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REPORT**



**STUDIES ON
INFORMATION MECHANICS**

by

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APRIL 1987

NORTH
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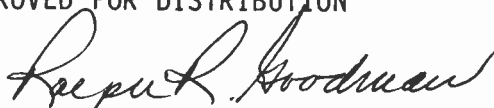
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April 1987

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Report no. changed (Mar 2006): SR-111-UU**Abstract**

Related issues connected with the transmission of information in a command and control environment are modelled and analyzed theoretically. These include search in an area rich in false targets, and surveillance. In addition, the mechanics are considered of the transmission of information from the field to the command level with particular emphasis on the use of redundant, fallible communication channels. A key connecting theme in the analyses is the concept of entropy and its role as a measure of the effectiveness of information transmission and the actions initiated upon receipt of information. The final chapters analyze simple message transmission systems with priority with particular reference to the considerations necessary to mitigate the potential damage caused by the misuse of priority to information transmitted at a lower, or non-priority level. Much of the analysis is either new, or an application of known work to new issues. The objective is to establish a theoretical framework for the study and improvement of command and control procedures in the widest context, including the interpretation of field data and the design of trials and experiments.

Keywords

CATASTROPHE THEORY
COMMAND AND CONTROL
CONTROL THEORY
ENTROPY
FALSE TARGET
FUZZY SET THEORY
INFORMATION THEORY
MULTIPLE
SEARCH
SURVEILLANCE
TWO CELL SEARCH

Report no. changed (Mar 2006): SR-111-UU**Introduction**

The formal study of military command and control as a scientific discipline has developed only in the past decade. It is still a young discipline, and its theoretical formulations are still the subject of considerable debate. A key issue in that debate is the choice among several extant mathematical formalisms as the proper vehicle for the description of the phenomena of command and control. Among those put forward are control theory, information theory, fuzzy set theory and catastrophe theory.

The first two of these have appeal at a fundamental theoretical level; the latter two, at a phenomenological level.

Control theory has a natural appeal because of the structural similarity of military command and control to the problems normally addressed by control engineering: a process (the battle) has some objective; deviations from the objective are measured; corrective action is then signalled through a feedback loop. In practice, the situations encountered in military command and control are extremely complex, and the ability of a commander to effect corrective action is very uncertain. Nonetheless, the structural analogy is valid, and some limited success has been achieved by applying the concepts and techniques of control theory to military command and control.

Information theory also has a natural appeal, but for quite different reasons. There is no structural analogy; rather, information, in the Shannon sense, is the fundamental quantity that pervades the entire process of command and control. Information is gathered by battlefield sensors; information is corrupted by noisy communication links; information decays as time changes the reality of the battlefield; information is the basis of command decisions; and information, again corrupted by noise, is the essence of directives from the commander to his combat forces.

Fuzzy set theory has been employed to express the inherent imprecision that is characteristic of most command and control problems. Catastrophe theory has been introduced, largely on an ad hoc basis, because certain sudden changes in the course of a battle that are related to breakdown of the command and control process appear to resemble certain catastrophes as defined by Thom <1>.

The analysis which follows in this report does not attempt to resolve the issue of the correct mathematical framework for the study of command and control. Instead we adopt the point of view of information theory as an assumption and attempt to elucidate certain operational processes in information-theoretic terms.

Attempts to apply information theory to various military operations that are embedded in the command and control process are hardly new, but earlier efforts met with indifferent success. Regarding the process of search for a target, Koopman <2> stated: " Ever since the mid-nineteen-forties when the theories of information and search became subjects of general interest, attempts have been made to apply the theory of information to problems of search. These have proved disappointing; neither the formulas nor the concepts of the former theory have found a place in clarifying the problems of the latter."

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A major contributing factor to this pessimistic view was the 1961 paper by Mela <3>. He presented numerical examples for simple search models, demonstrating that a search policy that is designed to maximize detection probability does not necessarily maximize either the gain in expected information or the probability of correctly committing forces on the basis of search outcomes. He concluded: "...it does not seem likely that there is any intimate connection between search theory and information theory."

Despite this seemingly negative result, other researchers continued to probe information-theoretic approaches, often obtaining results that contradicted those of Mela.

In 1973, Richardson <4> conducted a simulation study to explore alternative surveillance policies in a false- and moving- target environment. One of his policy options was a maximum information gain policy. He concluded: "... the maximum information gain policy appears to have desirable characteristics in the idealized surveillance scenario considered. Among these characteristics are ... good initial behaviour in the early stages [of surveillance] and good asymptotic behaviour in the later stages."

As a result of Richardson's findings, Barker <5> was able to prove a theorem that stated that "subject to a constraint on total search effort, the allocation of search effort that maximizes the probability of detection also maximizes the entropy of the posterior distribution." This theorem appeared to contradict the numerical results of Mela.

This contradictory situation was clarified in 1978 by one of the present authors <6>. In that paper, Pierce showed, among other things, that : (1) there are many different information or entropy quantities that can be calculated in a search situation, and that Barker's theorem referred to a different quantity from that calculated by Mela; and (2) Mela's examples included the single exception to an otherwise general rule that relates information gain and search effectiveness. The results showed that the connection between information theory and optimum search is strong but not universal. Two basic conclusions were:

"The search policy that maximizes the probability of detection is the one that uses up the information contained in the prior ensemble at the maximum rate.

In a wide range of practical cases [but not all cases] the search policy that maximizes the detection probability is the one that creates expected information at the maximum rate. We achieve that maximum rate of creation by using the existing information of the prior ensemble as fast as possible, thereby gaining an early detection, with its concomitantly large increase of expected information."

The work described in the present paper includes some extensions to this earlier work on information theory and search. It also covers applications of information-theoretic ideas to other processes that are part of the whole command and control process.

Chapter 1, as an extension of reference <6>, discusses the relationship of information theory and search when false targets are present. Chapter 2

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develops a mathematical description of the information created by periodic surveillance of a moving target, in which detection of the target is not at issue. Chapter 3 treats the merger in a filter centre of data about the same situation from several sources. The effects of missing or erroneous data are described in information-theoretic terms. Finally, Chapters 4 and 5 examine, on the basis of queueing theory, the effects of a priority system for transmitting information over a link with finite capacity.

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1 Simple models for search in presence of false targets

1.1 CONCEPT OF CLASSIFICATION

1.1.1 The model

The search model developed in this part introduces the concept of classification, that is the capability of identifying an object, rightly or wrongly. If a genuine target T is detected we shall say that with probability c it is declared correctly to be T , and with probability $\bar{c}(= 1-c)$ it is declared to be $\bar{T}$ (not T), that is to say F (false). Likewise, if a false target is detected there is a probability c that it is correctly identified as F , and $\bar{c}$ that it is stated to be T . (Here, as elsewhere, a bar denotes the logical complement of an event, or 1 minus the probability of the event).

We shall consider search in a single area, or cell.

Underlying the whole endeavour is the attempt to measure, accepting the limitations and idealization of the model, the corrupting effect of false targets on the information made available by the search of an area.

In the classical formulation of search theory a sensor-carrying vehicle can be thought of as sitting still in the centre of a search area. The sensor provides some kind of display of the area and this will, in general, be littered with multiple target-like images. Note that the stationary sensor concept corresponds to surveillance but that the same machinery can be made to apply to search by a moving vehicle in certain cases by placing it at relative rest.

We shall limit this introductory analysis to the case where, during the time of the search, the area may contain one genuine target T with probability t , and one target-like false target F with probability f . The two may coexist and are taken to be independent. Four fundamental subsets in the event space are thus TF , $\bar{T}\bar{F}$, $T\bar{F}$, $\bar{T}F$, with associated probabilities tf , $t\bar{f}$, $\bar{t}f$, $\bar{t}\bar{f}$, where $t+\bar{t}=1$, $f+\bar{f}=1$ ($0 \leq t, f \leq 1$). The probability that T/F is detected is denoted by d_t/d_f . There are 16 cases to distinguish. These are listed in Table 1.

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TABLE 1

EVENT SPACE AND PROBABILITIES

SET	CASE	PROBABILITY	SEARCH OUTCOME	JOINT EVENT
I(TF)	1	$tf\ d_t\ d_f\ c^2$	Y_1	(X_1, Y_1)
	2	$tf\ d_t\ d_f\ c\bar{c}$	Y_1	(X_1, Y_1)
	3	$tf\ d_t\ d_f\ \bar{c}c$	Y_2	(X_1, Y_2)
	4	$tf\ d_t\ d_f\ \bar{c}^2$	Y_1	(X_1, Y_1)
	5	$tf\ d_t\ \bar{d}_f\ c$	Y_1	(X_1, Y_1)
	6	$tf\ d_t\ \bar{d}_f\ \bar{c}$	Y_2	(X_1, Y_2)
	7	$tf\ \bar{d}_t\ d_f\ c$	Y_2	(X_1, Y_2)
	8	$tf\ \bar{d}_t\ d_f\ \bar{c}$	Y_1	(X_1, Y_1)
	9	$tf\ \bar{d}_t\ \bar{d}_f$	Y_0	(X_1, Y_0)
II($\bar{T}\bar{F}$)	10	$\bar{t}\bar{f}\ d_t\ c$	Y_1	(X_1, Y_1)
	11	$\bar{t}\bar{f}\ d_t\ \bar{c}$	Y_2	(X_1, Y_2)
	12	$\bar{t}\bar{f}\ \bar{d}_t$	Y_0	(X_1, Y_0)
III($\bar{T}F$)	13	$\bar{t}\bar{f}\ d_f\ c$	Y_2	(X_0, Y_2)
	14	$\bar{t}\bar{f}\ d_f\ \bar{c}$	Y_1	(X_0, Y_2)
	15	$\bar{t}\bar{f}\ \bar{d}_f$	Y_0	(X_0, Y_0)
IV($\bar{T}\ \bar{F}$)	16	$\bar{t}\ \bar{f}$	Y_0	(X_0, Y_0)

It is useful to set the cases out in detail and to check that the sum of the probabilities is unity. The notation is intended to be self-explanatory. For instance, case 1 is the event that both T and F are

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present, both detected, and both correctly identified. Case 2 is the event that both T and F are present and both detected, while T is correctly identified and F is not: in this case F would be thought to be a T, but the conclusion that T is present is true. If we took into account the loss in useful effort consequent on the prosecution of a false target this would be detrimental. Again, case 4 would lead to the correct conclusion that T is present, but for the wrong reason. Case 14 belongs to the same category.

Possible states of nature are:

- X_0 : no T present
- X_1 : T present

Possible outcomes of the search are:

- Y_0 : no detection
- Y_1 : at least one detection made and the conclusion drawn that X_1 is the state of nature
- Y_2 : at least one detection made and the conclusion drawn that X_0 is the state of nature

The cases giving rise to Y_0, Y_1, Y_2 are then:

- Y_0 : 9, 12, 15, 16
- Y_1 : 1, 2, 4, 5, 8, 10, 14
- Y_2 : 3, 6, 7, 11, 13

From the information-theoretic viewpoint there are six fundamental joint probabilities to assemble: these are listed below with the corresponding contributions from Table 1.

$$\begin{aligned}
 P(X_0, Y_0) &= 15+16 \\
 P(X_1, Y_0) &= 9+12 \\
 P(X_0, Y_1) &= 14 \\
 P(X_1, Y_1) &= 1+2+4+5+8+10 \\
 P(X_0, Y_2) &= 13 \\
 P(X_1, Y_2) &= 3+6+7+11
 \end{aligned}$$

For the record, these give:

$$P(X_0, Y_0) = \bar{t}(1 - f d_f) \quad (\text{Eq. 1})$$

$$P(X_1, Y_0) = t \bar{d}_t(1 - f d_f) \quad (\text{Eq. 2})$$

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$$P(Y_0) = (1 - t d_t)(1 - f d_f) \quad (\text{Eq. 3})$$

$$P(X_0 | Y_0) = \bar{t}/(1 - t d_t) \quad (\text{Eq. 4})$$

$$P(X_1 | Y_0) = t \bar{d}_t/(1 - t d_t) \quad (\text{Eq. 5})$$

$$P(X_0, Y_1) = \bar{t} f d_f \bar{c} \quad (\text{Eq. 6})$$

$$P(X_1, Y_1) = t(d_t c + f d_f \bar{c} (1 - c d_t)) \quad (\text{Eq. 7})$$

$$P(Y_1) = t d_t c + f d_f \bar{c} (1 - t d_t c) \quad (\text{Eq. 8})$$

$$P(X_0 | Y_1) = P(X_0, Y_1)/P(Y_1), \quad P(X_1 | Y_1) = P(X_1, Y_1)/P(Y_1) \quad (\text{Eq. 9})$$

$$P(X_0, Y_2) = \bar{t} f d_f c \quad (\text{Eq. 10})$$

$$P(X_1, Y_2) = t(\bar{f} d_t \bar{c} + f(d_f c(1 - d_t c) + \bar{d}_f d_t \bar{c})) \quad (\text{Eq. 11})$$

$$P(Y_2) = P(X_0, Y_2) + P(X_1, Y_2) \quad (\text{Eq. 12})$$

$$\begin{aligned} P(X_0 | Y_2) &= P(X_0, Y_2)/P(Y_2) \quad , \\ P(X_1 | Y_2) &= P(X_1, Y_2)/P(Y_2) \quad . \end{aligned} \quad (\text{Eq. 13})$$

The explicit expressions are readily capable of interpretation and could have been written down immediately from first principles.

Finally, the increase in information about X_j resulting from search outcome Y_k , is

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$$\log P(X_j | Y_k) - \log P(X_j) \quad (\text{Eq. 14})$$

and the mean value over all possibilities is

$$I_E = \sum_{j=0}^1 \sum_{k=0}^2 P(X_j, Y_k) \log(P(X_j | Y_k)/P(X_j)) \quad (\text{Eq. 15})$$

This will be tabulated in the next paragraph for a range of parameter values.

1.1.2 Behaviour of I_E

(a) Degradation of I_E as f increases

Intuitively one expects I_E to decrease as f increases, the values of the remaining parameters being fixed. Here is an example:

TABLE 2
 I_E AS A FUNCTION OF f

$$d_t = d_f = 0.5$$

$$c = 0.5, \quad t = 0.5$$

f	I_E
0.1	0.1547
0.3	0.1027
0.5	0.0738
0.7	0.0548
0.9	0.0413

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(b) Variation of I_E with respect to t and f

For fixed f it appears that I_E has a maximum near $t = 0.5$. This would mean that the search provides on average the greatest information gain when, a priori, no preferential statement can be made about whether or not T is present. Less information is yielded when the prior belief in the presence or absence of T is strong. This is illustrated by Table 3.

TABLE 3

I_E AS A FUNCTION OF f AND t

$$d_t = d_f = 0.5, \quad c = 0.5$$

f	0.1	0.5	0.9
t	I_E	I_E	I_E
0.1	0.0717	0.0279	0.0147
0.2	0.1167	0.0488	0.0262
0.3	0.1433	0.0632	0.0344
0.4	0.1552	0.0715	0.0395
0.5	0.1547	0.0738	0.0413
0.6	0.1431	0.0704	0.0398
0.7	0.1214	0.0613	0.0351
0.8	0.0901	0.0466	0.0270
0.9	0.0496	0.0262	0.0153

(c) Variation of I_E with respect to c and t

It appears that I_E decreases to a minimum as c increases, and that it then increases again. This can be interpreted as meaning that if one has high certainty about one's power of identification, or great uncertainty, the information gain is greater than otherwise. This statement needs to be viewed against the probability of drawing the correct conclusion from a search. This is, of course, c . Table 4 provides an illustration.

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TABLE 4

 I_E AS A FUNCTION OF t AND c

$$d_t = d_f = 0.5, \quad f = 0.5$$

$t \backslash c$	0.1	0.5	0.9
0.1	0.0573	0.0279	0.0771
0.3	0.1122	0.0632	0.1467
0.5	0.1202	0.0738	0.1545
0.7	0.0939	0.0613	0.1193
0.9	0.0383	0.0262	0.0481

(d) Variation of I_E with $d_t (=d_f)$ and c

Increase in d_t (here = d_f) procures a steady increase in I_E for each c . This is intuitively correct. Table 5 provides an illustration.

TABLE 5

 I_E AS A FUNCTION OF c AND d_t WITH $d_t = d_f$

$$f = 0.5, \quad t = 0.5$$

c	0.1	0.5	0.9
$d_t=d_f$	I_E	I_E	I_E
0.1	0.0248	0.0133	0.0259
0.5	0.1202	0.0738	0.1545
0.9	0.2255	0.1660	0.3668

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It is to be noted that increasing d_t and d_f may be interpreted as equivalent to increasing the search effort.

1.1.3 Behaviour of posterior probabilities

In addition to examining I_E we should also investigate posterior probabilities. Several of these are candidates, but one of great importance is $P(X_1 | Y_1 + Y_2)$, the probability that the state of nature is X_1 (a target is present) given that the search has yielded a detection, and whatever the conclusion that was drawn therefrom. We have

$$P(Y_1 + Y_2) = t d_t + f d_f (1 - t d_t) ,$$

which is increased by the presence of a false target and is independent of prowess in identification. Then

$$P(X_1 | Y_1 + Y_2) = \frac{t (1 - \bar{d}_t (1 - f d_f))}{t d_t + f d_f (1 - t d_t)} .$$

Before investigating this further we shall consider in Sect. 1.2 the partition of effort between two cells with the model structure of Sect. 1.1.

1.2 TWO CELL SEARCH: SIMPLE FALSE TARGET STRUCTURE

1.2.1 The model

Section 1.1 is extended to analyze the search in two cells. This is somewhat more complex and we shall, accordingly, provide a detailed account of the assumptions and arguments. The cells are designated C1 and C2 and it is supposed that C1 is the cell to be searched first: (see also (d) below).

(a) A target T is known to be either in C1 or C2. The probability that it lies in C1 is t ; consequently the probability that it is in C2 is $\bar{t} = 1 - t$.

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(b) There may be one false target (F) in either C1, C2 or both, that is, we are assuming possible false target patterns FF, $\overline{F}F$, $F\overline{F}$, $\overline{F}\overline{F}$ with respective probabilities f^2 , $\overline{f}f$, $f\overline{f}$, $\overline{f}^2$. f is the probability of a false target. Extension to an arbitrary number of false targets is the subject of Sect. 1.3.

(c) Detection and classification are dealt with by an extension of the classical theory. Let us suppose that the cell searched contains both T and F. The searching vehicle deploys a sensor which is supposed to make repeated sweeps of the area. Each time the "beam" sweeps over a target-like object constitutes a detection opportunity and, in the parlance of probability theory, is a Bernoulli trial whose outcome, by definition, is either success (detection), or failure (failure to detect). In this treatment the outcome of a trial is deemed to be independent of the outcome of previous trials. Let p_t , p_f be the probabilities of success when the object is, respectively, T and F, and let $q_t = 1-p_t$, $q_f = 1-p_f$. These should, in general, be variable from trial-to-trial but we shall treat them as fixed, equivalent to replacing a sequence of variable probabilities by a mean value.

An object which causes the sensor to register "detection" is then subjected to a classification or identification, procedure which is assumed to absorb no search effort or time. The effect of a contrary assumption is investigated elsewhere. With probability c the object is correctly identified, and with probability $\overline{c} = 1-c$ is stated to be what it is not. Thus an F which has been detected is with probability c identified as F, and with probability $\overline{c}$ is stated to be T.

1.2.2 Evaluation of probabilities

Let us suppose that the search effort applied permits N detection opportunities when a cell contains both T and F. The possible search outcomes are as follows:

- (1) T detected first, correctly classified, search stops. The search outcome is Y_1 : "T seems to have been detected".
- (2) T detected first, incorrectly classified, search continues: F detected next, correctly classified, search proceeds to C2, denoted by the symbol " $\rightarrow 2$ ".
- (3) T detected first, incorrectly classified, search continues: F detected next, incorrectly classified and thought to be T. This leads to the correct conclusion

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that T is in the cell, but the wrong contact would be prosecuted, a fact that we do not take into account, search stops (Y_1).

- (4) T detected first, incorrectly classified, search continues: F not detected ($\rightarrow 2$).
- (5) F detected first, correctly classified, search continues: T detected next, correctly classified, search stops (Y_1).
- (6) F detected first, correctly classified, search continues: T detected next, incorrectly classified ($\rightarrow 2$).
- (7) F detected first, correctly classified, search continues, T not detected ($\rightarrow 2$).
- (8) F detected first, incorrectly classified, search stops. This leads to the correct conclusion, but for the wrong reason, as in (3), search stops (Y_1).
- (9) Neither T nor F detected ($\rightarrow 2$).
- (10) Both T and F detected simultaneously. We shall see that this event can reasonably be neglected.

We now list the probabilities of (1) to (10). The assumption is that the search effort provides N detection opportunities.

Probability 1

Let T be detected at the n -th trial ($1 < n < N$). Then

$$P(1) = c \sum_{n=1}^N q_f^n q_t^{n-1} p_t = c p_t q_f \frac{(1-(q_f q_t)^N)}{1 - q_f q_t} \quad (\text{Eq. 16})$$

Probability 2

Let T be detected at the n -th trial ($1 < n < N$), and F at the $(n+r)$ -th trial ($1 < r < N-n$). Then

$$\begin{aligned} P(2) &= \sum_{n=1}^N q_f^n q_t^{n-1} p_t \bar{c} \sum_{r=1}^{N-n} q_f^{r-1} p_f c \\ &= q_f p_t \bar{c} \sum_{n=1}^N (q_f q_t)^{n-1} (1 - q_f^{N-n}) \\ &= q_f p_t \bar{c} \left[\frac{1 - (q_f q_t)^N}{1 - q_f q_t} - q_f^{N-1} \cdot \frac{1 - q_t^N}{1 - q_t} \right] \quad (\text{Eq. 17}) \end{aligned}$$

Probability 3

$$P(3) = q_f p_t \bar{c}^2 \left[\frac{1-(q_f q_t)^N}{1-q_f q_t} - q_f^{N-1} \cdot \frac{1-q_t^N}{1-q_t} \right] \quad (\text{Eq. 18})$$

Probability 4

$$P(4) = \sum_{n=1}^N q_f^n q_t^{n-1} p_t \bar{c} q_f^{N-n} = q_f^N \bar{c} (1-q_t^N) \quad (\text{Eq. 19})$$

Probability 5

$$\begin{aligned} P(5) &= \sum_{n=1}^N q_t^n q_f^{n-1} p_f c \sum_{r=1}^{N-n} q_t^{r-1} p_t c \\ &= q_t p_f c^2 \left[\frac{1-(q_f q_t)^N}{1-q_t q_f} - q_t^{N-1} \frac{1-q_f^N}{1-q_f} \right] \end{aligned} \quad (\text{Eq. 20})$$

Probability 6

$$P(6) = q_t p_f c \bar{c} \left[\frac{1-(q_f q_t)^N}{1-q_t q_f} - q_t^{N-1} \frac{1-q_f^N}{1-q_f} \right] \quad (\text{Eq. 21})$$

Probability 7

$$P(7) = \sum_{n=1}^N q_t^n q_f^{n-1} p_f c q_t^{N-n} = q_t^N c (1-q_f^N) \quad (\text{Eq. 22})$$

Probability 8

$$P(8) = \bar{c} \sum_{n=1}^N q_t^n q_f^{n-1} p_f = q_t \bar{c} p_f \frac{1-(q_t q_f)^N}{1-q_t q_f} \quad (\text{Eq. 23})$$

Probability 9

$$P(9) = (q_t q_f)^N \quad (\text{Eq. 24})$$

Probability 10

$$P(10) = \sum_{n=1}^N (q_t q_f)^{n-1} p_t p_f = p_t p_f \frac{(1-(q_t q_f)^N)}{(1-q_t q_f)} \quad (\text{Eq. 25})$$

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It is usual to suppose that the sensor scans fast, at time intervals h , say. If total finite time (or effort) z is allocated to the search, we have $z = Nh$. Since N Bernoulli trials with success probability p yield an average of Np successes, it is necessary to suppose p to be of the form $\alpha h + o(h)$ in order to have a finite mean as $N \rightarrow \infty$. This means that the probability of N failures, q^N , is given by $(1-\alpha h)^N = (1-\alpha h)^{z/h} \rightarrow \exp(-\alpha z)$ as $h \rightarrow 0$. On this basis we can now rewrite the probabilities $P(1)$ to $P(10)$ in exponential form. First we put

$$p_t = \alpha h + o(h), \quad p_f = \beta h + o(h). \quad (\text{Eq. 26})$$

Then, for example,

$$P(1) = c\alpha h(1 - \beta h) \left[\frac{1 - e^{-(\alpha+\beta)z}}{1 - (1-\alpha h)(1-\beta h)} \right] + o(h) \rightarrow \frac{c\alpha(1 - e^{-(\alpha+\beta)z})}{(\alpha+\beta)} \quad (\text{Eq. 27})$$

as $h \rightarrow 0$.

This and the other approximations to $P(n)$ can be obtained directly by supposing that T and F are members of two separate and independent Poisson streams with mean rates α and β respectively.

Then

$$P(1) = \alpha c \int_0^z e^{-\alpha s} e^{-\beta s} ds = \frac{\alpha c}{(\alpha+\beta)} (1 - e^{-(\alpha+\beta)z}). \quad (\text{Eq. 28})$$

Similarly we get

$$P(2) = \alpha c \bar{c} \left[\frac{1 - e^{-(\alpha+\beta)z}}{\alpha+\beta} - \frac{e^{-\beta z}(1 - e^{-\alpha z})}{\alpha} \right] \quad (\text{Eq. 29})$$

$$P(3) = \alpha c \bar{c}^2 \left[\frac{1 - e^{-(\alpha+\beta)z}}{\alpha+\beta} - \frac{e^{-\beta z}(1 - e^{-\alpha z})}{\alpha} \right] \quad (\text{Eq. 30})$$

$$P(4) = \bar{c} e^{-\beta z} (1 - e^{-\alpha z}) \quad (\text{Eq. 31})$$

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$$P(5) = \beta c^2 \left[\frac{1 - e^{-(\alpha+\beta)z}}{\alpha+\beta} - \frac{e^{-\alpha z}(1 - e^{-\beta z})}{\beta} \right] \quad (\text{Eq. 32})$$

$$P(6) = \beta c \bar{c} \left[\frac{1 - e^{-(\alpha+\beta)z}}{\alpha+\beta} - \frac{e^{-\alpha z}(1 - e^{-\beta z})}{\beta} \right] \quad (\text{Eq. 33})$$

$$P(7) = c e^{-\alpha z} (1 - e^{-\beta z}) \quad (\text{Eq. 34})$$

$$P(8) = \beta \bar{c} \left[\frac{1 - e^{-(\alpha+\beta)z}}{\alpha+\beta} \right] \quad (\text{Eq. 35})$$

$$P(9) = e^{-(\alpha+\beta)z} \quad (\text{Eq. 36})$$

Finally $P(10) = o(h)$ and is accordingly negligible in this approximation.

In the foregoing we have considered in detail the case where the first cell searched contains T and F. If it contains nothing the action must be to continue the search in another cell, or agree to an inconclusive outcome of the search.

The remaining cases are:

CELL CONTENT	OUTCOME	PROBABILITY	CLASSIFICATION	ACTION
T	Detected	$1 - e^{-\alpha z}$	Correct (c)	Stop
T	Detected	$1 - e^{-\alpha z}$	Incorrect ($\bar{c}$)	Continue
T	Not detected	$e^{-\alpha z}$	---	Continue
F	Detected	$1 - e^{-\beta z}$	Correct (c)	Continue
F	Detected	$1 - e^{-\beta z}$	Incorrect ($\bar{c}$)	Stop
F	Not detected	$e^{-\beta z}$	---	Continue

(d) The final assumption of this analysis is a search protocol. We have to designate a cell to be searched first and that will be defined to

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be cell 1 (C1). This is implicit in the foregoing analysis. To C1 will be allocated search effort z : to the other, $\bar{z} = 1-z$. One of our problems is the optimal allocation of z . The search procedure is as follows: C1 is searched and if a target is detected, or thought to be detected, the search stops, rightly or wrongly, with the conclusion that the state of nature is X_1 : = the target is in C1, whether or not there is a false target as well. To complement this is the state X_2 of nature stating the corresponding thing about C2.

If the search in C1 yields no detection at all, or the detection, apparent or otherwise, of a false target, the search continues in C2. The searcher expects to find the target in C2 and perhaps a false target as well.

At the end of the search the following situations may obtain:

- Y_0 : the target is not known to have been detected. One or more false targets may have been detected and correctly identified, or the target may itself have been detected, but misclassified.
- Y_1 : an object has been declared to be the genuine target and the cell in which it has been identified is the correct one, even if for the wrong reason (misidentification of a false target).
- Y_2 : as above, but the supposed target is declared to be in the cell where it is not.

The possible configurations awaiting the searcher are shown in the following table:

CELL 1	PROBABILITY	CELL 2	PROBABILITY
TF	tf	F $\bar{T}$	f
TF	tf	$\bar{F}$ $\bar{T}$	$\bar{f}$
$\bar{T}\bar{F}$	$t\bar{f}$	F $\bar{T}$	f
$\bar{T}\bar{F}$	$t\bar{f}$	$\bar{F}$ $\bar{T}$	$\bar{f}$
F $\bar{T}$	f	TF	$\bar{t}f$
$\bar{F}$ $\bar{T}$	$\bar{f}$	TF	$\bar{t}f$
F $\bar{T}$	f	$\bar{T}\bar{F}$	$\bar{t} \bar{f}$
$\bar{F}$ $\bar{T}$	$\bar{f}$	$\bar{T}\bar{F}$	$\bar{t} \bar{f}$
Search effort	z	$\bar{z} = 1-z$	

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1.2.3 Detailed enumeration

It is worthwhile, if tedious, to list all the possible cases and to indicate to which outcomes Y_k ($k = 0, 1, 2$) they lead. To preserve generality we shall write $\alpha = \alpha_1$, $\beta = \beta_1$ for C1, with subscript 2 for C2. Moreover let

$$d_t = 1 - e^{-\alpha_1 Z}, \quad d'_t = 1 - e^{-\alpha_2 \bar{Z}}, \quad d_f = 1 - e^{-\beta_1 Z}, \quad d'_f = 1 - e^{-\beta_2 \bar{Z}},$$

$$g = \frac{\alpha_1}{\alpha_1 + \beta_1}, \quad g' = \frac{\alpha_2}{\alpha_2 + \beta_2}.$$

The following diagrams show all the possible cases, probabilities and outcomes.

CELL 1 (C1)				CELL 2 (C2)			
Content	Case	Probability	Search outcome in C1	Content	Case	Probab.	Search outcome in C2
TF	1	$tf[g(1-\bar{d}_t \bar{d}_f) c$	Y_1	F or $\bar{F}$	(a')	$f d'_f c$	Y_0
TF	2	$+ c\bar{c}\{g(1-\bar{d}_t \bar{d}_f) - \bar{d}_f d_t\}$	+ 2	F or $\bar{F}$	(b')	$f d'_f \bar{c}$	Y_2
TF	3	$+ c^2\{g(1-\bar{d}_t \bar{d}_f) - \bar{d}_f d_t\}$	Y_1	F or $\bar{F}$	(c')	$f \bar{d}'_f$	Y_0
TF	4	$+ \bar{c} \bar{d}_f d_t$	+ 2	F or $\bar{F}$	(d')	$\bar{f}$	Y_0
TF	5	$+ c^2\{g(1-\bar{d}_t \bar{d}_f) - \bar{d}_t d_f\}$	Y_1	F or $\bar{F}$			
TF	6	$+ c\bar{c}\{g(1-\bar{d}_t \bar{d}_f) - \bar{d}_t d_f\}$	+ 2				
TF	7	$+ c \bar{d}_t d_f$	+ 2				
TF	8	$+ \bar{c} \bar{g}(1-\bar{d}_t \bar{d}_f)$	Y_1				
TF	9	$+ \bar{d}_t \bar{d}_f]$	+ 2				
$T\bar{F}$	10	$+ t\bar{f} [d_t c$	Y_1				
$T\bar{F}$	11	$+ d_t \bar{c}$	+ 2				
$T\bar{F}$	12	$+ \bar{d}_t]$	+ 2				

Notes: (1) In Cell 1 (C1) cases 1 to 9 are described in words on pages 12 and 13.

Cases 10, 11, 12 correspond to T in C1 but no F. The description and outcomes are easily inferred from the probabilities.

(2) "+ 2" indicates that search was inconclusive in C1 and passes therefore to C2.

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The next diagram is the mirror image of the first and deals with the case where C1, by protocol searched first, contains F or $\bar{F}$, while T is in C2 with or without an F.

CELL 1 (C1)				CELL 2 (C2)			
Content	Case	Probability	Search Outcome in C1	Content	Case	Probability	Search Outcome in C2
F or $\bar{F}$	a	$f d_f c$	+ 2	TF	1'	$\bar{t}f[g'(1-\bar{d}'_t \bar{d}'_f)c$	Y_1
F or $\bar{F}$	b	$f d_f \bar{c}$	Y_2	TF	2'	$+ c\bar{c}[g'(1-\bar{d}'_t \bar{d}'_f)-\bar{d}'_f d'_t]$	Y_0
F or $\bar{F}$	c	$f \bar{d}_f$	+ 2	TF	3'	$+ \bar{c}^2[g'(1-\bar{d}'_t \bar{d}'_f)-\bar{d}'_f d'_t]$	Y_1
F or $\bar{F}$	d	$\bar{F}$	+ 2	TF	4'	$+ \bar{c} \bar{d}'_f d'_t$	Y_0
				TF	5'	$+ c^2[\bar{g}'(1-\bar{d}'_t \bar{d}'_f)-\bar{d}'_t d'_f]$	Y_1
				TF	6'	$+ c\bar{c}[\bar{g}'(1-\bar{d}'_t \bar{d}'_f)-\bar{d}'_t d'_f]$	Y_0
				TF	7'	$+ c \bar{d}'_t d'_f$	Y_0
				TF	8'	$+ \bar{c} \bar{g}'(1-\bar{d}'_t \bar{d}'_f)$	Y_1
				TF	9'	$+ \bar{d}'_t \bar{d}'_f]$	Y_0
				TF	10'	$+ \bar{t} \bar{f}[d'_t c$	Y_1
				TF	11'	$+ d'_t \bar{c}$	Y_0
				TF	12'	$+ \bar{d}'_t]$	Y_0

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The advantage of these diagrams is that it shows the construction of the probabilities and, moreover, enables simple checks to be made for consistency. Using the diagram we now list $P(Y_0)$, $P(Y_1)$, $P(Y_2)$:

$$P(Y_0) = t(1-f d'_f \bar{c}) B_{00} + \bar{t}(1-f d_f \bar{c}) B_{01}$$

$$P(Y_1) = t B_{10} + \bar{t}(1-f d_f \bar{c}) B_{11}$$

$$P(Y_2) = tf d'_f c B_{20} + \bar{t} B_{21}$$

where

$$B_{00} = f\{1-\bar{d}_t \bar{d}_f\} c\bar{c} + \bar{c}^2 \bar{d}_f d_t + c^2 \bar{d}_t d_f + \bar{d}_t \bar{d}_f\} + \bar{f}(1-d_t c)$$

$$B_{01} = B'_{00} \text{ (that is, } B_{00} \text{ with primable entities primed)}$$

$$B_{10} = f\{(1-\bar{d}_t \bar{d}_f)(c + \bar{c}^2) - \bar{d}_f d_t \bar{c}^2 - \bar{d}_t d_f c^2\} + \bar{f} d_t c$$

$$B_{11} = B'_{10}$$

$$B_{20} = B_{00}$$

$$B_{21} = f d_f \bar{c} \quad (\text{Eq. 37})$$

As a check we see readily that the coefficient of t in $P(Y_0) + P(Y_1) + P(Y_2)$ is:

$$(1-f d'_f \bar{c}) B_{00} + B_{10} + tf d'_f c B_{20} = B_{00} + B_{10} = f(1-\bar{d}_t \bar{d}_f + \bar{d}_t \bar{d}_f) + \bar{f} = 1,$$

and similarly for the coefficient of $\bar{t}$.

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1.2.4 Numerical calculations

A program was written to calculate $P(X_1 | Y_0)$ (the posterior probability that the target is in C1, given an inconclusive outcome of the search in both cells); $P(X_1 | Y_0)$ (the posterior probability of X_1 given that the search yielded, rightly, or wrongly, a detection identified as a target; $P(Y_1)$ the probability that the conclusion drawn from the search about which cell the target is in, is correct, whether for the right or wrong reason; $P(Y_0)$, the probability that the search was inconclusive; I_E , the average increase in information yielded by the search and calculated over the set Y_0, Y_1, Y_2 of search outcomes and states X_1, X_2 of nature. This is somewhat different from calculations in Sect. 1.1.

To check the program and also to compare with "Mela's first example", we tabulate values for $f = 0.001$ and $c = 0.999$, that is practically no chance of false contact and almost 100 percent assurance of correct classification. The numerical values can therefore be expected to be very close.

TABLE 6

COMPARISON WITH MELA'S FIRST EXAMPLE
Cited by Pierce <5>

z	$P(X_1 Y_0)$	$P(X_1 Y_0)$	$P(Y_1)$	$P(Y_0)$	I_E
0.1	0.6350	0.1261	0.2653	0.7347	0.1105
0.3	0.5688	0.3281	0.2858	0.7142	0.0240
0.5	0.5	0.5	0.2926	0.7074	0
0.7	0.4312	0.6719	0.2858	0.7142	0.0240
0.9	0.3650	0.8739	0.2653	0.7347	0.1105

Although Y_0 and Y_1 have different meanings in the two models, the difference is not significant in this case: the values can be compared directly and the agreement noted.

