free surface Lagrangian in (2.54) to give:

$$\mathcal{L}_{S_0} = \frac{\partial^2}{\partial t^2} \left[\frac{1}{2} \boldsymbol{\nu}^T [\mathbf{A}_{S_0} + \mathbf{B}_{S_0}(\mathbf{q}) + \mathbf{C}_{S_0}(\mathbf{q})] \boldsymbol{\nu} + \int_0^t \boldsymbol{\nu}^T \mathbf{D}_{S_0}(\mathbf{q}, \mathbf{q}(\tau), t - \tau) \boldsymbol{\nu}(\tau) d\tau + \varepsilon_{S_0}(t) \right]$$

Now, the free surface added mass matrix can be defined as

$$\mathbf{M}_{S_0}(\mathbf{q}, \boldsymbol{\nu}) = \frac{\partial^2}{\partial t^2} [\mathbf{A}_{S_0} + \mathbf{B}_{S_0}(\mathbf{q}) + \mathbf{C}_{S_0}(\mathbf{q})]$$

If the following terms are additionally defined,

$$\begin{aligned} \mathbf{K}_{S_{01}}(\mathbf{q}, \boldsymbol{\nu}, \mathbf{q}(\tau), t - \tau) &= \frac{\partial^2}{\partial t^2} \mathbf{D}_{S_0}(\mathbf{q}, \mathbf{q}(\tau), t - \tau) \\ &= \frac{\partial^2 \mathbf{D}_{S_0}(\mathbf{q}, \mathbf{q}(\tau), t - \tau)}{\partial (t - \tau)^2} \\ &\quad + \sum_{i=1}^6 \sum_{j=1}^6 \left[\frac{\partial^2 \mathbf{D}_{S_0}(\mathbf{q}, \mathbf{q}(\tau), t - \tau)}{\partial q_i \partial q_j} (\mathbf{J}_i^T \boldsymbol{\nu}) (\mathbf{J}_j^T \boldsymbol{\nu}) \right. \\ &\quad \left. + \frac{\partial \mathbf{D}_{S_0}(\mathbf{q}, \mathbf{q}(\tau), t - \tau)}{\partial q_i} \frac{\partial \mathbf{J}_i^T}{\partial q_j} \boldsymbol{\nu} (\mathbf{J}_j^T \boldsymbol{\nu}) \right] \\ \gamma_{S_0}(\mathbf{q}, \boldsymbol{\nu}, t) &= \frac{\partial^2 \varepsilon_s(t)}{\partial t^2} \\ &= 2 \iint_{S_0} \left[\left(\frac{\partial \phi_m(t)}{\partial t} \right)^2 + \phi_m(t) \int_0^t \frac{\partial^2 \boldsymbol{\varphi}_m^T(\mathbf{q}(\tau), t - \tau)}{\partial t^2} \boldsymbol{\nu}(\tau) d\tau \right. \\ &\quad \left. + \phi_m(t) \frac{\partial \boldsymbol{\varphi}_m^T(\mathbf{q}, 0)}{\partial t} \boldsymbol{\nu} \right] dS \end{aligned} \quad (2.55)$$

then the free surface Lagrangian can be rewritten as

$$\mathcal{L}_{S_0} = \frac{1}{2} \boldsymbol{\nu}^T \mathbf{M}_{S_0}(\mathbf{q}, \boldsymbol{\nu}) \boldsymbol{\nu} + \boldsymbol{\nu}^T \int_0^t \mathbf{K}_{S_{01}}(\mathbf{q}, \boldsymbol{\nu}, \mathbf{q}(\tau), t - \tau) \boldsymbol{\nu}(\tau) d\tau + \gamma_{S_0}(t) \quad (2.56)$$

Removing the first term in the second row of (2.55), which is a scalar function of t , does not impact the system dynamics since the resulting equations of motion remain the same.

Therefore, term γ_{S_0} can be redefined as

$$\gamma_{S_0}(\mathbf{q}, \boldsymbol{\nu}, t) = 2 \iint_{S_0} \boldsymbol{\nu}^T \frac{\partial \boldsymbol{\varphi}_m(\mathbf{q}, 0)}{\partial t} \int_0^t \boldsymbol{\varphi}_m^T(\mathbf{q}(\tau), t - \tau) \boldsymbol{\nu}(\tau) d\tau dS$$

By defining a matrix

$$\mathbf{K}_{S_{02}}(\mathbf{q}, \mathbf{q}(\tau), t - \tau) = 2 \iint_{S_0} \frac{\partial \boldsymbol{\varphi}_m(\mathbf{q}, 0)}{\partial t} \boldsymbol{\varphi}_m^T(\mathbf{q}(\tau), t - \tau) dS$$

the term γ_{S_0} can be expressed as

$$\gamma_{S_0}(\mathbf{q}, \boldsymbol{\nu}, t) = \boldsymbol{\nu}^T \int_0^t \mathbf{K}_{S_{02}}(\mathbf{q}, \mathbf{q}(\tau), t - \tau) \boldsymbol{\nu}(\tau) d\tau$$

and by defining the complete free surface kernel matrix $\mathbf{K}_{S_0}$:

$$\mathbf{K}_{S_0}(\mathbf{q}, \boldsymbol{\nu}, \mathbf{q}(\tau), t - \tau) = \mathbf{K}_{S_{01}}(\mathbf{q}, \boldsymbol{\nu}, \mathbf{q}(\tau), t - \tau) + \mathbf{K}_{S_{02}}(\mathbf{q}, \mathbf{q}(\tau), t - \tau)$$

The free surface Lagrangian becomes

$$\mathcal{L}_{S_0} = \frac{1}{2} \boldsymbol{\nu}^T \mathbf{M}_{S_0}(\mathbf{q}, \boldsymbol{\nu}) \boldsymbol{\nu} + \boldsymbol{\nu}^T \int_0^t \mathbf{K}_{S_0}(\mathbf{q}, \boldsymbol{\nu}, \mathbf{q}(\tau), t - \tau) \boldsymbol{\nu}(\tau) d\tau \quad (2.57)$$

Complete System Lagrangian

Combining the rigid body, fluid body and free surface Lagrangians yields the following form of the complete system Lagrangian of a rigid body moving below the surface in a semi-infinite

volume of ideal fluid:

$$\begin{aligned}\mathcal{L} = & \frac{1}{2} \boldsymbol{\nu}^T (\mathbf{M}_b + \mathbf{M}_B + \Delta \mathbf{M}_B(\mathbf{q}) + \mathbf{M}_{S_0}(\mathbf{q}, \boldsymbol{\nu})) \boldsymbol{\nu} \\ & + \boldsymbol{\nu}^T \int_0^t [\mathbf{K}_B(\mathbf{q}(\tau), t - \tau) + \mathbf{K}_{S_0}(\mathbf{q}, \boldsymbol{\nu}, \mathbf{q}(\tau), t - \tau)] \boldsymbol{\nu}(\tau) d\tau \\ & + g(m_b - \rho V) z_b + m_b g \mathbf{r}_{cm} \cdot (\mathbf{R}^T \mathbf{i}_3)\end{aligned}\quad (2.58)$$

2.4.2 Near-Surface Maneuvering Equations of Motion

The equations of motion for a vessel moving in otherwise calm water can be obtained by applying the complete system Lagrangian from the previous subsection to the Boltzmann-Hamel equations (2.5). For the sake of simplicity, the vessel is assumed to be neutrally buoyant.

Contributions of the Rigid Body Lagrangian

We begin the derivation for the equations of motion by computing the terms in the Boltzmann-Hamel equations (2.5) for the rigid body Lagrangian (2.9). First, we compute

$$\begin{aligned}\frac{\partial \mathcal{L}_b}{\partial \boldsymbol{\nu}} &= \mathbf{M}_b \boldsymbol{\nu} \\ -\mathbf{J}^T \frac{\partial \mathcal{L}_b}{\partial \mathbf{q}} &= \begin{pmatrix} 0 \\ -mg \mathbf{r}_{cm} \times (\mathbf{R}^T \mathbf{i}_3) \end{pmatrix}\end{aligned}$$

Therefore, the rigid body contribution to the equations of motion is

$$\mathbf{M}_b \dot{\boldsymbol{\nu}} + \mathbf{G} \mathbf{M}_b \boldsymbol{\nu} + \begin{pmatrix} 0 \\ -mg \mathbf{r}_{cm} \times (\mathbf{R}^T \mathbf{i}_3) \end{pmatrix} \quad (2.59)$$

Contributions of the Fluid Body Lagrangian

Next, in order to apply the fluid body Lagrangian (2.50) to the Boltzmann-Hamel equations (2.5), we compute the following terms:

$$\begin{aligned}\frac{\partial \mathcal{L}_B}{\partial \boldsymbol{\nu}} &= (\mathbf{M}_B + \Delta \mathbf{M}_B(\mathbf{q})) \boldsymbol{\nu} + \int_0^t \mathbf{K}_B(\mathbf{q}(\tau), t - \tau) d\tau \\ \frac{d}{dt} \left(\frac{\partial \mathcal{L}_B}{\partial \boldsymbol{\nu}} \right) &= (\mathbf{M}_B + \Delta \mathbf{M}_B(\mathbf{q})) \dot{\boldsymbol{\nu}} + \left[\sum_{i=1}^6 (\mathbf{J}_i^T \boldsymbol{\nu}) \frac{\partial \Delta \mathbf{M}_B}{\partial q_i} \right] \boldsymbol{\nu} \\ &\quad + \int_0^t \frac{\partial (\mathbf{K}_B(\mathbf{q}(\tau), t - \tau))}{\partial (t - \tau)} \boldsymbol{\nu}(\tau) d\tau \\ \mathbf{J}^T \frac{\partial \mathcal{L}_B}{\partial \mathbf{q}} &= \frac{1}{2} \left[\sum_{i=1}^6 (\mathbf{J}_i \boldsymbol{\nu}^T) \frac{\partial \Delta \mathbf{M}_B}{\partial q_i} \right] \boldsymbol{\nu}\end{aligned}$$

where $\mathbf{J}_i$ is the i^{th} row of $\mathbf{J}$ and q_i is the i^{th} element of $\mathbf{q}$.

Therefore, the fluid body contribution to the equations of motions is

$$\begin{aligned}(\mathbf{M}_B + \Delta \mathbf{M}_B(\mathbf{q})) \dot{\boldsymbol{\nu}} &+ \mathbf{G}(\mathbf{M}_B + \Delta \mathbf{M}_B(\mathbf{q})) \boldsymbol{\nu} + \mathbf{G} \int_0^t \mathbf{K}_B(\mathbf{q}(\tau), t - \tau) d\tau \\ &+ \left[\sum_{i=1}^6 \left(\mathbf{J}_i^T \boldsymbol{\nu} \mathbf{I} - \frac{1}{2} \mathbf{J}_i \boldsymbol{\nu}^T \right) \frac{\partial \Delta \mathbf{M}_B}{\partial q_i} \right] \boldsymbol{\nu} + \int_0^t \frac{\partial (\mathbf{K}_B(\mathbf{q}(\tau), t - \tau))}{\partial (t - \tau)} \boldsymbol{\nu}(\tau) d\tau\end{aligned}\tag{2.60}$$

where $\mathbf{I}$ is a 6×6 identity matrix.

Contributions of the Free Surface Lagrangian

The terms in the Boltzmann-Hamel equations (2.5) are computed using the free surface Lagrangian in (2.57).

$$\begin{aligned}
\frac{\partial \mathcal{L}_{S_0}}{\partial \boldsymbol{\nu}} &= \mathbf{M}_{S_0} \boldsymbol{\nu} + \frac{1}{2} \left[\sum_{i=1}^6 (\mathbf{e}_i \boldsymbol{\nu}^T) \frac{\partial \mathbf{M}_{S_0}}{\partial \nu_i} \right] \boldsymbol{\nu} \\
&\quad + \int_0^t \mathbf{K}_{S_0}(\mathbf{q}, \boldsymbol{\nu}, \mathbf{q}(\tau), t - \tau) \boldsymbol{\nu}(\tau) d\tau \\
&\quad + \int_0^t \sum_{i=1}^6 (\mathbf{e}_i \boldsymbol{\nu}^T) \frac{\partial \mathbf{K}_{S_0}(\mathbf{q}, \boldsymbol{\nu}, \mathbf{q}(\tau), t - \tau)}{\partial \nu_i} \boldsymbol{\nu}(\tau) d\tau \\
\frac{d}{dt} \left(\frac{\partial \mathcal{L}_{S_0}}{\partial \boldsymbol{\nu}} \right) &= \\
&\quad \left[\mathbf{M}_{S_0} + \sum_{i=1}^6 \left((\mathbf{e}_i \boldsymbol{\nu}^T) \frac{\partial \mathbf{M}_{S_0}}{\partial \nu_i} + \frac{\partial \mathbf{M}_{S_0}}{\partial \nu_i} (\boldsymbol{\nu} \mathbf{e}_i^T) \right) + \frac{1}{2} \sum_{i=1}^6 \sum_{j=1}^6 (\mathbf{e}_i \boldsymbol{\nu}^T) \frac{\partial^2 \mathbf{M}_{S_0}}{\partial \nu_i \partial \nu_j} (\boldsymbol{\nu} \mathbf{e}_j^T) \right] \dot{\boldsymbol{\nu}} \\
&\quad + \left[\sum_{i=1}^6 (\mathbf{J}_i^T \boldsymbol{\nu}) \frac{\partial \mathbf{M}_{S_0}}{\partial q_i} + \frac{1}{2} \sum_{i=1}^6 \sum_{j=1}^6 (\mathbf{e}_i \boldsymbol{\nu}^T) \frac{\partial^2 \mathbf{M}_{S_0}}{\partial \nu_i \partial q_j} (\mathbf{J}_j^T \boldsymbol{\nu}) \right] \boldsymbol{\nu} \\
&\quad + \int_0^t \sum_{i=1}^6 \left(\mathbf{e}_i \boldsymbol{\nu}^T(\tau) \frac{\partial \mathbf{K}_{S_0}^T(\mathbf{q}, \boldsymbol{\nu}, \mathbf{q}(\tau), t - \tau)}{\partial \nu_i} + \frac{\partial \mathbf{K}_{S_0}(\mathbf{q}, \boldsymbol{\nu}, \mathbf{q}(\tau), t - \tau)}{\partial \nu_i} \boldsymbol{\nu}(\tau) \mathbf{e}_i^T \right) d\tau \dot{\boldsymbol{\nu}} \\
&\quad + \int_0^t \sum_{i=1}^6 \sum_{j=1}^6 \mathbf{e}_i \boldsymbol{\nu}^T \frac{\partial^2 \mathbf{K}_{S_0}(\mathbf{q}, \boldsymbol{\nu}, \mathbf{q}(\tau), t - \tau)}{\partial \nu_i \partial \nu_j} \boldsymbol{\nu}(\tau) \mathbf{e}_j^T d\tau \dot{\boldsymbol{\nu}} \\
&\quad + \int_0^t \frac{\partial \mathbf{K}_{S_0}(\mathbf{q}, \boldsymbol{\nu}, \mathbf{q}(\tau), t - \tau)}{\partial (t - \tau)} \boldsymbol{\nu}(\tau) d\tau \\
&\quad + \int_0^t \sum_{i=1}^6 \left(\frac{\partial \mathbf{K}_{S_0}(\mathbf{q}, \boldsymbol{\nu}, \mathbf{q}(\tau), t - \tau)}{\partial q_i} \mathbf{J}_i^T \boldsymbol{\nu} + \mathbf{e}_i \boldsymbol{\nu}^T \frac{\partial^2 \mathbf{K}_{S_0}(\mathbf{q}, \boldsymbol{\nu}, \mathbf{q}(\tau), t - \tau)}{\partial \nu_i \partial (t - \tau)} \right) \boldsymbol{\nu}(\tau) d\tau \\
&\quad + \int_0^t \sum_{i=1}^6 \sum_{j=1}^6 (\mathbf{J}_j^T \boldsymbol{\nu}) \mathbf{e}_i \boldsymbol{\nu}^T \frac{\partial^2 \mathbf{K}_{S_0}(\mathbf{q}, \boldsymbol{\nu}, \mathbf{q}(\tau), t - \tau)}{\partial q_j \partial \nu_i} \boldsymbol{\nu}(\tau) d\tau \\
\mathbf{J}^T \frac{\partial \mathcal{L}_{S_0}}{\partial \mathbf{q}} &= \frac{1}{2} \left[\sum_{i=1}^6 (\mathbf{J}_i \boldsymbol{\nu}^T) \frac{\partial \mathbf{M}_{S_0}}{\partial q_i} \right] \boldsymbol{\nu}
\end{aligned}$$

$$+ \int_0^t \sum_{i=1}^6 (\mathbf{J}_i \boldsymbol{\nu}^T) \frac{\partial \mathbf{K}_{S_0}(t-\tau)}{\partial q_i} \boldsymbol{\nu}(\tau) d\tau$$

Therefore, the free surface contribution to the equations of motions are

$$\begin{aligned}
& \left[\mathbf{M}_{S_0} + \sum_{i=1}^6 \left((\mathbf{e}_i \boldsymbol{\nu}^T) \frac{\partial \mathbf{M}_{S_0}}{\partial \nu_i} + \frac{\partial \mathbf{M}_{S_0}}{\partial \nu_i} (\boldsymbol{\nu} \mathbf{e}_i^T) \right) + \frac{1}{2} \sum_{i=1}^6 \sum_{j=1}^6 (\mathbf{e}_i \boldsymbol{\nu}^T) \frac{\partial^2 \mathbf{M}_{S_0}}{\partial \nu_i \partial \nu_j} (\boldsymbol{\nu} \mathbf{e}_j^T) \right] \dot{\boldsymbol{\nu}} \\
& + \left[\sum_{i=1}^6 (\mathbf{J}_i^T \boldsymbol{\nu}) \frac{\partial \mathbf{M}_{S_0}}{\partial q_i} + \frac{1}{2} \sum_{i=1}^6 \sum_{j=1}^6 (\mathbf{e}_i \boldsymbol{\nu}^T) \frac{\partial^2 \mathbf{M}_{S_0}}{\partial \nu_i \partial q_j} (\mathbf{J}_j^T \boldsymbol{\nu}) \right] \boldsymbol{\nu} \\
& + \int_0^t \sum_{i=1}^6 \left(\mathbf{e}_i \boldsymbol{\nu}^T(\tau) \frac{\partial \mathbf{K}_{S_0}^T(\mathbf{q}, \boldsymbol{\nu}, \mathbf{q}(\tau), t-\tau)}{\partial \nu_i} + \frac{\partial \mathbf{K}_{S_0}(\mathbf{q}, \boldsymbol{\nu}, \mathbf{q}(\tau), t-\tau)}{\partial \nu_i} \boldsymbol{\nu}(\tau) \mathbf{e}_i^T \right) d\tau \dot{\boldsymbol{\nu}} \\
& + \int_0^t \sum_{i=1}^6 \sum_{j=1}^6 \mathbf{e}_i \boldsymbol{\nu}^T \frac{\partial^2 \mathbf{K}_{S_0}(\mathbf{q}, \boldsymbol{\nu}, \mathbf{q}(\tau), t-\tau)}{\partial \nu_i \partial \nu_j} \boldsymbol{\nu}(\tau) \mathbf{e}_j^T d\tau \dot{\boldsymbol{\nu}} \\
& + \int_0^t \frac{\partial \mathbf{K}_{S_0}(\mathbf{q}, \boldsymbol{\nu}, \mathbf{q}(\tau), t-\tau)}{\partial(t-\tau)} \boldsymbol{\nu}(\tau) d\tau \\
& + \int_0^t \sum_{i=1}^6 \left(\frac{\partial \mathbf{K}_{S_0}(\mathbf{q}, \boldsymbol{\nu}, \mathbf{q}(\tau), t-\tau)}{\partial q_i} \mathbf{J}_i^T \boldsymbol{\nu} + \mathbf{e}_i \boldsymbol{\nu}^T \frac{\partial^2 \mathbf{K}_{S_0}(\mathbf{q}, \boldsymbol{\nu}, \mathbf{q}(\tau), t-\tau)}{\partial \nu_i \partial(t-\tau)} \right) \boldsymbol{\nu}(\tau) d\tau \\
& + \int_0^t \sum_{i=1}^6 \sum_{j=1}^6 (\mathbf{J}_j^T \boldsymbol{\nu}) \mathbf{e}_i \boldsymbol{\nu}^T \frac{\partial^2 \mathbf{K}_{S_0}(\mathbf{q}, \boldsymbol{\nu}, \mathbf{q}(\tau), t-\tau)}{\partial q_j \partial \nu_i} \boldsymbol{\nu}(\tau) d\tau \\
& + \mathbf{M}_{S_0} \boldsymbol{\nu} + \frac{1}{2} \left[\sum_{i=1}^6 (\mathbf{e}_i \boldsymbol{\nu}^T) \frac{\partial \mathbf{M}_{S_0}}{\partial \nu_i} \right] \boldsymbol{\nu} + \int_0^t \mathbf{K}_{S_0}(\mathbf{q}, \boldsymbol{\nu}, \mathbf{q}(\tau), t-\tau) \boldsymbol{\nu}(\tau) d\tau \\
& + \int_0^t \sum_{i=1}^6 (\mathbf{e}_i \boldsymbol{\nu}^T) \frac{\partial \mathbf{K}_{S_0}(\mathbf{q}, \boldsymbol{\nu}, \mathbf{q}(\tau), t-\tau)}{\partial \nu_i} \boldsymbol{\nu}(\tau) d\tau - \frac{1}{2} \left[\sum_{i=1}^6 (\mathbf{J}_i \boldsymbol{\nu}^T) \frac{\partial \mathbf{M}_{S_0}}{\partial q_i} \right] \boldsymbol{\nu} \\
& - \int_0^t \sum_{i=1}^6 (\mathbf{J}_i \boldsymbol{\nu}^T) \frac{\partial \mathbf{K}_{S_0}(t-\tau)}{\partial q_i} \boldsymbol{\nu}(\tau) d\tau
\end{aligned} \tag{2.61}$$

Maneuvering Equations of Motion

The contributions from the rigid body (2.4.2), fluid body (2.4.2) and free surface (2.4.2) to the equations of motion can be collected to yield the complete set of equations of motion for a vessel maneuvering near a free surface.

$$\begin{aligned}
 \mathbf{F} = & (\mathbf{M}_b + \mathbf{M}_i(\mathbf{q}, \boldsymbol{\nu}) + \mathbf{M}_m(\mathbf{q}, \boldsymbol{\nu}, t)) \dot{\boldsymbol{\nu}} \\
 & + (\mathbf{C}_b(\boldsymbol{\nu}) + \mathbf{C}_i(\mathbf{q}, \boldsymbol{\nu}) + \mathbf{C}_m(\mathbf{q}, \boldsymbol{\nu}, t)) \boldsymbol{\nu} \\
 & + (\mathbf{D}_i(\mathbf{q}, \boldsymbol{\nu}) + \mathbf{D}_m(\mathbf{q}, \boldsymbol{\nu}, t)) \boldsymbol{\nu} \\
 & + \mathbf{g}(\boldsymbol{\theta}_b) + \int_0^t \mathbf{K}_m(\mathbf{q}, \boldsymbol{\nu}, \mathbf{q}(\tau), t - \tau) \boldsymbol{\nu}(\tau) d\tau
 \end{aligned} \tag{2.62}$$

