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**STRUCTURAL PROPERTIES OF I-GRAPHS: THEIR  
INDEPENDENCE NUMBERS AND CAYLEY GRAPHS**

by

Zachary J. Klein

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Co-Advisors:

Thor Martinsen  
Raluca Gera  
Pantelimon Stanica

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**STRUCTURAL PROPERTIES OF I-GRAPHS:  
THEIR INDEPENDENCE NUMBERS AND CAYLEY GRAPHS**

Zachary J. Klein  
Ensign, United States Navy  
BS, United States Naval Academy, 2019

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June 2020**

Approved by: Thor Martinsen  
Co-Advisor

Raluca Gera  
Co-Advisor

Pantelimon Stanica  
Co-Advisor

Wei Kang  
Chair, Department of Applied Mathematics

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## **ABSTRACT**

We discuss in this paper the independence numbers and algebraic properties of I-graphs. The I-graphs are a further generalization of the Generalized Petersen graphs whose independence numbers have been previously researched. Specifically, we give bounds for the independence number of different I-graphs and sub-classes of I-graphs, and exactly determine the independence number for other I-graphs and sub-classes of I-graphs.

We also analyze the automorphism groups of the I-graphs. These groups have been characterized in previous papers; in this paper, we examine them via their Cayley graphs. These Cayley graphs are characterized completely and examined according to their graph theoretical and algebraic properties.

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# CHAPTER 1: Introduction

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Graph Theory is a relatively new branch of mathematics, comparatively that is, but the study of objects and their relationships to each other is perhaps universal. It is therefore no surprise that the mathematical laws and theorems that have flowered from the study of graphs have myriad applications. Indeed the subject is so large (and at times intractable) that it is quite difficult to generate broad sweeping theorems and axioms as can often be found in other disciplines. It is in this sense that focusing on specific classes of graphs and their properties becomes essential to the progression of our knowledge in graph theory. In this spirit, this paper focuses on the I-graphs, and to a lesser extent, other generalizations of the Petersen graph.

The object of this thesis is to study the graphs collectively known as I-graphs, both in terms of the independence number of these graphs, and in terms of the Cayley graphs that represent their automorphism groups. The I-graphs themselves are in fact a generalization of a very famous graph in graph theory, the Petersen graph. In Chapter 1, we seek to explain more casually the topics that are defined with mathematical rigor in proceeding chapters, and give a general context for the work of this thesis.

The Petersen graph has been an object of great interest to graph theorists since its inception. In many cases, the Petersen graph provides an important counterexample for graph theoretical conjectures. This has given the Petersen graph an important place over time as a tool by which new graph theoretical conjectures may be tested [1]. The graph itself is relatively simple, it consists of ten vertices arranged as two five-cycles attached to one another at each of their vertices (it is 3-regular). In its normal embedding in the plane there is an inner cycle and an outer cycle, with the inner cycle forming the shape of a five-sided star. In this fashion we observe that in the “outer cycle” every consecutive vertex is connected by an edge and in the “inner cycle” every second vertex is connected by an edge (a skip of two). This notion of an “inner graph” and an “outer graph” along with associated edge patterns for each is important to generalizing the Petersen graph into the class of graphs we study in this thesis.

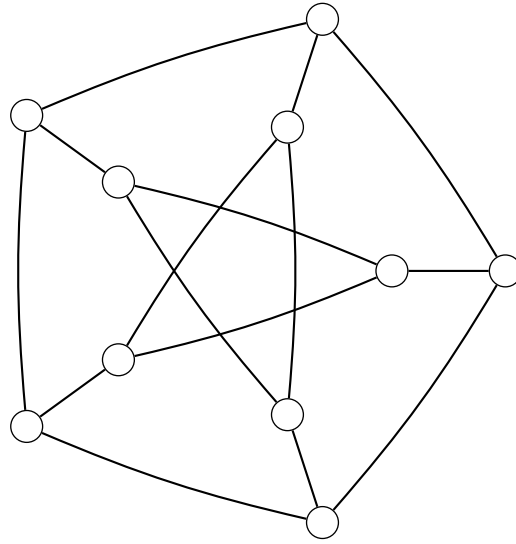


Figure 1.1. The Petersen graph.

The Generalized Petersen graph (introduced by [2]) preserves the basic structure of the Petersen graph in that it consists of two  $n$ -cycles attached to each other at every vertex (3-regular again). These graphs are no longer contained to just ten vertices however, and the vertices of each cycle need not be connected according to the pattern observed in the Petersen graph itself. This means that if we embedded a Generalized Petersen graph on the plane as before with an “outer cycle” where every consecutive vertex is attached by an edge, then the vertices of the “inner cycle” do not need to have a skip of two, but can in fact have a skip of any number. This is usually taken to be less than half the number of vertices in a cycle by convention.

Yet the class of I-graphs which we study in this thesis are an even further generalization of the Generalized Petersen graphs. These graphs viewed in the conventional embedding on the plane allow for not only a skip on the “inner cycle,” but also a skip on the “outer cycle.” If we try to construct a graph we may realize that the graph we have constructed could be re-arranged to a different embedding on the plane to resemble a Generalized Petersen graph. In fact, many I-graphs are actually isomorphic to Generalized Petersen graphs. This thesis however, is concerned with I-graphs that are not themselves Generalized Petersen graphs. As we shall see these are called “proper” I-graphs.

The first goal of this thesis is to study and understand independent sets in the I-graphs. In particular we find bounds and expressions for the size of maximum independent sets. But why is this important? From what problem did the question of independent sets arise? What is an independent set? Finding an independent set in a graph is essentially the same as solving some scheduling problem [3]. In other words we are seeking a set of objects that share no commonalities. In the graph this is simply a set of vertices where no two vertices have an edge between them. Thus giving parameters for a maximum size independent set in a graph or a class of graphs gives us a notion of the maximum sized set of objects with no redundancies for some system. There are many applications that rely on algorithms which find maximum size independent sets. Picking course loads or de-conflicting traffic are just two examples.

This thesis moves beyond independent sets to discuss the algebraic properties of the I-graphs. Graphs admit a great deal of algebraic structure. One such structure on graphs of interest in this thesis is the automorphism group of a graph. Notice that if we start from the conventional drawing of the Petersen graph, certain rotations or reflections of the Petersen graph are indistinguishable from our original starting point. In more mathematical terms, this means that there are isomorphisms between the Petersen graph and itself. These isomorphisms, in fact called automorphisms, form an algebraic structure called a group when paired with the operation of functional composition. The I-graphs being a generalization of the Petersen graphs admit both automorphisms of rotation and reflection, but also other more complex automorphisms described later in this thesis that work on the “inner” and “outer” cycles of the graph.

Why are these automorphism groups important to mathematicians? Well in some sense the automorphism group of any object encodes all of the various symmetries that object has. This notion of a group of symmetries has many applications and often allows physicists to describe the geometry of different objects and their symmetry properties as they move through space.

The automorphism groups of the I-graphs have been established in previous work, which we describe fully in Chapter 2, but no further work has been done to characterize properties of these automorphism groups. In this thesis, we study the algebraic properties of these automorphism groups via their associated Cayley graphs. These graphs represent the

structural properties of a given group. In our case, they succinctly represent the elements of the automorphism groups and the relations between those elements. We provide some important new results concerning the Cayley graphs for the automorphism groups of an I-graph, both in specific cases and in general.

We also move to examine the eigenvalues for our Cayley graphs. It is therefore that in coming up with eigenvalues for the adjacency matrix of our Cayley graphs we have traveled all the way from Graph Theory to Linear Algebra. Finding the spectrum of our Cayley graphs gives further insight into the automorphism groups for the I-graphs in this thesis.

One inadvertent consequence of studying the I-graphs in this thesis is that we have in reality gained a small window of insight into how thoroughly the mathematical world and all of its various fields are connected. The fact that an analysis of the structure of one small class of graphs can take us all the way from Graph Theory, to Algebra and Group Theory, and finally to a study of eigenvalues, shows this connection clearly.

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## CHAPTER 2: Background and Literature Review

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In this chapter, we outline prerequisite results that will contribute to our own work on independent sets and algebraic properties in I-graphs. Our work builds upon the results of Boben, Fox, Gera, Pisanski, Stănică, and Zitnik, in addition to others, as we detail and reference in this chapter.

