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THESIS

ANALYTIC MODELS FOR ACTIVE SHOOTER INCIDENTS

by

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June 2020

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Active shooter incidents in the United States are growing in frequency and scale. This increase has gained widespread attention, with mounting calls to authorities and policymakers to reduce the number and impact of such events. Unfortunately, studying active shooter incidents is particularly difficult. Unlike other social phenomena, the window for observation is brief, and realistic human experiments are practically unobtainable. The last decade has seen a flurry of computer simulation models attempt to fill this gap; however, without an abundance of data or corresponding analytic models for verification, their results are precarious. This thesis presents a series of analytic models that capture the central dynamics of active shooter incidents, allowing researchers to gain insight on which factors most affect the outcomes of these events. The models have potential to inform policy, enable analysis for decision-makers, and influence emergency response plans in order to ultimately save lives.

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ANALYTIC MODELS FOR ACTIVE SHOOTER INCIDENTS

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ABSTRACT

Active shooter incidents in the United States are growing in frequency and scale. This increase has gained widespread attention, with mounting calls to authorities and policymakers to reduce the number and impact of such events. Unfortunately, studying active shooter incidents is particularly difficult. Unlike other social phenomena, the window for observation is brief, and realistic human experiments are practically unobtainable. The last decade has seen a flurry of computer simulation models attempt to fill this gap; however, without an abundance of data or corresponding analytic models for verification, their results are precarious. This thesis presents a series of analytic models that capture the central dynamics of active shooter incidents, allowing researchers to gain insight on which factors most affect the outcomes of these events. The models have potential to inform policy, enable analysis for decision-makers, and influence emergency response plans in order to ultimately save lives.

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Executive Summary

The number and scale of active shooter incidents within the U.S. is increasing at an alarming rate. This grim reality has fueled an urgency to understand the dynamics of active shooter incidents in order to minimize their effects in our communities. By successfully modeling these events, we can better understand their dynamics and be more equipped as a society to make decisions which reduce their impact.

This thesis presents several analytic models to research active shooter incidents by transferring techniques often applied to other topics, such as radioactive decay and force-on-force warfare, to active shooter events. These models, which fill a large vacancy within academic literature, enable decision-makers to gain insight concerning policy changes and active shooter response strategies at little to no cost. The models are also useful for verifying the outputs of other modeling types such as agent-based computer simulation models. After establishing the generic versions of the models, the thesis investigates three case studies from recent real-world active shooter incidents, exploring the effects of hypothetical changes in the scenarios.

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CHAPTER 1:
Introduction

1.1 Motivation

The number and scale of active shooter incidents within the U.S. is increasing at an alarming rate. Figure 1.1 illustrates the growth in frequency of such events in the U.S. since the year 2000.

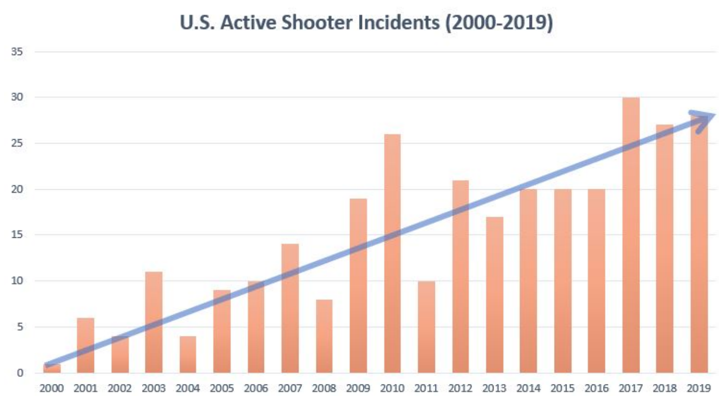

Figure 1.1. U.S. Active Shooter Incidents. Source: Blair (2014).

This grim reality has fueled an urgency to understand the dynamics of active shooter incidents in order to minimize their effects in our communities. Unfortunately, studying active shooter events is difficult. Unlike other social phenomena like the spread of disease or economic interactions, the window for observation is brief. The only attacks that occur are those we do not see coming; otherwise they would be prevented by the authorities. For this reason, no active shooter event is studied in real-time. We must, instead, rely upon eyewitness testimony and forensic evidence to recreate the events. After action reports are often used to intuit which factors were causal in the outcome of the event. Perhaps there were more casualties this time because of the low number of exits? Or perhaps it was due to the delayed police response? Unfortunately, these theories will not be conclusive unless the factors can be studied directly. One potential solution to this problem is to create and study models that adequately capture the dynamics of these events. Within the last decade, there has been a flurry of such models from academia, primarily consisting of agent-based computer simulation (ABS) models. While computer simulations are useful tools, they have many potential shortcomings and require comprehensive validation in order to prove their outcomes are well-founded. This thesis suggests using other modeling types, namely analytic models, to investigate the dynamics of active shooter incidents and corroborate the findings of other research.

1.2 Scope, Limitations and Assumptions

This thesis is not intended to be a comprehensive listing of all analytic models which are well-suited to active shooter scenarios. Instead, we aim to form a foundation which can be built upon by other researchers. In Chapter 3 we formulate seven analytic models as a toolbox of sorts, and in Chapter 4 we use the tools to analyze three real-world case studies.

Forensic data from active shooter incidents is often withheld from public record when the shooter is not put to trial. This limits the available details concerning the event to eyewitness interviews, which are known to be unreliable (Woocher 1977). While tactical drills and exercises can be studied directly, they are limited in their accuracy by the fact that subjects know they are participating in a simulation. We are, then, limited in our knowledge of the true dynamics of active shooter events. The outputs of models can be verified on the macro-level with the total amount of casualties from a given event, however, many of the micro-level phenomena must be presumed.

The assumptions which pertain to each individual model are noted throughout the thesis. Our aim is not to avoid making assumptions–they are unavoidable for all forms of modeling. Instead, we venture to make *reasonable* assumptions to simplify peripheral factors. We concur with Axelrod's statement concerning assumptions within models: "models that aim to explore fundamental processes should be judged by their fruitfulness, not by their accuracy. For this purpose, realistic representation of many details is unnecessary and even counterproductive" (Axelrod 1997b). Our models capture the elements of active shooter events which are necessary for fruitful analysis and leave many of the other details out. There are a number of assumptions that apply to all of the models within this thesis; they are listed below:

- No secondary casualties; i.e., bullets strike one person at most
- While exit rates from a shooting scene are capped at some maximum flow rate, exits

do not become jammed due to congestion

- People make decisions individually, without collaboration with other victims
- People are of equal size and, therefore, of equal probability of being hit; all else equal
- The attacker's goal is to indiscriminately murder as many people as possible

1.3 Research Questions

The primary objective of our research is to answer the following questions:

- Which features impact the outcome of active shooter scenarios the most? (e.g., Number/size of doors, number of bullets, firing rate, tactical proficiency of shooter, arena size, elapsed time of the event, etc.)
- Can analytic models achieve the complexity required to viably model active shooter incidents?
- Do the outcomes of analytic models align with real-world events and previous academic research?
- What operational insights and response strategies can be distilled from analytic active shooter models?

1.4 Thesis Outline

Chapter 2 establishes the background for active shooter incidents within the U.S. and surveys the existing literature on the modeling of active shooter incidents. Chapter 3 presents six descriptive and one prescriptive models written in generic form so to be broadly applicable. Chapter 4 utilizes the models from Chapter 3 to analyze recent active shooter incidents and compare the model outputs to the actual event outcomes. Finally, Chapter 5 presents summary observations and concludes the thesis.

1.5 Important Note

The topic of active shooter incidents is a sensitive matter, particularly for those who have been personally affected by them. In many sections within this thesis, we speak frankly when describing the dark realities of how active shooter events play out. This language is not meant to create a flippant tone regarding the real human consequences of such events. It is instead intended to produce clarity of thought and reason--an academic approach. Our hope is to gain true insight which leads to improved systems and processes in order to reduce the impact of active shooter events. In places where we must either err on the side of bluntness or on the side of ambiguity, we have chosen bluntness with the belief that precise language will create a higher possibility for true understanding.

CHAPTER 2:
Literature Review & Background

2.1 U.S. Active Shooter Background

In 2012, President Barack Obama signed into law the Investigative Assistance for Violent Crimes Act of 2012, which gave the Department of Justice the authority to assist in the investigation of "violent acts and shootings occurring in a place of public use" and "mass killings and attempted mass killings at the request of appropriate law enforcement officials of a state or political subdivision" (Sheldon 2012). While the term "active shooter" is now commonplace within U.S. culture, there are several varying definitions among researchers, government agencies, non-profit organizations and the like. In this thesis we adopt the U.S. government's definition for active shooter events, scenarios and incidents as "an individual actively engaged in killing or attempting to kill people in a confined and populated area" (U.S. Department of Homeland Security 2014). "Mass killings" were further defined by Congress in an amendment to this act the following year as "active shooter incidents which have three or more deaths" (Sheldon 2012). In response to the Violent Crimes Act, in 2014 the Federal Bureau of Investigation (FBI) launched a study of active shooter incidents between 2000 and 2013. Since then, the FBI has published active shooter studies biennially (Blair 2014).

Statistical analysis of historical data has been the most common tool for developing insights pertaining to active shooter incidents. The FBI United States Active Shooter Incident Reports have been cited most among active shooter historical analyses. Within these reports, the FBI identified 277 active shooter incidents on U.S. soil during the years 2000 to 2018 (Blair 2014). These incidents have included 2,430 total casualties with 884 killed and 1,546 wounded. The first seven years of this study (not including 2000, as December was the only month recorded) held an average of 8.3 active shooter incidents per year, while the final seven years held an average of 22.4 incidents–an increase of nearly 200%. Of these incidents, 108 were categorized as mass shootings, that is, having three or more fatalities in a single event. The study found active shooter incidents had very short durations with 69% of events lasting less than five minutes and 36% lasting less than two minutes.

Approximately half of these incidents ended with the shooter taking their own life and one in ten shooters were stopped by unarmed citizens. Nearly half of the incidents occurred at businesses open to pedestrian traffic or in educational settings.

The New York Police Department (NYPD) Counter Terrorism Bureau also released analyses of historical active shooter incidents in 2010, 2012 and 2016 (O'Neill et al. 2016). The data for these reports spanned the years 1966 to 2016. As the author notes, information from the years prior to 2000 may not be comprehensive as this predates widespread internet news reporting. While much of the information from these studies mirrored the findings of the FBI reports, they did include several additional emphases. During the 51 years assessed, 97% of perpetrators were men acting alone. The age distribution of shooters was bi-modal, reaching its first peak at age 15-19 and its second peak at age 40-44. Forty-two percent of attackers use multiple weapons with handguns as the most preferred type of firearm. In two thirds of the attacks, the shooter had personal or professional ties to their victims.

Despite increased social and governmental attention to the topic, active shooter and mass killing incidents continue to grow in number and in magnitude. The unfortunate truth spanning the U.S. is that active shooter incidents show little sign of waning within the foreseeable future. For this reason, we must collaborate to minimize the prevalence and scale of these attacks to every degree possible. Much research has been accomplished to determine the predictive factors which lead someone to carry out such an attack with the intent to preempt and deter individuals before a shot is fired. This thesis, however, focuses on the window of time between the first and last shots of an active shooter incident. We believe that by successfully modeling these events, we can better understand their dynamics and be more equipped as a society to make decisions which reduce their impact.

2.2 Literature Review

The field of operations research has had a markedly low contribution to active shooter modeling and analysis. Most existing academic works have developed agent-based computer simulation models, while a select few have modeled them analytically. This review surveys the field, noting the positive and negative attributes of each modeling and analysis type, then explains how our research enriches and expands upon the existing literature.

2.2.1 Computer-Based Simulation Models

Computer simulation models began in the early 1950's alongside the advent of pseudo-random number generators. Since then, as computer science and engineering have grown at an incredible pace, the capabilities of simulation models have far exceeded the expectations of the early developers. Advances in computer hardware and software have enabled highly complex models which give windows into real-world systems and processes (Nance and Sargent 2002). These broadly applicable tools have been used to gain insight in many fields of study and sectors of society, to include active shooter scenarios. Over the last decade, there have been several agent-based computer simulations (ABS) used to answer questions concerning active shooter incidents. ABS models are used to simulate complex systems using autonomous, interactive "agents" which are programmed to exhibit certain behaviors (Macal and North 2010).

Hayes and Hayes (2014) utilized an ABS to analyze proposed firearm legislation. Primarily, their study concluded that legislation aimed at reducing firing rate, such as limiting magazine size, would be far more effective at reducing casualties than policies which are designed to reduce accuracy, such as an assault weapons ban. Interestingly, their model indicated that "in crowded mass shooting events, lowering a weapon's accuracy may actually increase the number of people shot" (Hayes and Hayes 2014).

The Purdue Homeland Security Institute has used ABS's in three separate studies. Anklam et al. (2015) developed an ABS to research the effectiveness of armed security guards and civilians with concealed carry weapons during an active shooter event. Their model was able to quantify the benefit of armed staff within a school shooting scenario, showing that the presence of a resource officer reduced casualties by an average of 66% (Anklam et al. 2015). This may be an optimistic conclusion, however, considering the model instantly neutralizes the shooter upon entry to the room of an armed individual. Their research also emphasizes the importance of a timely police response. Second, Glass et al. (2018) used an ABS to find the point of diminishing returns for armed, off-duty law enforcement officials as a percentage of the population in a large event active shooter incident. Replicating the effective casualty rate of the Aurora, Colorado movie theater shooting, the team modeled a large event venue with 4,000 fans. Using these parameters, the model produced a negative logarithmic relationship between the number of casualties and police officers. When 38 law enforcement officers (or 0.7% of the total population) were present, the model predicted an average of three casualties (Glass et al. 2018). Finally, Lee et al. (2018) used an ABS to explore which factors reduce expected casualties the most during active shooter incidents. Their results, perhaps unsurprisingly, showed that immediate evacuation of civilians and high response effectiveness (including early detection of the shooter, accuracy of the responder and timeliness of the response) significantly contributed to a reduction in casualties (Lee et al. 2018).

Briggs and Kennedy (2016) employed an ABS to evaluate the effect of unarmed resistance by civilians during an active shooter event. Their results suggested that "even with a miniscule probability of overcoming a shooter, fighters may save lives but put themselves at increased risk" (Briggs and Kennedy 2016). Notably, if too few civilians choose to fight, there is a low chance of overcoming the attacker. However, according to their model, if greater than 0.4% of potential victims fought, the shooter was subdued more than 50% of the time, leading to significantly reduced casualties. This response is particularly effectual at the beginning of active shooter events, as the largest percentage of casualties occur within the initial confused moments after the attacker opens fire (Briggs and Kennedy 2016).

Stewart (2017) used an ABS to research how civilian response impacted expected casualties. All agents in the simulation would run, hide or some proportion of the agents would run and the others would hide. Her results indicated the optimal civilian response strategy was for some civilians to run and some hide, independent of police response time. This conclusion, however, appears to be highly dependent upon the design of her simulation as she attests to within her article, "If there were multiple exits from the classrooms, such as windows, or if the civilians were exiting the building out of multiple doorways, the all-run strategy might then be the best strategy" (Stewart 2017). For this reason, Stewart's paper is perhaps best utilized as a proof of concept, showing that an ABS model can be used to gain insight for active shooter incidents at a particular location.

Advantages of Computer-Based Simulation Models

There are many advantages to using computer simulation as an approach to research, here I will focus on the three which most apply to active shooter models. First, and most importantly, computer models allow a system to be analyzed without recreating the physical system. This limits the cost of research and often reduces risks associated with physical experimentation (Osais 2017). In the case of active shooter scenarios, ethical concerns are alleviated by using computer agents rather than human subjects. Neurological stressors are simulated in lieu of exposing participants to potentially traumatic scenarios. Next, the parameters and conditions for computer models are easily manipulated to study the effects of particular factors. Techniques for designs of experiments, such as orthogonal or nearly orthogonal Latin Hypercubes, allow the researcher to essentially isolate the effect of any given factor (Hernandez et al. 2012). This requires a large amount of uncorrelated design points which are impossible to produce with human experiments, but easily completed using simulation. Finally, simulation models reduce the amount of simplifying assumptions required for tractability compared to analytic models. While there are limitations for the complexity of computer models due to computational power and software capabilities, this threshold is higher than that of a corresponding analytic model. Although greater complexity does not necessarily translate to superior performance, a high ceiling is less likely to leave the researcher wanting.