Here, terms are collected depending on whether they are related to the instantaneous response or memory effects of the fluid. The matrices related to the instantaneous response of the fluid are,

$$\begin{aligned}
 \mathbf{M}_i(\mathbf{q}, \boldsymbol{\nu}) = & \mathbf{M}_B + \Delta \mathbf{M}_B + \mathbf{M}_{S_0} \\
 & + \sum_{i=1}^6 \left((\mathbf{e}_i \boldsymbol{\nu}^T) \frac{\partial \mathbf{M}_{S_0}}{\partial \nu_i} + \frac{\partial \mathbf{M}_{S_0}}{\partial \nu_i} (\boldsymbol{\nu} \mathbf{e}_i^T) \right) \\
 & + \frac{1}{2} \sum_{i=1}^6 \sum_{j=1}^6 (\mathbf{e}_i \boldsymbol{\nu}^T) \frac{\partial^2 \mathbf{M}_{S_0}}{\partial \nu_i \partial \nu_j} (\boldsymbol{\nu} \mathbf{e}_j^T)
 \end{aligned} \tag{2.63}$$

$$\mathbf{C}_i(\mathbf{q}, \boldsymbol{\nu}) = \mathbf{G} \left[\mathbf{M}_B + \Delta \mathbf{M}_B + \mathbf{M}_{S_0} + \frac{1}{2} \sum_{i=1}^6 (\mathbf{e}_i \boldsymbol{\nu}^T) \frac{\partial \mathbf{M}_{S_0}}{\partial \nu_i} \right] \tag{2.64}$$

$$\begin{aligned}
 \mathbf{D}_i(\mathbf{q}, \boldsymbol{\nu}) = & \sum_{i=1}^6 \left(\mathbf{J}_i^T \boldsymbol{\nu} \mathbf{I} - \frac{1}{2} \mathbf{J}_i \boldsymbol{\nu}^T \right) \left(\frac{\partial \Delta \mathbf{M}_B}{\partial q_i} + \frac{\partial \mathbf{M}_{S_0}}{\partial q_i} \right) \\
 & + \frac{1}{2} \sum_{i=1}^6 \sum_{j=1}^6 (\mathbf{J}_j^T \boldsymbol{\nu}) (\mathbf{e}_i \boldsymbol{\nu}^T) \frac{\partial^2 \mathbf{M}_{S_0}}{\partial \nu_i \partial q_j}
 \end{aligned} \tag{2.65}$$

and the matrices related to the memory effects of the fluid are,

$$\begin{aligned} \mathbf{M}_m(\mathbf{q}, \boldsymbol{\nu}, t) = & \int_0^t \left[\sum_{i=1}^6 \mathbf{e}_i \boldsymbol{\nu}^T(\tau) \frac{\partial \mathbf{K}_{S_0}^T(t-\tau)}{\partial \nu_i} + \frac{\partial \mathbf{K}_{S_0}(t-\tau)}{\partial \nu_i} \boldsymbol{\nu}(\tau) \mathbf{e}_i^T \right] d\tau \\ & + \int_0^t \left[\sum_{i=1}^6 \sum_{j=1}^6 \mathbf{e}_i \boldsymbol{\nu}^T(\tau) \frac{\partial^2 \mathbf{K}_{S_0}(t-\tau)}{\partial \nu_i \partial \nu_j} \boldsymbol{\nu}(\tau) \mathbf{e}_j^T \right] d\tau \end{aligned} \quad (2.66)$$

$$\begin{aligned} \mathbf{C}_m(\mathbf{q}, \boldsymbol{\nu}, t) = & \mathbf{G} \int_0^t \sum_{i=1}^6 \mathbf{e}_i \boldsymbol{\nu}^T(\tau) \frac{\partial \mathbf{K}_{S_0}^T(t-\tau)}{\partial \nu_i} d\tau \\ & + \mathbf{G} \int_0^t [\mathbf{K}_B(\mathbf{q}(\tau), t-\tau) + \mathbf{K}_{S_0}(\mathbf{q}, \boldsymbol{\nu}, \mathbf{q}(\tau), t-\tau)] \boldsymbol{\nu}(\tau) d\tau \end{aligned} \quad (2.67)$$

$$\begin{aligned} \mathbf{D}_m(\mathbf{q}, \boldsymbol{\nu}, t) = & \int_0^t \sum_{i=1}^6 \left(\frac{\partial \mathbf{K}_{S_0}(t-\tau)}{\partial q_i} \boldsymbol{\nu}(\tau) \mathbf{J}_i^T - \mathbf{J}_i \boldsymbol{\nu}(\tau)^T \frac{\partial \mathbf{K}_{S_0}^T(t-\tau)}{\partial q_i} \right) d\tau \\ & + \int_0^t \sum_{i=1}^6 \sum_{j=1}^6 \mathbf{e}_i \boldsymbol{\nu}^T(\tau) \left[(\mathbf{J}_j^T \boldsymbol{\nu}) \frac{\partial^2 \mathbf{K}_{S_0}^T(t-\tau)}{\partial q_j \partial \nu_i} + \frac{\partial^2 \mathbf{K}_{S_0}^T(t-\tau)}{\partial \nu_i \partial (t-\tau)} \right] d\tau \end{aligned} \quad (2.68)$$

$$\mathbf{K}_m(\mathbf{q}, \boldsymbol{\nu}, \mathbf{q}(\tau), t-\tau) = \frac{\partial (\mathbf{K}_B(t-\tau) + \mathbf{K}_{S_0}(t-\tau))}{\partial (t-\tau)} \quad (2.69)$$

and finally,

$$\mathbf{g}(\boldsymbol{\theta}_b) = \begin{pmatrix} 0 \\ -m_b g \mathbf{r}_{cm} \times (\mathbf{R}^T \mathbf{i}_3) \end{pmatrix} \quad (2.70)$$

2.4.3 Application of the LNM Model to a Special Case

As an application of the LNM model (2.62) to a special case, analytical expressions for approximating the hydrodynamic forces experienced by a rigid body moving near a free surface

are derived from the model [36]. For this particular application, the analytical expressions are derived for a 6:1 prolate spheroid moving at a constant speed and orientation parallel to the free surface. By applying the method of images, a “double body flow” is constructed as shown in Figure 2.2, whereby the undisturbed water surface becomes a plane of symmetry and therefore the deformation of the free surface is neglected. The resulting analytical expressions provide an approximation of the hydrodynamic forces acting on a moving body. For the prolate spheroid in steady forward motion, the analytical force expression is computed using a simple potential flow solver without a free surface. The validity of the analytical approximation is then assessed by comparing the approximated hydrodynamic force values to forces computed by a seakeeping potential flow code for double body flow as well as for a deformable free surface.

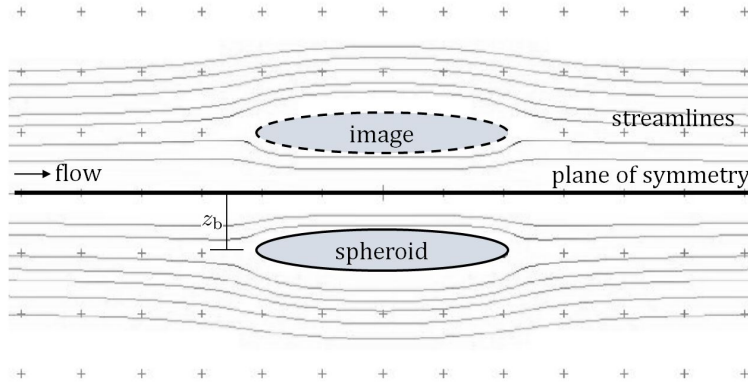


Figure 2.2: The “double body flow” past a submerged prolate spheroid.

The problem is setup as shown in Table 2.1, where forward speed and submergence depth are varied in order to assess their effects on the hydrodynamic force approximations.

Now, the total velocity potential in (2.43) is modified to include the effects of double body flow as follows:

$$\phi(\mathbf{q}, \boldsymbol{\nu}, t) = \phi_K(\boldsymbol{\nu}) + \bar{\phi}_s(\mathbf{q}, \boldsymbol{\nu}) + \phi'_s(\mathbf{q}, \boldsymbol{\nu}) + \phi_m(\mathbf{q}, \boldsymbol{\nu}, t) \quad (2.71)$$

Table 2.1: Problem Setup - Analytical Expressions for Hydrodynamic Forces

Variable	Value
Length-to-diameter ratio	6:1
Body length, L_b	2 m
Gravitational acceleration, g	9.81 m/s ²
Fluid density, ρ	1000 kg/m ³
Froude number, $Fr = u/\sqrt{gL_b}$	0.115 - 0.4 (20 equally spaced values)
Depth-to-length ratio, z_b/L_b	{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.8, 1}

In this revised potential function, the free surface potential ϕ_s is decomposed into $\bar{\phi}_s$, the double body flow potential, and ϕ'_s , a correction that accounts for the free surface deformation. Going through the derivation with the new decomposition, we arrive at the modified complete system Lagrangian,

$$\begin{aligned} \mathcal{L} = & \frac{1}{2} \boldsymbol{\nu}^T (\mathbf{M}_b + \mathbf{M}_B + \Delta \bar{\mathbf{M}}_B(\mathbf{q}) + \Delta \mathbf{M}_r(\mathbf{q}, \boldsymbol{\nu}) + \mathbf{M}_{S_0}(\mathbf{q}, \boldsymbol{\nu})) \boldsymbol{\nu} \\ & + \boldsymbol{\nu}^T \int_0^t [\mathbf{K}_B(\mathbf{q}(\tau), t - \tau) + \mathbf{K}_{S_0}(\mathbf{q}, \boldsymbol{\nu}, \mathbf{q}(\tau), t - \tau)] \boldsymbol{\nu}(\tau) d\tau \end{aligned} \quad (2.72)$$

where $\Delta \mathbf{M}_B$ is replaced by the $\mathbf{q}$ -dependent term $\Delta \bar{\mathbf{M}}_B$, and the $\mathbf{q}$ and $\boldsymbol{\nu}$ -dependent term $\Delta \mathbf{M}_r$. Adding $\Delta \bar{\mathbf{M}}_B$ to $\mathbf{M}_B$ yields the added mass of the body in double body flow. On the other hand, $\Delta \mathbf{M}_r$ is the collection of residual terms due to the decomposition of the free surface potential ($\phi_S = \bar{\phi}_S(\mathbf{q}, \boldsymbol{\nu}) + \phi'_S(\mathbf{q}, \boldsymbol{\nu})$). We assume that the spheroid is neutrally buoyant and the center of mass coincides with the geometric center.

The Euler-Lagrange equations generate force prediction equations using the modified com-

plete system Lagrangian (2.72). For a spheroid with zero incidence angle moving at a constant forward speed parallel to the free surface, the formulas for surge force, heave force and pitch moment, respectively, take the form,

$$X = f_X(\mathbf{q}, \boldsymbol{\nu}, t) \quad (2.73)$$

$$Z = -\frac{1}{2} \frac{\partial \Delta \bar{M}_{B11}(\mathbf{q})}{\partial z_b} u^2 + f_Z(\mathbf{q}, \mathbf{v}, t) \quad (2.74)$$

$$M = -\left(\Delta \bar{M}_{B31}(\mathbf{q}) + \frac{1}{2} \frac{\partial \Delta \bar{M}_{B11}(\mathbf{q})}{\partial \theta_b} \right) u^2 + f_M(\mathbf{q}, \boldsymbol{\nu}, t) \quad (2.75)$$

The functions f in each of the formulas are the adjustments that correct for the double body flow assumption, as well as the memory effects. The subscripts in $\Delta \bar{M}_{Bij}$ indicate the entry in the i^{th} row and j^{th} column of matrix $\Delta \bar{\mathbf{M}}_B$.

The potential flow solutions from Aegir, a time-domain seakeeping code introduced later in this dissertation in Section 4.1, show that $\Delta \bar{M}_{B31}$ is equal to zero for the axisymmetric body moving horizontally along the longitudinal axis, regardless of $\mathbf{q}$. Also, the gradient term $\partial \Delta \bar{M}_{B11}(\mathbf{q}) / \partial \theta_b$, has been confirmed to equal zero when evaluated at $\theta_b = 0$ for various depths z_b . The observation is consistent with the fact that the body in double body flow is fore and aft symmetric, with zero surge force and only a heave force acting through the geometric center. Therefore, (2.73-2.75) reduce to,

$$X = f_X(\mathbf{q}, \boldsymbol{\nu}, t) \quad (2.76)$$

$$Z = -\frac{1}{2} \frac{\partial \Delta \bar{M}_{B11}(\mathbf{q})}{\partial z_b} u^2 + f_Z(\mathbf{q}, \boldsymbol{\nu}, t) \quad (2.77)$$

$$M = f_M(\mathbf{q}, \boldsymbol{\nu}, t) \quad (2.78)$$

Now we can focus solely on the first term in (2.77), since all f terms disappear for the double body flow. The term $\partial\Delta\overline{M}_{B11}(\mathbf{q})/\partial z_b$ is numerically evaluated via the finite difference method by solving for the double body flow using a simple potential flow code that does not include a free surface. As shown in Figure 2.2, the varying values of nondimensional depth z_b/L_b in Table 2.1 can be realized in the double body flow by adjusting the distance from the plane of symmetry (undisturbed water surface) to the spheroid and its image. With the numerically evaluated values, heave force on the spheroid is easily computed for any Froude number using the analytical expression in (2.77). Heave force from the free surface flow, the double body flow and the model prediction (2.77) are shown in Figure 2.3 for various speeds at different depths. The nondimensional heave force in the plots is the ratio of the heave force to the weight of displaced water. We can also plot the error in the predicted heave force values for all nondimensional depth values z_b/L_b in Table 2.1, taking the heave force from free surface flow simulations as the reference value (Figure 2.4).

The plots in Figure 2.3 show that heave force computed using the first term in equation (2.77) matches the double body flow simulation values. Moreover, for low Froude numbers, these values agree well with simulations in which the surface is free to deform. On the other hand, for a submerged vehicle traveling at $Fr \simeq 0.25$, prediction error is nearly 50% when very close to the free surface, at $z_b/L_b = 0.1$ (Figure 2.4). But as the depth increases to $z_b/L_b = 0.2$, the force prediction error decreases rapidly to approximately 30 - 35%. For depths below $z_b/L_b = 0.8$, prediction error is less than 20%. At high speeds or when near the surface, discrepancies are much larger due to larger deformations of the free surface. These results indicate that the analytical approximation can be used to estimate heave force near the surface for moderately low speeds, or at least provide a lower bound. The proposed approach can also be applied to predict heave force and pitch moment for a spheroid moving at an incidence angle to the surface.

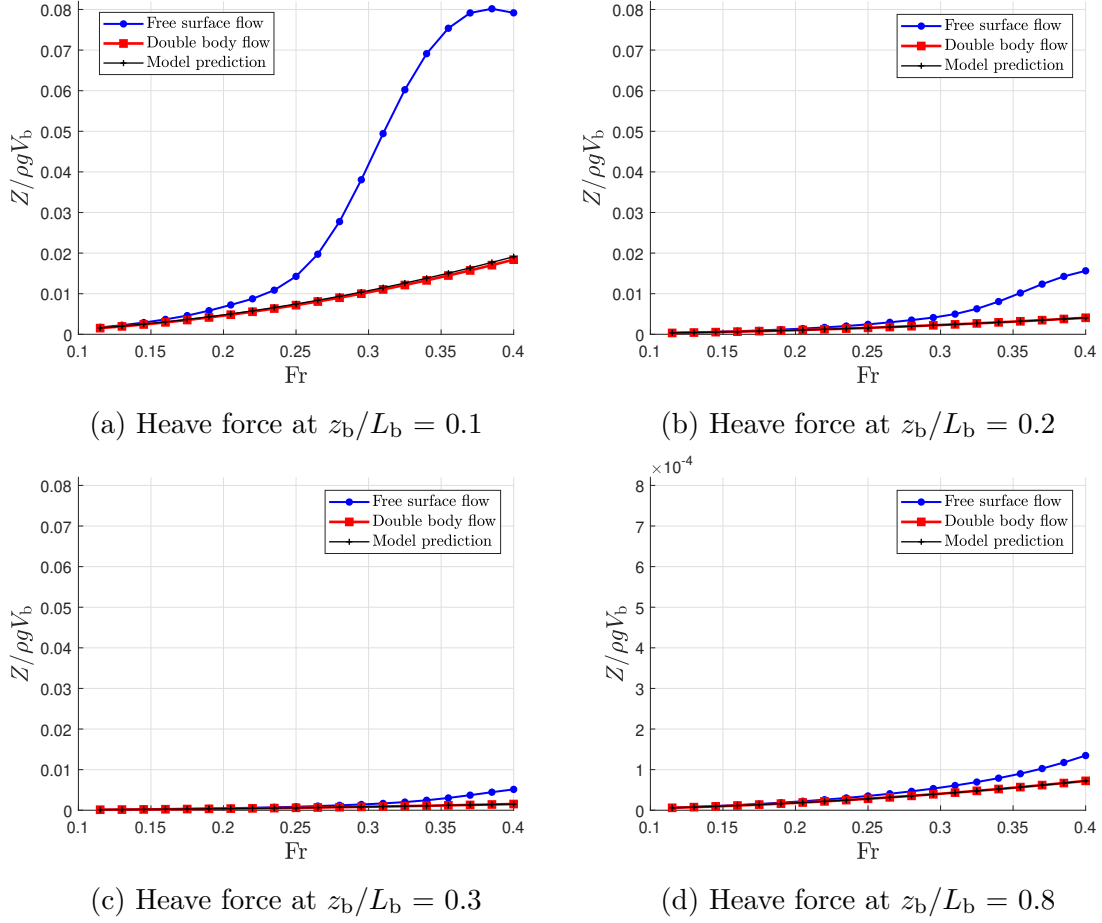


Figure 2.3: Comparison of heave force values obtained from different methods.

2.5 A Lagrangian Nonlinear Maneuvering and Seakeeping (LNMS) Model

The effects due to externally generated fluid motion acting on the vehicle can be incorporated to the LNM model developed in Section 2.4. Based on the method proposed by Thomasson and Woolsey [46] for defining a Lagrangian that incorporates the ambient flow field, the Lagrangian can be updated to yield a disturbance Lagrangian that accounts for the free surface by employing the method proposed by Battista [1]. The disturbance Lagrangian is

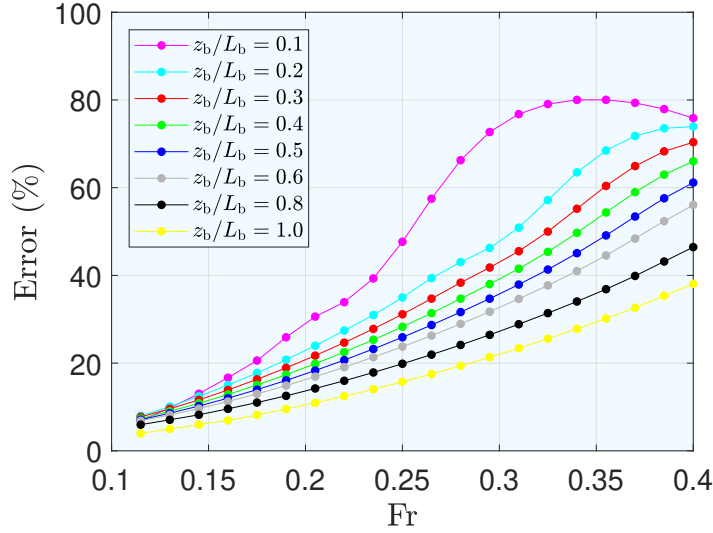


Figure 2.4: Heave force prediction error.

expressed as

$$\begin{aligned}
 \mathcal{L}_d = & \frac{1}{2} \boldsymbol{\nu}_r^T (\mathbf{M}_b + \mathbf{M}_B + \Delta \mathbf{M}_B(\mathbf{q}) + \mathbf{M}_{S_0}(\mathbf{q}, \boldsymbol{\nu})) \boldsymbol{\nu}_r \\
 & - \frac{1}{2} \boldsymbol{\nu}^T (\mathbf{M}_b + \mathbf{M}_B + \Delta \mathbf{M}_B(\mathbf{q}) + \mathbf{M}_{S_0}(\mathbf{q}, \boldsymbol{\nu})) \boldsymbol{\nu} \\
 & + m_b \boldsymbol{\omega}^T \hat{\mathbf{r}}_{cm} \mathbf{v}_f + (m_b - \rho V) \left(\mathbf{v}^T \mathbf{v}_f - \frac{1}{2} \mathbf{v}_f^T \mathbf{v}_f \right)
 \end{aligned} \tag{2.79}$$

where the flow-relative velocity of the vessel is

$$\boldsymbol{\nu}_r = \boldsymbol{\nu} - \boldsymbol{\nu}_f$$

and the fluid velocity in the body-fixed frame is

$$\boldsymbol{\nu}_f(\mathbf{q}, t) = \mathbf{J}(\boldsymbol{\theta}_b) \begin{pmatrix} \mathbf{V}_f(\mathbf{x}_b, t) \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{v}_f(\mathbf{q}, t) \\ \mathbf{0} \end{pmatrix}$$

Now, the following terms in the Boltzmann-Hamel equations (2.5) are computed by applying

the disturbance Lagrangian $\mathcal{L}_d$.

$$\begin{aligned}
\frac{\partial \mathcal{L}_d}{\partial \boldsymbol{\nu}} &= \left(-M_b - M_B - \Delta M_B(\mathbf{q}) - M_{S_0}(\mathbf{q}, \boldsymbol{\nu}) + \frac{1}{2} \sum_{i=1}^6 \mathbf{e}_i \boldsymbol{\nu}_f^T \frac{\partial M_{S_0}}{\partial \nu_i} \right) \boldsymbol{\nu}_f \\
&\quad - \sum_{i=1}^6 \left(\mathbf{e}_i \boldsymbol{\nu}_f^T \frac{\partial M_{S_0}}{\partial \nu_i} \right) \boldsymbol{\nu} + \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ m_b \hat{\mathbf{r}}_{cm} & \mathbf{0} \end{pmatrix} \boldsymbol{\nu}_f \\
\frac{d}{dt} \frac{\partial \mathcal{L}_d}{\partial \boldsymbol{\nu}} &= - \sum_{i=1}^6 \left(\mathbf{e}_i \boldsymbol{\nu}_f^T \frac{\partial M_{S_0}}{\partial \nu_i} + \frac{\partial M_{S_0}}{\partial \nu_i} \boldsymbol{\nu}_f^T \mathbf{e}_i \right) \dot{\boldsymbol{\nu}} + \sum_{i=1}^6 \sum_{j=1}^6 \left(\mathbf{e}_i \boldsymbol{\nu}_f^T \frac{\partial^2 M_{S_0}}{\partial \nu_i \partial \nu_j} \left(\frac{1}{2} \boldsymbol{\nu}_f - \boldsymbol{\nu} \right) \right) \dot{\boldsymbol{\nu}} \\
&\quad + \left(-M_b - M_B - \Delta M_B - M_{S_0} + \frac{1}{2} \sum_{i=1}^6 \mathbf{e}_i \boldsymbol{\nu}_f^T \frac{\partial M_{S_0}}{\partial \nu_i} + \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ m_b \hat{\mathbf{r}}_{cm} & \mathbf{0} \end{pmatrix} \right) \dot{\boldsymbol{\nu}}_f \\
&\quad - \sum_{i=1}^6 \left(\mathbf{e}_i \dot{\boldsymbol{\nu}}_f^T \frac{\partial M_{S_0}}{\partial \nu_i} + \frac{\partial (\Delta M_B + M_{S_0})}{\partial q_i} \boldsymbol{\nu}_f \mathbf{J}_i^T \right) \boldsymbol{\nu} \\
&\quad + \sum_{i=1}^6 \sum_{j=1}^6 \left(\mathbf{e}_i \boldsymbol{\nu}_f^T \frac{\partial^2 M_{S_0}}{\partial \nu_i \partial q_j} \left(\frac{1}{2} \boldsymbol{\nu}_f - \boldsymbol{\nu} \right) \mathbf{J}_j^T \right) \boldsymbol{\nu} \\
-\mathbf{J}^T \frac{\partial \mathcal{L}_d}{\partial \mathbf{q}} &= -\mathbf{J}^T \left(\frac{\partial \boldsymbol{\nu}_f^T}{\partial \mathbf{q}} (M_b + M_B + \Delta M_B + M_{S_0}) + \frac{1}{2} \sum_{i=1}^6 \mathbf{e}_i \boldsymbol{\nu}_f^T \frac{\partial (\Delta M_B + M_{S_0})}{\partial q_i} \right) \boldsymbol{\nu}_f \\
&\quad + \mathbf{J}^T \left(\frac{\partial \boldsymbol{\nu}_f^T}{\partial \mathbf{q}} (M_b + M_B + \Delta M_B + M_{S_0}) + \sum_{i=1}^6 \mathbf{e}_i \boldsymbol{\nu}_f^T \frac{\partial (\Delta M_B + M_{S_0})}{\partial q_i} \right) \boldsymbol{\nu}
\end{aligned}$$

In order to conform with the structure of the near-surface maneuvering equations of motion (2.62), the following external flow perturbation matrices are defined to collect terms from the disturbance Lagrangian contribution to the equations of motion.

$$\begin{aligned}
\mathbf{M}_e(\mathbf{q}, \boldsymbol{\nu}, t) &= - \sum_{i=1}^6 \left(\mathbf{e}_i \boldsymbol{\nu}_f^T \frac{\partial M_{S_0}}{\partial \nu_i} + \frac{\partial M_{S_0}}{\partial \nu_i} \boldsymbol{\nu}_f^T \mathbf{e}_i \right) \\
&\quad + \sum_{i=1}^6 \sum_{j=1}^6 \left(\mathbf{e}_i \boldsymbol{\nu}_f^T \frac{\partial^2 M_{S_0}}{\partial \nu_i \partial \nu_j} \left(\frac{1}{2} \boldsymbol{\nu}_f - \boldsymbol{\nu} \right) \right)
\end{aligned} \tag{2.80}$$

$$\mathbf{C}_e(\mathbf{q}, \boldsymbol{\nu}, t) = -\mathbf{G} \sum_{i=1}^6 \left(\mathbf{e}_i \boldsymbol{\nu}_f^T \frac{\partial \mathbf{M}_{S_0}}{\partial \nu_i} \right) \quad (2.81)$$

$$\begin{aligned} \mathbf{D}_e(\mathbf{q}, \boldsymbol{\nu}, t) = & - \begin{pmatrix} \Phi & 0 \\ -\hat{\mathbf{v}}_f & 0 \end{pmatrix} (\mathbf{M}_b + \mathbf{M}_B + \Delta \mathbf{M}_B + \mathbf{M}_{S_0}) - \sum_{i=1}^6 \left(\mathbf{e}_i \dot{\boldsymbol{\nu}}_f^T \frac{\partial \mathbf{M}_{S_0}}{\partial \nu_i} \right) \\ & + \sum_{i=1}^6 \left(\mathbf{J}_i \boldsymbol{\nu}_f^T \frac{\partial (\Delta \mathbf{M}_B + \mathbf{M}_{S_0})}{\partial q_i} - \frac{\partial (\Delta \mathbf{M}_B + \mathbf{M}_{S_0})}{\partial q_i} \boldsymbol{\nu}_f \mathbf{J}_i^T \right) \\ & + \sum_{i=1}^6 \sum_{j=1}^6 \left(\mathbf{e}_i \boldsymbol{\nu}_f^T \frac{\partial^2 \mathbf{M}_{S_0}}{\partial \nu_i \partial q_j} \left(\frac{1}{2} \boldsymbol{\nu}_f - \boldsymbol{\nu} \right) \mathbf{J}_j^T \right) \end{aligned} \quad (2.82)$$