The value of I_E reflects our choice of search outcomes.

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TABLE 7
t = 0.1

f = 0.1

z	c = 0.1			c = 0.5			c = 0.9		
	$P(X_1 Y_0)$	$P(X_1 \bar{Y}_0)$	$P(Y_1)$	$P(X_1 Y_0)$	$P(X_1 \bar{Y}_0)$	$P(Y_1)$	$P(X_1 Y_0)$	$P(X_1 \bar{Y}_0)$	$P(Y_1)$
0.1	0.1037	0.0613	0.0784	0.9120	0.0052	0.1227	0.0253	0.2278	0.7670
					I_E				I_E
					$P(Y_0)$				$P(Y_0)$
0.3	0.1018	0.0804	0.0670	0.9145	0.0015	0.1108	0.0582	0.1954	0.7944
0.5	0.1000	0.1000	0.0536	0.9202	0	0.1000	0.1000	0.1568	0.8287
0.7	0.0982	0.1238	0.0379	0.9294	0.0020	0.0901	0.1666	0.1111	0.8708
0.9	0.0964	0.1590	0.0196	0.9423	0.0100	0.0811	0.3219	0.0573	0.9216

f = 0.5

z	c = 0.1			c = 0.5			c = 0.9		
	$P(X_1 Y_0)$	$P(X_1 \bar{Y}_0)$	$P(Y_1)$	$P(X_1 Y_0)$	$P(X_1 \bar{Y}_0)$	$P(Y_1)$	$P(X_1 Y_0)$	$P(X_1 \bar{Y}_0)$	$P(Y_1)$
0.1	0.1037	0.0897	0.2180	0.7348	0.0294	0.1227	0.0507	0.2892	0.6847
					I_E				I_E
					$P(Y_0)$				$P(Y_0)$
0.3	0.1018	0.0950	0.1789	0.7295	0.0043	0.1108	0.0742	0.2451	0.7044
0.5	0.1000	0.1000	0.1386	0.7317	0.0003	0.1000	0.1000	0.1952	0.7331
0.7	0.0982	0.1052	0.0962	0.7414	0.0092	0.0901	0.1335	0.1379	0.7721
0.9	0.0964	0.1114	0.0506	0.7591	0.0370	0.0811	0.1877	0.0716	0.8228

f = 0.9

z	c = 0.1			c = 0.5			c = 0.9		
	$P(X_1 Y_0)$	$P(X_1 \bar{Y}_0)$	$P(Y_1)$	$P(X_1 Y_0)$	$P(X_1 \bar{Y}_0)$	$P(Y_1)$	$P(X_1 Y_0)$	$P(X_1 \bar{Y}_0)$	$P(Y_1)$
0.1	0.1037	0.0951	0.3507	0.5652	0.0489	0.1227	0.6530	0.3488	0.6045
					I_E				I_E
					$P(Y_0)$				$P(Y_0)$
0.3	0.1018	0.0976	0.2746	0.5626	0.0054	0.1108	0.0824	0.2905	0.6191
0.5	0.1000	0.1000	0.2041	0.5648	0.0016	0.1000	0.1000	0.2282	0.6434
0.7	0.0982	0.1024	0.1379	0.5718	0.0210	0.0901	0.1209	0.1600	0.6786
0.9	0.0964	0.1051	0.0745	0.5839	0.0703	0.0811	0.1501	0.0838	0.7264

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TABLE 8

t = 0.5

f = 0.1

z	$\frac{c = 0.1}{c = 0.5}$				$\frac{c = 0.9}{c = 0.5}$			
	$P(X_1 Y_0)$	$P(X_1 \bar{Y}_0)$	$P(Y_1)$	$P(Y_0)$	I_E	$P(X_1 Y_0)$	$P(X_1 \bar{Y}_0)$	$P(Y_1)$
0.1	0.5102	0.3702	0.0492	0.9272	0.0228	0.5772	0.1897	0.1429
0.3	0.5051	0.4402	0.0529	0.9219	0.0044	0.5287	0.3573	0.1541
0.5	0.5000	0.5000	0.0542	0.9202	0	0.5000	0.5000	0.1578
0.7	0.4949	0.5598	0.0529	0.9219	0.0053	0.4713	0.6427	0.1541
0.9	0.4898	0.6298	0.0492	0.9272	0.0238	0.4428	0.8103	0.1429

f = 0.5

z	$\frac{c = 0.1}{c = 0.5}$				$\frac{c = 0.9}{c = 0.5}$			
	$P(X_1 Y_0)$	$P(X_1 \bar{Y}_0)$	$P(Y_1)$	$P(Y_0)$	I_E	$P(X_1 Y_0)$	$P(X_1 \bar{Y}_0)$	$P(Y_1)$
0.1	0.5102	0.4699	0.1374	0.7469	0.0769	0.5572	0.3248	0.1827
0.3	0.5051	0.4859	0.1446	0.7355	0.0118	0.5287	0.4191	0.1968
0.5	0.5000	0.5000	0.1469	0.7317	0.0008	0.5000	0.5000	0.2013
0.7	0.4949	0.5141	0.1444	0.7355	0.0251	0.4713	0.5809	0.1966
0.9	0.4898	0.5301	0.1373	0.7469	0.0921	0.4428	0.6752	0.1825

f = 0.9

z	$\frac{c = 0.1}{c = 0.9}$				$\frac{c = 0.9}{c = 0.9}$			
	$P(X_1 Y_0)$	$P(X_1 \bar{Y}_0)$	$P(Y_1)$	$P(Y_0)$	I_E	$P(X_1 Y_0)$	$P(X_1 \bar{Y}_0)$	$P(Y_1)$
0.1	0.5102	0.4862	0.2217	0.5746	0.1267	0.5572	0.3861	0.2216
0.3	0.5051	0.4934	0.2272	0.5677	0.0149	0.5287	0.4470	0.2372
0.5	0.5000	0.5000	0.2288	0.5648	0.0044	0.5000	0.5000	0.2419
0.7	0.4949	0.5066	0.2268	0.5672	0.0567	0.4713	0.5530	0.2364
0.9	0.4898	0.5138	0.2213	0.5746	0.1740	0.4428	0.6139	0.2210

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TABLE 9

t = 0.9

f = 0.1

z	c = 0.1				c = 0.5				c = 0.9			
	$P(X_1 Y_0)$	$P(X_1 \bar{Y}_0)$	$P(Y_1)$	$P(Y_0)$	I_E	$P(X_1 Y_0)$	$P(X_1 \bar{Y}_0)$	$P(Y_1)$	$P(Y_0)$	I_E	$P(X_1 Y_0)$	$P(X_1 \bar{Y}_0)$
0.1	0.9036	0.8410	0.0200	0.9423	0.0096	0.9189	0.6781	0.0579	0.9216	0.0243	0.9356	0.5826
0.3	0.9018	0.8762	0.0389	0.9294	0.0017	0.9099	0.8334	0.1127	0.8708	0.0042	0.9196	0.8172
0.5	0.9000	0.9000	0.0547	0.9202	0	0.9000	0.9000	0.1587	0.8287	0	0.9000	0.9000
0.7	0.8982	0.9196	0.0679	0.9145	0.0019	0.8892	0.9418	0.1970	0.7944	0.0042	0.8763	0.9477
0.9	0.8963	0.9387	0.0788	0.9120	0.0086	0.8773	0.9747	0.2285	0.7670	0.0180	0.8480	0.9831

f = 0.5

z	c = 0.1				c = 0.5				c = 0.9			
	$P(X_1 Y_0)$	$P(X_1 \bar{Y}_0)$	$P(Y_1)$	$P(Y_0)$	I_E	$P(X_1 Y_0)$	$P(X_1 \bar{Y}_0)$	$P(Y_1)$	$P(Y_0)$	I_E	$P(X_1 Y_0)$	$P(X_1 \bar{Y}_0)$
0.1	0.9036	0.8886	0.8568	0.7591	0.0304	0.9189	0.8123	0.8763	0.8228	0.0332	0.9356	0.6389
0.3	0.9018	0.8948	0.1102	0.7414	0.0043	0.9099	0.8665	0.1485	0.7721	0.0050	0.9186	0.8262
0.5	0.9000	0.9000	0.1552	0.7317	0.0003	0.9000	0.9000	0.2074	0.7331	0.0002	0.9000	0.9000
0.7	0.8982	0.9050	0.1927	0.7295	0.0092	0.8892	0.9258	0.2552	0.7044	0.0082	0.8763	0.9446
0.9	0.8963	0.9103	0.2239	0.7348	0.0361	0.8773	0.9493	0.2934	0.6847	0.0323	0.8480	0.9786

f = 0.9

z	c = 0.1				c = 0.5				c = 0.9			
	$P(X_1 Y_0)$	$P(X_1 \bar{Y}_0)$	$P(Y_1)$	$P(Y_0)$	I_E	$P(X_1 Y_0)$	$P(X_1 \bar{Y}_0)$	$P(Y_1)$	$P(Y_0)$	I_E	$P(X_1 Y_0)$	$P(X_1 \bar{Y}_0)$
0.1	0.9036	0.8949	0.0928	0.5839	0.0495	0.9189	0.2499	0.0944	0.7264	0.0416	0.9356	0.6796
0.3	0.9018	0.8976	0.1798	0.5718	0.0054	0.9099	0.8791	0.1838	0.6786	0.0054	0.9196	0.8337
0.5	0.9000	0.9000	0.2534	0.5648	0.0016	0.9000	0.9000	0.2555	0.6414	0.0007	0.9000	0.9000
0.7	0.5152	0.9024	0.3156	0.5626	0.0211	0.8892	0.9176	0.3128	0.6191	0.0134	0.8763	0.0417
0.9	0.8943	0.9049	0.3682	0.5652	0.0700	0.8773	0.9347	0.3581	0.06045	0.0483	0.8480	0.9745

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1.2.5 Comments

Tables 7, 8, 9 are concerned respectively with the cases $t = 0.1, 0.5$ and 0.9 . In all cases it has been assumed that $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0.6931$. For the effect of different detection performance in the cells there is another set of tables available which are not given here.

The tables are set out in such a way that the effects of varying f and c are rendered as obvious as possible. Recall that t is the prior probability that the target is in C1 and that the independent variable z is the fraction of the total search effort available in C1. Let us examine briefly the columns of the tables in turn.

(1) $P(X_1 | Y_0)$

(i) This is seen to be independent of f , for given t and c .

This may be something of a surprise. However, even taking different $\alpha_1, \alpha_2, \beta_1, \beta_2$ we find that

$$P(X_1 | Y_0) = \frac{t(1 - cd_t)}{t(1 - cd_t) + \bar{t}(1 - cd'_t)}$$

where $\bar{d}_t = e^{-\alpha_1 z}$, $\bar{d}'_t = \exp(-\alpha_2(1-z))$. Thus, not only is $P(X_1 | Y_0)$ independent of f , but also of β_1 and β_2 which relate to the detection of false targets.

(ii) $P(X_1 | Y_0)$ increases with c for $z < 0.5$, and decreases with c for $z > 0.5$. For $z = 0.5$ it is independent of c and indeed is equal to t . This is a special case. From the general formula we see that $P(X_1 | Y_0)$ is independent of c for z such that $d_t = d'_t$, that is

$$e^{-\alpha_1 z} = e^{-\alpha_2 + \alpha_2 z},$$

that is where

$$z = \alpha_2 / (\alpha_1 + \alpha_2).$$

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The value of $P(X_1 | Y_0)$ is then equal to the prior value t . In the particular case $\alpha_1 = \alpha_2$, $z = 0.5$.

(2) $P(X_1 | \bar{Y}_0)$

(i) As z increases this posterior increases through the prior value t for given f and c . Here the prior value occurs where $z = 0.5$.

(ii) For fixed f it decreases as c increases for $z < 0.5$, and increases as c increases for $z > 0.5$.

(iii) For fixed c it increases with f for $z < 0.5$ and decreases for $z > 0.5$.

(3) $P(Y_1)$

is the probability that the search concludes with the correct identification of which cell the target is in. One might expect it to increase with c for fixed f and z . However, it is possible to arrive at the correct conclusions for the wrong reason, by identifying a false target as a true one. This seems more likely to happen when the false target probability is high, a tendency which might be corrected by classification capability. In fact, one observes when most effort is put into searching the cell in which the target is most likely to be a priori, that when f is high, $P(Y_1)$ decreases to a minimum and then increases again as c increases. Thus, in some conditions, to increase c does not necessarily increase $P(Y_1)$.

(4) $P(Y_0)$

(i) This is the probability that the search ends without the target having apparently been detected. In all cases given, $P(Y_0) + P(Y_1)$ is nearly unity and this means that $P(Y_2)$, the probability that a target is apparently identified and stated to be in the wrong cell, is rather small.

(ii) As a function of z , $P(Y_0)$, sometimes goes to a minimum. The associated value of z may have operational importance in the design of a search.

(iii) As a function of c , $P(Y_0)$ sometimes has a maximum.

(iv) To increase f decreases $P(Y_0)$, as would be expected.

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(5) I_E

(i) As a function of z the average gain I_E in information yielded by the search characteristically decreases to a minimum as z increases, and then increases. It seems always least in the tables in the neighbourhood of $z = 0.5$.

(ii) I_E increases as f increases.

(iii) Increasing c has complex effects and needs closer investigation.

(6) Symmetry for $t = 0.5$

Not surprisingly, there are many symmetries to be seen. Note also that Tables 7 and 9 present certain antisymmetries, as would be expected. For example, for $c = f = 0.1$, $P(X_1 | Y_0)$ for $t = 0.1$ and given z , has the value $1 - P(X_1 | Y_0)$ for $t = 0.9$ and $1 - z$. Similar observations can be made about the remaining entities.

In the next Sect. 1.3 the attempt will be made to furnish an analysis that will enable calculations similar to the above to be made for arbitrary numbers of false targets in either of the two cells.

1.3 TWO CELL SEARCH: SINGLE TARGET AND ARBITRARY NUMBERS OF FALSE TARGETS

1.3.1 Basic assumptions

In this part it will be assumed that false contacts in an area constitute a spatial Poisson process whose parameter depends on the area and not on time. Writing the parameters of C_1 , C_2 , a_1 , a_2 respectively, we have

$$f_n = e^{-a_1} a_1^n / n! , \quad f'_m = e^{-a_2} a_2^m / m! \quad (m, n \geq 0) , \quad (\text{Eq. 38})$$

where f_n is the probability of n false targets in C_1 , f'_m that of m false targets in C_2 . Thus, in Sect. 1.2 we have put $a_1 = a_2 = a$, say, and $\bar{f} = f_0 = e^{-a}$, $f = 1 - e^{-a}$, where a is such that the probability of the

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presence of more than one false target is negligible. This can be expressed by saying that in Sect. 1.2 a has the property that $e^{-a}(1+a) > 0.99$, or, roughly, $a < 0.1$.

The next major step in the analysis concerns the detection process. As seen in Sect. 1.2, the analysis requires apparently an ordering of the detection opportunities given to the sensor by the (assumed single) target and the train of false targets existing in the particular cell under search. Thus, if the cell contains a target (T) and n false targets F_r ($r = 1, 2, \dots, n$), there are $n+1$ possible orderings corresponding to placing T before F_1 , after F_n , or in the $n-1$ gaps between successive F_r . In the case of $n = 3$ we have

$$(1) \quad TF_1F_2F_3 \quad (2) \quad F_1TF_2F_3 \quad (3) \quad F_1F_2TF_3 \quad (4) \quad F_1F_2F_3T$$

The contribution of each of these configurations to the possible search outcomes Y_0, Y_1, Y_2 , as defined in Sect. 1.2 may seem to be different.

To deal with the actual detection process it is convenient to extend the concept, also hinted at in Sect. 1.2, that the time from the beginning of the search required to detect an object is an exponential random variable with mean $1/\alpha$ in the case of T , and $1/\beta$ in the case of F . Let s be the time to detection of the first object encountered, $s+u_1$ to the second, ..., $s+u_1+u_2+\dots+u_{n-1}$ to the n -th. Then the joint probability density (JPD) of times to detection of configurations (JPD 1) to (JPD 4) above are:

$$\underline{\text{JPD 1}} : \quad 3! \alpha e^{-\alpha s} \beta e^{-\beta(s+u_1)} \beta e^{-\beta(s+u_1+u_2)} \beta e^{-\beta(s+u_1+u_2+u_3)}$$

$$\underline{\text{JPD 2}} : \quad 3! \beta e^{-\beta s} \alpha e^{-\alpha(s+u_1)} \beta e^{-\beta(s+u_1+u_2)} \beta e^{-\beta(s+u_1+u_2+u_3)}$$

$$\underline{\text{JPD 3}} : \quad 3! \beta e^{-\beta s} \beta e^{-\beta(s+u_1)} \alpha e^{-\alpha(s+u_1+u_2)} \beta e^{-\beta(s+u_1+u_2+u_3)}$$

$$\underline{\text{JPD 4}} : \quad 3! \beta e^{-\beta s} \beta e^{-\beta(s+u_1)} \beta e^{-\beta(s+u_1+u_2)} \alpha e^{-\alpha(s+u_1+u_2+u_3)}$$

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The factorial is required because the F's may be ordered in $3!$ ways. The ranges of s, u_1, u_2, u_3 are all $(0, \infty)$ here, and to show that all possibilities have been taken into account we integrate out over these variables and show that this sum is unity.

JPD 1 gives

$$3! \alpha \beta^3 \int_0^\infty e^{-(\alpha+3\beta)s} ds \int_0^\infty e^{-3\beta u_1} du_1 \int_0^\infty e^{-2\beta u_2} du_2 \int_0^\infty e^{-\beta u_3} du_3 = \alpha/(\alpha+3\beta)$$

JPD 2 gives

$$3! \alpha \beta^3 \int_0^\infty e^{-(\alpha+3\beta)s} ds \int_0^\infty e^{-(\alpha+2\beta)u_1} du_1 \int_0^\infty e^{-2\beta u_2} du_2 \int_0^\infty e^{-\beta u_3} du_3 = \frac{3\alpha\beta}{(\alpha+3\beta)(\alpha+2\beta)}$$

JPD 3 gives

$$3! \alpha \beta^3 \int_0^\infty e^{-(\alpha+3\beta)s} ds \int_0^\infty e^{-(\alpha+2\beta)u_1} du_1 \int_0^\infty e^{-(\alpha+\beta)u_2} du_2 \int_0^\infty e^{-\beta u_3} du_3 = \frac{3! \alpha \beta^3}{(\alpha+3\beta)(\alpha+2\beta)(\alpha+\beta)}$$

JPD 4 gives

$$3! \alpha \beta^3 \int_0^\infty e^{-(\alpha+3\beta)s} ds \int_0^\infty e^{-(\alpha+2\beta)u_1} du_1 \int_0^\infty e^{-(\alpha+\beta)u_2} du_2 \int_0^\infty e^{-\alpha u_3} du_3 = \frac{3! \beta^3}{(\alpha+3\beta)(\alpha+2\beta)(\alpha+\beta)}$$

Then

$$(\text{JPD 3}) + (\text{JPD 4}) = 3! \beta^2 / ((\alpha+3\beta)(\alpha+2\beta))$$

$$(\text{JPD 2}) + (\text{JPD 3}) + (\text{JPD 4}) = 3\beta / (\alpha+3\beta)$$

$$(\text{JPD 1}) + (\text{JPD 2}) + (\text{JPD 3}) + (\text{JPD 4}) = 1$$

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1.3.2 Calculation of $P(Y_0)$, $P(Y_1)$, $P(Y_2)$

The criterion by which we determine how many targets are detected during a search of duration t (this is the "effort", previously denoted by z) is then merely how many of the particular ordering under consideration fall within the interval $(0, t)$. If none falls in $(0, t)$, no detection can take place and search would then pass to the next cell, or a contribution to Y_0 would have been made.

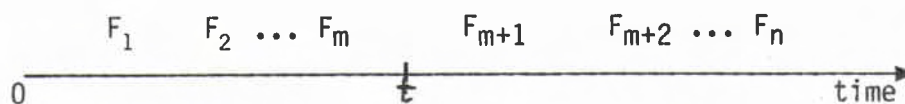
It is convenient for the development of $P(Y_0)$, $P(Y_1)$, $P(Y_2)$ to consider the contributions made by search in

- (i) a cell containing F 's and no T
- (ii) a cell containing F 's and one T

- (i) Search for time t in a cell containing $F_1, F_2, \dots, F_n$ and no T

We shall find the contribution to $P(Y_0)$. Recall that Y_0 is the outcome of a search which states that no target, apparently, has been detected. The alternative to Y_0 in a cell containing F 's only is then a contribution to Y_2 , the outcome of a search which states that a target is present when it really is not.

The times to detection from the beginning of search will in general be such that there exist m detection opportunities, $m = 0, 1, 2, \dots, n$ ($n > 1$). This can be visualized as a sequence of m points F_r on a time axis to the left of the epoch t which denotes the end of the search, and $n - m$ points to the right of this epoch.



The probability of this distribution is

$$\binom{n}{m} \gamma^m \bar{\gamma}^{n-m}$$

where

$$\gamma = 1 - e^{-\beta t},$$

$$\bar{\gamma} = 1 - \gamma = e^{-\beta t}$$

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and the contribution to $P(Y_0)$ is the above multiplied by c^m , since all of $F_1, F_2, \dots, F_m$ have to be classified correctly. The order of occurrence is irrelevant. Thus, the contribution to $P(Y_0)$ for $n > 1$ is

$$\sum_{m=0}^n \binom{n}{m} (\gamma c)^m \bar{\gamma}^{n-m} = (\bar{\gamma} + \gamma c)^n = (1 - c\bar{\gamma})^n .$$

The interpretation of the last form is that it is the probability of the complement of the event that each F is detected and misclassified. The contribution to $P(Y_0)$ for $n = 0$ is, of course, simply the probability that no F is present. Then, deconditioning with respect to n , we have for the contribution to $P(Y_0)$

$$e^{-a\bar{C}} \quad (\text{Eq. 39})$$

where

$$\bar{C} = c\bar{\gamma} . \quad (\text{Eq. 40})$$

The contribution to $P(Y_2)$ is then

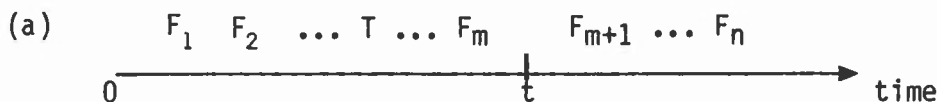
$$1 - e^{-a\bar{C}} . \quad (\text{Eq. 41})$$

(ii) Search for time t in a cell containing n F 's and one T

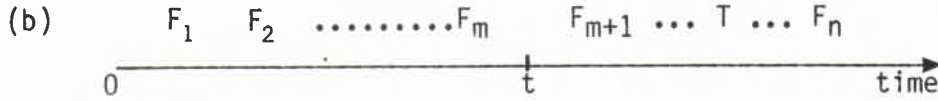
The argument is similar. For each disposition of $F_1, F_2, \dots, F_n$ with respect to t there are two cases:

- (a) The detection epoch of T is before t ;
- (b) It is after t .

Diagrammatically we have:



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and, of course, T may lie before or after the sequence of F 's, or may lie between two of them. In either case the overall probabilities of the dispositions are

$$(a) \quad \delta \binom{n}{m} \gamma^m \bar{\gamma}^{n-m} \quad (\delta = 1 - e^{-\alpha t}) \quad (\text{Eq. 42})$$

$$(b) \quad \bar{\delta} \binom{n}{m} \gamma^m \bar{\gamma}^{n-m} \quad . \quad (\text{Eq. 43})$$

The contribution to $P(Y_0)$ in this case is that all detected F 's are correctly classified and that T is incorrectly classified. Moreover, in this case, the search continues until all detectable objects have been classified. Thus, conditionally, we have from (a) the contribution

$$\delta \binom{n}{m} \gamma^m \bar{\gamma}^{n-m} c^m \bar{c} \quad , \quad (\text{Eq. 44})$$

and from (b)

$$\bar{\delta} \binom{n}{m} \gamma^m \bar{\gamma}^{n-m} c^m \quad . \quad (\text{Eq. 45})$$

It is readily verified that these results are independent of the order of occurrence of the contacts. Hence, for $n > 1$, (a) and (b) give the following contributions to $P(Y_0)$.

$$(a) \quad \bar{c} \delta (c\gamma + \bar{\gamma})^n = \bar{c} \delta c^n \quad (\text{Eq. 46})$$

$$(b) \quad \bar{\delta} c^n \quad . \quad (\text{Eq. 47})$$

This is for $n > 1$, but clearly holds also for $n = 0$. Deconditioning we get the following contribution to $P(Y_0)$:

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$$De^{-aC} \quad (\text{Eq. 48})$$

where

$$D = 1 - c\delta \quad (\text{Eq. 49})$$

is the probability of the complement of the event that T is detected and correctly classified. Naturally we also write

$$\bar{D} = c\delta \quad (\text{Eq. 50})$$

We can now construct $P(Y_0)$ for a two-cell search using the protocol of Sect. 1.2 for search in Cell 1 first. Let p, q be respectively the probabilities that T lies in C_1, C_2 . Then

$$P(Y_0) = pD_1 e^{-a_1 \bar{C}_1 - a_2 \bar{C}_2} + qD_2 e^{-a_1 \bar{C}_1 - a_2 \bar{C}_2} = e^{-a_1 \bar{C}_1 - a_2 \bar{C}_2} (pD_1 + qD_2) \quad (\text{Eq. 51})$$

where $\alpha_1, \beta_1, \alpha_2, \beta_2$ are the detection parameters in C_1, C_2 , a_1, a_2 are the parameters of the false target stream, $\bar{t} = 1 - t$, and

$$\bar{C}_1 = c(1 - e^{-\beta_1 t}) = \bar{C}_{Y_1}, \quad \bar{C}_2 = \bar{C}_{Y_2}, \quad (\text{Eq. 52})$$

$$\bar{D}_1 = c(1 - e^{-\alpha_1 t}) = c\delta_1, \quad \bar{D}_2 = c\delta_2. \quad (\text{Eq. 53})$$

Note that the search time in C_2 is $\bar{t} = 1 - t$.

Next we find $P(Y_2)$. If T is in C_1 the contribution of C_1 is the same as that to $P(Y_0)$, while the contribution of C_2 is given by Eq. 41; if T is in C_2 , search in C_1 must give an apparent target detection although C_1 contains nothing, or F 's. This is again the complement of Y_0 . Thus

$$P(Y_2) = pD_1 e^{-a_1 \bar{C}_1} (1 - e^{-a_2 \bar{C}_2}) + q(1 - e^{-a_1 \bar{C}_1}) \quad (\text{Eq. 54})$$

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Using

$$P(Y_0) + P(Y_1) + P(Y_2) = 1 \quad (\text{Eq. 55})$$

we deduce that

$$P(Y_1) = p(1-D_1 e^{-a_1 \bar{C}_1}) + qe^{-a_1 \bar{C}_1} (1-D_2 e^{-a_2 \bar{C}_2}) . \quad (\text{Eq. 56})$$

It will be noticed that $P(X_1 | Y_0)$ is independent of the "false target rate", as observed in Sect. 1.2.

An independent verification of Eq. 56 is of interest. The contributions of (a) and (b) are $(1-c^m \bar{c})$ and $(1-c^m)$, whatever the order of the F's and T. Thus the overall contributions of (a) and (b) for given n are the sum over m from zero to n of

$$(a) \quad \delta \binom{n}{m} \gamma^m \bar{\gamma}^{n-m} (1-c^m \bar{c})$$

and

$$(b) \quad \delta \binom{n}{m} \gamma^m \bar{\gamma}^{n-m} (1-c^m)$$

giving

$$(a) \quad \delta (1-\bar{c}(\bar{\gamma} + c\gamma)^n) = \delta (1-\bar{c} C^n)$$

and

$$(b) \quad e^{-\alpha t} (1-C^n) .$$

Combining these gives

$$\delta (1-\bar{c}e^{-a\bar{C}}) + \bar{\delta} (1-e^{-a\bar{C}}) = 1 - De^{-a\bar{C}} .$$

Thus, putting it into the actual two cell search context we get

$$P(Y_1) = p(1-D_1 e^{-a_1 \bar{C}_1}) + qe^{-a_1 \bar{C}_1} (1-D_2 e^{-a_2 \bar{C}_2}) .$$

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In this form the problem of allocation of search effort t in such a way as to maximize $P(Y_1)$, for example, seems possibly capable of an analytic approach. Leaving this on one side for a future instalment we next examine the results of some calculations, primarily to determine to what extent this generalization of the model by comparison with Sect. 1.2 has changed the picture.

The numerical values given in the tables below have been calculated using the program ENTROPY11 which is available if required. The intention is to facilitate comparison with Sect. 1.2. The program requires values of the following parameters: p (previously t), $a_1, a_2, c, \alpha_1, \alpha_2, \beta_1, \beta_2$. In what follows $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0.6931$, as in the sequence of Tables 7, 8, 9 of Sect. 1.2.

To compare the output of the model with that of Sect. 1.2, first we put $a_1 = a_2 = 0.001$. This means that the probability of no F in a cell is $\exp(-0.001) = 0.999$. This is $\bar{f}$ of the earlier parts and, in particular, enables a table to be generated to compare with Table 6 of Sect. 1.2. The agreement is exact. c is, of course, 0.999.

We turn now to the value $f = 0.1$ in Sect. 1.2. This means that $\bar{f} = 0.9 = e^{-a}$, giving $a = 0.1054$. It is therefore a comfort to find that values yielded by ENTROPY11 for $a = 0.1054$ are very close to those given by ENTROPY10, the program used to compute Tables 7 to 9 of Sect. 1.2.

To obtain values to compare with $f = 0.5, 0.9$ of Sect. 1.2 we have chosen a such that $e^{-a} = \bar{f}$. This gives $a = 0.6931, 2.3026$ respectively.

The tables follow. We make only a few comments since the pattern is similar to that shown in Sect. 1.2 with the changes to be expected by an allowance for more than one false target in the cells.

$P(X_1 | Y_0)$: This is independent of a and has exactly the same value as in Sect. 1.2.

$P(X_1 | Y_0)$: This is less than in Sect. 1.2 for small c . Otherwise it shows less variation with respect to t . The most remarkable difference is that it does not pass through the prior value when $t = 0.5$, but a value rather less.

$P(Y_1)$: This is greater than in Sect. 1.2, one of the results of higher false contact incidence and hence higher chance of mistaken identity leading to the identification of an object as a target when it is not. At high false target levels $P(Y_1)$ decreases as c increases, the increased capability decreasing the number of misidentifications, a tendency noted also in Sect. 1.2.

$P(Y_0)$: The values tend to be less than in Sect. 1.2.

I_E : The average gain is again minimized with respect to effort near $t = 0.5$ for all p (previously denoted by t) considered.

p = 0.1

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TABLE 11

$p = 0.5$

$a = 0.6931$ ($\bar{F} = 0.5$)

z	$c = 0.1$				$c = 0.5$				$c = 0.9$						
	$P(X_1 Y_0)$	$P(X_1 \bar{Y}_0)$	$P(Y_1)$	$P(Y_0)$	I_E	$P(X_1 Y_0)$	$P(X_1 \bar{Y}_0)$	$P(Y_1)$	$P(Y_0)$	I_E	$P(X_1 Y_0)$	$P(X_1 \bar{Y}_0)$	$P(Y_1)$	$P(Y_0)$	I_E
0.1	0.5102	0.4672	0.1667	0.6989	0.8650	0.5572	0.3468	0.2009	0.7215	0.0866	0.6174	0.1761	0.2504	0.7335	0.1028
0.2	0.5076	0.4678	0.1777	0.6877	0.0392	0.5430	0.3851	0.2133	0.7105	0.0407	0.5888	0.2677	0.2627	0.7219	0.0518
0.3	0.5051	0.4695	0.1853	0.6798	0.0142	0.5287	0.4191	0.2219	0.7028	0.0155	0.5596	0.3486	0.2714	0.7137	0.0214
0.4	0.5025	0.4725	0.1899	0.6752	2.6E-03	0.5144	0.4510	0.2271	0.6983	3.2E-03	0.5299	0.4233	0.2767	0.7087	5.1E-03
0.5	0.5000	0.4767	0.1914	0.6736	8E-04	0.5000	0.4824	0.2288	0.6967	7E-04	0.5000	0.4953	0.2784	0.7071	2E-04
0.6	0.4975	0.4824	0.1899	0.6752	7-8E-03	0.4856	0.5150	0.2271	0.6983	7E-03	0.4701	0.5675	0.2767	0.7087	6.1E-03
0.7	0.4949	0.4900	0.1853	0.6798	0.0239	0.4713	0.5504	0.2219	0.7028	0.0226	0.4404	0.6432	0.2714	0.7137	0.0234
0.8	0.4924	0.5001	0.1777	0.6877	0.0517	0.4570	0.5907	0.2133	0.7105	0.0497	0.4112	0.7259	0.2627	0.7219	0.0543
0.9	0.4898	0.5135	0.1667	0.6989	0.0980	0.4428	0.6388	0.2009	0.7215	0.0948	0.3826	0.8201	0.2504	0.7335	0.1049

$a = 2.3026$ ($\bar{F} = 0.1$)

z	$c = 0.1$				I_E	$c = 0.5$				I_E	$c = 0.9$				I_E
	$P(X_1 Y_0)$	$P(X_1 \bar{Y}_0)$	$P(Y_1)$	$P(Y_0)$		$P(X_1 Y_0)$	$P(X_1 \bar{Y}_0)$	$P(Y_1)$	$P(Y_0)$		$P(X_1 Y_0)$	$P(X_1 \bar{Y}_0)$	$P(Y_1)$	$P(Y_0)$	
0.1	0.5102	0.4667	0.3855	0.3239	0.1543	0.5572	0.4325	0.3276	0.4706	0.1397	0.6174	0.2550	0.2757	0.6734	0.1162
0.2	0.5076	0.4504	0.4244	0.3078	0.0552	0.5430	0.4356	0.3560	0.4546	0.1024	0.5888	0.3213	0.2916	0.6602	0.0566
0.3	0.5051	0.4407	0.4508	0.2968	0.0801	0.5287	0.4402	0.3755	0.4435	0.0182	0.5596	0.3797	0.3027	0.6509	0.0224
0.4	0.5025	0.4357	0.4660	0.2904	6.8E-03	0.5144	0.4465	0.3870	0.4370	4.1E-03	0.5299	0.4337	0.3094	0.6453	5.1E-03
0.5	0.5000	0.4346	0.4709	0.2883	0.0168	0.5000	0.4550	0.3907	0.4349	6.9E-03	0.5000	0.4861	0.3116	0.6435	9E-04
0.6	0.4975	0.4372	0.4660	0.2904	0.0412	0.4856	0.4662	0.3870	0.4370	0.0234	0.4701	0.5395	0.3094	0.6453	8.9E-03
0.7	0.4949	0.4440	0.4508	0.2968	0.0801	0.4713	0.4811	0.3755	0.4435	0.0539	0.4404	0.5963	0.3027	0.6509	0.2950
0.8	0.4924	0.4581	0.4244	0.3078	0.1387	0.4570	0.5013	0.3560	0.4546	0.1024	0.4112	0.6597	0.2916	0.6602	0.0654
0.9	0.4898	0.4755	0.3855	0.3239	0.2334	0.4428	0.5291	0.3276	0.4706	0.1815	0.3826	0.7336	0.2757	0.6734	0.1241

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TABLE 12

$p = 0.9$

$a = 0.6931$ ($\bar{r} = 0.5$)

z	$c = 0.1$				$c = 0.5$				$c = 0.9$			
	$P(X_1 Y_0)$	$P(X_1 \bar{Y}_0)$	$P(Y_1)$	I_E	$P(X_1 Y_0)$	$P(X_1 \bar{Y}_0)$	$P(Y_1)$	I_E	$P(X_1 Y_0)$	$P(X_1 \bar{Y}_0)$	$P(Y_1)$	I_E
0.1	0.9036	0.8876	0.0712	0.0342	0.9189	0.8269	0.0847	0.7875	0.9356	0.6580	0.1018	0.8713
0.2	0.9027	0.8878	0.1071	0.0149	0.9189	0.8269	0.0847	0.7875	0.9280	0.7669	0.1520	0.0245
0.3	0.9018	0.8885	0.1388	5.3E-03	0.9099	0.8666	0.1651	0.7351	5.9E-03	0.9196	0.8281	0.7817
0.4	0.9009	0.8896	0.1668	1E-03	0.9051	0.8809	0.1989	0.7143	1.2E-03	0.9103	0.8685	0.2401
0.5	0.9000	0.8913	0.1914	3E-04	0.9000	0.8935	0.2288	0.6967	3E-04	0.9000	0.8983	0.2784
0.6	0.8991	0.8935	0.2130	3E-03	0.8947	0.9053	0.2554	0.6822	2.6E-03	0.8887	0.9219	0.3132
0.7	0.8982	0.8963	0.2318	9.1E-03	0.8892	0.9168	0.2788	0.6706	8.4E-03	0.8763	0.9419	0.3448
0.8	0.8972	0.9000	0.2482	0.0201	0.8834	0.9285	0.2993	0.6617	0.0185	0.8627	0.9597	0.3773
0.9	0.8963	0.9048	0.2622	0.0394	0.8773	0.9409	0.3171	0.6554	0.0354	0.8480	0.9763	0.3990

$a = 2.3026$ ($\bar{r} = 0.1$)

z	$c = 0.1$				$c = 0.5$				$c = 0.9$			
	$P(X_1 Y_0)$	$P(X_1 \bar{Y}_0)$	$P(Y_1)$	I_E	$P(X_1 Y_0)$	$P(X_1 \bar{Y}_0)$	$P(Y_1)$	I_E	$P(X_1 Y_0)$	$P(X_1 \bar{Y}_0)$	$P(Y_1)$	I_E
0.1	0.9036	0.8873	0.1854	0.3291	0.9189	0.8727	0.1497	0.5136	0.9356	0.7550	0.1149	0.7999
0.2	0.9027	0.8806	0.2810	0.3115	0.9145	0.8741	0.2266	0.4858	0.9280	0.8099	0.1723	0.7540
0.3	0.9018	0.8764	0.3582	0.2992	0.9099	0.8762	0.2911	0.4639	0.9196	0.8463	0.2238	0.7129
0.4	0.9009	0.8742	0.4206	0.2916	0.9051	0.8789	0.3453	0.4470	0.9103	0.8733	0.2701	0.6762
0.5	0.9000	0.8737	0.4709	0.2883	0.9000	0.8825	0.3907	0.4349	0.9000	0.0949	0.3116	0.6435
0.6	0.8991	0.8749	0.4709	0.2083	0.8947	0.8871	0.4287	0.4270	0.8887	0.9134	0.3487	0.6145
0.7	0.8982	0.8779	0.5433	0.2944	0.8892	0.8930	0.4600	0.4231	0.8763	0.9300	0.3816	0.5888
0.8	0.8972	0.8830	0.5678	0.3040	0.8834	0.9005	0.4854	0.4233	0.8627	0.9451	0.4108	0.5664
0.9	0.8963	0.8908	0.5855	0.3186	0.8773	0.9100	0.5054	0.4275	0.8480	0.9612	0.4365	0.5469

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1.4 TWO CELL SEARCH: TWO TARGETS AND ARBITRARY NUMBERS OF FALSE TARGETS

1.4.1 Introduction

It is interesting as well as important to assess the effect of the presence of additional genuine targets. To generalize Sect. 1.3 completely in the same detail seems at present to be quite complicated, but it is manageable for the case of two targets and the details and results are worth recording for the illumination shed on the general problems of search and information. However, the increased complexity of specifying an appropriate inference space is illustrated, together with the need for care in doing so: it is possible to specify a space which makes the prospect of the searcher quite dim, and another which gives a distinct basis for optimism.