Limitations of Computer-Based Simulation Models

In 1973, the U.S. Comptroller General presented a report to Congress titled *Advantages and Limitations of Computer Simulation in Decisionmaking*. Despite the enormous growth in computer capabilities within the last 50 years, the conclusions of this report are still fundamentally valid to this day. The author asserts that "the choices of the scenarios, equipment performance, and personnel operations are based somewhat upon unknowns and uncertainties. The extent that the model reflects the real-world situation depends on the accuracy of the model builders' judgment" (U.S. Government Accountability Office 1973). While this is true of all models, simulation models are particularly prone to inter-dependencies from the model builder's judgements which are not readily apparent to the audience, or even the model builder at times. When used as a tool for decision-making, these "hidden" assumptions can have significant consequences. Developing useful simulation models is not a trivial task, it often requires powerful and expensive computer hardware and software as well as very skilled professionals. For this reason, these models are often very challenging to replicate, and therefore, difficult to validate. If the model is not excessively complex, it may be validated analytically. However, if the system can be represented analytically, an equation-based model would most often be preferred to a computer simulation. This is true because simulation models do not yield analytic insights. In an ideal circumstance, the outcomes of computer models can be compared to real-world events to establish relative congruence. Though, while the number of active shooter incidents is alarmingly high when seen through a societal lens, the prevalence is low for the purpose of validation.

2.2.2 Analytic Models

During the mid-1950s, the field of operations research, which had previously been vaguely defined as a broad application of scientific investigation, began to adopt emerging mathematical methods into its repertoire. Within a handful of years, these analytic methods such as linear programming, inventory theory, search theory and queuing theory, began to transform and even redefine the identity of operations research. Warfare analysis, which was born out of the World Wars, provided the foundation for applying deterministic and probabilistic models to real-world phenomena with the aim of enhancing decision making and predicting outcomes (Thomas 2015). The well-known Lanchester equations are an example of this. These elegant and relatively simple formulae describe the power relationship between two opposing forces in battle (Braun et al. 1983). Models like these have served as the groundwork not just for military application, but for analysis spanning business, biology and even cultural sectors.

Within their widely cited Global Survey of International Perspectives on Modelling in Mathematics Education, Kaiser and Sriraman proposed categorizing analytic models into six primary classifications: realistic or applied modeling, contextual modeling, educational modeling, socio-critical modeling, epistemological modeling and cognitive modeling (Kaiser and Sriraman 2006). The models in this thesis fall within the realistic or applied modeling category. According to Kaiser and Sriraman (2006), the central aim of realistic modeling is to "solve real world problems, understand the real world and promote modeling competencies." These types of models are primarily concerned with solving authentic problems rather than developing mathematical theory. A common difficulty among applied models, then, is caused by the interaction between mathematical abstraction and the real world. In many cases, particularly in those involving human interaction, there are no precise mathematical formulae to describe all elements of the given process. While math is clean and exact, the real world is messy and unpredictable. The challenge is to capture the central elements of a real-world system or process without allowing its complexity to exceed the limits of mathematical tractability. Just as the statistician George Box once famously said, "All models are wrong, but some are useful." In the modern world of "big data," large-scale computer simulation and black-box algorithms, we must not lose sight of the tremendous value of applied analytic modeling.

Gunn et al. (2017) developed a network flow problem formulation in order to optimize real-time evacuation guidance for potential victims during an active shooter incident. The model incorporates crowd dynamics and parameters from active shooter reports to predict how much danger evacuees are in at a given time. Essentially, when given the design of the arena and the location of the shooter and evacuees, the model repeatedly solves an optimization problem minimizing the risk to evacuees over the planning horizon. Using an ABS, the authors validated the effectiveness of their guidance formulation when compared to a control where no guidance was given (Gunn et al. 2017). The article does not, however, discuss the difference in expected casualties or any measurement to convey the magnitude of this outcome.

Kress (2005) used analytic modeling to examine the effect of crowd density on the expected number of casualties in a suicide bomber attack. While this article does not explicitly discuss active shooter incidents, Kress' model has many features which are generally applicable to attacks on passive targets. In the event of a suicide bombing attack, one would expect the number of casualties to increase as the density of the affected crowd increased. Kress' model showed that due to crowd blocking, this expectation is not well-founded. At a certain density, the effect of crowd blocking near the attacker overcomes the effect of increased targets and the expected casualties begins to decrease as crowd density increases (Kress 2005).

Advantages of Analytic Modeling

Occam's razor is a well-known heuristic which states that "plurality should not be posited without necessity" (Occam's Razor 2015). In other words, the principle favors simplicity: of two competing theories (or models in this instance), the least complex is to be preferred. In fact, this inclination was found to be nearly universal among Nobel laureates in economics through a questionnaire in 1995 where all but one agreed; simplicity is a desirable characteristic for a model (Zellner et al. 2001). While the functions of computer simulation models are often easier to conceptualize, their corresponding analytic models are fundamentally less complex. The dynamics of analytic models are evident allowing the model builder to gain insight concerning the interactions within the system. This contrasts with computer models, which frequently produce outcomes which are not easily traced. This is especially important during the verification and validation phase of model development. Analytic models are easily tested for correctness and allow for rapid iterations of validation and adjustment. Determining their credibility, then, is relatively straightforward. As mentioned before, computer models, by comparison, are often very difficult to assess. Some may suggest, then, to simply utilize statistical analysis of historical active shooter incidents, rather than models, for the purposes of research. While this technique is effective for gaining insights, it does not allow for sensitivity analysis or "what-if" hypotheticals as analytic models do.

Limitations of Analytic Modeling

Models are simply abstractions or imitations which help explain a system. They operate on a spectrum with simplicity on one end and accuracy on the other. No model is entirely accurate, perfectly resembling the true system, otherwise the model would be unnecessary. Analytic models are generally high in abstraction due to the limitations of mathematical tractability leading them to be particularly challenging to conceptualize. Untrained audiences, therefore, have difficulty interpreting the techniques and unwieldy mathematical syntax which often accompany these models. Many systems are far too complex to capture with a closed-form mathematical representation and, therefore, must be simplified in order to be modeled analytically. In many cases, this abstraction still captures the essential components of the system allowing for valuable analysis. In others, the restrictions render the model fruitless.

2.3 Our Contribution to Existing Literature

As previously stated, agent-based simulation models have made up the bulk of active shooter modeling in operations research. Even in circumstances when authors are willing to share their code base for these models, the simulations are often carried out on proprietary software. This barrier reduces the ability for others to replicate and validate findings. When outcomes from various simulation models differ, pinpointing the cause of these differences can be a formidable task, particularly when the models utilize separate software packages. Every degree of increased complexity produces another friction point where modeling artifacts can affect the outcome. In Advancing the Art of Simulation in the Social Sciences, Axelrod (1997a) confirms this principle:

> Achieving internal validity is harder than it might seem. The problem is knowing whether an unexpected result is a reflection of a mistake in the programming, or a surprising consequence of the model itself. For example, in one of my own models, a result was so counterintuitive that careful analysis was required to confirm that this result was a consequence of the model, and not due to a bug in the program. (Axelrod 1997a)

When computer bugs or unknown assumptions create large and counter-intuitive outcomes, they are at least identified and, in fortunate cases, are rectified. When the deficiencies create subtle changes, or worse yet, sizeable changes which align with the model builder's preconceived notions, they are often unidentified. One solution to this problem involves comparing all systems and their derivative operations to real-world data to confirm the validity of the model and surface all potential discrepancies. Unfortunately, information on the dynamics of active shooter incidents is limited and that which does exist is likely unreliable considering the sources are often eyewitness accounts from victims. Historical analysis does not allow for the modification of parameters and scenarios, which is a function frequently required for validation. Another option could be to conduct an experiment with human subjects, though a single-blind experiment when subjects believe they are in legitimate danger is morally untenable. Alternatively, an open trial where the subjects know the make-up of the test beforehand can lead to inauthentic outcomes. The best available option for validation is to model active shooter scenarios using separate techniques and compare outcomes. Analytic models can be used to examine varying levels of resolution, from interactions between individuals to more holistic approaches which capture the high-level outcomes of systems. This thesis presents multiple descriptive and prescriptive analytic models which explore active shooter incidents at the aggregate level. The set of models is not meant to be comprehensive, spanning the entirety of what analytic modeling has to contribute to the active shooter discussion. Rather, the models can form a foundation for others to build upon and can equip model builders with tools for verifying outcomes.

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CHAPTER 3:
Analytic Models

Analytic models are those that can be expressed as a closed-form mathematical expression. When discussing the relative merits of analytic modeling versus simulation modeling, Ignall et al. (1978) stated, "Whenever applicable, one prefers to use a simple analytic model yielding closed-form algebraic expressions relating system inputs and outputs," and "use of simulations to develop and test other mathematical models is conceptually analogous to use of experiments by physical scientists to develop new theory." Analytic models cannot be used to analyze all systems. However, when they can be used, they are generally preferred to simulation. Ideally, we can utilize both types and substantiate the analytic model with a simulation model.

This Chapter offers a set of analytic models which can be applied to numerous active shooter scenarios or used to verify the credibility of other models. We form, in effect, a toolbox to work from in Chapter 3, then illustrate how the tools may be utilized by performing case studies on recent real-world active shooter incidents in Chapter 4.

3.1 Definition of Terms

One can broadly categorize analytic models by their purpose or by their process. The purpose of an analytic model falls into one of three distinct groups: descriptive, predictive and prescriptive (Davenport 2013), while the process of an analytic model is either deterministic or stochastic. We begin by defining these terms then structure the remainder of the chapter using these broad categories.

3.1.1 Descriptive vs. Prescriptive Models

The models within this thesis are either descriptive or prescriptive. According to Friedenthal (2014), "Descriptive models describe a system or other entity and its relationship to its environment. It is generally used to help specify or understand what the system is, what it does, and how it does it." In our case, the descriptive models define certain active shooter scenarios capturing the essential interactions between the attacker, the victims and the environment (or arena). Prescriptive analytic models recommend particular actions to achieve a goal. That is, they connect analytic theory to real-world optimization (Gröger et al. 2014). The prescriptive models within this thesis emerge from the decisions a potential victim is required to make during an active shooter scenario, namely whether to run or hide.

3.1.2 Deterministic vs. Stochastic Models

A deterministic model is one that does not incorporate any randomness. Stochastic models, in contrast, vary based upon a set of random variables which are sampled from some defined probability distribution. The outputs, then, fall within some range of probable outcomes (Farinha 2018). When the random variables within a stochastic model are replaced by their expected values, the model becomes deterministic.

3.2 Descriptive Models

This section details six models which describe active shooter events using various techniques and levels of abstraction. Within each subsection, the models progress from least complex to most complex, that is, higher abstraction to lower abstraction.

3.2.1 Deterministic Models

Randomness is pervasive within the real world. Any model, then, which does not incorporate randomness is, at least for this reason, an abstraction. This approach is common within the field of applied mathematics. Real world systems, which are known to vary, are treated as functionally constant processes in order to simplify their operations. This subsection describes two such active shooter models.

Constant Kill and Exit Rates

Consider a basic scenario where N people are inside a room with no exits. A shooter fires at some constant rate b and hits new targets with some probability p. Provided the shooter only hits targets once and has a sufficient supply of ammunition, the effective casualty rate is $pb = r$. The shooting elapses until some time T when all the targets are hit. The number of casualties, k, at time t is simply the effective casualty rate multiplied by the elapsed time or, if all targets are hit, N:

$$k(t) = \begin{cases} rt & \text{if } rt < N \\ N & \text{otherwise} \end{cases}$$

and the total elapsed time of the event, $T = \frac{N}{r}$.

We now introduce exits into the model. If people exit at some constant rate ℓ, the casualties at time t are still rt, however, the elapsed time of the event is:

$$T = \frac{N}{r + \ell},$$

and the total number of casualties is:

$$k(T) = rT = \frac{rN}{r + \ell}.$$

This model is clearly simplistic, as it does not account for hiding or for shooting targets more than once. Additionally, casualty and exit rates will wane as the number of people within an arena is reduced. The casualty rate will be lower when the shooter must seek out targets rather than simply shooting into a crowd and the aggregate exit rate will be reduced as the arena becomes more sparse.

Dynamic Kill and Exit Rates

We now incorporate hiding into the model. In order to account for the dynamic nature of the rates, we define a set of differential equations for the number of casualties (K), number of people attempting to escape (E) and the number of people hidden (H). Using the Laplace Transformation, we solve the differential equations to convert them into functions of time. The parameters for the model are:

- α = Effective Casualty rate of exposed targets,
- δ = Casualty rate of hiding targets ($\delta < \alpha$),
- β = Exit rate,
- λ = Exposure rate of hiding targets.

The change in the number of casualties, K, is a combination of those who are shot while attempting to escape and those who are shot while hidden. Hence,

$$\dot{K} = \alpha E + \delta H. \tag{3.1}$$

The number of people attempting to escape, E, at a given time is reduced by the number of people who succeed in escaping and by those who are shot before escaping. Those who are hiding and then choose to attempt to escape increase this number. Hence,

$$\dot{E} = -(\alpha + \beta)E + \lambda H. \tag{3.2}$$

The number of people hiding, H, at a given time is reduced by those who choose to attempt to escape as well as those who are shot while hiding. So,

$$\dot{H} = -(\lambda + \delta)H. \tag{3.3}$$

At the very beginning of an active shooter incident, in the moments after the initial shot is fired, there is a small window of time when all targets are exposed. During this window, people choose to attempt to escape or hide and some are shot while doing so. At the conclusion of this initial period, we have K_0 casualties, E_0 targets attempting to escape and H_0 hidden targets.

The solution to Equation 3.3 is,

$$H(t) = H_0 \exp(-(\lambda + \delta)t). \tag{3.4}$$

Next, taking the Laplace Transform of Equation 2 we obtain,

$$E(t) = \left(\frac{\lambda H_0}{\lambda + \delta - \alpha - \beta} + E_0\right) \exp(-(\alpha + \beta)t) - \frac{\lambda H_0}{\lambda + \delta - \alpha - \beta} \exp(-(\lambda + \delta)t).$$

In order to simplify the notation, let,

$$\mu = \alpha + \beta,$$

$$\nu = \lambda + \delta,$$
$$\rho = \frac{\lambda H_0}{\nu - \mu},$$

then,

$$E(t) = (\rho + E_0)\exp(-\mu t) - \rho\exp(-\nu t). \tag{3.5}$$

Finally, taking the Laplace Transform of Equation 1 we obtain,

$$K(t) = K_0 + \frac{\alpha(\rho + E_0)}{\mu} - \frac{\alpha\rho - \delta H_0}{\nu} - \frac{\alpha(\rho + E_0)}{\mu}\exp(-\mu t) + \frac{\alpha\rho - \delta H_0}{\nu}\exp(-\nu t). \tag{3.6}$$

Assuming the shooter has sufficient bullets and is not terminated or otherwise halted, $K(T)$ is the total number of casualties from this model. Time T is the moment the final individual is either shot or escapes.

See Appendix A.1 for the Laplace Transform evaluations of $E(t)$ and $K(t)$.

3.2.2 Stochastic Models

As mentioned in Section 3.1.2, randomness is incorporated into the design of stochastic models. Random variables are sampled from probability distributions which fall into one of two general categories: discrete or continuous. Discrete random variables have a finite number of possible values, such as the faces of a die or a deck of cards. Continuous random variables have an infinite number of potential values, such as time or weight. Active shooter scenarios have many potential sources of randomness, to include: hit probability, elapsed time of the event, time between shots, time between escapes, quality of hiding place and many more. Though these values are all variable in real life active shooter incidents, our models will hold several of them constant (parameters) in order to trace the effects of randomness from the others. Within this subsection, we present four stochastic models which differ in their assumptions. The salient characteristics of each model make them more or less relevant to various types or parts of active shooter scenarios as we will demonstrate in Chapter 4.

