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On the Proper Formulation of Maxwellian Electrodynamics for Continuum Mechanics

by Daniel S Weile, David A Hopkins, George A Gazonas, and Brian M Powers

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14. ABSTRACT
Despite the importance of electromagnetomechanical physics to processes ranging from piezoelectricity to the dynamics of electron beams, confusion abounds in the continuum mechanics literature as to how Maxwell's equations of electrodynamics should be formulated in the material frame of continuum mechanics. Current formulations in the literature conflict as to the manner in which the authors define fields, derive constitutive relations, and interpret contradictory formulations. The difficulties persist even when the phenomena described are electrostatic. This paper will demonstrate that the perplexity arises from two sources: a misunderstanding of the limitations of material frame descriptions, and the failure to appreciate the centrality of relativity theory to the formulation of electrodynamic equations in the vicinity of mechanical motion. Two new formulations of Maxwell's equations are provided that avoid the paradoxes of earlier formulations and thus describe the physics clearly and without self-contradiction.

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On the proper formulation of Maxwellian electrodynamics for continuum mechanics

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Abstract Despite the importance of electromagnetomechanical physics to processes ranging from piezoelectricity to the dynamics of electron beams, confusion abounds in the continuum mechanics literature as to how Maxwell's equations of electrodynamics should be formulated in the material frame of continuum mechanics. Current formulations in the literature conflict as to the manner in which the authors define fields, derive constitutive relations, and interpret contradictory formulations. The difficulties persist even when the phenomena described are electrostatic. This paper will demonstrate that the perplexity arises from two sources: a misunderstanding of the limitations of material frame descriptions, and the failure to appreciate the centrality of relativity theory to the formulation of electrodynamic equations in the vicinity of mechanical motion. Two new formulations of Maxwell's equations are provided that avoid the paradoxes of earlier formulations and thus describe the physics clearly and without self-contradiction.

Keywords Convective coordinates · Continuum mechanics · Electrodynamics · Relativity theory

1 Introduction

On boundaries between media, electromagnetic fields obey complicated boundary conditions. The electric and magnetic fields (usually denoted by the letters **e** and **h**, and measured in V/m and A/m, respectively) must always have continuous tangential components. On the other hand, the displacement current density and magnetic flux density (denoted by **d** and **b**, and measured in C/m^2 and T, respectively) have continuous normal components, despite their close constitutive relationships with **e** and **h** [3,10]. These complicated relationships can be especially confounding in the presence of deforming media, and therefore, many authors have sought to express Maxwell's equations in a language compatible with continuum mechanics either explicitly or implicitly [6,7,9,12,19]. Very often, this goal is accomplished by formulating electromagnetic theory in the material

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or reference coordinates of continuum mechanics, so that the coordinate system (and hence the boundary description) stays fixed during the deformation.

Amazingly, no consistent formulation of electromagnetics for continuum mechanics can be found in the literature. For instance, both the approaches taken and the results related differ in [6,7,9,12,19], even when the problem is limited to electrostatics. Different formulas are given for the same quantities, and different meanings are attached to the same variables. Some sources argue that the formulation is arbitrary, and that multiple correct definitions of, say, the material frame electric field are possible; others imply that only one formulation is correct. Source [19] ensures the material frame invariance of the constitutive laws, while source [12] forces the invariance of Maxwell's equations themselves. None of the approaches result in a formulation that can be transformed to another coordinate systems via standard tensor manipulations while preserving its form.

Two primary sources of difficulty have seemed to confuse the combination of continuum mechanics and electromagnetic theory. The first of these is the nature of "material frame indifference" when Maxwell's equations are considered. Many sources (notably, but by no means exclusively, [12]) formulate Maxwell's equations directly in the material frame and use the resulting formulation to guide their definition of the reference frame fields. For reasons that will be discussed shortly, not all of the equations of electromagnetics can be successfully so transformed while preserving their form. Upon demonstrating this, these papers will generally comment that their results are not form invariant and leave it at that, or argue that variance in the form of the constitutive laws is somehow preferable to variance in Maxwell's equations. Others (for instance, [7]) seem to imply that the choice of invariance violation is a choice to be made by the needs of a particular application. This approach forces consideration of the possible objective reality of any theory whatsoever.

