Connection Between the Solutions of the Helmholtz and Parabolic Equations for Sound Propagation

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ABSTRACT

Using a conformal mapping technique in a rectangular waveguide, we present an exact integral relation between the solutions of the Helmholtz equation whose sound speed $c(x,y)$ varies as a function of both depth y and range x and the solutions of a parabolic equation whose sound speed varies in the mapped depth coordinate. The relation of the corresponding boundary value problems is also discussed, as well as the use of the parabolic approximation in underwater sound propagation problems. The conformal transformation interrelates the sound speeds of the two equations. Several examples are discussed. When $c(x,y) = c(y)$ is only a function of depth we get the recent result of Polyanskii. Other examples for a general conformal transformation are functions $c(x,y)$ which are sinusoidal in depth and exponentially decrease to a constant in range. Several alternative methods of using these results are also discussed.

INTRODUCTION

We wish to relate the solutions of the Helmholtz equation (an elliptic partial differential equation) to the solutions of a related parabolic partial differential equation for the problem of wave propagation in an inhomogeneous waveguide. Recently, Polyanskii[1] presented a short paper on this problem where the sound speed inhomogeneity was assumed to vary in only one direction, that of depth. A more general result is possible. In particular we present here, using a conformal mapping technique, an exact integral relation between the solution of a Helmholtz equation, ψ , whose sound speed $c(x,y)$ varies as a function of depth y and range x, and the solution of a corresponding parabolic equation, p, whose sound speed varies in the transformed depth coordinate. The method further clarifies the use of the parabolic approximation for underwater sound propagation problems.[2-4]

In Section 1 we present the notation and derive the exact integral relation between ψ and p. In Section 2 we show that, since p is separable, it is possible to derive a separable solution for ψ in the transformed coordinate, while relating the sound speeds for the elliptic and parabolic equations via the conformal map. The fact that ψ must satisfy a radiation condition yields a restriction on the available transformations. Hence ψ can be written in terms of parabolic eigenfunctions which depend on the transformed coordinates.

In Section 3 the use of the parabolic method as an approximation is studied using the integral relation and modal representation developed in Sections 1 and 2, and asymptotic properties of the Hankel function. Here, our main new conclusion is that for multi-mode propagation in cylindrical geometry the parabolic approximation can preserve at most one modal amplitude. The rest are scaled by an eigenfrequency-dependant factor. In addition, because of the exact relation between ψ and p, a given incident field in the parabolic boundary value problem yields a different incident field in the elliptic problem.

Section 4 contains several examples, one of which is the result of Polyanskii[1]. It is possible to assume a general form for the conformal transformation compatible with the radiation condition, and a method is discussed for constructing the profile in terms of the transformation and vice versa. The examples include profiles which are sinusoidal in depth and exponentially decreasing to a constant in range.

Section 5 concludes with a discussion of several alternative methods of viewing the problem and using the results, and a summary.

1. RELATION BETWEEN ELLIPTIC AND PARABOLIC SOLUTIONS

The velocity potential $\psi(x,y)$ describing sound propagation satisfies the scalar Helmholtz equation (an elliptic partial differential equation[5])

$$\psi_{xx} + \psi_{yy} + K(x,y)\,\psi = 0 \tag{1}$$

in the waveguide region $x \geq 0$ and $0 \leq y \leq L$. Here $K(x,y) = [\omega / c(x,y)]^2$ where ω is the circular frequency of the sound and $c(x,y)$ its speed. See Fig. 1. Define the coordinate transformation

$$\xi = u(x,y)\ ,\ \eta = v(x,y) \tag{2}$$

and use the functional definition

$$\phi(\xi,\eta) = \psi(x,y). \tag{3}$$

If we assume that the transformation is conformal[6], then we can write it as $(z = x + iy)$

$$u + iv = f(z) \tag{4}$$

where u and v satisfy the Cauchy-Riemann conditions

$$u_x = v_y,\ u_y = -v_x. \tag{5}$$

Here $f(z)$ is a regular analytic function and $f'(z) \neq 0$ in the waveguide. Then ϕ satisfies the Helmholtz equation

$$\phi_{\xi\xi} + \phi_{\eta\eta} + K_2(\xi, \eta)\phi = 0 \tag{6}$$

where

$$K_2(\xi, \eta) = K(x,y)/ |f'(z)|^2 \tag{7}$$

Next, assume the function $p(\xi,\eta)$ satisfies the parabolic partial differential equation[5]