Binomial Model

Consider a generic active shooter scenario where the attacker fires b bullets at n static targets with a hit probability of p. Presume the shooter only hits targets once and $n > b$. The number of targets hit, X, follows a binomial distribution with parameters b and p, hence,

$$Pr(X = x) = \binom{b}{x} p^x (1-p)^{b-x}. \tag{3.7}$$

The expected number of targets hit during this period is $E[X] = p \times b$.

We now relax the assumption that targets are only hit once. Now, let X be the total number of bullets that fall on any target; these are the bullets that hit a body. Then, as before, X has a binomial distribution with parameters b and p.

Let Y be the number of casualties, that is, the number of people hit by one or more bullets. For $y = 0, 1, 2, \ldots, n$,

$$P(Y = y) = E_X[P(Y = y|X)] = \sum_{x=0}^{b} P(X = x)P(Y = y|X = x).$$

Note that:

- P(no casualties if at least one bullet hits a target) $= P(Y = 0|X > 0) = 0$,
- P(no casualties if no bullet hits a target) $= P(Y = 0|X = 0) = 1$,
- P(more casualties than effective bullets) $= P(Y > X) = 0$. That is, $X \geq Y$ with probability 1.

This scenario is analogous to the classic "occupancy problem" in probability theory. Within this thought exercise, b balls are randomly assigned to n cells and one must answer the question, "what is the probability a given cell is occupied by at least one ball?" In our scenario, occupied cells are targets who have been struck by atleast one bullet, that is, they are a casualty.

By a stars and bars argument, the number of arrangements of b balls into n cells is $\binom{b+n-1}{n-1}$. See Appendix A.2 for a proof of the stars and bars argument.

Hence,

$$P(Y = 0) = P(X = 0) = (1 - p)^b,$$

and for $y = 1, \ldots, \min\{x, n\}$,

$$P(Y = y) = \sum_{x=y}^{b} P(X = x)P(Y = y|X = x).$$

Let $x > 0$. For $y = 1, 2, \ldots, \min\{x, n\}$,

$$P(Y = y|X = x) = P(n - y \text{ targets are not hit } \textit{and } y \text{ targets are hit})$$

$$= P(n - y \text{ targets are not hit; 1 bullet hits each of } y \text{ targets;}$$
$$x - y \text{ bullets randomly allocated between } y \text{ targets})$$

$$= \frac{\binom{n}{n-y}\binom{x-y+y-1}{y-1}}{\binom{n+x-1}{x-1}} = \frac{\binom{n}{y}\binom{x-1}{y-1}}{\binom{n+x-1}{x-1}}. \quad (3.8)$$

This distribution can be treated as a worst case scenario, as targets are static for the duration of the event. In a real-life scenario, targets would flee the shooter and would no longer be vulnerable to attack after departing the arena.

Exponential Decay Model

We now relax the assumption that targets are static. At the beginning of the incident, the arena contains some number of targets and, as time goes on, the number of remaining targets is reduced as targets exit the arena or are shot. This scenario can be modeled using exponential decay.

Perhaps the best known application of exponential decay is in describing the lifetimes of radioisotopes in chemistry (Radioactivity 2015). Radioactive material, such as Carbon-14, breaks down by shedding atoms over time. Though this process describes discrete elements (atoms), the aggregate process is approximated as a continuous variable, allowing the use of differential calculus. The expected change in atoms, N, over an infinitesimally small slice of time dt is some unique decay coefficient λ for all isotopes, so,

$$\frac{dN}{N} = -\lambda dt.$$

The solution for this differential equation is:

$$N(t) = N_0 \exp(-\lambda t).$$

This idea can be applied to an active shooter scenario where N_0 is the initial number of people and λ is a combination of the rates at which people exit or become casualties. Let the arena be a grid with m cells, where some cells contain a person and others do not. The arena has N people and the shooter has a firing rate r. Assign a random clock time, T, to each person which, upon completion, "escapes" them from the arena. So, for $n = 1, \ldots, N$,

$$P(\text{target with clock } T_n \text{ is alive by time } t) = (1-1/m)^{r\min\{T_n,t\}} = \exp(-r\min\{T_n,t\}\log(\frac{1}{1-1/m})).$$

If the clock T is random with p.d.f. $f(\cdot)$ and c.d.f. $F(\cdot)$ then,

$$P(\text{target is alive by time } t) = \int f(T)\exp((-r\min\{T,t\}\log(\frac{1}{1-1/m}))dT$$

$$= (1-F(T))\exp((-tr\log(\frac{1}{1-1/m})) + \int_0^t f(T)\exp((-Tr\log(\frac{1}{1-1/m}))dT.$$

And the expected number of targets remaining in the arena at time t is,

$$E[\text{number of people alive by time } t] =$$

$$N[(1-F(T))\exp((-tr\log(\frac{1}{1-1/m})) + \int_0^t f(T)\exp((-Tr\log(\frac{1}{1-1/m}))dT]. \quad (3.9)$$

This model is most accurate when N is very large. Therefore, it would be most useful when the number of potential victims is large, such as a concert or sporting event. The smaller the number of potential victims, the less relevant this model becomes.

Continuous Time Markov Chain

A Markov Chain is a stochastic model which probabilistically describes a sequence of events. In these systems, the probability of transitioning from one state to another is exclusively dependent upon its current state. As the name implies, Continuous Time Markov Chains (CTMC) define systems which may transition to another state at any time along some continuous interval. A CTMC can be used to model an active shooter incident by defining a state as the number of casualties and exits which have occurred by a given time. There are three processes which dominate the situation:

- Fire on targets in the arena: the shooter uses a semi-automatic weapon and the shots form a Poisson process (similar to the assumption in classic stochastic duel (Williams and Ancker 1963)).
- Successful exits of targets from the arena: the exit process from the arena follows a Poisson process. The exit is not in a constant fixed rate because people may not find the escape route or can get jammed in the doorway(s).
- Stopping the shooter: The shooter stops shooting after an exponential length of time because of weapon failure or being captured/killed.

Consider a generic active shooter scenario where there are N targets in the arena at the start of the shooting. Let E be the number of people who escaped at the time of the current state and K be the number of casualties. So, a state is defined as (E, K). The effective firing rate of the shooter, r, may be a function of the shape of the arena and the number of people remaining in the arena during the current state, namely $N - K - E$. The exit rate of the targets, μ, may be a function of the number and size of exits and the number of targets remaining in the arena. Finally, let $1/\lambda$ be the expected time until the shooter is stopped. The transitions for this Markov Chain will be,

$$(E, K) \rightarrow (E+1, K) \;\; w.p. \;\; \frac{\mu}{r+\mu+\lambda},$$

$$(E, K) \rightarrow (E, K+1) \;\; w.p. \;\; \frac{\phi}{r+\mu+\lambda},$$

$$(E, K) \rightarrow END \;\; w.p. \;\; \frac{\lambda}{r+\mu+\lambda}.$$

The scenario ends when all targets have either been killed or have exited the arena, hence:

Absorbing States: END, $(n-k,k)$,$k=0$,. . . ,n.

Marked Poisson Process

Consider a scenario where targets are in a hallway attempting to escape through a single exit, and are arranged in rows of fixed length. To account for target blocking, shooting will be modeled as a series of bursts which are defined as the number of bullets fired in the time it takes for a casualty to collapse after being struck by a bullet. During a burst, a victim may be struck multiple times, but at the conclusion of the burst, those who are hit effectively disappear (fall to the ground). Assume the number of bullets per burst, b, is less than or equal to the number of targets per row, and that all the bullets fall on a target. Hence, the number of casualties per burst, Y, has the distribution described in Section 3.2.2,

$$P(Y=y)=\frac{\binom{n}{y}\binom{b-1}{y-1}}{\binom{n+b-1}{b-1}}.$$

The expected number of casualties, μ_y, per burst is,

$$E[Y]=\sum_{y=1}^{b} y\times\frac{\binom{n}{y}\binom{b-1}{y-1}}{\binom{n+b-1}{b-1}}, \tag{3.10}$$

and the variance of casualties, σ_y^2, per burst is,

$$var[Y]=\sum_{y=1}^{b}(y-\mu_y)^2 p[Y=y]$$

$$=\sum_{y=1}^{b}\left(y-y\times\frac{\binom{n}{y}\binom{b-1}{y-1}}{\binom{n+b-1}{b-1}}\right)^2\frac{\binom{n}{y}\binom{b-1}{y-1}}{\binom{n+b-1}{b-1}}. \tag{3.11}$$

Also, assume that people exit through the door at some fixed rate, ℓ people per period. Then, the number of live targets per period is a Markov chain, $Z=(Z_t:t\geq 0)$, with state space the number of live targets remaining. The transition matrix is,

$$P(Z_{t+1} = z_{t+1} | Z_t = z_t) = P(Y_{t+1} = z_{t+1} - \ell). \tag{3.12}$$

Given z_0 initial live targets, the number of remaining targets after t periods is,

$$Z_{t+1} = z_0 - \ell t - \sum_{i=1}^{t} Y_i. \tag{3.13}$$

The number of periods until all the targets are gone or killed,

$$\tau = \min\{t \geq 0 : Z_t \leq 0\},$$

can be characterized using random walk techniques. The value of τ can be used to compute the time available to have an effective police intervention.

Now suppose bursts follow a Poisson process with rate λ. More specifically, let:

1. Y has mean μ_Y and standard deviation σ_Y.
2. $N(t)$: The number of bursts by time t.
3. $\ell > 0$: The exit rate through a single door.
4. M: The initial number of targets.
5. τ: The time until there are no more targets to shoot,

$$\tau = \min\{t > 0 : Y_1 + Y_2 + \cdots + Y_{N(t)} + \ell t > M\}.$$

We can approximate the distribution of τ using a CLT approximation and an approximation for the tails of a Gaussian random variable when y is large. Hence,

$$P(\tau > t) \approx \sum_{n=0}^{\infty} \frac{\exp(-\lambda t)(\lambda t)^n}{n!} (1 - \sqrt{2\pi} \exp\left(-.5 \left(\frac{M - \ell t - n\mu_Y}{\sqrt{n}\sigma_Y}\right)^2\right))$$

$$= 1 - \sqrt{2\pi} \sum_{n=0}^{\infty} \frac{\exp(-\lambda t)(\lambda t)^n}{n!} \exp\left(-.5 \left(\frac{M - \ell t - n\mu_Y}{\sqrt{n}\sigma_Y}\right)^2\right). \tag{3.14}$$

See Appendix A.3 for an explanation for these approximations.

3.3 Prescriptive Models

In most cases, potential victims of active shooter events must choose between two viable options: run or hide. Running will often cause the target to be more exposed, and therefore more likely to be shot. However, upon exiting the arena, the target's probability of being shot is reduced to zero. Hiding lowers a target's probability of being shot initially, but affords the target no opportunity to leave the arena.

3.3.1 Game Theory Model

Consider a generic active shooter scenario. When the shooting starts, people instantaneously decide whether to attempt to escape or hide until the shooting stops. The shooting evolves in two phases:

In Phase 1 the shooter targets the people attempting to escape, who are easier to kill. In Phase 2, when there are no more exposed people to target, the shooter aims at the people hiding with the remaining bullets (if any). In both phases, the shooter fires bursts of c bullets at each target.

Next, we develop a model for each phase.

Phase 1: Since the shooter chooses targets randomly, each individual who attempts to escape has the same chance of survival. The survival probability depends on the number of people who attempt to escape (E) and number of bullets (b). Viewing E as the main decision variable for targets, we call the latter probability $P(E)$.
If E people attempt to escape, then the first phase evolves in bursts of gunfire where $P(E)$ starts very low (if only a few people run, the shooter is likely to kill them all) then increases due to the "shelter effect" of the crowd. All other factors the same, the larger the number of escapees, the smaller the chance a particular individual will be targeted. As E gets large, the marginal increase in $P(E)$ decreases due to a "crowding effect" at the exit points. As E gets very large, the "shelter effect" and the "crowding effect" offset one another causing $P(E)$ to asymptotically approach some probability a. The assumptions regarding the behavior of $P(E)$ lead to a logistic differential

equation. Hence,

$$\frac{dP}{dE} = kP(1 - P/a), \tag{3.15}$$

and after solving this differential equation we have,

$$P(E) = \frac{a}{1 + \exp(-kE)}. \tag{3.16}$$

See Appendix A.4 for an explanation of this solution.
k can be re-written as two constants, u and v, which will represent the effect of crowd blocking (based on the arena) and the number of escapees, E^* for which $P(E^*) = 0.5\ (Max)\ P(E)$. So,

$$P(E) = \frac{a}{1 + \exp(-u(E - v))}. \tag{3.17}$$

Now, the shooter fires bursts of size c at each target. So, if there is a single target, and the single-shot kill probability is p, the target's probability of survival is $(1 - p)^c$. Using this single target probability of survival as an initial condition for Equation 3.16, we can solve for u. Hence,

$$P(1) = (1 - p)^c = \frac{a}{\exp(uv) - 1} \rightarrow v = \frac{\log\left(1 + \frac{a}{(1-p)^c}\right)}{u}.$$

If $b/c < m$, then when E exceeds b/c, the probability of survival begins approaching 1 with every additional escapee. This is the case when the shooter runs out of bullets before all the escapees exit the arena. So,

$$P(E) = \begin{cases} \frac{a}{1+\exp(-u(E-v))}, \text{ if } b \geq cE, \\ 1 - \left(1 - \frac{a}{1+\exp(-u(E-v))}\right) \times \frac{b}{cE}, \text{ if } b < cE. \end{cases} \tag{3.18}$$

Phase 2: Individuals who choose to hide are killed with a probability that depends on the number of people who attempt to escape (E) and bullets available (b); we call this probability $Q(E)$. The shooter seeks out targets and fires a burst of c bullets at each target. Assuming all targets have equally sufficient cover,

$$Q(E) = \begin{cases} (1-p)^c, \text{ if } b \geq cm, \\ (1-p)^c \frac{b/c-E}{m-E} + \frac{m-b/c}{m-E}, \text{ if } cE \leq b < cm, \\ 1 \text{ if } b < cE. \end{cases} \tag{3.19}$$

As you can see in Equation 3.19, if the shooter runs out of bullets during the first phase, the probability of survival for hiders is 1.

3.3.2 Cases

The instant the shooter pulls the gun, each individual attempts to escape or hide. People attempt to escape if the probability of survival when doing so is larger than the survival probability when hiding; that is, if $P(E) > Q(E)$. The points where $P(E) = Q(E)$ make people indifferent between attempting to escape or hiding, but there may exist several such points depending on the scenario parameters. Of these intersection points, the only one that is an equilibrium (meaning that no individual has a strict incentive to deviate) is the one with largest $P(E)$. Call this point E^* (if it exists). Then we can interpret the dynamics as follows: A person attempts to escape with probability E^*/m, and to hide with probability $1 - E^*/m$.

There are three possible scenarios for the equilibrium, depending on the parameters.

Case 1 – For all $E \leq m$, $P(E) > Q(E)$: Everybody attempts to escape.

In this case, the shooter has a large number of bullets allowing him to fire at escapees until all exit then fire c bullets at each hider, regardless of the number of people who choose to attempt to escape. This is reflected in Equation 3.19, which holds $Q(E)$ constant at $(1-p)^c$ when b and p are large. As shown in Figure 3.1, the probability of survival for escapees is greater than hiders for all E, so, in this case, everyone will attempt to escape.