The second source of misunderstandings in the combination of continuum mechanics and electromagnetics arises from the proper provenance of relativity theory. While Newtonian mechanics is self-consistent without any appeal to Einstein's ideas, Maxwell's equations do not obey Galilean relativity. This is true regardless of the speed of the objects producing or observing radiation and has nothing to do with the velocity of the object relative to the speed of light. Even the most humble predictions of electromagnetics (such as the production of magnetic fields by currents) involve terms which are of first order in v/c: the ratio of the speed of an object to that of light. Indeed, in the opening paragraph of his very first paper on the special theory of relativity, "On the electrodynamics of moving bodies," Einstein argues that the purpose of relativity theory is to harmonize the description of electromagnetics and mechanics in moving frames of reference [8]:

> It is known that Maxwell's electrodynamics—as usually understood at the present time—when applied to moving bodies leads to asymmetries which do not appear to be inherent in the phenomena. Take, for example, the reciprocal electrodynamic action of a magnet and a conductor. The observable phenomenon here depends only on the relative motion of the conductor and the magnet, whereas the customary view draws a sharp distinction between the two cases in which either the one or the other of these bodies is in motion. For if the magnet is in motion and the conductor at rest, there arises in the neighborhood of the magnet an electric field with a certain definite energy, producing a current at the places where parts of the conductor are situated. But if the magnet is stationary and the conductor in motion, no electric field arises in the neighborhood of the magnet. In the conductor, however, we find an electromotive force, to which in itself there is no corresponding energy, but which gives rise—assuming equality of relative motion in the two cases discussed—to electric currents of the same path and intensity as those produced by the electric forces in the former case.

Indeed, this observation of Einstein's is entirely germane to the difficulties faced here: Every single "material frame" formulation cited above assumes that the electric field $\mathbf{e}'$ in one frame depends on both the electric field $\mathbf{e}$ and the magnetic flux density $\mathbf{b}$ in the other. This sort of metamorphosis of one physical quantity into another is utterly alien to tensor algebra and cannot preserve the laws of physics upon a change of frame unless $\mathbf{e}$ and $\mathbf{b}$ are joined somehow into a larger quantity that does obey a tensor transformation.

In this paper, we will develop two consistent formulations of Maxwell's equation in continua by combining relativistic electromagnetics with Newtonian kinematics. In the first of the theories presented, time and space are entangled, but the resulting formulation resembles that of Lax and Nelson [12]. In the second theory, time and space are made orthogonal so that the result more closely mimics the usual practice. In both theories, we assume that $v/c \ll 1$ and retain only terms to first order so that the clock used to time events may be deemed universal. Relativistic time dilation is a second-order effect.

2 Convective coordinates and relativistic notation

Relativistic theories are often formulated in two different ways depending upon how measurements are taken in the two systems in relative motion. In the first of these, both systems assume a Euclidean spatial subspace measured in some fixed length unit, augmented by a temporal dimension orthogonal to it. The theory then relates how measurements taken by clocks and rulers in one system differ from those in the other. This approach is familiar from most formulations of special relativity described by Lorentz transformations between coordinate systems. Unfortunately, it is limited to constant velocity Cartesian coordinate transformations, implying that it is not flexible enough to describe deforming continua.