We start in the first section by introducing common graph theory terminology. We subsequently examine the links between graph theory and groups in the second section. The third section states prior results for finding bounds of the independence number in Generalized Petersen graphs. The Generalized Petersen graphs are a sub-class of the I-graphs introduced in the fourth section of this chapter.

This paper uses common graph theory terminology found in reference [4]. For the benefit of the reader, we list some of these common definitions and concepts in Section 2.1. We build upon these definitions to understand I-graphs.

Gross and Tucker's work on the connection between Graph and Group theory forms the core of Section 2.2 in this chapter [5]. Our own work seeks to explore the algebraic properties of the I-graphs in the form of automorphism groups. Boben, Pisanski and Zitnik discuss the automorphism groups of I-graphs in their paper [6].

The basis of the Section 2.4 is formed by Fox, Gera, and Stănică's results on bounding the independence number in Generalized Petersen graphs [7]. This work is helpful to our own goal of finding bounds on the independence number of I-graphs.

Boben, Pisanski and Zitnik demonstrate many properties of I-graphs in their work that inform the final section of this chapter [6]. The properties of I-graphs are integral for finding bounds on the independence number and exploring automorphism groups in I-graphs. This section is integral to the new work of this thesis.

## 2.1 Graph Theory Basics

As stated, we now provide some basic graph theory definitions for the benefit of the reader. We start with the definition of a graph itself. The definition appears here as in [4].

**Definition 2.1.1.** A *graph* is a pair  $G = (V, E)$  of sets satisfying  $E \subset V^2$ . Thus the elements of  $E$  called *edges* of  $G$  are two element subsets of  $V$ . The elements of  $V$  are known as *vertices* of  $G$ .

For the purposes of this paper we limit ourselves to the study of simple graphs. This means that our graphs avoid *loops* and *multi-edges*. All I-graphs and Generalized Petersen graphs are simple. For convenience, edges of graphs are often denoted by their incident vertices in alphanumeric order and not in set form.

In this paper, we concern ourselves with independent sets in graphs. These sets are useful in applications, and give us a graph invariant. We now give the definition of an independent set. An example of an independent set in a graph is found in Figure 2.1. This definition appears here as found in previous work [7].

**Definition 2.1.2.** Given a graph  $G$ , an *independent set*  $I(G)$  is a subset of the vertices of  $G$  such that no two vertices in  $I(G)$  are adjacent.

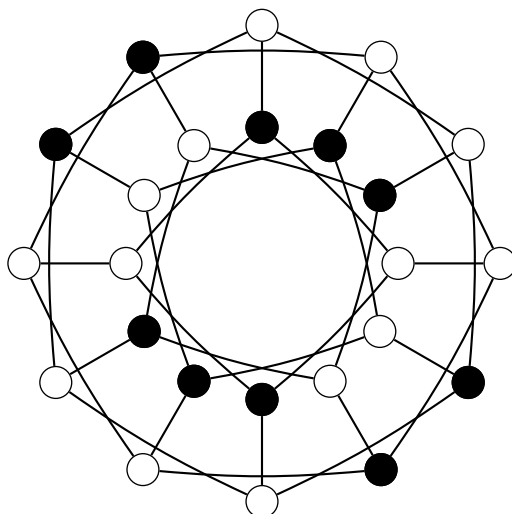


Figure 2.1. The filled in vertices identify an independent set in  $I(12,2,3)$ .

Clearly, these sets have many interesting applications. The independent sets that we are interested in are the maximum independent sets. Note that independent sets are not unique, not even maximum independent sets. The size of the largest independent set in a graph is known as the independence number as we introduce in Definition 2.1.3. Finding the independence number of I-graphs is what this paper is chiefly concerned with. Finding the independence number is an NP-hard problem [8], and we therefore rely on finding bounds for the independence number in some cases. We formally define the independence number below.

**Definition 2.1.3.** The *independence number*,  $\alpha(G)$  of the graph  $G$ , is the cardinality of a largest set of independent vertices.

The independent set shown in the previous figure is in fact maximum, and therefore we may say that  $\alpha(I(12,2,3)) = 10$ .

## 2.2 Algebraic Perspectives on Graphs

Another important goal of this project is to find a well-defined Cayley graph that can represent the automorphism group of an I-graph. To that end, we introduce some basic

definitions here such as the notions of isomorphism, automorphism and Cayley graphs. Much work has been done on the connection between group and graph theory. We rely largely on the work of Gross and Tucker, and specifically [5]. Another great source for this paper was the work of Joyner and Melles in [4]. Many definitions below are taken from these works.

**Definition 2.2.1.** We say  $f : G_2 \rightarrow G_1$  is a *map* from the graph  $G_2 = (V_2, E_2)$  to  $G_1 = (V_1, E_1)$  if  $f$  is associated to the maps  $f_V : V_2 \rightarrow V_1$  and  $f_E : E_2 \rightarrow E_1$  where  $f_E(uv) = f_V(u)f_V(v) \in E_1$ . We say  $f$  is an *isomorphism* if both  $f_V$  and  $f_E$  are bijections.

Isomorphisms are important for understanding the characteristics of graphs that are invariant. The notion of isomorphism between graphs motivates the assigning of a group structure to a graph, or likewise, the assigning of a graph to a group structure. Gross and Tucker give us the following definition in [5].

**Definition 2.2.2.** An isomorphism,  $f : G \rightarrow G$ , from a graph to itself is called an *automorphism*. Under the operation of composition, the family of all automorphisms of a graph  $G$  forms a group called *the automorphism group of  $G$* , and is denoted  $Aut(G)$ .

Note that automorphisms must by definition preserve graph invariants. Only certain vertex/edge re-labelings are automorphisms of the graph. To emphasize this point we introduce the definitions of vertex-transitivity and edge-transitivity in graphs. These are important notions that further ground our understanding of Cayley graphs and the connections between group and graph theory. We take these definitions from Joyner and Melles [4].

**Definition 2.2.3.** A *vertex-transitive* graph is a graph  $G$  such that, given any two vertices  $v_1$  and  $v_2$  of  $G$ , there is an automorphism of  $G$  that maps  $v_1$  to  $v_2$ .

**Definition 2.2.4.** Likewise, an *edge-transitive* graph is a graph  $G$  such that, given any two edges  $e_1$  and  $e_2$  of  $G$ , there is an automorphism of  $G$  that maps  $e_1$  to  $e_2$ .

The group of automorphisms of a graph essentially encodes the structure of a graph algebraically. In other words, it is the group of vertex/edge re-labelings that do not change the fundamental structure of the graph. Likewise, we would like to be able to represent groups via a graph that preserves the structure of the group. This is done with Cayley graphs. These too will be important to our work. We give the definition of the Cayley graph as defined in [4] by Joyner and Melles.

**Definition 2.2.5.** Let  $\mathcal{G}$  be a finite multiplicative group. Let  $S \subset \mathcal{G}$  be a subset which satisfies the condition  $S = S^{-1}$  and  $1 \notin S$ . The *Cayley graph* of  $(\mathcal{G}, S)$  is the graph  $G = \text{Cay}(\mathcal{G}, S)$  whose vertices are  $V = \mathcal{G}$  and whose edges  $E$  are defined by pairs  $(g_1, g_2)$  such that  $g_2g_1^{-1} \in S$ . Note that  $G = \text{Cay}(\mathcal{G}, S)$  is  $|S|$ -regular, and is connected if and only if  $S$  generates  $\mathcal{G}$ .

Gross and Tucker go on to explain the connection of Cayley graphs to group theory by linking maps between Cayley graphs to group homomorphisms. This is done by assigning both colors and directions to the Cayley graph and thus making a Cayley color graph. To get the Cayley color graph we say the positive direction of an edge is from  $g_1$  to  $g_2$ , and a coloring is assigned (see [5]). This link between group theory and graph theory is important because it allows us to shift seamlessly between algebraic and graph theoretical observations of the same structure. This is outlined in the following result from [5].