1.4.2 The model

The notation of Sect. 1.3 is continued. For completeness we first recall the definition of the frequently recurring symbols. The first cell searched is referred to as C1. It is given in this case that two targets are present and the possible target distribution is accordingly.

STATE OF NATURE	C1	C2	WITH PROBABILITY
x_1	$T_1 T_2$	-	p^2
x_2	T_1	T_2	$2pq$ (the order is immaterial)
x_3	-	$T_1 T_2$	q^2

If the search outcome is two or more detections in C1 classified as target (though they may not be), the search stops: otherwise it continues into C2.

The time (effort) allocated to C1 is t : that to C2 is $\bar{t} = 1 - t$.

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False targets are assumed to occur in a spatial Poisson stream independently in C1 and C2 with parameters a_1, a_2 respectively. Given that a cell contains n false targets, a search for finite time t will provide a random number of false target detection opportunities, a detection opportunity being an "encounter" with the false target before search time in the cell is exhausted. The probability that a false target detection opportunity occurs in a search of duration t is $\gamma = 1 - e^{-\beta t}$ and we write $\bar{\gamma} = 1 - \gamma = e^{-\beta t}$. By a well known property of the Poisson distribution the conditional probability of there being m false target detection opportunities in a search of duration t is given by

$$\binom{n}{m} \gamma^m \bar{\gamma}^{n-m}.$$

For generality a distinction is made between C1 and C2 by providing for different values of the parameter β , β_1 and β_2 respectively. Note that whereas $\bar{\gamma}_1 = e^{-\beta_1 t}$, $\bar{\gamma}_2 = e^{-\beta_2 \bar{t}}$.

A similar argument applies to genuine target detection opportunities. A cell containing two targets provides two detection opportunities in a search of duration t with probability δ^2 , one with probability $2\delta\bar{\delta}$, and none with probability $\bar{\delta}^2$, where $\delta = 1 - e^{-\alpha t}$, $\bar{\delta} = 1 - \delta$. The values of α and t for C1 and C2 are respectively $\alpha_1, t, \alpha_2, \bar{t}$.

The analysis also incorporates an assumed binary classification capability. With probability c an object is correctly identified: with probability $\bar{c}$ it is identified as the complement of what it is.

There are six mutually exclusive and exhaustive outcomes of the two cell search. These are illustrated below: the numbers are those of detections classified rightly or wrongly as "target".

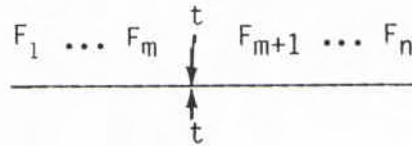
OUTCOME	C1	C2
(1)	> 2*	-
(2)	1	> 1
(3)	1	0
(4)	0	0
(5)	0	1
(6)	0	> 2*

* If there were an excess over 2 it would naturally be assumed that false targets were the cause. By virtue of there having occurred two or more apparent target detections in C1 the assumption is that search would not continue into C2 it being given that the total number of targets available is two.

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We now list the probabilities of (1) to (6) corresponding to the three states of nature X_1, X_2, X_3 enumerated on page 39.

The diagram shows the three possible occurrences in time of the detection opportunities in a search of duration t with m out of n false target detection opportunities.



- (a) $T_1 \quad T_2$
 (b) $T_1 \quad T_2$
 (c) $- \quad T_1 \quad T_2$

The conditional probability of (1) is accordingly:

$$\delta_1^2 \{ c^2 + (2c\bar{c})(1-c^m) + \bar{c}^2(1-c^m - mc^{m-1}\bar{c}) \} + \\ + 2\delta_1 \bar{\delta}_1 \{ c(1-c^m) + \bar{c}(1-c^m - mc^{m-1}\bar{c}) \} + \bar{\delta}_1^2 \{ 1 - c^m - mc^{m-1}\bar{c} \} .$$

The notation makes the argument transparent. Note that the order in which the opportunities occur is discounted. It is as though in case (a) 2 T's and m F's are placed in a box, picked out one by one in any order, examined, and identified correctly with probability c , incorrectly with probability $\bar{c}$. The expression can be reduced to

$$\delta_1^2 \{ 1 - c^m(1-c^2) - \bar{c}^2 m c^{m-1}\bar{c} \} + 2\delta_1 \bar{\delta}_1 (1-c^m - \bar{c} mc^{m-1}\bar{c}) + \bar{\delta}_1^2 (1-c^m - mc^{m-1}\bar{c}) \\ = 1 - c^m \{ \delta_1^2 (1-c^2) + 2\delta_1 \bar{\delta}_1 + \bar{\delta}_1^2 \} - (\bar{\delta}_1 + \bar{c} \delta_1)^2 mc^{m-1}\bar{c} \\ = 1 - c^m(1-\delta_1^2 c^2) - (1-\delta_1 c)^2 mc^{m-1}\bar{c} .$$

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We vary the notation of Sect. 1.3 to avoid confusion of c and C . Thus we shall write

$$c\delta = \bar{D} \quad , \quad \bar{c}\gamma = \bar{r}$$

with subscripts according to the cell. The expression can then be written

$$1 - c^m(1 - \bar{D}_1^2) - D_1^2 m c^{m-1} \bar{c} \quad .$$

Recalling that

$$\sum_{m=0}^n \binom{n}{m} \bar{\gamma}^m \gamma^{n-m} c^m = (\bar{\gamma} + \gamma c)^n = r^n$$

and

$$\sum_{m=0}^n \binom{n}{m} m c^{m-1} \gamma^m \bar{\gamma}^{n-m} = n \gamma \bar{\gamma} r^{n-1} \quad ,$$

and deconditioning the expression with respect to m gives

$$1 - (1 - \bar{D}_1^2) r_1^n - D_1^2 \bar{c} n \gamma \bar{\gamma} r_1^{n-1} = 1 - (1 - \bar{D}_1^2) r_1^n - D_1^2 n \bar{r}_1^n r_1^{n-1} \quad .$$

Finally, deconditioning with respect to n gives for the probability $P((1) | X_1)$ of (1), given that the state of nature is X_1 ,

$$\begin{aligned} P((1) | X_1) &= e^{-a_1} \sum_{n \geq 0} \frac{(a_1)^n}{n!} \{1 - (1 - \bar{D}_1^2) r_1^n - D_1^2 \bar{r}_1^n r_1^{n-1}\} \\ &= 1 - (1 - \bar{D}_1^2) e^{-a_1 \bar{r}_1} - D_1^2 \bar{r}_1 a_1 e^{-a_1 \bar{r}_1} \quad . \end{aligned}$$

It is not necessary to describe the remaining cases in such detail. We list below all eighteen $P((i) | X_j)$ for $i = 1, 2, \dots, 6, j = 1, 2, 3$.

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State of nature X_1

$$P((1) \mid X_1) = 1 - (1 - \overline{D_1^2}) e^{-a_1 \overline{\Gamma_1}} - \overline{D_1^2} \overline{\Gamma_1} a_1 e^{-a_1 \overline{\Gamma_1}} \quad (\text{Eq. 57})$$

$$P((2) \mid X_1) = \overline{D_1} e^{-a_1 \overline{\Gamma_1}} (1 - e^{-a_2 \overline{\Gamma_2}}) (2 \overline{D_1} + a_1 \overline{\Gamma_1} \overline{D_1}) \quad (\text{Eq. 58})$$

$$P((3) \mid X_1) = \overline{D_1} e^{-a_1 \overline{\Gamma_1} - a_2 \overline{\Gamma_2}} (2 \overline{D_1} + a_1 \overline{\Gamma_1} \overline{D_1}) \quad (\text{Eq. 59})$$

$$P((4) \mid X_1) = \overline{D_1^2} e^{-a_1 \overline{\Gamma_1} - a_2 \overline{\Gamma_2}} \quad (\text{Eq. 60})$$

$$P((5) \mid X_1) = \overline{D_1^2} \overline{\Gamma_2} a_2 e^{-a_1 \overline{\Gamma_1} - a_2 \overline{\Gamma_2}} \quad (\text{Eq. 61})$$

$$P((6) \mid X_1) = \overline{D_1^2} e^{-a_1 \overline{\Gamma_1}} (1 - e^{-a_2 \overline{\Gamma_2}} - a_2 \overline{\Gamma_2} e^{-a_2 \overline{\Gamma_2}}) \quad (\text{Eq. 62})$$

State of nature X_2

$$P((1) \mid X_2) = 1 - e^{-a_1 \overline{\Gamma_1}} - \overline{D_1} \overline{\Gamma_1} a_1 e^{-a_1 \overline{\Gamma_1}} \quad (\text{Eq. 63})$$

$$P((2) \mid X_2) = e^{-a_1 \overline{\Gamma_1}} (\overline{D_1} + \overline{D_1} \overline{\Gamma_1} a_1) (1 - \overline{D_2} e^{-a_2 \overline{\Gamma_2}}) \quad (\text{Eq. 64})$$

$$P((3) \mid X_2) = \overline{D_2} e^{-a_1 \overline{\Gamma_1} - a_2 \overline{\Gamma_2}} (\overline{D_1} + \overline{D_1} \overline{\Gamma_1} a_1) \quad (\text{Eq. 65})$$

$$P((4) \mid X_2) = \overline{D_1} \overline{D_2} e^{-a_1 \overline{\Gamma_1} - a_2 \overline{\Gamma_2}} \quad (\text{Eq. 66})$$

$$P((5) \mid X_2) = \overline{D_1} e^{-a_1 \overline{\Gamma_1} - a_2 \overline{\Gamma_2}} (\overline{D_2} + a_2 \overline{D_2} \overline{\Gamma_2}) \quad (\text{Eq. 67})$$

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$$P((6) \mid \chi_2) = D_1 e^{-a_1 \bar{\Gamma}_1} (1 - e^{-a_2 \bar{\Gamma}_2} - D_2 \bar{\Gamma}_2 a_2 e^{-a_2 \bar{\Gamma}_2}) . \quad (\text{Eq. 68})$$

State of nature χ_3

$$P((1) \mid \chi_3) = 1 - e^{-a_1 \bar{\Gamma}_1} - a_1 \bar{\Gamma}_1 e^{-a_1 \bar{\Gamma}_1} \quad (\text{Eq. 69})$$

$$P((2) \mid \chi_3) = a_1 \bar{\Gamma}_1 e^{-a_1 \bar{\Gamma}_1} (1 - D_2^2 e^{-a_2 \bar{\Gamma}_2}) \quad (\text{Eq. 70})$$

$$P((3) \mid \chi_3) = a_1 \bar{\Gamma}_1 D_2^2 e^{-a_1 \bar{\Gamma}_1 - a_2 \bar{\Gamma}_2} \quad (\text{Eq. 71})$$

$$P((4) \mid \chi_3) = D_2^2 e^{-a_1 \bar{\Gamma}_1 - a_2 \bar{\Gamma}_2} \quad (\text{Eq. 72})$$

$$P((5) \mid \chi_3) = D_2 e^{-a_1 \bar{\Gamma}_1 - a_2 \bar{\Gamma}_2} (2 \bar{D}_2 + a_2 \bar{\Gamma}_2 D_2) \quad (\text{Eq. 73})$$

$$P((6) \mid \chi_3) = e^{-a_1 \bar{\Gamma}_1} \{1 - (1 - \bar{D}_2^2) e^{-a_2 \bar{\Gamma}_2} - D_2^2 \bar{\Gamma}_2 a_2 e^{-a_2 \bar{\Gamma}_2}\} . \quad (\text{Eq. 74})$$

It is of interest to identify the probabilistic meanings of the components of these expressions, but we leave this to the reader.

An important check is to verify that

$$\sum_{i=1}^6 P((i) \mid \chi_j) = 1 \quad (j = 1, 2, 3) , \quad (\text{Eq. 75})$$

which has been done.

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1.4.3 Decision state space

The next step is to define a suitable decision state space. We consider two possibilities.

- (i) Y_0 Search inconclusive because either no, or at most one, target-like identifications were made when it is known that two targets are present.
- Y_1 At least two target-like identifications altogether, and in the correct cell.
- Y_2 At least two target-like identifications altogether in the wrong cell, or with the wrong distribution.
- (ii) Z_0 No target-like identification in either cell.
- Z_1 At least one target-like identification in a cell where there is a target.
- Z_2 No target-like identification in a cell where there is a target.

The following Table shows how the six possible search outcomes contribute to these spaces.

TABLE 13
CONTRIBUTIONS OF ALL COMBINATIONS OF OBSERVED AND TRUE STATES
IN Y AND Z SPACES

OBSERVED STATE			TRUE STATE					
Number	Content of		X_1		X_2		X_3	
			C1	2	1		0	
	C1	C2	C2	0	1		2	
			Y	Z	Y	Z	Y	Z
(1)	> 2	-	Y_1	Z_1	Y_2	Z_1	Y_2	Z_2
(2)	1	> 1	Y_2	Z_1	Y_1	Z_1	Y_2	Z_1
(3)	1	0	Y_0	Z_1	Y_0	Z_1	Y_0	Z_2
(4)	0	0	Y_0	Z_0	Y_0	Z_0	Y_0	Z_1
(5)	0	1	Y_0	Z_2	Y_0	Z_1	Y_0	Z_1
(6)	0	> 2	Y_2	Z_2	Y_2	Z_1	Y_1	Z_1

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The set Y_i ($i = 0, 1, 2$) was defined as a direct extension of the set Y_i defined in Sect. 1.3 in the case of a single target. In that case Y_0 meant "an inconclusive search" and its extension to two targets is unexceptionable. Y_1 meant "a target-like detection in the cell where the target is": the extension to two targets seems more restrictive since it requires not only the identification of the number of targets but also the correct distribution. Y_2 meant "a target like detection but in the wrong cell". In the two target case Y_2 arises also because of incorrect observation of the distribution.

The set Z_i ($i = 0, 1, 2$) is less severe. Z_1 gives positive indication that there is a target present. However an ambiguity remains when the observed state is (2), whereas the true state is X_1 or X_3 .

1.4.4 Numerical comparisons

The right-hand side of Table 14 gives some numerical values of:

$$P(Y_j) = \sum_{i=1}^3 P(X_i, Y_j)$$

$$P(Z_j) = \sum_{i=1}^3 P(X_i, Z_j), \quad j = 0, 1,$$

and the corresponding mutual information measures

$$I_E = \sum_{i=1}^3 \sum_{j=0}^2 P(X_i, Y_j) \ln \left\{ \frac{P(X_i, Y_j)}{P(X_i)P(Y_j)} \right\}$$

$$I_F = \sum_{i=1}^3 \sum_{j=0}^2 P(X_i, Z_j) \ln \left\{ \frac{P(X_i, Z_j)}{P(X_i)P(Z_j)} \right\}$$

for the two target case with parameter $p = 0.5$, $a = 0.6931$, $c = 0.5$. For comparison a portion of Table 11 from Sect. 1.3 is appended on the left-hand side. It is emphasized that Y_0 and Y_1 are not the same in the two target case as in the single target case so care must be taken in the comparison. It is not surprising that $P(Y_0)$ is always greater than $P(Z_0)$ because the definitions make it that way. One sees the expected symmetries about $t = 0.5$ in the two target case and the minimization of the information measures near $t = 0.5$. In conclusion we wish to remind

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the reader that this series of studies in an information-theoretic context of the effect of multiple targets, true and false, on the conduct of search, is both methodological and exploratory. Models and analysis have been developed to formulate important probabilities and information measures, both for one and two targets, against a background of unlimited false targets. Generalization to more targets is in principle possible while further systematic investigation in connection with optimal distribution of search effort, and the link between this topic and the information generated by a search, remain to be tackled.

TABLE 14

A COMPARISON BETWEEN THE TWO TARGETS AND SINGLE TARGET CASES

$$p = 0.5, \quad a = 0.6931, \quad (\bar{f} = 0.5), \quad c = 0.5$$

ONE TARGET				TWO TARGETS					
t	P(Y ₀)	P(Y ₁)	I _E	P(Y ₀)	P(Z ₀)	P(Y ₁)	P(Z ₁)	I _E	I _F
0.1	0.7215	0.2009	0.0866	0.9324	0.6257	0.0379	0.3375	0.0252	0.0918
0.2	0.7105	0.2133	0.0407	0.9268	0.6119	0.0422	0.3534	0.0117	0.0918
0.3	0.7028	0.2219	0.0155	0.9228	0.1023	0.0422	0.3645	0.0045	0.0376
0.4	0.6983	0.2271	0.0032	0.9203	0.5966	0.0470	0.3710	0.0009	0.0269
0.5	0.6967	0.2288	0.0007	0.9195	0.5947	0.0474	0.3731	0.0001	0.0237
0.6	0.6983	0.2271	0.0070	0.9203	0.5966	0.0467	0.3709	0.0018	0.0273
0.7	0.7028	0.2219	0.0226	0.9228	0.6023	0.0447	0.3642	0.0062	0.0383
0.8	0.7105	0.2133	0.0497	0.9268	0.6119	0.0416	0.3531	0.0143	0.0585
0.9	0.7215	0.2009	0.0948	0.9324	0.6257	0.0374	0.3373	0.0280	0.0926
Y ₀	no target-like response from search, at least not apparent			Y ₀	at most one target-like response				
Y ₁	apparent target detection in correct cell			Y ₁	at least two target-like responses in correct cell(s)				
				Z ₀	no target-like response at all				
				Z ₁	at least one target-like response in a cell where there is a target				

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2 Information provided by regular surveillance of a moving target

One of the simplest and purest information-transforming processes in military operations is the surveillance of a target. Each observation of the target provides information about its location. If the target is moving, the information decays between observations and must be replenished periodically. These characteristics of the information process can be captured in some very simple models that describe the essential elements without undue detail. A quantitative description of the information dynamics can be developed from the following simple assumptions.

A target moves randomly in one dimension. At regular periodic intervals, measurements of its position are made, with a variance, Σ^2 . The target is assumed to be detected at each observation, so there are no irregularities in the measurements.

Initially the target position is normally distributed about the origin with variance Σ^2 (resulting, for example, from the zeroth measurement). As time passes, the density flattens, but remains normal. The flattening may be described by a function of the elapsed time, t .

$$\sigma^2(t) = f(t) \Sigma^2 \quad (\text{Eq. 76})$$

with

$$\begin{aligned} f(0) &= 1, \\ df/dt &> 0; \text{ all } t. \end{aligned}$$

Consequently, in the absence of measurement, the variance of the position density is always increasing.

Let the first position measurement take place at $t = \tau$. At that time, before the measurement, the prior density of the position x is

$$p(x) = N(x, 0, \sigma) \quad (\text{Eq. 77})$$

where

$$\sigma^2 = \sigma^2(\tau) = \Sigma^2 f(\tau), \quad N(x, \mu, \sigma) = (2\pi\sigma^2)^{-1/2} \exp -1/2 \left(\frac{x-\mu}{\sigma} \right)^2. \quad (\text{Eq. 78})$$

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The posterior density, as a result of a measurement m , is

$$\begin{aligned} p(x | m) &= \left(p(m | x) p(x) \right) / p(m) , \\ &= \left(p(m | x) p(x) \right) / \int p(m | x') p(x') dx' . \end{aligned} \quad (\text{Eq. 79})$$

The probability density of obtaining a measured value m , given a target at x , is $N(m, x, \Sigma)$ and

$$\begin{aligned} p(m) &= \int_{x=-\infty}^{\infty} N(m, x, \Sigma) N(x, 0, \sigma) dx \\ &= N(m, 0, \sqrt{\Sigma^2 + \sigma^2}) . \end{aligned} \quad (\text{Eq. 80})$$

Combining Eqs. 79 and 80, we get for the posterior density:

$$p(x | m) = N(m, x, \Sigma) N(x, 0, \sigma) / N(m, 0, \sqrt{\sigma^2 + \Sigma^2}) . \quad (\text{Eq. 81})$$

which turns out to be normal also, namely

$$p(x | m) = N(x, m \sigma^2 / (\sigma^2 + \Sigma^2) , \sigma \Sigma / \sqrt{\sigma^2 + \Sigma^2}) . \quad (\text{Eq. 82})$$

This implies that the posterior density is centred at a weighted average of the prior position, 0, and the measured position, m , and has a variance $\sigma_a < \sigma_b$ (The subscripts b and a refer to before and after the measurements.)

The measurement process thus serves to reduce the variance of the position density, in contrast to the increase due to target motion.

The probability densities can be used to derive expressions for the information. Two types of information prove to be useful. Self information of a single probability density can be used to express the decrease of information due to target motion. Mutual information between two densities can be used to express the increase in information due to the measurement process. Taken together, these two quantities define the information dynamics of the surveillance of a moving target.

The entropy of a probability density $p(x)$ is

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$$H = - \int dx p(x) \ln p(x) . \quad (\text{Eq. 83})$$

For a Gaussian density such as $N(x,0,\sigma)$ the entropy is

$$H = - \int_{-\infty}^{\infty} dx \frac{e^{-x^2/2\sigma^2}}{\sigma \sqrt{2\pi}} \left[- \frac{x^2}{2\sigma^2} - \ln \sigma \sqrt{2\pi} \right] = 1/2 + \ln \sqrt{2\pi} + 1/2 \ln \sigma^2 . \quad (\text{Eq. 84})$$

When σ^2 has the form specified by Eq. 78,

$$H = 1/2 + \ln \sqrt{2\pi} + 1/2 \ln \Sigma^2 + 1/2 \ln f = H_0 + 1/2 \ln f . \quad (\text{Eq. 85})$$

Information is the negative of the entropy, so

$$I = I_0 - 1/2 \ln f(t) . \quad (\text{Eq. 86})$$

This expresses the decrease in information due to target motion between measurements. The function $f(t)$ has the general characteristics implied by Eq. 76, but its specific form remains to be defined. Much of the analysis can be carried out with f as an unspecified function.

To evaluate the change in information resulting from measurement, we need to consider the probability densities both before and after the measurements, and to compute the mutual information between them.

The mutual information is $\ln p(x|m)/p(x)$, and its ensemble average is

$$I = \int dx dm p(x|m) p(m) \ln(p(x|m)/p(x)) . \quad (\text{Eq. 87})$$

This reduces to

$$I = 1/2 \ln (1 + \sigma_b^2 / \Sigma^2) , \quad (\text{Eq. 88})$$

$$= 1/2 \ln (1 + f(t)) , \quad (\text{Eq. 89})$$

using the notation of Eq. 78.

Consider now a repeated series of measurements. Each supplies an amount of information

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$$I_n = 1/2 \ln (1 + \sigma_{bn}^2 / \Sigma^2) , \quad (\text{Eq. 90})$$

where σ_{bn}^2 is the variance before the n^{th} measurement. By Eq. 82, the variance after the n^{th} measurement is

$$\sigma_{an}^2 = (\sigma_{bn}^2 \Sigma^2) / (\sigma_{bn}^2 + \Sigma^2) ,$$

or,

$$\sigma_{an}^2 / \Sigma^2 = (\sigma_{bn}^2 / \Sigma^2) / (1 + \sigma_{bn}^2 / \Sigma^2) . \quad (\text{Eq. 91})$$

In addition, we assume that $f(t)$ is a universal law of density flattening in the sense that

$$\sigma_{b\ n+1}^2 = \sigma_{an}^2 f(t) , \quad (\text{Eq. 92})$$

(all $n > 1$).

Combining Eqs. 91 and 92, we get

$$\frac{\sigma_{b\ n+1}^2}{\Sigma^2} = f(\tau) (\sigma_{bn}^2 / \Sigma^2) / (1 + (\sigma_{bn}^2 / \Sigma^2)) . \quad (\text{Eq. 93})$$

where τ is the time interval between measurements.

This recurrence relation can be solved explicitly in terms of f and n , with the initial condition: $\sigma_{n1}^2 / \Sigma^2 = f(\tau)$.

Putting $f(\tau) = f$ we get

$$\sigma_{bn}^2 / \Sigma^2 = (f^{n+1} - f^n) / (f^n - 1) . \quad (\text{Eq. 94})$$

It is of interest to examine this expression in the limit $n \rightarrow \infty$. Consider first the limit $f \rightarrow 1$, corresponding to a stationary target. Putting $f = 1 + \epsilon$, we have, to first order in ϵ ,

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$$\sigma_{bn}^2 / \Sigma^2 = \frac{1 + (n+1) \epsilon - 1 - n\epsilon}{1 + n\epsilon - 1} = \frac{1}{n} . \quad (\text{Eq. 95})$$

This is the classical result,

$$\sigma_{bn}^2 \sim \Sigma^2 / n \rightarrow 0 \quad (\text{Eq. 96})$$

for the variance of n independent, identically distributed measurements, which states that a fixed position may be specified with any desired precision, given enough measurements.

For $f > 1$, we obtain a different result:

$$\sigma_{bn}^2 / \Sigma^2 = (f-1) \frac{1}{1 - \left(\frac{1}{f}\right)^n} \rightarrow f-1 . \quad (\text{Eq. 97})$$

Alternatively, using Eq. 92, we can write the variance after the n^{th} measurement:

$$\sigma_{an}^2 / \Sigma^2 = \left(1 - \frac{1}{f}\right) \left(\frac{1}{1 - \left(\frac{1}{f}\right)^{n+1}} \right) \rightarrow 1 - \frac{1}{f} . \quad (\text{Eq. 98})$$

This expresses the fact that for a moving target, even with an unlimited number of measurements, the posterior variance does not approach zero. The residual imprecision is, to first order, $\sim (f-1)\Sigma^2$, proportional to the amount of motion that occurs between measurements.

These ideas can be expressed in information terms with the use of Eq. 90. We shall be interested in three different information values: the marginal information of the n^{th} measurement, I_n ; the total information of the first N measurements,

$$I_N \equiv \sum_{n=1}^N I_n ;$$

and the average information per measurement, $\bar{I}_N = I_N / N$.

Equations 90 and 94 can be combined to give I_n :

$$I_n = \frac{1}{2} \ln \left[\frac{(f^{n+1} - 1)}{(f^n - 1)} \right] . \quad (\text{Eq. 99})$$

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This, in turn, can be summed exactly to give

$$I_N = 1/2 \ln [(f^{N+1} - 1)/(f - 1)] \quad (\text{Eq. 100})$$

In the stationary target limit ($f=1$),

$$I_N = 1/2 \ln(1 + \frac{1}{n}) \quad (\text{Eq. 101})$$

$$I_N = 1/2 \ln(N + 1) \quad (\text{Eq. 102})$$

$$\bar{I}_N = 1/2 \frac{\ln(N + 1)}{N} . \quad (\text{Eq. 103})$$

Thus, as n and $N \rightarrow \infty$, both the marginal information and the average information per measurement approach zero. However, the total information $\rightarrow \infty$ as $\ln(N+1)$ reflecting the fact that an infinite amount of information is required to specify a position with arbitrarily great precision.

For the moving target, $f > 1$, and the limiting values are

$$I_N, \bar{I}_N \rightarrow 1/2 \ln f \quad (\text{Eq. 104})$$

and

$$I_N \rightarrow \frac{N}{2} \ln f . \quad (\text{Eq. 105})$$

In this case, the marginal and average values approach a finite common limit. By comparison with Eq. 86, that limit can be interpreted as the amount of information about the target location which is lost because of target motion between measurements, and which must be regained by each succeeding measurement. Similarly, the fact that $I_N \rightarrow \infty$ as N reflects the cumulative information losses that must be made up, even though only a finite amount of information is needed to express the target position with the precision implied by Eq. 98 at any particular time.

The sort of limiting processes discussed above imply that measurements are taken at constant intervals for an unspecified length of time. This is implicit in the assumption that f has a fixed value for given sampling interval τ .

A different problem arises in the situation for which the total length of time is fixed, and we wish to know the results of more frequent sampling within that interval. Let the total period be T , divided into N intervals. Then f is no longer a function of T , but of T/N . The total information gathered in T is

$$I_N(T) = 1/2 \ln[(f(T/N)^{N+1} - 1)/(f(T/N) - 1)] . \quad (\text{Eq. 106})$$

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To evaluate this in the limit $N \rightarrow \infty$, we need some explicit form for f . On physical grounds it would be reasonable to take

$$f(T/N) = 1 + a(T/N)^\alpha \quad (\text{Eq. 107})$$

with

$$1 < \alpha < 2 .$$

This form for f would correspond to a random walk when $\alpha = 1$ and to a "farthest-on" interval when $\alpha = 2$. Intermediate values of α would correspond to intermediate growth rates for the position variance. However, we do not have complete freedom to choose the form of f , because it is implicitly constrained by assumption (Eq. 92), that f describes a universal law of density flattening. If Eq. 92 is applied successively we must have

$$f(2\tau) = f^2(\tau) . \quad (\text{Eq. 108})$$

This in turn implies that f has the form

$$f = e^{at} . \quad (\text{Eq. 109})$$

For small at , this is approximately $1 + at$, which describes the random walk. Using

$$f(T/N) = e^{aT/N} \quad (\text{Eq. 110})$$

in Eq. 106 we get

$$I_N(T) \sim 1/2 \ln N + 1/2 \ln \left[\frac{eaT - 1}{aT} \right] . \quad (\text{Eq. 111})$$

The leading term is the same as that determined in Eq. 102 for the stationary-target limit. This leads to the following interpretation. By increasing the sampling rate indefinitely, we essentially freeze the target. The principal contribution to the information rate is equivalent to the information obtained by repeatedly measuring a fixed target.

The other term,

$$I_M(T) = 1/2 \ln \left[\frac{eaT - 1}{aT} \right] , \quad (\text{Eq. 112})$$

represents the information about the motion obtained by continuous surveillance of a moving target for a period T .

The results of this simple model can be summarized as follows.

Target motion between regular measurements causes a decrease of information in the amount:

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$$\Delta I_{TM} = - 1/2 \ln f \quad . \quad (\text{Eq. 113})$$

The measurement process causes an increase in the amount of information that depends on the sequential number of the measurement:

$$\Delta I_M = + 1/2 \ln [(f^{n+1} - 1)/(f^n - 1)] \quad . \quad (\text{Eq. 114})$$

For small values of the index, n , $|\Delta I_M| > |\Delta I_{TM}|$, and a net gain of information ensues. As $n \rightarrow \infty$, $|\Delta I_M| \rightarrow |\Delta I_{TM}|$, and the process comes into balance, with the information gain from the measurements equalling the information loss due to target motion. The net information gain from a long series of measurements is

$$\sum_{n=1}^{\infty} \{ 1/2 \ln [(f^{n+1} - 1)/(f^n - 1)] - 1/2 \ln f \} = 1/2 \ln \left(\frac{f}{f-1} \right) \quad . \quad (\text{Eq. 115})$$

These results can form the basis for the exploration of more complex and realistic cases, such as those involving irregular measurements, missed detections, and false targets.

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3 Merger of data in a filter centre

3.1 BASIC CONCEPTS

The merger of data from diverse sources is a common feature of military command and control. Filter centres are designed to collect surveillance and reconnaissance information from a variety of sensors, to merge and reconcile that information, and to forward a consolidated picture to a tactical commander. The sensors that observe the battle situation may overlap to a certain extent, but more often they are covering different aspects of a complex situation. They provide limited pieces of a puzzle that are usually complementary, occasionally contradictory, and always incomplete.

The process of merging data of this type is one of many information-modifying processes that occur within a command and control system. It is an important one, and, to understand how it interacts with the other information-modifying processes, it is necessary to have a quantitative description. This chapter develops one such quantitative description in information theoretic terms, using a relatively simple model.

The features that we wish to model can be summarized quite simply. At some instant of time, $t = 0$, the relevant tactical factors of a battle situation can be totally specified by N_0 bits of information. The battle situation is observed by several sensors, $s_1, s_2, \dots, s_k$, which obtain n_j ($j=1\dots k$) bits of data respectively. Usually $n_j \ll N_0$ for all k . The observations from the sensors are sent to a filter centre, where they are processed, either by a human or by some automated algorithms to create N_1 bits of information about the real battle situation.

The dominant features of the sensors are their limited coverage and their propensity for error. Thus, if the description of the real battle situation is viewed as a message N_0 bits long, then the bits reported by the sensors may be either missing (limited coverage) or incorrect (error). A convenient way to model a process with these characteristics is the use of the so-called erasure channel from communication theory.

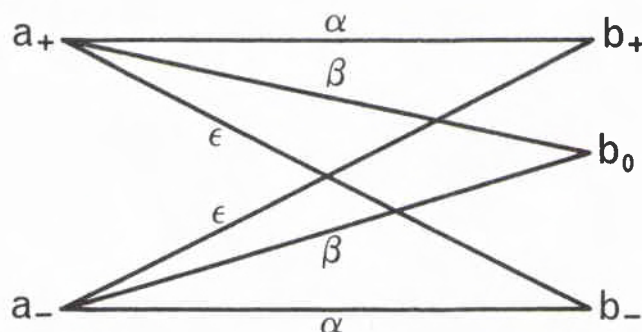


FIG. 1 ERASURE CHANNEL WITH ERRORS

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Figure 1 is a symbolic representation of the erasure channel. The input to the channel is binary, represented as a_+ or a_- . The output is ternary, with the possibilities b_+ , b_0 , or b_- . The b_0 output is the "erasure", a null result that carries no information about the input. The channel as shown is symmetric. This is not a necessary condition, but it is a realistic representation of many practical cases, and it vastly simplifies the analysis.

The quantities α , β , and ϵ are transition probabilities. α is the probability that an input will be correctly represented by the output; ϵ is the probability of a false output; and β is the probability of an inconclusive output. Naturally

$$\alpha + \beta + \epsilon = 1.$$

In the analogy between the abstract channel model and a military command and control system, ϵ represents the probability that a sensor will return false data about the battle situation, whereas β is the probability that no data at all will be returned. This is, in fact, the effective "coverage" provided by the sensor.

The fundamental quantity that we wish to compute on the basis of the channel model is the mutual information between the output and the input. If I_0 is the initial information content of the input (the true battle situation), I_1 is the mutual information between the output of one channel and the input, and I_2 is the mutual information between the input and the output of two parallel, independent channels, then the comparisons among I_0 , I_1 and I_2 will give the necessary quantitative insights into the effects of sensor coverage (β), sensor error (ϵ), and fusion of data from two channels. The plan of this paper is thus to calculate I_0 , I_1 and I_2 and to compare them for various ranges of the parameters α , β , and ϵ .

Many probability expressions are required in the computation of the mutual information. Generally, the mutual information between two sets $\{x_i\}$, $\{y_j\}$ is:

$$I = \sum_{i,j} p(x_i, y_j) \log \frac{p(x_i, y_j)}{p(x_i) p(y_j)} \quad (\text{Eq. 110})$$

We identify x_i with the inputs, a_+ and a_- , and y_j with the outputs b_+ , b_0 , b_- .