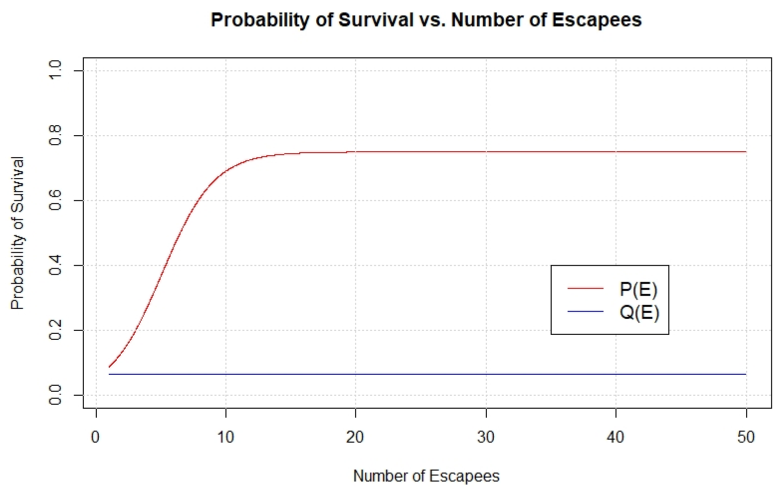

Figure 3.1. All Attempt to Escape: The probability of survival for targets who attempt to escape is higher than the probability of survival for those who hide, independent of E.

Case 2 – For some $E \leq m$ we have $P(E) = Q(E)$: Some people attempt to escape. The number of escapees will settle on the largest root of $P(E) = Q(E)$, we will call this E^*.

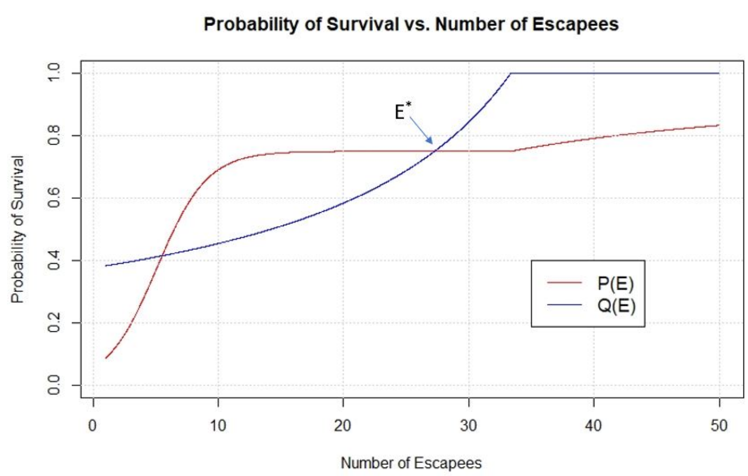

Figure 3.2. Some Attempt to Escape: Targets will continue to attempt to escape until the number of escapees reaches E^*.

In this case, if $E > E^*$ then both escapees and hiders could have a higher chance of survival, but no one wants to be in the escaping group, since its survival probability is lower than that of the hiders. If some people are willing to behave sub-optimally by choosing the action with a smaller chance of survival, then no escapee is worse off, and the hiders are better off. If a sufficient number of people attempt to escape in this manner, then the escapees are also strictly better off. Hence this equilibrium is unstable, and the people who deviate from it by attempting to escape are called *weak altruists*.

Case 3 – For all $E \leq m$, $q(E) > p(E)$: In this case, everybody hides. However, the same story as in the earlier scenario applies: If some people were willing to attempt to escape, then the hiders would be better off, and the escapees no worse off.

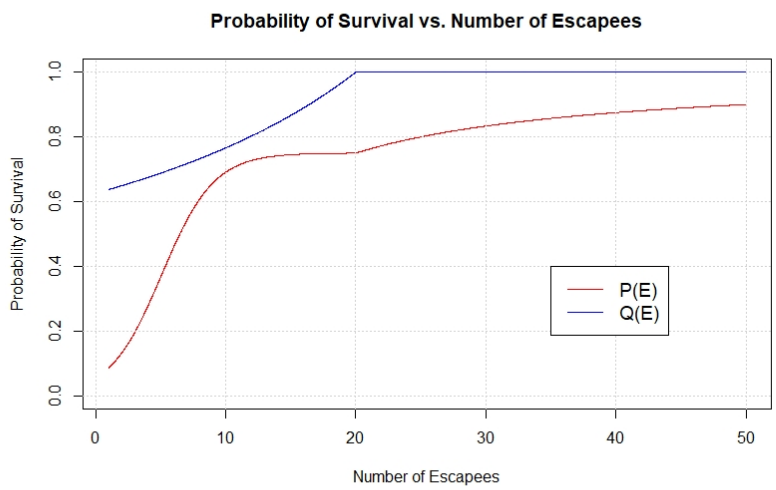

Figure 3.3. All Targets Hide: The probability of survival for targets who hide is higher than the probability of survival for those who attempt to escape, independent of E.

In this case, the shooter has a low number of bullets. Attempting to escape will make you a target and will give you a lower probability of survival than hiding, independent of the number of people who attempt to escape with you. As shown in Figure 3.3, the probability of survival for hiders is greater than escapees for all E, so everyone will choose to hide. This, however, is also an unstable equilibrium. If some people are willing to behave sub-optimally by choosing to be in the group with a smaller chance of survival (that is, by attempting to escape), the probability of survival will be higher for hiders and, with enough escapees, will be higher for the escapees as well.

CHAPTER 4:
Numerical Analysis

4.1 Introduction

Once an active shooter scenario starts, it continues until the shooter runs out of bullets, runs out of targets or runs out of time--we will call these *limiting factors*. Any attempt to reduce the expected number of casualties during an active shooter event will hasten one or more limiting factor. When policy and decision makers ask "what-ifs" concerning active shooter scenarios, they are likely to flow from these purposes. Below we find common questions in each category.

Shooter runs out of bullets:

- What if we outlaw weapons with detachable magazines?

Shooter runs out of targets:

- What if we increase the required size or number of exits within public places?
- What if we improve our emergency response training so people know safe escape routes?

Shooter runs out of time:

- What if we improve police response time, hire a security guard or train/equip employees to mitigate the attacker?
- What if we limit the firing rate of legal firearms? (E.g., limit legal magazine size or outlaw weapons with certain action types)
- What if we install an improved locking mechanism for rooms?

Effective analytic models are capable of addressing one or more of these questions. This chapter begins by describing the various "types" of active shooter incidents then applies the techniques outlined in Chapter 3 to recent scenarios. Records of active shooter incidents in the U.S. have varying amounts of available data and information. Those events, which end with the shooter's arrest and subsequent trial, rather than by suicide or otherwise, often provide the clearest picture of the incident. Unfortunately, shooters are only apprehended 41% of the time (Blair 2014) and only a small fraction of them plead not guilty, leading to full court proceedings. We have selected three incidents to use as reference cases which either went to trial or garnered a large amount of public interest and are, therefore, well-documented. These three incidents encompass the primary types of active shooter incidents which are described below. Using the models from Chapter 3, we analyze the events and present observations from the results.

The case studies within this chapter are intended to present how our models may be utilized to analyze active shooter incidents. While we examine real-world events, the same tools could be used to verify a simulation model or explore potential events before they occur. Some scenarios are defined by a single modeling technique, while others are broken into parts in order to combine techniques. The questions we answer in this Chapter are not meant to be exhaustive, but rather to show the types of insights these models can provide.

Note: Whether or not a policy decision will have its desired effect is outside the scope of this thesis. Our analysis explores how outcomes change *given* some adjustment in parameters.

4.2 Scenario Types

It is helpful for the process of model selection to categorize active shooter incidents by their common, general characteristics. A model that fits a movie theater shooting, for instance, is more likely to be applicable to a night club shooting than to a school shooting. This chapter utilizes three categories: complex arena, open arena and closed arena scenarios. All active shooter incidents are within one of these three broad categories.

4.2.1 Complex Arena

Complex arena active shooter incidents are those which people may exit from a finite number of exits. In these scenarios, the exit rate is often restricted due to the size and amount of exits. Victims may choose to either hide or attempt to escape, and those who previously hid may subsequently attempt to escape. When a large number of people attempt to escape from the same exit simultaneously, crowd blocking occurs leaving the blocked victims exposed to attack. A recent example of this scenario is the Aurora Movie Theater Shooting on July 20, 2012 (The Washington Post 2012).

4.2.2 Open Arena

Open arena active shooter scenarios are those which people may exit in an unconstrained manner. These generally take place outdoors at festivals, concerts or open shopping areas. In our models, these events will practically have infinite exit points and, therefore, the individual exit rates are defined by the distance to the closest exit. A recent example of this scenario type is the Las Vegas Strip Massacre on October 1, 2017 (Las Vegas Metropolitan Police Department 2018).

4.2.3 Closed Arena

Closed arena active shooter incidents are those in which people can not exit. These generally take place indoors in classrooms, religious gatherings, and work places. When an active shooter enters a room by its only entrance and effectively blocks the exit point, this is considered a closed arena. In this scenario, the only options for victims are to hide or fight back against the attacker. A school shooting is often a series of closed arena scenarios where each classroom is a closed arena. A recent example of this scenario is the Sandy Hook Elementary School Shooting on December 14, 2012 (Sedensky 2013).

4.3 Complex Arena: Aurora, CO, Movie Theater Shooting

4.3.1 Description of the Event

On July 20, 2012, 24-year-old James Holmes attended a midnight showing of "The Dark Knight Rises" at a movie theater in Aurora, Colorado. Approximately 20 minutes into the movie, he exited through a rear door and left it propped open. When he returned, he was equipped with full protective gear, a rifle, a 12-gauge shotgun and a .40-caliber handgun. He released two tear gas canisters then began firing randomly into the crowd with the shotgun. Next, he fired from his rifle which was accompanied by a 100-round drum magazine. He targeted people who attempted to flee before those who sought cover (CBS News 2012). Once the rifle malfunctioned, he switched to the handgun. All-in-all, Holmes fired 76 shots in the theater: six shotgun shells, 65 bullets from the rifle and five from the handgun. The total elapsed time of this event was approximately seven minutes and had ended before first responders arrived on the scene. Holmes was arrested next to his car in the parking lot at 12:45 am without incident. Of the 400 people who were initially in the theater, 12 were killed and 70 were injured with 58 of the injuries due to gunfire. Holmes' trial took place nearly three years later where he was found guilty and sentenced to life in prison without the possibility of parole (The Washington Post 2012).

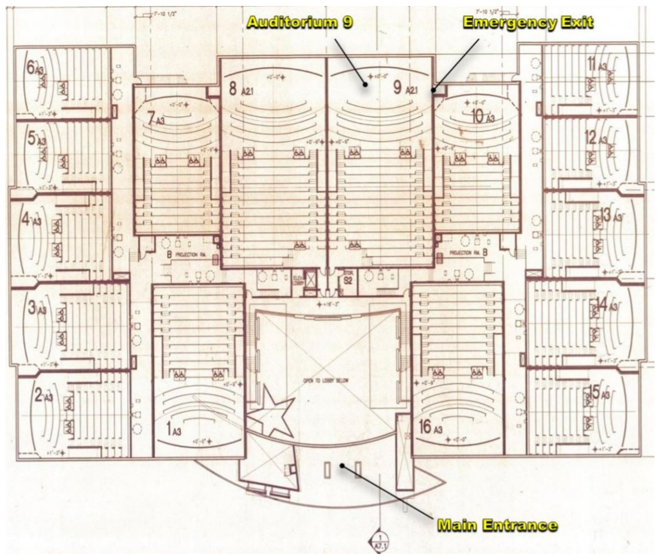

Figure 4.1. Century 16 Theater Schematic. Source: TriData (2014). The shooting took place in Auditorium 9.

4.3.2 Model

This scenario is modeled by dividing it into three distinct phases: Initiation, maximum capacity and seek and shoot. Phase 0: Initiation, captures the confused moments following the first gunshot. People are treated as static entities who are vulnerable to fire for a short duration before choosing to either run or hide. Phase 1: Maximum Capacity, describes the interval of time when the potential escapees are exiting at the highest possible rate allowed by the theater exits and the shooter is firing indiscriminately into the crowd, limited only by the speed at which he can pull the trigger. Phase 2: Seek and Shoot, begins when the amount of remaining evacuees is low enough to exit the arena freely. During this phase, the attacker must *seek out* potential targets before firing on them, whether they are hidden or attempting to escape.

Phase 0: Initiation

Eye-witness reports stated that there was confusion during the first moments of the attack. Many people believed the shooter was an actor who was participating in a hyper-realistic promotion stunt for the movie (The Washington Post 2012). For this reason, the model begins with a 10 second window after firing the first shot in which all targets are vulnerable to being hit. Since the time window is short, the shooter does not have to reload. The effective casualty rate, then, is only limited by how quickly the shooter can fire his weapon. According to the *U.S. Army Rifle Marksmanship M16-/M4-Series Weapons Field Manual (FM 3-22.9)*, the maximum effective semi-automatic rifle firing rate, r_{semi}, is 45 rounds per minute (U.S. Army 2016). Assuming the worst case scenario where each bullet hits a different target, the number of casualties at the conclusion of Phase 0 will be:

$$10 \text{ sec } \times \text{ .75 casualty/sec} = 7.5 \text{ casualties.}$$

See Appendix A.6 for specification tables from the *US Army Rifle Marksmanship M16-/M4-Series Weapons Field Manual (FM 3-22.9)*.

We will presume that half of the remaining people hide and the other half attempt to escape. So, at the conclusion of Phase 0, the number of casualties, $K_0 = 7.5$, the number of exposed targets, $E_0 = 196.25$ and the number of hidden targets, $H_0 = 196.25$. The following notation will be used throughout the remainder of the section: $X_z(t)$ where $X \in \{K, E, H\}$ represents the number of casualties, exposed and hidden targets respectfully, $z \in \{0, 1, 2\}$ indicating the Phase, and t indicating the time elapsed since the beginning of the Phase. So, for example, the initial number of casualties, $K_0 = K_1(0)$.

Phase 1: Maximum Capacity

The casualty rate during Phase 1 is linear because the shooter has many exposed targets. The rate, then, is only limited by how quickly he can fire his weapon and how long it takes to reload and change weapons. This is the sustained firing rate. The maximum sustained semi-automatic rifle firing rate, $r_{\text{sustained}}$, is 15 rounds per minute (U.S. Army 2016).

According to the *Society of Fire Protection Engineers (SFPE) Handbook*, the maximum flow rate through an exit door is 24 persons per minute per foot of effective width of the doorway (Hurley et al. 2015). Since the shooter is standing at the front of the theater, the emergency exit doors are effectively blocked leaving a single, double-door exit at the rear of the theater. With an effective width of five feet, this allows for a maximum exit rate, ℓ (for leave), of 120 persons per minute. As long as the number of people attempting to escape exceeds the capacity of the doorway, the exit rate will remain linear.

So, the number of casualties at time t during Phase 1 is:

$$K_1(t) = K_1(0) + r_{\text{sustained}} t.$$

The number of exposed people (those who are attempting to escape) is:

$$E_1(t) = E_1(0) - (r_{\text{sustained}} + \ell)t.$$

None of the people who are hiding attempt to escape because they see the exits are congested. So the number of people hiding remains constant:

$$H_1(t) = H_1(0).$$

Phase 1 concludes when the casualty and exit rates become dependent upon the number of exposed individuals, and therefore, less than the max rates. This occurs at the time t^* when the number of people exposed E_1 equals the max exit rate ℓ.

So,

$$\ell = E_1(t^*),$$

$$\ell = E_1(0) - (r_{sustained} + \ell)t^*,$$

finally,

$$t^* = \frac{E_1(0) - \ell}{r_{sustained} + \ell}.$$

In our case,

$$t^* = \frac{196.25 - 1.6}{.25 + 1.6} = 105.22 \text{ seconds},$$

$$K_1(105.22) = 7.5 + .25(105.22) = 33.8 \text{ casualties},$$

$$E_1(105.22) = 196.25 - (.25 + 1.6)105.22 = 1.6 \text{ exposed},$$

$$H_1(105.08) = H_1(0) = 196 \text{ hidden}.$$

Phase 2: Seek and Shoot

As targets become sparse due to casualties and exits from the arena, the kill rate begins to be limited by a lack of targets. The rate of exposed targets which become casualties, then, is $r_{\text{sustained}}/(\ell)$, we will call this rate α. A similar phenomenon takes place concerning the exit rate. As the number of exposed targets become sparse due to casualties and exits from the arena, the aggregate exit rate is limited by the number of people attempting to escape and the rate of escape is $1 - \alpha$.