**Theorem 2.2.1** (Gross, Tucker). *Let  $f : \text{Cay}(\mathcal{G}, S) \rightarrow \text{Cay}(\mathcal{G}', S')$  be a color-consistent, direction-preserving, identity preserving map. Then its vertex function coincides with a group homomorphism  $\mathcal{G} \rightarrow \mathcal{G}'$ . Conversely, any group homomorphism  $h : \mathcal{G} \rightarrow \mathcal{G}'$  such that  $h(S) \subset S'$  coincides with the vertex function of a color-consistent, direction-preserving, identity preserving map  $\text{Cay}(\mathcal{G}, S) \rightarrow \text{Cay}(\mathcal{G}', S')$ . If our map is a graph isomorphism then the theorem implies coincidence with a group isomorphism.*

Gross and Tucker go on to the following result which is helpful to our work in that it sheds light on a connection to I-graphs.

**Theorem 2.2.2** (Gross, Tucker). *Any Cayley graph is vertex transitive.*

Gross and Tucker move on to note that the converse of this does not hold, and that a notable exception is the Petersen graph itself. This is important to our work here.

## 2.3 Generalized Petersen Graphs

To understand fully the class of I-graphs we first introduce the notion of Generalized Petersen graphs as found in reference [7]. The I-graphs are a further generalization of this class of graphs, and work has already been done to determine bounds for the independence number of Generalized Petersen graphs. This work was done by Fox, Gera, and Stănică [7], and some of their results appear here in this section.

The Petersen graph is quite a famous graph in itself. It is often cited as a counter-example to many graph theoretical results. The Petersen graph is made of outer and inner  $C_5$  sub-graphs connected such that the inner sub-graph is in a pentagram form. This graph will become the base of our notion of Generalized Petersen graphs. See Figure 2.2.

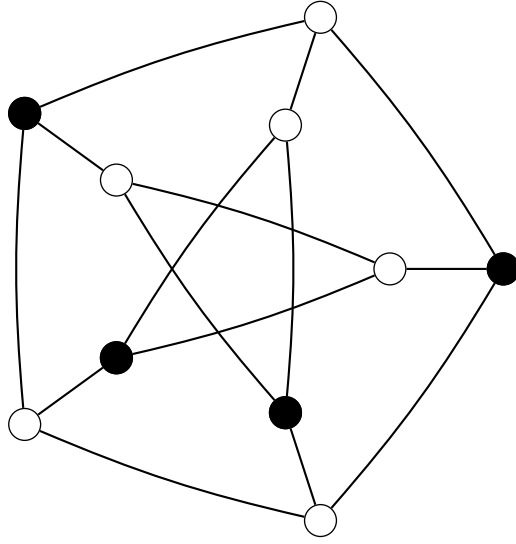


Figure 2.2. The Petersen graph, with an independent set.

The Generalized Petersen graphs expand on the Petersen graph by allowing a generic skip in the inner sub-graph and a generic number of vertices. It is hard to pin down the independence number of these graphs. Indeed, finding the independence number is a NP hard problem [8].

**Definition 2.3.1.** A *Generalized Petersen graph*,  $P(n, k)$  has vertices and edges given by

$$V(P(n, k)) = \{a_i, b_i | 0 \leq i \leq n - 1\},$$

$$E(P(n, k)) = \{a_i a_{i+1}, a_i b_i, b_i b_{i+k} | 0 \leq i \leq n - 1\},$$

where the subscripts are expressed as integers modulo  $n$  ( $n \geq 5$ ). We define  $A(n, k)$ , the *outer sub-graph*, as the sub-graph of  $P(n, k)$  consisting of the vertices  $\{a_i | 0 \leq i \leq n - 1\}$  and edges  $\{a_i a_{i+1} | 0 \leq i \leq n - 1\}$ . Similarly,  $B(n, k)$ , the *inner sub-graph*, is the sub-graph of  $P(n, k)$  restricted to the  $b$  vertices.

We now move to give some basic properties of the Generalized Petersen graphs. These results are recorded in [7] as introduction to the results of Fox, Gera, and Stănică as well.

**Theorem 2.3.1** (Fox, Gera, Stănică). *The following are true for the Generalized Petersen graphs  $P(n, k)$ ,  $1 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$ . The restriction on the “skip”  $k$  follows from (1).*

1.  $P(n, k) \cong P(n, n - k)$ .
2.  $P(n, k)$  is a 3-regular graph with  $2n$  vertices and  $3n$  edges.
3.  $P(n, k)$  is bipartite if and only if  $n$  is even and  $k$  is odd.
4. Assume that  $n, k, s$  are positive integers satisfying  $k \not\equiv \pm s \pmod{n}$ . Then  $P(n, k) \cong P(n, s)$  if and only if  $ks \equiv \pm 1 \pmod{n}$ .
5.  $P(n, k)$  is vertex-transitive if and only if  $(n, k) = (10, 2)$  or  $k^2 \equiv \pm 1 \pmod{n}$ .
6.  $P(n, k)$  is edge-transitive only in the following cases  $(n, k) = (4, 1), (5, 2), (8, 3), (10, 2), (10, 3), (12, 5), (24, 5)$ .
7.  $P(n, k)$  is a Cayley graph if and only if  $k^2 \equiv \pm 1 \pmod{n}$ .

Finding bounds for the I-graphs is an important aspect of our work. Fox, Gera, and Stănică go on in their noteworthy paper (reference [7]) to give sharp bounds and exact equations for the independence number of Generalized Petersen graphs. This is important to our work because this class of graphs is in fact a sub-class of the I-graphs. We will rely on many of Fox, Gera, and Stănică’s methods for finding bounds of the independence number in this work. We therefore record some of the results for independence number bounds in the class of Generalized Petersen graphs below. This work can be found in [7].

**Theorem 2.3.2** (Fox, Gera, Stănică). *The following results apply to a Generalized Petersen graph  $P(n, k)$ :*

1. For any  $n$  and  $k$  we have,  $\alpha(x) \leq \begin{cases} n & \text{if } n \text{ is even} \\ n - 1 & \text{if } n \text{ is odd.} \end{cases}$   
Equality is achieved when  $k = 1$ .

2.  $\alpha(P(n, k)) = n$  if and only if  $P(n, k)$  is bipartite.
3. If  $n \geq 5$ , then  $\alpha(P(n, 2)) = \lfloor \frac{4n}{5} \rfloor$ .
4. If  $n > 6$ , then  $\alpha(P(n, 3)) = \begin{cases} n & \text{if } n \text{ is even} \\ n - 2 & \text{if } n \text{ is odd.} \end{cases}$
5. If  $n > 10$ , then  $\alpha(P(n, 5)) = \begin{cases} n & \text{if } n \text{ is even} \\ n - 3 & \text{if } n \text{ is odd.} \end{cases}$
6. If  $n$  and  $k$  are odd integers then  $\alpha(P(n, k)) \geq \frac{2n-k-1}{2}$ . With equality if  $k$  divides  $n$ .
7. If  $n$  and  $k$  are integers with  $n$  odd and  $k$  even, then
$$\alpha(P(n, k)) \geq \frac{n-1}{2} + \left\lfloor \frac{\lceil \frac{n}{k} \rceil + 1}{2} \right\rfloor \cdot \left\lfloor \frac{n}{2d \lceil \frac{n}{k} \rceil} \right\rfloor + \frac{d-1}{2} \cdot \left\lfloor \frac{1}{2} \left( \frac{n}{d} \pmod{\lceil \frac{n}{k} \rceil} \right) \right\rfloor.$$

This equation approaches  $\frac{(2d+1)n}{4d} + \frac{d-1}{2}$  asymptotically, where  $d = \gcd(n, k)$ .
8. If  $n$  and  $k$  are even integers, then  $\alpha(P(n, k)) \geq \frac{n}{2} + \frac{d}{2} \lfloor \frac{n}{2d} \rfloor \asymp \frac{3n}{4}$ , where  $d = \gcd(n, k)$ .

In order to find these sharp bounds Fox, Gera, and Stănică developed techniques that we use in this paper. Most notably are the techniques of breaking a graph into sub-graphs and examining independent sets locally. Another interesting method discussed in [7] that we place particular emphasis on is the so called “greedy” approach to finding a maximum independent set where you start at a particular vertex and iteratively expand the set as quickly as possible as you move away. This approach does not necessarily guarantee a maximum sized independent set, but we discuss this in this paper.