Define

$$\begin{aligned} p(a_+) &= q \\ p(a_-) &= 1 - q \end{aligned} \quad (\text{Eq. 111})$$

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The conditional probabilities of b given a are taken from the channel diagram

$$\begin{aligned}\alpha &= p(b_+ | a_+) = p(b_- | a_-) \\ \beta &= p(b_0 | a_+) = p(b_0 | a_-) \\ \epsilon &= p(b_- | a_+) = p(b_+ | a_-)\end{aligned}\tag{Eq. 112}$$

These are then combined to give the joint probabilities

$$\begin{aligned}p(b_+, a_+) &= \alpha q \\ p(b_+, a_-) &= \epsilon(1 - q) \\ p(b_0, a_+) &= \beta q \\ p(b_0, a_-) &= \beta(1 - q) \\ p(b_-, a_+) &= \epsilon q \\ p(b_-, a_-) &= \alpha(1 - q)\end{aligned}\tag{Eq. 113}$$

Finally, the probabilities for the b 's are computed using Bayes' rule.

$$\begin{aligned}p(b_+) &= \alpha q + \epsilon(1 - q) \\ p(b_0) &= \beta \\ p(b_-) &= \epsilon q + \alpha(1 - q)\end{aligned}\tag{Eq. 114}$$

All of these expressions can be combined to give the mutual information between input and output for the channel shown in Fig. 1.

$$\begin{aligned}I_1 &= \alpha \log \alpha + \epsilon \log \epsilon \\ &\quad - (\alpha q + \epsilon[1 - q]) \log(\alpha q + \epsilon[1 - q]) \\ &\quad - (\epsilon q + \alpha[1 - q]) \log(\epsilon q + \alpha[1 - q])\end{aligned}\tag{Eq. 115}$$

This is a completely general expression, which reduces to various familiar special cases for particular values of the parameters.

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a) When $\alpha = 1$, $\beta = 0$, $\epsilon = 0$, we have a perfect channel with no errors or erasures.

Then

$$I_1 \rightarrow -q \log q - (1 - q) \log(1 - q) \equiv H(q) \quad (\text{Eq. 116})$$

which is just the source entropy. For $q = 1/2$

$$I_1 = \log 2 = 1 \text{ bit | symbol.}$$

b) When $q = 1/2$

$$I_1 \rightarrow \alpha \log \alpha + \epsilon \log \epsilon - (\alpha + \epsilon) \log(\alpha + \epsilon) + (\alpha + \epsilon) \log 2 \quad (\text{Eq. 117})$$

This further reduces to special cases:

$$(i) \quad \alpha + \epsilon = 1, \quad \beta = 0$$

This is the binary symmetric channel, for which

$$I_1 = \log 2 - H(\alpha) = 1 - H(\alpha) \quad (\text{Eq. 118})$$

$$(ii) \quad \epsilon = 0$$

This is the traditional binary erasure channel, for which

$$I_1 = \alpha \log 2 = \alpha \quad (\text{Eq. 119})$$

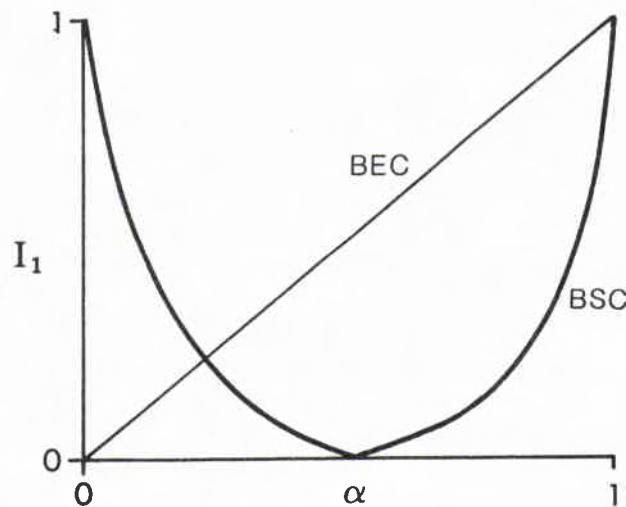


FIG. 2 $I_1(x)$ FOR SPECIAL CASES

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Figure 2 shows $I_1(\alpha)$ for these two special cases. More generally it can be shown that $I_1 > 0$ for all values of the parameters $\alpha, \beta, \epsilon, q$. As in the two examples, equality occurs for $\epsilon = \alpha$, independently of β and q .

To see this, note that

$$\frac{\partial I_1}{\partial \alpha} = \log \alpha - q \log(\alpha q + \epsilon[1 - q]) - (1 - q) \log(\epsilon q + \alpha[1 - q]) \quad (\text{Eq. 120})$$

and

$$\frac{\partial I_1}{\partial \epsilon} = \log \epsilon - (1 - q) \log(\alpha q + \epsilon[1 - q]) - q \log(\epsilon q + \alpha[1 - q]) \quad (\text{Eq. 121})$$

When we set $\partial I_1 / \partial \alpha = \partial I_1 / \partial \epsilon = 0$ we get

$$q(1 - q)(\alpha - \epsilon)^2 = 0 \quad (\text{Eq. 122})$$

as the condition for stationarity. It is then easily verified that $I_1 = 0$ when $\alpha = \epsilon$.

Table 15 shows representative values of I_1 for various values of α, ϵ for $q = 0.4$.

TABLE 15

REPRESENTATIVE VALUES OF I_1
 $q = 0.4$

$\epsilon \backslash \alpha$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0	0	0.097	0.194	0.291	0.388	0.485	0.583	0.680	0.777	0.874	0.971
0.1	0.097	0	0.023	0.073	0.134	0.202	0.276	0.352	0.431	0.513	
0.2	0.194	0.023	0	0.139	0.047	0.092	0.145	0.204	0.268		
0.3	0.291	0.073	0.014	0	0.010	0.035	0.071	0.114			
0.4	0.388	0.134	0.047	0.010	0	0.008	0.028				
0.5	0.485	0.202	0.092	0.035	0.008	0					
0.6	0.583	0.276	0.145	0.071	0.028						
0.7	0.680	0.352	0.204	0.114							
0.8	0.777	0.431	0.268								
0.9	0.874	0.513									
1.0	0.971										

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The calculation of I_2 proceeds in the same manner, although the details become more complicated. We assume that two channels each provide outputs. If those are considered pairwise, the relevant probabilities of the possible outcomes are:

$$\begin{aligned}
 p(b_{++} | a_+) &= p(b_{--} | a_-) = \alpha_1 \alpha_2 \\
 p(b_{+0} | a_+) &= p(b_{-0} | a_-) = \alpha_1 \beta_2 + \alpha_2 \beta_1 \\
 p(b_{+-} | a_+) &= p(b_{+-} | a_-) = \alpha_1 \epsilon_2 + \alpha_2 \epsilon_1 \\
 p(b_{00} | a_+) &= p(b_{00} | a_-) = \beta_1 \beta_2 \\
 p(b_{0-} | a_+) &= p(b_{+0} | a_-) = \beta_1 \epsilon_2 + \beta_2 \epsilon_1 \\
 p(b_{--} | a_+) &= p(b_{++} | a_-) = \epsilon_1 \epsilon_2
 \end{aligned}
 \tag{Eq. 123}$$

The joint probabilities of the a's and b's and the unconditional probabilities of the b's can then be formed. When they are all incorporated properly in Eq. 110, the result is:

$$\begin{aligned}
 I_2 &= \alpha_1 \alpha_2 \log \alpha_1 \alpha_2 + \epsilon_1 \epsilon_2 \log \epsilon_1 \epsilon_2 \\
 &\quad - B \log B - C \log C + D \log D + E \log E - F \log F - G \log G,
 \end{aligned}
 \tag{Eq. 124}$$

where

$$\begin{aligned}
 B &= \alpha_1 \alpha_2 q + \epsilon_1 \epsilon_2 (1 - q) \\
 C &= \alpha_1 \alpha_2 (1 - q) + \epsilon_1 \epsilon_2 q \\
 D &= \alpha_1 \beta_1 + \alpha_2 \beta_1 \\
 E &= \beta_1 \epsilon_2 + \beta_2 \epsilon_1 \\
 F &= (\alpha_1 \beta_2 + \alpha_2 \beta_1)q + (\beta_1 \epsilon_2 + \beta_2 \epsilon_1)(1 - q) \\
 G &= (\beta_1 \epsilon_2 + \beta_2 \epsilon_1)q + (\alpha_1 \beta_2 + \alpha_2 \beta_1)(1 - q)
 \end{aligned}$$

This is a form that allows numerical calculations for specific cases to be made. However, with seven parameters, it is too complicated to yield any general results, so we will examine more restricted cases.

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First, when the two channels are identical, I_2 reduces to

$$\begin{aligned}
 I_2 = & 2 \beta I_1 + \alpha^2 \log \alpha^2 + \epsilon^2 \log \epsilon^2 \\
 & - (\alpha^2 q + \epsilon^2 (1 - q)) \log(\alpha^2 q + \epsilon^2 (1 - q)) \\
 & - (\epsilon^2 q + \alpha^2 (1 - q)) \log(\epsilon^2 q + \alpha^2 (1 - q))
 \end{aligned}
 \tag{Eq. 125}$$

For general comparison with I_1 , we can then look at the same special cases as were considered above.

a) $\alpha = 1, \beta = \epsilon = 0$. The perfect channel.

Again

$$I_2 \rightarrow H(q)$$

b) $q = 1/2$

(i) $\beta = 0$

$$\begin{aligned}
 I_2 \rightarrow & (\alpha^2 + (1-\alpha)^2) \log 2 + \alpha^2 \log \alpha^2 + (1-\alpha)^2 \log(1-\alpha)^2 \\
 & - (\alpha^2 + (1-\alpha)^2) \log(\alpha^2 + (1-\alpha)^2)
 \end{aligned}
 \tag{Eq. 126}$$

This has a shape similar to that of I_1 , but differs slightly. Figure 3 plots both $I_1(\alpha)$ and $I_2(\alpha)$ for this special case.

(ii) $\epsilon = 0$

In this case $I_2 \rightarrow 2\alpha - \alpha^2$ which is compared with I_1 in Fig. 4.

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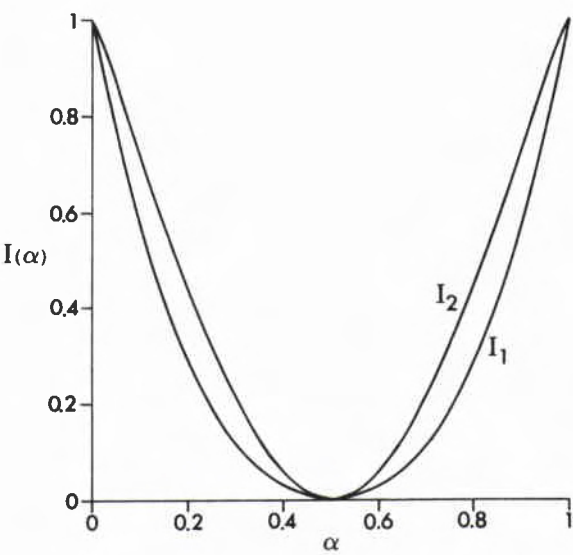


FIG. 3 PLOT OF I_1 AND I_2

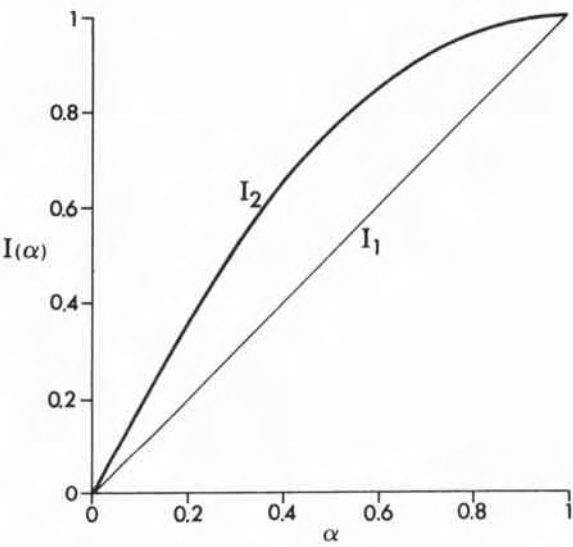


FIG. 4 PLOT OF I_1 AND I_2

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Figures 3 and 4 both show that $I_2 > I_1$ for these special cases. These are suggestive of the general result.

Direct calculation of I_2 is shown in Table 16 for a range of values of α and ϵ . Table 17 shows $\Delta = I_2 - I_1$. From these, we can see that $\Delta > 0$ for all parameter values, with equality occurring only at boundaries and where $\alpha = \epsilon$.

TABLE 16

REPRESENTATIVE VALUES OF I_2 $q = 0.4$

$\epsilon \backslash \alpha$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0	0	0.184	0.350	0.495	0.621	0.728	0.816	0.883	0.932	0.961	0.971
0.1	0.184	0	0.046	0.138	0.245	0.354	0.459	0.557	0.644	0.719	
0.2	0.350	0.046	0	0.028	0.091	0.173	0.263	0.355	0.445		
0.3	0.495	0.138	0.028	0	0.020	0.068	0.135	0.211			
0.4	0.621	0.245	0.091	0.020	0	0.015	0.056				
0.5	0.728	0.354	0.173	0.068	0.015	0					
0.6	0.816	0.459	0.263	0.135	0.056						
0.7	0.883	0.557	0.355	0.211							
0.8	0.932	0.644	0.445								
0.9	0.961	0.719									
1.0	0.971										

TABLE 17

REPRESENTATIVE VALUES OF $\Delta = I_2 - I_1$ $q = 0.4$

$\epsilon \backslash \alpha$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0	0	0.087	0.156	0.204	0.233	0.243	0.233	0.203	0.155	0.087	0
0.1	0.087	0	0.023	0.066	0.111	0.152	0.184	0.205	0.213	0.207	
0.2	0.156	0.023	0	0.014	0.044	0.081	0.118	0.151	0.177		
0.3	0.204	0.066	0.014	0	0.010	0.033	0.064	0.097			
0.4	0.233	0.111	0.044	0.010	0	0.008	0.027				
0.5	0.243	0.152	0.081	0.033	0.008	0					
0.6	0.233	0.184	0.118	0.064	0.027						
0.7	0.203	0.205	0.151	0.097							
0.8	0.155	0.213	0.177								
0.9	0.087	0.207									
1.0	0										

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Consequently, by the criterion of information, merger of two data streams is always advantageous, regardless of the size of the errors or the number of erasures. When other criteria are used, the situation is not so clearcut, however.

In the practical situation of merging tactical data and trying to make tactical decisions, it is important to know whether better decisions can be made on the basis of the merged or the unmerged data. Thus, the probability of correctly interpreting a received symbol is important, as is the probability of making an error. Since we allow the possibility of erasures, and, in merged data, of contradictions, the probabilities of being right and of being wrong do not add to 1.

The calculation of these probabilities presupposes some decoding algorithm which assigns a unique value of the *a* symbol to each value of the observed *b* symbol or combinations thereof. We choose for illustration a very simple and highly practical assignment algorithm.

For one channel we use:

<u>Observed</u>	<u>Assigned</u>
+	+
0	None
-	-

For two channels we use:

<u>Observed</u>	<u>Assigned</u>
++	+
+0	+
+-	0
0+	+
00	0
0-	-
-+	0
-0	-
--	-

For higher numbers of channels we follow the same basic rules, of assigning a *+* when the number of observed plusses is greater than the number of observed minuses, assigning a *-* when the number of observed minuses is greater, and making no assignment when *+*'s and *-*'s are equal or all zero.

When these rules are combined with the probabilities of Eqs. 112 and 123 and with the analogue for three channels, we get:

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No. of channels	Probability of correct decoding P_c	Probability of error P_e
1	α	ϵ
2	$\alpha(\alpha + 2\beta)$	$\epsilon(\epsilon + 2\beta)$
3	$\alpha(\alpha^2 + 3\alpha\epsilon + 3\alpha\beta + 3\beta^2)$	$\epsilon(\epsilon^2 + 3\epsilon(1-\epsilon) + 3\beta^2)$

If merger of data is uniformly superior to the use of unmerged data, we would expect

$$P_{3c} > P_{2c} > P_{1c}$$

and

$$P_{3e} < P_{2e} < P_{1e}$$

That is, as data from more channels are added, we would ideally want the probability of being correct to increase and the probability of error to decrease. To determine the conditions under which this is the case, we examine the relationships among the pairwise probabilities in the α, β, ϵ space. For this purpose, it is most convenient to eliminate α and to work in the β - ϵ plane.

In that plane, we get, by equating the various probabilities

For $P_{1c} = P_{2c}$:

$$\alpha + 2\beta = 1$$

or

$$\beta = \epsilon$$

(Eq. 127)

For $P_{2c} = P_{3c}$:

$$\alpha + 2\beta = \alpha^2 + 3\alpha(\epsilon + \beta) + 3\beta^2$$

or

$$\epsilon = 1/2 - \beta + \sqrt{(3/2)\beta^2 - \beta + 1/4}$$

(Eq. 128)

For $P_{1c} = P_{3c}$:

$$\alpha^2 + 3\alpha(\epsilon + \beta) + 3\beta^2 = 1$$

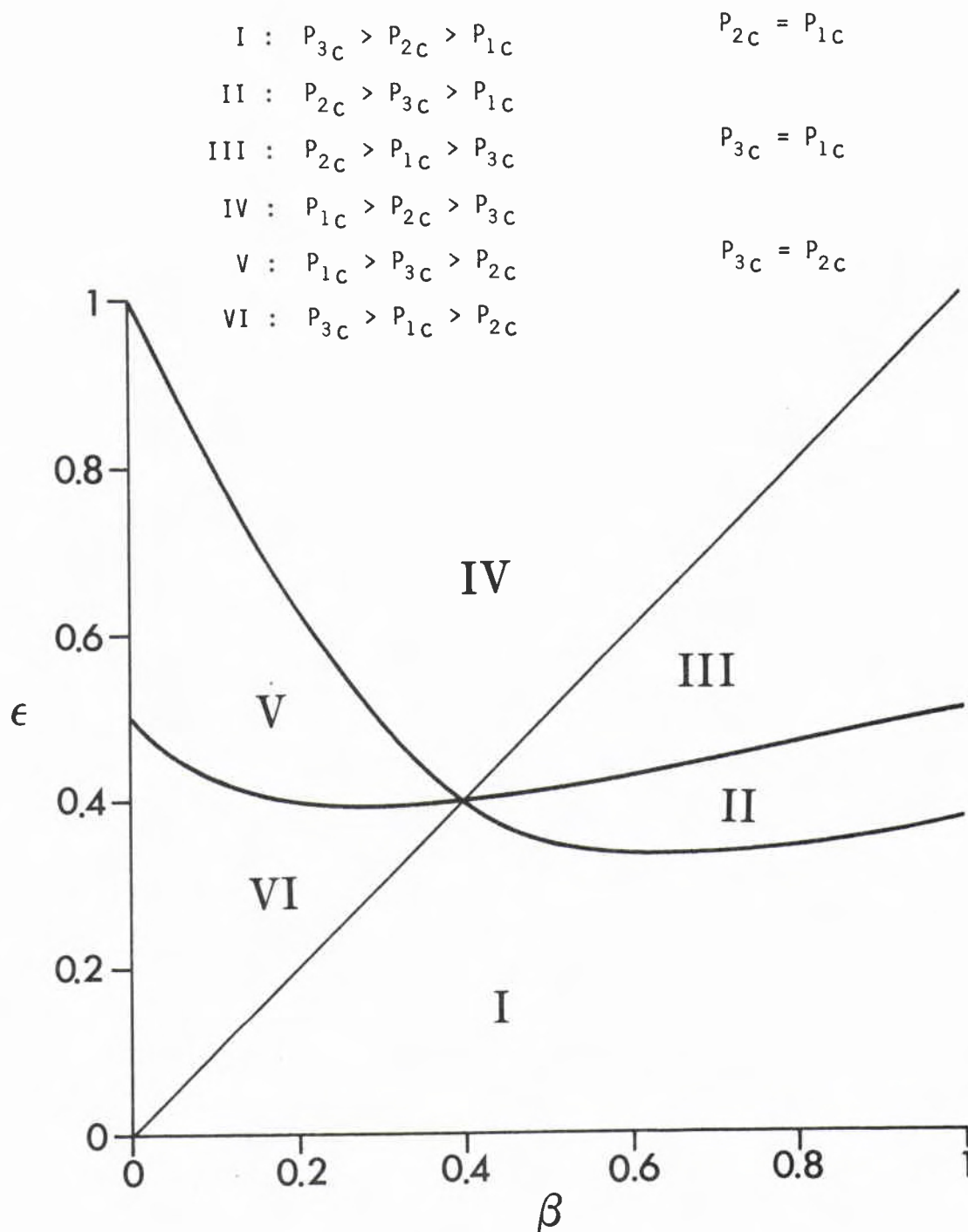
or

$$\epsilon = 1/4 - \beta + 1/4 \sqrt{(1 + 24\beta^2)}$$

(Eq. 129)

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These relations are plotted in Fig. 5 in the region $0 < \beta < 1$, $0 < \epsilon < 1$. The figure shows that the boundary lines divide the area into six segments, corresponding to the six possible orderings of P_{1c} , P_{2c} , and P_{3c} . In only one of these segments, for $\beta > \epsilon$ and for $\epsilon < 0.33$, is the desired ordering achieved.

FIG. 5 RELATIONS BETWEEN P_{1c} , P_{2c} , P_{3c}

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The relations among the P_e values can be used in a similar manner.

For $P_{1e} = P_{2e}$:

$$\epsilon = 1 - 2\beta \quad (\text{Eq. 130})$$

For $P_{2e} = P_{3e}$:

$$\begin{aligned} \epsilon + 2\beta &= \epsilon^2 + 3\epsilon(1 - \epsilon) + 3\beta^2 \\ \text{or} \\ \epsilon &= 1/2 \pm 1/2 \sqrt{6\beta^2 - 4\beta + 1} \end{aligned} \quad (\text{Eq. 131})$$

For $P_{1e} = P_{3e}$:

$$\begin{aligned} 1 &= \epsilon^2 + 3\epsilon(1 - \epsilon) + 3\beta^2 \\ \text{or} \\ \epsilon &= 3/4 - 1/4 \sqrt{1 + 24\beta^2} \end{aligned} \quad (\text{Eq. 132})$$

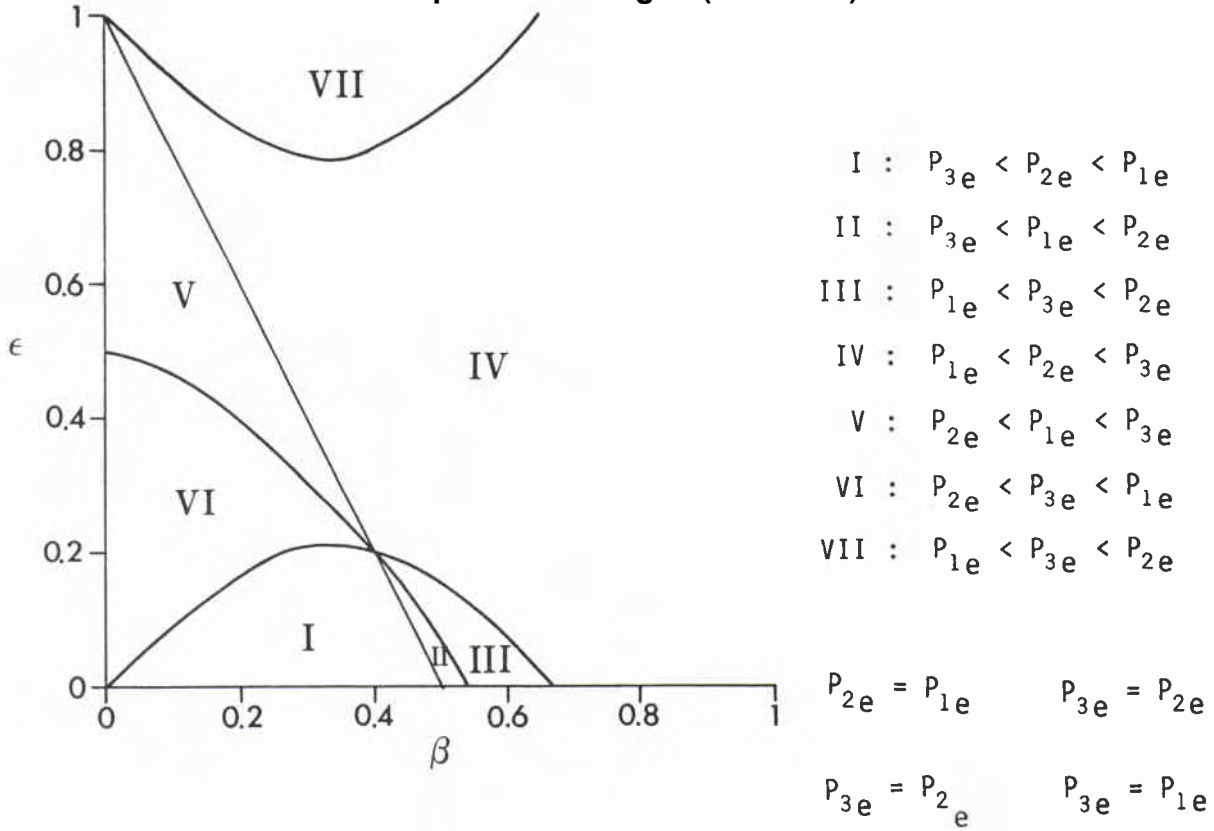
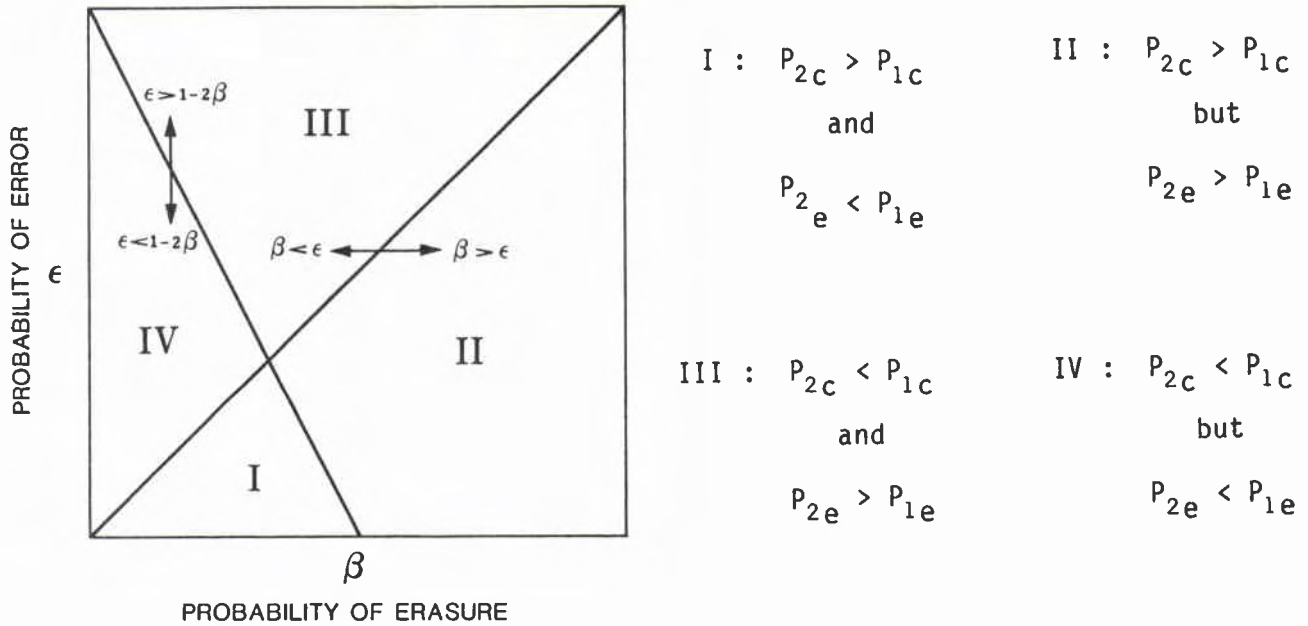
These curves are used to divide the β - ϵ plane as shown in Fig. 6. Again, only one relatively small region has the desired ordering.

Figure 7 shows the intersection of the areas for which both desired orderings are found for the case of two channels:

$$\begin{aligned} &P_{1e} < P_{2e} \\ \text{and} \quad &P_{2e} < P_{1e} . \end{aligned}$$

That is, a small part of the allowed space for β and ϵ , and we can conjecture that the corresponding segment for a larger number of channels would be even smaller.

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FIG. 6 RELATIONS BETWEEN P_{1e} , P_{2e} , P_{3e} FIG. 7 JOINT ORDERING OF P_{1c} , P_{2c} , P_{1e} , P_{2e} (TWO CHANNELS)

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Consequently, if we try to impose a strict criterion based on the ordering of probabilities of being right and wrong, we find that the merging of data is rarely universally superior to the use of unmerged data from a single source.

The use of such a criterion should not be imposed blindly, however. In practical applications, one is always concerned with the degree of change in the probabilities achievable through merging of data, not necessarily with strict ordering. Tables 18 to 20 give some examples.

Table 18 shows a case for which the probability of being right increases significantly as additional channels are merged. The probability of being wrong also increases, but both the absolute magnitude and the amount of change are small. In this case, a prudent decision maker would choose to use the added data.

Table 19 shows the ideal case in which the probability of being right increases and the probability of being wrong decreases as more channels are merged.

Table 20 is the problematic case. The probability of being right decreases as the second channel is added and then increases above the initial value when the third channel is added. Conversely, the probability of being wrong decreases as the second channel is added but increases again with the third channel. Moreover, the absolute magnitude of P_e is quite significant in all cases.

TABLE 18

CASE: $\beta = 0.8$; $\alpha = 0.18$; $\epsilon = 0.02$

CHANNELS	P_c	P_e	P_r
1	0.18	0.02	0.80
2	0.32	0.03	0.65
3	0.43	0.04	0.53

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TABLE 19

CASE: $\beta = 0.4$; $\alpha = 0.54$; $\epsilon = 0.06$

CHANNELS	P_c	P_e	$P_?$
1	0.54	0.06	0.40
2	0.72	0.05	0.23
3	0.82	0.04	0.14

TABLE 20

CASE: $\beta = 0.20$; $\alpha = 0.50$; $\epsilon = 0.30$

CHANNELS	P_c	P_e	$P_?$
1	0.50	0.30	0.20
2	0.45	0.21	0.34
3	0.56	0.25	0.19

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3.2 GENERALIZED THEORY

In this section we generalize the model developed in Sect. 3.1 and provide an analysis which will facilitate further investigation. The conclusions reported above hold good concerning the sometimes curious response to added information of the probabilities of correct, incorrect and null inferences regarding the message. Some tables are given which include results for more observers with their corresponding independent channels.

In addition we investigate the effect of a single observer who chooses to transmit his interpretation of a message down a number of independent channels in the belief that this should somehow enhance the credibility of what arrives.

It is convenient to build more detail into the model. This clarifies the structure of the probabilities α, β, ϵ , introduced above according to a particular interpretation of the processes of observation and transmission.

3.2.1 Case 1: n independent observers each with an independent communication channel to the fusion centre

In the following we introduce notation different from that used in Sect. 3.1. This is to identify the details of the structure. It is supposed that an observer in the field perceives a message X which has either the value $X_0 = 0$, or $X_1 = 1$. With probability c the observer decodes, or identifies, the message correctly: with probability $\bar{c} = 1 - c$ the message is identified as the complement of what it actually is, namely X_0 is decoded as X_1 , and conversely. The decoded message is then passed down an information channel to a fusion centre whose task is to collect and reconcile information received from diverse sources and to pass an interpreted version to a decision-maker. The information channel may be serviceable with probability s , or unserviceable with probability $\bar{s} = 1 - s$. A channel may be unserviceable because the equipment is out of order, because a part of the route on which it depends is out of action or destroyed, or because of jamming; a host of other reasons can be imagined according to the context. If serviceable, a channel may be defective in another fashion; it may "garble" the information supplied to it in such a way that the recipient receives a false impression. Degrees of garbling and identification belong to the fashionable and improbable realm of so-called fuzzy logic. Here we merely say that with probability g the sense of the message imparted by the observer at the input is reversed, so that if X_0 were transmitted X_1 would be received, and vice versa. With probability $\bar{g}$ the message transmitted is received, and $\bar{g} = 1 - g$. It is thus seen that the probabilities α, β, ϵ introduced in Sect. 3.1 for a single channel are given by

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$$\alpha = sr, \quad \epsilon = s\bar{r}, \quad \beta = \bar{s}, \quad (\text{Eq. 133})$$

where

$$r = c\bar{g} + \bar{c}g, \quad \bar{r} = cg + \bar{c}\bar{g},$$

and r is the probability that the correct message is received, whether or not it was correctly interpreted by the observer, conditional on the channel being in service. $\bar{r} = 1 - r$ is the probability that the wrong message is received.

Given that there are n independent observers and corresponding channels all possessing the foregoing characteristics the number of correct messages received at the fusion centre is a random variable K whose probability $p_k = P(K = k)$ is the coefficient of x^k in the generating function

$$\begin{aligned} E(x^K) &= \bar{s}^n + \binom{n}{1}\bar{s}^{n-1}s(rx+\bar{r}) + \binom{n}{2}\bar{s}^{n-2}s^2(rx+\bar{r})^2 + \dots + \binom{n}{n}s^n(rx+\bar{r})^n \\ &= (\bar{s} + s(rx+\bar{r}))^n = (1-sr+srx)^n. \end{aligned} \quad (\text{Eq. 134})$$

The independence of the channels guarantees the correctness of this result. It is not, however, quite what we want for the purpose of this analysis.

If at least one message is received (that is, if at least one channel is serviceable) an inference can be made at the fusion centre concerning the nature of the message X . The algorithm prescribed in Sect. 3.1 is to accept the majority, that is to accept the interpretation provided by the greater number of channels. The third possibility, referred to in Sect. 3.1 as "erasure", is the acknowledgement of impossibility to make an inference; this occurs if either all channels are unserviceable, so that no transmission at all is received, or that an even number is serviceable and impart an equal number of either interpretation of the message. We call the probability of this event M_n and it is easily seen from Eq. 134, or otherwise, to be given by

$$\begin{aligned} M_n &= \bar{s}^n + \binom{n}{2}\bar{s}^{n-2}s^2\binom{2}{1}(r\bar{r}) + \binom{n}{4}\bar{s}^{n-4}s^4\binom{4}{2}(r\bar{r})^2 + \dots + \\ &+ \binom{n}{n_1}\bar{s}^{n-n_1}s^{n_1}\binom{n_1}{n_1/2}(r\bar{r})^{n_1/2} \end{aligned} \quad (\text{Eq. 135})$$

where n_1 is the largest even integer which does not exceed n . Let L_n be the probability that the incorrect inference is drawn. This is the

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sum of terms obtained by considering in turn the possibility that $m (= 1, 2, \dots, n)$ channels are serviceable. Thus

$$L_n = \sum_{m=1}^n \binom{n}{m} \bar{s}^{n-m} s^m (r^m + \binom{m}{1} \bar{r}^{m-1} r + \binom{m}{2} \bar{r}^{m-2} r^2 + \dots + \binom{m}{m_2} \bar{r}^{m-m_2} r^{m_2}) \quad (\text{Eq. 136})$$

where $m_2 = (m/2) - 1$ if m is even and $(m-1)/2$ if m is odd. There is a corresponding symmetrical expression for N_n , the probability that the correct inference is drawn, in which r and $\bar{r}$ are interchanged. Clearly $L_n + M_n + N_n = 1$.

Let us now introduce the generating function $g_n(x)$ defined by

$$g_n(x) = (s\bar{r}/x + \bar{s} + sr x)^n = \sum_{m=-n}^n g_m^{(n)} x^m, \quad (\text{Eq. 137})$$

say. Then it is easily checked that

$$L_n = \sum_{m=-n}^{-1} g_m^{(n)}, \quad M_n = g_0^{(n)}, \quad N_n = \sum_{m=1}^n g_m^{(n)}. \quad (\text{Eq. 138})$$

The conditional probabilities PR , PW of making a correct or incorrect inference are then

$$PW = L_n / (1 - M_n), \quad PR = N_n / (1 - M_n). \quad (\text{Eq. 139})$$

Finally we need the mean increase in information induced by any transmissions received. This is based on the joint probabilities $P(X_i, Z_j)$, where X_i is the message and Z_j is the inference. We define the inferences as Z_0 that the message is X_0 , Z_1 that it is X_1 , and Z_2 is reserved for "don't know".

Let

$$P(X = X_1) = p, \quad P(X = X_0) = q = 1 - p. \quad (\text{Eq. 140})$$

Then the mean increase in information induced by the transmissions is given by

$$I_e = N_n \log N_n + L_n \log L_n - (qN_n + pL_n) \log(qN_n + pL_n) - (qL_n + pN_n) \log(qL_n + pN_n). \quad (\text{Eq. 141})$$

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Given the form (Eq. 137) which generates the probabilities of interest it is particularly easy to evaluate them numerically. Unfortunately the same cannot be said of the algebra. Put

$$f_n(\theta) = g_n(e^{i\theta}) \quad . \quad (\text{Eq. 142})$$

Then $f_n(\theta)$ is a finite Fourier series and therefore

$$g_m^{(n)} = \frac{1}{2\pi} \int_0^{2\pi} e^{-im\theta} f_n(\theta) d\theta \quad , \quad (-n \leq m \leq n) \quad . \quad (\text{Eq. 143})$$

Since the series is finite it follows that if we choose any positive integer t greater than n , the integral in Eq. 143 is exactly equal to a discrete Fourier transform type of series, that is

$$g_m^{(n)} = \frac{1}{t} \sum_{k=0}^{t-1} e^{-2\pi imk/t} f(2\pi k/t) \quad (-n \leq m \leq n) \quad . \quad (\text{Eq. 144})$$

This means that fast Fourier transform algorithms provide a quick way of evaluating the fundamental probabilities.