At the beginning of Phase 2, the number of people exposed $E_2(0) = \ell$, the number of casualties $K_2(0) = K_1(0) + r_{sustained}t^*$, and the number of people hiding $H_2(0) = H_1(0)$. During this phase, the rate of those hiding who attempt to escape is λ, joining the pool of exposed individuals before either being shot or escaping. λ is naturally bounded because people will stop attempting to escape when the exit is congested, so, the maximum value for λ is the maximum exit rate divided by the maximum number of people hiding, or, $\lambda < \ell/H_0$. The number of hiders who are shot per second is much smaller than the number of exposed shot per second. So, the rate of hiders shot per second, δ, is less than $r_{sustained}/H_0$. We assume both λ and δ are half of their maximum values, or, $\lambda = \ell/(2 \times H_0)$ and $\delta = r_{sustained}/(2 \times H_0)$.

So,

- $\alpha = r_{sustained}/\ell$,
- $\delta = r_{sustained}/(2 \times H_0)$,
- $\lambda = \ell/(2 \times H_0)$.

Utilizing Equations 3.4, 3.5 and 3.6 from Section 3.2.1, we have,

$$H_2(t) = 196 \exp(-(0.0153 + 0.0006)t),$$

$$E_2(t) = (-3.05 + 1.6) \exp(-t) - (-3.05) \exp(-0.01594t),$$

and

$$K_2(t) = 33.8 + 0.04167 \times (-3.05 + 1.6) - \frac{0.04167 \times (-3.05) - 0.00064 \times 196}{0.01594}$$
$$- 0.04167 \times (-3.05 + 1.6) \exp(-t) + \frac{0.04167 \times (-3.05) - 0.00064 \times 196}{0.01594} \exp(-0.01594t).$$

The elapsed time of the Aurora Shooting was approximately 7 minutes (or 420 seconds). The total expected casualties, then, according to this model is $K2(420) = 72$ people. Recall, the actual number of casualties was 70.

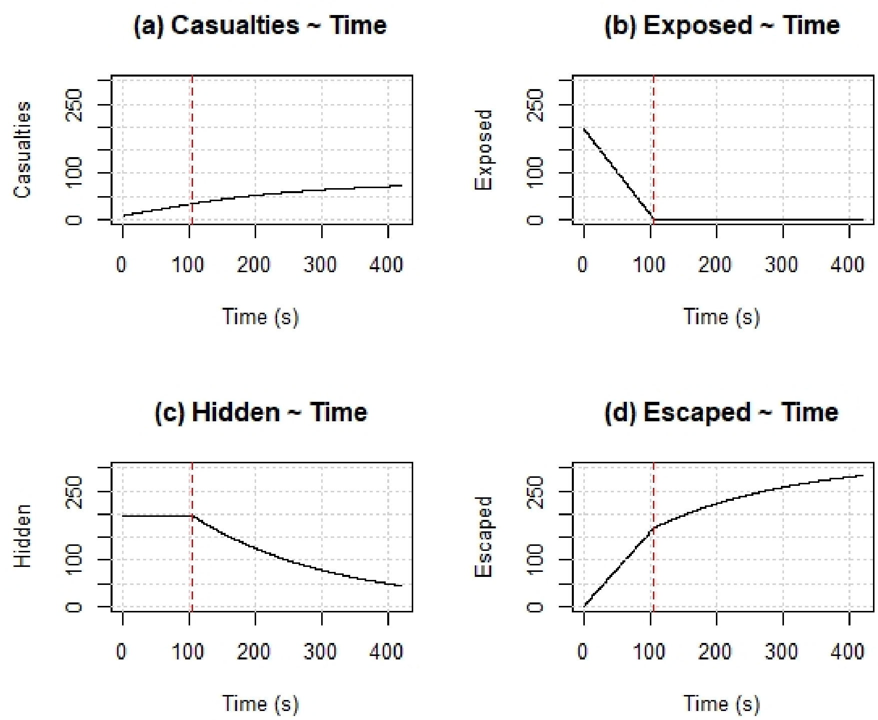

Figure 4.2. Aurora Shooting Model Plots. (a): Number of casualties as a function of time, (b): Number of exposed targets as a function of time, (c): Number of hidden targets as a function of time, (d): Number of escaped targets as a function of time

4.3.3 Observations and Analysis

Now that we have demonstrated the model produces reasonable outputs when given approximately actual inputs, we can perform sensitivity analysis on the parameters and answer basic questions concerning the incident. Here, we analyze two questions in order to illustrate the insights which may be gained by such a model.

Question 1: What if the theater had additional accessible emergency exits?

Assuming additional exits are the same size as their current exit, we would simply increase the exit rate, ℓ, by 1.6 exits per second for every additional exit.

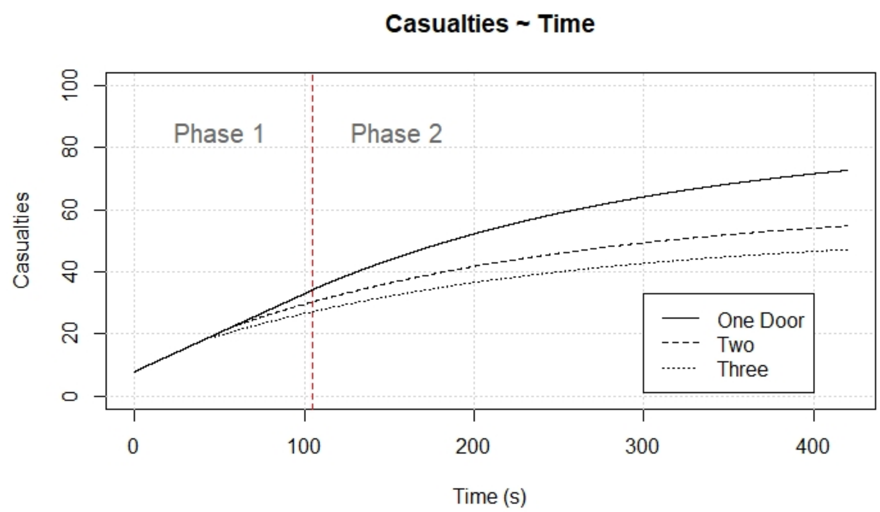

Figure 4.3. Aurora Shooting with Additional Exits. The three curves show the number of casualties as a function of time with no additional doors, one addition door and two additional doors.

We can see in Figure 4.3, there are diminishing returns from adding additional emergency exits. One additional exit results in 24 less casualties, and two additional exits will save 36 people.

Question 2: What if the theater had a security guard who responded one minute after the shooting began and had a 50% chance of neutralizing the shooter? What if the response time was two minutes?

In order to calculate the expected number of lives saved by a security guard, we multiply the remaining expected casualties at the response time by the probability of neutralizing the shooter, so:

$$E[\text{less casualties}|\text{one minute response}] = \frac{K2(420) - K1(60)}{2}$$

$$= (72 - 22.5)/2 \approx 25 \text{ lives.}$$

$$E[\text{less casualties}|\text{two minute response}] = \frac{K2(420) - K2(120)}{2}$$
$$= (72 - 37)/2 = 17.5 \text{ lives.}$$

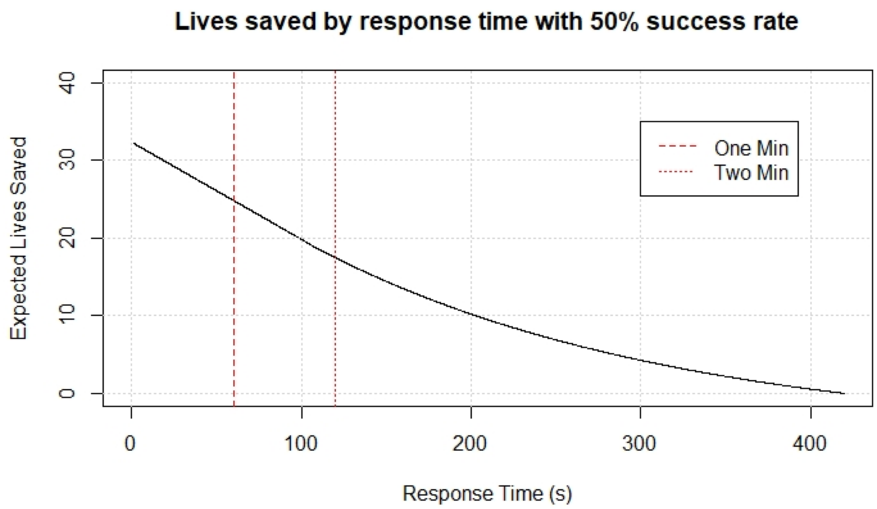

Figure 4.4. Aurora Shooting with a Security Guard. The curve shows the expected number of lives saved given a 50% success rate as a function of time. The red dotted lines show 60 second and 120 second response times respectively.

As we would expect, there are diminishing returns for response time. The beginning moments of an active shooter incident are nearly always the deadliest due to the high number of targets. As reported within the FBI Crime data, 69% of active shooter incidents end within five minutes. In the case of the Aurora Shooting, one additional exit door would have saved approximately the same number of lives as a quick responding and courageous security guard, assuming the given parameters. The model can be used to assess how these outcomes change given different probabilities of success and response times as well.

4.4 Open Arena: Las Vegas, NV, Harvest Music Festival Shooting

4.4.1 Description of the Event

On October 1, 2017, 64-year-old Stephen Paddock began firing bullets from his 43rd story hotel room onto the concertgoers at the Route 91 Harvest Music Festival below. There were approximately 22,000 festival attendees and Paddock fired 1,058 bullets from 15 separate firearms. Though the firearms were all semi-automatic, they were equipped with bump stocks which effectively converted them into automatic firearms. Ultimately, 58 people were killed and 413 were injured for a total of 471 casualties. The event elapsed over a total of 10 minutes and ended when Paddock used a revolver on himself (Las Vegas Metropolitan Police Department 2018).

The Harvest Music Festival was located at the Las Vegas Village, a 15-acre lot utilized for outdoor concerts and festivals. The event area, where the main stage is, is approximately 25,000 square meters. Within this model, we approximate the arena to be circular and the area occupied by each individual to be one square meter.

Figure 4.5. Las Vegas Village. Adapted from Google Maps (2020).

4.4.2 Model

The event is modeled as a circular grid with $m = 25,000$ cells and $N_0 = 22,000$ targets uniformly distributed over the arena. The effective firing rate, r, is equal to the number of bullets utilized divided by the elapsed time, so, $r = 1058/(10 \times 60) = 1.76$ bullets per second. Presume the conservative case where bullets fall randomly within the arena, i.e., the shooter is not more likely to fire toward a crowd. Each target is assigned a random clock time, T, for the time it takes them to exit the arena. The average effective velocity of fleeing targets must incorporate time for maneuvering around obstacles and aiding injured persons and will, therefore, be much slower than directly running toward safety. Let the average effective velocity be the speed required for someone directly in the middle of the arena to exit by the end of the event, that is, at $t = 600$. The probability that a target with clock time T is still in the arena, then, can be understood as the ratio between the area of two circles: the circle representing the entire arena and the smaller circle whose radius decreases with time. The proportion of targets who began in the smaller circle will still be in the arena at time t. So,

$$P(T < t) = \frac{2\pi(600 - t/600)^2}{2\pi 600^2} = \frac{2t}{600} - \frac{t^2}{600^2}.$$

So, the CDF for clock times, $F(t)$, is,

$$F(t) = \begin{cases} 0, \text{ if } t \leq 0, \\ 2t/600 - t^2/600^2, \text{ if } 0 < t < 600, \\ 1, \text{ if } t \geq T. \end{cases} \tag{4.1}$$

To find the PDF for clock times, $f(t)$, we simply take the derivative of the CDF, so,

$$f(t) = \begin{cases} 0, \text{ if } t \leq 0, \\ 2/600 - 2t/600^2, \text{ if } 0 < t < 600, \\ 0, \text{ if } t \geq 600. \end{cases} \tag{4.2}$$

Figures 4.6 and 4.7 show $F(t)$ and $f(t)$, respectively:

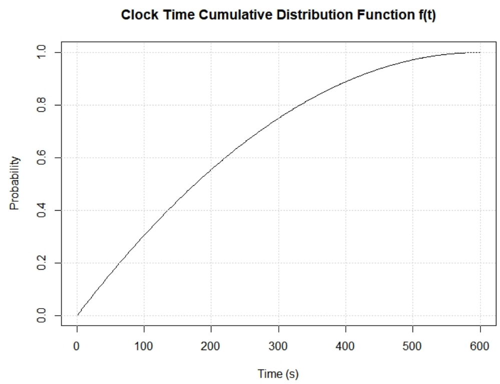

Figure 4.6. Clocks CDF–Equation 4.1

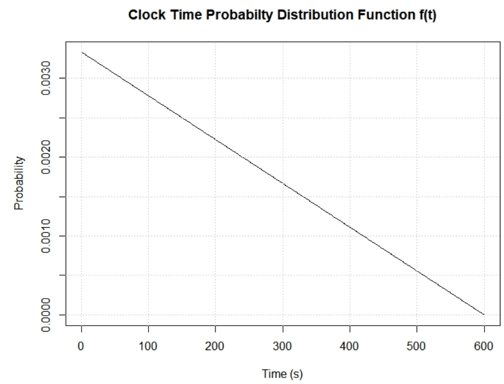

Figure 4.7. Clocks PDF–Equation 4.2

Now, in order to calculate the number of casualties as a function of time, we use Equation 3.9,

$$E[\text{number of people alive by time t}] =$$

$$N[P(T > t)\exp(-tr\log(\frac{m}{m-1})) + \int_0^t f(T)\exp(-Tr\log(\frac{m}{m-1})dT].$$

Inserting the PDF for clocks and evaluating the integral, we get:

$$N\left[(\frac{2t}{600} - \frac{t^2}{600^2})\exp(-tr\log(\frac{m}{m-1}) - \frac{2(\frac{m-1}{m})^{rt}(r\log(\frac{m-1}{m})(t-600)-1)}{(600r\log(\frac{m-1}{m}))^2} - C\right],$$

where C is the integral evaluated at t=0, so,

$$C = \frac{-1200r\log(\frac{m-1}{m}) - 2}{(600r\log(\frac{m-1}{m}))^2}.$$

With $N = 22,000$, $m = 25,000$ and $r = 1.76$, we have,

$E[\text{number of people alive by time t}] =$

$$22,000\left[(1 - (\frac{2t}{600} - \frac{t^2}{360,000}))\exp(-0.00007t) - \frac{2(\frac{24,999}{25,000})^{1.76t}(-0.00007t - 0.958)}{0.00179} + 1,069.403\right].$$

See Appendix A.5 for an explanation of the integral.

$$E[\text{number of casualties by time t}] = N - E[\text{number of people alive by time t}].$$

Figure 4.8 shows the expected number of casualties at time t:

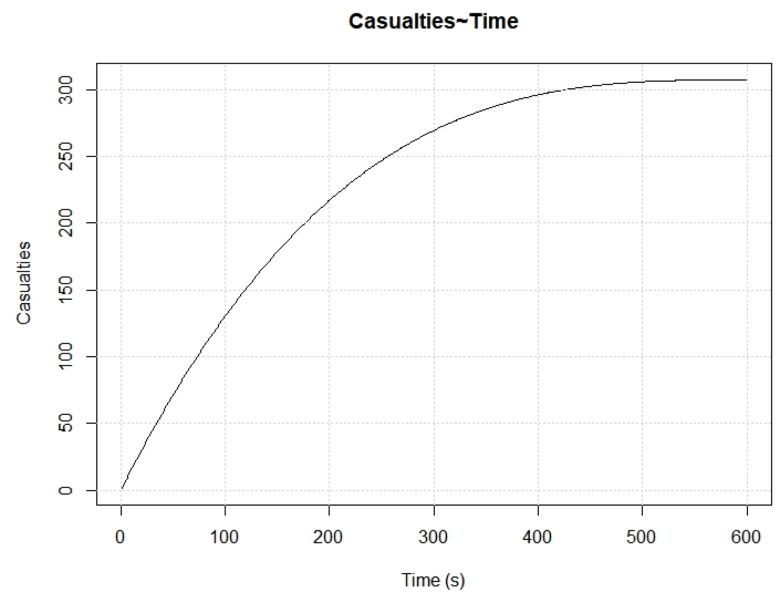

Figure 4.8. Las Vegas Casualties as a Function of Time: The number of casualties per second drastically decreases over time as targets either escape or are shot, reducing the remaining number of targets.