$$\alpha p_\xi + p_{\eta\eta} + K_3(\eta)p = 0 \tag{8}$$

where α is a constant and K_3 is an arbitrary function of the transformed depth coordinate η. We wish to establish a relation between ϕ and p. If we assume the general form[1]

$$\phi(\xi, \eta) = C \exp[A(\xi)] \int_0^\infty p[g(t),\eta]\exp[B(\xi,t)]dt \tag{9}$$

where C is a constant, substitute (9) into (6) and integrate by parts using (8), then it is easy to show that (9) satisfies (6) provided we choose

$$A(\xi) = a \ln(\xi) , \tag{10}$$

$$B(\xi, t) = \beta\xi^2 t + (a - 3/2)\ln(t) , \tag{11}$$

$$g(t) = \alpha /4\beta t , \tag{12}$$

and

$$K_2(\xi, \eta) = K_3(\eta) + a(1-a)/\xi^2 , \tag{13}$$

where β and a are constants. Here the surface terms resulting from the partial integration can be neglected.

Substituting (10), (11) and (12) into (9) and using (3) yields the integral relation

$$\psi(x,y) = C\,\xi^{a}\int_{0}^{\infty} p\left(\frac{\alpha}{4\beta t},\,\eta\right)\exp\,(\beta\xi^{2}t)\,t^{a-3/2}\,dt \tag{14}$$

between ψ and p. For $\beta = i$ this can be viewed as a combined Mellin-Fourier-inversion transformation on the parabolic solution. Combining (7) and (13) yields the relation

$$K(x,y) = |f'(z)|^{2}[K_{3}(\eta) + a(1 - a)/\xi^{2}] \tag{15}$$

between the sound speeds.

Thus we have a general exact integral relation between the elliptic and parabolic solutions given by (14), where their respective sound speeds are related using (15).

2. MODAL REPRESENTATION

The parabolic equation (8) is separable. We can write its solutions as

$$p(\xi,\eta) = \sum_{j=0}^{\infty} p_{j}N_{j}(\eta)\exp\,(-\lambda_{j}^{2}\ \xi/\alpha) \tag{16}$$

where the $\{p_j\}$ are constants (and can be determined from the known incident field) and where the eigenfunctions $N_j(\eta)$ satisfy, in the transformed depth coordinate, the ordinary differential equations

$$N_{j}''(\eta) + [K_{3}(\eta) - \lambda_{j}^{2}]\,N_{j}(\eta) = 0 \tag{17}$$

and boundary conditions which are discussed later.[7] The $\{\lambda_j\}$ are the discrete eigenvalues.[8] If we substitute (17) into (15) and use the integral representation[9]

$$\int_0^\infty \exp\,(it\rho^2 - \delta^2/4t)t^{-\nu-1}\,dt$$

$$= \pi i(2\rho/\delta)^\nu \exp(\pi i\nu/4)\, H_\nu^{(1)}\,(\delta\rho \exp\,(\pi i/4)) \tag{18}$$

where $H_\nu^{(1)}$ is the Hankel function of first kind and ν^{th} order, then, for $\beta = i$, ψ can be written as

$$\psi(x,y) = C\pi i\ \xi^{1/2} \sum_{j=0}^{\infty}{}' \, p_j N_j(\eta)(\lambda_j/2i)^{a-1/2}\ H_{1/2-a}^{(1)}\ (\lambda_j\xi) \tag{19}$$

Hence (19) is a separable expansion for ψ in terms of the transformed coordinates ξ and η. In addition to satisfying boundary conditions at $x = 0$ and $y = 0$ and L, ψ must satisfy a radiation condition as $x \to \infty$ (this was the reason for the choice $\beta = i$). That is, each partial mode in (19) must behave like $\exp(ix\lambda_j)$ as $x \to \infty$. This holds if we use the asymptotic representation of the Hankel function and the requirement that as $x \to \infty$, $\xi \sim x$. This asymptotic restriction enables us to write the transformation as

$$f(z) = z + f_1(z) \tag{20}$$

where $f_1(z) \to 0$ as $z \to \infty$ in the waveguide, and, since the transformation is conformal, $f_1'(z) \neq -1$ also in the waveguide region. Unfortunately we can determine no more properties of the transformation simply and it is easiest to proceed by considering some examples. We do this in Section 4. First, however, we briefly consider the parabolic solution when it is used as an approximation.