Finally, of mathematical interest is the fact that $g_m^{(n)}$ can be expressed in terms of associated Legendre functions, at least formally.

Hobson's definition of the function concerned is given in <7>, Chapter XV. Thus

$$P_n^m(z) = (z^2 - 1)^{m/2} d^m P_n(z)/dz^m$$

where m is a positive integer and n is unrestricted, while $P_n(z)$, the coefficient of h^n in $(1 - 2hz + h^2)^{-1/2}$, is the Legendre Polynomial given by

$$P_n(z) = \sum_{r=0}^m \frac{(-)^r (2n - 2r)!}{2^n r! (n-r)! (n-2r)!} z^{n-2r} \quad ,$$

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where $m = n/2$ or $(n-1)/2$, whichever is an integer. Whittaker and Watson <7> show that $P_n^m(z)$ has the integral representation

$$P_n^m(z) = \frac{(n+1)(n+2) \dots (n+m)}{2\pi} \int_0^{2\pi} (z+(z^2-1)^{1/2} \cos\phi)^n e^{-im\phi} d\phi ,$$

for all z .

After some transformation of Eq. 143 we obtain the formula

$$g_m^{(n)} = \frac{R^n}{(n+1)(n+2) \dots (n+m)} (r/\bar{r})^{m/2} P_n^m(\bar{s}/R) \quad (m = 0, 1, \dots, n)$$

where

$$R^2 = \bar{s}^2 - 4r\bar{r}s^2 . \quad (\text{Eq. 145})$$

$g_m^{(n)}$ for negative m is obtained by merely interchanging r and $\bar{r}$.

Since the asymptotic properties of $P_n^m(z)$ are available these are of potential help in providing information about $g_m^{(n)}$.

Equation 143 provides a way of investigating the effect on erasure probability of adding an extra channel. This can be done by examining the sign of $M_{n+1} - M_n$, the difference between erasure probabilities, or loss. Recalling that $M_n = g_0^{(n)}$, we have

$$\begin{aligned} M_{n+1} - M_n &= g_0^{(n+1)} - g_0^{(n)} = \frac{1}{2\pi} \int_0^{2\pi} f_n(\theta) (s\bar{r}e^{-i\theta} + \bar{s} + sre^{i\theta} - 1) d\theta \\ &= s(\bar{r} g_1^{(n)} + r g_{-1}^{(n)} - M_n) . \end{aligned}$$

Because of symmetry this can be written

$$M_{n+1} - M_n = s(2\bar{r} g_1^{(n)} - M_n) . \quad (\text{Eq. 146})$$

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Next we record $g_1^{(n)}$ for even and odd values of n , $g_0^{(n)}$ having been given by Eq. 135.

$n = 2N$:

$$g_1^{(2N)} = \binom{2N}{N} \binom{N}{1} s^{2N-1} \bar{s} \bar{r}^{N-1} r^N + \binom{2N}{N+1} \binom{N+1}{3} s^{2N-3} \bar{s}^3 \bar{r}^{N-2} r^{N-1} +$$

$$+ \binom{2N}{N+2} \binom{N+2}{5} s^{2N-5} \bar{s}^5 \bar{r}^{N-3} r^{N-2} + \dots + \binom{2N}{2N-1} s \bar{s}^{2N-1} r$$

(Eq. 147)

$n = 2N+1$:

$$g_1^{(2N+1)} = \binom{2N+1}{N} s^{2N+1} \bar{r}^N r^{N+1} + \binom{2N+1}{N+1} \binom{N+1}{2} s^{2N-1} \bar{s}^2 \bar{r}^{N-1} r^N +$$

$$+ \binom{2N+1}{N+2} \binom{N+2}{4} s^{2N+3} \bar{s}^4 \bar{r}^{N-2} + \dots + \binom{2N+1}{2N} s \bar{s}^{2N} r$$

(Eq. 148)

Suppose that the channels are highly reliable. This means that $s \rightarrow 1$ and the dominant terms in Eqs. 147 and 148 are those corresponding to the higher powers of s . Writing $\stackrel{D}{=}$ to mean "dominated by", we find that

$$M_{2N+1} - M_{2N} \stackrel{D}{=} \binom{2N}{N} (r\bar{r}s^2)^N (2N\bar{s} - s),$$

and

$$M_{2N+2} - M_{2N+1} \stackrel{D}{=} \binom{2N+1}{N} s (r\bar{r}s^2)^N (2r\bar{r}s - (N+1)\bar{s}). \quad (\text{Eq. 149})$$

If adding an extra observer results in a reduction in the erasure probability it could be said that the expenditure was worthwhile. However, it is seen for high s (~ 0.9) and moderate r (~ 0.5), that the addition of an observer to an already odd number eventually makes even worse than odd and adding a channel to an even number results in an odd number superior to even. An example is given below with $s = 0.9$, $r = \bar{r} = 0.5$.

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Then $s/\bar{s} = 9$, $2r\bar{r}s/\bar{s} = 4.5$.

n			
1			
2	$M_2 > M_1$	since	$1 < 4.5$: Bad
3	$M_3 < M_2$		$2 < 9$: Good
4	$M_4 > M_3$		$2 < 4.5$: Bad
5	$M_5 < M_4$		$4 < 9$: Good
6	$M_6 > M_5$		$3 < 4.5$: Bad
7	$M_7 < M_6$		$6 < 9$: Good
8	$M_8 > M_7$		$4 < 4.5$: Good
9	$M_9 < M_8$		$8 < 9$: Good
10	$M_{10} < M_9$		$5 < 4.5$: Good
11	$M_{11} > M_{10}$		$10 > 9$: Bad
12	$M_{12} < M_{11}$		$6 > 4.5$: Good
13	$M_{13} > M_{12}$		$12 > 9$: Bad!

A similar procedure can be used to make rough statements about the effect on N_n of using the messages of additional observers. We get

$$N_{n+1} - N_n = s(r M_n - \bar{r} g_1^{(n)}) .$$

Adding an extra channel is by this criterion "better" if $N_{n+1} > N_n$ since N_n is the probability of making a correct inference at the fusion centre. The values of s and r used above give results consistent with those noted above.

An explanation for this apparently bizarre behaviour can be furnished if we assume that the network is almost fault-free ($s \doteq 1$), though garbling may still occur as well as observer unreliability (c). Now

$$M_{2N} \doteq s^{2N} \binom{2N}{N} (r\bar{r})^N ,$$

$$M_{2N+1} \doteq (2N+1) \bar{s} s^{2N} \binom{2N}{N} (r\bar{r})^N ,$$

and putting $s = 1$ we have $M_n \neq 0$ for even channels, and yet zero for an odd number. The reason is clear. There can only be an erasure when the number of channels is even. It is therefore not surprising that when s is very close to unity $M_{n+1} - M_n$ should alternate in sign : nor, moreover, that for a highly reliable system an odd number of channels is preferable to an even number in yielding an extremely low probability of erasure.

For large values of N and s close to 1, we also have by use of Stirling's formula,

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$$M_{2N} \sim (4r\bar{r}s^2)^N / (\pi N)^{1/2}, \quad M_{2N+1} \sim (2N+1) \bar{s} M_{2N}. \quad (\text{Eq. 150})$$

Similarly, when $s = 1$,

$$N_{2N} = r^{2N} + \binom{2N}{1} r^{2N-1} \bar{r} + \dots + \binom{2N}{N-1} r^{N+1} \bar{r}^{N-1}.$$

$$N_{2N+1} = r^{2N+1} + \binom{2N+1}{1} r^{2N} \bar{r} + \dots + \binom{2N+1}{N} r^{N+1} \bar{r}^N.$$

The sums can be approximated for large N by using the normal approximation to the binomial. For even $n = 2N$ we have as $N \rightarrow \infty$,

$$P\left(\alpha < \frac{m-2Nr}{\sqrt{2Nrr}} < \beta\right) \sim \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\beta} e^{-1/2x^2} dx.$$

Now

$$N_{2N} = P(N+1 < m < 2N) \sim \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\beta} e^{-1/2x^2} dx$$

where

$$\beta = \left(\frac{2Nr}{r}\right)^{1/2}, \quad \alpha = \frac{N(\bar{r}-r) + 1}{(2Nrr)^{1/2}}.$$

Then, if

$$r > 1/2, \quad N_{2N} \rightarrow 1$$

$$r = 1/2, \quad N_{2N} \rightarrow 1/2$$

$$r < 1/2, \quad N_{2N} \rightarrow 0$$

and for s very close to unity a factor s^{2N} can be inserted on the right hand sides for evaluation for particular finite but large N .

Similar results are obtained for odd n , and let it not be forgotten that the model provides for a very bad observer who, with an equally bad case of noise on the channel, can obtain as good results as an excellent observer with crystal transmission clarity.

Conversely, when the channels in the network have a large unerviceability probability ($\bar{s} \rightarrow 1$), M_n is dominated by $\bar{s}^n$, monotonic decreasing with n . On the other hand,

$$N_n \underset{D}{=} \bar{s}^{n-1} s r n,$$

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and the probability of a correct inference at the fusion centre, conditional on there having been no erasure, is for even $n = 2N$,

$$PR = \frac{{N \choose 2N}}{1 - M_{2N}} = \frac{2Ns^{2N-1} s r}{1 - s^{2N} \binom{2N}{N} (rr)^N},$$

and for odd $n = 2N+1$,

$$PR = \frac{(2N+1) \bar{s}^{2N} s r}{1 - (2N+1) s s^{2N} \binom{2N}{N} (rr)^N}.$$

In both cases it is the numerator which dominates for $\bar{s} \doteq 1$.

For non-extreme s and $n \rightarrow \infty$ we can conveniently use an exact representation for Legendre functions given by Abramowitz and Stegun <8>. The formula is

$$P_\nu^\mu(z) = (2\pi)^{-1/2} (z^2 - 1)^{-1/4} \frac{\Gamma(\nu + \mu + 1)}{\Gamma(\nu + 3/2)} \left\{ [z + (z^2 - 1)^{1/2}]^{\nu+1/2} F(1/2 + \mu, 1/2 - \mu; 3/2 + \nu; Z_1) + \right. \\ \left. + i e^{-i\mu\pi} [z - (z^2 - 1)^{1/2}]^{\nu+1/2} F(1/2 + \mu, 1/2 - \mu; 3/2 + \nu; -Z_2) \right\},$$

where

$$Z_1 = \frac{z + (z^2 - 1)^{1/2}}{2(z^2 - 1)^{1/2}}, \quad Z_2 = \frac{z - (z^2 - 1)^{1/2}}{2(z^2 - 1)^{1/2}}. \quad (\text{Eq. 151})$$

This enables any of the coefficients $g_m^{(n)}$ to be expressed in this way.

In particular, the leading terms in M_n are obtained from Eq. 151 with $\mu = 0$. Then

$$M_n = R^n P_n(\bar{s}/R) = \frac{1}{\{2\pi(4r\bar{r}s^2)^{1/2}\}^{1/2}} \frac{\Gamma(n+1)}{\Gamma(n+3/2)} \{ \bar{s} + 2s(r\bar{r})^{1/2} \} \left\{ 1 + \frac{1/4 Z_1}{(3/2 + n)} + \dots \right\}$$

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where

$$Z_1 = \frac{1}{2} \frac{\overline{s} + 2s(\overline{r}\overline{r})^{1/2}}{2s(\overline{r}\overline{r})^{1/2}} = \frac{1}{2} \left\{ 1 + \frac{\overline{s}}{2s(\overline{r}\overline{r})^{1/2}} \right\}, \quad (\text{Eq. 152})$$

and R is given by Eq. 145.

3.2.2 Case 2: A single observer sends his message through a network of n independent channels in parallel

Two questions arise:

- (i) Does the use of several channels enhance system performance in the sense of giving an improved probability of making the correct inference and/or a reduced probability of erasure?
- (ii) Does it pay to use several observers, each with his own independent channel, rather than one?

Note that we do not discuss the loss of channels which might be used for other possibly more important messages.

If the single observer makes a correct interpretation of the message in the field the generating function corresponding to Eq. 137 is $(sg/x + \overline{s} + sgx)^n$: otherwise it is $(s\overline{g}/x + \overline{s} + s\overline{g}x)^n$. Thus, all information about the stochastic parallel information network is embodied in a function $h_n(x)$ given by

$$h_n(x) = c(sg/x + \overline{s} + sgx)^n + \overline{c}(s\overline{g}/x + \overline{s} + s\overline{g}x)^n. \quad (\text{Eq. 153})$$

In terms of the generating function $g_n(x)$ given by Eq. 137 we have

$$h_n(x) = c g_n(x | c=1) + \overline{c} g_n(x | c=0) \quad (\text{Eq. 154})$$

since $r = c\overline{g} + \overline{c}g$, $\overline{r} = cg + \overline{c}\overline{g}$ reduce to the required values when $c = 0$ and 1 . The notation $g_n(x | c = c')$ means put $c = c'$ in $g_n(x)$.

We shall now give specific expressions for the basic quantities of interest in cases 1 and 2 for $n = 1, 2, 3, 4$. Some numerical values will then be calculated and discussed briefly.

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The quantities of interest are:

- M_n Probability of erasure
- N_n Probability of correct inference
- PR Conditional probability of correct inference, given no erasure
- I_E The average gain in information.

We shall differentiate the single observer case by using a prime.

In the first place it is clear that M'_n is in form identical with M_n , r replacing g . It follows that M_n exceeds M'_n as long as

$$r\bar{r} > g\bar{g} .$$

Now

$$r\bar{r} - g\bar{g} = c\bar{c}(g - \bar{g})^2$$

and thus $M_n > M'_n$ for any c and g except $g = 0.5$, when they are equal. They are also equal for any g when $c = 0$ or 1 . Thus, except in these rare circumstances, there is a higher probability of erasure of messages passed by several observers compared with that of a single message passed down the same number of channels.

It is more difficult to make such general statements about $N_n - N'_n$.

The particular values are as follows:

$n = 1$

$$\begin{aligned} M_1 &= \bar{s} , & M'_2 &= \bar{s} \\ N_1 &= sr , & N'_1 &= sr \end{aligned}$$

$n = 2$

$$\begin{aligned} M_2 &= \bar{s}^2 - 2s^2 r\bar{r} , & M'_2 &= \bar{s}^2 + 2s^2 g\bar{g} \\ N_2 &= 2s\bar{s}r + s^2 r^2 , & N'_2 &= 2s\bar{s}r + s^2 (c\bar{g}^2 + \bar{c}g^2) \end{aligned}$$

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$$n = 3$$

$$M_3 = \overline{s}^3 + 6\overline{s}s^2 \overline{r}, \quad M'_3 = \overline{s}^3 + 6\overline{s}s^2 \overline{g}$$

$$N_3 = 3\overline{s}^2 s r + 3\overline{s}s^2 r^2 + s^3 (r^3 + 3r^2 \overline{r}) ,$$

$$N'_3 = 3\overline{s}^2 s r + 3\overline{s}s^2 (c\overline{g}^2 + \overline{c}g^2) + s^3 (c\overline{g}^3 + \overline{c}g^3 + 3rg\overline{g})$$

$$n = 4$$

$$M_4 = \overline{s}^4 + 12\overline{s}^2 s^2 \overline{r} + 6s^4 (\overline{r})^2 , \quad M'_4 = \overline{s}^4 + 12\overline{s}^2 s^2 \overline{g} + 6s^4 (\overline{g})^2$$

$$N_4 = 4\overline{s}^3 s r + 6\overline{s}^2 s^2 r^2 + 4\overline{s}s^3 (r^3 + 3r^2 \overline{r}) + s^4 (r^4 + 4r^3 \overline{r})$$

$$N'_4 = 4\overline{s}^3 s r + 6\overline{s}^2 s^2 (c\overline{g}^2 + \overline{c}g^2) + 4\overline{s}s^3 (c\overline{g}^3 + \overline{c}g^3 + 3g\overline{g}r) + s^4 (c\overline{g}^4 + \overline{c}g^4 + 4g\overline{g}(c\overline{g}^2 + \overline{c}g^2))$$

After some algebra we find that

$$N_2 - N'_2 = -c\overline{c}s^2 (\overline{g} - g)^2 < 0 ,$$

while

$$N_3 - N'_3 = -c\overline{c}(\overline{g} - g)^2 s^2 ((1-2r)s + 3\overline{s}) ,$$

whose sign is not unequivocal.

The following tables give values of three quantities.

PR_n : the probability that the fusion centre makes the correct inference, given that an inference is made.

M_n : the probability of erasure, that is, insufficient information available for an inference to be made using the "majority algorithm" described in Sects. 3.1 and 3.2.

IE : the mean increase in information resulting from the messages.

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The independent variables are, in order, p , c , s , g , n .

- p : the probability that the message presented to the observer is the binary digit 1. The alternative is that it is 0, with probability $\bar{p} = 1-p$. This is less restrictive than may seem. For example:
- (a) the area searched contains less than two targets = 1; at least two = 0;
 - (b) the landing will take place between Calais and Boulogne = 1; elsewhere = 0.
- c : The probability that the observer correctly identifies, or decodes the message.
- s : The probability that each channel has of being in a condition to transmit information.
- g : The probability that each channel has of reversing the sense of the message sent.
- n : The number of information channels available to the observers, $n = 1, 2, 3, 4$.

In point of fact p is irrelevant to the calculation of PR and M_n and only enters as a prior probability whose value is modified by the information (in the technical sense) received at the fusion centre.

For each value of n , two values are given. The first column refers to a single observer using n channels, the second to n independent observers each using an independent channel.

3.2.2 Comments on the tables

Comment (1): For fixed s and c and each n :

$$\left. \begin{array}{l} (a) \quad PR(g) = 1 - PR(\bar{g}) \\ (b) \quad M_n(g) = M_n(\bar{g}) \\ (c) \quad I_E(g) = I_E(\bar{g}) \end{array} \right\} \text{ for both a single observer and many.}$$

Note that the arguments are introduced for emphasis. Of course, PR and the others depend on n , c , s as well as g .

(1b) is obvious theoretically since M_n in both cases is a function of $\bar{r}\bar{r} = (c\bar{g} + \bar{c}g)(cg + \bar{c}\bar{g})$ which is symmetrical in g and $\bar{g}$. It is also symmetrical in c and $\bar{c}$, a reference to which will be made below.

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TABLE 21

COMPARISON BETWEEN ONE OBSERVER USING n CHANNELS AND n OBSERVERS USING ONE CHANNEL EACH

P = 0.5													
c = 0.1													
c = 0.9													
PR _n													
M _n													
I _E													
PR													
M _n													
I _E													
n	one	n	one	n	one	n	one	n	one	n	one	n	one
s = 0.1													
g = 0.1	1	0.18	0.18	0.9	0.9	0.0222	0.0222	0.82	0.82	0.9	0.9	0.0222	0.0222
	2	0.1769	0.1750	0.8118	0.8130	0.0426	0.0429	0.8231	0.8251	0.8118	0.8130	0.0426	0.0429
	3	0.1739	0.1691	0.7339	0.7370	0.0615	0.0624	0.8261	0.8301	0.7339	0.7370	0.0615	0.0624
	4	0.1710	0.1650	0.6649	0.6705	0.0790	0.0808	0.8290	0.8350	0.6649	0.6705	0.0790	0.0808
g = 0.5	1	0.5	0.5	0.9	0.9	0	0	0.5	0.5	0.9	0.9	0	0
	2	0.5	0.5	0.815	0.815	0	0	0.5	0.5	0.815	0.815	0	0
	3	0.5	0.5	0.7425	0.7425	0	0	0.5	0.5	0.7425	0.7425	0	0
	4	0.5	0.5	0.6804	0.6804	0	0	0.5	0.5	0.6804	0.6804	0	0
g = 0.9	1	0.82	0.82	0.9	0.9	0.0222	0.0222	0.18	0.18	0.9	0.9	0.0222	0.0222
	2	0.8231	0.8250	0.8118	0.8130	0.0426	0.0429	0.1769	0.1750	0.8118	0.8130	0.0426	0.0429
	3	0.8267	0.8301	0.7339	0.7370	0.0615	0.0624	0.1739	0.1699	0.7339	0.7370	0.0615	0.0624
	4	0.8290	0.8350	0.6649	0.6705	0.0790	0.0808	0.1710	0.1650	0.6649	0.6705	0.0790	0.0808
s = 0.5													
g = 0.1	1	0.18	0.18	0.5	0.5	0.1109	0.1109	0.82	0.82	0.5	0.5	0.1109	0.1109
	2	0.1596	0.1451	0.295	0.3238	0.1792	0.1887	0.8404	0.8548	0.295	0.3238	0.1792	0.1887
	3	0.1449	0.1182	0.1925	0.2357	0.2264	0.2521	0.8557	0.8818	0.1925	0.2357	0.2264	0.2521
	4	0.1332	0.0975	0.1330	0.1814	0.2607	0.3059	0.8668	0.9025	0.1330	0.1814	0.2607	0.9058
g = 0.5	1	0.5	0.5	0.5	0.5	0	0	0.5	0.5	0.5	0.5	0	0
	2	0.5	0.5	0.375	0.375	0	0	0.5	0.5	0.375	0.375	0	0
	3	0.5	0.5	0.3125	0.3125	0	0	0.5	0.5	0.3125	0.3125	0	0
	4	0.5	0.5	0.2734	0.2734	0	0	0.5	0.5	0.2734	0.2734	0	0
g = 0.9	1	0.82	0.82	0.5	0.5	0.1109	0.1109	0.18	0.18	0.5	0.5	0.1109	0.1109
	2	0.8404	0.8549	0.295	0.3238	0.1792	0.1887	0.1596	0.1451	0.295	0.3238	0.1792	0.1887
	3	0.8557	0.8818	0.1925	0.2957	0.2264	0.2521	0.1443	0.1182	0.1925	0.2357	0.2264	0.2521
	4	0.8668	0.8025	0.1330	0.1814	0.2607	0.3059	0.1332	0.0975	0.1330	0.1814	0.2607	0.3059
s = 0.9													
g = 0.1	1	0.18	0.18	0.1	0.1	0.1996	0.1996	0.82	0.82	0.1	0.1	0.1996	0.1996
	2	0.1247	0.0781	0.1558	0.2491	0.2675	0.3147	0.8753	0.8219	0.1558	0.2491	0.2675	0.3147
	3	0.1214	0.0810	0.0447	0.0727	0.3090	0.3820	0.8786	0.9190	0.0447	0.0727	0.3090	0.3820
	4	0.1095	0.0449	0.0407	0.1002	0.3334	0.4588	0.8905	0.9557	0.0407	0.1002	0.3334	0.4588
g = 0.5	1	0.5	0.5	0.1	0.1	0	0	0.5	0.5	0.1	0.1	0	0
	2	0.5	0.5	0.415	0.415	0	0	0.5	0.5	0.415	0.415	0	0
	3	0.5	0.5	0.1225	0.1225	0	0	0.5	0.5	0.1245	0.1245	0	0
	4	0.5	0.5	0.2704	0.2704	0	0	0.5	0.5	0.2704	0.2704	0	0
g = 0.9	1	0.82	0.82	0.1	0.1	0.1996	0.1996	0.18	0.18	0.1	0.1	0.1996	0.1996
	2	0.8753	0.9219	0.1558	0.2491	0.2675	0.3147	0.1247	0.0781	0.1558	0.2491	0.2675	0.3147
	3	0.8786	0.9190	0.0447	0.0727	0.3090	0.3820	0.1214	0.0810	0.0447	0.0727	0.3090	0.3828
	4	0.8905	0.9537	0.0407	0.1002	0.3334	0.4588	0.1095	0.0449	0.0407	0.1002	0.3334	0.4588

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(1a) We have $PR = N/(1-M)$ for fixed n , which is omitted as a subscript. For (a) to be true

$$N(g)/(1 - M(g)) = 1 - N(\bar{g})/(1 - M(\bar{g})) = 1 - N(\bar{g})/(1 - M(g)) \quad \text{by (b),}$$

or

$$N(g) = 1 - M(g) - N(\bar{g}) = 1 - M(\bar{g}) - N(\bar{g}) = L(\bar{g}).$$

This is, by definition, true, and so (a) could have been predicted.

(1c) This symmetry is independent of the probabilities p of the message. By definition (see Eq. 141) we have

$$I_E(g) = N(g)\log(N(g)) + L(g)\log(L(g)) - J(g)\log(J(g)) - K(g)\log(K(g))$$

where

$$J = \bar{p}N + pL, \quad K = \bar{p}L + pN.$$

Since

$$N(\bar{g}) = L(g)$$

the first two terms interchange when g is replaced by $\bar{g}$. Also

$$J(\bar{g}) = \bar{p}N(\bar{g}) + pL(\bar{g}) = \bar{p}L(g) + pN(g) = K(g).$$

The result (c) follows.

Note that the symmetries 1(a), (b), (c) hold also with respect to c for fixed s and g and each n . The proofs are similar. Similar proofs substantiate the results for a single observer.

Comment (2): There is another symmetry in c . We see that

$$(a) \quad PR(g, c) = PR(\bar{g}, \bar{c})$$

$$(b) \quad M_n(\bar{g}, \bar{c}) = M_n(g, c)$$

$$(c) \quad I_E(g, c) = I_E(\bar{g}, \bar{c})$$

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These observations emphasize the fact that the model evaluates the performance of a bad observer with a noisy channel as highly as that of a good observer with a noise-free channel.

(2b) Replacing $\bar{c}$ by c as well as $\bar{g}$ by g leaves the value of $r\bar{r}$ unaltered. Hence 2(b) is to be expected.

(2a) If $PR(g,c) = PR(\bar{g},\bar{c})$,

$$\frac{N(g,c)}{L(g,c) + N(g,c)} = \frac{N(\bar{g},\bar{c})}{L(g,c) + N(g,c)} = \frac{L(g,\bar{c})}{N(g,c) + L(g,c)} .$$

But

$$r(c) = c\bar{g} + \bar{c}g , \quad \bar{r}(c) = cg + \bar{c}\bar{g}$$

so that

$$r(\bar{c}) = \bar{c}\bar{g} + cg = \bar{r}(c) , \quad \bar{r}(\bar{c}) = \bar{c}g + c\bar{g} = r(c) .$$

Thus, putting $\bar{c}$ for c in L exchanges $\bar{r}(c)$ for $r(\bar{c})$, and $r(c)$ for $\bar{r}(\bar{c})$. Thus $L(g,\bar{c}) = N(g,c)$ and

$$PR(g,c) = \frac{L(g,\bar{c})}{L(g,c)+N(g,c)} = \frac{N(g,c)}{L(g,c) + N(g,c)} = PR(\bar{g},\bar{c}) ,$$

quod erat demonstrandum.

(2c) Similar argument substantiates 2(c) theoretically.

In what follows the prime denotes the value corresponding to a single observer and n channels.

Comment (3): The tables are consistent with the proven fact that for fixed $n > 1$ and $g \neq \bar{g}$, $M_n > M'_n$ and that when $g = \bar{g} = 0.5$, $M_n = M'_n$.

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Comment (4): The figures suggest that, uniformly with respect to g and for each $n > 1$,

$$(a) \quad PR' > PR \quad \text{for} \quad g < \bar{g} \quad \text{and} \quad c < \bar{c}$$

$$g > \bar{g} \quad \text{and} \quad c > \bar{c}$$

$$(b) \quad PR' = PR = 0.5 \quad \text{for} \quad g = \bar{g}$$

$$(c) \quad PR' < PR \quad \text{for} \quad g > \bar{g}, \quad c > \bar{c}$$

$$g < \bar{g}, \quad c < \bar{c}$$

This will be shown below to be true theoretically.

Comment (5): The figures suggest that

(a) As n increases, both I_E and I'_E increase. In other words, to add channels is to increase the mean gain in information.

(b) For fixed n , $I_E > I'_E$, that is, adding observers increases the mean gain in information. This accords with intuition.

Comment (6): As $s \rightarrow 1$ the phenomenon of oscillating M_n as well as M'_n is seen as n increases. This was explained above for M_n and the same argument holds for M'_n . The tabular values of PR oscillate also in the same region as expected from earlier discussion; this is not seen to be the case for PR' . However, a finer grain tabulation does reveal the phenomenon, as might be expected.

Analytic examination of the sign of $PR' - PR$ is of interest. First, omitting subscripts, we have

$$PR' - PR = \frac{N'}{N'+L} - \frac{N}{N+L} = \frac{\Delta}{(N'+L')(N+L)}$$

where

$$\Delta \equiv \Delta(r, g) = L(r) N'(g) - L'(g) N(r) \quad . \quad (\text{Eq. 155})$$

Thus, the sign of $PR' - PR$ is the sign of Δ .

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To facilitate the analysis we recall that

$$\begin{aligned}
 N(x) = & \binom{n}{1} \overline{s}^{n-1} s x + \binom{n}{2} \overline{s}^{n-2} s^2 x^2 + \binom{n}{3} \overline{s}^{n-3} s^3 (x^3 + \binom{3}{1} x^2 \overline{x}) + \\
 & + \binom{n}{4} \overline{s}^{n-4} s^4 (x^4 + \binom{4}{1} x^3 \overline{x}) + \dots + \binom{n}{n} s^n (x^n + \binom{n}{1} x^{n-1} \overline{x} + \\
 & + \dots + \binom{n}{n_2} x^{n-n_2} \overline{x}^{n_2})
 \end{aligned}$$

where $n_2 = n/2 - 1$ if n is even, and $(n-1)/2$ if n is odd.

Also,

$$L(r) = N(\overline{r})$$

$$N'(g) = cN(\overline{g}) + \overline{c}N(g)$$

$$L'(g) = cN(g) + N(\overline{g})\overline{c}$$

$$r = c\overline{g} + \overline{c}g$$

$$\overline{r} = cg + \overline{c}\overline{g} = 1 - r$$

Putting $g = \overline{g}$ we have $\overline{r} = \overline{g}$, $r = g$ and

$$\begin{aligned}
 \Delta(r, g) \rightarrow \Delta(g, g) &= L(g)N'(g) - L'(g)N(g) \\
 &= N(\overline{g})(cN(g) + \overline{c}N(g)) - N(g)(cN(g) + \overline{c}N(\overline{g})) = 0.
 \end{aligned}$$

Thus $g - \overline{g}$ is a factor of $\Delta(r, g)$.

Similarly, putting $c = \overline{c}$ we have

$$r = c$$

$$\overline{r} = \overline{c}$$

and

$$\Delta(r, g) \rightarrow \Delta(c, g) = cN(c)(N(\overline{g}) + N(g)) - cN(c)(N(g) + N(\overline{g})) = 0$$

and $c - \overline{c}$ is also a factor of $\Delta(r, g)$.

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Next put $c = 0$ and $\Delta(r, g) \rightarrow \Delta(g, g) = N(\bar{g})N(g) - N(g)N(\bar{g}) = 0$.

Similarly, if $c = 1$, $\Delta = 0$ and thus $\bar{c}\bar{c}$ divides $\Delta(r, g)$. Hence we can write

$$\Delta(r, g) = c\bar{c}(c-\bar{c})(g-\bar{g}) E(r, g) . \quad (\text{Eq. 156})$$

Finally we need information about the sign of $E(r, g)$. This can be looked at for small c and g , and then also for c and g nearly equal to unity. Take the former case. First note that $N(0) = 0$ and $N(1) = 1 - \bar{s}^n$. Next

$$\begin{aligned} \frac{dN(x)}{dx} &= \binom{n}{1} \bar{s}^{n-1} s + n(n-1) \bar{s}^{n-2} s^2 x + \binom{n}{3} \bar{s}^{n-3} s^3 (3x^2 + 2\binom{3}{1} x \bar{x} - \binom{3}{1} x^2) + \\ &+ \binom{n}{4} \bar{s}^{n-4} s^4 (4x^3 + \binom{4}{1} 3x^2 \bar{x} - \binom{4}{1} x^3) + \dots + \binom{n}{n} s^n n x^{n-1} + \\ &+ \binom{n}{1} (n-1) x^{n-2} \bar{x} - \binom{n}{1} x^{n-1} + \binom{n}{2} (n-2) x^{n-3} \bar{x}^2 - 2\binom{n}{2} x^{n-2} \bar{x} + \\ &+ \dots + \binom{n}{n_2} (n-n_2) x^{n-n_2-1} \bar{x}^{n_2-n_2} \binom{n}{n_2} x^{n-n_2} \bar{x}^{n_2-1} . \end{aligned}$$

Thus

$$\begin{aligned} (dN(x)/dx)_{x=0} &= B_0 , \quad \text{where} \quad B_0 = n \bar{s}^{n-1} s , \\ (dN(x)/dx)_{x=1} &= B_0 + B_1 , \quad \text{where} \quad B_1 = n(n-1) \bar{s}^{n-2} s^2 . \end{aligned}$$

For small x we then have to the first order in x ,

$$\begin{aligned} N(x) &= B_0 x \\ N(\bar{x}) &= N(1) - x(B_0 + B_1) . \end{aligned}$$

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Now let c and g be so small that c^2 , g^2 , cg and higher order terms can be neglected. To this order, $r = c+g$, $\bar{r} = 1-c-g$,

$$L(r) = N(\bar{r}) = N(1) - r(B_0 + B_1)$$

$$N'(g) = cN(\bar{g}) + \bar{c}n(g) = cN(1) + B_0g$$

$$L'(g) = cN(g) + \bar{c}N(\bar{g}) = N(1) - cN(1) - g(B_0 + B_1)$$

$$N(r) = B_0(c+g) \quad .$$

To the same order

$$\Delta(r,g) = cN(1)(N(1)-B_0) = c(1-\bar{s}^n)(1-\bar{s}^n-n\bar{s}^{n-1}s) > 0 \quad \text{when } 0 < s < 1 \quad .$$

From Eq. 156, to the same order,

$$\Delta(r,g) = cE(r,g)$$

and it can be concluded that $E(r,g) > 0$ over the range of practical interest and, moreover, that $E(r,g)$ contains $N(1) = 1-\bar{s}^n$ as a factor. Note in particular that for $n = 2$

$$\Delta(r,g) = c\bar{c}(c-\bar{c})(g-\bar{g})^3(1-\bar{s})^2 \quad .$$

This is exact and there may be a simple exact general formula for arbitrary n , only we not yet found it if there is.

In conclusion we have shown that

$$\Delta(r,g) > 0 \quad \text{when } 0 < c < 1/2, \quad 0 < g < 1/2, \quad \text{and } 1/2 < c < 1, \quad 1/2 < g < 1$$

$$\Delta(r,g) = 0 \quad \text{when } c = 0, 1/2, 1 \quad \text{and when } g = 1/2$$

$$\Delta(r,g) < 0 \quad \text{when } 0 < c < 1/2, \quad 1/2 < g < 1, \quad \text{and } 1/2 < c < 1, \quad 0 < g < 1/2 \quad .$$

Thus, for fixed $0 < c < 1/2$, $\Delta(r,g)$ increases from zero to a maximum as g increases from zero, returning to zero when $g = 1/2$; it then decreases to a minimum as g increases from $1/2$ and returns to zero when $g = 1$. The behaviour for fixed g and variable c is similar.

The fact that when $g = \bar{g} = 0.5$, $PR = PR' = 0.5$ follows because in this case $r = 0.5$, $N(r) = L(r)$ and $PR = N(r)/(N(r)+L(r)) = 0.5$. Similarly for PR' .

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4 Management of an information channel with a priority facility

4.1 INTRODUCTION TO "HEAD-OF-THE-LINE" PRIORITY

The models and analysis presented in the next two chapters are addressed to the problem of quantifying the degradation that may be imposed on a communication system by the abuse of priority. There is quite a long road to travel before the goal is reached and the mathematical details will require patience on the part of those readers unacquainted with queuing theory, even though the methods are quite routine. Some technical notes are given in App. A. The practitioner concerned with the management of information flow may wish to jump straight to Sects. 4.7.2, 4.8.3, 4.8.5, 4.9, where the findings are presented and discussed.

The main principles and lessons can be extracted from a simple model. This consists of a single channel service (or queuing) system. Messages are supposed to arrive for onward transmission in a Poisson stream with constant rate λ requests per unit time. The lengths of messages are random variables with a common exponential distribution, parameter μ : μ is also a rate, and its reciprocal is the mean time required to transmit the message. It is also taken to be time-independent. Two further assumptions are made:

(i) The stream of requests and the service mechanism are statistically independent;

(ii) A state of statistical equilibrium reigns.
Roughly speaking, assumption (ii) means that the service is not swamped by the demand and that the system has been in operation long enough for the various probabilities of interest, for example, the number of messages awaiting transmission, to be time-independent. The condition for this is that $\lambda < \mu$. In real life there will be periods when $\lambda \geq \mu$. In this case the probability that a finite number of messages awaits tends to zero. But there is little point in trying to understand chaos before coming to terms with order which itself, as we shall see, holds alarming prospects for non-priority messages.

To silence, at least temporarily, potential critics of the model's simplicity, we note that:

(i) Multiple transmission channels can be simulated roughly by appropriate multiplication of μ ;

(ii) A "loss" system can be analyzed, albeit with some trouble. Such a system imposes a ceiling on the total number of messages allowed to wait. When the system is full, new messages are "turned away". This models management of the system by adjustment of demand. There are other methods.

Finally, let it be said that the model and assumptions are essentially those underlying the studies made by Erlang in the early years of this century of the Copenhagen telephone service. This work laid the foundations of modern queueing theory. A thorough discussion of the principles of priority queues is given in <9>.