The total expected number of casualties from the model, 307, is significantly lower than what actually occurred. This was expected, as we assumed the conservative circumstance where bullets fell randomly upon the arena. In reality, the shooter was likely to have aimed toward areas where people were gathered and away from open space. An increased hit probability does not change the essential dynamics of the model, however, allowing us to examine the effects of changes in parameters as a proportion of the total.

4.4.3 Observations and Analysis

Question 1: What if the shooter was not equipped with bump stocks?

In response to the Las Vegas shooting, on December 18, 2018, the U.S. Department of Justice amended the "regulations of the Bureau of Alcohol, Tobacco, Firearms, and Explosives (ATF) to clarify that bump stock-type devices... are 'machineguns' as defined by the National Firearms Act of 1934 and the Gun Control Act of 1968" (U.S. Department of Justice 2018). This effectively banned their use throughout the United States. As stated before, whether or not legislation will have its intended effect is outside the scope of this paper. However, we can use our model to predict the reduction in casualties if the shooter had not equipped his semi-automatic firearms with bump stocks. Using the maximum effective semi-automatic rifle firing rate of 45 shots per minute as we did before, and keeping all else the same, we find the expected number of casualties to be 57% less than the original as shown in Figure 4.9.

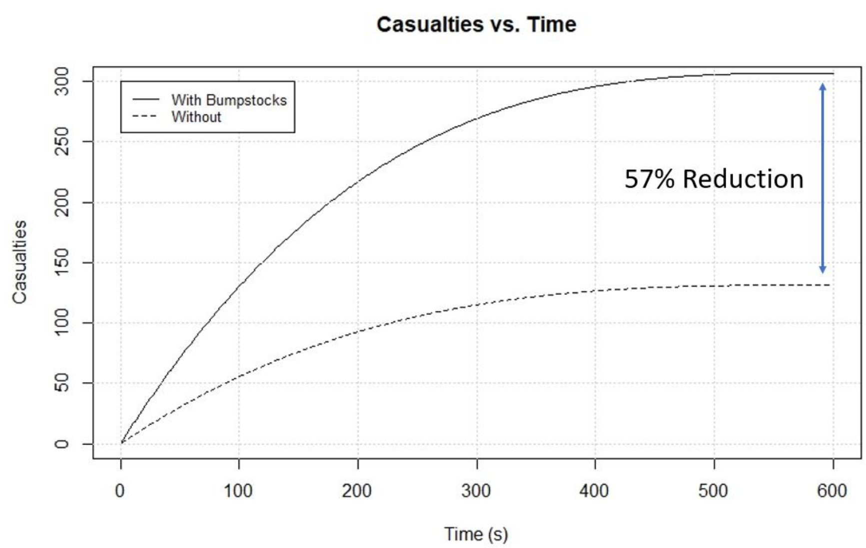

Figure 4.9. Reduction in Casualties Without Bump Stocks: The reduction in firing rate has a drastic impact on the expected number of casualties, particularly within the first five minutes.

Question 2: What if a gunfire locating device triangulated the position of the shooter and reduced the elapsed time of the incident?

Gunshot detection systems have gained popularity within the last decade among civilian law enforcement agencies throughout the U.S. Using acoustic signatures of firearms, they triangulate the location of perceived gunfire and reduce the response time of law enforcement (Aguilar 2015). One of the reasons the Las Vegas shooting had a high number of casualties was that it lasted much longer than average active shooter incidents. What if the event had lasted half as long; that is, five minutes instead of 10?

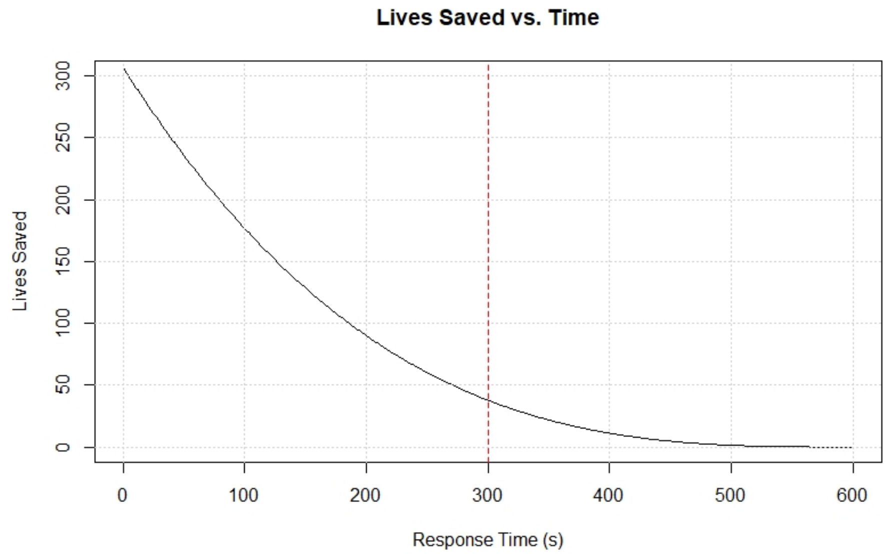

Figure 4.10. Reduction in Casualties as a Function of Response Time: The large majority of casualties happen within the first five minutes, limiting the affect of a modest or even moderate reduction to response time.

As shown in Figure 4.10, a 50% reduction in the response time only caused a 12% reduction in expected casualties. At the beginning of the incident, the shooter has many targets allowing for a high hit probability. Our model indicates that 50% of the total casualties happened within the first two minutes.

4.5 Closed Arena: Sandy Hook School Shooting

4.5.1 Description of the Event

At 9:35am on December 14, 2012, 20-year-old Adam Lanza shot out an exterior window and entered Sandy Hook Elementary School (SHES). The school had approximately 450 enrolled kindergarten through 4th-graders with 30 classrooms and nearly 50 staff members. Lanza was equipped with a rifle and handgun as well as ten 30-round magazines. After entering the school, Lanza encountered four staff members, shooting three and killing two. He then progressed toward the classrooms, firing a shot that ricocheted into a teacher's foot in the hallway while she was closing her door. He first entered a first grade classroom and shot and killed all fifteen students as well as two teachers. He then continued onto another near-by first grade classroom where he shot and killed five students and two teachers. The incident ended when Lanza turned his handgun on himself at approximately 9:40am, shortly after the first police officers arrived on the scene. In total, there were 26 deaths and two wounded, not including Lanza. 50 expended bullet casings were found in the school including the one used on himself (Sedensky 2013).

At the conclusion of the event, the shooter had hundreds of bullets remaining and there were still hundreds of children hiding within the school. This indicates the event ended because the shooter ran out of time, potentially because he heard police sirens outside. Our model, then, will use time as the limiting factor. SHES is a rectangular building, with four primary hallways as shown in Figure 4.11.

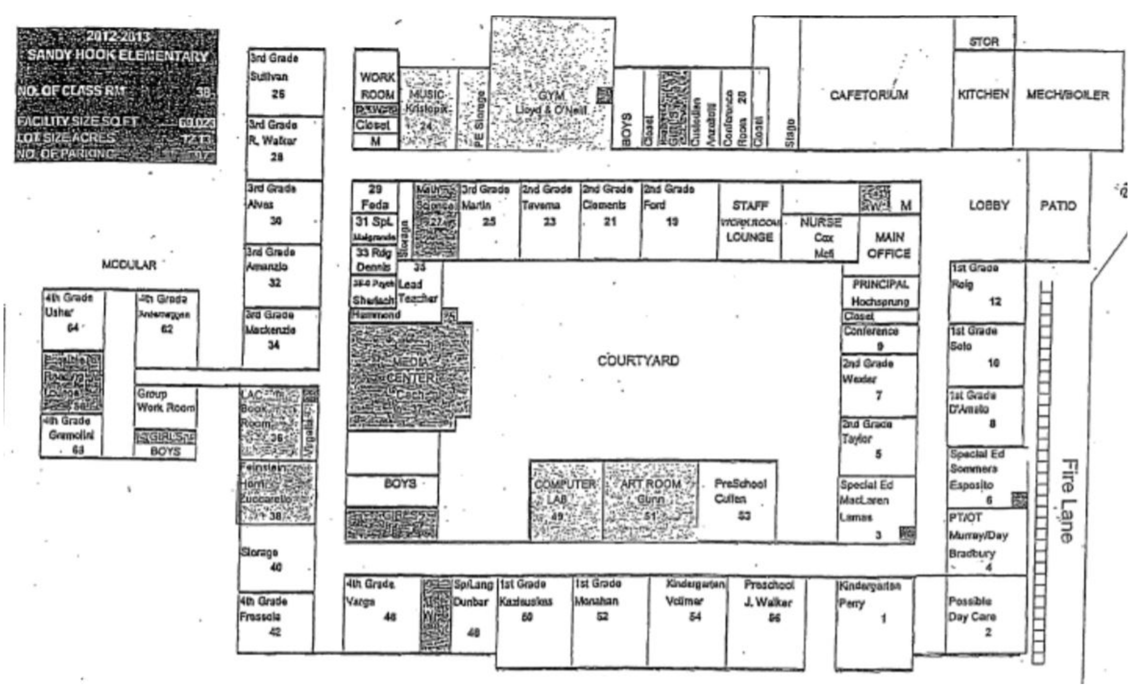

Figure 4.11. Sandy Hook Elementary Map. Source: Sedensky (2013). The shooting primarily took place within the classrooms along the north-east corner, that is, the lower-right corner in the diagram.

4.5.2 Model

Presume teachers choose to either hide or flee with their classes. The incident, then, happens in two phases. The first consists of hallway interactions, if there are any, and the second a series of closed-arena scenarios as the shooter goes from classroom to classroom.

Phase 1: Hallway Marked Poisson Process

The number of classes the shooter meets in the hallway, X, is a binomial distribution with parameters p = the probability a class comes into contact with the shooter and n = the number of classes which flee.

$$f(x; n, p) \equiv Pr(X = x) = \binom{b}{x} p^x (1 - p)^{n-x}.$$

Let classes who flee have a 25% chance of coming into contact with the shooter, as the shooter will be in one of the four primary hallways, and a 75% chance of escaping successfully. So, if five classes chose to flee, the probability that x classes meet the shooter in the hallway is represented in Figure 4.12.

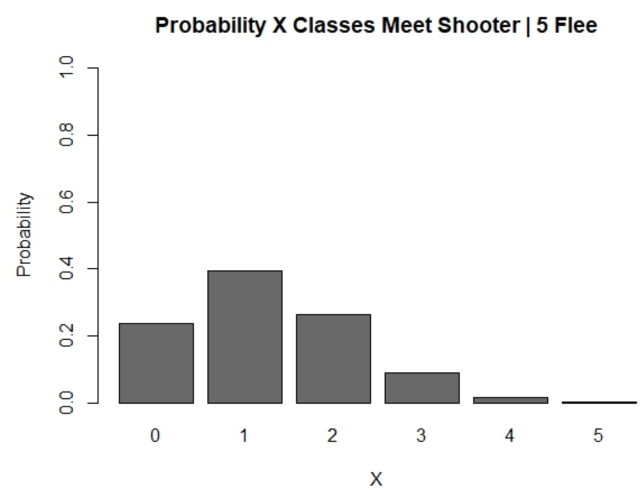

Figure 4.12. Probability X Classes Meet Shooter Given Five Flee.

The number of targets in the hallway will be (15 students + 1 teacher) $\times X$. If the shooter meets students in the hallway, the incident plays out similarly to the scenario described in Section 3.2.2. Victims attempt to exit through a single exit, and are arranged in rows of fixed length; in this case, let the rows be four targets wide. To account for target blocking, shooting will be modeled as a series of bursts which are defined as the number of bullets fired in the time it takes for a casualty to collapse after being struck by a bullet; in this case, the shooter will fire bursts of four bullets over five seconds. During a burst, a victim may be struck multiple times, but at the conclusion of the burst, those who are hit effectively disappear (fall to the ground). Since the shooter is firing into a crowd, assume that all the bullets fall on a target. Hence, the number of casualties per burst, Y, has the distribution described in Section 3.2.2,

$$P(Y = y) = \frac{\binom{n}{y}\binom{b-1}{y-1}}{\binom{n+b-1}{b-1}}.$$

Also, assume that targets exit at a constant rate ℓ. In this case, ℓ will be the maximum flow rate through a door defined by the SFPE Handbook over the 5 seconds per burst (Hurley et al. 2015). That is,

24 people per minute per foot of effective width $\times$ 3 ft $\times$ 5/60 seconds = 6 people per burst

Then, the number of live targets per period is a Markov chain, $Z = (Z_t : t \geq 0)$, where Z_t is state space the number of live targets remaining and the transition matrix is the same as Equation 3.12,

$$P(Z_{k+1} = z_{k+1} | Z_k = z_k) = P(Y_{k+1} = z_{k+1} - \ell).$$

Given z_0 initial live targets, the number of remaining targets after k periods is,

$$Z_{k+1} = z_0 - \ell k - \sum_{i=1}^{k} Y_i. \tag{4.3}$$

By presuming bursts follow a Poisson process with rate λ, we can use Equation 3.14 to approximate the number of bursts in the hallway.

$$P[\tau = t] = 1 - \sqrt{2\pi} \sum_{n=0}^{\infty} \frac{\exp(-\lambda t)(\lambda t)^n}{n!} \exp\left(-.5\left(\frac{M - \ell t - n\mu_Y}{\sqrt{n}\sigma_Y}\right)^2\right).$$

The expected amount of time spent in the hallway is $\tau \times 5$ seconds and the expected number of casualties in the hallway is $\tau \times \mu_y$.

Phase 2: Classrooms

During Phase 2, the shooter randomly selects classrooms which may contain people, or may be empty depending on if the teacher chose to flee. If the shooter chooses an occupied classroom, he fires at all targets within the room until they have all been hit. He then begins searching again. This process can be modeled as a Discrete-Time Markov chain with states defined by the number of occupied rooms remaining and the number of vacant rooms remaining. At the beginning of Phase 2, the number of vacant rooms, V_0, is equal to the number of teachers who chose to flee, X. The number of remaining occupied rooms $M_0 = 30 - V_0$.

So, a state is (M, V).

Transitions

$$(M,V) \to (M-1,V) \;\; w.p. \;\; \frac{M}{M+V},$$
$$(M,V) \to (M,V-1) \;\; w.p. \;\; \frac{V}{M+V}.$$

The scenario ends when all classrooms have been opened by the shooter, hence:
Absorbing state: (0,0).

Presume it takes 30 seconds to travel between and check rooms. If the shooter selects a vacant room, he continues searching for an occupied room without delay. If the shooter selects an occupied room, he fires upon the targets until they have all been hit. Let $p = 0.8$ be the effective probability of hitting a target, b the number of bullets, and n the number of targets within the room, in this case each room will have 15 students and 1 teacher. The probability of hitting a target does not change as the number of targets is reduced because the shooter is at a relatively close range allowing him to aim at individuals rather than groups. If B_i is the number of bullets shot within room i, then B_i follows a negative binomial distribution with probability mass function:

$$P(B_i = b|n, p) = \binom{b-1}{n-1} p^n (1-p)^{b-n}, b = n, n+1, \ldots, \tag{4.4}$$

And the expected number of bullets shot in each room is,

$$E[B_i] = \frac{n}{p}. \tag{4.5}$$

If the shooter has a firing rate of r bullets per second, the total expected time spent in room i is $E[B_i]/r$. In this case, we will use the maximum sustained semi-automatic rifle firing rate of 15 rounds per minute (U.S. Army 2016).

If the shooter selects a vacant room during period i, the starting time of the following period, $t_{i+1} = t_i + 30$ seconds and the expected number of casualties after the transition does not change, so $k_{i+1} = k_i$. If the shooter selects an occupied room, the expected starting time of the following period, $t_{i+1} = t_i + 30$ seconds $+ E[B_i]/r$ and the number of casualties after the transition is $k_{i+1} = k_i + 16$.