3. PARABOLIC APPROXIMATION

For $\alpha = ik$ the parabolic solution (16) is

$$p(x,y) = p(\xi,\eta) = \sum_{j=0}^{\infty} p_j N_j(\eta) \exp(i\,\lambda_j^2\,\xi/k) \tag{21}$$

This is used as an approximation to the asymptotic value of ψ which is, using the asymptotic representation of the Hankel function in (19)

$$\psi(x,y) \sim C(\pi i)^{1/2} \sum_{j=0}^{\infty} p_j(\lambda_j/2)^{a-1} N_j(\eta) \exp(i\lambda_j\xi) \tag{22}$$

comparison of (21) and (22) shows immediately that the parabolic approximation does not preserve the phase of the asymptotic elliptic solution. This is well known, as is the fact that the mode shapes are preserved by the approximation.[3,4] However, the question of the amplitudes of the modes is somewhat different. If $a = 1$, and the normalization constant C is chosen as $C = (\pi i)^{-1/2}$, then p preserves all the modal amplitudes of ψ. The case $a = 1$ is that of cartesian coordinates and is considered as Example 1 in the next section. For $a \neq 1$, all the modal amplitudes are not preserved by the approximation because of the λ_j factor in (22). It is possible to preserve one of them, say the m^{th} modal amplitude, by choosing $C = (\pi i)^{-1/2}(2/\lambda_m)^{a-1}$. Then all the remaining amplitudes will be scaled in the parabolic approximation by the factor $(\lambda_m/\lambda_j)^{a-1}$. For example, the case $a = 1/2$ is the case of cylindrical coordinates (see Ex. 2 in Sec. 4) which is most often used in sound propagation problems. Our conclusions on the preservation of the amplitude in the parabolic approximation for multi-mode propagation thus differ from the corresponding results in the literature.[10] For multi-mode propagation the parabolic approximation preserves all the amplitudes in cartesian coordinates, but can preserve at most one amplitude for cylindrical coordinates.

There is a further question having to do with the incident field. If we let $a = 1/2$ and choose cylindrical coordinates $(\xi = r, \eta = y)$, then the incident parabolic field which is known at say $r = 1$ can be used in (21) to determine the set $\{p_j\}$. Substituting $\{p_j\}$ into (19) and noting that (Ex. 2, Sec. 4) the cylindrical field in the elliptic problem is $r^{-\frac{1}{2}}\psi(r, y)$, we see that at $r = 1$, $\psi(1,y) \neq p(1,y)$. Again one point can be matched by proper choice of the constant C (which is thereby unavailable to match an asymptotic modal amplitude), but the incident fields and hence the corresponding boundary value problems are different for ψ and p. The two incident fields cannot be chosen independently.

From the exact relation (14) we thus conclude that for cylindrical coordinates the parabolic approximation in general doesn't preserve the amplitudes of the asymptotic elliptic solution, nor does it match the incident field.

4. EXAMPLES

In this section we present some examples of the exact formalism developed in Secs 1 and 2.

(Example 1)

Choose cartesian coordinates $\xi = x$ and $\eta = y$, then the transformation is $f(z) = z$. Then if $a = 1$, $\alpha = ik$, and $\beta = i$, (14), (15) and (19) yield

$$K(x,y) = K_3(y) \tag{23}$$

$$\psi(x,y) = Cx \int_0^\infty p\left(\frac{k}{4t}, y\right) \exp(ix^2 t)\, t^{-1/2}\, dt \tag{24}$$

$$= \sum_{j=0}^{\infty} P_j N_j(y) \exp(i\lambda_j x) \tag{25}$$

where $C = (\pi i)^{-1/2}$ in (25) and where we have used the definition[11]

$$H^{(1)}_{-1/2}(z) = (2/\pi z)^{1/2} \exp(iz) \tag{26}$$

This is the result of Polyanskii[1] and by (23) only holds for sound speeds which vary in depth. Comparing (21) and (25) shows that the parabolic solution preserves all the modal amplitudes.