Also, the fluid disturbance matrices are defined as

$$\mathbf{M}_d(\mathbf{q}, \boldsymbol{\nu}, t) = -\mathbf{M}_b - \mathbf{M}_B - \Delta \mathbf{M}_B - \mathbf{M}_{S_0} + \sum_{i=1}^6 \mathbf{e}_i \boldsymbol{\nu}_f^T \frac{\partial \mathbf{M}_{S_0}}{\partial \nu_i} + \begin{pmatrix} 0 & 0 \\ m_b \hat{\mathbf{r}}_{cm} & 0 \end{pmatrix} \quad (2.83)$$

$$\begin{aligned} \mathbf{C}_d(\mathbf{q}, \boldsymbol{\nu}, t) = & \mathbf{G} (-\mathbf{M}_b - \mathbf{M}_B - \Delta \mathbf{M}_B - \mathbf{M}_{S_0}) \\ & + \mathbf{G} \left(\frac{1}{2} \sum_{i=1}^6 \mathbf{e}_i \boldsymbol{\nu}_f^T \frac{\partial \mathbf{M}_{S_0}}{\partial \nu_i} + \begin{pmatrix} 0 & 0 \\ m_b \hat{\mathbf{r}}_{cm} & 0 \end{pmatrix} \right) \end{aligned} \quad (2.84)$$

$$\begin{aligned} \mathbf{D}_d(\mathbf{q}, \boldsymbol{\nu}, t) = & \begin{pmatrix} \Phi & 0 \\ -\hat{\mathbf{v}}_f & 0 \end{pmatrix} (\mathbf{M}_b + \mathbf{M}_B + \Delta \mathbf{M}_B + \mathbf{M}_{S_0}) \\ & - \frac{1}{2} \sum_{i=1}^6 \left(\mathbf{J}_i \boldsymbol{\nu}_f^T \frac{\partial (\Delta \mathbf{M}_B + \mathbf{M}_{S_0})}{\partial q_i} \right) \end{aligned} \quad (2.85)$$

Finally, a Lagrangian nonlinear maneuvering and seakeeping (LNMS) model can be repre-

sented by the following equations of motion.

$$\begin{aligned}
\mathbf{F} = & (\mathbf{M}_b + \mathbf{M}_i(\mathbf{q}, \boldsymbol{\nu}) + \mathbf{M}_m(\mathbf{q}, \boldsymbol{\nu}, t) + \mathbf{M}_e(\mathbf{q}, \boldsymbol{\nu}, t)) \dot{\boldsymbol{\nu}} \\
& + (\mathbf{C}_b(\boldsymbol{\nu}) + \mathbf{C}_i(\mathbf{q}, \boldsymbol{\nu}) + \mathbf{C}_m(\mathbf{q}, \boldsymbol{\nu}, t) + \mathbf{C}_e(\mathbf{q}, \boldsymbol{\nu}, t)) \boldsymbol{\nu} \\
& + (\mathbf{D}_i(\mathbf{q}, \boldsymbol{\nu}) + \mathbf{D}_m(\mathbf{q}, \boldsymbol{\nu}, t) + \mathbf{D}_e(\mathbf{q}, \boldsymbol{\nu}, t)) \boldsymbol{\nu} \\
& + \mathbf{g}(\boldsymbol{\theta}_b) + \int_0^t \mathbf{K}_m(\mathbf{q}, \boldsymbol{\nu}, \mathbf{q}(\tau), t - \tau) \boldsymbol{\nu}(\tau) d\tau \\
& + \mathbf{M}_d(\mathbf{q}, \boldsymbol{\nu}, t) \dot{\boldsymbol{\nu}}_f + \mathbf{C}_d(\mathbf{q}, \boldsymbol{\nu}, t) \boldsymbol{\nu}_f + \mathbf{D}_d(\mathbf{q}, \boldsymbol{\nu}, t) \boldsymbol{\nu}_f
\end{aligned} \tag{2.86}$$

where the matrices $\mathbf{M}_i$, $\mathbf{C}_i$ and $\mathbf{D}_i$ are defined in (2.63-2.65), the matrices $\mathbf{M}_m$, $\mathbf{C}_m$, $\mathbf{D}_m$ and $\mathbf{K}_m$ are defined in (2.66-2.69), the matrices $\mathbf{M}_e$, $\mathbf{C}_e$ and $\mathbf{D}_e$ are defined in (2.80-2.82), the matrices $\mathbf{M}_d$, $\mathbf{C}_d$ and $\mathbf{D}_d$ are defined in (2.83-2.85), and $\mathbf{g}$ is defined in (2.70).

This chapter summarizes the derivation of a LNMS model using Lagrangian mechanics [1]. In addition, an application of the LNM model derived by Battista [1] is presented in Section 2.4.3, where analytical expressions are derived for approximating the hydrodynamic force experienced by a rigid body moving near a free surface [36].

Chapter 3

Reformulation of Model

In this chapter, a reformulated Lagrangian nonlinear maneuvering and seakeeping (LNMS) model is derived for parameter computation using data from a potential flow code that has been modified to output the necessary numerically computed values [16]. To begin, the Lagrangian nonlinear maneuvering (LNM) model (2.62) portion of the complete LNMS model (2.86) is reformulated by applying two methods, namely the differentiation under the integral sign and the computation of partial derivatives of potential functions. The reformulation process avoids additional numerical discretization that would otherwise be necessary to approximate the derivatives that appear in the Boltzmann-Hamel equations (2.5). A reformulated hydrodynamic force model for motion in calm water in terms of the total velocity potential is introduced first. Then, a reformulated LNM model, in a form analogous to the LNM model (2.62) from the previous chapter, is derived by decomposing the total velocity potential into physically meaningful components. Finally, incident wave effects are incorporated to the reformulated LNM model to yield the reformulated LNMS model.

3.1 Differentiation Under the Integral Sign

In the reformulation process, differentiation under the integral sign is applied in order to interchange the order of differentiation and integration, which results in some terms that are readily computed by potential flow codes. First, the partial differentials in the Boltzmann-

Hamel equations (2.5), namely $\frac{\partial}{\partial \boldsymbol{\nu}}$ and $\frac{\partial}{\partial \mathbf{q}}$, are interchanged with the volume integral over the fluid volume $\mathcal{F}$. The divergence theorem (see Appendix A.1) is then applied to the volume integral to yield the surface integrals over $\mathcal{B}$ and $\mathcal{S}$ that appear in the Lagrangian $\mathcal{L}$ (2.40). Finally, the time derivative $\frac{d}{dt}$ is interchanged with the surface integrals over $\mathcal{B}$ and $\mathcal{S}_0$.

3.1.1 Derivatives with respect to $\boldsymbol{\nu}$

Since the limits of the volume integral over the fluid volume $\mathcal{F}$ do not depend on $\boldsymbol{\nu}$, the order of $\frac{\partial}{\partial \boldsymbol{\nu}}$ with the integral over $\mathcal{F}$ can simply be interchanged. Therefore,

$$\frac{\partial}{\partial \boldsymbol{\nu}} \iiint_{\mathcal{F}} f dV = \iiint_{\mathcal{F}} \frac{\partial f}{\partial \boldsymbol{\nu}} dV \quad (3.1)$$

for a differentiable scalar function f in the integrand.

3.1.2 Derivatives with respect to $\mathbf{q}$

The Boltzmann-Hamel equations (2.5) are used to derive equations of motion that accommodate the set of quasivelocities $\boldsymbol{\nu}$ that are expressed in the body-fixed frame. However, the set of generalized coordinates $\mathbf{q}$, which also appears in the same equation, is composed of the body position $\mathbf{x}_b$ and Euler angles $\boldsymbol{\theta}_b$, neither of which is expressed in the body-fixed frame. By defining a set of quasicordinates [40] expressed in the body-fixed frame, denoted $\mathbf{q}' = (\mathbf{x}_b^T, \boldsymbol{\theta}_b^T)^T$ and satisfying $\boldsymbol{\nu} = \dot{\mathbf{q}}'$, the relation between $\frac{\partial}{\partial \mathbf{q}}$ and $\frac{\partial}{\partial \mathbf{q}'}$ can be used to derive the differentiation with respect to $\mathbf{q}'$ under the integral sign.

From the definition of the kinematic transformation matrix $\mathbf{J}$ in equation (2.2),

$$\dot{\mathbf{q}} = \mathbf{J}(\boldsymbol{\theta}_b) \boldsymbol{\nu}$$

the following expression can be derived.

$$\frac{dq_i}{dt} = \sum_{j=1}^6 J_{ij} \frac{dq'_j}{dt} \quad (3.2)$$

Now, if we consider the equality $\bar{\mathcal{L}}(\mathbf{q}', \boldsymbol{\nu}, t) = \mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t)$, where the Lagrangians have the same value but different forms depending on whether generalized coordinates or quasicordinates are used [39],

$$\frac{\partial \bar{\mathcal{L}}}{\partial q'_j} = \sum_{i=1}^6 \left(\frac{\partial \mathcal{L}}{\partial q_i} \right) \left(\frac{\partial q_i}{\partial q'_j} \right)$$

and by applying equation (3.2),

$$\frac{\partial \bar{\mathcal{L}}}{\partial q'_j} = \sum_{i=1}^6 J_{ij} \left(\frac{\partial \mathcal{L}}{\partial q_i} \right)$$

Therefore,

$$\frac{\partial \bar{\mathcal{L}}}{\partial \mathbf{q}'} = \mathbf{J}^T \frac{\partial \mathcal{L}}{\partial \mathbf{q}}$$

The Boltzmann-Hamel equations (2.5) can now be written as

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \boldsymbol{\nu}} + \mathbf{G}(\boldsymbol{\nu}) \frac{\partial \mathcal{L}}{\partial \boldsymbol{\nu}} - \frac{\partial \bar{\mathcal{L}}}{\partial \mathbf{q}'} = \mathbf{J}(\mathbf{q})^T \mathbf{Q} \quad (3.3)$$

The overbar on the Lagrangian function is omitted hereon, since $\mathcal{L}$ and $\bar{\mathcal{L}}$ denote the same value. The focus here is on how $\frac{\partial}{\partial \mathbf{q}'}$ can be interchanged with a surface integral over the body $\mathcal{B}$, where the limits of the integral depend on the configuration of the body $\mathbf{q}'$.

The translational component of $\mathbf{q}'$ is examined first. In Figure 3.1, suppose that the body $\mathcal{B}$ undergoes translational displacement.

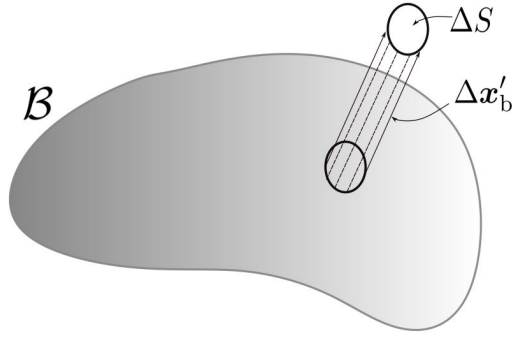


Figure 3.1: Displacement of an element of the surface.

For a small translational displacement $\Delta \mathbf{x}'_b$ of an element ΔS of the surface $\mathcal{B}$, the points $\mathbf{x}'$ of ΔS sweep out a volume

$$\Delta V \simeq \Delta \mathbf{x}'_b \cdot \mathbf{n} \Delta S$$

where $\mathbf{n}$ is a unit vector normal to ΔS . If we multiply the above volume by a function f and sum the products over the entire surface $\mathcal{B}$, we obtain

$$\Delta I \equiv \sum_{\mathcal{B}} f \Delta \mathbf{x}'_b \cdot \mathbf{n} \Delta S$$

Now, if we consider displacement along only the x' -axis,

$$\Delta I \equiv \sum_{\mathcal{B}} f \Delta x'_b \cos(n, x'_b) \Delta S$$

Dividing both sides by $\Delta x'_b$ and passing to the limit $\Delta x'_b \rightarrow 0$ yields

$$\frac{\partial I}{\partial x'_b} = \iint_{\mathcal{B}} f \cos(n, x'_b) dS$$

We can perform the same derivation for displacement along the y' and z' -axes, to conclude

$$\frac{\partial I}{\partial \mathbf{x}'_b} = \iint_{\mathcal{B}} f \mathbf{n} dS$$

This is the term that should be added when taking the derivative of the volume integration over the fluid domain while the body surface is undergoing translational displacement:

$$\frac{\partial}{\partial \mathbf{x}'_b} \iiint_{\mathcal{F}} f(\mathbf{x}'_b, t) dV = \iiint_{\mathcal{F}} \frac{\partial f(\mathbf{x}'_b, t)}{\partial \mathbf{x}'_b} dV + \iint_{\mathcal{B}} f(\mathbf{x}'_b, t) \mathbf{n} dS \quad (3.4)$$

The boundaries of the fluid volume $\mathcal{F}$ include the body surface $\mathcal{B}$ and the free surface $\mathcal{S}$, which is approximated by the undisturbed free surface $\mathcal{S}_0$. The limits of the integral over $\mathcal{S}_0$ are independent of the configuration of the body $\mathbf{q}'$, so there are no additional terms that involve the integral over $\mathcal{S}_0$.

A similar approach can be taken for the rotational component of $\mathbf{q}'$. Differentiation with respect $\boldsymbol{\theta}'_b$ can be derived as follows. In Figure 3.2, suppose that the body $\mathcal{B}$ undergoes rotational displacement.

For a small rotational displacement $\Delta \boldsymbol{\theta}'_b$ of an element ΔS of the surface $\mathcal{B}$, the points $\mathbf{x}'$ of ΔS sweep out a volume

$$\begin{aligned} \Delta V &\simeq (\Delta \boldsymbol{\theta}'_b \times \mathbf{r}) \cdot \mathbf{n} \Delta S \\ &= \Delta \boldsymbol{\theta}'_b \cdot (\mathbf{r} \times \mathbf{n}) \Delta S \end{aligned}$$

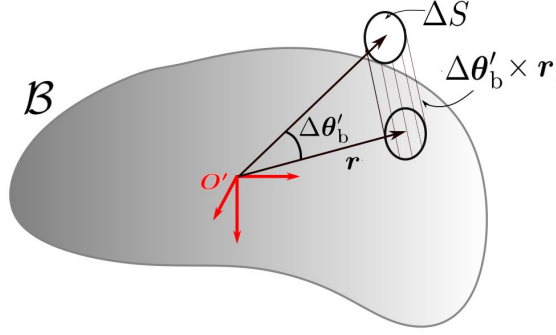


Figure 3.2: Displacement of an element of the surface (rotation).

where $\mathbf{r}$ is the position of a point on the body $\mathcal{B}$ expressed with respect to the body-fixed frame. If we multiply the above volume by a function f and sum the products over the entire surface $\mathcal{B}$,

$$\Delta I \equiv \sum_{\mathcal{B}} f \Delta \theta'_b \cdot (\mathbf{r} \times \mathbf{n}) \Delta S$$

we can conclude the following:

$$\frac{\partial I}{\partial \theta'_b} = \iint_{\mathcal{B}} f(\mathbf{r} \times \mathbf{n}) dS$$

The resulting differential under the integral sign for a body undergoing rotational displacement is,

$$\frac{\partial}{\partial \theta'_b} \iiint_{\mathcal{F}} f(\theta'_b, t) dV = \iiint_{\mathcal{F}} \frac{\partial f(\theta'_b, t)}{\partial \theta'_b} dV + \iint_{\mathcal{B}} f(\theta'_b, t)(\mathbf{r} \times \mathbf{n}) dS \quad (3.5)$$

3.1.3 Derivatives with respect to t

Once the order of $\frac{\partial}{\partial \mathbf{v}}$ and $\frac{\partial}{\partial \mathbf{q}}$ and the volume integral over the fluid volume $\mathcal{F}$ have been switched as described in the previous sections, the divergence theorem (see Appendix A.1)

is applied to the volume integral to yield the surface integrals over $\mathcal{B}$ and $\mathcal{S}$. With regard to the integral over $\mathcal{B}$, although the body surface $\mathcal{B}$ is in motion, if the limits of the integral and the scalar function in the integrand use the same time and spatial coordinates, the order of the derivative and the integral can simply be switched as shown below.

$$\frac{d}{dt} \iint_{\mathcal{B}} f \, dS = \iint_{\mathcal{B}} \frac{\partial f}{\partial t} \, dS \quad (3.6)$$

Once again, f is a differentiable scalar function.

As for the integral over $\mathcal{S}$, once it is approximated as the undisturbed free surface $\mathcal{S}_0$, the surface is now fixed so the order of the derivative and the integral can simply be interchanged.

Therefore,

$$\frac{d}{dt} \iint_{\mathcal{S}_0} f \, dS = \iint_{\mathcal{S}_0} \frac{\partial f}{\partial t} \, dS \quad (3.7)$$

3.2 Partial Derivatives of the Integrand

In this section, the partial derivatives of the velocity potential with respect to $\boldsymbol{\nu}$ and $\mathbf{q}$ are determined. The physical properties of the velocity potential that composes the integrand are used to obtain equivalent expressions for each of the partial derivatives.

3.2.1 Computation of $\frac{\partial \phi}{\partial \boldsymbol{\nu}}$

The decomposition of the total velocity potential in equation (2.44) can be rewritten in an alternative form:

$$\phi = \underbrace{\boldsymbol{\varphi}_i(\mathbf{q}) \cdot \boldsymbol{\nu}}_{\text{impulsive flow potential}} + \underbrace{\int_0^t \boldsymbol{\varphi}_m(\mathbf{q}(\tau), t - \tau) \cdot \boldsymbol{\nu}(\tau) \, d\tau}_{\text{memory flow potential}} \quad (3.8)$$

where $\varphi_i = \varphi_K + \varphi_s$ is the normalized potential vector for impulsive flow. In equation (3.8), we can see that the memory component is independent of $\nu(t)$, so the partial derivative of the integral term with respect to $\nu(t)$ is zero. Therefore,

$$\frac{\partial \phi}{\partial \nu} = \varphi_i \quad (3.9)$$

3.2.2 Computation of $\frac{\partial \phi}{\partial \mathbf{q}'}$ - The relative displacement

The set of quasicoordinates $\mathbf{q}'$ is composed of the vectors $\mathbf{x}'_b$ and $\boldsymbol{\theta}'_b$. A small increment in $\mathbf{x}'_b$ is a small translational displacement $\Delta \mathbf{x}'_b$ of the body $\mathcal{B}$ that can, as shown in Figure 3.3, be equivalently represented by a translational displacement $-\Delta \mathbf{x}'$ of the fluid domain while the body is fixed. The equivalent displacement of the fluid domain includes the displacement of an arbitrary fluid particle P and also the displacement of the undisturbed free surface $\mathcal{S}_0$.

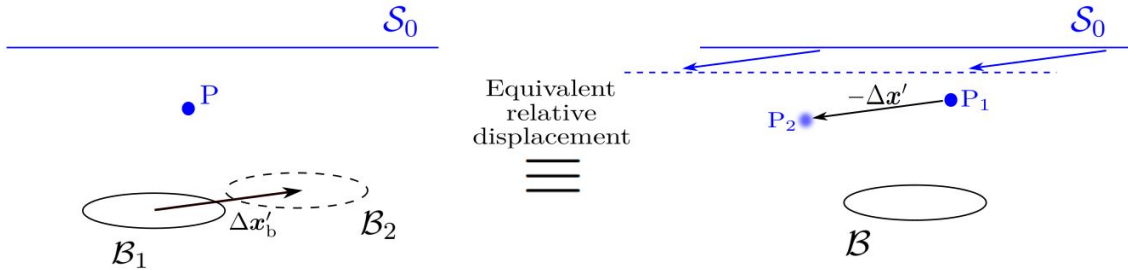


Figure 3.3: Relative displacement of fluid and free surface.

Relative Displacement of a Fluid Particle

When the relative displacement of a fluid particle is applied as shown in Figure 3.3, the following equivalence relation can be derived, where $\nabla \phi(\mathbf{x}')$ is an equivalent expression

for $\frac{\partial \phi}{\partial \mathbf{x}'_b}$.

$$\begin{aligned}
\frac{\partial \phi(\mathbf{x}'_b)}{\partial \mathbf{x}'_b} &= \lim_{\Delta \mathbf{x}'_b \rightarrow 0} \frac{\phi(\mathbf{x}'_b + \Delta \mathbf{x}'_b) - \phi(\mathbf{x}'_b)}{\Delta \mathbf{x}'_b} \\
&\equiv \lim_{-\Delta \mathbf{x}' \rightarrow 0} \frac{\phi(\mathbf{x}' - \Delta \mathbf{x}') - \phi(\mathbf{x}')}{-\Delta \mathbf{x}'} \\
&= \lim_{\Delta \mathbf{x}' \rightarrow 0} \frac{\phi(\mathbf{x}') - \phi(\mathbf{x}' - \Delta \mathbf{x}')}{\Delta \mathbf{x}'} \\
&= \frac{\partial \phi(\mathbf{x}')}{\partial \mathbf{x}'} \\
&= \nabla \phi(\mathbf{x}')
\end{aligned} \tag{3.10}$$

An analogous equivalence relation can be derived for rotational displacement, where the relative displacement of the fluid particle at $\mathbf{x}'$ for a small rotational displacement $\Delta \boldsymbol{\theta}'_b$ of the body $\mathcal{B}$ can be derived. The displacement of a point on the body surface $\mathcal{B}$ due to a small rotational displacement of the body is

$$\begin{aligned}
\Delta \boldsymbol{\theta}'_b \times \mathbf{r} &= (\Delta \theta'_b r_z - \Delta \psi'_b r_y) \mathbf{b}_1 \\
&\quad + (\Delta \psi'_b r_x - \Delta \phi'_b r_z) \mathbf{b}_2 \\
&\quad + (\Delta \phi'_b r_y - \Delta \theta'_b r_x) \mathbf{b}_3
\end{aligned} \tag{3.11}$$

where $\boldsymbol{\theta}'_b = (\phi'_b \ \theta'_b \ \psi'_b)^T$. Also, the displacement of a fluid particle expressed in the body-fixed frame is

$$\Delta \mathbf{x}' = \Delta x' \mathbf{b}_1 + \Delta y' \mathbf{b}_2 + \Delta z' \mathbf{b}_3 \tag{3.12}$$

Therefore, the equivalence relation between the displacement of the body $\mathcal{B}$ and the relative

displacement of a fluid particle while fixing the body $\mathcal{B}$ is

$$\Delta \boldsymbol{\theta}'_{\mathbf{b}} \times \mathbf{r} \equiv -\Delta \mathbf{x}'$$

By using equations (3.11) and (3.12),

$$\begin{aligned} (\Delta \theta'_{\mathbf{b}} r_z - \Delta \psi'_{\mathbf{b}} r_y) \mathbf{b}_1 + (\Delta \psi'_{\mathbf{b}} r_x - \Delta \phi'_{\mathbf{b}} r_z) \mathbf{b}_2 \\ + (\Delta \phi'_{\mathbf{b}} r_y - \Delta \theta'_{\mathbf{b}} r_x) \mathbf{b}_3 \equiv -\Delta x' \mathbf{b}_1 - \Delta y' \mathbf{b}_2 - \Delta z' \mathbf{b}_3 \end{aligned} \quad (3.13)$$

By applying the chain rule to the first element of $\frac{\partial \phi}{\partial \boldsymbol{\theta}'_{\mathbf{b}}}$,

$$\frac{\partial \phi}{\partial \phi'_{\mathbf{b}}} = \frac{(-\partial x')}{\partial \phi'_{\mathbf{b}}} \frac{\partial \phi}{(-\partial x')} + \frac{(-\partial y')}{\partial \phi'_{\mathbf{b}}} \frac{\partial \phi}{(-\partial y')} + \frac{(-\partial z')}{\partial \phi'_{\mathbf{b}}} \frac{\partial \phi}{(-\partial z')}$$

and by applying the equivalence relation in equation (3.10),

$$\frac{\partial \phi}{\partial \phi'_{\mathbf{b}}} \equiv \frac{(-\partial x')}{\partial \phi'_{\mathbf{b}}} \frac{\partial \phi}{\partial x'} + \frac{(-\partial y')}{\partial \phi'_{\mathbf{b}}} \frac{\partial \phi}{\partial y'} + \frac{(-\partial z')}{\partial \phi'_{\mathbf{b}}} \frac{\partial \phi}{\partial z'}$$

Using equation (3.13),

$$\frac{\partial \phi}{\partial \phi'_{\mathbf{b}}} \equiv r_y \frac{\partial \phi}{\partial z'} - r_z \frac{\partial \phi}{\partial y'}$$

The remaining elements of $\frac{\partial \phi}{\partial \boldsymbol{\theta}'_{\mathbf{b}}}$ can be derived through the same process. Therefore,

$$\begin{aligned} \frac{\partial \phi}{\partial \boldsymbol{\theta}'_{\mathbf{b}}} &= \frac{\partial \phi}{\partial \phi'_{\mathbf{b}}} \mathbf{b}_1 + \frac{\partial \phi}{\partial \theta'_{\mathbf{b}}} \mathbf{b}_2 + \frac{\partial \phi}{\partial \psi'_{\mathbf{b}}} \mathbf{b}_3 \\ &\equiv \left(r_y \frac{\partial \phi}{\partial z'} - r_z \frac{\partial \phi}{\partial y'} \right) \mathbf{b}_1 + \left(r_z \frac{\partial \phi}{\partial x'} - r_x \frac{\partial \phi}{\partial z'} \right) \mathbf{b}_2 + \left(r_x \frac{\partial \phi}{\partial y'} - r_y \frac{\partial \phi}{\partial x'} \right) \mathbf{b}_3 \end{aligned}$$

$$\begin{aligned}
&= \mathbf{r} \times \frac{\partial \phi}{\partial \mathbf{x}'} \\
&= \mathbf{r} \times \nabla \phi
\end{aligned} \tag{3.14}$$

Relative displacement of the Undisturbed Free Surface

Applying the relative displacement of the undisturbed free surface $\mathcal{S}_0$ affects the differentiation with respect to $\mathbf{q}'$ under the integral sign, described in Section 3.1. In equations (3.4) and (3.5), the additional term (integral over $\mathcal{B}$) is replaced with an integral over $\mathcal{S}_0$, since the undisturbed free surface $\mathcal{S}_0$ is now relatively displaced while the body $\mathcal{B}$ is fixed.