In the second approach, the transformation between systems may be more general than a relationship between two identical Cartesian systems in relative motion; for instance, the two systems may be curvilinear. Time becomes a fourth dimension to be treated on a par with space, except that space and time have weightings of opposite sign in the measurement of four-dimensional (also known as *world-*) distance. This approach was invented by Minkowski [15], so the space described is called *Minkowski space* to differentiate it from a four-dimensional space with a positive definite metric. In the Minkowski formulation, the Lorentz transformation arises out of "rotations" (i.e., orthogonal, metric-preserving transformations) in four-dimensional space. The test of a law of physics, then, is that it maintains the same form under legal tensor transformations.

The most basic description of phenomena in space-time is rendered in what is called the "spatial frame" in standard continuum mechanics [5,13,17]. We assume this laboratory frame to be described spatially by a right-handed Cartesian coordinate system with coordinates denoted by x^1, x^2, and x^3. Here, we use superscripts to differentiate contravariant components from covariant components, and coordinates are always contravariant [13,14]. We eschew Cartesian notation, as the coordinate systems we introduce evolve with the material, and cannot therefore be assumed rectilinear or orthogonal. Moreover, we let $\mathsf{x}^0 = ct$, where $c = 2.99792458 \times 10^8$ m/s is the speed of light and t is the time on a laboratory clock in seconds. Sans serif print is used to indicate Minkowskian four-vectors, whereas standard Roman font is used to indicate other variables including three-component systems. Thus, a world vector in space-time is denoted by the four-vector x^α, where Greek indices always range from 0 to 3. To indicate purely spatial quantities, we let Latin indices range from 1 to 3, so that we may write sensibly that $\mathsf{x}^i = x^i$ indicating the spatial part of the world vector is composed of the usual Cartesian coordinates. We assume the Einstein summation convention for an index repeated in both a superscript and a subscript throughout.

World distance in the Lorentzian spatial frame can be computed with the help of the metric tensor

$$\mathsf{g}_{\alpha\beta} \doteq \begin{cases} 0 & \text{if } \alpha \neq \beta, \\ 1 & \text{if } \alpha = \beta = 0, \\ -1 & \text{if } \alpha = \beta = i. \end{cases} \tag{1}$$

With this metric, the space-time inner product between two vectors a^α and b^β may be computed according to the formula $\mathsf{g}_{\alpha\beta}\mathsf{a}^\alpha\mathsf{b}^\beta$, and the "length" ds of a differential space-time interval $d\mathsf{x}^\alpha$ can be found from the formula

$$ds^2 = \mathsf{g}_{\alpha\beta} d\mathsf{x}^\alpha d\mathsf{x}^\beta = (cdt)^2 - \left(dx^1\right)^2 - \left(dx^2\right)^2 - \left(dx^3\right)^2. \tag{2}$$

Relativistic indifference requires the laws of the universe retain their form upon applying the standard tensor transformation rules, and the metric tensor is no different; the tensor described here is twice-covariant and changes coordinate systems following standard tensor procedure. A twice-contravariant tensor, $\mathsf{g}^{\alpha\beta}$, is defined to be its inverse [4,14] (which, for this particular tensor, has the same set of components). Finally, we will denote the determinant of a twice-covariant tensor by the same letter used to denote the tensor, but without indices. Thus, it is clear from the definition that

$$\mathsf{g} = \det\left(\mathsf{g}_{\alpha\beta}\right) = -1. \tag{3}$$

To describe a continuum, we need to further introduce reference coordinates X^I which serve to name the particles. These variables must represent a potential configuration of the continuum, in the sense that the mapping from the X^I to the x^i must be one-to-one and map right-handed triads to right-handed triads (at any fixed time for any observer), but in general need not represent any actual state of the body. For this reason alone, differentiating between "reference coordinate" formulations of Maxwell's equations is impossible—since the reference coordinates need not refer to any physical state of the body, the mapping of physical variables into the reference frame is completely arbitrary.