4.5.3 Observations and Analysis

As mentioned previously, time was the limiting factor for the Sandy Hook active shooter incident. Once the police arrived, the shooter turned his weapon on himself. Response time by the authorities varies between schools throughout the U.S. depending on location (e.g., urban, suburban, rural) and local response capabilities. In 2010, Buerger identified this reality within an FBI Law Enforcement Bulletin:

> Many rural schools are located in small, isolated towns served by only state police or sheriff's departments. The far-flung patrol responsibilities and limited staff levels of those agencies make a 20- to 30-minute response time an optimistic best-case scenario; in reality, it may take 45 minutes to an hour before authorities arrive. (Buerger and Buerger 2010)

Question 1: How many casualties would have occurred during the Sandy Hook shooting if it had been a rural school?

The two previous scenarios we analyzed, the Aurora movie theater shooting and the Las Vegas music festival shooting, had non-linear aggregate casualty rates, with the beginning of the incident producing a larger amount of casualties than the end. This occurred because the number of targets decreased with time causing the expected interval between casualties to increase. Closed arena incidents function differently.

We can see in Figure 4.13 that the casualty rate remains linear with time.

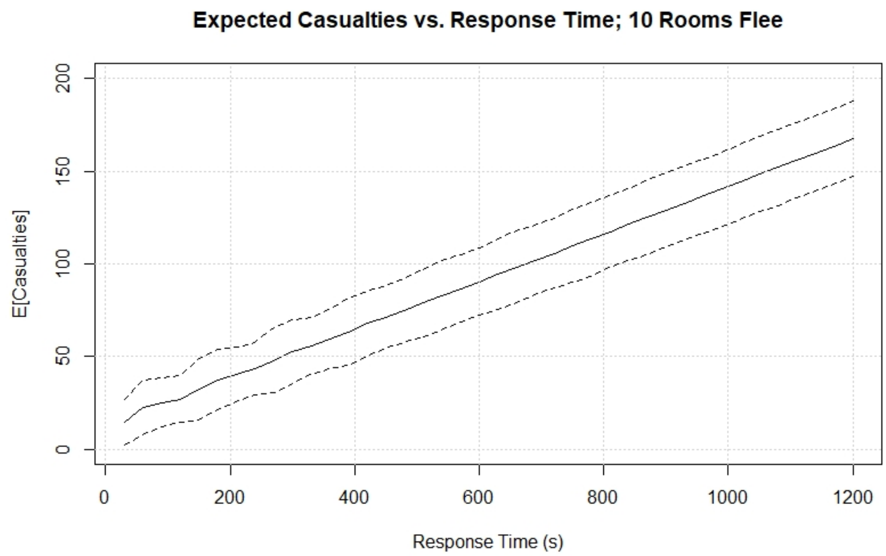

Figure 4.13. Expected Casualties by Response Time: The dotted lines represent two standard deviations above and below the expected number of casualties.

The casualty rate is linear because instead of firing into a crowd which is becoming increasingly sparse, the shooter travels from classroom to classroom firing at all targets he finds. If the shooter had unlimited time and ammunition, choosing to hide in a classroom would be a poor choice, as the shooter would eventually find all those who are hiding. Since 2013, *The Federal Guide for Developing High-Quality School Emergency Operations Plans* has explicitly identified the "Run, Hide, Fight" option as the standard for school response plans (U.S. Department of Justice 2013). This procedure encourages students and staff to run if possible, hide if running is not possible and fight the attacker as a last resort. This replaced the "lockdown-only" method which schools across the country had previously embraced. Our model aligns with this conclusion--in many cases, hiding is not the best option. As response times increase, running becomes more favorable to hiding. Buerger notes this as well within his aforementioned FBI Law Enforcement Bulletin: "A longer wait for police response extends the period of vulnerability. The smaller size of rural schools compresses both distance and time, making confrontations more intimate and dramatically altering the dynamics of refuge and escape. The advantage of lockdown quickly evaporates, tipping the advantage to the armed invader" (Buerger and Buerger 2010). Figure 4.14 demonstrates this reality. When the response time is four minutes, the expected number of casualties is practically independent of the number of classes which flee. However, when the response time is large, as the number of rooms which flee increases, the expected number of casualties decreases. At a 16-minute response time, the expected number of casualties if no one flees is 146 and the expected number of casualties when everyone flees is 32. In other words, a "lockdown-only" approach would be a grievous mistake for a rural school with a large expected response time.

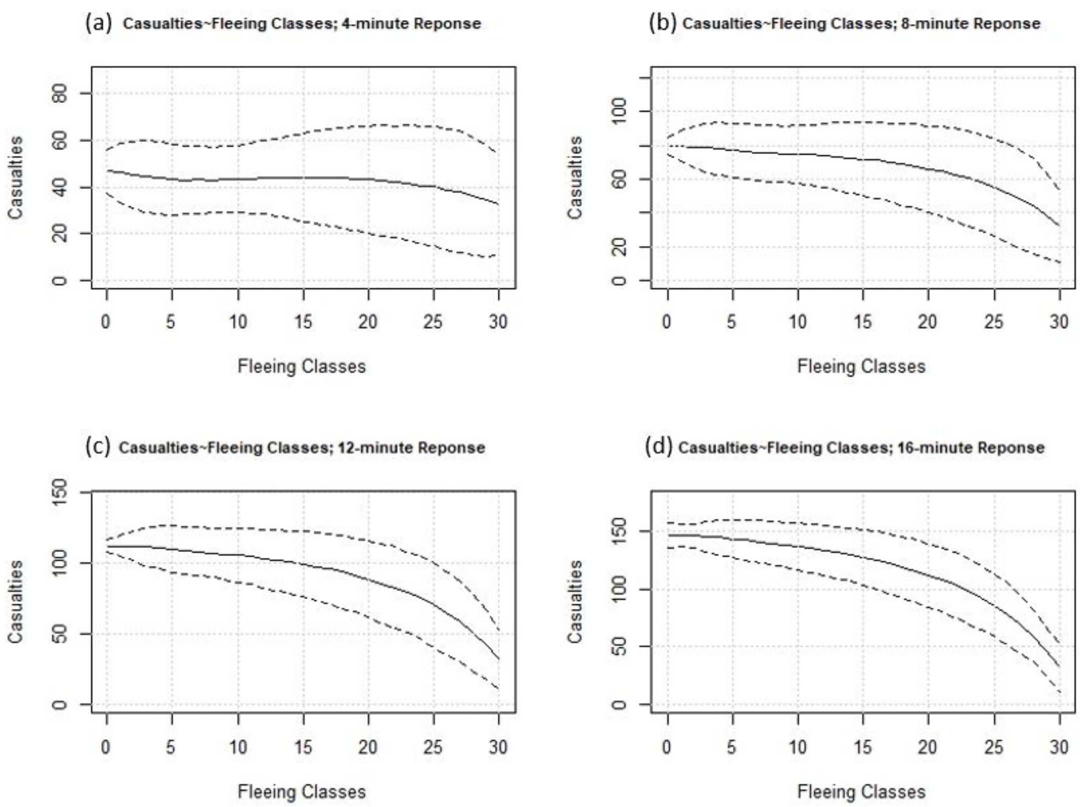

Figure 4.14. Progression in the Importance of Running as Response Time Increases: (a) Shows the expected number of casualties as a function of the number of fleeing classes when the response time is 4 minutes; (b),(c) and (d) show the same with a 8,12 and 16 minutes response time, respectively.

Some schools have begun installing steel classroom doors and advanced locking mechanisms to protect students in the event of an active shooter, causing entrance into classrooms by the shooter to be practically impossible. In-fact, SHES was demolished and rebuilt following the active shooter incident, and the new school has steel doors (Sedensky 2013). In this case, the "lock-down only" method may be preferable to the "Run, Hide, Fight" method.

Question 2: Would restricting magazine size or type reduce the expected number of casualties?

A firearm's magazine size effects two factors during active shooter incidents: firing rate and mobility. During the Aurora and Las Vegas attacks, the shooters were static. This allowed them to use a high number of weapons and ammunition. Since mobility was not a concern, both used very high-capacity magazines to increase their firing rate. When a weapon ran out of bullets or jammed, they simply moved to the next pre-loaded firearm (TriData 2014; Las Vegas Metropolitan Police Department 2018). During a school shooting, the shooter must be mobile, navigating around the arena to find targets. The size of their arsenal, then, exists in tension with their mobility. The more weapons and ammunition they carry, the slower and less mobile they are. During the Sandy Hook incident, the shooter solved this problem by wielding only two weapons, one of which he solely used on himself. Instead of carrying a single high-capacity magazine, which could jam and end his rampage, he carried ten 30-round magazines. In several cases, the shooter ejected magazines before they were expired. He was not, then, concerned with running out of bullets, but rather, with expending an entire magazine while in contact with targets. If a shooter walked into a closed-arena active shooter incident (a classroom) with a 30 round magazine and 16 targets, assuming a hit probability of 80% as we did in our model and that the weapon does not jam, his probability of hitting all targets before expending his entire magazine is nearly certain.

$$P[\text{hit 16 targets before expending 30-round magazine}]$$

$$= 1 - P[\text{hit } < \text{ 16 targets with 30 trials}]$$

Let X be the number of targets hit, so,

$$= 1 - P[X \leq 15] = 1 - \sum_{x=0}^{1} 5\binom{30}{x}.8^x(1-.8)^{30-x} = 0.999.$$

Even with a hit probability of 60%, the shooter would have a better than 70% chance of not having to reload in the classroom. If magazine size was restricted, this would increase the probability of having to reload within the classroom, however, this would have a low impact on total expected casualties, considering reloading a detachable magazine takes approximately 3 seconds. If magazine size was restricted to 10 bullets per magazine, the Sandy Hook shooter could have simply carried thirty 10-round magazines and had nearly equal killing capacity. For this reason, some states have banned detachable magazines in favor of fixed magazines which require them to be manually reloaded, thus taking more time. Assuming the shooter was only equipped with a fixed 10-round magazine and had to reload one bullet at a time (in contrast to using a fast-loading clip), how would this effect the expected number of casualties within our model?

Assuming the shooter could reload one bullet every three seconds, and he was not attacked while reloading, Figure 4.15 shows the impact to expected casualties.

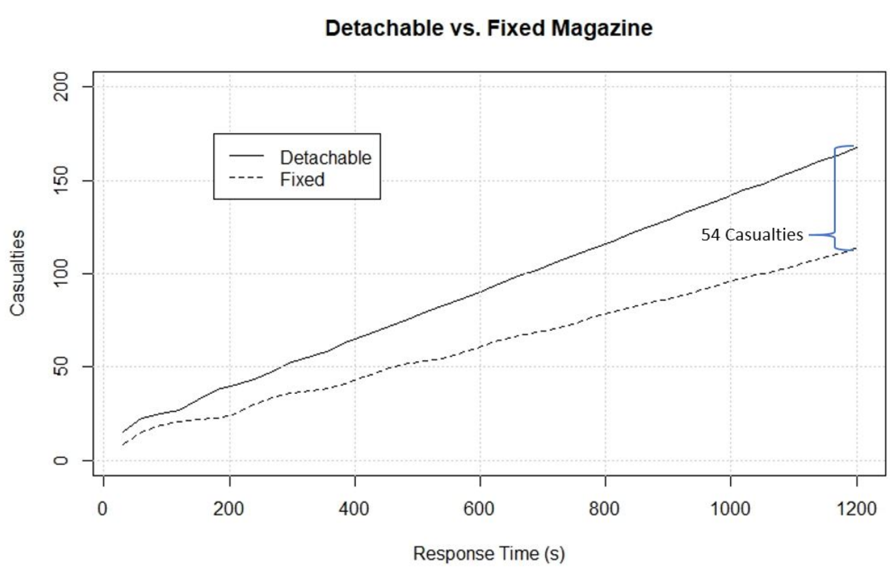

Figure 4.15. Impact of Magazine Type on Expected Casualties.

As expected, the impact of requiring fixed magazines increases as response time increases. At a response time of 20 minutes (1200 seconds), there is a decrease in expected casualties of 54, a reduction of 32%. At five minutes, the elapsed time of the Sandy Hook shooting, there is an expected reduction in casualties of 14, a decrease of 27%.

CHAPTER 5:
Conclusion and Recommendations

5.1 Conclusion

Since 2013, nearly twice as many American lives have been taken due to active shooter incidents than from the war in Afghanistan (iCasualties.org 2020; Blair 2014). The prevalence of these events shows little sign of waning. We must be diligent as a society to reduce the frequency and impact of active shooter incidents. Every step toward understanding them, and the individuals who carry them out, is another step toward reducing and someday eliminating their presence in our communities. The four primary questions posed in Chapter 1 are listed below with summary answers discovered through our research.

Research Question 1: Which features impact the outcome of active shooter scenarios the most? (e.g., number/size of doors, number of bullets, firing rate, tactical proficiency of shooter, arena size, elapsed time of the event, etc.)

We have clearly shown the answer to this question varies based on circumstance. There is no one-size-fits-all strategy that applies to all active shooter scenarios. However, when we split scenarios into their various types, we are able to discern several common threads between like events.

Complex-arena active shooter incidents, like the Aurora movie theater shooting, happen very quickly. Barring the presence of an off-duty police officer in the crowd or an armed guard in the lobby, police response is not likely to occur in time to reduce the number of casualties. This means the incident will likely end due to the shooter running out of bullets or targets. Detachable magazines allow shooters to carry large amounts of ammunition with them, so, the best option to reduce casualties will likely be to increase the number and size of doors. When people become trapped in egress routes because the maximum flow rate is exceeded, they become easy targets for shooters. If the number and size of exits allowed the maximum capacity of occupants for a room to exit freely, the expected number of casualties would be significantly reduced.

When the shooter is able to remain static during the incident, such as the open arena Las Vegas active shooter incident, they are likely to have a large supply of weapons and ammunition, since mobility is not a concern. This means, the event will likely not end due to a lack of bullets, but rather a lack of time or targets. There will be three main drivers to the number of casualties from this event: rate of fire, total elapsed time and egress of targets. We showed in Chapter 4 the effects of reducing the rate of fire (by limiting access to bump-stocks) and reducing total elapsed time (using a gunfire locating device). The last factor, and the only one which the victims are able to influence, is egress time. There were reports from the Las Vegas shooting that many concert-goers laid down when they initially realized there was an active shooter (The Washington Post 2018). Though this aligns with what many of us have been taught, it happened to be one of the worst response options in this particular case. Exiting the arena as quickly as possible while avoiding crowds would likely improve their personal probability of survival and minimize the total number of casualties from the event. This judgment is difficult to make in real-time, particularly when your knowledge of the situation is limited and your life is in danger. However, emergency planners should prepare for the possibility of an active shooter situation at a large event, such as the Route 91 Harvest Music Festival. If a shooting happens, the event coordinators should direct concertgoers to run rather than take cover in place.

Closed arena active shooter incidents, with no practical means of escape, are the worst circumstance for victims. When the shooter entered into classrooms during the Sandy Hook shooting, the occupants had very few options and, therefore, a low probability of survival. In this case, increasing the number of doors in order to turn a closed arena incident into a complex arena incident would likely improve the outcome the most. If all classrooms were on exterior walls, allowing for multiple exit points, the teachers could more readily evacuate their classes. Staff members should also have a firm understanding of the expected time of response for the local authorities. The longer the expected response time, the more likely running will be the best option. If the school can afford to, installing impenetrable, auto-locking doors on classrooms would also cause a large expected reduction in casualties. If this were the case, a "lock-down only" strategy may be the best option.

<u>Research Question 2:</u> Can analytic models achieve the complexity required to viably model active shooter incidents?

Though our analytic models required numerous simplifying assumptions, the insights they provide are invaluable. The transparent nature of equation-based models allow analysts to more readily distill the interactions between factors and their dynamics. While no active shooter scenario is the same, there are common characteristics and driving elements which are accessible despite the level of abstraction that is required. It is not our desire for the models we presented to be used in isolation, but rather to be tested and compared to other model types and historical events to ultimately gain a more holistic understanding of these scenarios.