For translational displacement, instead of the body $\mathcal{B}$ being displaced by $\Delta \mathbf{x}'_b$ with a fixed undisturbed free surface boundary $\mathcal{S}_0$, an equivalent relative displacement of the undisturbed free surface $\mathcal{S}_0$ in the opposite direction $-\Delta \mathbf{x}'$ can be applied while the body $\mathcal{B}$ is fixed. Therefore, substituting $\Delta \mathbf{x}'_b$ with $-\Delta \mathbf{x}'$, an element ΔS of the undisturbed free surface $\mathcal{S}_0$ sweeps out a region of space whose volume is

$$\Delta V \simeq (-\Delta \mathbf{x}') \cdot \mathbf{n} \Delta S$$

Going through the same process as in Section 3.1.2 yields

$$\frac{\partial}{\partial \mathbf{x}'_b} \iiint_{\mathcal{F}} f(\mathbf{x}'_b, t) dV = \iiint_{\mathcal{F}} \frac{\partial f(\mathbf{x}'_b, t)}{\partial \mathbf{x}'_b} dV - \iint_{\mathcal{S}_0} f(\mathbf{x}', t) \mathbf{n} dS \tag{3.15}$$

A similar approach can be taken for the rotational component of $\mathbf{q}'$. For relative rotational displacement, instead of the body being displaced by $\Delta \boldsymbol{\theta}'_b$ with a fixed undisturbed free surface boundary, an equivalent relative displacement $-\Delta \boldsymbol{\theta}'_b$ can be applied to the undisturbed free surface while the body is fixed. Replacing $\Delta \boldsymbol{\theta}'_b \cdot (\mathbf{r} \times \mathbf{n})$ with $-\Delta \boldsymbol{\theta}'_b \cdot (\mathbf{r} \times \mathbf{n})$, an element

ΔS of the undisturbed free surface $\mathcal{S}_0$ sweeps out a volume

$$\Delta V \simeq -\Delta \boldsymbol{\theta}'_{\text{b}} \cdot (\mathbf{r} \times \mathbf{n}) \Delta S$$

Going through the same process as in Section 3.1.2,

$$\frac{\partial}{\partial \boldsymbol{\theta}'_{\text{b}}} \iiint_{\mathcal{F}} f(\boldsymbol{\theta}'_{\text{b}}, t) dV = \iiint_{\mathcal{F}} \frac{\partial f(\boldsymbol{\theta}'_{\text{b}}, t)}{\partial \boldsymbol{\theta}'_{\text{b}}} dV - \iint_{\mathcal{S}_0} f(\boldsymbol{\theta}'_{\text{b}}, t) (\mathbf{r} \times \mathbf{n}) dS \quad (3.16)$$

Hereon, the vector $\mathbf{r}$ in the integral over $\mathcal{S}_0$ denotes the position of a point on $\mathcal{S}_0$ in the body-fixed frame. Also, it is important to note that both the relative displacement of the fluid and the relative displacement of the undisturbed free surface, described in the previous sections, need to be applied simultaneously to the mathematical model. The alternative formulations resulting from the application of the methods described in Sections 3.1 and 3.2 simplify the numerical computation of parameter values, as is discussed in the context of a numerical example explored in Section 4.2.

3.3 A Reformulated Hydrodynamic Force Model for Motion in Calm Water

The reformulation methods proposed in Sections 3.1 and 3.2 can be applied to the motion model derivation described by Battista [1] to yield a reformulated Lagrangian mechanics maneuvering model of which the parameters can be computed using a potential flow code. In this section, the proposed methods are applied to derive a hydrodynamic force model as a first step toward developing the reformulated LNM model.

The hydrodynamic force model for motion in calm water is derived using the undecomposed

total velocity potential ϕ . For a body moving near a free surface, the fluid Lagrangian is

$$\begin{aligned}\mathcal{L}_{\mathcal{F}} &= T_{\mathcal{F}} - V_{\mathcal{F}} \\ &= \iiint_{\mathcal{F}} \frac{1}{2} \rho \nabla \phi \cdot \nabla \phi \, dV - \iiint_{\mathcal{F}} \rho g z \, dV\end{aligned}$$

where $T_{\mathcal{F}}$ and $V_{\mathcal{F}}$ are the kinetic and potential energies of the fluid domain. If we assume neutral buoyancy and homogeneous mass distribution of the body, the potential energy associated with the body and fluid cancel each other out. In addition, if the free surface boundary condition is linearized, potential energy due to wave elevation can be neglected. Then, the fluid Lagrangian simply becomes

$$\mathcal{L}_{\mathcal{F}} = \iiint_{\mathcal{F}} \frac{1}{2} \rho \nabla \phi \cdot \nabla \phi \, dV \quad (3.17)$$

The terms in the Boltzmann-Hamel equations (3.3) can be derived as follows by applying the methods proposed in equations (3.1), (3.15) and (3.16) along with the divergence theorem (See Appendix A.1) to the fluid Lagrangian in equation (3.17).

$$\frac{\partial \mathcal{L}_{\mathcal{F}}}{\partial \boldsymbol{\nu}} = \rho \iint_{\mathcal{B}} \phi \mathbf{b} \, dS + \rho \iint_{S_0} \phi \frac{\partial \boldsymbol{\varphi}_i}{\partial n} \, dS \quad (3.18)$$

$$\frac{\partial \mathcal{L}_{\mathcal{F}}}{\partial \mathbf{q}'} = \rho \iint_{S_0} \phi \tilde{\nabla} \left(\frac{\partial \phi}{\partial n} \right) \, dS - \frac{1}{2} \rho \iint_{S_0} (\nabla \phi \cdot \nabla \phi) \mathbf{b} \, dS \quad (3.19)$$

where

$$\tilde{\nabla} = \begin{pmatrix} \nabla \\ \mathbf{r} \times \nabla \end{pmatrix}$$

Equations (3.18) and (3.19) can be applied to equation (3.3) to derive the resulting hydrodynamic force model for motion in calm water:

$$\begin{aligned}
\mathbf{F}_h = & -\rho \frac{d}{dt} \iint_{\mathcal{B}} \phi \mathbf{b} dS - \rho \frac{d}{dt} \iint_{S_0} \phi \frac{\partial \boldsymbol{\varphi}_i}{\partial n} dS \\
& - \rho \mathbf{G} \iint_{\mathcal{B}} \phi \mathbf{b} dS - \rho \mathbf{G} \iint_{S_0} \phi \frac{\partial \boldsymbol{\varphi}_i}{\partial n} dS \\
& + \rho \iint_{S_0} \phi \tilde{\nabla} \left(\frac{\partial \phi}{\partial n} \right) dS - \frac{1}{2} \rho \iint_{S_0} (\nabla \phi \cdot \nabla \phi) \mathbf{b} dS
\end{aligned} \tag{3.20}$$

where $\mathbf{F}_h$ denotes the vector of hydrodynamic forces and moments on the surface-affected body.

Now, based on equation (3.6), the first term in the first row of equation (3.20) can be expressed as

$$\frac{d}{dt} \iint_{\mathcal{B}} \phi \mathbf{b} dS = \iint_{\mathcal{B}} \frac{\partial \phi}{\partial t} \mathbf{b} dS \tag{3.21}$$

As for the second term in the first row of equation (3.20), the undisturbed free surface is fixed, so the order of the derivative and the integral can simply be interchanged as shown in equation (3.7). Therefore,

$$\frac{d}{dt} \iint_{S_0} \phi \frac{\partial \boldsymbol{\varphi}_i}{\partial n} dS = \iint_{S_0} \left[\frac{\partial \phi}{\partial t} \frac{\partial \boldsymbol{\varphi}_i}{\partial n} + \phi \frac{\partial}{\partial t} \left(\frac{\partial \boldsymbol{\varphi}_i}{\partial n} \right) \right] dS \tag{3.22}$$

Here, the first term in the integral can be removed since $\frac{\partial \phi}{\partial t}$ can be approximated to zero at the undisturbed free surface (See Appendix B).

With respect to the second term in the integral of equation (3.22), the normalized potential vector $\boldsymbol{\varphi}_i$, which is expressed in the body-fixed frame, can be expressed in the earth-fixed frame as $\boldsymbol{\varphi}_{S_0i}$ using the following relation:

$$\boldsymbol{\varphi}_i = \mathbf{J}^T \boldsymbol{\varphi}_{S_0i}$$

since

$$\begin{aligned}
 \phi_i &= \boldsymbol{\varphi}_i^T \boldsymbol{\nu} \\
 &= \boldsymbol{\varphi}_i^T \mathbf{J}^{-1} \mathbf{J} \boldsymbol{\nu} \\
 &= \boldsymbol{\varphi}_i^T \mathbf{J}^{-1} \dot{\mathbf{q}} \\
 &= \boldsymbol{\varphi}_{S_0i}^T \dot{\mathbf{q}}
 \end{aligned}$$

Now, if the translational and rotational components of the normalized potential vector are considered individually,

$$\boldsymbol{\varphi}_i = \begin{pmatrix} \boldsymbol{\varphi}_{it} \\ \boldsymbol{\varphi}_{ir} \end{pmatrix} \quad (3.23)$$

The time derivative of the translational component is

$$\begin{aligned}
 \frac{\partial \boldsymbol{\varphi}_{it}}{\partial t} &= \frac{\partial (\mathbf{R}^T \boldsymbol{\varphi}_{S_0it})}{\partial t} \\
 &= \hat{\boldsymbol{\omega}}^T \mathbf{R}^T \boldsymbol{\varphi}_{S_0it} \\
 &= -\hat{\boldsymbol{\omega}} \boldsymbol{\varphi}_{it} \\
 &= -\boldsymbol{\omega} \times \boldsymbol{\varphi}_{it}
 \end{aligned}$$

Here, $\frac{\partial \boldsymbol{\varphi}_{S_0it}}{\partial t} = \mathbf{0}$ for an arbitrary $\dot{\mathbf{q}}$ (See Appendix B).

Therefore,

$$\frac{\partial}{\partial t} \left(\frac{\partial \boldsymbol{\varphi}_{it}}{\partial n} \right) = -\boldsymbol{\omega} \times \frac{\partial \boldsymbol{\varphi}_{it}}{\partial n} \quad (3.24)$$

Care is required when considering the rotational component of (3.23). Following the procedure in equation (3.14), the rotational component can be expressed as

$$\boldsymbol{\varphi}_{ir} = \frac{\partial \phi}{\partial \boldsymbol{\omega}}$$

$$= \mathbf{r} \times \frac{\partial \phi}{\partial \mathbf{v}} \quad (3.25)$$

where $\mathbf{r}$ in the integral over $\mathcal{S}_0$ is the position of a point on the undisturbed free surface $\mathcal{S}_0$ in the body-fixed frame. If the position of a point on $\mathcal{S}_0$ in the earth-fixed frame is denoted as $\mathbf{r}_{\mathcal{S}_0}$, then

$$\mathbf{r}_{\mathcal{S}_0} = \mathbf{R}\mathbf{r} + \mathbf{x}_b$$

Taking the time derivative,

$$\dot{\mathbf{r}}_{\mathcal{S}_0} = \dot{\mathbf{R}}\mathbf{r} + \mathbf{R}\dot{\mathbf{r}} + \dot{\mathbf{x}}_b$$

Since $\mathbf{r}_{\mathcal{S}_0}$ is constant, $\dot{\mathbf{r}}_{\mathcal{S}_0} = 0$. In addition, some of the terms above can be replaced with ones shown below [1].

$$\dot{\mathbf{x}}_b = \mathbf{R}\mathbf{v}$$

$$\dot{\mathbf{R}} = \mathbf{R}\hat{\boldsymbol{\omega}}$$

Therefore,

$$\dot{\mathbf{r}} = -\mathbf{v} - \boldsymbol{\omega} \times \mathbf{r} \quad (3.26)$$

With respect to the rotational component of the second term in the integral of equation (3.22), by applying equations (3.25), (3.26) and (3.24),

$$\begin{aligned} \frac{\partial}{\partial t} \left(\frac{\partial \varphi_{\text{ir}}}{\partial n} \right) &= \frac{\partial}{\partial t} \left(\mathbf{r} \times \frac{\partial \varphi_{\text{it}}}{\partial n} \right) \\ &= (-\mathbf{v} - \boldsymbol{\omega} \times \mathbf{r}) \times \frac{\partial \varphi_{\text{it}}}{\partial n} + \mathbf{r} \times \frac{\partial}{\partial t} \left(\frac{\partial \varphi_{\text{it}}}{\partial n} \right) \\ &= (-\mathbf{v} - \boldsymbol{\omega} \times \mathbf{r}) \times \frac{\partial \varphi_{\text{it}}}{\partial n} - \mathbf{r} \times \left(\boldsymbol{\omega} \times \frac{\partial \varphi_{\text{it}}}{\partial n} \right) \end{aligned} \quad (3.27)$$

Therefore, applying equations (3.21), (3.24) and (3.27) to the first row of equation (3.20)

yields

$$\begin{aligned}
& -\rho \frac{d}{dt} \iint_{\mathcal{B}} \phi \mathbf{b} dS - \rho \frac{d}{dt} \iint_{\mathcal{S}_0} \phi \frac{\partial \varphi_i}{\partial n} dS \\
& = -\rho \iint_{\mathcal{B}} \frac{\partial \phi}{\partial t} \mathbf{b} dS - \rho \iint_{\mathcal{S}_0} \phi \left(\begin{array}{c} -\boldsymbol{\omega} \times \frac{\partial \varphi_{it}}{\partial n} \\ (-\mathbf{v} - \boldsymbol{\omega} \times \mathbf{r}) \times \frac{\partial \varphi_{it}}{\partial n} - \mathbf{r} \times \left(\boldsymbol{\omega} \times \frac{\partial \varphi_{it}}{\partial n} \right) \end{array} \right) dS \\
& = -\rho \iint_{\mathcal{B}} \frac{\partial \phi}{\partial t} \mathbf{b} dS \\
& \quad - \rho \iint_{\mathcal{S}_0} \phi \left(\begin{array}{c} -\boldsymbol{\omega} \times \frac{\partial \varphi_{it}}{\partial n} \\ -\mathbf{v} \times \frac{\partial \varphi_{it}}{\partial n} - \boldsymbol{\omega} \times \left(\mathbf{r} \times \frac{\partial \varphi_{it}}{\partial n} \right) + \mathbf{r} \times \left(\boldsymbol{\omega} \times \frac{\partial \varphi_{it}}{\partial n} \right) - \mathbf{r} \times \left(\boldsymbol{\omega} \times \frac{\partial \varphi_{it}}{\partial n} \right) \end{array} \right) dS \\
& = -\rho \iint_{\mathcal{B}} \frac{\partial \phi}{\partial t} \mathbf{b} dS - \rho \iint_{\mathcal{S}_0} \phi \left(\begin{array}{c} -\boldsymbol{\omega} \times \frac{\partial \varphi_{it}}{\partial n} \\ -\mathbf{v} \times \frac{\partial \varphi_{it}}{\partial n} - \boldsymbol{\omega} \times \frac{\partial \varphi_{ir}}{\partial n} \end{array} \right) dS \tag{3.28}
\end{aligned}$$

The second term in the second row of equation (3.20) can be rewritten as

$$\begin{aligned}
-\rho \mathbf{G} \iint_{\mathcal{S}_0} \phi \frac{\partial \varphi_i}{\partial n} dS & = -\rho \begin{pmatrix} \hat{\boldsymbol{\omega}} & \mathbf{0} \\ \hat{\mathbf{v}} & \hat{\boldsymbol{\omega}} \end{pmatrix} \iint_{\mathcal{S}_0} \phi \frac{\partial \varphi_i}{\partial n} dS \\
& = -\rho \iint_{\mathcal{S}_0} \phi \left(\begin{array}{c} \boldsymbol{\omega} \times \frac{\partial \varphi_{it}}{\partial n} \\ \mathbf{v} \times \frac{\partial \varphi_{it}}{\partial n} + \boldsymbol{\omega} \times \frac{\partial \varphi_{ir}}{\partial n} \end{array} \right) dS \tag{3.29}
\end{aligned}$$

Now, the integrals over $\mathcal{S}_0$ in the first row (3.28) and second row (3.29) cancel each other out, and the resulting reformulated hydrodynamic force model can now be expressed as

$$\begin{aligned}
\mathbf{F}_h & = -\rho \iint_{\mathcal{B}} \frac{\partial \phi}{\partial t} \mathbf{b} dS - \rho \mathbf{G} \iint_{\mathcal{B}} \phi \mathbf{b} dS \\
& \quad + \rho \iint_{\mathcal{S}_0} \phi \tilde{\nabla} \left(\frac{\partial \phi}{\partial n} \right) dS - \frac{1}{2} \rho \iint_{\mathcal{S}_0} (\nabla \phi \cdot \nabla \phi) \mathbf{b} dS \tag{3.30}
\end{aligned}$$

3.4 A Reformulated Lagrangian Nonlinear Maneuvering (LNM) Model

Combining the rigid body dynamics with the hydrodynamic forces on the body (3.30) results in the following reformulated LNM model:

$$\begin{aligned}
 \mathbf{F} = & \mathbf{M}_b \dot{\boldsymbol{\nu}} + \mathbf{G} \mathbf{M}_b \boldsymbol{\nu} \\
 & + \rho \iint_{\mathcal{B}} \frac{\partial \phi}{\partial t} \mathbf{b} dS + \rho \mathbf{G} \iint_{\mathcal{B}} \phi \mathbf{b} dS \\
 & - \rho \iint_{S_0} \phi \tilde{\nabla} \left(\frac{\partial \phi}{\partial n} \right) dS + \frac{1}{2} \rho \iint_{S_0} (\nabla \phi \cdot \nabla \phi) \mathbf{b} dS
 \end{aligned} \tag{3.31}$$

Now we revisit the decomposition of the total velocity potential in equations (2.44) and (3.8):

$$\begin{aligned}
 \phi(\mathbf{q}, \boldsymbol{\nu}, t) &= (\boldsymbol{\varphi}_K + \boldsymbol{\varphi}_s) \cdot \boldsymbol{\nu} + \phi_m \\
 &= \boldsymbol{\varphi}_i \cdot \boldsymbol{\nu} + \phi_m
 \end{aligned} \tag{3.32}$$

Equation (3.32) can be applied to each term of the reformulated LNM model (3.31):

$$\begin{aligned}
 \rho \iint_{\mathcal{B}} \frac{\partial \phi}{\partial t} \mathbf{b} dS &= \rho \iint_{\mathcal{B}} \frac{\partial}{\partial t} [(\boldsymbol{\varphi}_K + \boldsymbol{\varphi}_s) \cdot \boldsymbol{\nu} + \phi_m] \mathbf{b} dS \\
 &= \rho \iint_{\mathcal{B}} \left[\left(\frac{\partial \boldsymbol{\varphi}_s}{\partial t} \cdot \boldsymbol{\nu} \right) \mathbf{b} + (\boldsymbol{\varphi}_i \cdot \dot{\boldsymbol{\nu}}) \mathbf{b} + \frac{\partial \phi_m}{\partial t} \mathbf{b} \right] dS \\
 &= \rho \iint_{\mathcal{B}} \left[\left(\left(\sum_{i=1}^6 \nu_i \frac{\partial \boldsymbol{\varphi}_s}{\partial q'_i} \right) \cdot \boldsymbol{\nu} \right) \mathbf{b} + (\boldsymbol{\varphi}_i \cdot \dot{\boldsymbol{\nu}}) \mathbf{b} + \frac{\partial \phi_m}{\partial t} \mathbf{b} \right] dS \\
 &= \rho \iint_{\mathcal{B}} \left[(\mathbf{b} \boldsymbol{\varphi}_K^T) \dot{\boldsymbol{\nu}} + (\mathbf{b} \boldsymbol{\varphi}_s^T) \dot{\boldsymbol{\nu}} + \left(\sum_{i=1}^6 \nu_i \mathbf{b} \tilde{\nabla}_i \boldsymbol{\varphi}_s^T \right) \boldsymbol{\nu} + \frac{\partial \phi_m}{\partial t} \mathbf{b} \right] dS \\
 \\
 \rho \mathbf{G} \iint_{\mathcal{B}} \phi \mathbf{b} dS &= \rho \mathbf{G} \iint_{\mathcal{B}} [(\boldsymbol{\varphi}_K + \boldsymbol{\varphi}_s) \cdot \boldsymbol{\nu} + \phi_m] \mathbf{b} dS \\
 &= \rho \mathbf{G} \iint_{\mathcal{B}} [(\mathbf{b} \boldsymbol{\varphi}_K^T) \boldsymbol{\nu} + (\mathbf{b} \boldsymbol{\varphi}_s^T) \boldsymbol{\nu} + \phi_m \mathbf{b}] dS
 \end{aligned}$$

$$\begin{aligned}
-\rho \iint_{S_0} \phi \tilde{\nabla} \left(\frac{\partial \phi}{\partial n} \right) dS &= -\rho \iint_{S_0} (\boldsymbol{\varphi}_i \cdot \boldsymbol{\nu} + \phi_m) \tilde{\nabla} \left[\frac{\partial (\boldsymbol{\varphi}_i \cdot \boldsymbol{\nu} + \phi_m)}{\partial n} \right] dS \\
&= -\rho \iint_{S_0} \left(\phi_m \tilde{\nabla} \frac{\partial \boldsymbol{\varphi}_i^T}{\partial n} \boldsymbol{\nu} + \phi_m \tilde{\nabla} \frac{\partial \phi_m}{\partial n} \right) dS \\
\frac{1}{2} \rho \iint_{S_0} (\nabla \phi \cdot \nabla \phi) \mathbf{b} dS &= \frac{1}{2} \rho \iint_{S_0} (\nabla \phi_i \cdot \nabla \phi_i + \nabla \phi_i \cdot \nabla \phi_m + \nabla \phi_m \cdot \nabla \phi_i + \nabla \phi_m \cdot \nabla \phi_m) \mathbf{b} dS \\
&= \frac{1}{2} \rho \iint_{S_0} \left\{ \left[\sum_{i=1}^6 \left(\frac{\partial \varphi_{ii}}{\partial n} \mathbf{b} \frac{\partial \boldsymbol{\varphi}_i^T}{\partial n} \right) \boldsymbol{\nu} \mathbf{e}_i^T \right] \boldsymbol{\nu} \right. \\
&\quad \left. + (\nabla \phi_i \cdot \nabla \phi_m + \nabla \phi_m \cdot \nabla \phi_i + \nabla \phi_m \cdot \nabla \phi_m) \mathbf{b} \right\} dS
\end{aligned}$$

Therefore, the resulting decomposed equations of motion can be expressed as

$$\begin{aligned}
\mathbf{F} &= \left(\mathbf{M}_b + \rho \iint_{\mathcal{B}} \mathbf{b} \boldsymbol{\varphi}_K^T dS + \rho \iint_{\mathcal{B}} \mathbf{b} \boldsymbol{\varphi}_s^T dS \right) \dot{\boldsymbol{\nu}} + \mathbf{G} \left(\mathbf{M}_b + \rho \iint_{\mathcal{B}} \mathbf{b} \boldsymbol{\varphi}_K^T dS + \rho \iint_{\mathcal{B}} \mathbf{b} \boldsymbol{\varphi}_s^T dS \right) \boldsymbol{\nu} \\
&\quad + \left(\rho \iint_{\mathcal{B}} \sum_{i=1}^6 \nu_i \mathbf{b} \tilde{\nabla}_i \boldsymbol{\varphi}_s^T dS - \rho \iint_{S_0} \phi_m \tilde{\nabla} \frac{\partial \boldsymbol{\varphi}_i^T}{\partial n} dS + \frac{1}{2} \rho \iint_{S_0} \sum_{i=1}^6 \left(\frac{\partial \varphi_{ii}}{\partial n} \mathbf{b} \frac{\partial \boldsymbol{\varphi}_i^T}{\partial n} \right) \boldsymbol{\nu} \mathbf{e}_i^T dS \right) \boldsymbol{\nu} \\
&\quad + \rho \iint_{\mathcal{B}} \frac{\partial \phi_m}{\partial t} \mathbf{b} dS + \rho \mathbf{G} \iint_{\mathcal{B}} \phi_m \mathbf{b} dS - \rho \iint_{S_0} \phi_m \tilde{\nabla} \frac{\partial \phi_m}{\partial n} dS \\
&\quad + \frac{1}{2} \rho \iint_{S_0} (\nabla \phi_i \cdot \nabla \phi_m + \nabla \phi_m \cdot \nabla \phi_i + \nabla \phi_m \cdot \nabla \phi_m) \mathbf{b} dS
\end{aligned} \tag{3.33}$$

Reorganizing the reformulated LNM model (3.33) yields the following equations of motion in matrix-vector form, a form motivated by robot dynamic formulations that are helpful in recognizing and exploiting physical system properties [47].

$$\mathbf{F} = (\mathbf{M}_b + \mathbf{M}_B + \mathbf{M}_2(\mathbf{q})) \dot{\boldsymbol{\nu}} + \mathbf{G}(\mathbf{M}_b + \mathbf{M}_B + \mathbf{M}_2(\mathbf{q})) \boldsymbol{\nu} + \mathbf{D}(\mathbf{q}, \boldsymbol{\nu}, t) \boldsymbol{\nu} + \mathbf{E}(\mathbf{q}, \boldsymbol{\nu}, t) \tag{3.34}$$

Here, $\mathbf{M}_b$, $\mathbf{M}_B$ and $\mathbf{M}_2$ are identical to the matrices in equations (2.37), (2.46) and (2.48),

and

$$\begin{aligned}
\mathbf{D} &= \rho \iint_{\mathcal{B}} \sum_{i=1}^6 \nu_i \mathbf{b} \tilde{\nabla}_i \boldsymbol{\varphi}_s^T dS - \rho \iint_{S_0} \phi_m \tilde{\nabla} \frac{\partial \boldsymbol{\varphi}_i^T}{\partial n} dS \\
&\quad + \frac{1}{2} \rho \iint_{S_0} \sum_{i=1}^6 \left(\frac{\partial \varphi_{ii}}{\partial n} \mathbf{b} \frac{\partial \boldsymbol{\varphi}_i^T}{\partial n} \right) \boldsymbol{\nu} \mathbf{e}_i^T dS \\
\mathbf{E} &= \rho \iint_{\mathcal{B}} \frac{\partial \phi_m}{\partial t} \mathbf{b} dS + \rho \mathbf{G} \iint_{\mathcal{B}} \phi_m \mathbf{b} dS - \rho \iint_{S_0} \phi_m \tilde{\nabla} \frac{\partial \phi_m}{\partial n} dS \\
&\quad + \frac{1}{2} \rho \iint_{S_0} (\nabla \phi_i \cdot \nabla \phi_m + \nabla \phi_m \cdot \nabla \phi_i + \nabla \phi_m \cdot \nabla \phi_m) \mathbf{b} dS
\end{aligned}$$

The right hand side of (3.34) is equivalent to the LNM model in (2.62); each term has a corresponding term in the other formulation. The correspondence is not apparent at first glance because equation (3.34) is obtained by performing the differentiation under the integral sign (Section 3.1) first and then applying the divergence theorem (See Appendix A.1), while the LNM model in (2.62) is derived in the opposite order. Once the equivalence in Appendix A.2 and hence the correspondence in terms is recognized, the physical insight and useful properties described in [9] for the LNM model can be carried over to the model in (3.34).