The motion of the body may now be specified by giving the location of each material point X^I at any given laboratory time t. This can be done with four equations

$$\begin{aligned} \mathsf{x}^0 &= ct, \\ \mathsf{x}^1 &= \chi^1\left(t, X^1, X^2, X^3\right), \\ \mathsf{x}^2 &= \chi^2\left(t, X^1, X^2, X^3\right), \\ \mathsf{x}^3 &= \chi^3\left(t, X^1, X^2, X^3\right). \end{aligned} \tag{4}$$

Because we are interested in preserving boundary descriptions that are presumably specified in terms of the X^I, we seek to create a formulation in which locations are specified in terms of these coordinates. Unfortunately, the X^I are merely point names, unconnected with the physical evolution of the body in time. Therefore, we need to more carefully specify the exact space-time coordinates we will use. The coordinates presented here are related to the convected coordinates described in passing in Aris [1] and in great detail in Kelly [11].

We define a new set of coordinates with spatial coordinates that match the reference coordinates. This makes these systems *spatially convected* coordinate systems. Specifically, by inverting the set of equations above, we may write

$$\begin{aligned} \tilde{\mathsf{x}}^0 &= \tilde{\chi}^0\left(\mathsf{x}^0, \mathsf{x}^1, \mathsf{x}^2, \mathsf{x}^3\right), \\ \tilde{\mathsf{x}}^1 &= \tilde{\chi}^1\left(\mathsf{x}^0, \mathsf{x}^1, \mathsf{x}^2, \mathsf{x}^3\right) = X^1, \\ \tilde{\mathsf{x}}^2 &= \tilde{\chi}^2\left(\mathsf{x}^0, \mathsf{x}^1, \mathsf{x}^2, \mathsf{x}^3\right) = X^2, \\ \tilde{\mathsf{x}}^3 &= \tilde{\chi}^3\left(\mathsf{x}^0, \mathsf{x}^1, \mathsf{x}^2, \mathsf{x}^3\right) = X^3. \end{aligned} \tag{5}$$

The temporal transformation has been left unspecified for the moment; different choices will lead to slightly different mathematical formulations, though all will of course have the same physical content. In any case, because of the functional relationship between the $\tilde{\mathsf{x}}^{\tilde{\alpha}}$ and the x^{α}, we can define the metric tensor for the curvilinear coordinate system:

$$\tilde{\mathsf{g}}_{\tilde{\alpha}\tilde{\beta}} = \frac{\partial \mathsf{x}^{\alpha}}{\partial \tilde{\mathsf{x}}^{\tilde{\alpha}}} \frac{\partial \mathsf{x}^{\beta}}{\partial \tilde{\mathsf{x}}^{\tilde{\beta}}} \mathsf{g}_{\alpha\beta} \tag{6}$$

Submatrices that break this matrix into its spatial and temporal parts will also be found useful. Following [16] and [18], we can define the three-component spatial/temporal part of the metric tensor as

$$\tilde{\mathsf{g}}_{\tilde{i}} \doteq \frac{\tilde{\mathsf{g}}_{0\tilde{i}}}{\sqrt{\tilde{\mathsf{g}}_{00}}}. \tag{7}$$

We also define a normalized spatial (i.e., three-dimensional) set of tensor components

$$\tilde{g}_{\tilde{i}\tilde{k}} \doteq \tilde{\mathsf{g}}_{\tilde{i}} \tilde{\mathsf{g}}_{\tilde{j}} - \tilde{\mathsf{g}}_{\tilde{i}\tilde{k}}, \tag{8}$$

and note that by elementary row operations, the determinants of these various systems are related by

$$-\tilde{\mathsf{g}} = \tilde{\mathsf{g}}_{00} \tilde{g}. \tag{9}$$

The normalized spatial metric tensor $\tilde{g}_{\tilde{i}\tilde{k}}$ is important, as it generally behaves as the purely spatial metric tensor in interpreting parts of equations.