(Example 2)

Again choose cartesian coordinates $\xi = x$ and $\eta = y$ so that the transformation is $f(z) = z$. Let $a = 1/2$. Then (15) becomes

$$K(x,y) = K_3(y) + (4x^2)^{-1} . \tag{27}$$

If we relable $x \to r$, then the function Ψ defined by

$$\Psi(r,y) = r^{-1/2}\psi(r,y) \tag{28}$$

satisfies the Helmholtz equation in cylindrical coordinates

$$\Psi_{rr} + \frac{1}{r}\Psi_r + \Psi_{yy} + K_3(y)\Psi = 0 \tag{29}$$

Thus the term involving a in (15) is like a centrifugal barrier term in potential scattering theory.[12]

(Example 3)

More generally, choose the transformation

$$f(z) = z + f\exp(-\varepsilon z) \qquad 0 < \varepsilon,\ f < 1 \tag{30}$$

with
$$\xi = \mathrm{Re}f(z) = x + f\cos(\varepsilon y)\exp(-\varepsilon x)$$
$$\eta = \mathrm{Im}f(z) = y - f\sin(\varepsilon y)\exp(-\varepsilon x) .$$

Further, let $a = 0$ and $K_3(\eta) = k^2$ where $k = \omega/c$ and c is an arbitrary sound speed used in the parabolic equation. Then (15) yields

$$K(x,y) = k^2\left\{1-2\varepsilon f\cos(\varepsilon y)\exp(-\varepsilon x) + (\varepsilon f)^2\exp(-2\varepsilon x)\right\}. \tag{31}$$

Since $K_3(\eta)$ is constant, the N_j eigenfunctions are by (17)

$$N_j(\eta) = D\sin(m_j\eta) + E\cos(m_j\eta) \tag{32}$$

where $m_j = (k^2 - \lambda_j^2)^{\frac{1}{2}}$. Further, assume the boundary value problem of a soft (Dirichlet) surface at $y = 0$, a hard (Neumann) bottom at $y = L$, and an arbitrary incident field at $x = 0$, i.e.,

$$\begin{aligned} \psi(x,0) &= 0 \\ \frac{\partial\psi}{\partial y}(x, L) &= 0 \\ \text{and} \quad \psi(0,y) &= \psi^{(0)}(y). \end{aligned} \tag{33}$$

Then since $\eta(x,0) = 0$ and $\eta(x,L) = L - f\sin(\varepsilon L)\exp(-\varepsilon x)$ the boundary conditions (33) are satisfied using (19) and (32) provided that

$$\begin{aligned} E &= 0 \\ \varepsilon &= \pi/L \\ \text{and} \quad \lambda_j &= \begin{cases} (k^2 - m_j^2)^{\frac{1}{2}} & k^2 \geq m_j^2 \\ +\,i(m_j^2 - k^2)^{\frac{1}{2}} & m_j^2 > k^2 \end{cases} \end{aligned} \tag{34}$$

where

$$m_j = (2j+1)\pi/2L.$$

The choice of λ_j is made to ensure that the outgoing radiation condition is fulfilled. Using (34) the parabolic boundary value problem is

$$\begin{aligned} p(\xi(x,0),0) &= 0 \\ \frac{\partial p}{\partial \eta}\,\xi(x,L),L) &= 0 \\ p(\xi,\eta)\Big|_{x=0} &= p(f\cos(\pi y/L),\ y-f\sin(\pi y/L)) = p^{(0)}(y) \end{aligned} \tag{35}$$

with $\xi(x,0)$ and $\xi(x,L)$ given by (30).

If we write (31) in terms of sound speeds using $K(x,y) = [\omega/c(x,y)]^2$ then we have

$$c(x,y)/c = \left\{ 1 - (2\pi f/L)\cos(\pi y/L)\exp(-\pi x/L) + (\pi f/L)^2 \exp(-2\pi x/L) \right\}^{-1/2}, \tag{36}$$

a one-parameter family of sound speed profiles which are sinusoidal in depth and exponentially decreasing in range. An example is given in Fig. 2.

Thus for the sound speed (36) the elliptic solution of (1), ψ, can be expressed exactly by (14) as an integral over the solution of the simple parabolic boundary value problem (35), or as an exact modal expansion by (19) using the transformed eigenfunctions (32) (using (34)) and a set $\{p_j\}$ which can be found by point matching the incident parabolic field in (35).