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4.2 THE MECHANISM

The arrival stream will be assumed composed of two parts. A fraction α of messages are assigned priority in the sense that, irrespective of other messages waiting, they are serviced as soon as the channel becomes free. We do not assume that they replace a non-priority message currently in service: that would be unrealistic in this context. This means that the mean arrival rate of priority messages is $\alpha\lambda$, and that of non-priority messages is $\bar{\alpha}\lambda$, where $\bar{\alpha} = 1 - \alpha$. In considering the waiting time processes of priority and non-priority messages, which is of central interest, it will be assumed that those messages which arrive when the channel is busy wait in a "queue" and are subsequently serviced in order of arrival. It is convenient to think thus of two queues, one of priority messages which is serviced as long as it contains members, and a second of non-priority messages which can only receive attention when the other is empty.

4.3 STATISTICAL DESCRIPTION OF SYSTEM STATE

The first task is to describe the system state. This is a vector random variable with three components (X, Y, Z) . X and Y are, respectively, the numbers of messages in the priority and non-priority queues, including the message in service. $Z = 1$ when a priority message is being transmitted and $Z = 0$ when a non-priority message is being transmitted. The value of Z when $X = Y = 0$ is immaterial. Let

$$\begin{aligned} a_{m,n} &= P(X = m, Y = n, Z = 1) & (m \geq 1, n \geq 0) \\ b_{m,n} &= P(X = m, Y = n, Z = 0) & (n \geq 1, m \geq 0) \end{aligned} \quad (\text{Eq. 157})$$

To an onlooker concerned only with the total number of messages this is a stationary M/M/1 (see Appendix A) system, and it is known that

$$p_n = P(X + Y = n) = r^n(1-r), \quad (n \geq 0) \quad (\text{Eq. 158})$$

where

$$r = \lambda/\mu.$$

Thus,

$$\sum_{m=1}^n a_{m,n-m} + \sum_{m=0}^{n-1} b_{m,n-m} = p_n. \quad (\text{Eq. 159})$$

The difference equations satisfied by the probabilities can be obtained by a technique called the "method of balance", or otherwise, according to taste. The method of balance is the expression of a conservation principle

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which must hold for a system of this kind in statistical equilibrium; the details can be found in reasonable elementary texts on queuing theory. We have:

$$a_{1,0} + b_{0,1} = p_1 = r p_0 \quad (\text{Eq. 160})$$

$$(1+r) a_{1,0} = a_{2,0} + b_{1,1} + \alpha r p_0 \quad (\text{Eq. 161})$$

$$(1+r) b_{0,1} = a_{1,1} + b_{0,2} + \bar{\alpha} r p_0 \quad (\text{Eq. 162})$$

More generally, for a total of $n > 1$ messages waiting, including that being transmitted,

$$(1+r) a_{n0} = a_{n+1 0} + b_{n 1} + \alpha r a_{n-1 0} \quad (n > 2)$$

$$(1+r) a_{n-1 1} = a_{n 1} + b_{n-1 2} + \alpha r a_{n-2 1} + \bar{\alpha} r a_{n-1 0}$$

$$(1+r) a_{n-2 2} = a_{n-1 2} + b_{n-2 3} + \alpha r a_{n-3 2} + \bar{\alpha} r a_{n-2 1} \quad (n > 3)$$

.....

$$(1+r) a_{2 n-2} = a_{3 n-2} + b_{2 n-1} + \alpha r a_{1 n-2} + \bar{\alpha} r a_{2 n-3}$$

(Eq. 163)

$$(1+r) a_{1 n-1} = a_{2 n-1} + b_{1 n} + \bar{\alpha} r a_{1 n-2}$$

$$(1+r) b_{0 n} = a_{1 n} + b_{0 n+1} + \bar{\alpha} r b_{0 n-1} \quad (n > 2)$$

(Eq. 164)

$$(1+r) b_{1 n-1} = \alpha r b_{0 n-1} + \bar{\alpha} r b_{1 n-2}$$

$$(1+r) b_{2 n-2} = \alpha r b_{1 n-2} + \bar{\alpha} r b_{2 n-3} \quad (n > 3)$$

.....

$$(1+r) b_{n-2 2} = \alpha r b_{n-3 2} + \bar{\alpha} r b_{n-2 1}$$

$$(1+r) b_{n-1 1} = \alpha r b_{n-2 1} \quad (n > 2)$$

(Eq. 165)

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Summation of these equations and use of Eq. 159 yields

$$(1+r) p_n = p_{n+1} + r p_{n-1} \quad (n \geq 1)$$

(Eq. 166)

the familiar difference equation satisfied by the M/M/1 equilibrium state probabilities with solution given by Eq. 158.

4.4 NON-PRIORITY SYSTEM TIME: GENERAL DISCUSSION

Before discussing the solution of the set (Eqs. 160 to 165), it is worthwhile examining briefly the plight of a non-priority message. For this purpose let us consider a random variable called here "system time", the sum of queueing time and transmission time, often called "waiting time including service" in queueing texts. The system time of a priority message is reasonably uncomplicated and is, in general, the sum of the residual transmission times of the priority messages ahead and own transmission time. Thus we are dealing with weighted sums of random variables whose densities are autoconvolutions of the basic transmission time density, $b(t) = \mu \exp(-\mu t)$, with weights derived from appropriate combinations of the solutions of Eqs. 160 to 165.

The case of a non-priority message is different and can be illuminated by designating special epochs in the evolution of the process as non-priority service opportunities. Such an epoch occurs when a message has just been transmitted and the priority queue is empty. This means that the next message in the non-priority queue can be processed.

Suppose that such a service opportunity has just occurred and that a non-priority message has begun transmission. Let X be the time to the next non-priority service opportunity. By definition the priority queue was empty at the start. If no priority message arrives during the transmission time S , $X = S$: if one priority message arrives X is equal to S plus the transmission time B_1 of the priority arrival and of all the priority arrivals which take place until the priority queue is next empty and the message in service has completed transmission. B_1 is called a "busy period" in the queueing literature. If n priority arrivals occur during S , $X = S + B_n$, where B_n is the sum of the busy periods initiated by each of the n priority arrivals.

Let B_1 have probability density function $k(t)$ and denote by $k_n(t)$ the n -fold autoconvolution of $k(t)$. Then, conditional on there being n priority arrivals during S , the probability density function of X is $b(t)*k_n(t)$, where $*$ denotes convolution of the functions it separates.

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Given that the Poisson arrival process of priority customers has mean rate $\alpha\lambda$ we then find for the probability density function $f(t)$ of X the expression

$$f(t) = \mu e^{-\alpha\lambda t - \mu t} + e^{-\alpha\lambda t} \sum_{n \geq 1} \frac{(\alpha\lambda t)^n}{n!} \mu e^{-\mu t} * k_n(t) .$$

Let

$$\phi(z) = \int_0^{\infty} e^{-zt} f(t) dt$$

be the Laplace transform of $f(t)$ and $\kappa(z)$ that of $k_1(t)$. Finally let

$$Z = z + \mu + \alpha\lambda$$

Then

$$\begin{aligned} \phi(z) &= \frac{\mu}{Z} \sum_{n \geq 0} \left(\frac{\alpha\lambda}{Z} \right)^n \frac{\Gamma(n+1)}{n!} \\ &= \frac{\mu}{Z - \alpha\lambda \kappa(z)} . \end{aligned} \quad (\text{Eq. 167})$$

It is known that, provided that $\alpha\lambda < \mu$,

$$\kappa(z) = (Z - R)/(2\alpha\lambda) \quad (\text{Eq. 168})$$

where

$$R^2 = Z^2 - 4\alpha\lambda\mu . \quad (\text{Eq. 169})$$

Note that when $\alpha\lambda < \mu$, B_1 is a proper random variable (finite with probability one). This means that the system as a whole could be operated with $\lambda > \mu$, which would mean that, overall, it would be in disequilibrium and that the number of messages waiting would grow indefinitely. But if, at the same time $\alpha\lambda < \mu$, the priority queue would be in equilibrium and priority messages would have finite waiting times, while the queue of the non-priority messages would grow indefinitely, giving rise to the increasing likelihood of their never being transmitted at all. This is a possible consequence of the abuse of priority to which allusion was made at the beginning.

Although X is only a component of the system time of a non-priority message it is of interest to calculate $E[X]$ and $\text{Var}[X]$. These can be deduced directly from $\phi(z)$. We have

$$\ln \phi(z) = \ln \mu - \ln(Z - \alpha\lambda \kappa(z)) .$$

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Thus

$$\frac{\phi'(z)}{\phi(z)} = - \frac{(1 - \alpha\lambda\kappa'(z))}{Z - \alpha\lambda\kappa(z)}$$

and

$$\frac{\phi''(z)}{\phi(z)} - \left[\frac{\phi'(z)}{\phi(z)} \right]^2 = \frac{\alpha\lambda\kappa''(z)}{Z - \alpha\lambda\kappa(z)} + \left[\frac{1 - \alpha\lambda\kappa'(z)}{Z - \alpha\lambda\kappa(z)} \right]^2$$

Since when $\alpha\lambda < \mu$, $\phi(0) = 1$,

$$\phi'(0) = -E[X] \quad , \quad \frac{\phi''(0)}{\phi(0)} - \left[\frac{\phi'(0)}{\phi(0)} \right]^2 = \text{Var}[X] \quad ,$$

we have

$$E[X] = \frac{1}{\mu(1-\alpha r)} \quad , \quad \text{Var}[X] = \frac{1 + \alpha r}{\mu^2(1-\alpha r)^3} \quad , \quad (\text{Eq. 170})$$

where $r = \lambda/\mu$. Notice that $\text{Var}[X]$ is inversely proportional to $(1 - \alpha r)^3$ which is worse than exponential and disastrous as $\alpha r \rightarrow 1$.

The corresponding statistics of X for priority messages are $1/\mu$ and $1/\mu^2$, since in this case X is statistically identical with the service time, and so, although X is but a component of system time it is clear that the priority system can disadvantage non-priority messages to such an extent that they never get transmitted at all.

4.5 GENERAL FORM OF PROBABILITY DENSITIES OF SYSTEM TIME

A full treatment of the system time problem calls for solution of Eqs. 160 to 165 which will be treated later.

Let $g_p(t)$, $g_{np}(t)$ denote respectively the system time probability density functions of priority and non priority messages. Consider first priority messages and write

$$A_m = \sum_{n \geq 0} a_{mn} \quad (m \geq 1) \quad ,$$

$$B_m = \sum_{n \geq 1} b_{mn} \quad (m \geq 0)$$
(Eq. 171)

and

$$f_n(t) = f_{n-1}(t) * f(t) \quad ,$$

so that f_n is the n -fold autoconvolution of f .

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Then

$$g_p(t) = \mu e^{-\mu t} [p_0 + \sum_{m \geq 1} (A_m + B_{m-1}) (\mu t)^m / m!] \quad . \quad (\text{Eq. 172})$$

The explanation is as follows.

For a priority message to arrive and find $m-1$ priority messages ahead in the queue, and one more in transmission, the probability is

$$\sum_{n \geq 0} a_{mn} = A_m \quad .$$

The time required to transmit the residue of the message in transmission, the next $m-1$ messages, and then the message which has just arrived, has a probability density function which is the $(m+1)$ -fold autoconvolution of $\mu \exp(-\mu t)$, namely

$$\mu e^{-\mu t} (\mu t)^m / m! \quad .$$

Similarly for a priority message to arrive and find $m (> 0)$ priority messages waiting and a non-priority message being transmitted has probability

$$\sum_{n \geq 1} b_{mn} = B_m \quad .$$

The wait is the residual transmission time of the current message in transmission plus the sum of the transmission times of the m priority messages already present plus own transmission time; again the $(m+1)$ -fold autoconvolution of $\mu \exp(-\mu t)$ is needed. This explains the terms under the summation sign in Eq. 172. The remaining term caters for the case where the new message finds an empty system.

For a non-priority message the situation is a little more complicated. The result is

$$\begin{aligned} g_{np}(t) = & \mu p_0 e^{-\mu t} + \sum_{m \geq 1} a_{m0} k_m(t) * \mu e^{-\mu t} + \sum_{n \geq 1} b_{0n} f_n(t) * \mu e^{-\mu t} \\ & + \sum_{m \geq 1} \sum_{n \geq 1} a_{mn} k_m(t) * f_n(t) * \mu e^{-\mu t} \\ & + \sum_{m \geq 1} \sum_{n \geq 1} b_{mn} k_{m+1} * f_{n-1}(t) * \mu e^{-\mu t} \quad . \quad (\text{Eq. 173}) \end{aligned}$$

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The case of an empty system is dealt with as above. If the arriving message finds no message ahead in the non-priority queue and $m(>1)$ messages in the priority part of the system (probability a_{m0}) it must await the clearance of the m and any other priority messages that arrive meanwhile. The associated probability density function is $k_m(t)$ and this must be convolved with $\mu \exp(-\mu t)$ to give system time. This accounts for the first sum in Eq. 173. If the arriving message finds $n(>0)$ ahead in the non-priority queue, one in transmission, and no priority message present, it must wait effectively through n intervals X between service opportunities before being transmitted. Note that lack of memory of the service time distribution of the message being transmitted permits the n -fold convolution entailed to be written $f_n(t)$. This accounts for the second term in Eq. 173. The last two terms correspond to the case where the newly arrived non-priority message finds at least one message of either kind present. Suppose that there are m priority, and n non-priority, messages. If a priority message is being transmitted, the time to the next service opportunity is a busy period beginning with m and has probability density function $k_m(t)$. The further wait is the time required to service the n non-priority messages ahead, and to this must be added own transmission time, giving probability density function

$$k_m(t) * f_n(t) * \mu \exp(-\mu t).$$

If a non-priority message is being transmitted on arrival, the time until the service next becomes available for a non-priority message is a busy period beginning with $m+1$ (that is the one in transmission and the m priority messages waiting), while the remaining wait is the sum of the $n-1$ interservice-opportunity intervals X and, finally, own transmission time. This completes the explanation of Eq. 173. We shall return to this in Sect. 4.8.4.

4.6 PROCESSING OF EQUATIONS 160 TO 165 AND THE EXPLICIT FORM OF $g_p(t)$

Next we turn to the problem of processing the set of difference Eqs. 160 to 165 for the state probabilities $a_{m,n-m}$ and $b_{m,n-m}$, with a view to finding $g_p(t)$.

To find $g_p(t)$ we must calculate A_m and B_{n-1} . It is in fact straightforward to construct the following difference equations for these quantities:

$$(1+\alpha r) A_1 = A_2 + B_1 + \alpha r p_0 \quad (\text{Eq. 174})$$

$$(1+\alpha r) A_2 = A_3 + B_2 + \alpha r A_1 \quad (\text{Eq. 175})$$

.....

$$(1+\alpha r) A_m = A_{m+1} + B_m + \alpha r A_{m-1} \quad (\text{Eq. 176})$$

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Since the form (Eq. 176) contains A_1 when $m = 2$, it follows that the solution found for A_m ($m > 2$) holds also for $m = 1$, and substitution into (Eq. 174) will determine a constant.

In addition

$$(1+r) B_0 = A_1 - a_{10} + B_0 - b_{01} + \bar{\alpha} r p_0 + \bar{\alpha} r B_0$$

from which, using Eq. 160,

$$\alpha r B_0 = A_1 - \alpha p_1 \quad . \quad (\text{Eq. 177})$$

For $n > 1$ we have also

$$(1+\alpha r) B_n = \alpha r B_{n-1} \quad (\text{Eq. 178})$$

so that

$$B_n = \sigma^n B_0 \quad (\text{Eq. 179})$$

where

$$\sigma = \alpha r / (1+\alpha r) \quad . \quad (\text{Eq. 180})$$

The solution of Eq. 176 is of the form

$$A_m = K_1 \xi_1^m + K_2 \xi_2^m$$

plus any particular solution, u_m , which satisfies Eq. 176, given B_m . ξ_1 and ξ_2 are the roots of $x^2 - (1+\alpha r)x + \alpha r = 0$, that is,

$$\xi_1, \xi_2 = \frac{(1+\alpha r) \pm (1-\alpha r)}{2} \quad .$$

The larger root is $\xi_1 = 1$, and the smaller, $\xi_2 = \alpha r (< 1)$. Since $\sum A_m$ must be finite (it is a probability), the ξ_1 term has to be excluded from the solution. Thus

$$\text{where } A_m = K(\alpha r)^m + u_m$$

$$\text{and } f(E) u_m = B_m = B_0 \sigma^m$$

$$f(E) = 1 + \alpha r - E - \alpha r/E \quad .$$

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E is the usual shift operator which has the effect $E^n u_m = u_{m+n}$ for positive and negative n .

To find u_m , consider

$$f(E) \sigma^m = \sigma^{m-1} (\alpha r - \sigma^2 - \alpha r) = -\sigma^{m+1}.$$

Thus

$$-\sigma^{m-1} = \frac{1}{f(E)} \sigma^m,$$

and the solution sought is

$$A_m = K(\alpha r)^m - B_0 \sigma^{m-1},$$

wherein K and B_0 have still to be found.

Since the above solution has to be valid for $m-1$ we obtain by substitution into Eq. 174.

$$K = \frac{B_0}{\sigma} + p_0.$$

Thus

$$A_m = B_0[(1+\alpha r)(\alpha r)^{m-1} - \sigma^{m-1}] + p_0(\alpha r)^m. \quad (\text{Eq. 181})$$

It remains to find B_0 , and this can be done by normalization or by using Eq. 177; pursuing the path of normalization we find that

$$1 = p_0 + \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} a_{mn} + \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} b_{mn} = p_0 + \sum_{m \geq 0} B_m + \sum_{m \geq 1} A_m.$$

After some calculation we get

$$B_0 = \bar{\alpha} r / (1+\alpha r) = (\bar{\alpha}/\alpha) \sigma, \quad (\text{Eq. 182})$$

and accordingly

$$A_m = \left(\frac{\bar{\alpha}}{\alpha}\right) [(\alpha r)^m - \sigma^m] + p_0(\alpha r)^m \quad (m > 1)$$

(Eq. 183)

$$B_n = \left(\frac{\bar{\alpha}}{\alpha}\right) \sigma^{n+1} \quad (n > 0)$$

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from which we see that

$$A_m + B_{m-1} = \left(\frac{\bar{\alpha}}{\alpha} + p_0\right)(\alpha r)^m . \quad (\text{Eq. 184})$$

Substituting into Eq. 172 and summing gives finally

$$g_p(t) = \mu \left[\left(\frac{1-\alpha r}{\alpha} \right) e^{-\mu(1-\alpha r)t} - \left(\frac{\bar{\alpha}}{\alpha} \right) e^{-\mu t} \right] . \quad (\text{Eq. 185})$$

A check on this is provided by putting $\alpha = 1$, so that all traffic is treated in the same way. In this case the system reduces to the standard M/M/1 system for which system time is known to have density

$$\mu(1-r) e^{-\mu(1-r)t} . \quad (\text{Eq. 186})$$

It is readily seen that Eq. 185 does indeed reduce to this form.

4.7 SOME STATISTICS OF SYSTEM TIME AND SYSTEM STATE

4.7.1 Formulae and the use of Little's Result

We can now establish the effect of a priority system for priority messages in terms of system time, and we can do the same for the less fortunate non-priority messages. Writing W_p for priority system time, we get

$$E[W_p] = \frac{(1+\bar{\alpha}r)}{\mu(1-\alpha r)} , \quad (\text{Eq. 187})$$

$$\text{Var}[W_p] = \frac{1+2\bar{\alpha}r - r^2(1-\alpha^2)}{\mu^2(1-\alpha r)^2} .$$

Little's formula (see App. A) applied to the priority system gives for the mean of the number N_p of priority messages waiting,

$$E[N_p] = \alpha \lambda E[W_p] = \frac{\alpha r(1+\bar{\alpha}r)}{(1-\alpha r)} . \quad (\text{Eq. 188})$$

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Since the mean number present in the total system is $r/(1-r)$, we find for the mean number $E[N_{np}]$ of messages in the non-priority system

$$E[N_{np}] = \frac{r}{1-r} - E[N_p] = \frac{\bar{\alpha}r(1-\alpha r + \alpha r^2)}{(1-r)(1-\alpha r)} . \quad (\text{Eq. 189})$$

Using Little's formula again, we get for the mean system time $E[W_{np}]$ of non-priority messages,

$$E[W_{np}] = E[N_{np}] / \bar{\alpha}\lambda . \quad (\text{Eq. 190})$$

Some numerical values are given in Table 22 which follows.

[Note: The validity of applying Little's formula to the priority subsystem alone can be verified by direct calculation. We observe that

$$P[N_p = 0] = p_0 + B_0, \quad P[N_p = n] = A_n + B_n \quad (n \geq 1).$$

It is easily verified that

$$p_0 + B_0 + \sum_{n \geq 1} (A_n + B_n) = 1$$

(this is indeed how the constants were determined) and that

$$E[N_p] = \sum_{n \geq 1} n(A_n + B_n)$$

does have the value given by Eq. 188. Moreover

$$E[N_p^2] = \sum_{n \geq 1} n^2(A_n + B_n) = \frac{r(1+\alpha r)}{(1-\alpha r)^2} - \bar{\alpha}r(1+2\alpha r) ,$$

and consequently

$$\text{Var}[N_p] = \frac{r}{(1-\alpha r)^2} - \bar{\alpha}r \frac{1+\alpha r}{1-\alpha r} + \frac{\alpha^2 r^3}{(1-\alpha r)^2} . \quad (\text{Eq. 191})$$

This confirms the validity of the simple approach to finding mean waiting time for the non-priority stream by means of Little's formula. No such simple approach is available for the variances.]

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Table 22

Mean equilibrium system times of priority and non-priority streams as a function of traffic intensity r and priority ratio α

		Priority	Non-priority			Priority	Non-Priority
		$r = 0.1$				$r = 0.5$	
α	.1	1.1010101	1.11223345	α	.1	1.52631579	2.05263158
	.2	1.10204032	1.11337868		.2	1.55555556	2.11111111
	.3	1.10309278	1.11454754		.3	1.58823529	2.17647059
	.4	1.10416667	1.11574074		.4	1.625 2.25	
	.5	1.10326216	1.11695906		.5	1.66666667	2.33333333
	.6	1.10638298	1.11820331		.6	1.71428572	2.42857143
	.7	1.10752638	1.11947431		.7	1.76923077	2.53846154
	.8	1.10869565	1.12077295		.8	1.83333333	2.66666667
	.9	1.10989011	1.12210012		.9	1.90909091	2.81818181

		Priority	Non-priority			Priority	Non-priority
		$r = 0.7$				$r = 0.9$	
α	.1	1.75268817	3.50894058	α	.1	1.98901099	10.8901099
	.2	1.81395349	3.7131783		.2	2.09756098	11.9756097
	.3	1.88607595	3.9535865		.3	2.23287671	13.3287671
	.4	1.97222222	4.24074074		.4	2.40625 15.0625	
	.5	2.07692308	4.58974359		.5	2.63636364	17.3636364
	.6	2.20689655	5.02298851		.6	2.95652174	20.5652174
	.7	2.37254902	5.5751634		.7	3.43243243	25.3243243
	.8	2.59090909	6.3030303		.8	4.21428572	33.1428571
	.9	2.89189189	7.30630630		.9	5.7368421	48.368421

		Priority	Non-priority			Priority	Non-priority
		$r = 0.95$				$r = 0.99$	
α	.1	2.04972376	21.9944751	α	.1	2.09877913	110.877911
	.2	2.17283951	24.4567571		.2	2.23441397	124.441394
	.3	2.32867133	27.5734266		.3	2.40825036	141.825032
	.4	2.53225806	31.6451613		.4	2.63907285	164.907281
	.5	2.80952381	37.1904762		.5	2.96039604	197.039599
	.6	3.20930233	45.1860465		.6	3.43842364	244.842359
	.7	3.8358209	57.7164179		.7	4.2247557	323.475562
	.8	4.95833334	80.1666666		.8	5.75761538	476.961527
	.9	7.55172413	132.034483		.9	10.0825688	909.256856

Note: the time unit is $1/\mu$, the mean time required to process a message

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4.7.2 Numerical values and comments

Traffic intensity is measured by r , the ratio of mean arrival rate to mean service rate. r also has an interpretation as the probability that the service is busy or, in this context, that messages are waiting and being transmitted. It is a convention to describe the region $0 < r \leq 0.5$ as "light traffic", $0.5 < r \leq 0.8$ as "medium traffic", and $0.8 < r \leq 1$ as "heavy traffic". The tables show what is intuitively obvious, that under light traffic conditions the assignment of any priority ratio α makes little difference, but as traffic becomes heavier the difference between mean system times becomes increasingly significant. At a level of exceptionally heavy traffic ($r = 0.99$) the assignment of priority to 10% of the traffic leads to a 55-fold increase, while if priority is given to 90% of the traffic the difference is 90-fold. The mean system time of non-priority traffic seems quite intolerable and indicates that measures are needed to control traffic intensity, either by reduction of demand, for example application of a severe criterion for the acceptance of messages, or by increase of mean processing rate, for example by introducing one or more additional communication channels.

Four more points are worth making:

- (i) To compare the effect of the priority system with one in which all messages are treated alike we note that in such a case the mean system time is $1/\mu(1-r)$. The general observation is that non-priority messages always have to wait longer than this, while the priority class always waits less, on average.
- (ii) Increasing the proportion of priority messages always increases their system time, as well as that of the non-priority class, though less severely.
- (iii) Increases in traffic intensity always increase mean system time for a given value of α .
- (iv) As we shall see, the system time variance of non-priority messages shows up even worse by comparison with that of priority messages.

4.8 NON-PRIORITY SYSTEM STATE AND THE FORM OF $g_{np}(t)$

We shall now address the task of obtaining more information about non-priority system time. We shall obtain the Laplace transform $\gamma_{np}(z)$ of the probability density function $g_{np}(t)$. Although it is not obvious how to obtain a simple expression for $g_{np}(t)$ by inversion, nevertheless, having $\gamma_{np}(z)$ means that all the moments of W_{np} can be calculated in principle. In particular we shall obtain an expression for $\text{Var}[W_{np}]$ to

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be compared numerically with $\text{Var}[W_p]$ in Table 24. The analysis also permits a direct evaluation of $E[N_{np}]$, which can be compared with Eq. 189 for checking, and $\text{Var}[N_{np}]$.

4.8.1 Generating functions of state probabilities

The analysis is based on the introduction of the generating functions

$$A_m(x) = \sum_{n \geq 0} a_{mn} x^n \quad (m \geq 1)$$

$$B_m(x) = \sum_{n \geq 1} b_{mn} x^n \quad (m \geq 0)$$

(Eq. 192)

Thus A_m and B_n calculated in Sect. 4.6 are respectively $A_m(1)$ and $B_n(1)$.

To get $B_0(x)$ we start in the usual way with the set.

$$\begin{aligned} x \quad (1+r) b_{01} &= a_{11} + b_{02} + \bar{\alpha} r p_0 \\ x^2 \quad (1+r) b_{02} &= a_{12} + b_{03} + \bar{\alpha} r b_{01} \\ x^3 \quad (1+r) b_{03} &= a_{13} + b_{04} + \bar{\alpha} r b_{02} \\ &\dots \end{aligned}$$

Multiplying by the power of x indicated in the margin and adding gives

$$(1+r) B_0(x) = A_1(x) - a_{10} + \frac{1}{x} (B_0(x) - b_{01}x) + \bar{\alpha} r p_0 x + \bar{\alpha} r x B_0(x)$$

which can be rearranged into

$$(1+r - \frac{1}{x} - \bar{\alpha} r x) B_0(x) = A_1(x) - p_1 + \bar{\alpha} x p_1 \quad (\text{Eq. 193})$$

after using Eq. 160.

Similarly, from the set

$$\begin{aligned}(1+r) b_{m1} &= \alpha r b_{m-1} \\ (1+r) b_{m2} &= \alpha r b_{m-1} + \bar{\alpha} r b_{m1} \\ (1+r) b_{m3} &= \alpha r b_{m-1} + \bar{\alpha} r b_{m2}\end{aligned}\quad (m > 1)$$

and so on, can be derived

$$B_m(x) = \sigma(x) B_{m-1}(x) \quad (\text{Eq. 194})$$

where

$$\sigma(x) = \alpha r / (1 + r - \bar{\alpha} r x) \quad (\text{Eq. 195})$$

leading to

$$B_m(x) = B_0(x) [\sigma(x)]^m \quad (m > 0) \quad (\text{Eq. 196})$$

Similar treatment yields expressions for $A_m(x)$.

Thus

$$(1 + r - \bar{\alpha} r x) A_1(x) = A_2(x) + \frac{1}{x} B_1(x) + \alpha r p_0 \quad (\text{Eq. 197})$$

and

$$(1 + r - \bar{\alpha} r x) A_m(x) = A_{m+1}(x) + \alpha r A_{m-1}(x) + \frac{1}{x} B_m(x) \quad (m > 2) \quad (\text{Eq. 198})$$

Since $A_1(x)$ enters Eq. 198 when $m = 2$, the general form of solution found holds also for $m = 1$, and then Eq. 197 provides a constraint from which a constant can be determined. As was the case for A_m , the solution of Eq. 198 is of the form

$$A_m(x) = K_1 \eta_1^m + K_2 \eta_2^m + \text{a particular solution,}$$

where η_1 and η_2 are the roots of

$$y^2 - (1 + r - \bar{\alpha} r x) y + \alpha r = 0, \quad (\text{Eq. 199})$$

namely

$$\eta_1, \eta_2 = 1/2(X \pm R) \quad (\text{Eq. 200})$$

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where

$$\begin{aligned} \chi &= 1 + r - \bar{\alpha} r x \quad , \\ R^2 &= \chi^2 - 4\alpha r \quad . \end{aligned} \quad (\text{Eq. 201})$$

Since

$$(\eta_1 - 1)(1 - \eta_2) = \bar{\alpha} r (1 - x)$$

is positive for $0 < x < 1$ (required to ensure convergence) it follows that the larger root is ineligible to form part of the solution. Thus we write

$$A_m(x) = K \eta^m(x) + \frac{B_0(x)}{x} \frac{1}{f(E)} \sigma^m(x)$$

where

$$f(E) = 1 + r - \bar{\alpha} r x - E - \frac{\alpha r}{E}$$

and $\eta(x)$ is the smaller root. K is independent of m .

Since

$$f(E) \sigma^m(x) = \sigma^{m-1}(x) (\alpha r - \sigma^2(x) - \alpha r) = -\sigma^{m+1}(x)$$

we have

$$A_m(x) = K \eta^m(x) - \frac{B_0(x)}{x} \sigma^{m-1}(x) \quad . \quad (\text{Eq. 202})$$

If this is to satisfy Eqs. 197 and 196 we get

$$K = p_0 + \frac{B_0(x)}{x\sigma(x)} \quad (\text{Eq. 203})$$

so that

$$A_m(x) = p_0 \eta^m(x) + \frac{B_0(x)}{x\sigma(x)} (\eta^m(x) - \sigma^m(x)) \quad , \quad (m > 1) \quad . \quad (\text{Eq. 204})$$

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To complete the solution $B_0(x)$ is needed. This can be obtained from Eq. 193. We get

$$B_0(x) = \frac{x\sigma(x) p_0(\eta(x) - r(1-\bar{\alpha}x))}{\alpha rx - \eta(x)} . \quad (\text{Eq. 205})$$

As a check this can be shown to reduce when $x = 1$ to Eq. 182. The operation is a little delicate since numerator and denominator contain $1-x$ as a factor.

4.8.2 Non-priority system state

Before dealing with w_{np} let us consider N_{np} , in particular with a view to calculating $\text{Var}[N_{np}]$ and checking $E[N_{np}]$ as given by Eq. 188. The generating function

$$C(x) = E[x^{N_{np}}] \quad (\text{Eq. 206})$$

is easily seen to be given by

$$\begin{aligned} C(x) &= p_0 + \sum_{m \geq 1} [A_m(x) + B_{m-1}(x)] \\ &= p_0 + \frac{B_0(x)}{x} \left[\frac{x-1}{1-\sigma(x)} + \frac{\eta(x)}{\sigma(x)(1-\eta(x))} \right] . \end{aligned} \quad (\text{Eq. 207})$$

As usual we can calculate the moments from

$$\begin{aligned} E[N_{np}] &= \left[\frac{dC(x)}{dx} \right]_{x=1} , \\ E[N_{np}(N_{np} - 1)] &= \left[\frac{d^2C(x)}{dx^2} \right]_{x=1} . \end{aligned} \quad (\text{Eq. 208})$$

The direct approach is very laborious and error-prone. A slight gain in manageability can be obtained by putting

$$x = 2(\alpha r)^{1/2} \cosh u = 1 + r - \bar{\alpha}rx , \quad (\text{Eq. 209})$$

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in which case

$$R(x) = 2(\alpha r)^{1/2} \sinh u ,$$

$$n(x) = (\alpha r)^{1/2} e^{-u} ,$$

$$\sigma(x) = (\alpha r)^{1/2} / (2 \cosh u) ,$$

$$B_0(x) = \frac{P_0}{2(\alpha r)^{1/2}} \left[\frac{1+r - 2(\alpha r)^{1/2} \cosh u}{\cosh u (1-(r/\alpha)^{1/2} e^{-u})} \right] . \quad (\text{Eq. 210})$$

The gain is that the common factor $1-x$ in the numerator and denominator of Eq. 205 can be cancelled explicitly when $B_0(x)$ is thus expressed in exponential form.

After further manipulations we arrive at the formula

$$C(x) = p_0(C_1(x) + C_2(x))$$

where

$$C_1(x) = \frac{1}{1 - (r/\alpha)^{1/2} e^{-u}} ,$$

$$C_2(x) = \frac{(e^u - (\alpha r)^{1/2})(e^{-u} - (\alpha r)^{1/2})}{(\alpha r)^{1/2}(2 \cosh u - (\alpha r)^{1/2}(1-(r/\alpha)^{1/2} e^{-u}))} . \quad (\text{Eq. 211})$$

The value u_1 of u corresponding to $x = 1$ is given by $e^{-u_1} = (\alpha r)^{1/2}$. This shows readily that $C_2(1) = 0$, $C_1(1) = (1-r)^{-1}$ giving $C(1) = 1$, as should be the case.

We shall not reproduce the intermediate steps because of their tedium. The final results are

$$E[N_{np}] = \frac{\bar{\alpha} r (1 - \alpha r + \alpha r^2)}{(1-r)(1-\alpha r)} \quad (\text{Eq. 212})$$

as obtained at Eq. 189 by a much more attractive method, and

$$\text{Var}[N_{np}] = \frac{\bar{\alpha}^2 r^2}{(1-r)^2 (1-\alpha r)^3} [2r(1-\alpha r^2) - (1-\alpha r)(1-\alpha r + \alpha r^2)^2] + \frac{\bar{\alpha} r (1+2\bar{\alpha} r)(1-\alpha r + \alpha r^2)}{(1-r)(1-\alpha r)} . \quad (\text{Eq. 213})$$

It is readily confirmed that Eqs. 212 and 213 reduce to the M/M/1 values $r/(1-r)$ and $r/(1-r)^2$ when $\bar{\alpha} = 1$.

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Table 23

Means and standard deviations of N_p and N_{np}

N_{np}			N_p		
<u>Mean</u>		<u>Sdvn</u>	<u>Mean</u>		<u>Sdvn</u>
$r = 0.1$			$r = 0.1$		
$\alpha .1$	.10010101	.331900797	$\alpha .1$	.011010101	.105496382
.2	.0890702948	.311550399	.2	.0220406163	.150066485
.3	.0780183276	.290130484	.3	.0330927835	.184863529
.4	.0669444445	.267389032	.4	.0441666667	.214701019
.5	.0558479532	.242961158	.5	.0552631579	.24143213
.6	.0447281324	.216283533	.6	.0663829788	.266002039
.7	.0335842294	.186402044	.7	.0775268018	.288969387
.8	.0224154509	.151444529	.8	.0886956522	.310696516
.9	.0112210012	.106545677	.9	.0998901099	.331434128
$r = 0.5$			$r = 0.5$		
$\alpha .1$	.92368421	1.05016036	$\alpha .1$	.0763157895	.284417566
.2	.844444445	1.28098899	.2	.155555556	.418108616
.3	.761764706	1.20530347	.3	.238235294	.533002528
.4	.675	1.12124373	.4	.325	.641774882
.5	.583333333	1.02627518	.5	.416666667	.75
.6	.485714286	.916833186	.6	.514285714	.86141791
.7	.380769231	.787644258	.7	.619230769	.979380615
.8	.266666667	.63010875	.8	.733333333	1.10754985
.9	.140909091	.426293498	.9	.859090909	1.25046272
$r = 0.7$			$r = 0.7$		
$\alpha .1$	2.21064516	2.72189147	$\alpha .1$	.122688172	.364340063
.2	2.07937985	2.6461936	.2	.253953488	.545590712
.3	1.93725739	2.5582188	.3	.39607594	.710266249
.4	1.78111111	2.4526799	.4	.552222222	.876496251
.5	1.60641026	2.32144426	.5	.726923077	1.05511574
.6	1.40643678	2.15147831	.6	.926896551	1.25717082
.7	1.17078431	1.92078898	.7	1.16254902	1.49755665
.8	.862424242	1.58964527	.8	1.45090909	1.79959959
.9	.511441442	1.07826228	.9	1.82189189	2.20367402

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Table 23 (cont'd)

Means and standard deviations N_p and N_{np}

N_{np}			N_p		
<u>Mean</u>		<u>Sdvn</u>	<u>Mean</u>		<u>Sdvn</u>
$r = 0.9$			$r = 0.9$		
$\alpha .1$	8.820989	9.45110064	$\alpha .1$	.179010989	.443615376
.2	8.62240902	9.4081404	.2	.377560976	.676509218
.3	8.39712326	9.35344737	.3	.602876712	.900670514
.4	8.13375	9.27970754	.4	.86625	1.14415567
.5	7.81363636	9.17320099	.5	1.18636364	1.43233757
.6	7.40347825	9.00611631	.6	1.59652174	1.80344472
.7	6.83756756	8.71487712	.7	2.16243243	2.33013668
.8	5.96571428	8.12647891	.8	3.03428571	3.1778622
.9	4.35315709	6.61759141	.9	4.6468421	4.82656319
$r = 0.95$			$r = 0.95$		
$\alpha .1$	18.8052762	19.4737303	$\alpha .1$	.194723757	.463400915
.2	18.5871605	19.4492404	.2	.412839506	.709805643
.3	18.3363287	19.4172098	.3	.663671329	.950464199
.4	18.0377419	19.3723526	.4	.962258064	1.21711516
.5	17.6654762	19.3042532	.5	1.33452381	1.5416538
.6	17.1706977	19.1900478	.6	1.82930233	1.97660705
.7	16.4491791	18.9713016	.7	2.5508209	2.63129547
.8	15.2316667	18.4594886	.8	3.76833333	3.7879601
.9	12.5432759	16.7333562	.9	6.45672413	6.47071465
$r = 0.99$			$r = 0.99$		
$\alpha .1$	98.7922185	99.4944196	$\alpha .1$	.207779134	.479220287
.2	98.5575836	99.4889878	.2	.442413965	.736588747
.3	98.2847473	99.4816988	.3	.715250356	.990922157
.4	97.9549248	99.4711369	.4	1.04507285	1.27737834
.5	97.5346016	99.4543562	.5	1.46539604	1.63435404
.6	96.9575739	99.4243866	.6	2.04242365	2.12968851
.7	96.0722419	99.3613431	.7	2.9277557	2.91599542
.8	94.4383822	99.1883032	.8	4.56161538	4.43688519
.9	90.0164288	96.3548064	.9	8.98356878	8.75157945

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4.8.3 Numerical values and comments

It is instructive, before making the final attack on W_{np} , to compare the means and standard deviations of N_p and N_{np} on the lines of Table 22. The results of the comparison are given in Table 23.