Linearizing the LNM model (3.34) by evaluating the body boundary condition along the mean position of the body surface and by removing higher order terms not included in the linearized Bernoulli equation, as described in [1], yields the equations of motion presented by Cummins [45]. Cummins' equations can be useful when studying the contribution of the impulsive and memory terms for small perturbations from the mean position of the body in slowly flowing surrounding fluid. On the other hand, the LNM model (3.34) will be able to provide additional nonlinear terms critical in computing hydrodynamic forces generated during aggressive maneuvers.

3.5 A Reformulated Lagrangian Nonlinear Maneuvering and Seakeeping (LNMS) Model

Incident wave forces are incorporated into the reformulated LNM model (3.31) as an external force $\mathbf{F}_I$, which accelerates the body and counteracts the forces due to radiation and diffraction of waves. Incident wave force can be expressed using the incident wave potential as

$$\mathbf{F}_I = -\rho \iint_{\mathcal{B}} \left(\frac{\partial \phi_I}{\partial t} + \frac{1}{2} \nabla \phi_I \cdot \nabla \phi_I \right) \mathbf{b} dS \quad (3.35)$$

And the reformulated LNMS model can be written as

$$\begin{aligned} \mathbf{F} + \mathbf{F}_I &= \mathbf{M}_b \dot{\boldsymbol{\nu}} + \mathbf{G} \mathbf{M}_b \boldsymbol{\nu} \\ &+ \rho \iint_{\mathcal{B}} \frac{\partial \phi}{\partial t} \mathbf{b} dS + \rho \mathbf{G} \iint_{\mathcal{B}} \phi \mathbf{b} dS \\ &- \rho \iint_{\mathcal{S}_0} \phi \tilde{\nabla} \left(\frac{\partial \phi}{\partial n} \right) dS + \frac{1}{2} \rho \iint_{\mathcal{S}_0} (\nabla \phi \cdot \nabla \phi) \mathbf{b} dS \end{aligned} \quad (3.36)$$

where the velocity potential ϕ now accounts for (i) radiation waves due to body motion in the absence of incident waves and (ii) *diffraction waves due to scattering of the incident waves*. Finally, the decomposed reformulated LNMS model is expressed in matrix-vector form as

$$\mathbf{F} + \mathbf{F}_I = (\mathbf{M}_b + \mathbf{M}_B + \mathbf{M}_2(\mathbf{q})) \dot{\boldsymbol{\nu}} + \mathbf{G} (\mathbf{M}_b + \mathbf{M}_B + \mathbf{M}_2(\mathbf{q})) \boldsymbol{\nu} + \mathbf{D}(\mathbf{q}, \boldsymbol{\nu}, t) \boldsymbol{\nu} + \mathbf{E}(\mathbf{q}, \boldsymbol{\nu}, t) \quad (3.37)$$

where

$$\begin{aligned} \mathbf{D} &= \rho \iint_{\mathcal{B}} \sum_{i=1}^6 \nu_i \mathbf{b} \tilde{\nabla}_i \boldsymbol{\varphi}_s^T dS - \rho \iint_{\mathcal{S}_0} \phi_m \tilde{\nabla} \frac{\partial \boldsymbol{\varphi}_i^T}{\partial n} dS \\ &+ \frac{1}{2} \rho \iint_{\mathcal{S}_0} \sum_{i=1}^6 \left(\frac{\partial \varphi_{ii}}{\partial n} \mathbf{b} \frac{\partial \boldsymbol{\varphi}_i^T}{\partial n} \right) \boldsymbol{\nu} \mathbf{e}_i^T dS \end{aligned}$$

$$\begin{aligned}
\mathbf{E} = & \rho \iint_{\mathcal{B}} \frac{\partial \phi_{\mathbf{m}}}{\partial t} \mathbf{b} \, dS + \rho \mathbf{G} \iint_{\mathcal{B}} \phi_{\mathbf{m}} \mathbf{b} \, dS - \rho \iint_{S_0} \phi_{\mathbf{m}} \tilde{\nabla} \frac{\partial \phi_{\mathbf{m}}}{\partial n} \, dS \\
& + \frac{1}{2} \rho \iint_{S_0} (\nabla \phi_{\mathbf{i}} \cdot \nabla \phi_{\mathbf{m}} + \nabla \phi_{\mathbf{m}} \cdot \nabla \phi_{\mathbf{i}} + \nabla \phi_{\mathbf{m}} \cdot \nabla \phi_{\mathbf{m}}) \mathbf{b} \, dS
\end{aligned}$$

Chapter 4

Parameter Determination

Although parameters of a parametric ODE model are typically estimated using system identification techniques, a different approach is taken in this section for the reformulated models in equations (3.30), (3.34) and (3.37). The parameters of the reformulated models, expressed as integro-differential functions of the velocity potential, can be computed and approximated by using data from a modified time-domain seakeeping panel code. The computation process offers a more accurate and less computationally expensive method of determining the parameters, compared to conventional parameter estimation techniques. At the beginning of this section, the potential flow code that is used, and the code modifications for the required calculations, are described. Then, validity of the proposed methods in Sections 3.1 and 3.2 are confirmed using the modified code. Subsequently, forces and moments are computed using the reformulated hydrodynamic force model (3.30), and the results are numerically validated against computations from the basic panel code solution. Finally, methods for computing the impulsive terms and approximating the memory terms in the parameters of the decomposed reformulated motion models in equations (3.34) and (3.37) are presented.

4.1 Time-Domain Seakeeping Code (Aegir)

The time-domain nonlinear seakeeping code Aegir (Figure 4.1) can be used to compute the velocity potential and its derivatives that appear in the reformulated models in equations

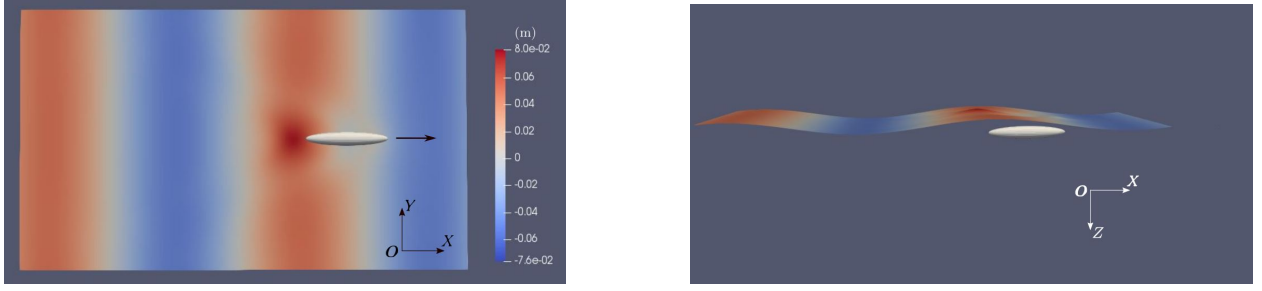


Figure 4.1: Surface elevation for a horizontal, prolate spheroid approaching the free surface at constant forward speed ($Fr = 0.35$). A scale factor of 3.0 was used in (b) to exaggerate the wave elevations.

(3.30), (3.34) and (3.37). Aegir is a medium-fidelity potential flow code that uses a boundary element method to compute steady and unsteady wave patterns, forces, and motions of vehicles operating at or near a free surface [48].

Aegir, as it stands, computes the memory potential ϕ , its time derivative $\frac{\partial \phi}{\partial t}$, as well as its gradient $\nabla \phi$, which appear in the reformulated models in equations (3.30), (3.34) and (3.37). Also, the code computes ϕ_I and its time derivative $\frac{\partial \phi_I}{\partial t}$ in equation (3.37). Other values, such as the second-order partial derivative $\tilde{\nabla} \left(\frac{\partial \phi}{\partial n} \right)$ or the normalized potential vector for impulsive flow φ_i , which appear in the model formulation in (3.34) and (3.37), require some modification of the code. With respect to $\tilde{\nabla} \left(\frac{\partial \phi}{\partial n} \right)$, the values needed to calculate the term can be readily obtained from the panel code. Spatial discretization in Aegir employs a bi-quadratic normal B-spline representation, and therefore the first and second derivatives of the function can be obtained directly without implementing any additional finite difference methods [19]. While $\tilde{\nabla} \left(\frac{\partial \phi}{\partial n} \right)$ can be obtained with minor modifications using computations readily available from the panel code [48, 49], computing φ_i requires more effort.

The normalized potential vector for impulsive flow φ_i is not obtained from the time-domain solutions of the flow along with the other parameters of the model, but they are derived separately by solving for the instantaneous response of the fluid due to an impulsive motion

of the body. For example, the surge element of the normalized potential vector φ_i is obtained by prescribing unit velocity in surge on a body while imposing $\phi_i = 0$ at the undisturbed free surface $\mathcal{S}_0$, a condition derived from the linearized free surface boundary condition for impulsive motion of the body [50]. The resulting velocity potential for the instantaneous reponse of the fluid will be equal to the first element of the normalized potential vector for impulsive flow φ_i . The remaining elements of the normalized potential vector can be obtained in the same manner but by prescribing unit velocity in the corresponding direction.

The parameters in the reformulated hydrodynamic force model (3.30), reformulated Lagrangian nonlinear maneuvering (LNM) model (3.34) and reformulated Lagrangian nonlinear maneuvering and seakeeping (LNMS) model (3.37) can be computed once the potentials ϕ_I and ϕ , their derivatives, and the normalized potential vector for impulsive flow φ_i are obtained and integrated over the surfaces bounding the fluid. As a matter of fact, any conventional potential flow code can be used to compute the terms in the reformulated models, as long as the aforementioned basic outputs are available and modifications are feasible. Strictly speaking, the code should be able to compute the total and incident wave potentials ϕ and ϕ_I , and their respective time derivatives $\frac{\partial \phi}{\partial t}$ and $\frac{\partial \phi_I}{\partial t}$, as well as the fluid velocity $\nabla \phi$. Also, modifications to the code must be possible in order to compute the second-order partial derivative $\tilde{\nabla} \left(\frac{\partial \phi}{\partial n} \right)$. In addition, the application of the condition $\phi_i = 0$ at the undisturbed free surface $\mathcal{S}_0$ must be possible in order to compute the normalized potential vector for impulsive flow φ_i . Finally, the integration of all values must be possible over the body $\mathcal{B}$ and the undisturbed free surface $\mathcal{S}_0$.

✱ Aegir is developed and managed by Navatek LLC. The source code that includes the modifications described in this section is stored in Navatek's archives.

4.2 Numerical Validation

Parameters of the reformulated models from the previous chapter are computed by using simulation data obtained from the modified version of the seakeeping code in Section 4.1. First, the validity of the methods proposed in Sections 3.1 and 3.2 is confirmed by examining the computation of two terms in the hydrodynamic force model (3.30) for impulsive motion in calm water. Next, the steady forces and moment on a spheroidal body in longitudinal motion near a free surface in steady flow are computed using the hydrodynamic force model (3.30), and the results are compared to simulation data. Then, the LNM (3.34) and LNMS (3.37) models are numerically validated by computing unsteady forces and moment acting on a body undergoing various motions in calm water and in waves, and comparing them to forces and moment from simulation data.

4.2.1 Confirmation of Proposed Methods

The two terms on the right-hand side of (3.19), obtained by applying the methods proposed in Sections 3.1 and 3.2, correspond to the last two terms in the hydrodynamic force model for motion in calm water (3.20). If we assume that the body-fixed frame is aligned with the earth-fixed frame, the translational components of these terms for impulsive motion are

$$\begin{aligned} \frac{\partial \mathcal{L}_{\mathcal{F}}}{\partial \mathbf{x}_b} &= \rho \iint_{S_0} \phi_i \tilde{\nabla} \left(\frac{\partial \phi_i}{\partial n} \right) dS - \frac{1}{2} \rho \iint_{S_0} (\nabla \phi_i \cdot \nabla \phi_i) \mathbf{n} dS \\ &= -\frac{1}{2} \rho \iint_{S_0} (\nabla \phi_i \cdot \nabla \phi_i) \mathbf{n} dS \end{aligned} \quad (4.1)$$

since ϕ_i , the impulsive flow component of the total velocity potential, is equal to zero at the undisturbed free surface S_0 [45, 50].

If we first consider the elements x_b and y_b of the body position vector $\mathbf{x}_b$, equation (4.1)

gives

$$\frac{\partial \mathcal{L}_{\mathcal{F}}}{\partial x_b} = -\frac{1}{2}\rho \iint_{S_0} (\nabla \phi_i \cdot \nabla \phi_i) n_x dS = 0 \quad (4.2)$$

and

$$\frac{\partial \mathcal{L}_{\mathcal{F}}}{\partial y_b} = -\frac{1}{2}\rho \iint_{S_0} (\nabla \phi_i \cdot \nabla \phi_i) n_y dS = 0 \quad (4.3)$$

since $n_x = n_y = 0$ at the undisturbed free surface S_0 .

The validity of equations (4.2) and (4.3) can be confirmed by examining the left hand side of equation (4.1) and applying the divergence theorem and the finite difference method, as shown below.

$$\begin{aligned} \frac{\partial \mathcal{L}_{\mathcal{F}}}{\partial \mathbf{x}_b} &= \frac{1}{2}\rho \frac{\partial}{\partial \mathbf{x}_b} \iiint_{\mathcal{F}} \nabla \phi_i \cdot \nabla \phi_i dV \\ &= \frac{1}{2}\rho \frac{\partial}{\partial \mathbf{x}_b} \iint_{B+S_0} \phi_i \frac{\partial \phi_i}{\partial n} dS \\ &= \frac{1}{2}\rho \frac{\partial}{\partial \mathbf{x}_b} \iint_B \phi_i \frac{\partial \phi_i}{\partial n} dS \\ &\simeq \frac{1}{2}\rho \frac{\Delta \left(\iint_B \phi_i \frac{\partial \phi_i}{\partial n} dS \right)}{\Delta \mathbf{x}_b} \end{aligned} \quad (4.4)$$

Now, the value of $\frac{\partial \mathcal{L}_{\mathcal{F}}}{\partial \mathbf{x}_b}$ evaluated at a certain body position $\mathbf{x}_b$ can be computed using equation (4.1) and compared to the finite difference method calculation in equation (4.4), which is computed using output data at the same position $\mathbf{x}_b$ as well as output data at position $\mathbf{x}_b + \Delta \mathbf{x}_b$.

The left hand side of equations (4.2) and (4.3) can be approximated by applying the finite

difference method in equation (4.4):

$$\frac{\partial \mathcal{L}_{\mathcal{F}}}{\partial x_b} \simeq \frac{1}{2} \rho \frac{\Delta \left(\iint_{\mathcal{B}} \phi_i \frac{\partial \phi_i}{\partial n} dS \right)}{\Delta x_b} = 0 \quad (4.5)$$

and

$$\frac{\partial \mathcal{L}_{\mathcal{F}}}{\partial y_b} \simeq \frac{1}{2} \rho \frac{\Delta \left(\iint_{\mathcal{B}} \phi_i \frac{\partial \phi_i}{\partial n} dS \right)}{\Delta y_b} = 0 \quad (4.6)$$

since ϕ_i does not depend on x_b nor y_b for a submerged body in motion near an otherwise calm free surface. Therefore, the calculations from the proposed method, shown in equations (4.2) and (4.3), and the finite difference method, shown in equations (4.5) and (4.6), are equal for elements x_b and y_b .

For the remaining element z_b , which denotes the depth of the body, equations (4.1) and (4.4) can be rewritten as follows.

$$\frac{\partial \mathcal{L}_{\mathcal{F}}}{\partial z_b} = -\frac{1}{2} \rho \iint_{\mathcal{S}_0} (\nabla \phi_i \cdot \nabla \phi_i) n_z dS \quad (4.7)$$

$$\frac{\partial \mathcal{L}_{\mathcal{F}}}{\partial z_b} \simeq \frac{1}{2} \rho \frac{\Delta \left(\iint_{\mathcal{B}} \phi_i \frac{\partial \phi_i}{\partial n} dS \right)}{\Delta z_b} \quad (4.8)$$

These equations are computed numerically for comparison by solving the impulsive flow for a 6:1 prolate spheroid in the conditions shown in Table 4.1 using the modified seakeeping code in Section 4.1. The free surface size (domain size) in the panel code simulations is determined by the distances to the upstream, downstream and exterior edges, as shown in Figure 4.2. In order to simulate impulsive motion, the boundary condition $\phi_i = 0$ is imposed at the undisturbed free surface $\mathcal{S}_0$.

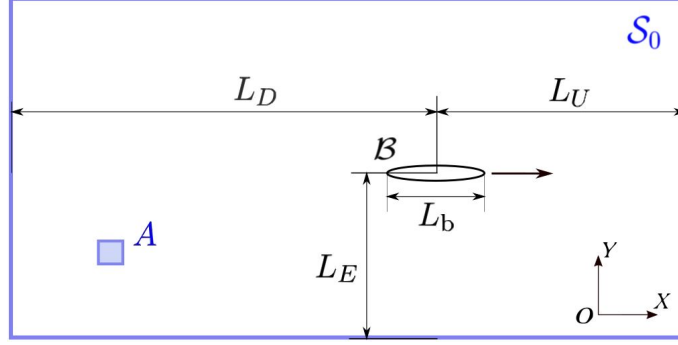


Figure 4.2: Horizontal dimensions of the computational domain.

The proposed method in equation (4.7) is computed using flow solutions at depth-to-body length ratios of 0.15 and 0.30. Values obtained using the finite difference method in equation (4.8), on the other hand, are computed using flow solutions at depth-to-body length ratios of 0.15 and 0.30 and at the slightly larger ratios of 0.1505 and 0.3005. The resulting computations for equations (4.7) and (4.8) at depth-to-body length ratios of 0.15 and 0.30 are shown in Table 4.2.

Overall, computations of $\frac{\partial \mathcal{L}_F}{\partial z_b}$ using the proposed method (4.7) agree well with the results from the finite difference method (4.8). More accurate results can be obtained by reducing the panel area A and the increment in depth Δz_b . These calculations illustrate the validity of the alternative formulation presented in Sections 3.1 and 3.2 for computing partial derivatives of the fluid Lagrangian with respect to translational configuration variables for a body undergoing impulsive motion. The main advantage in employing this alternative formulation is to increase accuracy and efficiency when computing the parameters of the Lagrangian mechanical system model.

Table 4.1: Input Variables - Confirmation of Methods

Variable	Value
Length-to-diameter ratio	6:1
Body length, L_b	2 m
Fluid density, ρ	1000 kg/m ³
Depth-to-length ratio, z_b/L_b	{0.1500, 0.1505, 0.3000, 0.3005}
Direction of motion	{ surge, sway, heave, roll, pitch, yaw }
Speed of motion	Translation: 1 m/s Rotation: 1 rad/s
Free surface dimensions (L_U, L_D, L_E)	{ $2.5L_b, 4.25L_b, 1.75L_b$ }
Panel area, A/L_b^2	0.0075

4.2.2 Computation of Hydrodynamic Forces on a Submerged Body Maneuvering in Calm Water

Steady Forces Acting on a Submerged Body

When a body is in steady translational motion parallel to the free surface, the flow around the body reaches a steady state, and as a consequence, constant forces and moments act on the body. These forces and moments can be derived from equation (3.30), where ϕ is now constant, and therefore the hydrodynamic force model can be rewritten as

$$\mathbf{F}_h = -\rho \mathbf{G} \iint_{\mathcal{B}} \phi \mathbf{b} dS + \rho \iint_{S_0} \phi \tilde{\nabla} \left(\frac{\partial \phi}{\partial n} \right) dS - \frac{1}{2} \rho \iint_{S_0} (\nabla \phi \cdot \nabla \phi) \mathbf{b} dS \quad (4.9)$$

Table 4.2: Computations of $\frac{\partial \mathcal{L}_F}{\partial z_b}$

	$z_b/L_b = 0.15$		$z_b/L_b = 0.3$	
Direction of motion	Proposed method (eq. 4.7)	FDM [†] (eq. 4.8)	Proposed method (eq. 4.7)	FDM (eq. 4.8)
surge	-0.0046	-0.0046	-0.0011	-0.0011
sway	-0.0693	-0.0688	-0.0076	-0.0070
heave	-0.0821	-0.0814	-0.0110	-0.0110
roll	-0.0062	-0.0062	-0.0027	-0.0027
pitch	-0.0081	-0.0079	-0.0003	-0.0004
yaw	-0.0068	-0.0067	-0.0004	-0.0004

FDM[†], Finite Difference Method

From equation (4.9), the following formulas for surge force, heave force and pitch moment acting on the body due to steady translational motion parallel to the free surface can be derived.

$$X = \rho \iint_{S_0} \phi \frac{\partial^2 \phi}{\partial x \partial n} dS - \frac{1}{2} \rho \iint_{S_0} (\nabla \phi \cdot \nabla \phi) n_x dS \quad (4.10)$$

$$Z = \rho \iint_{S_0} \phi \frac{\partial^2 \phi}{\partial n^2} dS - \frac{1}{2} \rho \iint_{S_0} (\nabla \phi \cdot \nabla \phi) n_z dS \quad (4.11)$$

$$M = \rho \iint_B \phi u n_z dS + \rho \iint_{S_0} \phi \left(r_z \frac{\partial^2 \phi}{\partial x \partial n} - r_x \frac{\partial^2 \phi}{\partial n^2} \right) dS - \frac{1}{2} \rho \iint_{S_0} (\nabla \phi \cdot \nabla \phi) (r_z n_x - r_x n_z) dS \quad (4.12)$$

Using the modified seakeeping code in Section 4.1 to compute equations (4.10) - (4.12), the constant forces and moment on a 6:1 prolate spheroid in steady motion along its longitudinal axis parallel to the free surface were derived. The effects of panel and free surface size on force and moment computations are examined in this section. In addition, force and moment computations at various speeds and depths are compared to panel code output values.

Table 4.3: Input Variables - Panel Area Variation

Variable	Value
Length-to-diameter ratio	6:1
Body length, L_b	2 m
Fluid density, ρ	1000 kg/m ³
Depth-to-length ratio, z_b/L_b	0.15
Direction of motion	surge
Froude number, $Fr = u/\sqrt{gL}$	{0.3, 0.35, 0.4, 0.45, 0.5}
Free surface dimensions (L_U , L_D , L_E)	$\{1.5L_b, 4L_b, 1.6L_b\}$
Panel area, A/L_b^2	{0.00125, 0.0025, 0.005}

Varying the dimensions of the panels over the body surface and the undisturbed free surface affects accuracy of the flow solution. However, when computing forces using equations (4.10) - (4.12), which are mostly the evaluation of integrals over the free surface, only changes in panel dimensions on the free surface have significant effect; changes in panel dimensions on the body have relatively minimal impact. The aspect ratio of the panels at the free surface is set to one, as recommended in [48]. Accordingly, the effect of free surface panel size on computed hydrodynamic forces is studied for the conditions shown in Table 4.3.

General guidelines for determining fluid domain dimensions in CFD simulations recommend that the upstream and exterior boundaries be at 1~2 body lengths from the hull and the downstream boundary at 3~5 body lengths [51]. The distances to the downstream (L_D) and exterior (L_E) edges shown in Table 4.3 are set to the median value of the recommended range. Reducing the distance to the downstream edge below $3L_b$ can have a detrimental effect on accuracy. This matter is investigated later in this section by examining the effect of free surface size on hydrodynamic force computations.

For now, free surface size is kept constant while panel size is varied. Figure 4.3 shows simulation results for the distribution of normal velocity over the undisturbed free surface $\mathcal{S}_0$. The normal velocity and its derivatives are key values used to compute the surge force, heave force and pitch moment in equations (4.10) - (4.12).