(Example 4)

For a multi-parameter family of sound speeds choose the transformation

$$f(z) = z + \sum_{m=1}^{M} f_m \exp(-\varepsilon_m z) \tag{37}$$

where

$$\xi = x + \sum_{m=1}^{M} f_m \cos(\varepsilon_m y)\exp(-\varepsilon_m x)$$

and

$$\eta = y - \sum_{m=1}^{M} f_m \sin(\varepsilon_m y)\exp(-\varepsilon_m x).$$

If we again choose as in Ex. 3, $K_3(\eta) = k^2$, the eigenfunctions $N_j(\eta)$ are the same as (32) and the boundary value problem (33) is satisfied provided (34) and $\varepsilon_m = m\pi/L$ hold. The values of ξ and η at $y = 0$ and L and at $x = 0$ are found from (37). If $a = 0$, then (15) and (37) yield

$$K(x,y) = k^2\left\{1 - 2\sum_{m=1}^{M}\varepsilon_m f_m \cos(\varepsilon_m y)\exp(-\varepsilon_m x)\right. \tag{38}$$

$$\left. + \sum_{m=1}^{M}\sum_{n=1}^{M}\varepsilon_m\varepsilon_n f_m f_n \cos\left[(\varepsilon_m-\varepsilon_n)y\right]\exp\left[-(\varepsilon_m+\varepsilon_m)x\right]\right\}$$

which is an M-parameter family of curves. Each additional term in the sum in (37) introduces an additional turning point in the sound speed curve. Example 3 with $M = 1$ had no turning points. For $M = 2$, (38) written in terms of sound speeds is

$$c(x,y)/c = \left\{1 + (\pi f_1/L)^2\exp(-2\pi x/L)\right.$$

$$+ (2\pi f_2/L)^2\exp(-4\pi x/L)$$

$$- (2\pi f_1/L)\left(1 - [2\pi f_2/L]\exp(-2\pi x/L)\right). \tag{39}$$

$$\cdot\cos(\pi y/L)\exp(-\pi x/L)$$

$$\left. - (4\pi f_2/L)\cos(2\pi y/L)\exp(-2\pi x/L)\right\}^{-\frac{1}{2}},$$

an example of which is plotted in Fig. 3. Note the fact that the sound speeds have one turning point. By multiplying $[c/c(x,y)]^2$ by $\cos(m\pi y/L)$ for $m = 1,2$ and then integrating over y from 0 to L it is possible to solve for f_1 and f_2 and fit them to various ranges.

Again, for the sound speed (39), ψ is either an exact integral relation (14) or an exact modal representation (19) over terms associated with the parabolic boundary value problem defined above.

(Example 5)

Some additional examples of other transformations which can be used are, from (20)

$$f_1(z) = \sum_{m=1}^{M} g_m(z)\exp(-\varepsilon_m z) \tag{40}$$

where the $g_m(z)$ can be finite polynomials (splines), oscillatory functions, etc. Indeed f_1 can be even more general. So long as $f_1(z)$ has no singularities in the waveguide, and vanishes as $z \to \infty$ in the waveguide, it can be quite arbitrary. This admits ratios of polynomials (Padé approximants) as well as functions with more complicated singularities outside the waveguide. Some of these examples are presently being pursued.

5. SUMMARY

There are several alternative ways to view the results in this paper. Firstly, one could take as the central issue the conformal transformation. By choosing various transformations one could construct a library of available profiles including those generated using more complicated profiles in the parabolic equation, those involving centrifugal barrier terms, and those involving more complicated transformation functions. Solution of the resulting parabolic boundary value problem in the transformation distorted waveguide (as in Eqs 3 and 4) and either continuous (Eq 14) or discrete (Eq 19) quadrature yield the solution of the full elliptic problem for the various profiles. The numerical solution of the parabolic boundary value problem can be accomplished quickly by using a marching algorithm in range (although this is complicated by the distortion of the waveguide) whereas the elliptic equation requires a much more involved and time consuming numerical discretization over a closed boundary.

Secondly, one could consider the solution of the parabolic equation as central and numerically solve the simplest non-trivial parabolic boundary value problem available. This will then lead to the transformation.

Finally, the profile may be regarded as fundamental. In particular, $K(x,y)$ may be given as a discrete set of points and the conformal transformation is constructed by fitting these points. The transformation then yields the parabolic boundary value problem, etc. These three interpretations of the ways to utilize the above results are, of course, intimately connected.