It is seen that increasing the proportion α of priority messages decreases both the mean number of non-priority messages and its standard deviation. The reverse is the case for priority messages: increasing the priority ratio α increases $E[N_p]$ and the standard deviation. Note that $E[N_p] + E[N_{np}] = r/(1-r)$, the value for the simple M/M/1 system.

The congestion induced for non-priority customers under heavy traffic is clearly in evidence. Each of the average number of 90 non-priority messages waiting for $r = 0.99$, $\alpha = 0.9$, could expect an average wait of 909 message transmission times before reaching its destination (see Table 22).

4.8.4 Laplace transform of $g_{np}(t)$ and statistics of W_{np}

We turn finally to the distribution and statistics of W_{np} . It seems improbable that as simple an expression as Eq. 185 can be obtained for the probability density function $g_{np}(t)$. However, the convolutions in Eq. 173 for $g_{np}(t)$ invite Laplace transformation. We write $\gamma_{np}(z)$ for this Laplace transform. Since it can in principle be made to yield all moments of the random variable W_{np} it is sufficient for practical purposes. Our immediate objective is to verify Eq. 190 and to find $\text{Var}[W_{np}]$. A table resembling Table 23 will then be made to examine the lot of non-priority messages from the point of view of standard deviation of system time.

In transforming Eq. 173 we use the fact that under Laplace transformation $k_n(t)$ and $f_n(t)$ pass respectively into $\kappa^n(z)$ and $\phi^n(z)$, where $\kappa(z)$ and $\phi(z)$ are the Laplace transforms respectively of the probability density functions of the busy period B_1 (see sect. 4.4), and the interval X separating successive non-priority service opportunities. $\kappa(z)$ is given by Eq. 168, and $\phi(z)$ by Eq. 167. Now it happens that for the system under consideration

$$\phi(z) = \kappa(z) \quad . \quad (\text{Eq. 214})$$

Probabilistically this means that X and B_1 are statistically identical. After a moment's thought this is not surprising, since X is actually the busy period initiated by the non-priority message that just began transmission. The fact that after the completion of transmission of that

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message the only messages to be transmitted before the next non-priority service opportunity will be priority message makes no difference statistically since the service mechanism is unaware of priorities, and the arrival mechanism in this case is entirely that of the priority stream. To confirm Eq. 214 analytically it is merely necessary to substitute Eq. 168 into Eq. 167. We then find

$$\phi(z) = \frac{\mu}{Z - (Z - R_1)^{1/2}} = \frac{2\mu}{Z + R_1} = \frac{Z - R_1}{2\alpha\lambda} = \kappa(z) \quad ,$$

as stated. There are other reasons why Eq. 214 is not surprising, but they are not in place here.

In the following analysis we shall record more detail than usual to help the reader check the steps. Applying the Laplace transformation to Eq. 173 according to the rules gives

$$\begin{aligned} \gamma_{np}(z) &= \frac{\mu}{(\mu+z)} \left[p_0 + \sum_{m \geq 1} a_{m0} \kappa^m(z) + \sum_{n \geq 1} b_{0n} \kappa^n(z) + \right. \\ &\quad \left. + \sum_{m \geq 1} \sum_{n \geq 1} a_{mn} \kappa^{n+m}(z) + \sum_{m \geq 1} \sum_{n \geq 1} b_{mn} \kappa^{m+n}(z) \right] \\ &= \frac{\mu}{\mu+z} \left[p_0 + \sum_{m \geq 1} a_{m0} \kappa^m(z) + B_0(\kappa(z)) + \right. \\ &\quad \left. + \sum_{m \geq 1} \kappa^m(z) \{A_m(\kappa(z)) - a_{m0}\} + \sum_{m \geq 1} \kappa^m(z) B_m(\kappa(z)) \right] \\ &= \frac{\mu}{\mu+z} \left[p_0 + \sum_{m \geq 0} B_m(\kappa(z)) \kappa^m(z) + \sum_{m \geq 1} A_m(\kappa(z)) \kappa^m(z) \right] . \end{aligned}$$

Hereafter we shall usually omit specific reference to the argument z of $\kappa(z)$. Substituting Eqs. 196 and 204 gives the considerably reduced expression

$$\gamma_{np}(z) = \frac{\mu}{(\mu+z)(1-\kappa\eta(\kappa))} \left[p_0 + \frac{B_0(\kappa) \eta(\kappa)}{\sigma(\kappa)} \right] . \quad (\text{Eq. 215})$$

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Substituting Eq. 205 we have then

$$\begin{aligned} \gamma_{np}(z) &= \frac{\mu p_0}{(\mu+z)(1-\kappa\eta(\kappa))} \left[1 + \frac{\kappa\eta(\kappa)\{\eta(\kappa) - r(1-\bar{\alpha}\kappa)\}}{\alpha r\kappa - \eta(\kappa)} \right] \\ &= \frac{\mu p_0[\kappa\{\eta^2(\kappa) - \eta(\kappa)\chi(\kappa) + \alpha r\} - \eta(\kappa)(1-\kappa)]}{(\mu+z)(1-\kappa\eta(\kappa))(\alpha r\kappa - \eta(\kappa))} . \end{aligned}$$

Since η satisfies Eq. 199 this reduces to

$$\gamma_{np}(z) = \frac{\mu p_0 \eta(\kappa)(1 - \kappa(z))}{(\mu+z)(1 - \kappa(z)\eta(\kappa))(\eta(\kappa) - \alpha r\kappa(z))} . \quad (\text{Eq. 216})$$

Naturally we should check this by putting $z = 0$ upon which we should get $\gamma_{np}(0) = 1$. However, there appears again the difficulty encountered already in the form (Eq. 205) that both numerator and denominator vanish when $z = 0$. This can be mitigated by using the exponential substitution introduced in Eq. 209 where in this case $x = \kappa(z)$. After some algebra we arrive at the considerably simplified form

$$\gamma_{np}(z) = \frac{p_0 \mu \bar{\alpha}}{\alpha(\mu+z)\left\{\left(\frac{r}{\alpha}\right)^{1/2} - e^{-u}\right\}\left\{e^u - \left(\frac{r}{\alpha}\right)^{1/2}\right\}} , \quad (\text{Eq. 217})$$

in which the relation between u , z and κ is given by

$$2(\alpha r)^{1/2} \cosh u = 1+r - \bar{\alpha} r \kappa(z) . \quad (\text{Eq. 218})$$

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After checking that $\gamma_{np}(0) = 1$ (recall that when $z \rightarrow 0$, $\kappa(z) \rightarrow 1$ ($\alpha\lambda < \mu$), $u \rightarrow u_1$ where $\exp(-u_1) = (\alpha r)^{1/2}$) we write

$$\ln \gamma_{np}(z) = \ln(\mu p_0 \frac{\bar{\alpha}}{\alpha}) + u - \ln(\mu+z) - \ln M - \ln N \quad (\text{Eq. 219})$$

where

$$M = \left(\frac{r}{\alpha}\right)^{1/2} e^u - 1, \quad N = e^u - \left(\frac{r}{\alpha}\right)^{1/2}. \quad (\text{Eq. 220})$$

Then

$$\gamma'_{np}(z)/\gamma_{np}(z) = \frac{du}{dz} - \frac{1}{(\mu+z)} - \left(\frac{M_u}{M} + \frac{N_u}{N}\right) \frac{du}{dz}. \quad (\text{Eq. 221})$$

Noting that

$$\frac{du}{dz} = \frac{-\bar{\alpha}r}{(\alpha r)^{1/2}(e^u - e^{-u})} \frac{d\kappa}{dz},$$

$$\frac{d\kappa}{dz} = \frac{-\kappa(z)}{R_1(z)}$$

$$M_u = \left(\frac{r}{\alpha}\right)^{1/2} e^u,$$

$$N_u = e^u$$

we get

$$\left(\frac{du}{dz}\right)_{z=0} = \frac{\bar{\alpha}r}{\mu(1-\alpha r)^2},$$

$$\left(\frac{M_u}{M}\right)_{z=0} = \frac{1}{\bar{\alpha}},$$

$$\left(\frac{N_u}{N}\right)_{z=0} = \frac{1}{1-r}.$$

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Inserting these in Eq. 221 we obtain

$$E[W_{np}] = -\gamma'_{np}(0) = \frac{1-\alpha r + \alpha r^2}{\mu(1-r)(1-\alpha r)} = \frac{E[N_{np}]}{\bar{\alpha}\lambda} \quad (\text{Eq. 222})$$

in agreement with Eqs. 189 and 190.

To get the variance we differentiate Eq. 221 again, obtaining

$$\begin{aligned} \frac{\gamma''_{np}(z)}{\gamma_{np}(z)} - \left[\frac{\gamma'_{np}(z)}{\gamma_{np}(z)} \right]^2 &= \frac{d^2 u}{dz^2} + \frac{1}{(\mu+z)^2} - \left(\frac{M_u}{M} + \frac{N_u}{N} \right) \frac{d^2 u}{dz^2} - \\ &\quad - \left(\frac{du}{dz} \right)^2 \left[\frac{M_{uu}}{M} - \left(\frac{M_u}{M} \right)^2 + \frac{N_{uu}}{N} - \left(\frac{N_u}{N} \right)^2 \right] . \end{aligned} \quad (\text{Eq. 223})$$

Then

$$\text{Var}[W_{np}] = \frac{1}{\mu^2} + \left(1 - \frac{1}{\bar{\alpha}} - \frac{1}{\bar{r}} \right) \left[\frac{d^2 u}{dz^2} \right]_{z=0} - \frac{\bar{\alpha}^2 r^2}{\mu^2 (1-\alpha r)^4} \left[\frac{1}{\bar{\alpha}} \left(1 - \frac{1}{\bar{\alpha}} \right) + \frac{1}{\bar{r}} \left(1 - \frac{1}{\bar{r}} \right) \right]$$

where $\bar{r} = 1-r$, and since $M_{uu} = M_u$, $N_{uu} = N_u$.

Now

$$\frac{d^2 u}{dz^2} = \frac{-(\bar{\alpha}r)}{\mu^2 (1-\alpha r)^5} (2-2\alpha r + \bar{\alpha}r + \alpha\bar{\alpha}r^2) ,$$

and

$$1 - \frac{1}{\bar{\alpha}} - \frac{1}{\bar{r}} = \frac{(\bar{\alpha}-1)(\bar{r}-1)-1}{\bar{\alpha}\bar{r}} = \frac{-(1-\alpha r)}{\bar{\alpha}\bar{r}} .$$

Consequently

$$\text{Var}[W_{np}] = \frac{1}{\mu^2} + \frac{r(2-2\alpha r + \bar{\alpha}r + \alpha\bar{\alpha}r^2)}{\mu^2 (1-\alpha r)^4 (1-r)} + \frac{\bar{\alpha}^2 r^2}{\mu^2 (1-\alpha r)^4} \left(\frac{\alpha}{\bar{\alpha}^2} + \frac{r}{\bar{r}^2} \right) . \quad (\text{Eq. 224})$$

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Table 24

Standard deviation of equilibrium system times W_p and W_{np}
of priority and non-priority streams as functions
of traffic intensity r and priority ratio α

Priority			Priority		
Non-priority			Non-priority		
$r = 0.1$			$r = 0.5$		
$\alpha .1$	1.09263799	1.11428041	$\alpha .1$	1.3531537	2.1073051
.2	1.09445614	1.11756966	.2	1.38777733	2.23115489
.3	1.09632765	1.12098447	.3	1.42760719	2.37502611
.4	1.09825461	1.12453075	.4	1.47372827	2.54337374
.5	1.10023921	1.12821473	.5	1.52752523	2.74199171
.6	1.10228373	1.13204302	.6	1.59078982	2.97854329
.7	1.10439059	1.13602255	.7	1.66587752	3.26336055
.8	1.10656231	1.1401607	.8	1.75594229	3.61068374
.9	1.10880156	1.14446523	.9	1.86529859	4.04064623

Priority			Priority		
Non-priority			Non-priority		
$r = 0.7$			$r = 0.9$		
$\alpha .1$	1.43253089	3.61892509	.1	1.4817246	11.0770277
.2	1.49345064	3.9620975	.2	1.57236697	12.4141082
.3	1.56783279	4.38049994	.3	1.69149089	14.1162662
.4	1.65994013	4.89951613	.4	1.8485108	16.3529342
.5	1.77590714	5.55689914	.5	2.06705764	19.4154043
.6	1.924867	6.41110934	.6	2.3829895	23.8479385
.7	2.12100315	7.55725836	.7	2.86906882	30.7901245
.8	2.38755382	9.15999461	.8	3.6915513	43.0596576
.9	2.7653549	11.5282095	.9	5.33139969	69.7786253

Priority			Priority		
Non-priority			Non-priority		
$r = 0.95$			$r = 0.99$		
$\alpha .1$	1.48926544	22.204391	$\alpha .1$	1.49388949	111.107411
.2	1.58756024	24.9544526	.2	1.59829948	124.990441
.3	1.71790444	28.4799369	.3	1.73874469	142.837636
.4	1.89603617	33.1597653	.4	1.93412351	166.628844
.5	2.14919697	39.6655668	.5	2.21828584	199.924834
.6	2.52879576	49.308742	.6	2.65820038	249.83735
.7	3.14458166	65.0270303	.7	3.4072175	332.91E223
.8	4.2799192	94.9589579	.8	4.91035578	498.540905
.9	6.96013789	172.150142	.9	9.22819496	988.849756

Note: time unit is mean processing time

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A check on this can be provided by putting $\bar{\alpha} = 1$. All traffic is then homogeneous and $\text{Var}[W_{np}]$ reduces, as it should, to $(\mu(1-r))^{-2}$. It is depressing for the non priority message that the standard deviation in this case should appear to behave like the inverse square of $1-\alpha r$ as well as the inverse of $1-r$.

Substitution of Eq. 218 back into Eq. 217 gives the even simpler form

$$\gamma_{np}(z) = \frac{\mu p_0}{(\mu+z)(1-r\kappa(z))},$$

which yields the moments much more easily than above. Moreover, expansion of the denominator in a series of ascending powers of $\kappa(z)$ and inversion term by term leads to an expression for the distribution function

$$G_{np}(t) = \int_0^t g_{np}(v) dv$$

of W_{np} when t is measured in units of μ^{-1} . This is

$$G_{np}(t) = r p_0 \int_0^t \{1 - e^{-(t-v)}\} e^{-v(1+\alpha r)} \sum_{n \geq 0} \left(\frac{r}{\alpha}\right)^{n/2} \{I_n(2v \sqrt{\alpha r}) - I_{n+2}(2v \sqrt{\alpha r})\} dv$$

entailing modified Bessel functions. In spite of its formidable appearance this expression is not hard to calculate numerically.

4.8.5 Numerical values and comments

Table 24 complements Table 22 by giving numerical values of the standard deviations $(\text{Var}[W_p])^{1/2}$, $(\text{Var}[W_{np}])^{1/2}$ calculated from Eqs. 187 and 224 respectively. After the discussion that has preceded it will be no surprise to observe the adverse effect of this priority discipline on the system time standard deviation of non-priority messages at heavy traffic levels. If there were no priority discipline, and customers were treated as homogeneous units with first come, first served discipline, the system times W would be exponentially distributed with parameter $\mu(1-r)$ and $E[W] = \text{SDVN}(W) = (\mu(1-r))^{-1}$. We can now see that the effect of priority for the fortunate customers is to confer on them a system time distribution that is "better" than exponential, and much better at practical traffic levels. The fate of the less fortunate non-priority customers is a W distribution that is worse - much worse - than exponential as $r \rightarrow 1$ and $\alpha \rightarrow 1$.

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4.9 MANAGEMENT OF A PRIORITY SYSTEM

4.9.1 Basic considerations

A simple model has exposed the principles with which management of a message system with priority facility has to be familiar. Without the knowledge of the features and requirements of particular systems the theoretician, on the other hand, has to abstain from sweeping pronouncements. Detailed advice can only be given in a rational manner after examination of the problems in the field, followed probably by further detailed analysis and consultation. Here we can only underline what is common to most demand and supply situations.

In the command and control context under operational conditions the demand to transmit messages is likely to be high in relation to the number of communication channels available, and the system will therefore be operating most of the time under heavy traffic conditions unless measures are taken. That is to say, the parameter r introduced in the text will tend to have a value in the neighbourhood of unity. In addition to their natural urge to pass on information, originators of messages are likely in many instances to possess an exaggerated notion of the importance of the content. This, coupled with the belief that it will arrive at the destination more quickly, encourages a possibly unjustified allocation of priority to the message. This tends to increase the ratio α of priority messages in the total message stream. Originators of messages who would normally avoid using priority, disturbed by the long delays that experience shows their diffidence produces, are then likely, in sheer self defence, themselves to request priority, thus making matters worse.

There are accordingly two major considerations:

- (i) management of traffic intensity;
- (ii) at a given level of traffic intensity, management of the priority mechanism.

4.9.2 Management of the traffic intensity

In the text traffic intensity has been denoted by the parameter r , whose numerical value, expressed in terms of a general service system, is the average number of demands for service made during the mean service time of a customer. r has also the convenient interpretation when less than unity as the probability that the service is busy. When $r > 1$ the service will virtually always be busy and the system is out of hand. In this case the system is said to be in a state of congestion and the number of customers waiting, or turned away, will tend to increase with the passage of time. The waiting time for service will also increase without limit.

The first management principle is accordingly to maintain r at a numerical value below unity.

There are only two ways of doing this: adjustment of the rate of providing service and management of the demand. In some contexts it is possible to cause the service to respond to increasing demand by working faster, but in

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the communication context this does not seem obviously a practical proposition. If technology permits it, it should not be overlooked as a possibility.

A simpler method is to provide more service channels which, in the communication context, means more circuits. A detailed analysis of a two-channel system with priority, on the lines of what has been done in the first part of this paper, would be illuminating and has not, to this writer's knowledge, ever been attempted in this detail. Qualitatively, however, we can infer what the effect might be. To add a second channel identical in service characteristics with the first to a non-priority system where the traffic intensity is r reduces the probability that both channels are busy (i.e. that the service is entirely occupied) from r to $r^2/(2+r)$. For example, if $r = 0.99$, $r^2/(2+r)$ is roughly 0.33, and the effect is to transform a system under heavy traffic to one with light traffic. Moreover, the mean waiting time is reduced from $(1-r)^{-1}$ to $(1-r^2/4)^{-1}$ in units of mean service time of a customer. Again, with $r = 0.99$, this means a reduction from 100 units to 1.325. We have seen that the effect of assigning priority to a proportion of customers is to reduce their mean waiting time relative to a non-priority system, and to increase that of non-priority customers. But, at light traffic levels there is little difference between the system times of either species of customer, so priority becomes less important as a mere factor in getting one's message to its destination promptly, and the psychological pressure to assign priority is reduced.

Restriction of demand is the second method available for the management of traffic. It is a common assumption that demand and service are independent, but, in real life, this is not true. It is common experience that to upgrade a computer system has the initial effect of reducing the load on the new system, but that as customers become aware of the new facilities available, demand appears where there was none before and the new system before long again becomes saturated. The lesson is that a very rigorous discipline must be imposed on demand, all the more in an operational context where the arrival of a message may be a matter of life and death. This may be done by the enforcement of strict protocols, the effect of which is to filter out and reject selectively a proportion of the would-be customers. Alternatively, and easier to apply, but potentially more dangerous in the sense of risking the rejection of important messages, a limit may be imposed on the number of messages permitted to be resident in the system. If this were 50, say, the numbers shown in Table 23 for $r = 0.99$ would not arise. Such systems are called "loss-systems" and were the subject of study by Erlang, the father of the subject, in the early twentieth century. Again, the detailed effect of such a measure in the context of a priority system has not, to this writer's knowledge, been evaluated.

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4.9.3 Management of priority mechanism

We have studied a particular mechanism described as "head-of-the-line priority" in the literature. Whether this applies or not, a rational procedure for the allocation of priority must take into account the loss in value of the commodity due to its late arrival, as well as the cost involved in implementing the system. Here we shall ignore the latter. The former might be evaluated by supposing that the value of the information transmitted decreases exponentially. That is to say we might suppose that the decrease is at a constant fraction z of current value per unit time late. It is as though interest is compounded negatively and continuously on the initial value. If the time late W has probability density function $g(t)$, the expected value of the information on arrival, given that the initial value was unity, is given by

$$E[e^{-zW}] = \int_0^{\infty} e^{-zt} g(t) dt = \gamma(z) ,$$

which will be recognized to be the Laplace transform of $g(t)$. The theory in the first part of the chapter provides an expression for $\gamma_{np}(z)$ in Eq. 217. $\gamma_p(z)$ can be obtained from Eq. 185: specifically

$$\gamma_p(z) = \frac{(1-\alpha r) \mu}{\alpha(z+\mu-\alpha r \mu)} - \frac{\mu \alpha}{\alpha(\mu+z)} .$$

Finally, from Eq. 186 for a system without priority we have

$$\gamma(z) = \frac{\mu(1-r)}{z+\mu-\mu r} .$$

4.9.4 Numerical values and comments

Table 25 gives values of $\gamma(z)$, $\gamma_p(z)$, $\gamma_{np}(z)$ and the weighted sum $\alpha\gamma_p(z) + \alpha\gamma_{np}(z)$, as functions of the priority ratio α for the values of r used in Tables 22 to 24, spanning light, medium and heavy traffic. The unit of time is fixed by putting $\mu = 1$ as was done in Tables 22 and 24. The values are repeated for $z = 0.1$ and $z = 0.5$. These are, as it were, interest rates. It will be seen that the second column $\gamma(z)$ is constant. It does not depend upon α and it tells us how much the value of a message can be expected to deteriorate under a single-channel, non-priority system. The next column $\gamma_p(z)$, shows how much the values of priority messages decrease on average and it is seen that, as they arrive earlier, their value has decreased less, as would be expected. The next column contains $\gamma_{np}(z)$ and shows that non-priority messages decrease in value on average more than if there were no priority system in operation.

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The weighted combination whose value is given in the next column shows the value of any message that enters the system at priority ratio value α . It is seen that the value rises to a maximum as α increases and then decreases again. This means that a value of α exists for this kind of priority system such that the devaluation of any message entering the system is least. A similar finding is likely to hold for other priority systems.

Although the criterion chosen for the evaluation of the consequence of delaying the arrival of information may not be the most appropriate, the argument shows that an optimal priority ratio may well exist for each level of traffic intensity r (in fact there seems to be little variation) which is such that the value of messages, priority and otherwise, suffers the least decrease. This is also a topic deserving a deeper analysis.

In conclusion we repeat that traffic intensity r must be controlled first. When this has been done, an optimal value should be sought upon which to base the selection of priority messages in such a way as to minimize the damage inflicted by delays.

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Table 25

Reduced value of message due to late arrival

$z = 0.1$ $r = 0.1$					$z = 0.1$ $r = 0.5$				
α	$\gamma(z)$	$\gamma_p(z)$	$\gamma_{np}(z)$	$\alpha\gamma_p(z) + \bar{\alpha}\gamma_{np}(z)$	α	$\gamma(z)$	$\gamma_p(z)$	$\gamma_{np}(z)$	$\alpha\gamma_p + \bar{\alpha}\gamma_{np}$
.1	.9	.90075	.89993	.90001	.1	.83333	.8658	.83036	.83391
.2	.9	.90067	.89985	.90001	.2	.83333	.86364	.82718	.83447
.3	.9	.90059	.89977	.90002	.3	.83333	.86124	.82377	.83501
.4	.9	.90051	.89969	.90002	.4	.83333	.85859	.82011	.8355
.5	.9	.90043	.89961	.90002	.5	.83333	.85561	.81618	.8359
.6	.9	.90035	.89953	.90002	.6	.83333	.85227	.81195	.83615
.7	.9	.90026	.89945	.90002	.7	.83333	.84848	.80741	.83616
.8	.9	.90018	.89937	.90002	.8	.83333	.84416	.80253	.83583
.9	.9	.90009	.89926	.90001	.9	.83333	.83916	.79729	.83497

$z = 0.1$ $r = 0.9$					$z = 0.1$ $r = 0.95$				
α	$\gamma(z)$	$\gamma_p(z)$	$\gamma_{np}(z)$	$\alpha\gamma_p + \bar{\alpha}\gamma_{np}$	α	$\gamma(z)$	$\gamma_p(z)$	$\gamma_{np}(z)$	$\alpha\gamma_p + \bar{\alpha}\gamma_{np}$
.1	.5	.82806	.4827	.51723	.1	.33333	.82316	.31642	.36709
.2	.5	.82016	.46423	.53541	.2	.33333	.81419	.29884	.40191
.3	.5	.81051	.44457	.55435	.3	.33333	.80312	.28067	.4374
.4	.5	.79853	.42376	.57366	.4	.33333	.78914	.26202	.47287
.5	.5	.78322	.40188	.59255	.5	.33333	.77091	.24309	.507
.6	.5	.76299	.37914	.60945	.6	.33333	.74614	.22412	.53733
.7	.5	.73501	.35583	.62126	.7	.33333	.71055	.20548	.55903
.8	.5	.69378	.33241	.62151	.8	.33333	.65508	.18755	.56158
.9	.5	.62696	.30943	.59521	.9	.33333	.55659	.17079	.51801

$z = 0.1$ $r = 0.99$				
α	$\gamma(z)$	$\gamma_p(z)$	$\gamma_{np}(z)$	$\alpha\gamma_p + \bar{\alpha}\gamma_{np}$
.1	.09091	.81918	.08422	.15772
.2	.09091	.80931	.07756	.22391
.3	.09091	.79701	.07098	.28879
.4	.09091	.78125	.06452	.35121
.5	.09091	.76033	.05827	.4093
.6	.09091	.73123	.0523	.45965
.7	.09091	.68796	.04671	.49559
.8	.09091	.61688	.0416	.50183
.9	.09091	.47847	.03706	.43433

Notes: z is the "compound interest" rate at which the value decreases r is the traffic intensity α is the proportion of traffic assigned priority $\gamma(z)$ is the average message value under a non-priority system $\gamma_p(z)$ is the average message value under a priority system $\gamma_{np}(z)$ is the average message value under a non-priority system $\alpha\gamma_p + \bar{\alpha}\gamma_{np}$ is the weighted average

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Table 25 (cont'd)

Reduced value of message due to late arrival

 $z = 0.5$ $r = 0.1$ $z = 0.5$ $r = 0.5$

	$\gamma(z)$	$\gamma_p(z)$	$\gamma_{np}(z)$	$\alpha\gamma_p + \bar{\alpha}\gamma_{np}$		$\gamma(z)$	$\gamma_p(z)$	$\gamma_{np}(z)$	$\alpha\gamma_p + \bar{\alpha}\gamma_{np}$
$\alpha .1$	.64286	.6443	.64275	.64291	$\alpha .1$	.5	.55172	.49721	.50266
.2	.64286	.64414	.64265	.64295	.2	.5	.54762	.49439	.50503
.3	.64286	.64399	.64255	.64298	.3	.5	.54321	.49155	.50704
.4	.64286	.64384	.64245	.643	.4	.5	.53846	.48868	.5086
.5	.64286	.64368	.64234	.64301	.5	.5	.53333	.48581	.50957
.6	.64286	.64352	.64224	.64301	.6	.5	.52778	.48293	.50984
.7	.64286	.64336	.64213	.64299	.7	.5	.52174	.48004	.50923
.8	.64286	.64319	.64203	.64296	.8	.5	.51515	.47715	.50755
.9	.64286	.64303	.64192	.64292	.9	.5	.50794	.47427	.50457

 $z = 0.5$ $r = 0.9$ $z = 0.5$ $r = 0.95$

$\alpha .1$	.16667	.4539	.16172	.19094	$\alpha .1$	.09091	.44128	.08765	.12311
.2	.16667	.43939	.15689	.21339	.2	.09091	.42494	.0845	.15259
.3	.16667	.42276	.15221	.23338	.3	.09091	.40604	.08148	.17885
.4	.16667	.40351	.14769	.25002	.4	.09091	.38393	.07859	.20073
.5	.16667	.38195	.14334	.26215	.5	.09091	.35772	.07586	.21679
.6	.16667	.35417	.13918	.26817	.6	.09091	.32616	.07327	.22501
.7	.16667	.32184	.13523	.26586	.7	.09091	.28743	.07084	.22245
.8	.16667	.28205	.13149	.25194	.8	.09091	.23874	.06856	.2047
.9	.16667	.23188	.12796	.22149	.9	.09091	.17571	.06644	.16478

 $z = 0.5$ $r = 0.99$

$\alpha .1$	.01961	.43112	.01879	.06002
.2	.01961	.41321	.01801	.09705
.3	.01961	.39235	.01726	.12979
.4	.01961	.36775	.01656	.15704
.5	.01961	.33831	.0159	.17711
.6	.01961	.30243	.01529	.18757
.7	.01961	.25774	.01471	.184
.8	.01961	.20056	.01418	.16329
.9	.01961	.12479	.01369	.11368

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5 Single-channel service with alternating priority

5.1 INTRODUCTION TO ALTERNATING PRIORITY

The following material is complementary to the analysis of a head-of-the-line priority message system reported in Ch. 4. The so-called "alternating priority" discipline is investigated. This is such that the service devotes itself entirely to a single priority class until no more of this class is left. As before, the model envisages two classes only, namely, a proportion α of the total traffic which we call the P-class, and the remaining proportion $\alpha = 1-\alpha$ constitutes the NP-class. The question of priority in this case is really non-existent. It arises only at the beginning of operations when there may be a rule which says "choose P-customers first". After that the service switches from one class to the other, or to the first arrival in the eventuality that the service becomes completely idle on completing the service of one type. This is a discipline encountered sometimes at road intersections where traffic in one direction has precedence until none is left. It is of special interest in being a self regulating mechanism and easy to implement.

The evolution of the system is a series of service periods each composed of a sum of busy periods. If this system has an advantage in comparison with head-of-the-line priority it is that it avoids the build-up of exceptionally long queues and waiting times of either traffic type under all but the heaviest traffic conditions when they are unavoidable. P customers are on the whole worse off than under head-of-the-line priority, but the other class is better favoured. In the context of messages, their information content will reach the destination in a more balanced manner.

5.2 STATISTICAL DESCRIPTION OF SYSTEM STATE

We begin, as before, by investigating the state probabilities on the assumption of statistical equilibrium. The underlying processes are X and Y , the numbers in the P and NP classes, respectively, and Z , a random variable equal to 1 when P messages are being processed, and equal to 0 when NP messages are being processed. Thus

$$p_0 = P[X=0, Y=0] = 1-r \quad (\text{Eq. 225})$$

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for the single channel system with Poisson arrivals, mean rate λ , and exponential service time with mean μ^{-1} . As usual $r = \lambda/\mu$. In this case Z does not enter: otherwise we write

$$\begin{aligned} a_{mn} &= P[X=m, Y=n, Z=1] & (m > 1, n > 0) & , \\ b_{mn} &= P[X=m, Y=n, Z=0] & (m > 0, n > 1) & . \end{aligned} \quad (\text{Eq. 226})$$

There are four sets of equations to be solved. These show what a moment's thought will confirm, namely that a_{mn} and b_{mn} are symmetrical with respect to α and $\bar{\alpha}$, and m and n . The equations are:

$$\begin{aligned} (1+r)a_{10} &= a_{20} + b_{11} + \alpha r p_0 \\ &\dots\dots\dots \\ (1+r)a_{1n} &= a_{2n} + \bar{\alpha} r a_{1n-1} & (n > 1) & \quad (\text{Eq. 227}) \end{aligned}$$

$$\begin{aligned} (1+r)a_{m0} &= a_{m+1,0} + b_{m1} + \alpha r a_{m-1,0} & (m > 2) & \\ &\dots\dots\dots \\ (1+r)a_{mn} &= a_{m+1,n} + \alpha r a_{m-1,n} + \bar{\alpha} r a_{m,n-1} & (n > 1) & \quad (\text{Eq. 228}) \end{aligned}$$

$$\begin{aligned} (1+r)b_{01} &= b_{02} + a_{11} + \bar{\alpha} r p_0 \\ &\dots\dots\dots \\ (1+r)b_{m1} &= b_{m2} + \alpha r b_{m-1,1} & (m > 1) & \quad (\text{Eq. 229}) \end{aligned}$$

$$\begin{aligned} (1+r)b_{0n} &= b_{0,n+1} + a_{1n} + \bar{\alpha} r b_{0,n-1} & (n > 2) & \\ &\dots\dots\dots \\ (1+r)b_{mn} &= b_{m,n+1} + \alpha r b_{m-1,n} + \bar{\alpha} r b_{m,n-1} & (m > 1) & . \quad (\text{Eq. 230}) \end{aligned}$$

An ignorant onlooker will merely observe the total number present in the system. Thus

$$a_{mn} + b_{mn} = p_{m+n} = (1-r)r^{m+n} , \quad (\text{Eq. 231})$$

a standard result for the M/M/1 system.