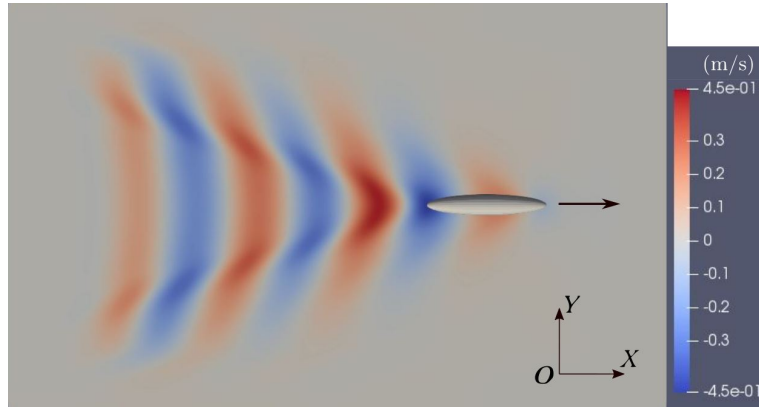
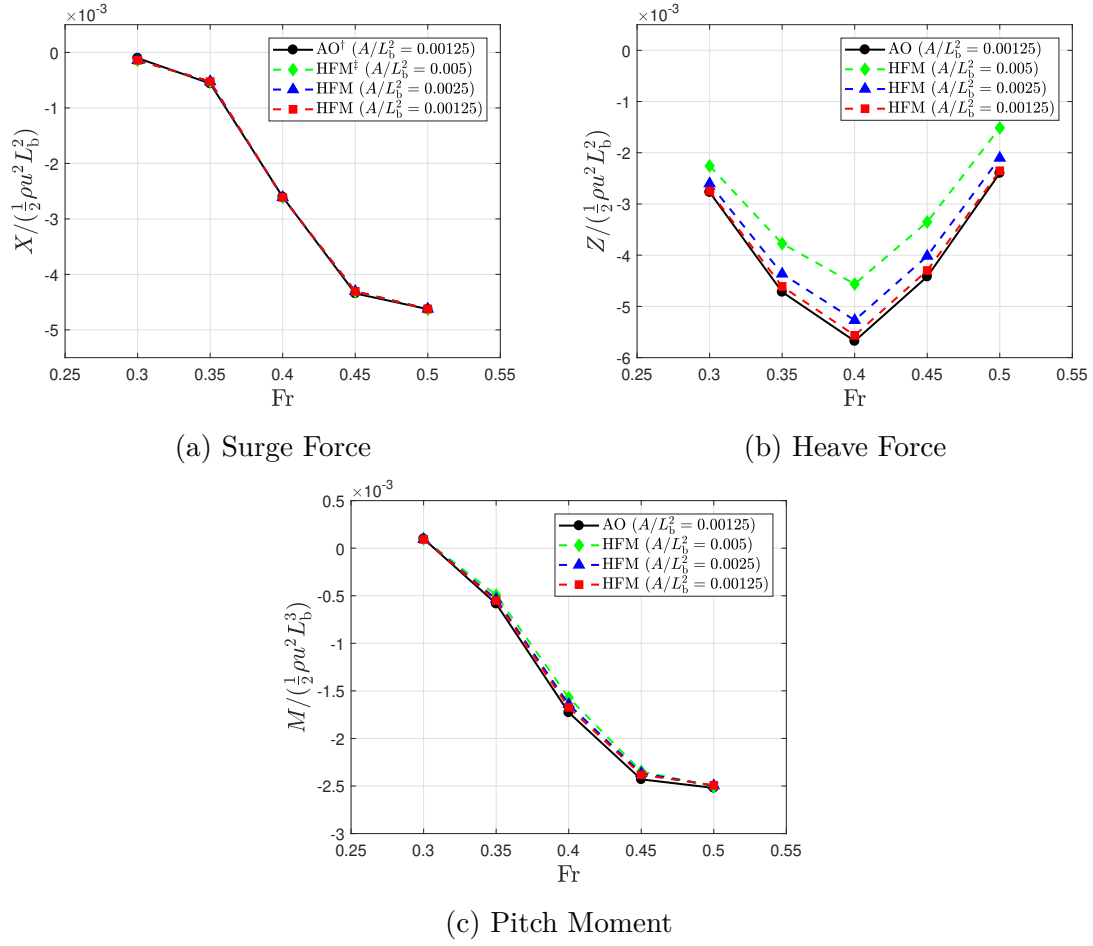


Figure 4.3: Normal velocity distribution over the undisturbed free surface ($Fr = 0.4$).

Figure 4.4 compares computations of force and moment values from the hydrodynamic force model for motion in calm water with values obtained directly from the seakeeping code in Section 4.1. The code computes the force and moment output values by integrating pressure over the body surface. These values for the several panel sizes in Table 4.3 vary within 1% of the output value for the smallest panel area of 0.00125. Therefore, only the panel code force and moment output values for panel area 0.00125 are plotted in Figure 4.4.



AO[†], Aegir Output
HFM[‡], Hydrodynamic Force Model

Figure 4.4: Force and moment comparisons for panel area variation.

Figure 4.4a shows that the surge force computations plotted as dashed lines agree well with the force values output by the code, which is plotted as a solid line. The surge force computations for the various panel areas overlap, indicating that surge force is barely affected by change in the given panel sizes. Figure 4.4c shows that, like surge force, the effect of panel size on pitch moment computations is not significant, with the largest discrepancies for the largest panel area ($A/L_b^2 = 0.005$). As for heave force, the computed values approach the values output by the code as panel area is decreased. The computed values for the smallest panel area ($A/L_b^2 = 0.00125$) agree well with the code output values. Ultimately,

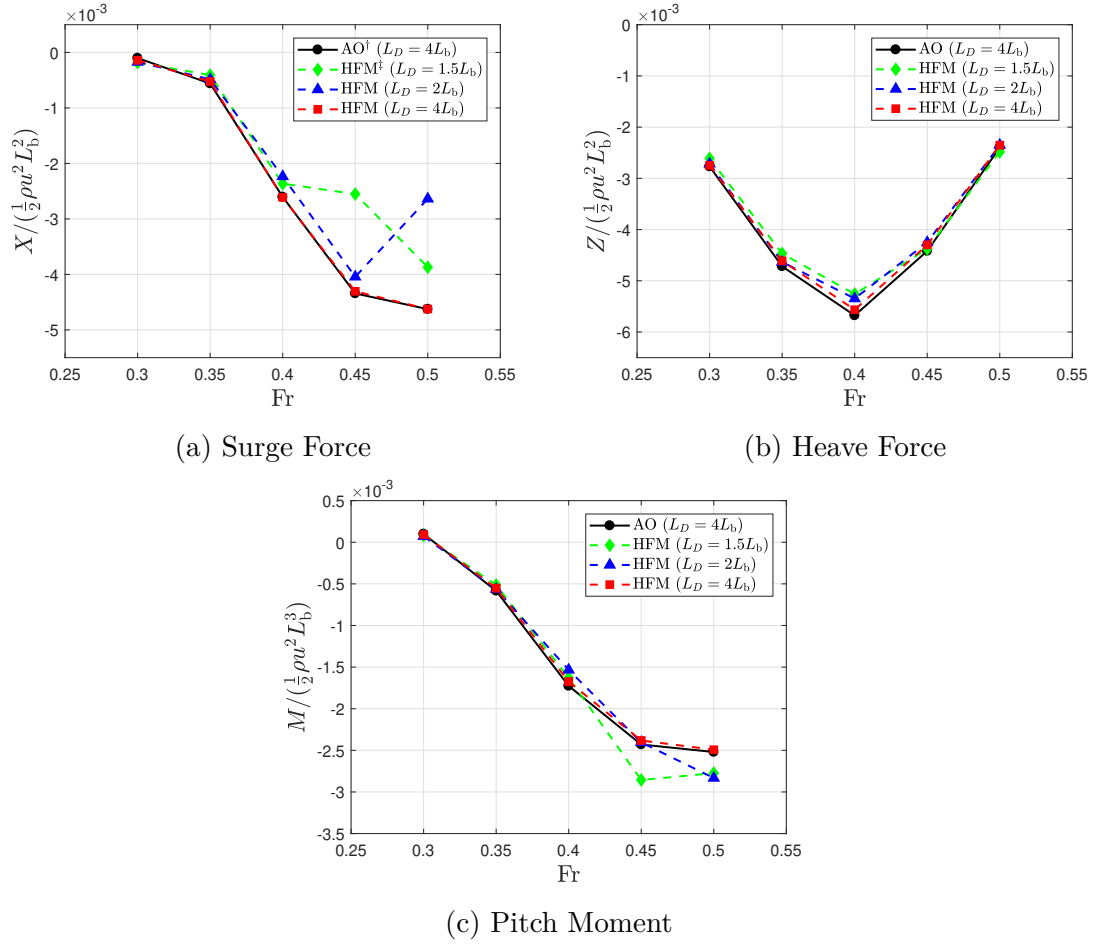
the required accuracy of the model will determine the panel size used in the simulations, noting that decreasing the panel size increases accuracy at higher computational cost.

Moving on to the effect of free surface size on force computations using equations (4.10) - (4.12), the conditions in Table 4.4 are applied for a uniform panel area of $A/L_b^2 = 0.00125$, while free surface size is varied. In order to vary free surface size, distance to the downstream edge (L_D) of the free surface is adjusted and the distance to the exterior (L_E) is varied accordingly in order for the free surface to enclose the Kelvin wave pattern that is generated with a half-angle of approximately 20 degrees. Distance to the upstream edge (L_U) is fixed since it has negligible effect on force computations in calm water.

Table 4.4: Input Variables - Free Surface Size Variation

Variable	Value
Length-to-diameter ratio	6:1
Body length, L_b	2 m
Fluid density, ρ	1000 kg/m ³
Depth-to-length ratio, z_b/L_b	0.15
Direction of motion	surge
Froude number, $Fr = u/\sqrt{gL_b}$	{0.3, 0.35, 0.4, 0.45, 0.5}
Free surface dimensions (L_U , L_D , L_E)	{ $1.5L_b$, $1.5L_b$, $0.6L_b$ }
	{ $1.5L_b$, $2L_b$, $0.8L_b$ }
	{ $1.5L_b$, $4L_b$, $1.6L_b$ }
Panel area, A/L_b^2	0.00125

As shown in Figure 4.5, surge force, heave force and pitch moment on the body are computed



AO[†], Aegir Output
HFM[‡], Hydrodynamic Force Model

Figure 4.5: Force and moment comparisons for free surface size variation.

at several forward speeds for varying surface sizes, and the computations are compared to force and moment values output by the seakeeping code in Section 4.1. The discrepancy in output values for free surface sizes at $L_D = 1.5L_b$ and $2L_b$ is approximately 10% or less of the output values when $L_D = 4L_b$. This discrepancy is negligible when compared to the discrepancies in the computed values. Therefore, for simplicity, only the panel code force and moment output values with free surface dimensions $L_U = 1.5L_b$, $L_D = 4L_b$ and $L_E = 1.6L_b$ are plotted in Figure 4.5.

As shown in Figure 4.5(b), the heave force computations plotted as dashed lines agree well with the force values output by the code, plotted as a solid line. The heave force computations for the various free surface sizes overlap, indicating that heave force is hardly affected by change in the given free surface sizes. However, as can be seen in Figs. 4.5(a) and (c), there is an evident discrepancy in surge force and pitch moment. Decreasing the free surface size from $L_D = 4L_b$ to $L_D = 2L_b$ results in a large disparity between computations and output values of surge force and pitch moment for the highest speed ($Fr = 0.5$). Further decreasing the free surface size to $L_D = 1.5L_b$ results in a large disparity for the two highest speeds ($Fr = 0.45$ and 0.5). These results indicate that as free surface size is decreased, the highest speeds with the longest wave lengths are affected first. On the other hand, increasing free surface size includes more wave cycles in the wave computations, leading to more accurate computations, but at a higher computational cost.

Table 4.5 shows the final set of steady flow simulation conditions for surge force, heave force and pitch moment computation. Simulations are performed with the smallest panel area and largest free surface size, the conditions that yield the most accurate computations, for a spheroid traveling at various depths and speeds.

As shown in Figure 4.6, force computations agree very well with force values output by the panel code. As expected, the magnitudes of the forces and moment decrease as the body moves away from the free surface. For the given range of depths and speeds in the plots, using the same panel area ($A/L_b^2 = 0.00125$) will yield satisfactory results for the various given depths. On the other hand, smaller panel areas will be required for smaller scale calculations at lower speeds or greater depths. Note that depth does not affect the number of wave cycles included in a given free surface size, so free surface area does not have to be adjusted according to depth.

Throughtout this section, the reformulated hydrodynamic force model (4.9) is numerically

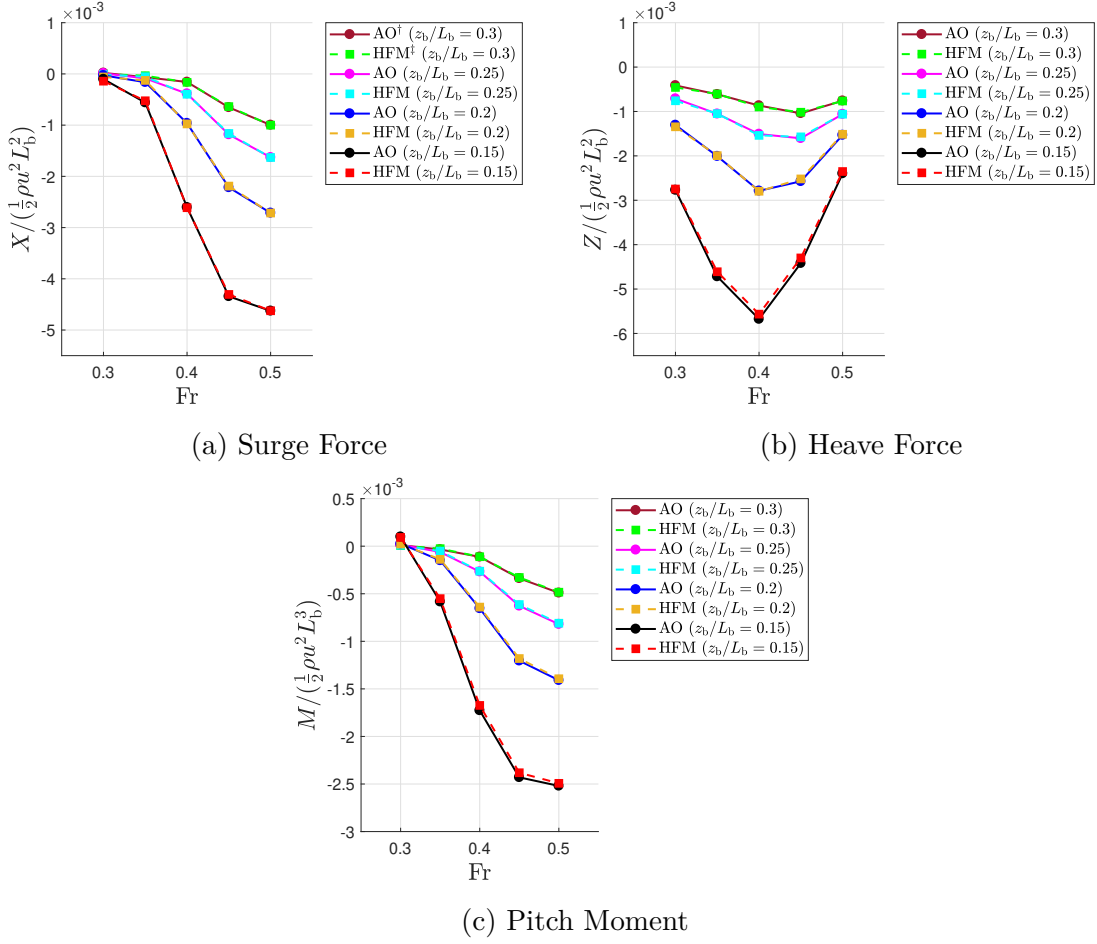
Table 4.5: Input Variables - Depth Variation

Variable	Value
Spheroid length-to-diameter ratio	6:1
Spheroid body length, L_b	2 m
Fluid density, ρ	1000 kg/m ³
Depth-to-body length ratio, z_b/L_b	{0.15, 0.2, 0.25, 0.3}
Direction of motion	surge
Froude number, $Fr = u/\sqrt{gL_b}$	{0.3, 0.35, 0.4, 0.45, 0.5}
Free surface dimensions (L_U , L_D , L_E)	{ $1.5L_b$, $4L_b$, $1.6L_b$ }
Panel area, A/L_b^2	0.00125

validated by computing steady forces on a body moving at a constant forward speed parallel to a calm free surface. The effect of panel area and free surface size variation on surge force, heave force and pitch moment calculations is examined, and the conditions that yield the most accurate results are used to study the effect of depth variation on forces. Results agree well in the range of $Fr = 0.3 \sim 0.5$ and depths $z_b/L_b = 0.15 \sim 0.3$. The following sections describe how hydrodynamic forces acting on a spheroid undergoing unsteady motion in calm water and in waves are computed using the reformulated LNMS model (3.36).

Unsteady Forces Acting on a Submerged Body

In this section, hydrodynamic forces and moment acting on a prolate spheroid exhibiting sinusoidal motion in calm water under the conditions shown in Table 4.6 are computed using



AO[†], Aegir Output
HFM[†], Hydrodynamic Force Model

Figure 4.6: Force and moment comparisons for depth variation.

the panel code in Section 4.1. The free surface dimensions are determined by the distances to the upstream, downstream and exterior edges, as shown in Figure 4.2. The domain size and panel area used in this section match those determined to produce reliable results in the previous subsection.

Formulas for the hydrodynamic force and moment acting in the vertical plane can be derived from the LNMS model (3.36). In calm water, $\mathbf{F}_I$ is set to zero, and the resulting formulas

Table 4.6: Input Variables - Sinusoidal Motion

Variable	Value
Spheroid length-to-diameter ratio	6:1
Spheroid body length, L_b	2 m
Fluid density, ρ	1000 kg/m ³
Depth-to-body length ratio, z_b/L_b	0.15
Motion type	sinusoidal
Direction of motion	{ surge, heave, pitch }
Oscillation period, T	1 s
Oscillation amplitude, a	0.05 m
Free surface dimensions (L_U , L_D , L_E)	$\{1.5L_b, 4L_b, 1.6L_b\}$
Panel area, A/L_b^2	0.00125

for the vertical plane force components in purely surging or purely heaving motion are

$$X = -\rho \iint_{\mathcal{B}} \frac{\partial \phi}{\partial t} n_x dS + \rho \iint_{S_0} \phi \frac{\partial^2 \phi}{\partial x \partial n} dS - \frac{1}{2} \rho \iint_{S_0} (\nabla \phi \cdot \nabla \phi) n_x dS \quad (4.13)$$

$$Z = -\rho \iint_{\mathcal{B}} \frac{\partial \phi}{\partial t} n_z dS + \rho \iint_{S_0} \phi \frac{\partial^2 \phi}{\partial n^2} dS - \frac{1}{2} \rho \iint_{S_0} (\nabla \phi \cdot \nabla \phi) n_z dS \quad (4.14)$$

The formula for pitching moment in purely surging motion is

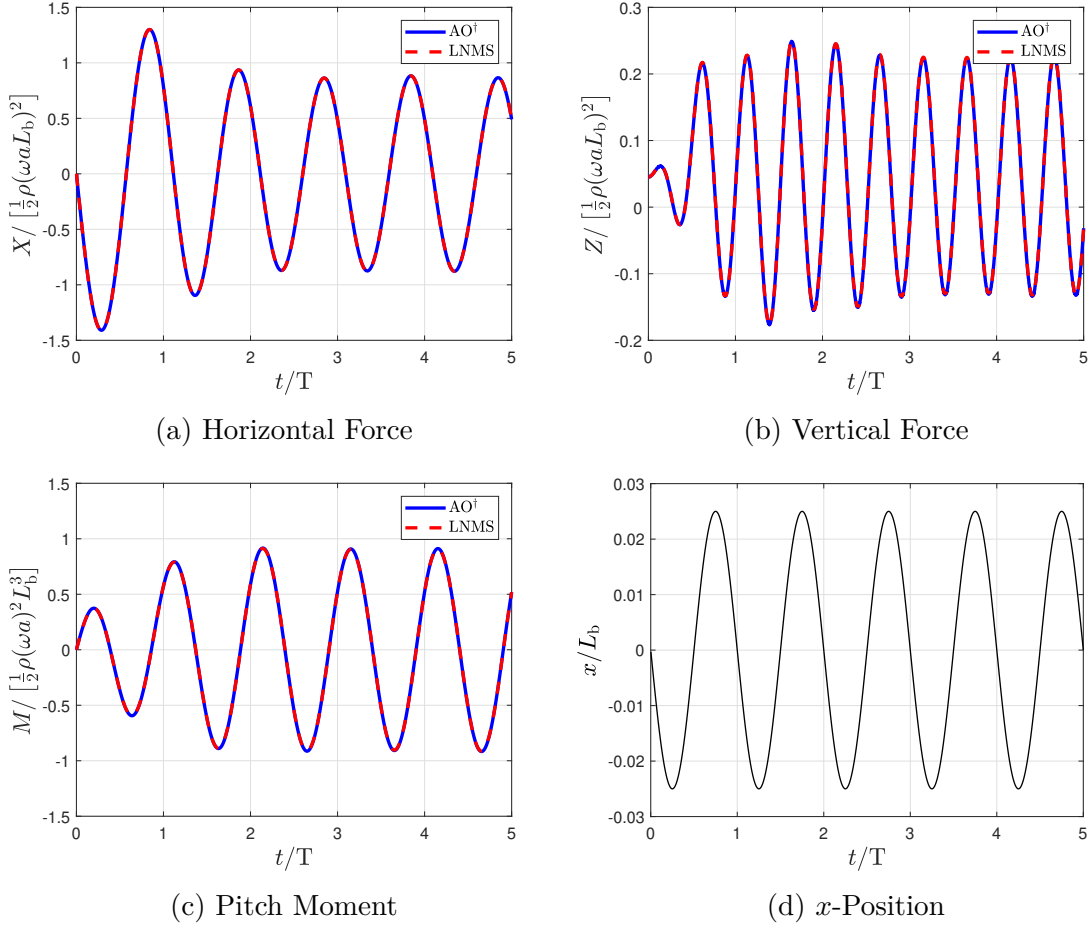
$$\begin{aligned}
 M = & -\rho \iint_{\mathcal{B}} \frac{\partial \phi}{\partial t} (r_z n_x - r_x n_z) dS + \rho \iint_{\mathcal{B}} \phi u n_z dS \\
 & + \rho \iint_{\mathcal{S}_0} \phi \left(r_z \frac{\partial^2 \phi}{\partial x \partial n} - r_x \frac{\partial^2 \phi}{\partial n^2} \right) dS - \frac{1}{2} \rho \iint_{\mathcal{S}_0} (\nabla \phi \cdot \nabla \phi) (r_z n_x - r_x n_z) dS
 \end{aligned} \tag{4.15}$$

And the formula for pitching moment in purely heaving motion is

$$\begin{aligned}
 M = & -\rho \iint_{\mathcal{B}} \frac{\partial \phi}{\partial t} (r_z n_x - r_x n_z) dS + \rho \iint_{\mathcal{B}} \phi w n_x dS \\
 & + \rho \iint_{\mathcal{S}_0} \phi \left(r_z \frac{\partial^2 \phi}{\partial x \partial n} - r_x \frac{\partial^2 \phi}{\partial n^2} \right) dS - \frac{1}{2} \rho \iint_{\mathcal{S}_0} (\nabla \phi \cdot \nabla \phi) (r_z n_x - r_x n_z) dS
 \end{aligned} \tag{4.16}$$

The hydrodynamic force and moment components are first computed for a prolate spheroid whose longitudinal axis remains parallel to the free surface when the body undergoes pure sinusoidal surging motion, starting from rest. The results are shown in Figure 4.7, where the horizontal force, vertical force, and pitch moment computations from the panel code in Section 4.1 are plotted as blue solid lines and the corresponding force and moment calculations using equations (4.13), (4.14), and (4.15) are plotted as red dashed lines. The position of the spheroid along the horizontal axis is included in Figure 4.7d to illustrate the frequency and phase shift between the input motion and the resulting hydrodynamic forces. It is evident that the values produced by the panel code and by the computations using the LNMS model (3.36) agree quite well. Note that the frequency of the horizontal force and the pitch moment oscillations matches that of the input motion, with a slight phase shift, but that the heave force frequency is double that of the input motion. Also note that transient effects decay within about 2 to 3 oscillations for this choice of motion parameters.

Next, we consider the case where the body undergoes a purely sinusoidal heaving motion. The hydrodynamic force and moment are computed using the formulas in equations (4.13), (4.14) and (4.16). Figure 4.8 shows that the vertical force produced by the panel code in



AO†, Aegir Output

Figure 4.7: Sinusoidal surging motion in calm water: (a-c) Vertical plane forces and moment acting on the body and (d) corresponding position in x -axis.

Section 4.1, plotted as a blue solid line, agrees very well with computations using equation (4.14), plotted as a red dashed line. The frequency of oscillation matches, with a slight phase shift between the input motion and the corresponding hydrodynamic force. The computed values of horizontal force and pitching moment, which are not included in the figure, are approximately 10 orders of magnitude smaller than the vertical force and decrease further with decreasing free surface panel areas. The horizontal force and pitch moment can be regarded as zero, which is consistent with expectations for a fore-aft symmetric spheroid undergoing pure heaving motion in calm water.