Thus we have presented an exact relation between the solution of the Helmholtz equation whose sound speed varies in both depth and range and the solution of a parabolic boundary value problem with sound speed varying in a conformally transformed depth coordinate. The relation can be expressed either as a Fourier-Mellin-inversion transformation or as a discrete modal sum. The sound speeds in both equations are themselves related via the conformal transformation. Several examples were presented, among them profiles which change sinusoidally in depth and decrease exponentially to a constant in range. When viewed as an approximation the parabolic method was found to preserve at most one of the modal amplitudes in a multi-mode propagation problem in cylindrical coordinates and, because of the exact relation between the two solutions their respective incident fields could not be independently chosen to be equal.

Footnotes

* Temporary Address

1. E. A. Polyanskii, Sov. Phys. Acoust. 20, 90 (1974)
2. See the article by F. Tappert and R. Hardin in "A Synopsis of the AESD Workshop on Acoustic-Propagation Modelling By Non-Ray Tracing Techniques", ed. by C. W. Spofford, AESD Tech. Note TN 73-05 (Nov. 1973) (Acoustic Environmental Support Detachment, Office of Naval Research, Arlington, Virginia).
3. S. T. McDaniel, J. Acoust. Soc. Am. 57, 307 (1975)
4. R. M. Fitzgerald, J. Acoust. Soc. Am. 57, 839 (1975)
5. P. R. Garabedian, "Partial Differential Equations", Wiley, New York (1964)
6. E. T. Copson, "Theory of Functions of a Complex Variable", Oxford University Press (1944), p.180
7. Usually in these problems $\alpha = ik$, where k is some average wave-number and $K_3(\eta) = k^2 K_4(\eta)$ where $K_4(\eta)$ is dimensionless. See Refs 2-4. We make these assumptions explicit later.
8. We are restricting the boundary value problem to one with a purely discrete spectrum for convenience only. Extension to a continuous spectrum is straightforward. To satisfy the radiation condition on ψ the λ_j are either positive real or positive imaginary. See Ex. 3 in Sec. 4.
9. Bateman Manuscript Project, "Tables of Integral Transforms, vol I", ed. by A. Erdelyi, McGraw-Hill, New York (1954), pg 16, no. 20 and pg 75, no. 31.
10. In Ref. 3, Eqs (11) and (12) are used as the iterative algorithms for a single mode propagating via the parabolic and elliptic equations respectively. If this single mode starts with the same aplitude for both parabolic and elliptic cases it will end up with the same amplitude. The same results are found in Ref. 4. For a single mode, as we indicated, this is always possible. But for the case of multi-mode propagation,

amplitude changes will occur. The authors of both references use in their respective multi-mode examples the fact that all the amplitudes are preserved by the approximation. They conclude this by neglecting the subtle effect the centrifugal barrier term $(4r^2)^{-1}$ has on the asymptotic amplitude (through the λ_j factor) effectively reducing their problems to cartesian coordinates where, as we indicated, all the amplitudes are preserved.

11. W. Magnus, F. Oberhettinger, "Functions of Mathematical Physics", Chelsea, New York (1949).

12. V. De Alfaro, T. Regge, "Potential Scattering", Wiley, New York (1965).

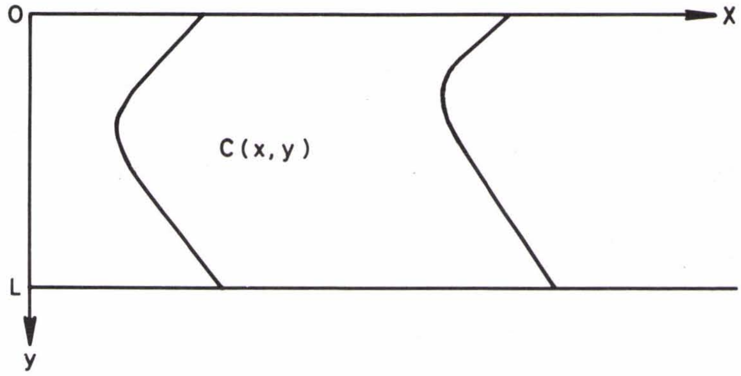

FIG. 1 THE WAVEGUIDE REGION IN WHICH WE SOLVE THE ORIGINAL ELLIPTIC (HELMHOLTZ) EQUATION FOR SOUND PROPAGATION. WE CONSTRUCT EXAMPLES OF SOLVABLE SOUND SPEEDS $c(x, y)$ DEPENDING ON BOTH RANGE x AND DEPTH y.