## 2.4 I-graphs

By now it is surely irritating to realize that none of the definitions and preliminary results we have discussed have been related to our desired class of graphs, that is, I-graphs. I-graphs are a generalization of Generalized Petersen graphs. In fact, the Generalized Petersen graphs are themselves a special sub-class of I-graphs. We further generalize the Generalized Petersen graphs by also allowing a skip in the outer sub-graph. The authors Boben, Pisanski, and

Zitnik, give a definition for the class of I-graphs in reference [6], and we replicate the definition here.

**Definition 2.4.1.** The *I-graph*  $I(n, j, k)$  is a graph with vertex set

$$V(I(n, j, k)) = \{a_0, a_1, \dots, a_{n-1}, b_0, b_1, \dots, b_{n-1}\}$$

and edge set

$$E(I(n, j, k)) = \{a_i a_{i+j}, a_i b_i, b_i b_{i+k} | 0 \leq i \leq n-1\}.$$

Since  $I(n, j, k) \cong I(n, k, j)$  we assume that  $j \leq k$ . Note that we define the outer and inner sub-graphs as we did before with the Generalized Petersen graphs. It should also be obvious that  $P(n, k) \cong I(n, 1, k)$ .

Boben, Pisanski and Zitnik term the vertices  $a_i$  of the outer sub-graph  $A(n, j, k)$  the *vertices on the outer rim*. Likewise, the vertices  $b_i$  of  $B(n, j, k)$  are the *vertices on the inner rim*. The edges running between  $A(n, j, k)$  and  $B(n, j, k)$  are termed *spokes*.

As with the Generalized Petersen graphs we start with some basic properties of the I-graphs. These properties appear here as they are in reference [6], where they introduce the work of Boben, Pisanski, and Zitnik. These properties are crucial for this paper, and we draw special attention to (5) and (6). These properties distinguish when an I-graph falls into the sub-class of the Generalized Petersen graphs.

**Theorem 2.4.1** (Boben, Pisanski, Zitnik). *The following are true for I-graphs  $I(n, j, k)$ :*

1. *The graph  $I(n, j, k)$  is connected if and only if  $\gcd(n, j, k) = 1$ . If  $\gcd(n, j, k) = d > 1$ , then the graph  $I(n, j, k)$  consists of  $d$  copies of  $I(n/d, j/d, k/d)$ .*

2. A connected graph  $I(n, j, k)$  is bipartite if and only if  $n$  is even and  $j, k$  are odd.
3. If  $j \neq \pm k$  then  $I(n, j, k)$  has a cycle of length 8. If  $j = \pm k$  then in  $I(n, j, k)$  there exists a cycle of length 4.
4. Let  $n, j, k$  and  $a$  be positive integers such that  $\gcd(n, j, k) = 1$  and  $\gcd(n, a) = 1$ . Then the graph  $I(n, aj, ak) \cong I(n, j, k)$ .
5. Let  $n, j$  and  $k$  be positive integers such that  $\gcd(n, j, k) = 1, \gcd(n, j) \neq 1$  and  $\gcd(n, k) \neq 1$  then the graph  $I(n, j, k)$  is neither vertex-transitive nor edge-transitive.
6. A corollary of (5). A graph  $I(n, j, k)$  is a Generalized Petersen graph if and only if  $\gcd(n, j) = 1$  or  $\gcd(n, k) = 1$ . If  $\gcd(n, j) = 1$  then  $I(n, j, k) \cong P(n, s)$ , where  $s$  is the solution of the equation  $k \equiv s \cdot j \pmod{n}$  (this equation is always solvable, for  $s > n/2$  take instead  $P(n, n - s)$ ).
7. Let  $n, j, k, j',$  and  $k'$  be positive integers such that  $\gcd(j, k) = \gcd(j', k') = 1$  and  $\gcd(n, j) = \gcd(n, j') \neq 1$ , and  $\gcd(n, k) = \gcd(n, k') \neq 1$ . Then the graph  $I(n, j, k) \cong I(n, j', k')$  if and only if  $k \cdot j' \equiv \pm k' \cdot j \pmod{n}$ .

Boben, Pisanski, and Zitnik note that specifically items (4), (6) and (7), as well as (4) from Theorem 2.4.1 point to the fact that it is entirely possible to have I-graphs with different parameters that are in fact isomorphic graphs. Understanding this overlap and its implications is important to the work of this paper. Note that (5) defines a class of the I-graphs that are not isomorphic to Generalized Petersen graphs themselves. Boben, Pisanski, and Zitnik term these *proper I-graphs*. We are chiefly concerned with this set of I-graphs in this paper. We give the formal definition below.

**Definition 2.4.2.** An I-graph is called *proper* if it is connected and not isomorphic to a Generalized Petersen graph.

These are the graphs that we are chiefly concerned with in this work and most of our results are tailored for application to this sub-class of the I-graphs. The class is narrower than might

be thought. The two smallest proper I-graphs, shown below in Figure 2.5 are  $I(12, 2, 3)$  and  $I(12, 3, 4)$ .

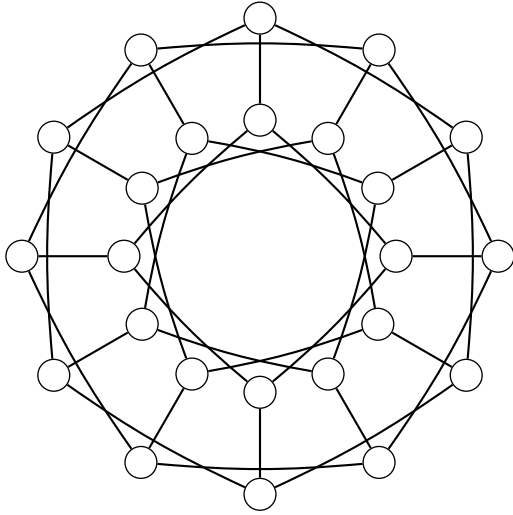


Figure 2.3.  $I(12, 2, 3)$

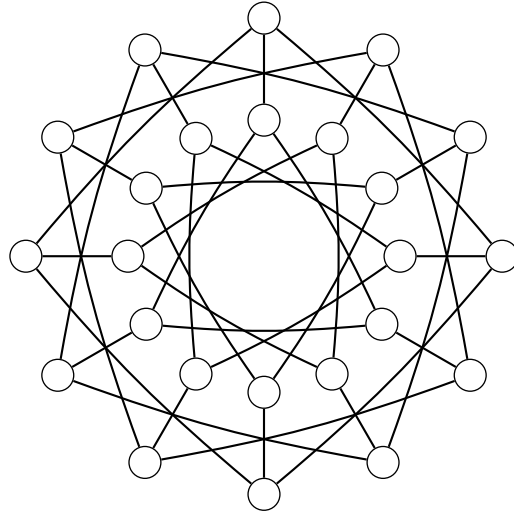


Figure 2.4.  $I(12, 3, 4)$

Figure 2.5. The smallest proper I-graphs.

In summary, this chapter gave a background of both the graph theory and group theory surrounding I-graphs. Most notably we discussed previous work on Cayley graphs, Generalized Petersen graphs, sharp bounds on the independence number for such graphs, and properties of the I-graphs. We will go on to explore these topics as applied to I-graphs in our work.

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## CHAPTER 3: The Results

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In this chapter, we provide some results that state or bound the independence number for specific classes of proper I-graphs. We then move on to characterize the Cayley graphs for the automorphism groups of proper I-graphs.

### 3.1 Bounds on the Independence Number of Specific I-graphs

We begin this section by introducing bounds for specific proper I-graphs, generalizing these in some direction, further in the section. We note from the introduced definitions and Theorem 2.4.1 that for a proper I-graph  $I(n, j, k)$  we have that  $n$  is a multiple of  $(j, k)$ . This will be very useful for the following theorems involving independence numbers in I-graphs.

Since the decision problem associated with maximum independent set is NP-Complete [8], it is in general difficult to show that an independent set is maximum even for a given I-graph. In the next theorem we determine precisely the independence number for a specific subclass of the I-graphs.