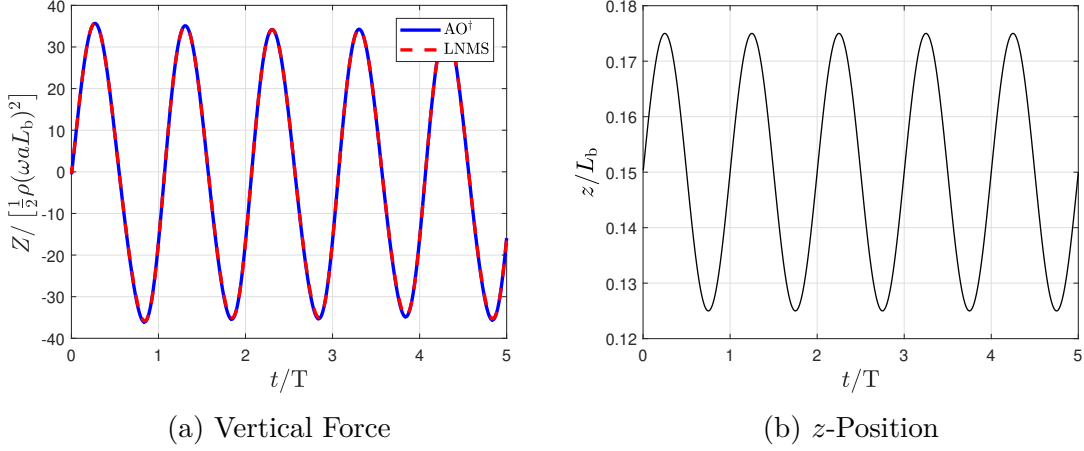
AO[†], Aegir Output

Figure 4.8: Sinusoidal heaving motion in calm water: (a) Vertical force acting on the body and (b) corresponding position in z -axis.

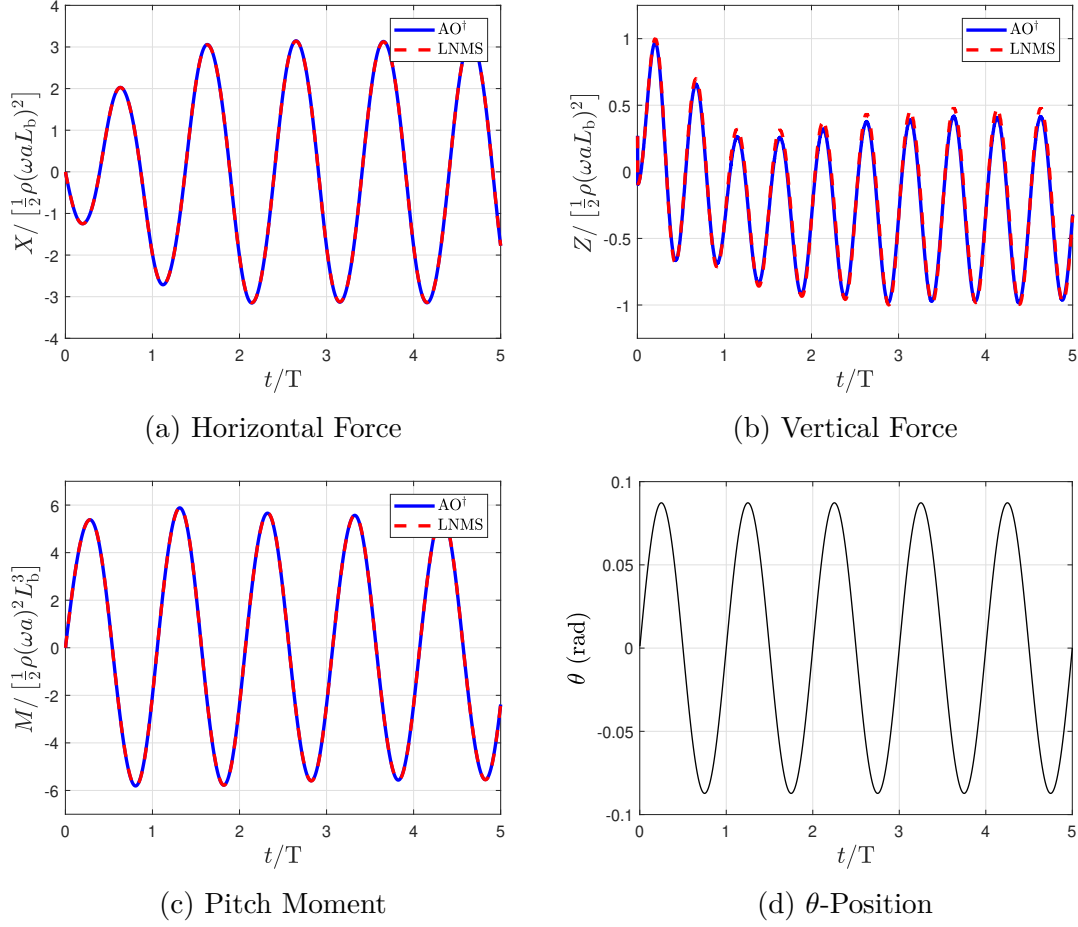
Moving on to purely pitching motion, formulas for the hydrodynamic force and moment in the vertical plane derived from the LNMS model (3.36) are shown in equations (4.17 - 4.19).

$$\begin{aligned}
 X = & -\rho \iint_B \frac{\partial \phi}{\partial t} n_x dS - \rho \iint_B \phi q n_z dS + \rho \iint_{S_0} \phi \frac{\partial^2 \phi}{\partial x \partial n} dS \\
 & - \frac{1}{2} \rho \iint_{S_0} (\nabla \phi \cdot \nabla \phi) n_x dS
 \end{aligned} \tag{4.17}$$

$$\begin{aligned}
 Z = & -\rho \iint_B \frac{\partial \phi}{\partial t} n_z dS + \rho \iint_B \phi q n_x dS + \rho \iint_{S_0} \phi \frac{\partial^2 \phi}{\partial n^2} dS \\
 & - \frac{1}{2} \rho \iint_{S_0} (\nabla \phi \cdot \nabla \phi) n_z dS
 \end{aligned} \tag{4.18}$$

$$\begin{aligned}
 M = & -\rho \iint_B \frac{\partial \phi}{\partial t} (r_z n_x - r_x n_z) dS + \rho \iint_{S_0} \phi \left(r_z \frac{\partial^2 \phi}{\partial x \partial n} - r_x \frac{\partial^2 \phi}{\partial n^2} \right) dS \\
 & - \frac{1}{2} \rho \iint_{S_0} (\nabla \phi \cdot \nabla \phi) (r_z n_x - r_x n_z) dS
 \end{aligned} \tag{4.19}$$

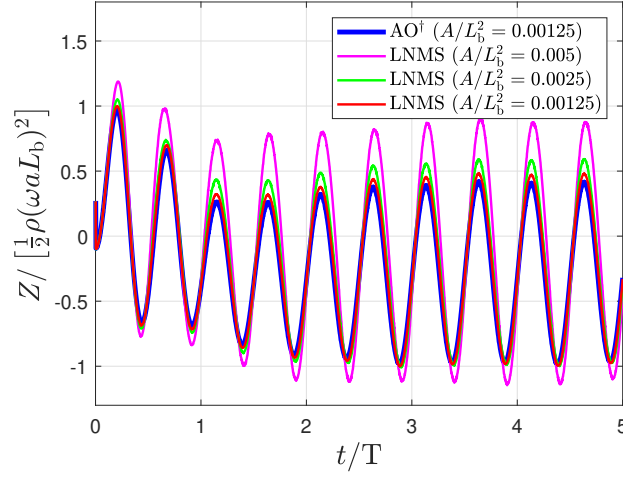
Figure 4.9 shows the force and moment produced by the panel code in Section 4.1, plotted in blue solid lines, and hydrodynamic force computations using equations (4.17), (4.18), and (4.19), plotted as red dashed lines. Horizontal force and pitch moment comparisons agree



$AO^\dagger$, Aegir Output

Figure 4.9: Sinusoidal pitching motion in calm water: (a-c) Vertical plane forces and moment acting on the body and (d) corresponding angular position in y -axis.

very well, though some discrepancy can be seen in vertical force. As shown in Figure 4.10, the discrepancy in vertical force is mainly due to a large panel area on the free surface. It is evident that the vertical force due to purely sinusoidal pitching motion derived from equation (4.18) approaches the panel code results as panel area is decreased. However, panel area should be determined carefully according to the required accuracy of the model, recognizing that smaller panel areas increase accuracy but at higher computational cost.



AO[†], Aegir Output

Figure 4.10: Comparison of heave force between varying panel areas for sinusoidal pitching motion in calm water.

4.2.3 Computation of Hydrodynamic Forces on a Submerged Body Maneuvering in Waves

In this section, a 6:1 prolate spheroid approaching the free surface in calm water and in plane progressive waves (Figure 4.11) has been simulated under the conditions specified in Table 4.7. The prolate spheroid starts from rest at depth z_i with its longitudinal axis parallel to the free surface, begins moving horizontally at constant speed, and then begins pitching nose up and ascending at constant speed while continuing at the given constant horizontal speed. When the spheroid approaches the shallower depth z_f , it begins pitching down until the longitudinal axis is again parallel to the free surface and the vertical motion is arrested so that the spheroid once again travels horizontally at constant speed and depth. Horizontal speed is constant throughout the surfacing motion while the vertical and pitch motion profiles are as shown in Figure 4.12. The simulation is carried out in calm water and then under plane progressive waves.

Remark: The prescribed motion was constructed to mimic a near-surface maneuver, however

the given motion profile may not be the most natural for a given vessel. An important application of the motion model described in this paper is to generate dynamically feasible maneuvers through trajectory optimization, for example, which can be used as reference trajectories for a motion control system.

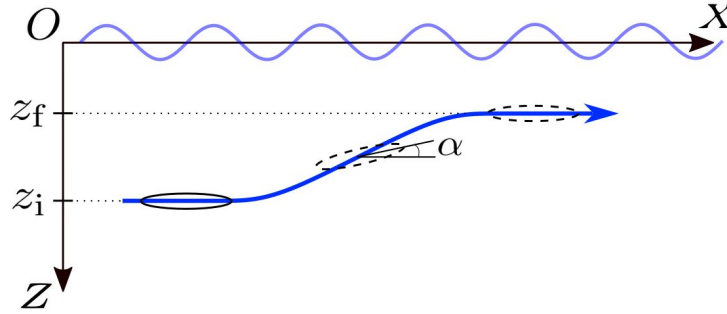


Figure 4.11: Prolate spheroid surfacing under waves (axes not to scale).

Results for a prolate spheroid surfacing under an otherwise undisturbed free surface are shown in Figure 4.13, where values output by the panel code in Section 4.1 are plotted as blue solid lines and the corresponding force and moment calculations using equation (3.36) are plotted as red dashed lines. Horizontal and vertical force comparisons agree very well. There is some discrepancy in pitch moment, however, which increases as the spheroid approaches the free surface. This discrepancy can be reduced by refining the panel on the free surface and increasing the overall free surface size, but at the expense of computational resources. Note that vertical force (Figure 4.13b) and pitch moment (Figure 4.13c) are similar in form, but opposite in sign, to vertical acceleration (Figure 4.12a) and acceleration in pitch (Figure 4.12b), respectively. On the other hand, there is no horizontal acceleration during the surfacing motion. The horizontal force (Figure 4.13a), which would vanish in an unbounded volume of ideal fluid, is due to surface waves being radiated along the free surface.

Moving on to results for a prolate spheroid surfacing under plane progressive waves, hy-

Table 4.7: Input Variables - Surfacing Motion

Variable	Value
Spheroid length-to-diameter ratio	6:1
Spheroid body length, L_b	2 m
Fluid density, ρ	1000 kg/m ³
Motion type	Surfacing (Figs. 4.12a and 4.12b)
Depth-to-body length ratio	z_i/L_b : 0.4 (initial depth) z_f/L_b : 0.2 (final depth)
Incidence angle, α	0 $\sim$ 0.0873 rad
Forward speed, U_0	1.55031 m/s
Free surface type	Calm water
	Incident waves Wave type: monochromatic Direction: head seas Period: 2 s Amplitude: 0.02 m
Free surface dimensions (L_U , L_D , L_E)	$\{1.5L_b, 4L_b, 1.6L_b\}$
Panel area, A/L_b^2	0.00125

drodynamic force values output by the panel code in Section 4.1 are plotted as blue solid lines and the corresponding forces and moment calculations using equation (3.36) are plotted as red dashed lines; the results are shown in Figure 4.14. Similar to the results in calm water (Figure 4.13), horizontal and vertical force comparisons agree very well, while there is some

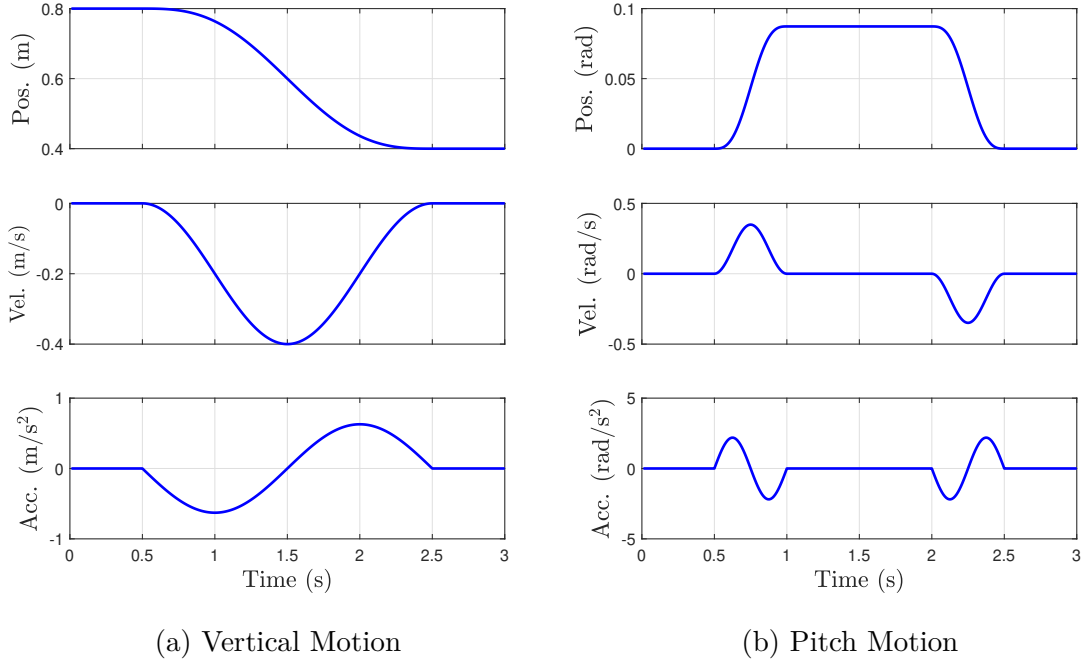
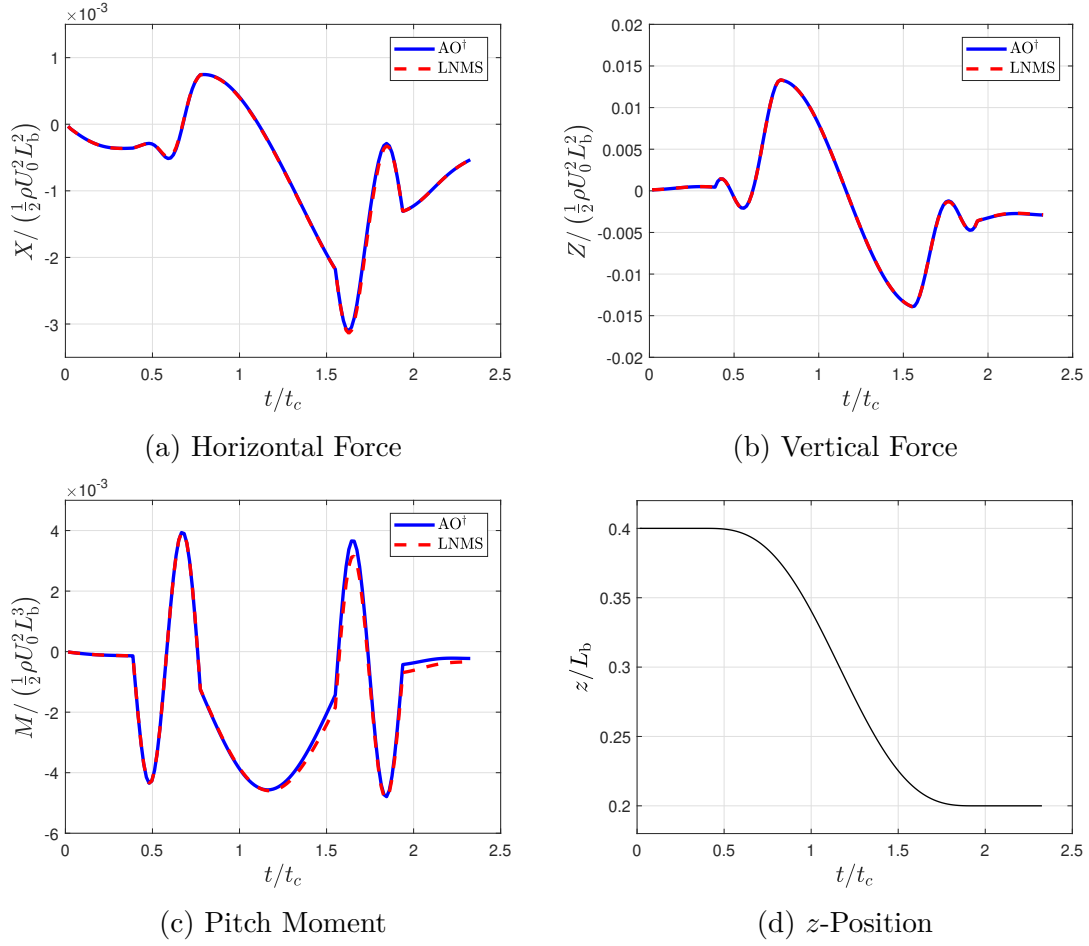


Figure 4.12: Motion profile: Position, velocity and acceleration along (a) z -axis and (b) y -axis during surfacing motion.

discrepancy in pitch moment that increases as the spheroid approaches the free surface. The forces and moment acting on the body due to the incident wave forces (3.35) are included in Figure 4.14 as solid green lines. It is evident that horizontal force (Figure 4.14a) is predominantly affected by the incident waves, especially because there is no acceleration in the horizontal direction and horizontal force is therefore mainly affected by surface effects. On the other hand, vertical force (Figure 4.14b) and pitch moment (Figure 4.14c) are affected to some extent by the incident wave at the beginning and end of the surfacing motion, where there is no displacement in the vertical direction or in pitch angle. But during ascent and pitching, vertical force and pitch moment are much less affected by incident waves, and the resulting values are similar to those in Figure 4.13.

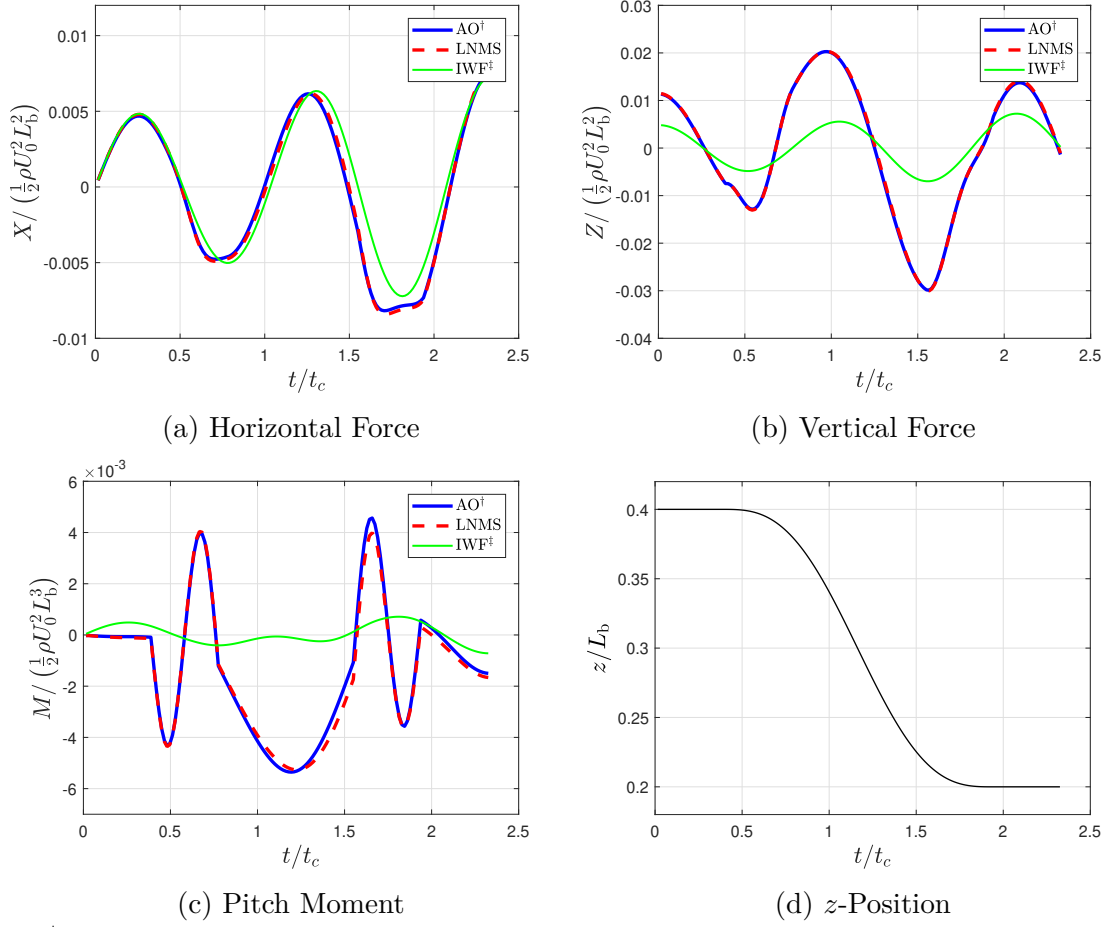


AO†, Aegir Output

Figure 4.13: Surfacing motion in calm water: (a-c) Vertical plane forces and moments acting on the body and (d) corresponding position in z -axis. <On the horizontal axis, time constant $t_c = L_b/U_0$ >

4.3 Parameter Determination for Decomposed Reformulated Models

The following subsections describe procedures for determining impulsive and memory terms in the decomposed LNM model in (3.34) and decomposed LNMS model in (3.37). The normalized potential vectors φ_K and φ_s and their derivatives, which appear in the decomposed version of the reformulated models, can be computed by solving the flow for impulsive mo-



AO[†], Aegir Output

IWF[†], Incident Wave Force

Figure 4.14: Surfacing motion near waves: (a-c) Vertical plane forces and moments acting on the body and (d) corresponding position in z -axis. <On the x -axis, time constant $t_c = L_b/U_0$ >

tion of the body using a potential flow panel code. In addition, the memory potential ϕ_m and its derivatives can be approximated using the impulse response of the system.

4.3.1 Computation of Impulsive Terms

As described in Section 4.1, the $\mathbf{q}$ -dependent normalized potential vector for impulsive flow φ_i can be computed by solving for the instantaneous response of the fluid due to an impul-

sive motion of the body using a potential flow panel code. On the other hand, Kirchhoff's normalized potential vector $\boldsymbol{\varphi}_K$ can be obtained by considering a moving body in an unbounded fluid, independent of $\mathbf{q}$. In addition, the $\mathbf{q}$ -dependent $\boldsymbol{\varphi}_s$ can be computed by subtracting $\boldsymbol{\varphi}_K$ from $\boldsymbol{\varphi}_i$. The derivatives of $\boldsymbol{\varphi}_K$ and $\boldsymbol{\varphi}_s$ can be obtained through the same procedure. Finally, by computing a set of impulsive terms in the operating range of the system, a look-up table can be created to form a discrete parameter model, or a data fit can be performed on the $\mathbf{q}$ -dependent impulsive terms to yield a continuously parameterized model.

4.3.2 Approximation of Memory Terms

The memory potential ϕ_m is the sum of a diffraction potential ϕ_D and a radiation potential ϕ_R . Based on the derivation introduced by Falnes [52], the diffraction potential can be expressed as the following convolution integral:

$$\phi_D(t) = \int_{-\infty}^{\infty} h(\mathbf{q}(\tau), t - \tau) \eta(\tau) d\tau \quad (4.20)$$

where h is the impulse response to the wave elevation η . In addition, the radiation potential can be expressed as the following convolution integral described by Battista [1]:

$$\phi_R(t) = \int_0^t \boldsymbol{\varphi}_R(\mathbf{q}(\tau), t - \tau) \cdot \boldsymbol{\nu}(\tau) d\tau \quad (4.21)$$

Here, the normalized potential vector $\boldsymbol{\varphi}_R$ is the impulse response to the body's velocity $\boldsymbol{\nu}$. A set of $\mathbf{q}$ -dependent impulse response functions h and $\boldsymbol{\varphi}_R$ can be obtained within the operating envelope of the system using a potential flow panel code, and the resulting sets can then be used to create a look-up table or to form a data fit. The resulting discrete

set of impulse response functions, or the continuous, fitted impulse response function, can be used to approximate the convolution integrals in equations (4.20) and (4.21) using finite-dimensional state equations according to a method proposed by Kung [53]. That method has been implemented by several authors, including Yu & Falsen [54] and Damaren [55] for the diffraction problem and Kristiansen & Egeland [56], Taghipour et al. [57], Perez & Fossen [58] and Chen et al. [37] for the radiation problem. The same procedure can be applied to approximate the derivatives of ϕ_D and ϕ_R in order to obtain the derivatives of ϕ_m in vector $\mathbf{E}$ of equation (3.34).