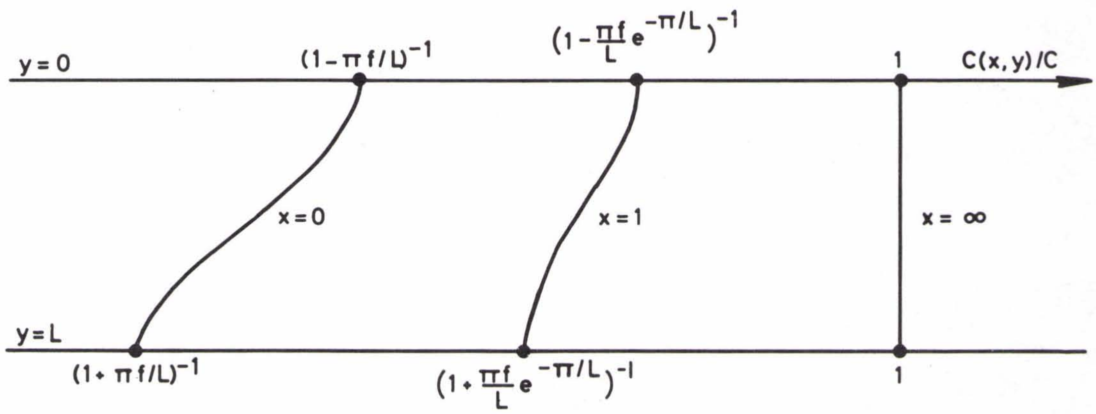

FIG. 2 AN EXAMPLE OF A 1-PARAMETER FAMILY OF SOUND SPEED PROFILES FOR WHICH THE ELLIPTIC EQUATION (1) IS EXACTLY SOLVABLE. THEY ARE SINUSOIDAL IN DEPTH, EXPONENTIALLY DECREASING IN RANGE, AND HAVE NO TURNING POINTS. THE FIGURE REFERS TO EXAMPLE 3 IN SEC. 4.

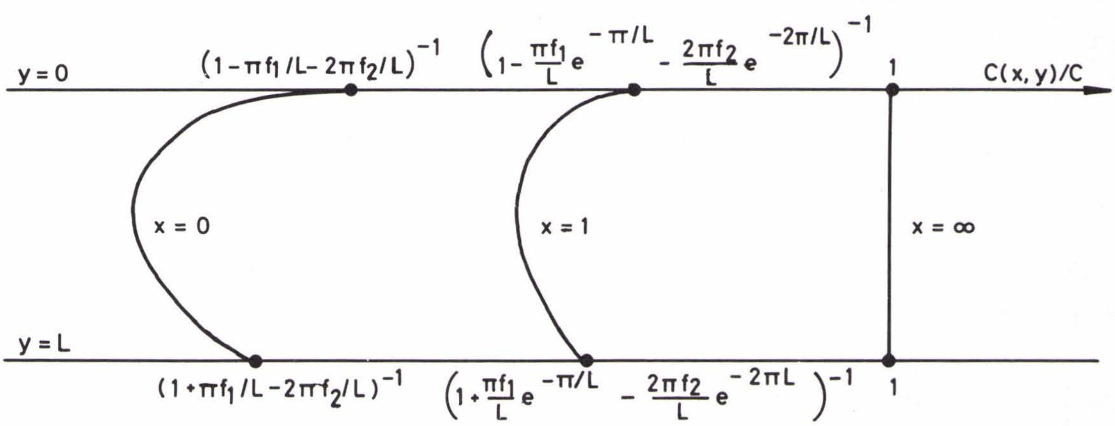

FIG. 3 AN EXAMPLE OF A 2-PARAMETER FAMILY OF SOUND SPEED PROFILES FOR WHICH THE ELLIPTIC EQUATION (1) IS EXACTLY SOLVABLE. THEY ARE SINUSOIDAL IN DEPTH, EXPONENTIALLY DECREASING IN RANGE, AND HAVE ONE TURNING POINT. THE FIGURE REFERS TO EXAMPLE 4 IN SEC. 4.