**Theorem 3.1.1.** *Consider the proper I-graphs of the form  $I(6r, 2, 3)$ , with  $2 \leq r \in \mathbb{Z}$ . Then  $\alpha(I(6r, 2, 3)) = 5r$ .*

*Proof.* We first demonstrate by construction that  $\alpha(I(6r, 2, 3)) \geq 5r$ . Take an independent set  $Q$  via the following construction. Let

$$Q_1 = \{b_0, b_1, b_2, b_6, b_7, b_8, \dots, b_{n-6}, b_{n-5}, b_{n-4}\}.$$

As  $k = 3$ , this is an independent set. Take also

$$Q_2 = \{a_3, a_4, a_9, a_{10}, \dots, a_{n-3}, a_{n-2}\}.$$

As  $j = 2$ , these vertices are not adjacent to themselves. Because no index value appears twice in  $Q = Q_1 \cup Q_2$ , none of the vertices are adjacent.

Note that  $6r$  is an even multiple of 3 for all  $r \geq 2$ . We see  $Q$  thus includes half the vertices of the inner subgraph  $B(n, j, k)$ . Furthermore,  $Q$  includes  $\frac{6r}{2k}$  sets of consecutively indexed vertices. We can conclude that for every consecutively indexed set of  $B(n, j, k)$  vertices,  $Q$  includes two vertices from the outer subgraph  $A(n, j, k)$ . Thus  $|Q| = \frac{6r}{2} + \frac{2 \cdot 6r}{2k} = 5r$ .

We now demonstrate that  $\alpha(I(6r, 2, 3)) \leq 5r$ . We define a subset  $Q_i$  of  $V(I(6r, 2, 3))$  by  $Q_i = \{b_i, b_{i+1}, \dots, b_{i+5}, a_i, a_{i+1}, \dots, a_{i+5}\}$ , where  $0 \leq i \leq n-1$ . Then  $Q_i$  must contain either 5 or 6 independent vertices in a maximal independent set denoted by  $Q'_i$  as we show next. We then show  $Q'_i \cup Q'_{i+6} \cup \dots \cup Q'_{i+(6(r-1))}$  is only an independent set if  $|Q'_{i+j}| = 5$  for all  $j \in \{0, 6, \dots, 6(r-1)\}$ , where  $r$  is the parameter in  $I(6r, 2, 3)$ .

We next show by contradiction that this is the case when  $r = 2$ . Suppose that the vertices of an I-graph can be partitioned into two sets  $Q_0$  and  $Q_6$  as defined. Note that  $\max |Q'_i| = 6$  for all  $i$ . This can be realized by taking  $Q'_i = \{b_{i+2}, b_{i+3}, b_i, b_{i+1}, a_{i+4}, a_{i+5}\}$  and since every index value of  $Q_i$  is represented in the set no other vertex from  $Q_i$  can be included in the independent set  $Q'_i$ . Thus in order to beat the bound of  $5r$  either  $|Q'_0| = 5$  and  $|Q'_6| = 6$  or  $|Q'_6| = 5$  and  $|Q'_0| = 6$ . Without loss of generality we suppose  $|Q'_0| = 6$ , then  $Q'_0 = \{b_2, b_3, a_0, a_1, a_4, a_5\}$ . Note that the vertices  $b_6, b_{11}, a_6, a_7, a_{10}, a_{11}$  are all adjacent to vertices in  $Q'_0$ , therefore; since  $Q'_0 \cup Q'_6$  is an independent set, none of these vertices are elements of  $Q'_6$ . This leaves only  $b_7, b_8, b_9, b_{10}, a_8, a_9$ , but we can take at most one vertex (either  $b_i$  or  $a_i$ ) for every index value  $i$  as an element of  $Q'_6$ . It is therefore impossible to have  $|Q'_6| = 5$ .

Due to the fact that  $k = 3$  we can have at most three vertices from the inner-subgraph in any  $Q'_i$ . There are three possibilities for this, either  $b_p, b_{p+1}, b_{p+2} \in Q'_i$  where  $p \in \{0, 1, 2, 3\}$ ,  $b_i, b_{i+1}, b_{i+5} \in Q'_i$  or  $b_i, b_{i+4}, b_{i+5} \in Q'_i$ . In every case where this occurs we can include only two additional vertices from the outer-subgraph in  $Q'_i$ . We pick these from the three index values not adjacent to some  $b_p \in Q'_i$ . We thus have that if  $Q'_i$  has three vertices from the inner-subgraph (the most possible) then  $\max |Q'_i| = 5$ . Since  $j = 2$ , we can have at most four independent  $a$  vertices in  $Q_i$ . This can only occur in one way, where  $Q'_i = \{b_{i+2}, b_{i+3}, a_i, a_{i+1}, a_{i+4}, a_{i+5}\}$ . In this case, we have  $|Q'_i| = 6$ .

We now show by contradiction that our claim holds in the case when  $r \geq 3$ . Suppose that we partition  $I(6r, 2, 3)$  into  $r$  subsets of  $Q_i$  and there exists  $j \in \{0, 6, \dots, 6(r-1)\}$  such that  $Q'_{i+j} = \{b_{i+j+2}, b_{i+j+3}, a_{i+j}, a_{i+j+1}, a_{i+j+4}, a_{i+j+5}\}$ . Without loss of generality (simply a

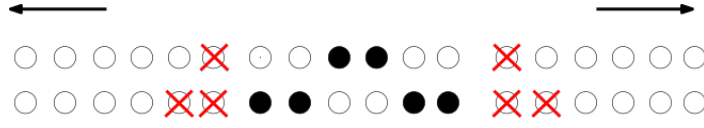


Figure 3.1. This image displays a  $Q'_i$  with six vertices.

consequence of labeling) say this occurs at  $j = 0$  and  $i = 0$ .

Then  $Q'_0 = \{b_2, b_3, a_0, a_1, a_4, a_5\}$ , and the vertices  $b_6, b_{n-1}, a_6, a_7, a_{n-1}, a_{n-2}$  are all adjacent to vertices in  $Q'_0$ . In order to achieve an independent set larger than the bound of  $5r$ , it must be the case that  $\frac{|Q'_6| + |Q'_{12}| + \dots + |Q'_{6(r-1)}|}{r-1} \geq 5$ . If there exists some  $Q'_j$  ( $j \in \{6, \dots, 6(r-1)\}$ ) such that  $|Q'_j| \leq 4$  then it necessitates that there is some  $Q'_p$  ( $p \in \{6, \dots, 6(r-1)\}$ , and  $p \neq j$ ) such that  $|Q'_p| = 6$ . Having both  $|Q'_p| = 6$  and  $|Q'_0| = 6$  leads to an immediate contradiction because it would imply two  $Q'_s$  ( $s \in \{6, \dots, 6(r-1)\}$ , and  $s \neq j, k$ ) such that  $|Q'_s| \leq 4$ . This implies further sets  $Q'_i$  of size six, this process continues to compound until we arrive at the contradiction of two adjacent subsets say without loss of generality  $Q'_j$  and  $Q'_{j+6}$  with size six. Thus in order to beat the bound of  $5r$  it must be the case that  $|Q'_6|, |Q'_{12}|, \dots, |Q'_{6(r-1)}| \geq 5$ . Due to the vertices included in  $Q'_0$ , we note that there is only one possibility for choosing  $Q'_6$  such that  $|Q'_6| \geq 5$ . That is having  $Q'_6 = \{b_7, b_8, b_9, a_{10}, a_{11}\}$ . This then forces  $Q'_{12} = \{b_{13}, b_{14}, b_{15}, a_{16}, a_{17}\}$  and so forth. Thus if we have  $Q'_j = \{b_{j+1}, b_{j+2}, b_{j+3}, a_{j+4}, a_{j+5}\}$  then it must be that  $Q'_{j+6} = \{b_{j+7}, b_{j+8}, b_{j+9}, a_{j+10}, a_{j+11}\}$ . We therefore conclude that it must be the case that  $Q'_{6(r-1)} = \{b_{n-5}, b_{n-4}, b_{n-3}, a_{n-2}, a_{n-1}\}$ , but this set includes vertices adjacent to  $Q'_0$ . We have a contradiction.