To summarize, applying the decomposed form of the velocity potential in equation (3.32) to the LNMS model (3.36) produces expressions that may provide more physical insight concerning vessel motion [9]. Computing the impulsive terms (Section 4.3.1) and replacing the memory terms (Section 4.3.2) yields a model formulation (3.37) well suited for control design and simulation.

4.3.3 An Example: Determination of Motion Model Parameters for a Prolate Spheroid Maneuvering in Calm Water

This section describes an example included in [37], where the parameters of a motion model for a prolate spheroid are determined, and then the model is used to develop a gain-scheduled controller. The problem of a 2 m long 6:1 prolate spheroid undergoing longitudinal motion in a perfect fluid under an otherwise undisturbed free surface is considered; neutral buoyancy and evenly distributed mass is assumed for the body. To begin the parameter determination process, only the impulsive portion of the LNM model (3.31) is taken into account by setting the memory potential ϕ_m in equation (3.32) to zero. The resulting impulsive motion model

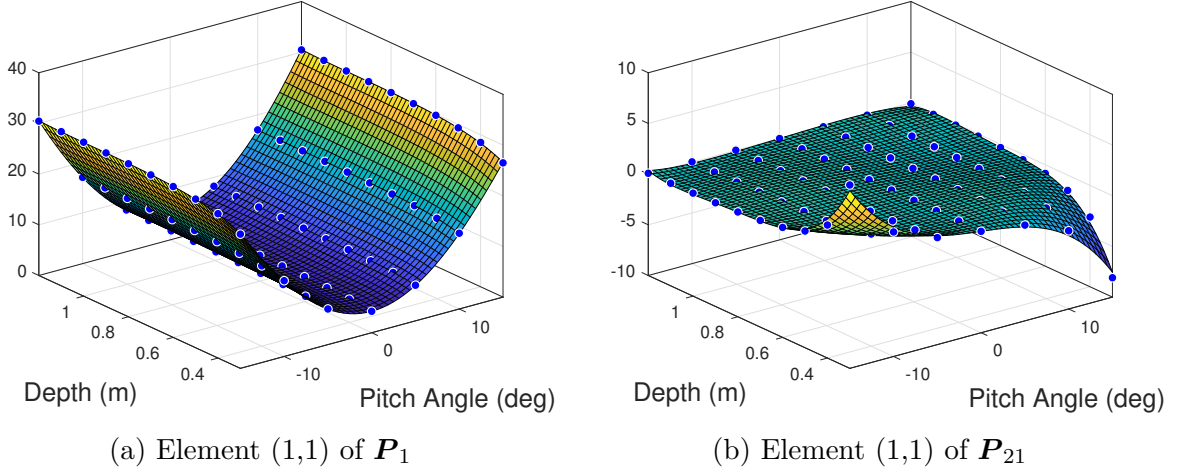


Figure 4.15: Plots of surfaces fits to element values of parameter matrices.

is expressed as

$$\begin{aligned}
 \mathbf{F} = & \left(\mathbf{M}_b + \underbrace{\rho \iint_{\mathcal{B}} \boldsymbol{\varphi}_i \mathbf{b}^T dS}_{\mathbf{P}_1} \right) \dot{\boldsymbol{\nu}} + \mathbf{G} \left(\mathbf{M}_b + \rho \iint_{\mathcal{B}} \boldsymbol{\varphi}_i \mathbf{b}^T dS \right) \boldsymbol{\nu} \\
 & + \left(\underbrace{\frac{1}{2} \sum_{i=1}^6 \rho \iint_{\mathcal{S}_0} \left(\frac{\partial \varphi_{ii}}{\partial n} \mathbf{b} \frac{\partial \varphi_i^T}{\partial n} \right) dS}_{\mathbf{P}_{2i}} \boldsymbol{\nu} \mathbf{e}_i^T \right) \boldsymbol{\nu}
 \end{aligned} \tag{4.22}$$

where φ_{ii} is the i^{th} element of $\boldsymbol{\varphi}_i$.

A set of $\mathbf{q}$ -dependent values for each matrix element of the parameters in equation (4.22) are computed using the panel code in Section 4.1. For instance, a set of (1,1) entries for parameter matrices $\mathbf{P}_1$ and $\mathbf{P}_{21}$ can be computed as shown in Figure 4.15. In the figure, the operating range is set to $z_b = 0 \sim 1.2$ for depth, and $\theta = -15 \sim 15$ deg for pitch angle, and piecewise polynomials are used to fit the computed values shown in blue dots.

Now, the impulsive portion of the model is extended to include memory effects by making additional simplifying assumptions of steady surge motion with small deviations from a mean

body depth and pitch angle. The hydrodynamic force due to memory effects is then written as described by Perez [24]:

$$\int_0^t \mathbf{K}(t - \tau) \boldsymbol{\nu}(\tau) d\tau \quad (4.23)$$

where $\mathbf{K}(t)$ is a matrix of kernel function that is obtained by using the frequency-dependent added mass or damping from a panel code. The convolution integral can then be approximated with the state space realization [59]:

$$\begin{aligned} \dot{\boldsymbol{\mu}} &= \mathbf{A}_m \boldsymbol{\mu} + \mathbf{B}_m \boldsymbol{\nu}, \quad \boldsymbol{\mu} = 0 \\ \boldsymbol{\gamma} &= \mathbf{C}_m \boldsymbol{\mu} + \mathbf{D}_m \boldsymbol{\nu} \end{aligned} \quad (4.24)$$

Here, $\boldsymbol{\mu}$ is an auxiliary state vector and $\boldsymbol{\gamma}$ is the force.

Therefore, the extended model can be expressed as

$$\mathbf{F} = (\mathbf{M}_b + \mathbf{P}_1) \dot{\boldsymbol{\nu}} + \mathbf{G} (\mathbf{M}_b + \mathbf{P}_1) \boldsymbol{\nu} + \left(\frac{1}{2} \sum_{i=1}^6 \mathbf{P}_{2i} \boldsymbol{\nu} \mathbf{e}_i^T \right) \boldsymbol{\nu} + \boldsymbol{\gamma} \quad (4.25)$$

Finally, the motion model can be used to design a gain-scheduled controller, where local control algorithms are built, an interpolation scheme is constructed, and scheduling variables are prescribed, as described in [37].

Chapter 5

Conclusions

5.1 Summary

In this paper, parameters of a Lagrangian mechanical system model of a submerged vessel maneuvering in waves are determined using a medium fidelity potential flow code. To begin with, the derivation of the first principles nonlinear Lagrangian mechanical system model for a surface-affected marine craft constructed by Battista [1] is briefly reviewed. Then, terms in the Lagrangian nonlinear maneuvering (LNM) model are reformulated to allow computation of parameters using data that are readily available from the potential flow panel code. Subsequently, incident wave effects are incorporated to the reformulated LNM model to yield a Lagrangian nonlinear maneuvering and seakeeping (LNMS) model. Numerical validation is performed by comparing computations using the methods proposed in this work against basic panel code solutions for various motions in calm water and in waves. Finally, a control oriented formulation of the LNMS model is presented.

Subtleties arise in reformulating the original Lagrangian mechanical system model, concerned with changing the order of integration and differentiation for the integro-differential parameters and computing partial derivatives of the Lagrangian function. The parameters of the reformulated model can then be derived accurately by direct computation. The reformulation process circumvents the need for additional numerical discretization, thus increasing accuracy and decreasing computational cost. In contrast to parameter estimation methods

using experimental or CFD data, the parameter computation method proposed in this dissertation allows every model parameter to be computed directly. Often, having knowledge of how every component in a model is computed can provide valuable physical insight about that parameter's role in the physics.

A numerical investigation is carried out to confirm the validity of the proposed approach to model parameter computation involving partial derivatives with respect to translational configuration variables for a body undergoing impulsive motions. The results show that terms computed using the proposed method agree well with numerical calculations output by the panel code. The parameters, expressed as integro-differential functions of the velocity potential, are then computed for a prolate spheroid in steady surge motion using a potential flow code that has been modified to output the necessary functions of the flow potential. Hydrodynamic force computations for the reformulated LNM model, obtained using the modified potential flow software, are then compared with computations obtained directly from the panel code. The effect of computational parameters, such as panel area and domain size, are first examined and then the effect of depth. Results agree well in the range of $Fr = 0.3 \sim 0.5$ and depths $z_b/L_b = 0.15 \sim 0.3$. Also, the reformulated LNM model is used to compute hydrodynamic forces on a prolate spheroid undergoing sinusoidal motion in the vertical plane, and the results are compared to forces obtained from the potential flow panel code. In addition, the LNMS model is numerically validated using hydrodynamic force calculations for a prolate spheroid surfacing in calm water and in plane progressive waves. The results for hydrodynamic force calculations for sinusoidal and surfacing motions in the vertical plane using the reformulated model agree well with force values output by the panel code.

Finally, an approach for determining the parameters of the control oriented formulation of the LNMS model is presented, where methods for computing the impulsive terms and

replacing the memory convolution terms with their state space realizations is introduced. The determined model is capable of predicting nonlinear hydrodynamic forces due to aggressive maneuvers or significant wave forcing that other linear models fail to capture, and therefore provide greater accuracy in stability analysis, nonlinear control design and simulations.

5.2 Future Work

The overarching goal of the collaborative research effort specified in Section 1.2 is to expand the operating envelope and provide robust control for a submerged vessel maneuvering in a seaway. In order to achieve this goal, milestones beyond the scope of this dissertation have to be met. First, a LNMS model with a complete set of determined parameters, including a parameter look up table or data-fitted parameters for the operating range of the system must be constructed, as described in Sections 4.3.1 and 4.3.2. Then, the constructed potential flow model should be numerically validated using a potential flow code. Next, **non-potential real fluid** effects should be incorporated into the model using empirical methods or otherwise, and the resulting model should be validated against high-fidelity CFD and/or experimental data. The model can then be used for model-based nonlinear control algorithm design. The component Lagrangians and parameter calculations of the resulting model could prove useful in the design process, as shown in the energy Casimir method applied in [60]. Finally, the performance of the nonlinear controller can be assessed by carrying out controller-in-the-loop free running tests using high-fidelity CFD or experimental tests. **A simple yet practical application of the approach described above can be, for instance, the design of a nonlinear controller for a Joubert BB2 submarine aggressively maneuvering near a free surface. The parameters of the LNM model for the Joubert submarine bare hull can be determined by applying the methods suggested in this dissertation. Then, the Joubert submarine LNM**

model can be numerically validated using a potential flow code, and a nonlinear control strategy can be designed by applying energy-based methods such as in [61]. Although there exists a gap between a real system and the bare hull maneuvering in ideal fluid, rapid simulations can be carried out in order to design and test the controller by indentifying the dominant variables in the model that affect hydrodynamic performance. This example is one of many possible model-based control design applications, and ultimately, when it comes to vessels undergoing aggressive maneuvers or significant wave forcing, where nonlinear hydrodynamic effects are critical, the model-based nonlinear controller is expected to show superior performance compared to linear controllers.

Bibliography

- [1] T. Battista, *Lagrangian Mechanics Modeling of Free Surface-Affected Marine Craft*. PhD thesis, Virginia Tech, 2018.
- [2] L. Ljung and T. Glad, *Modeling of Dynamic Systems*. PTR Prentice Hall, 1994.
- [3] L. Ljung, *System Identification*. PTR Prentice Hall, 1999.
- [4] S. Venkateshan and P. Swaminathan, *Computational Methods in Engineering*. Elsevier, Inc., 2013.
- [5] R. Beck, W. Cummins, J. Dalzell, P. Mandel, and W. Webster, “Motions in waves,” in *Principles of Naval Architecture*, vol. 3, ch. 8, pp. 1–190, SNAME, 1989.
- [6] T. Fossen, *Guidance and Control of Ocean Vehicles*. Wiley, 1994.
- [7] International Towing Tank Conference, *ITTC-Recommended Procedures and Guidelines: Validation of Manoeuvring Simulation Models*, 2008.
- [8] P. Bailey, W. Price, and P. Temarel, “A unified mathematical model describing the manoeuvring of a ship travelling in a seaway,” *Transactions of the Royal Institution of Naval Architects*, vol. 140, pp. 131–149, 1998. DOI: 10.3940/rina.sbt.1998.a7.
- [9] T. Battista, F. Valentini, and C. Woolsey, “A maneuvering model for an underwater vehicle near a free surface - part I: Motion without memory effects,” *IEEE Journal of Oceanic Engineering*, 2019 (to appear). DOI: 10.1109/JOE.2018.2871650.
- [10] C. Crane, H. Eda, and A. Landsburg, “Controllability,” in *Principles of Naval Architecture*, vol. 3, ch. 9, pp. 191–429, SNAME, 1989.

- [11] H. Lamb, *Hydrodynamics*. Dover, 1932.
- [12] M. Abkowitz, “Lectures on ship hydrodynamics—steering and manoeuvrability,” tech. rep., Hydro-and Aerodynamics Laboratory, 1964. Lyngby, Denmark.
- [13] N. Norrbin, “Theory and observations on the use of a mathematical model for ship manoeuvring in deep and confined waters,” tech. rep., Swedish State Shipbuilding Experimental Tank Goteborg, 1971.
- [14] K. Son and K. Nomoto, “On the coupled motion of steering and rolling of a high-speed container ship,” *the Society of Naval Architects of Japan*, vol. 150, pp. 232–244, 1981.
- [15] A. Christensen and M. Blanke, “A linearized state-space model of steering and roll of a high-speed container ship,” tech. rep., Technical University of Denmark, 1986. Technical Report 86-D-574.
- [16] S. Jung, S. Brizzolara, and C. Woolsey, “Parameter computation for a Lagrangian mechanical system model of a submerged vessel moving near a free surface,” *Journal of Ship Research*, 2020. (In review).
- [17] G. Weinblum and M. St. Denis, “On the motions of ships at sea,” in *Trans. SNAME*, vol. 58, 1950.
- [18] J. Newman, “Theory of ship motions,” in *Advances in Applied Mechanics*, vol. 18, pp. 221–283, Academic Press, Inc., 1978.
- [19] D. C. Kring, *Time Domain Ship Motions by a Three-Dimensional Rankine Panel Method*. PhD thesis, MIT, 1994.
- [20] T. Crook, “An initial assessment of free surface effects on submerged bodies,” Master’s thesis, Naval Postgraduate School, Monterey CA, USA, Sept. 1994.

- [21] T. Ogilvie, “Singular-perturbation problems in ship hydrodynamics,” in *Advances in Applied Mechanics*, vol. 17, pp. 91–188, Academic Press, Inc., 1977.
- [22] O. Grim, “A method for a more precise computation of heaving and pitching motions both in smooth water and in waves,” in *Proceedings of the Third Symposium on Naval Hydrodynamics*, 1960.
- [23] H. Eda, “Directional stability and control of ships in waves,” *Journal of Ship Research*, vol. 16, pp. 205–218, 1972.
- [24] T. Perez, *Ship Motion Control: Course keeping and roll stabilisation using rudder and fins*. Springer, 2005.
- [25] T. I. Fossen, “A nonlinear unified state-space model for ship maneuvering and control in a seaway,” *International Journal of Bifurcation and Chaos*, vol. 15, no. 9, pp. 2717–2746, 2005. DOI: 10.1142/S0218127405013691.
- [26] M. Chislett and J. Strom-Tejsen, “Planar motion mechanism tests and full-scale steering and manoeuvring predictions for a mariner class vessel,” *International Shipbuilding Progress*, vol. 12, no. 129, pp. 201–224, 1965. DOI: 10.3233/ISP-1965-1212902.
- [27] M. Gertler, “The DTMB planar-motion-mechanism system,” tech. rep., Naval Ship Research and Development Center, 1967.
- [28] E. Lewis, ed., *Principles of Naval Architecture Volume III: Motion in Waves and Controllability*. The Society of Naval Architects and Marine Engineers, 1989.
- [29] S. Jung, T. Battista, E. Paterson, and C. A. Woolsey, “Effects of depth on added mass obtained from virtual PMM tests of a submerged prolate spheroid,” in *OCEANS 2017*, (Anchorage, USA), 2017.

- [30] Y. Pan, H. Zhang, and Q. Zhou, “Numerical prediction of submarine hydrodynamic coefficients using CFD simulation,” *Journal of Hydrodynamics*, vol. 24, pp. 840–847, 2012. DOI: 10.1016/S1001-6058(11)60311-9.
- [31] M. Araki, H. Sadat-Hosseini, Y. Sanada, K. Tanimoto, N. Umeda, and F. Stern, “System identification using CFD captive and free running tests in severe stern waves,” in *13th International Ship Stability Workshop*, (Brest, France), 2013.
- [32] M. Araki, H. Sadat-Hosseini, Y. Sanada, K. Tanimoto, N. Umeda, and F. Stern, “Estimating maneuvering coefficients using system identification methods with experimental, system-based, and CFD free-running trial data,” *Ocean Engineering*, vol. 51, pp. 63–84, 2012. DOI: 10.1016/j.oceaneng.2012.05.001.
- [33] J. Angeles, *Dynamic Response of Linear Mechanical Systems*. Springer, 2011.
- [34] A. Trujillo and H. Thurman, *Essentials of Oceanography*. Pearson, 2016.
- [35] S. Hoerner, *Fluid Dynamic Drag*. Midland Park, N.J., 1958.
- [36] S. Jung, T. Battista, F. Valentinis, S. Brizzolara, E. Paterson, and C. Woolsey, “An analytical approximation of a force prediction model for a prolate spheroid moving at a constant forward speed parallel to a calm free surface,” in *The 13th International Conference on Hydrodynamics*, (Incheon, South Korea), Sept. 2018.
- [37] Y. Chen, S. Jung, and C. Woolsey, “Longitudinal, near-surface maneuvering of a prolate spheroid,” in *IFAC CAMS 2019*, pp. 412–417, Sept. 2019.
- [38] H. Goldstein, C. Poole, and J. Safko, *Classical Mechanics*. Addison Wesley, 2000.
- [39] D. T. Greenwood, *Advanced Dynamics*. Cambridge University Press, 2003.

- [40] H. Schaub and J. L. Junkins, *Analytical Mechanics of Space Systems*. American Institute of Aeronautics and Astronautics, Inc., 2009.
- [41] J. N. Newman, *Marine Hydrodynamics*. MIT Press, 1972.
- [42] K. Karamcheti, *Principles of Ideal-Fluid Aerodynamics*. Wiley, 1966.
- [43] R. G. Dean and R. A. Dalrymple, *Water Wave Mechanics for Engineers and Scientists*. World Scientific, 1991.
- [44] E. M. Lewandowski, *The Dynamics of Marine Craft*. World Scientific, 2004.
- [45] W. E. Cummins, “The impulse response function and ship motions,” tech. rep., David Taylor Model Basin, 1962.
- [46] P. Thomasson and C. Woolsey, “Vehicle motion in currents,” *IEEE Journal of Ocean Engineering*, vol. 38, no. 2, pp. 226–242, 2013.
- [47] T. Fossen, *Handbook of Marine Craft Hydrodynamics and Motion Control*. Wiley, 2011.
- [48] Navatek, *Aegir User Guide*, 2018.
- [49] D. C. Kring, *Aegir Theory Guide*, 2017.
- [50] G. P. Weinblum, “On hydrodynamic masses,” tech. rep., David Taylor Model Basin, 1952.
- [51] Twenty Sixth ITTC Specialist Committee on CFD in Marine Hydrodynamics, *ITTC - Recommended Procedures and Guidelines*, 2011.
- [52] J. Falnes, “On non-causal impulse response functions related to propagating water waves,” *Applied Ocean Research*, vol. 17, pp. 379–389, 1995. [https://doi.org/10.1016/S0141-1187\(96\)00007-7](https://doi.org/10.1016/S0141-1187(96)00007-7).

- [53] S. Kung, “A new identification and model reduction algorithm via singular value decompositions,” in *Proc. 12th Asilomar Conf. on Circuits, Systems and Computers*, 1978.
- [54] Z. Yu and J. Falnes, “State-space modelling of a vertical cylinder in heave,” *Applied Ocean Research*, vol. 17, pp. 265–275, 1995. [https://doi.org/10.1016/0141-1187\(96\)00002-8](https://doi.org/10.1016/0141-1187(96)00002-8).
- [55] C. Damaren, “A causal transient water wave diffraction formulation,” *Applied Ocean Research* 2, vol. 22, pp. 267–280, 2000. [https://doi.org/10.1016/S0141-1187\(00\)00020-1](https://doi.org/10.1016/S0141-1187(00)00020-1).
- [56] E. Kristiansen and O. Egeland, “Frequency-dependent added mass in models for controller design for wave motion damping,” in *IFAC Proceedings*, 2003.
- [57] R. Taghipour, T. Perez, and T. Moan, “Hybrid frequency-time domain models for dynamic response analysis of marine structures,” *Ocean Engineering*, vol. 35, pp. 685–705, 2008. <https://doi.org/10.1016/j.oceaneng.2007.11.002>.
- [58] T. Perez and T. Fossen, “A matlab toolbox for parametric identification of radiation-force models of ships and offshore structures,” *Modeling, Identification and Control*, vol. 30, no. 1, pp. 1–15, 2009. doi:10.4173/mic.2009.1.1.
- [59] E. Kristiansen, A. Hjulstad, and O. Egeland, “State-space representation of radiation forces in time-domain vessel models,” *Ocean Engineering*, vol. 32, pp. 2195–2216, 2005. <https://doi.org/10.1016/j.oceaneng.2005.02.009>.
- [60] T. Battista, S. Jung, C. Woolsey, and E. Paterson, “An energy-Casimir approach to underwater vehicle depth and heading regulation in short crested waves,” in *2017 IEEE Conference on Control Technology and Applications (CCTA)*, (Mauna Lani, USA), 2017. DOI: 10.1109/CCTA.2017.8062466.

- [61] F. Valentinis and C. Woolsey, “Nonlinear control of a subscale submarine in emergency ascent,” *Ocean Engineering*, vol. 171, no. 1, pp. 646–662, 2019.
<https://doi.org/10.1016/j.oceaneng.2018.11.029>.

Appendices

Appendix A

Concepts from Multivariate Calculus

A.1 Divergence theorem

The divergence theorem, also known as Gauss's theorem, relates the flux of a vector field across a surface to the integral of the divergence over its interior. For a closed region V bounded by a closed surface S , if the vector field $\mathbf{A}$ is defined and continuously differentiable throughout V , then

$$\iiint_V \nabla \cdot \mathbf{A} \, dV = \iint_S \mathbf{A} \cdot \mathbf{n} \, dS$$

where $\mathbf{n}$ is a unit vector normal to S .

A.2 Equivalent expressions

Changing the order in which the differential under the integral sign (Section 3.1) and the divergence theorem (See Appendix A.1) are applied to the fluid kinetic energy below leads to different yet equivalent expressions.

$$\frac{1}{2}\rho \iiint_V \nabla \phi \cdot \nabla \phi \, dV \tag{A.1}$$

where V is the region enclosing the fluid.

For example, if the differential under the integral sign (Section 3.1) is applied to the kinetic energy (A.1) with respect to some variable e , where the limits of the integral are independent of e , and if the divergence theorem (See Appendix A.1) is subsequently implemented,

$$\begin{aligned}
 \frac{1}{2}\rho\frac{\partial}{\partial e}\iiint_V \nabla\phi \cdot \nabla\phi \, dV &= \rho\iiint_V \nabla\phi \cdot \frac{\partial}{\partial e}\nabla\phi \, dV \\
 &= \rho\iiint_V \nabla \cdot \left(\phi \frac{\partial}{\partial e}\nabla\phi \right) \, dV \\
 &= \rho\iint_S \left(\phi \frac{\partial}{\partial e}\nabla\phi \right) \cdot \mathbf{n} \, dS
 \end{aligned} \tag{A.2}$$

Here, the fluid is bounded by S and $\mathbf{n}$ is a unit vector normal to S .

Now, if the order is reversed so that the divergence theorem (See Appendix A.1) is applied first and then the differential under the integral sign (Section 3.1) is performed,

$$\begin{aligned}
 \frac{1}{2}\rho\frac{\partial}{\partial e}\iiint_V \nabla\phi \cdot \nabla\phi \, dV &= \frac{1}{2}\rho\frac{\partial}{\partial e}\iint_S (\phi\nabla\phi) \cdot \mathbf{n} \, dS \\
 &= \frac{1}{2}\rho\iint_S \left(\frac{\partial\phi}{\partial e}\nabla\phi + \phi\frac{\partial}{\partial e}\nabla\phi \right) \cdot \mathbf{n} \, dS
 \end{aligned} \tag{A.3}$$

Therefore, the resulting expressions in equations (A.2) and (A.3) are equivalent.

Appendix B

Linearization of $\frac{\partial\phi}{\partial t}$

Recall the unsteady Bernoulli equation:

$$-\frac{1}{\rho}(p - p_a) = \frac{\partial\phi}{\partial t} + \frac{1}{2}\nabla\phi \cdot \nabla\phi - gz$$

Now, the unsteady Bernoulli equation is reorganized as an expression of $\frac{\partial\phi}{\partial t}$ at the exact free surface,

$$\frac{\partial\phi(z)}{\partial t} = \left(gz - \frac{1}{2}\nabla\phi \cdot \nabla\phi \right) \Big|_{-z=\eta}$$

This expression is linearized about $z = 0$ by using its Taylor series expansion,

$$\begin{aligned} \frac{\partial\phi(z)}{\partial t} &= \left(gz - \frac{1}{2}\nabla\phi \cdot \nabla\phi \right) \Big|_{z=0} + \frac{\partial}{\partial z} \left(gz - \frac{1}{2}\nabla\phi \cdot \nabla\phi \right) \Big|_{z=0} z + \dots \\ &= gz + (\text{higher order terms}) \\ &\approx gz \end{aligned}$$

Therefore,

$$\frac{\partial\phi}{\partial t} \Big|_{z=0} = 0$$