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It is also convenient sometimes to rewrite Eqs. 227 to 230 in such a way as to emphasize the constancy of the total number present. Thus we have

$$\sum_{m=1}^n a_{m \ n-m} + \sum_{m=0}^{n-1} b_{m \ n-m} = p_n = (1-r)r^n, \quad (n > 1) \quad (\text{Eq. 232})$$

and

$$\begin{aligned} (1+r)a_{n0} &= a_{n+1 \ 0} + b_{n1} + \alpha a_{n-1 \ 0} \\ (1+r)a_{n-1 \ 1} &= a_{n1} + \alpha a_{n-2 \ 1} + \bar{\alpha} a_{n-1 \ 0} \\ (1+r)a_{n-2 \ 2} &= a_{n-1 \ 2} + \alpha a_{n-3 \ 2} + \bar{\alpha} a_{n-2 \ 1} \\ &\dots\dots\dots \\ (1+r)a_{1 \ n-1} &= a_{2 \ n-1} + \bar{\alpha} a_{1 \ n-2} \\ (1+r)b_{n-1 \ 1} &= b_{n-1 \ 2} + \alpha b_{n-2 \ 1} \\ (1+r)b_{n-2 \ 2} &= b_{n-2 \ 3} + \alpha b_{n-3 \ 2} + \bar{\alpha} b_{n-2 \ 1} \\ &\dots\dots\dots \\ (1+r)b_{0n} &= a_{1n} + b_{0 \ n+1} + \bar{\alpha} b_{0 \ n-1} \end{aligned} \quad (\text{Eq. 233})$$

Addition of these equations and use of Eq. 232 gives

$$(1+r)p_n = p_{n+1} + r p_{n-1}, \quad (n > 1) \quad (\text{Eq. 234})$$

the familiar state equations for M/M/1 with solution

$$p_n = (1-r)r^n, \quad (n > 0). \quad (\text{Eq. 235})$$

5.3 A GENERATING FUNCTION APPROACH TO THE STATE PROBABILITIES

There are various ways of attempting to find the state probabilities. One which we used in the Head-of-the-Line analysis is to attempt to solve the basic difference equations directly. But, perversely, in spite of the symmetry, this approach seems fraught with difficulties owing to awkward boundary conditions. An alternative is to use a generating function technique. For this purpose we introduce

$$A_m(x) = \sum_{n \geq 0} a_{mn} x^n \quad (m > 1, \ |x| < 1) \quad (\text{Eq. 236})$$

$$B_n(x) = \sum_{m \geq 0} b_{mn} x^m \quad (n > 1, \ |x| < 1).$$

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From Eq. 227 we get

$$(1+r)A_1(x) = A_2(x) + b_{11} + rp_0 + \bar{\alpha}rx A_1(x) , \quad (\text{Eq. 237})$$

and from Eq. 228 for $m > 2$,

$$(1+r) A_m(x) = A_{m+1}(x) + b_{m1} + \alpha r A_{m-1}(x) + \bar{\alpha}rx A_m(x) \quad (\text{Eq. 238})$$

These can be rearranged to give:

$$(1+r-\bar{\alpha}rx) A_1(x) = A_2(x) + b_{11} + \alpha rp_0$$

$$(1+r-\bar{\alpha}x) A_m(x) = A_{m+1}(x) + b_{m1} + \alpha r A_{m-1}(x) \quad (m > 2) . \quad (\text{Eq. 239})$$

By symmetry these have companions

$$(1+r-\alpha rx) B_1(x) = B_2(x) + a_{11} + \bar{\alpha}rp_0$$

$$(1+r-\alpha x) B_n(x) = B_{n+1}(x) + a_{1n} + \bar{\alpha}r B_{n-1}(x) \quad (n > 2) . \quad (\text{Eq. 240})$$

Equations 239 and 240 can be combined if we next introduce the double generating functions:

$$A(x,y) = \sum_{m \geq 1} A_m(x) y^m , \quad |y| < 1 . \quad (\text{Eq. 241})$$

$$B(x,y) = \sum_{n \geq 1} B_n(x) y^n ,$$

We then obtain

$$(1+r-\bar{\alpha}rx) A(x,y) = (A(x,y)-yA_1(x))/y + (B_1(y)-b_{01}) + \alpha rp_0 y + \alpha ry A(x,y) \quad (\text{Eq. 242})$$

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giving

$$A(x,y) = \frac{y(A_1(x) - B_1(y) + b_{01} - \alpha r p_0 y)}{\alpha r y^2 - y(1+r-\bar{\alpha} r x) + 1} . \quad (\text{Eq. 243})$$

By symmetry we have

$$B(x,y) = \frac{y(B_1(x) - A_1(y) + a_{10} - \bar{\alpha} r p_0 y)}{\alpha r y^2 - y(1+r-\alpha r x) + 1} . \quad (\text{Eq. 244})$$

Equations 243 and 244 in principle contain all information about the system state. This statement can be rendered more precise by noting that $A(x,y)$ and $B(x,y)$ are finite for $|x| < 1$ and $|y| < 1$. Moreover, the following relation must hold:

$$p_0 + A(1,1) + B(1,1) = 1 . \quad (\text{Eq. 245})$$

This does not appear to be immediately helpful since the denominators of Eqs. 243 and 244 both vanish when $x = y = 1$ and thus

$$A_1(1) - B_1(1) = \alpha r p_0 - b_{01} = a_{10} - \bar{\alpha} r p_0 , \quad (\text{Eq. 246})$$

or

$$a_{10} + b_{01} = r p_0 = p_1 ,$$

which is Eq. 232 with $m = n = 1$.

Let

$$X = 1+r-\bar{\alpha} r x , \quad (\text{Eq. 247})$$

$$R = (X^2 - 4\alpha r)^{1/2} ,$$

and let $\underline{X}$, $\underline{R}$ denote the corresponding quantities with α and $\bar{\alpha}$ interchanged. Similarly let

$$Y(x) = (X-R)/2\alpha r , \quad (\text{Eq. 248})$$

$$\underline{Y}(x) = (\underline{X}-\underline{R})/2\bar{\alpha} r .$$

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The denominators of Eqs. 243 and 244 can then be written respectively as

$$\alpha r(y-Y)(y-1/\alpha r Y)$$

$$\bar{\alpha} r(y-\underline{Y})(y-1/\bar{\alpha} r \underline{Y}) \quad .$$

Since

$$(1/\alpha r Y - 1)(1-Y) = 1/\alpha r Y - 1/\alpha r - 1 + Y = \frac{1+r-\bar{\alpha} r x}{\alpha r} - \frac{1}{\alpha r} - 1 = \frac{\bar{\alpha} r(1-x)}{\alpha} \quad ,$$

it is clear that when $x < 1$, $Y < 1$. Moreover, provided that $\alpha r < 1$, $Y \rightarrow 1$ as $x \rightarrow 1$. We can accordingly write

$$A(x,y) = \frac{y(B_1(y)-A_1(x) + \alpha r p_0 y - b_{01})}{\alpha r(y-Y)(1/\alpha r Y - y)} \quad (\text{Eq. 249})$$

and, mutatis mutandis,

$$B(x,y) = \frac{y(A_1(y)-B_1(x) + \bar{\alpha} r p_0 y - a_{10})}{\bar{\alpha} r(y-\underline{Y})(1/\bar{\alpha} r \underline{Y} - y)} \quad (\text{Eq. 250})$$

Since $A(1,1)$ and $B(1,1)$ are finite the numerators of A and B must vanish when $x \leq 1$ respectively when $y = Y, \underline{Y}$. Thus we conclude that

$$A_1(x) = B_1(Y(x)) - b_{01} + \alpha r Y(x) p_0 \quad (\text{Eq. 251})$$

$$B_1(x) = A_1(\underline{Y}(x)) - a_{10} + \bar{\alpha} r \underline{Y}(x) p_0 \quad (\text{Eq. 252})$$

$A_1(x)$ can be interpreted as the generating function of the number of non-priority messages waiting just before the end of a priority class service period, for the probability that such a service period terminates in any small time interval $(t, t+h)$ is $\mu h + \sum_{n \geq 0} a_{1n} + o(h)$. Apart from a

constant $A_1(x)$ serves the same purpose for the epoch just after the end of a priority class service period. A similar interpretation holds for $B_1(x)$ with respect to the number of priority messages waiting at the end of a non-priority class service period.

It is also worth pointing out that $Y(x)$ and $\underline{Y}(x)$ have a concrete interpretation in terms of the system dynamics. $\bar{Y}(x)$ is the generating

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function of the number of NP arrivals during the busy period initiated by the service of a single P message. For, from M/M/1 theory, that busy period has probability density function $k(t)$ with Laplace transform

$$\kappa(z) = \frac{z + \mu + \alpha\lambda - ((z + \mu + \alpha\lambda)^2 - 4\alpha\lambda\mu)^{1/2}}{2\alpha\lambda},$$

while the said generating function is

$$\int_0^{\infty} e^{-\bar{\alpha}\lambda(1-x)t} k(t) dt = \kappa(\bar{\alpha}\lambda(1-x)) = Y(x).$$

With this interpretation of $Y(x)$ we can see that Eqs. 251 and 252 summarize the change of events over a service period. $A_1(x)$ generates the number of NP messages present just before the end of a PSP. The number of P messages at the beginning depends on how many were left at the end of the preceding service period. If the preceding SP was a NPSP, the probability that it ended leaving $m > 1$ P messages is μb_{m1} . The number of NP arrivals in this case has generating function $Y^m(x)$ and the contribution to μa_{1n} , the probability that the PSP terminates with n NP messages waiting, is the coefficient of x^n in $\mu b_{m1} Y^m(x)$. Summing over $m > 1$ gives the contribution $\mu(B_1(Y(x)) - b_{01})$ to $\mu A_1(x)$. The alternative is that the preceding service period, whether NPSP or PSP, ended leaving an empty system. The next PSP must then start with the arrival of a P message (probability αp_0) which generates NP messages waiting according to $Y(x)$. Thus

$$\mu A_1(x) = \mu B_1(Y(x)) - \mu b_{10} + \alpha p_0 Y(x)$$

which is Eq. 251 when μ is divided out.

In conclusion, substitution of Eqs. 251, 252 into Eqs. 249, 250 gives

$$A(x, y) = \frac{y(B_1(y) - B_1(Y) + \alpha r p_0(y - Y))}{\alpha r(y - Y)(1/\alpha r Y - y)} \quad (\text{Eq. 253})$$

$$B(x, y) = \frac{y(A_1(y) - A_1(Y) + \bar{\alpha} r p_0(y - Y))}{\bar{\alpha} r(y - Y)(1/\bar{\alpha} r Y - y)} \quad (\text{Eq. 254})$$

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A partial check on Eqs. 253, 254 can be obtained by evaluating $A(1,1)$ and $B(1,1)$. These should sum to $1 - p_0 = r$. Writing

$$B_1(y) = \sum_{j=0}^{\infty} \frac{(y-Y)^j}{j!} B_1^{(j)}(Y)$$

we get from Eq. 253

$$A(x,y) = \frac{y \left(\alpha p_0 + \sum_{j=1}^{\infty} \frac{(y-Y)^{j-1}}{j!} B_1^{(j)}(Y) \right)}{(1/Y - \alpha r y)} \quad (\text{Eq. 255})$$

and so

$$A(1,y) = \frac{y \left(\alpha p_1 + \sum_{j=1}^{\infty} \frac{(y-1)^{j-1}}{j!} B_1^{(j)}(1) \right)}{(1 - \alpha r y)} \quad (\text{Eq. 256})$$

since $Y(1) = 1$. $B_1^{(j)}(Y)$ denotes the j -th derivative of $B_1(y)$ evaluated at $y = Y$. From Eqs. 251 and 252 we also get

$$A_1^{(1)}(x) = (B_1^{(1)}(Y) + \alpha p_1) Y'(x)$$

$$B_1^{(1)}(x) = (A_1^{(1)}(Y) + \bar{\alpha} p_1) \underline{Y}'(x) \quad .$$

Now

$$Y'(x) = \bar{\alpha} r Y(x) / R$$

and therefore

$$Y'(1) = \bar{\alpha} r / (1 - \alpha r) \quad , \quad \underline{Y}'(1) = \alpha r / (1 - \bar{\alpha} r) \quad . \quad (\text{Eq. 257})$$

Hence

$$A'(1) = \frac{\bar{\alpha} r}{1 - \alpha r} \left(\alpha p_1 + \frac{\alpha r (\bar{\alpha} p_1 + A'(1))}{1 - \alpha r} \right)$$

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giving

$$A'_1(1) = \alpha \bar{\alpha} r^2 = B'_1(1) \quad (\text{Eq. 258})$$

by symmetry.

It follows from Eq. 256 with $y = 1$ that

$$A(1,1) = \alpha r \quad , \quad B(1,1) = \bar{\alpha} r \quad , \quad (\text{Eq. 259})$$

symmetry giving the second.

Now

$$A(1,y) = \sum_{m \geq 1} A_m(1) y^m$$

and

$$A_m(1) = \sum_{n \geq 0} a_{mn}$$

so that $A(1,y)$ generates the probabilities $A_m(1)$ that the number X of P-messages waiting at an arbitrary moment during a PSP is m . Similarly $B(1,y)$ generates the probabilities $B_n(1)$ that the number Y of NP-messages waiting during a NPSP is n .

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5.4 SYSTEM STATE PERFORMANCE

5.4.1 Formulae for mean and variance

It is of interest to develop expressions for the conditional means and variances. The means are given by

$$[dA(x,1)/dx]_{x=1} = A(1,1) E[Y | Z=1] = A(1,1) \frac{\bar{\alpha}r}{1-\alpha r} E[X | Z=1] \quad (\text{Eq. 260})$$

$$[dB(1,y)/dy]_{y=1} = B(1,1) E[Y | Z=0] = \frac{1/2 A''(1) + \bar{\alpha}r}{1-\bar{\alpha}r} \quad (\text{Eq. 261})$$

$$[dA(1,y)/dy]_{y=1} = A(1,1) E[X | Z=1] = \frac{1/2 B''(1) + \alpha r}{1-\alpha r} \quad (\text{Eq. 262})$$

$$[dB(x,1)/dx]_{x=1} = B(1,1) E[X | Z=0] = B(1,1) \frac{\alpha r}{1-\bar{\alpha}r} E[Y | Z=0]. \quad (\text{Eq. 263})$$

Here $A(1,1) = \alpha r$, $B(1,1) = \bar{\alpha}r$. The right-hand sides of Eqs. 261 and 262 can be deduced immediately from Eq. 256 since the left-hand side, for example in Eq. 262, is the coefficient of y in $A(1,y)$. Equation 261 follows by symmetry. The relationship in Eq. 260 also comes directly from Eq. 255 by differentiating first with respect to x and then setting $x,y = 1$. Equation 263 is the symmetrical counterpart.

The variances can be obtained similarly but are more cumbersome to evaluate explicitly.

The means require $A_1''(1)$ and $B_1''(1)$. Another differentiation of $A_1'(x)$ gives

$$A_1''(1) = \frac{2\alpha\bar{\alpha}^2 r^3 + \bar{\alpha}^2 r^2 B_1''(1)}{(1-\alpha r)^2} \quad (\text{Eq. 264})$$

and by symmetry there follows

$$B_1''(1) = \frac{2\bar{\alpha}\alpha^2 r^3 + \alpha^2 r^2 A_1''(1)}{(1-\bar{\alpha}r)^2}. \quad (\text{Eq. 265})$$

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By elimination and symmetry we then have

$$A_1''(1) = \frac{2\bar{\alpha}\bar{\alpha}^2 r^3 (1-2\bar{\alpha}r+\bar{\alpha}r^2)}{(1-r)(1-r+2\bar{\alpha}\bar{\alpha}r^2)} \quad (\text{Eq. 266})$$

$$B_1''(1) = \frac{2\bar{\alpha}\alpha^2 r^3 (1-2\alpha r+\alpha r^2)}{(1-r)(1-r+2\bar{\alpha}\alpha r^2)} \quad (\text{Eq. 267})$$

5.4.2 Numerical values and comments

Based on the above, Tables 26 and 27 have been constructed. Table 26 gives all four conditional expectations: $E[X | Z=1]$, $E[X | Z=0]$, $E[Y | Z=0]$, $E[Y | Z=1]$ for $r = 0.1, 0.5, 0.9, 0.95, 0.99$ and $\alpha = 0.1 (0.2) 0.9$. Naturally there are symmetries to be exploited but it is convenient to be able to scan the whole range of values. Table 27 gives the unconditional mean values $E[X]$ and $E[Y]$ waiting and processing in the P and NP streams respectively. These last means were calculated from Table 26 by the weighted sums

$$\begin{aligned} E[X] &= A(1,1) E[X | Z=1] + B(1,1) E[X | Z=0] \\ E[Y] &= A(1,1) E[Y | Z=1] + B(1,1) E[Y | Z=0] \end{aligned} \quad (\text{Eq. 268})$$

This provides a check on the algebra. For a Peeping Tom would, at an arbitrary moment, observe on average what is shown by Table 27. The sum of the means is the mean number of messages in an M/M/1 system, whose value is known to be $r/(1-r)$. It will be seen that the pairs of columns in Table 27 do satisfy this relation though it is not so easy to make the check algebraically without error.

Table 27 is more interesting as a basis for comment and may be compared with Table 23 of Ch. 4 which gives similar means for head-of-the-line priority. Under heavy traffic the monstrously unbalanced waiting line of NP messages is avoided and the load is fairly evenly spread with more or less equal misery for all. We shall turn to the system time later, but it is clear that shorter lines mean shorter waiting times. Thus, if a mistake in classifying a message as P or NP is made, when it should be the other, the damage is less serious. On the other hand genuine P messages which ought to arrive without delay are in fact delayed about half as much again as if no such regulatory mechanism were in operation. And here we should recall once more that this is not a true priority system but one which, like roundabouts rather than traffic lights at a road intersection, strives to facilitate the flow of vehicles by adapting to the needs of the traffic.

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TABLE 26

VALUES OF $E[X | Z=1]$, $E[X | Z=0]$, $E[Y | Z=1]$, $E[Y | Z=0]$

α	$r = 0.1$				$r = 0.5$			
	$E[X Z=1]$	$E[X Z=0]$	$E[Y Z=0]$	$E[Y Z=1]$	$E[X Z=1]$	$E[X Z=0]$	$E[Y Z=0]$	$E[Y Z=1]$
0.1	1.01120	0.01209	1.09991	1.09193	0.13303	1.16972	1.86697	0.53670
0.3	1.03344	0.03476	1.07767	0.07458	1.33471	0.38430	1.66529	0.54959
0.5	1.05556	0.05556	1.05556	0.05556	1.5	0.5	1.5	0.5
0.7	1.07767	0.07458	1.03344	0.03476	1.66529	0.54959	1.33471	0.38430
0.9	1.09991	0.09193	1.01120	1.01209	1.86697	0.53670	1.13303	0.16972

α	$r = 0.9$				$r = 0.95$			
	$E[X Z=1]$	$E[X Z=0]$	$E[Y Z=0]$	$E[Y Z=1]$	$E[X Z=1]$	$E[X Z=0]$	$E[Y Z=0]$	$E[Y Z=1]$
0.1	4.03539	3.29902	6.96461	3.59194	8.71134	8.05119	12.28866	8.23005
0.3	5.09109	4.31190	5.90891	4.39368	10.05716	9.30958	10.94284	9.35386
0.5	5.5	4.5	5.5	4.5	10.5	9.5	10.5	9.5
0.7	5.90891	4.39368	5.09109	4.31190	10.94284	9.35386	10.05716	9.30958
0.9	6.96461	3.59194	4.03539	3.29902	12.28866	8.23005	8.71134	8.05119

α	$r = 0.99$			
	$E[X Z=1]$	$E[X Z=0]$	$E[Y Z=0]$	$E[Y Z=1]$
0.1	48.37544	47.79635	52.62426	47.83883
0.3	50.03041	49.30935	50.96959	49.31874
0.5	50.5	49.5	50.5	49.5
0.7	50.96959	49.31874	50.03041	49.30935
0.9	52.62426	47.83883	48.37544	47.79634

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TABLE 27

UNCONDITIONAL MEAN VALUES $E[X]$ $E[Y]$ OF P AND NP MESSAGE NUMBERS IN SYSTEM

α	<u>$r = 0.1$</u>		<u>$r = 0.5$</u>		<u>$r = 0.9$</u>		<u>$r = 0.95$</u>	
	$E[X]$	$E[Y]$	$E[X]$	$E[Y]$	$E[X]$	$E[Y]$	$E[X]$	$E[Y]$
0.1	0.01120	0.09991	0.13303	0.86697	3.03539	5.96461	7.71134	11.28866
0.3	0.03344	0.07767	0.33471	0.66529	4.09109	4.90891	9.05716	9.94284
0.5	0.05556	0.05556	0.5	0.5	4.5	4.5	9.5	9.5
0.7	0.07767	0.03344	0.66529	0.33471	4.90891	4.09109	9.94284	9.05716
0.9	0.09991	0.01120	0.86697	0.13303	5.96461	3.03539	11.28866	7.71134

α	<u>$r = 0.99$</u>	
	$E[X]$	$E[Y]$
0.1	47.37574	51.62426
0.3	49.03041	49.96959
0.5	49.5	49.5
0.7	49.96959	49.03041
0.9	51.62426	47.37574

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5.5 SYSTEM TIME INCLUDING PROCESSING

5.5.1 Probability density functions and their Laplace transforms

Let W_p and W_{np} be the system times of P and NP messages respectively. Recalling that system time is the sum of waiting time and service (processing and transmission and decoding time), and writing $f(t) = \mu \exp(-\mu t)$, $f_n(t)$ its n -fold autoconvolution, $g_p(t)$, $g_{np}(t)$ the probability density functions of W_p and W_{np} respectively, we have

$$\begin{aligned}
 g_p(t) = & p_0 f(t) + \sum_{n \geq 1} A_n(1) f_{n+1}(t) + \sum_{m \geq 0} b_{m1} k_{\bar{\alpha}}(t) * f_{m+1}(t) \\
 & + \sum_{m \geq 0} b_{m2} k_{\bar{\alpha}}^{(2)}(t) * f_{m+1}(t) + \dots + \sum_{m \geq 0} b_{mn} k_{\bar{\alpha}}^{(n)}(t) * f_{m+1}(t) \\
 & + \dots
 \end{aligned} \tag{Eq. 269}$$

Here $k_{\bar{\alpha}}^{(n)}(t)$ is the n -fold autoconvolution of an $\bar{\alpha}$ -busy period density function. If $g_p(t)$ has Laplace transform $\gamma_p(z)$ then, with obvious notation,

$$\gamma_p(z) = \phi(z) [p_0 + A(1, \phi(z)) + B(\phi(z), \kappa_{\bar{\alpha}}(z))] \tag{Eq. 270}$$

and by symmetry

$$\gamma_{np}(z) = \phi(z) [p_0 + B(1, \phi(z)) + A(\phi(z), \kappa_{\alpha}(z))]. \tag{Eq. 271}$$

It suffices to explain Eq. 269. The p_0 term arises when an empty system is found upon arrival. If n P-messages are already in the system and a P-message is being processed (probability $A_n(1)$) the new message joins the P-queue and the system time is the sum of the residual system time of the message being processed which has density $f(t)$ by the lack of memory property of the exponential distribution, the $n-1$ system times of the messages ahead in the queue and own system time. If NP messages are being processed and there are n present the first component of the wait is the time to clear the NP queue. This is the sum of the residual busy period corresponding to the NP message being processed, which again by lack of memory has probability density function $k_{\alpha}(t)$, and the $n-1$ busy periods initiated by each of the remaining NP messages found, and taking place in sequence: if $m (>0)$ P customers are present already, the new arrival has

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a further system time composed of the sum of $m+1$ processing times. The contribution is accordingly

$$b_{mn}f_{m+1}(t)*\frac{k^{(n)}(t)}{\bar{\alpha}}.$$

The properties of convolutions under the Laplace transformation and the definitions of $A(x,y)$ and $B(x,y)$ give Eq. 270. We can immediately check that $\gamma_p(0)=1$, as should be the case.

5.5.2 Mean system time: numerical values and comments

$\gamma_p(z)$ can now be calculated directly in principle to give a measure of the reduction in value of α -stream information resulting from the alternating priority procedure. It can also be used to deduce the moments of W_p .

Consider, for example, $E[W_p]$. If Little's formula holds we should have

$$E[W_p] = E[X]/\alpha\lambda,$$

and we shall see later that this indeed is the case. But directly from Eq. 270 we obtain

$$E[W_p] = -\gamma_p'(0) = \frac{1}{\mu} \left[1 + \left\{ \frac{dA(1,y)}{dy} \right\}_{y=1} + \left\{ \frac{dB(x,1)}{dx} \right\}_{x=1} + \frac{1}{1-\bar{\alpha}r} \left\{ \frac{dB(1,y)}{dy} \right\}_{y=1} \right]$$

giving

$$\mu E[W_p] = 1 + \alpha r E[X | Z=1] + \bar{\alpha} r E[X | Z=0] + \frac{\bar{\alpha} r}{1-\bar{\alpha} r} E[Y | Z=0] \quad (\text{Eq. 272})$$

which can be expressed in various forms, one of which is

$$\mu E[W_p] = E[X] + \frac{(1/2A_1''(1)+\bar{\alpha}r)}{(1-\bar{\alpha}r)^2} + 1, \quad (\text{Eq. 273})$$

bearing a resemblance to Little's formula without actually being in the right form. It can be shown to reduce to Little's formula by showing that $E[X] - \alpha r E[W_p] = 0$, using Eq. 273 for $E[W_p]$ and setting $\mu = 1$. Equations 266 and 267 are needed.

Table 28 gives $E[W_p]$ in units of $1/\mu$ by direct calculation for $r = 0.9, 0.95, 0.99$ and we see readily that the values could have been obtained from Table 27 by use of Little's formula. In view of the symmetry there is no need to repeat this story for W_{np} , that is $E[W_p, \alpha] = E[W_{np}, \bar{\alpha}]$.

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TABLE 28MEAN P-CLASS SYSTEM TIME $E[W_p]$

(Time unit is mean time required to process a message)

<u>α</u>	<u>$r = 0.9$</u>	<u>$r = 0.95$</u>	<u>$r = 0.99$</u>
0.1	33.72661	81.17204	478.54284
0.2	20.82405	45.57261	245.36511
0.3	15.1522	31.77951	165.08555
0.4	11.98854	24.48262	124.47965
0.5	10	20	100
0.6	8.6743	17.01159	83.68023
0.7	7.79191	14.95164	72.10619
0.8	7.29399	13.60685	63.65872
0.9	7.36371	13.20311	57.93968

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Table 28 shows that at these traffic levels there is a substantial variation in mean system time between $\alpha = 0.1$ and 0.9 . In this sense the traffic with $\alpha > \bar{\alpha}$ appears to have a distinct advantage.

To complete the comparison fairly it will be necessary to examine the variances too. They will suffer the disadvantage under heavy traffic of behaving like the inverse cube of a small quantity. Some values are given in App. C.

If the Laplace transform is to be used as a measure of degradation of information due to delay in arrival we need to be able to calculate $A(x,y)$ and $B(x,y)$ for arbitrary x and y . This topic is addressed in App. B, included for completeness. Table 29 gives values of $\gamma_p(z)$, $\gamma_{np}(z)$ and the weighted average $\alpha\gamma_p(z) + \bar{\alpha}\gamma_{np}(z)$.

Table 29 gives the value of message content degraded by a penalty factor equivalent to paying compound interest at 10% per time unit of delay in arrival. If the message were to arrive as soon as generated its value would be unity. Taking the P-stream on its own, we see from the column headed γ_p that P-traffic is degraded most for small α : that is to say, if P-traffic forms a small proportion of total traffic the value of its information is degraded more than if it formed the larger proportion of total traffic. A completely symmetrical statement can be made about NP-traffic and so we can generalize the statement to the point of obviousness:

Under an alternating priority system it is that class of message which forms the smaller proportion of total traffic whose value is degraded the more.

To evaluate the performance of the system as a whole we can consider the weighted mean $\alpha\gamma_p + \bar{\alpha}\gamma_{np}$, given in the third column of the table. In this case we see that the system performs worst when the two categories of traffic form equal proportions ($\alpha = 1/2$), and best when there is a class of traffic which constitutes the larger proportion.

The advantage of the system compared with a no-priority single-server system can be gauged by comparing the $\gamma(z)$ that would apply in such a case. The values are given by $(1-r)/(z+1-r)$ at the foot of the table and it is seen that the alternating priority system is in this sense more efficient.

To compare on the same basis with head of the line priority see Table 25 of Ch. 4. At its best this degrades information to a lesser extent through late arrival, but if other factors to which attention has been drawn are taken into account the favourable balance may be reduced. Furthermore, the advantage of head of the line priority wanes as traffic intensity r increases.

TABLE 29
LOSS IN VALUE OF INFORMATION DUE TO LATE ARRIVAL

α	$r = 0.9$				$r = 0.95$				$r = 0.99$			
	γ_p	γ_{np}	$\alpha\gamma_p + \alpha\gamma_{np}$		γ_p	γ_{np}	$\alpha\gamma_p + \alpha\gamma_{np}$		γ_p	γ_{np}	$\alpha\gamma_p + \alpha\gamma_{np}$	
0.1	0.36614	0.56792	0.54774		0.22565	0.42487	0.40495		0.06263	0.16199	0.15206	
0.3	0.46658	0.57188	0.54029		0.31852	0.42717	0.39457		0.10588	0.1646	0.14698	
0.5	0.5331	0.53311	0.53311		0.38503	0.38502	0.38502		0.14047	0.14048	0.14047	
0.7	0.57186	0.46658	0.54028		0.42718	0.31852	0.39458		0.1646	0.10589	0.14699	
0.9	0.56792	0.36614	0.54774		0.42487	0.22565	0.40495		0.162	0.06263	0.15206	
	$\frac{1-r}{z+1-r} = 0.5$				$\frac{1-r}{z+1-r} = 0.33$				$\frac{1-r}{z+1-r} = 0.09$			

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Appendix A

TECHNICAL NOTES ON TEXT

The analysis in the text is based on the single server queueing system usually designated in the literature by the notation $M/M/1$. The first M means that the arrival stream is Poisson and, therefore, that the random time intervals between successive arrivals are exponentially distributed: the mean arrival rate is usually denoted by λ . The second M means that the service times are exponentially distributed. The mean service time is usually denoted by μ^{-1} . The third symbol, in this case 1, designates the number of service channels. It is implicit here that the number of potential customers is infinite.

There is a wide choice of books describing this system and the associated stochastic processes of interest. This text uses results and notation based on <A.2>.

The literature on priority queueing systems began sporadically in 1954 with an elegant paper by Cobham <A.1>. It has grown considerably, as is the fashion in queueing theory. The book by Jaiswal <A.4> is probably the first connected account but although it too is elegant it is couched in very general terms and is somewhat academic in its orientation. Nevertheless some of the results we have derived independently are to be found in <A.4> as special cases. Indeed, we do not claim originality for anything in the text except that it has been developed specifically and independently in the context of a potential communication application. The type of priority described in Ch. 4 is called "head-of-the-line" in the literature. There are many other types one of which is "pre-emptive" meaning that when a priority customer arrives he can interrupt the service of any non-priority customer who may be receiving attention. This did not seem appropriate in the context of communication. In the text we have considered only two classes of priority, but obviously a multiclass system is possible. Interesting in the message context is a system called "alternating priority". In this case, when one priority class is exhausted the service turns to the other class and serves it to exhaustion. This would tend to smooth out system times but one wonders how much better it might be than the first come, first served discipline often adopted for non-priority systems. Some analysis is given in Ch. 5.

Conway, Maxwell and Miller <A.3> is a masterful and interesting text aimed at sequencing and processing jobs in an industrial context. Kleinrock <A.5> is a text aimed primarily at computer systems and networks, and is full of wisdom and common-sense. It is of potential interest in the study of communication systems.

Reference is made in Sect. 4.3 to the "method of balance". This is a convenient and easily comprehended method for the derivation of the state

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probabilities for simple systems with exponential characteristics in equilibrium. In the normal M/M/1 system, for example, the system state is a random variable N capable of taking non-negative integer values n with probability p_n . One might think of the total probability, whose mass is unity, as being divided into boxes labelled $n = 0, 1, 2, \dots$. If the dynamics of arrival and service cause probability to flow, as it were, out of a box, this flow must, if the system is in equilibrium, be balanced by a total flow in of the same amount. The flow into and from other than adjacent boxes has negligible probability. As an example consider box n ($n > 1$) and its two neighbours labelled $n-1$ and $n+1$. The mass p_n of probability in box n will increase if there is flow from boxes $n-1$ and $n+1$. Flow from box $n-1$ occurs at rate λ and its amount per unit time is λp_{n-1} due to arrivals, and from box $n+1$ come μp_{n+1} per unit time due to departures. Flow out of box n is likewise of amount λp_n into box $n+1$ due to arrivals, and μp_n into box $n-1$ due to service completions. Thus, for equilibrium, we must have a balance equation

$$(\lambda + \mu)p_n = \lambda p_{n-1} + \mu p_{n+1} \quad (n > 1) .$$

For $n = 0$, a moment's thought shows that the balance equation is

$$\lambda p_0 = \mu p_1 .$$

The whole system can be seen easily to have solution $p_n = (1-r)r^n$ ($r = \lambda/\mu$). Try substitution if all else fails. The condition for equilibrium is $r < 1$. Otherwise all $p_n = 0$ and with probability one the system contains infinitely many clients. The result above is quoted in Sect. 4.3 of the text, at Eqs. 158, and 166 is the equation we have indicated above.

Reference is made in Sect. 4.7 to the very useful and versatile Little's Formula. This links mean system time $\hat{W}$ with mean system state $\hat{N}$. If we were to imagine customers entering a large room upon arrival and moving through it one after another in a line until their service is completed we could see that the average time a typical customer spends in the room is $\hat{W}$, and, moreover, that at any one time the mean number of customers in the room is $\hat{N}$. The mean spacing between customers is $\hat{T}$, the mean time between successive arrivals, and thus we would expect to find that

$$\hat{N} \hat{T} = \hat{W} .$$

This is Little's Formula, Result, or Theorem.

We have checked that Eqs. 188 and 190 are obtainable as special cases of results in <A.4>. Jaiswal does not appear to derive formulae for the

Appendix B

CALCULATION OF $A(x,y)$, $B(x,y)$ FOR $x,y \in (0,1)$

Reference to Eqs. 249, 250 or to Eqs. 253, 254 shows that $A(x,y)$ and $B(x,y)$ are completely determined if we know $A_1(x)$, $B_1(x)$. These are given by Eqs. 251 and 252. Substituting from Eq. 252 into Eq. 251 and vice versa we obtain the pair

$$A_1(x) = A_1(\underline{Y}(Y(x))) - p_1(1 - \overline{\alpha}\underline{Y}(Y(x)) - \alpha Y(x)) \quad (\text{Eq. B.1})$$

$$B_1(x) = B_1(Y(\underline{Y}(x))) - p_1(1 - \alpha Y(\underline{Y}(x)) - \overline{\alpha}\underline{Y}(x))$$

for which we have used $a_{10} + b_{01} = p_1$.

Define sequences u_n , v_n as follows:

$$u_0 = x, v_1 = Y(u_0), u_2 = \underline{Y}(v_1), \dots, u_{2n} = \underline{Y}(v_{2n-1}), v_{2n+1} = Y(u_{2n}). \quad (\text{Eq. B.2})$$

It is easily seen that $\{u_{2n}\}$ and $\{v_{2n+1}\}$ are monotonically increasing sequences in n for $0 < x < 1$ provided that $r < 1$ (condition also for statistical equilibrium). For if $x = Y(x)$ has a root it satisfies

$\alpha r x^2 - (1+r-\alpha r x)x+1 = 0$, that is $r x^2 - (1+r)x+1 = 0$ of which the roots are 1 and $1/r$. Now $Y(0)$, $\underline{Y}(0)$ are both positive and less than unity and so $Y(x)-x$ and $\underline{Y}(x)-x$ are positive when $x = 0$. Both $Y(x)$ and $\underline{Y}(x)$ increase steadily as x increases and accordingly $Y(x)-x$, $\underline{Y}(x)-x$ remain positive over $0 < x < 1$, decreasing to zero for the first time when $x = 1$ provided that $r < 1$. Thus the sequences $\{u_{2n}\}$ and $\{v_{2n+1}\}$ both tend to unity for every x in $0 < x < 1$.

The first equation of the pair B.1 then gives, successively,

$$A_1(u_0) = A_1(u_2) - w_0$$

$$A_1(u_2) = A_1(u_4) - w_2$$

$$A_1(u_4) = A_1(u_6) - w_4$$

.....

$$A_1(u_{2n}) = A_1(u_{2n+2}) - w_{2n}$$

where

$$w_{2n} = p_1 (1 - \bar{\alpha} u_{2n+2} - \alpha v_{2n+1}) .$$

Addition gives

$$A_1(u_0) = A_1(u_{2n+2}) - \sum_{i=0}^n w_{2i} .$$

The series has to be convergent since $u_{2n} \rightarrow 1$ and $A_1(1)$ is finite by its very nature. The practical implication is that if the series is computed using enough terms to give a prescribed accuracy we shall have $A_1(u_0)$ to the same accuracy since u_{2n+2} will be unity to the same accuracy. All that accordingly remains is to calculate $A_1(1)$. This can be done in the same way by starting with $u_0 = 0$. Then $A_1(0) = a_{10} = \alpha p_1$ since by balance we must have $\alpha \lambda p_0 = \mu a_{10}$ and $p_1 = r p_0$. Accordingly

$$A_1(1) = \alpha p_1 + \lim_{n \rightarrow \infty} \sum_{i=0}^{2n} w_{2i} . \quad (\text{Eq. B.3})$$

$B(1)$ can be obtained similarly, or by using Eq. 246, namely

$$B_1(1) = A_1(1) + (\bar{\alpha} - \alpha) p_1 . \quad (\text{Eq. B.4})$$

This is the basis for the calculations made for Table 29. The program is available on request.

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Appendix C

SOME VALUES OF THE STANDARD DEVIATIONS FOR ALTERNATING PRIORITY

A considerable amount of labour is needed to evaluate the second order statistics and the details will not be given here. Some numerical values are, however, given in Table C.1 below to facilitate comparison with the head-of-the-line priority system analyzed in Ch. 4.

The Table gives standard deviations σ_X and σ_Y of the numbers X and Y of P- and NP-messages in the system, and the standard deviation $\sigma_{W_{np}}$ of W_{np} , the NP-message system time. σ_{W_p} can be inferred by symmetry since $\sigma_{W_p}(\alpha) = \sigma_{W_{np}}(\bar{\alpha})$. The values of σ_N and σ_W for a non-priority system are given also for each r .

Comments

1. Comparisons are most interesting under heavy traffic conditions. The variability of P- and NP-message numbers is less extreme and they are nearer to each other than under a head-of-the-line priority regime.

2. It is possible to show that as $\alpha \rightarrow 1$, $\sigma_X \rightarrow r^{1/2}/(1-r)$, the non-priority system value, and as $\alpha \rightarrow 0$ it must tend to infinity. There is accordingly at least one value of α for which σ_X is least. For $r = 0.9$, an interesting phenomenon is observed where more than one local minimum occurs.

3. The values of $\sigma_{W_{np}}$ can be compared with Table 24 of Ch. 4. The variability of W_p and W_{np} is greatly reduced by the alternating priority system though, again, the P-messages are less well favoured than under head-of-the-line priority. Since $\sigma_{W_{np}} \rightarrow 1/(1-r)$ in units of mean processing time as $\alpha \rightarrow 0$, we again see that a value of α exists for which $\sigma_{W_{np}}$ is minimized, a value which corresponds to assigning priority to a rather large proportion of the total traffic.

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