Thus  $Q'_i \cup Q'_{i+6} \cup \dots \cup Q'_{i+(6(r-1))}$  is only an independent set if  $|Q'_{i+j}| = 5$  for all  $j \in \{0, 6, \dots, 6(r-1)\}$ . We have shown by contradiction that if this is not the case then the union is not an independent set. If  $|Q'_{i+j}| = 5$  for all  $j \in \{0, 6, \dots, 6(r-1)\}$ , then the size of  $Q'_i \cup Q'_{i+6} \cup \dots \cup Q'_{i+(6(r-1))}$  is  $5r$ . Thus  $\alpha(I(6r, 2, 3)) \leq 5r$ . Since  $5r \leq \alpha(I(6r, 2, 3)) \leq 5r$ , we have  $\alpha(I(6r, 2, 3)) = 5r$ .  $\square$

### 3.2 K-groupings and J-groupings

We have developed (in conjunction with our colleague Matthew Dods [9]) methods for constructing an independent set in an I-graph that will always provide a decent lower bound for the independence number. For the construction we use the terms “K-groupings” and

“J-groupings.”

**Definition 3.2.1.** Take a sequence of  $k$  consecutively indexed vertices in the inner subgraph  $B(n, j, k)$  we call this a  $K$ -grouping. Likewise, we call a sequence of  $j$  consecutively indexed vertices from  $A(n, j, k)$  a  $J$ -grouping.

We give now a loose description of constructing an independent using  $K$ -groupings, that we employ in our Theorems. The construction partitions the vertices of the inner subgraph  $B(n, j, k)$  into  $K$ -groupings. For the construction of an independent set we take a sequence of  $k$  consecutively indexed vertices from  $B(n, j, k)$  in an independent set as our first  $K$ -grouping. We then exclude the adjacent  $K$ -groupings from our set, as we examine each consecutive vertex of  $B(n, j, k)$  we include every other  $K$ -grouping in our independent set until we arrive at our original  $K$ -grouping or at a  $K$ -grouping adjacent to the original. We say two  $K$ -groupings are adjacent when every edge  $b_i b_{i+k} \in E(B(n, j, k))$  is such that  $b_i$  belongs to the first  $K$ -grouping and  $b_j$  belongs to the second. In the second case we have two adjacent  $K$ -groupings not in our independent set. We then fill in as many vertices from  $A(n, j, k)$  as possible between each  $K$ -grouping in our independent set. This method gives an independent set. The construction for  $J$ -groupings is identical except that we start by taking  $J$ -groupings from  $A(n, j, k)$ . Note that in the construction we partitioned the inner subgraph.

Note that in the previous theorem we used the method of  $K$ -groupings to construct the lower bound in our proof, and in fact, that independent set was maximum. We now provide a theorem that pins down a lower bound for the independence number of  $I$ -graphs with  $k = j + 1$  using the method of  $K$ -groupings.

**Theorem 3.2.1.** Consider the proper  $I$ -graphs of the form  $I(r \cdot j(j + 1), j, j + 1)$ ,  $j, j + 1$ , where  $r > j$  are both positive integers. Then

$$\alpha(I(r \cdot j(j + 1), j, j + 1)) \geq (2j + 1) \cdot \left\lfloor \frac{rj}{2} \right\rfloor + 2\delta(rj),$$

where for  $x$  an integer

$$\delta(x) = \begin{cases} 0 & x \text{ is even} \\ 1 & x \text{ is odd.} \end{cases}$$

*Proof.* This is shown by constructing an independent set via the method of K-groupings. We form a partition of  $B(r \cdot j(j+1), j, j+1)$  with K-groupings. We start by taking  $j+1$  consecutively indexed vertices  $K_1 = \{b_0, \dots, b_j\}$ . Without loss of generality this is the first K-grouping, and we let  $K_1 \subset Q$ . Note now that  $K_2 = \{b_{j+1}, \dots, b_{2j+1}\}$  consists entirely of vertices adjacent to vertices from  $K_1$ . We therefore let  $K_2 \not\subset Q$ . Moving consecutively through  $B(r \cdot j(j+1), j, j+1)$  we take  $\bigcup_{i \text{ odd}} K_i \subset Q$ . Note that if  $r \cdot j$  is odd then  $K_{rj}, K_{rj-1} \not\subset Q$ , and that if  $r \cdot j$  is even then  $K_{rj} \not\subset Q$  and  $K_{rj-1} \subset Q$ . Thus we see that the number of K-groupings in  $Q$  is exactly  $\lfloor \frac{rj}{2} \rfloor$ . Now note that since  $k = j+1$  we can include a J-grouping from  $A(r \cdot j(j+1), j, j+1)$  for every K-grouping that is a subset of  $Q$  into the set  $Q$  and it will still be an independent set. For example, we can take the J-grouping  $J_1 = \{a_{j+1}, \dots, a_{2j}$  which are the indices in the outer-subgraph that are indexed consecutively to indices in  $K_1$ . With these vertices we now have  $|Q| = (2j+1) \cdot \lfloor \frac{rj}{2} \rfloor$ . Now suppose that  $rj$  is odd. In this case we have that  $K_{rj}, K_{rj-1} \not\subset Q$ . We can therefore include more vertices from  $A(r \cdot j(j+1), j, j+1)$  since there are two consecutive K-groupings that are not included in  $Q$ . We can in fact include  $J' = \{a_{(r \cdot j(j+1)) - 1}, a_{(r \cdot j(j+1)) - 2}\}$  into  $Q$ . These two vertices are not adjacent to any vertices already in  $Q$ . Any other  $a_i$  adjacent to a vertex in  $K_{rj}$  is adjacent to  $J_{rj-2}$ . Thus we see that  $|Q| = (2j+1) \cdot \lfloor \frac{rj}{2} \rfloor + 2\delta(rj)$ , where we define the  $\delta$  function over the integers as

$$\delta(x) = \begin{cases} 0 & x \text{ is even} \\ 1 & x \text{ is odd.} \end{cases}$$

Since  $Q$  is an independent set by construction, we have the claim. □

In summation, using K-groupings and J-groupings to construct an independent set in I-graphs provides a lower bound for the independence number. Independent sets constructed in this fashion include vertices in a way that is evenly distributed across the graph. It is in some way a greedy approach that seeks to maximize the number of independent vertices in a localized subgraph.

### 3.3 Cayley Graphs for $Aut(I(n, j, k))$

We have already briefly discussed the connection between graphs and groups. We now move to solidify this connection for I-graphs in particular.

**Theorem 3.3.1.** *No proper I-graph is itself a Cayley graph for any group.*

*Proof.* This is an immediate result of the definition of a proper I-graph and a Cayley graph. Let  $G = I(n, j, k)$  be an I-graph. We note that all Cayley graphs are by definition vertex transitive. The I-graph  $G$  is only vertex transitive if  $\gcd(n, j) = 1$  or  $\gcd(n, k) = 1$ . Now, we see by Theorem 2.4.1 that  $G$  is not a proper I-graph in this case.  $\square$

Boben, Pisanski and Zitnik [6] have characterized the automorphism groups of the proper I-graphs completely. That when taken in combination with the work of Frucht, Graver, and Watkins gives us a complete characterization for the automorphism group of any I-graph [6].

This paper will focus on the automorphism groups of proper I-graphs. To define these groups we first define the necessary automorphisms [6].

**Proposition 3.3.1** (Boben, Pisanski, Zitnik). The following mappings are automorphisms of  $I(n, j, k)$ :

1. The “rotation” mapping  $\rho$ , such that  $\rho(x_a) = x_{a+1}$ .
2. The “reflection” mapping  $\tau$ , such that  $\tau(x_a) = x_{-a}$ .

This implies that the automorphism group of  $G = I(n, j, k)$ , denoted  $Aut(G)$ , has the dihedral group  $D_n$  as a subgroup.

These automorphisms of the dihedral group do not represent all possible automorphisms of  $I(n, j, k)$ . The restrictions on the proper I-graphs, means there are also I-graphs that have automorphisms which rotate cycles from  $A(n, j, k)$  and reflect cycles in  $B(n, j, k)$  or vice versa.

We note that since the I-graphs under consideration are proper we have for  $I(n, j, k)$  that  $\gcd(n, j) = j_1 \neq 1$ ,  $\gcd(n, k) = k_1 \neq 1$ . This notation is borrowed [6].