Research Question 3: Do the outcomes of analytic models align with real-world events and previous academic research?

In many instances, the models aligned very well with the outcomes of agent-based simulations and real-world incidents. Where they did not, it was often the case that certain simplifying assumptions caused the deviation and they were noted. Ultimately, analytic models are often best used to investigate "what-if" scenarios rather than making point estimates on the outcomes of future events. Since real-world incidents can not be replayed after toggling input parameters, we believe the models we developed are useful tools for gaining this type of insight.

Research Question 4: What operational insights and response strategies can be distilled from analytic active shooter models?

We have assumed, throughout this thesis, that people will function in alignment with their own self-interests--that they will function as "rational actors." The "Run, Hide, Fight" strategy is, perhaps, the most broadly applicable procedure for maximizing the probability of survival for each individual. However, if the goals of individuals shift to instead pursue the least harm for the group, even at their own expense, the strategy would also change. One factor which significantly reduces the number of expected casualties, regardless of scenario type, is the termination of the shooter. While the idea of civilian resistance was not discussed explicitly within this thesis, it should be noted that this form of altruism has the potential to severely limit the impact of most active shooter incidents. No emergency plan can expect this during an active shooter incident. Any plan whose success teeters on the altruism of its participants, especially in the face of grave danger, is precarious at best. For this reason, you will not often find an emergency planner, scholar or authority of any kind who will outright encourage altruism in the event of an active shooter incident. We will refrain from doing so as well and instead say this:

The best hope a rational actor has in the event of an active shooter incident is not perfect information, but rather that he is surrounded by a group of irrational actors ready to operate in his best interest at the expense of their own.

5.2 Future Work

There is much work still to be done in the arena of active shooter incidents and more specifically in how models can aid in our understanding of their dynamics. We have listed three natural extensions to our research which we believe would be particularly beneficial.

5.2.1 Civilian Resistance

Our prescriptive model investigated the choice between running and hiding during an active shooter incident. The obvious extension to this model would be to incorporate civilian resistance as an option. We predict there are some desperate circumstances when fighting the shooter, as an unarmed civilian is, in fact, the best option for maximizing your own probability of survival. When an individual's goal is to minimize the expected number of casualties rather than maximize their own probability of survival, we anticipate that the vast majority of circumstances would have them resist. Investigating armed resistance from civilians carrying a concealed weapon (CCW) could also offer insight such as how many CCW individuals must be present in a given circumstance in order to reach some probability threshold for terminating the attacker.

5.2.2 Multiple Shooters

All of our models presumed the shooter was acting alone during the incident. If there were multiple shooters, like there was during the 2015 San Bernardino attack, some of the dynamics of the event would likely change. It may also alter the recommended civilian response strategy during common active shooter scenario types. Fortunately, only 1.5% of U.S. active shooter incidents have included multiple shooters since the year 2000 (Blair 2014).

5.2.3 Human Experimentation with Simulated Ammunition

While no human experimentation could, or ethically should, reproduce the distress victims experience during active shooter incidents, limited tactical drills with simulated ammunition could be a very useful tool for validating the outputs of models. Incentivizing "survival" with meaningful rewards could produce an adequate imitation of the decisions made during an active shooter scenario. This would likely be the closest academics and law enforcement officials could realistically get to studying a real-time active shooter incident. Additionally, Slater et al. (2006) showed promising research indicating human subjects respond realistically while participating in virtual reality scenarios, despite their knowledge that the environment is, in-fact, virtual. Utilizing this technology to research active shooter incidents may provide a more tenable option for validating the findings within this thesis.

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APPENDIX

A.1 Solutions to Differential Equation Model

Solve:

$$\dot{K} = \alpha E + \delta H, \tag{A.1}$$

and

$$\dot{E} = -(\alpha + \beta)E + \lambda H, \tag{A.2}$$

given,

$$H(t) = H_0 \exp(-(\lambda + \delta)t). \tag{A.3}$$

Taking the Laplace Transform of Equation A.2 we have,

$$sf(s) - E_0 = -(\alpha + \beta)f(s) + \frac{\lambda H_0}{s + \lambda + \delta}$$

or

$$\begin{aligned} f(s) &= \frac{\lambda H_0}{(s + \alpha + \beta)(s + \lambda + \delta)} + \frac{E_1}{s + \alpha + \beta} \\ &= \frac{\lambda H_1}{\lambda + \delta - \alpha - \beta}\left(\frac{1}{s + \alpha + \beta} - \frac{1}{s + \lambda + \delta}\right) + \frac{E_1}{s + \alpha + \beta} \end{aligned}$$

It follows that,

$$E(t) = \left(\frac{\lambda H_0}{\lambda + \delta - \alpha - \beta} + E_0\right)\exp(-(\alpha + \beta)t) - \frac{\lambda H_0}{\lambda + \delta - \alpha - \beta}\exp(-(\lambda + \delta)t)$$

In order to simplify the notation, let,

$$\begin{aligned} \mu &= \alpha + \beta \\ \nu &= \lambda + \delta \\ \rho &= \tfrac{\lambda H_0}{\nu - \mu} \end{aligned}$$

then,

$$E(t) = (\rho + E_0)\exp(-\mu t) - \rho \exp(-\nu t). \tag{A.4}$$

The Laplace transform of Equation A.1 is:

$$sf(s) - K_0 = \frac{\alpha(\rho + E_0)}{s + \mu} - \frac{\alpha\rho - \delta H_0}{s + \nu}$$

so,

$$f(s) =$$

$$\frac{K_0}{s} + \frac{\alpha(\rho + E_0)}{s(s + \mu)} - \frac{\alpha\rho - \delta H_0}{s(s + \nu)} = \frac{K_0}{s} + \frac{\alpha(\rho + E_0)}{\mu}\left(\frac{1}{s} - \frac{1}{s + \mu}\right) - \frac{\alpha\rho - \delta H_0}{\nu}\left(\frac{1}{s} - \frac{1}{s + \nu}\right)$$

$$= \left(K_0 + \frac{\alpha(\rho + E_0)}{\mu} - \frac{\alpha\rho - \delta H_0}{\nu}\right)\frac{1}{s} - \frac{\alpha(\rho + E_0)}{\mu}\left(\frac{1}{s + \mu}\right) + \frac{\alpha\rho - \delta H_0}{\nu}\left(\frac{1}{s + \nu}\right).$$

Finally,

$$K(t) = K_0 + \frac{\alpha(\rho + E_0)}{\mu} - \frac{\alpha\rho - \delta H_0}{\nu} - \frac{\alpha(\rho + E_0)}{\mu}\exp(-\mu t) + \frac{\alpha\rho - \delta H_0}{\nu}\exp(-\nu t). \quad \text{(A.5)}$$

A.2 Occupancy Problem - Stars and Bars

There are r *indistinguishable* balls which are to be placed randomly in n cells. The occupancy of the kth cell is denoted as $r_1, r_2, \ldots, r_n$, where r_k stands for the number of balls in the kth cell. So,

$$r_1 + r_2 + \ldots + r_n = r \quad \text{(A.6)}$$

The number of distinguishable distributions is:

$$A_{r,n} = \left(\frac{n + r - 1}{n - 1}\right) \quad \text{(A.7)}$$

Proof:

Use $*$'s to represent the r balls and $|$'s to represent the n cells which contain the balls, thus there are $n+1$ bars ($|$). So, if $r = 7$ and $n = 5$, $| * * | * | | * * * | * |$ would represent a combination where the occupancy of the cells is 2,1,0,3,1. Each combination begins and ends with a $|$, but the remaining $*$'s and $|$'s can appear in an arbitrary order. The number of distinguishable distributions, then, equals the number of ways of choosing

r positions out of the $n + r - 1$ total places. Hence, Equation 2. (Feller 2008)

A.3 Marked Poisson Process Approximation

Given z_0 initial live targets, the number of remaining targets after t periods is,

$$Z_{t+1} = z_0 - \ell t - \sum_{i=1}^{t} Y_i. \tag{A.8}$$

The number of periods, τ, until all the targets are gone or killed is,

$$\tau = \min\{t \geq 0 : Z_t \leq 0\}.$$

Now suppose bursts follow a Poisson process with rate λ. More specifically, let

1. Y has mean μ_Y and standard deviation σ_Y.
2. $N(t)$: The number of bursts by time t.
3. $\ell > 0$: The exit rate through a single door.
4. M: The initial number of targets.
5. τ: The time until there are no more targets to shoot,

$$\tau = \min\{t > 0 : Y_1 + Y_2 + \cdots + Y_{N(t)} + \ell t > M\}.$$

$$\begin{aligned}
P(\tau > t) &= P(Y_1 + Y_2 + \cdots + Y_{N(t)} + \ell t \leq M) \\
&= E[P(Y_1 + Y_2 + \cdots + Y_{N(t)} + \ell t \leq M | N(t) = n)] \\
&= \sum_{n=0}^{\infty} P(N(t) = n) P(Y_1 + Y_2 + \cdots + Y_n + \ell t \leq M) \\
&= \sum_{n=0}^{\infty} \frac{\exp(-\lambda t)(\lambda t)^n}{n!} P(Y_1 + Y_2 + \cdots + Y_n + \ell t \leq M) \\
&\approx \sum_{n=0}^{\infty} \frac{\exp(-\lambda t)(\lambda t)^n}{n!} P(n\mu_Y + \sigma_Y n^{1/2} N(0,1) + \ell t \leq M)
\end{aligned}$$

after using the CLT approximation. Then,

$$P(\tau > t) \approx \sum_{n=0}^{\infty} \frac{\exp(-\lambda t)(\lambda t)^n}{n!} \Phi\left(\frac{M - \ell t - n\mu_Y}{\sqrt{n}\sigma_Y}\right).$$

We can use the approximation for the tails of a Gaussian random variable,

$$1 - \Phi(y) \approx \sqrt{2\pi} \exp(-y^2/2)/y \approx \sqrt{2\pi} \exp(-y^2/2),$$

so,

$$P(\tau > t) \approx 1 - \sqrt{2\pi} \sum_{n=0}^{\infty} \frac{\exp(-\lambda t)(\lambda t)^n}{n!} \exp\left(-.5\left(\frac{M - \ell t - n\mu_Y}{\sqrt{n}\sigma_Y}\right)^2\right) \tag{A.9}$$

A.4 Solving Differential Equation for P(E)

Solve for P:

$$\frac{dP}{dE} = kP(1 - P/a)$$

$$\frac{1}{P(1 - P/a)} \frac{dP}{dE} = k$$

Rewrite as:

$$\left(\frac{1}{P} - \frac{-\frac{1}{a}}{1 - \frac{P}{a}}\right) \frac{dP}{dE} = k$$

Integrate both sides w.r.t. E (where $0 < p(E) < a$),

$$\log(N) - \log(1 - \frac{P}{a}) = kE + C$$

Combine logs:

$$\log\left(\frac{P}{1 - \frac{P}{a}}\right) = kE + C$$

Exponentiate both sides:

$$\frac{P}{1 - \frac{P}{a}} = C \exp(kE)$$

Solve for P:

$$P(E) = \frac{1}{C\exp(-kE) + \frac{1}{a}}$$

Solve for C using initial condition:

$$P(0) = \frac{1}{C + \frac{1}{a}} \rightarrow C = \frac{1}{E_0} - \frac{1}{a}$$

$$P(E) = \frac{a}{1 + \exp(-kE)} \tag{A.10}$$

A.5 Integration of Exponential Decay Model

Let t_{max} be the time it takes for a person directly at the center of the arena to escape and $f(t) = \frac{2}{t_{max}} - \frac{2t}{t_{max}^2}$.

Solve:

$$\int_0^t f(T)\exp(-Tr\log(\frac{1}{1-1/m}))dT \rightarrow \int_0^t (\frac{2}{t_{max}} - \frac{2T}{t_{max}^2})\exp(-Tr\log(\frac{m}{m-1}))dT$$

$$= \int_0^t (\frac{2}{t_{max}} - \frac{2t}{t_{max}^2})(1-\frac{1}{m})^{rT}dT = -\frac{2}{t_{max}^2}\int_0^t (T - t_{max})(1-\frac{1}{m})^{rT}dT$$

Solve:

$$\int_0^t (T - t_{max})(1-\frac{1}{m})^{rT}dT$$

Integration by Parts:

- $u = T - t_{max}$
- $v = \frac{(1-\frac{1}{m})^{rT}}{r\log(1-\frac{1}{m})}$
- $du = 1$
- $dv = (1-\frac{1}{m})^{rT}dT$

$$\int u\ dv = uv - \int v\ du$$

$$\int_0^t (T - t_{max})(1 - \frac{1}{m})^{rT} dT$$

$$= (t - t_{max}) * (\frac{(1-\frac{1}{m})^{rt}}{r\log(1-\frac{1}{m})}) - \int \frac{(1-\frac{1}{m})^{rT}}{r\log(1-\frac{1}{m})} dT$$

Solve:

$$\int \frac{(1-\frac{1}{m})^{rT}}{r\log(1-\frac{1}{m})} dT$$

U Substitution:

$u = rT,\ \frac{du}{dT} = r,\ dT = \frac{1}{r}du$

$$= \frac{1}{r^2\log(1-\frac{1}{m})} \int (1 - \frac{1}{m})^u du$$

$$= \frac{(1-\frac{1}{m})^u}{\log(1-\frac{1}{m})} \rightarrow \frac{(1-\frac{1}{m})^{rt}}{(r\log(1-\frac{1}{m}))^2}$$

Combine and simplify:

$$\int_0^t (\frac{2}{t_{max}} - \frac{2T}{t_{max}^2}) \exp(-Tr\log(\frac{m}{m-1}))dT$$
$$= -\frac{2(\frac{m-1}{m})^{rt}(r\log(\frac{m-1}{m})(t - t_{max}) - 1)}{(rt_{max}\log(\frac{m-1}{m}))^2} + C \qquad \text{(A.11)}$$

A.6 Army Rifle Marksmanship M16-/M4-Series Weapons Field Manual (FM 3-22.9) Specifications Table

Variations of the Colt AR-15 and its derivatives were used in all three of the case studies we reviewed. The M-16 rifle was originally designed as the military counterpart to the AR-15 rifle and has similar specifications when used in the semi-automatic setting (Defense Technical Information Center 1968).

CHARACTERISTICS OF M16-/M4-SERIES WEAPONS

2-1. Table 2-1 describes the general characteristics of M16-/M4-series weapons.

Table 2-1. Characteristics of M16-/M4-series weapons.

CHARACTERISTICS	M4-SERIES	M16A2/A3	M16A4	M16A1
WEIGHT (lb)				
Without magazine and sling	6.49	7.78	9.08	6.35
With sling and loaded:				
20-round magazine	7.19	8.48	9.78	6.75
30-round magazine	7.50	8.79	10.09	8.06
Bayonet knife, M9	1.50	1.50	1.50	1.50
Scabbard	0.30	0.30	0.30	0.30
Sling, M1	0.40	0.40	0.40	0.40
LENGTH (in)				
Rifle w/bayonet knife	N/A	44.88	44.88	44.25
Overall rifle length	N/A	39.63	39.63	39.00
Buttstock closed	29.75	N/A	N/A	N/A
Buttstock open	33.0	N/A	N/A	N/A
OPERATIONAL CHARACTERISTICS				
Barrel rifling-righthand 1 twist (in)	7	7	7	12
Muzzle velocity (fps)	2,970	3,100	3,100	3,250
Cyclic rate of fire (rounds per min)	700-900	700-900	800	700-800
MAXIMUM EFFECTIVE RATE OF FIRE (rounds per min)				
Semiautomatic	45	45	45	45-65
3-round burst	90	90 (A2)	90	N/A
Automatic	150-200 A1	150-200 A3	N/A	150-200
Sustained	12-15	12-15	12-15	12-15
RANGE (m)				
Maximum range	3,600	3,600	3,600	2,653
Maximum effective range:				
Point target	500	550	550	460
Area target	600	800	600	N/A

Figure A.1. Rifle Specification Table. Source: U.S. Army (2016).

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