**Proposition 3.3.2** (Boben, Pisanski, Zitnik). Let  $n = j_1 k_1$  or  $n = 2j_1 k_1$ , since  $j, k$  are relatively prime we label a vertex  $x_a$  of  $I(n, j, k)$  as  $x_{ij+pk}$  where  $ij+pk \equiv a \pmod{n}$ . Then the graph  $I(n, j, k)$  has also the following automorphisms:

1. The mapping  $\phi$ , such that  $\phi(x_{ij+pk}) = x_{-ij+pk}$ .
2. The mapping  $\psi$ , such that  $\psi(x_{ij+pk}) = x_{ij-pk}$ .

We can now completely characterize the automorphism group of any I-graph. This is shown in the following theorem.

**Theorem 3.3.2** (Boben, Pisanski, Zitnik). If  $\frac{n}{j_1 k_1} > 2$ , then the automorphism group of the graph  $I(n, j, k)$  is isomorphic to the dihedral group  $D_n$  otherwise it is isomorphic to the group  $\Gamma$  defined by:

$$\Gamma = \langle \rho, \phi, \psi \mid \rho^n = \phi^2 = \psi^2 = 1, \phi\psi = \psi\phi, \rho\phi = \phi\rho^a, \rho\psi = \psi\rho^b \rangle$$

where  $b \equiv -a \pmod{n}$  and  $a$  is the inverse in  $\mathbb{Z}_n$  of the element  $sk - tj$  such that  $sk + tj \equiv 1 \pmod{n}$ .

**Theorem 3.3.3.** Consider the proper I-graphs,  $G = I(n, j, k)$ , with their automorphism groups  $\text{Aut}(G)$ . The following constructions are Cayley graphs that represent  $\text{Aut}(G)$  using the fewest group generators:

1. If  $\text{Aut}(G) = D_n$  then we have for the Cayley graph  $C_1(D_n, S_1)$  that

$$V(C) = \{1, \rho, \rho^2, \dots, \rho^{n-1}, \tau, \tau\rho, \tau\rho^2, \dots, \tau\rho^{n-1}\}$$

$$E(C) = \{(a, \rho a), (a, \tau a) \mid a \in D_n\}$$

$$S_1 = \{\rho, \tau\}.$$

2. If  $Aut(G) = \Gamma$  then we have for the Cayley graph  $C_2(\Gamma, S_2)$  that

$$V(C) = \{1, \rho, \rho^2, \dots, \rho^{n-1}, \phi, \phi\rho, \phi\rho^2, \dots, \phi\rho^{n-1}, \\ \psi\rho, \psi\rho^2, \dots, \psi\rho^{n-1}, \psi\phi\rho, \psi\phi\rho^2, \dots, \psi\phi\rho^{n-1}\}$$

$$E(C) = \{(a, \rho a), (a, \phi a), (a, \psi a) | a \in \Gamma\}$$

$$S_2 = \{\rho, \phi, \psi\}.$$

*Proof.* We construct the Cayley graphs as possible. Take the vertices of the graphs as the automorphisms on an I-graph,  $G = I(n, j, k)$ , which form the group  $Aut(G)$ . These automorphisms are the elements of  $D_n$  or  $\Gamma$  respectively. The smallest generating set for  $D_n$  is  $S_1 = \{\rho, \tau\}$ . This is because  $\tau$  cannot be represented as a composition of  $\rho$  functions. The smallest generating set for  $\Gamma$  is  $S_2 = \{\rho, \phi, \psi\}$ . This is because none of these automorphisms can be represented as compositions of the others via their definition. For two vertices  $g_1, g_2 \in D_n$  we create the directed edge  $(g_1, g_2)$  if  $g_2 \cdot g_1^{-1} \in S_1$ . For two vertices  $g_1, g_2 \in \Gamma$  we create the directed edge  $(g_1, g_2)$  if  $g_2 \cdot g_1^{-1} \in S_2$ . We see that these are by definition Cayley graphs.

□

### 3.4 The Eigenspectrum of the Cayley Graphs for $Aut(I(n, j, k))$

We know from the previous section that the automorphism groups for I-graphs break into two categories, either it is the dihedral group of appropriate order, or it is the group  $\Gamma$  of appropriate order described previously. In this section, we move to characterize the eigenvalues for the Cayley graphs of these automorphism groups.

**Example 3.4.1.** We begin by computing the eigenvalues for  $Cay(Aut(I(12, 2, 3)), S_2)$ . Since  $Aut(I(12, 2, 3))$  has 48 vertices the eigenvalues for its associated 48 x 48 adjacency matrix were computed via the online computing platform COCALC. To facilitate this computation  $Cay(Aut(I(12, 2, 3)), S_2)$  was created in the graph making program Gephi. The code used and the results are shown, the eigenspectrum of the Cayley graph is shown as outputted from the computer program.

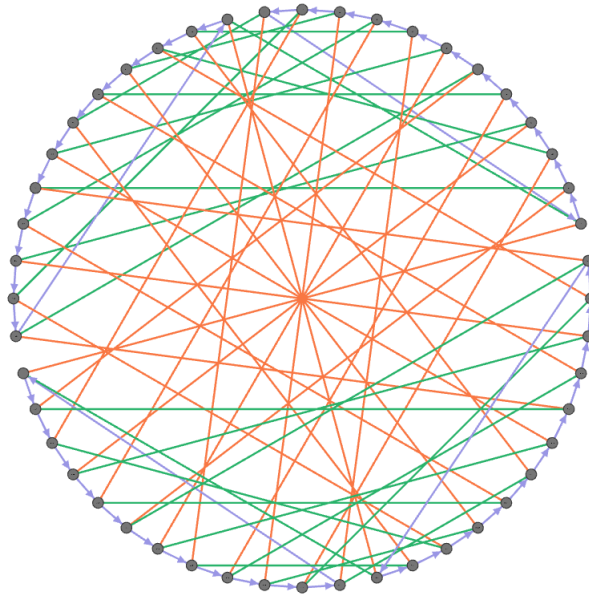


Figure 3.2. The Cayley graph for the automorphism group of  $I(12,2,3)$

```
G = nx.read_gml('CayleyGraph.gml')
nx.draw_spring(G)
Adj = nx.to_numpy_matrix(G)
nx.write_gexf(G, 'CayleyGraphAdjacencyMat.gexf')
eigvals = np.linalg.eigvals(Adj)
```

There were 12 eigenvalues in this case each of multiplicity four. They are:

$$-1, -0.866 + 0.5i, -0.866 - 0.5i, -0.5 + 0.866i, -0.5 - 0.866i, -1.388 \times 10^{-16} + i, \\ -1.388 \times 10^{-16} - i, 0.5 + 0.866i, 0.5 - 0.866i, 1, 0.866 + 0.5i, 0.866 - 0.5i.$$

The next and final section of this thesis uses a lot of terminology from Group Theory and Representation Theory. We point the reader to [10] and [11] for any unfamiliar definitions. Some definitions are included here for the benefit of the reader as well.

We now seek to characterize the eigenspectrum of the Cayley graphs representing the group  $\Gamma = \text{Aut}(G)$ , or  $D_n = \text{Aut}(G)$  of an arbitrary I-graph,  $G = I(n, j, k)$ . To do this we employ Cayley's Theorem to represent the automorphism groups as an isomorphic symmetric group. We use the language of a "group acting on  $G$ " to describe this. This allows us to embed the group into the general linear group,  $GL(4n, \mathbb{C})$  or  $GL(2n, \mathbb{C})$  respectively, via group homomorphisms. Such homomorphisms are called *representations* of the group. Only with these can we completely characterize the eigenspectrum of our Cayley graphs in the general case.

We know by Cayley's Theorem that every order  $t$  finite group is isomorphic to a subgroup of  $S_t$ . In our specific case we know that the automorphism group of an I-graph  $I(n, j, k)$  is either the group  $D_n$  with order  $2n$ , or the group  $\Gamma$  of order  $4n$ . Thus we have that the automorphism group being subgroups of the following,

$$\begin{aligned}\Gamma &< S_{4n} \\ D_n &< S_{2n}.\end{aligned}$$

But the question remains, what does this look like as a subgroup? Well Cayley's Theorem is often termed in the language of a group acting on a set. That is, we say that the automorphism groups in question are isomorphic to a subset of a symmetric group acting on the elements of  $\text{Aut}(G)$ . That is the automorphisms that are elements of  $\text{Aut}(G)$  are in fact simply permutations that act on themselves. We say  $\text{Aut}(G)$  is an  $\text{Aut}(G)$ -set.

More formally we have, for all  $g \in \text{Aut}(G)$ , let  $f_g : \text{Aut}(G) \rightarrow \text{Aut}(G)$  be defined by  $f_g(x) = g \circ x$ . These are permutations and  $X = \{f_g | g \in \text{Aut}(G)\} \cong \text{Aut}(G)$  by Cayley's Theorem. Note that  $X$  is an  $\text{Aut}(G)$ -set.

We are now ready to determine a representation of  $\Gamma = \text{Aut}(G)$ . Take the standard ordered basis for  $\mathbb{C}^n$  given by  $\mathcal{B} = \{e_1, e_2, \dots, e_{4n}\}$ . Take a permutation  $\alpha \in \text{Aut}(G)$ . We now define a linear transformation  $T_\alpha : \mathbb{C}^{4n} \rightarrow \mathbb{C}^{4n}$  where  $T_\alpha(e_i) = e_{\alpha(i)}$  for all  $i$ . Thus we now have

that  $[T_\alpha]_{\mathcal{B}}$  is a matrix for the permutation  $\alpha$ . This is a permutation matrix and we denote it  $P_\alpha$ .

Thus we set up the following representation of  $\Gamma$  via a group homomorphism.

$$\begin{aligned}\pi : \Gamma &\rightarrow GL(4n, \mathbb{C}) \\ \pi(\alpha) &= P_\alpha, \text{ for all } \alpha \in \Gamma.\end{aligned}$$

For our I-graph with automorphism group in the form  $\Gamma$ , we note that since the set  $\{\phi, \psi, \rho\}$  generates  $\Gamma$  so to does  $\{P_\phi, P_\psi, P_\rho\}$  generate the image under the group representation homomorphism  $\pi$ .

But in order to completely characterize the eigenspectrum of an automorphism group we need the *irreducible representations* of  $\Gamma$ . These are simply representations that are not the direct sum of other group representations. Luckily, it is well known that given a group representation, the irreducible representations of the group can be found in polynomial time [12]. It is not however the irreducible representations themselves but their corresponding *irreducible characters* that is needed to characterize the eigenspectrum. We provide the definition of an irreducible character as it appears in [10].

**Definition 3.4.1.** The *character* of a representation  $\pi$  of group  $\Gamma$  is the mapping  $\chi_\pi : \Gamma \rightarrow \mathbb{C}$  defined as  $\chi_\pi(g) = \text{Tr}(\pi(g)), g \in \Gamma$ . Where “Tr” is the trace of a matrix. If the representation is such that  $\pi \cong \pi_1 \oplus \pi_2$  then it and its associated character mapping are said to be *reducible*. They are said to be *irreducible* if no such non-trivial representations exist.

In order to characterize the eigenspectrum for the Cayley graph of an automorphism group  $\Gamma$  in the general case we need one final piece of the puzzle. That is the fact that given a representation  $\pi$  of  $\Gamma$  that is based on a generating set, we can compute the irreducible characters of  $\pi$  in polynomial time. The necessary theorem is quoted here from [12].

**Theorem 3.4.1** (Babai, Rónyai). *Let  $S \subset \Gamma$  be a set of generators of the group  $\Gamma$ . Assume that a representation  $\pi : \Gamma \rightarrow GL(n, F)$  is given by the list of matrices  $\{\pi(g); g \in S\}$ , where  $F$  is an algebraic number field. Then the irreducible characters involved in  $\pi$  and their multiplicities can be determined in time polynomial in the input size.*

Thus we see that our representation  $\pi$  can be used to find all the irreducible characters for the automorphism group  $\Gamma$  in polynomial time. The generating set  $S$  guarantees that this set of irreducible characters is maximal. This is because we have captured all the conjugacy classes of  $\Gamma$ . Though we do not show the calculation of all the irreducible characters here it is possible in every case.

Finally, we can completely characterize the eigenspectrum for the Cayley graph of an automorphism group  $\Gamma$  in the general case. This is shown in the following theorem as it appears in [10], and is originally the work of Babai as found in [13].

**Theorem 3.4.2** (Babai). *Let  $\Gamma$  be a finite group. Let  $\chi_1, \chi_2, \dots, \chi_h$  be the irreducible characters of  $\Gamma$  and let  $n_1, n_2, \dots, n_h$  be their respective degrees. Then the eigenvalues  $\lambda_{i,j}$  with  $(1 \leq i \leq h, 1 \leq j \leq n_i)$  of a Cayley graph  $\text{Cay}(\Gamma, S)$  of  $\Gamma$  satisfy*

$$\lambda_{i,1}^t + \dots + \lambda_{i,n_i}^t = \sum_{A \subset S, |A|=t} \chi_i \prod_{g \in A} g$$

*for any positive integer  $t$ .*

Thus we note that it is possible to characterize the eigenspectrum for the Cayley graph of the group  $\text{Aut}(I(n, j, k))$  in a general case. Though we worked through the process for an automorphism group of the form  $\Gamma$  we note that it is also possible when the automorphism group of the given I-graph is a dihedral group. Though we found a representation of an automorphism group of the form  $\Gamma$  and showed it is possible to find the necessary irreducible character functions from such a representation, we did not do the required calculations here.

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## CHAPTER 4: Conclusion

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In this chapter we give a brief summary and conclusion of the work of this thesis, and of the subject matter as a whole. We also seek to lay out potential areas of future work in regards to the independence number of I-graphs, and the algebraic properties of the I-graphs.

As explained in Chapters 1 and 2, the work of this thesis began by generalizing the problem of finding the independence number of a Generalized Petersen graph to the I-graphs. These graphs, especially in their proper forms, have a more complicated edge set than the Generalized Petersen graphs. This means that finding the independence number of I-graphs with equality is quite challenging in the general case. We therefore relied on applying bounds to the independence number of I-graphs via the method of “K-groupings.”

The method of “K-groupings” is simply a way of constructing an independent set in an I-graph that relies upon knitting together a series of locally maximal independent sets. These sets are started by taking the maximum possible number of consecutively indexed vertices from the inner subgraph of an I-graph to include in the independent set and filling in as many remaining vertices from the outer subgraph as possible. The method of “J-groupings” is the same except that we begin by taking consecutively indexed vertices on the outer subgraph.

As indeed is clear from the proofs, these constructed independent sets are not necessarily always maximum and do not necessarily have cardinality equal to the independence number of an I-graph. It is however conjectured that one of these constructions either “K-groupings” or “J-groupings” will yield a maximum independent set for a given I-graph.

In some more specific I-graph classes, the independence number was able to be determined exactly. This was done by examining subgraphs and determining all of the different locally maximal independent sets possible. Thus it was proven that I-graphs of the form  $I(6r, 2, 3)$  have an independence number of  $|5r|$ . We note that the construction of an independent set of this size was achieved via the method of “K-groupings” in the proof.

The algebraic properties of the I-graphs were of great interest in this thesis as well. More specifically, creating a Cayley graph for the automorphism of an I-graph in a general case.

We did this from the definition of the Cayley graph and from its automorphism group. These Cayley graphs have eigenvalues. Eigenvalues were calculated in the final portion of this thesis and we discussed how this may be done in the general case theoretically.

## 4.1 Future Work

Naturally, it would be beneficial to continue the work of this thesis for other classes of I-graphs and determine their independence numbers in general cases. Future work focusing specifically on the independence number of I-graphs,  $I(n, j, k)$ , where there is a large difference between  $j$  and  $k$  may be beneficial, it is in this case that we believe the constructive bound based on K-groupings does not give equality. We conjecture that in the case of skips of  $j$  and  $j + 1$  we have in fact equality, and not just a lower bound using K-groupings.

There is also potential for future algebraic work with the I-graphs. We note that, in seeking to completely characterize the eigenspectrum of the Cayley graphs covered, we did not actually compute the irreducible characters that arise from the given representation of the automorphism group  $\Gamma$ , and gave little treatment for the automorphism groups that are dihedral groups. Future work calculating the necessary irreducible characters would be beneficial.

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