3 Relativistic electrodynamics

The introduction described how Einstein introduced relativity theory by discussing how electric fields and magnetic fields must be interrelated in the presence of mechanical motion. This effect is easily observed even if the motion involved is very slow: it is the cause of all electrically generated magnetic fields. All of this implies that under coordinate changes, the electric field and the magnetic flux density cannot exist independently, and indeed, in a relativistic formulation, the electric and magnetic fields combine to form a four-dimensional, second-order, antisymmetric tensor. This formulation is presented in many books (e.g., [10,14,16,18]), so we merely present the results here to establish notation.

In an arbitrary curvilinear system (including the convected system), the electromagnetic tensor containing the primary fields (i.e., the electric field and magnetic flux density, which are responsible for the force) is antisymmetric and has elements given by

$$\tilde{\mathsf{n}}_{0\tilde{k}} = \frac{\tilde{e}_{\tilde{k}}}{c}, \tag{10}$$

$$\tilde{\mathsf{n}}_{\tilde{i}\tilde{k}} = -\sqrt{\tilde{g}}\,\epsilon_{\tilde{i}\tilde{k}\tilde{\ell}}\tilde{b}^{\tilde{\ell}}, \tag{11}$$

where $\epsilon_{\tilde{i}\tilde{k}\tilde{\ell}}$ is the permutation symbol [13,14]

$$\epsilon_{ik\ell} \doteq \begin{cases} 1 & \text{if } (i,k,\ell) = (1,2,3),\ (3,1,2),\ \text{or } (2,3,1), \\ -1 & \text{if } (i,k,\ell) = (2,1,3),\ (1,3,2),\ \text{or } (3,2,1), \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

In Eq. 10, the subscripts and superscripts on the field variables $\tilde{e}_{\tilde{k}}$ and $\tilde{b}^{\tilde{\ell}}$ indicate the basis set to which the coefficients pertain and do not imply that these systems transform as vectors or tensors. Indeed, where the indices of tensors can generally be "raised" or "lowered" by multiplication with an appropriate metric tensor, this is not the case with $\tilde{e}_{\tilde{k}}$ and $\tilde{b}^{\tilde{\ell}}$ because they do not have an independent existence upon coordinate system changes. On the other hand, the electromagnetic tensor $\tilde{\mathsf{n}}_{\tilde{\alpha}\tilde{\beta}}$ is an unambiguous tensor quantity that changes coordinate systems and bases in the standard manner.

The electromagnetic tensor solves the Faraday–Gauss Law

$$\frac{\partial \tilde{\mathsf{n}}_{\tilde{\alpha}\tilde{\beta}}}{\partial \tilde{\mathsf{x}}^{\tilde{\theta}}} + \frac{\partial \tilde{\mathsf{n}}_{\tilde{\beta}\tilde{\theta}}}{\partial \tilde{\mathsf{x}}^{\tilde{\alpha}}} + \frac{\partial \tilde{\mathsf{n}}_{\tilde{\theta}\tilde{\alpha}}}{\partial \tilde{\mathsf{x}}^{\tilde{\beta}}} = 0. \tag{13}$$

In particular, Gauss's Law for the magnetic field obtains with $\tilde{\alpha} = 1$, $\tilde{\beta} = 2$, and $\tilde{\theta} = 3$. Substituting these index values gives

$$\frac{\partial \tilde{\mathsf{n}}_{12}}{\partial \tilde{\mathsf{x}}^3} + \frac{\partial \tilde{\mathsf{n}}_{23}}{\partial \tilde{\mathsf{x}}^1} + \frac{\partial \tilde{\mathsf{n}}_{31}}{\partial \tilde{\mathsf{x}}^2} = -\frac{\partial\left(\sqrt{\tilde{g}}\,\tilde{b}^1\right)}{\partial \tilde{x}^1} - \frac{\partial\left(\sqrt{\tilde{g}}\,\tilde{b}^2\right)}{\partial \tilde{x}^2} - \frac{\partial\left(\sqrt{\tilde{g}}\,\tilde{b}^3\right)}{\partial \tilde{x}^3} = 0, \tag{14}$$

and then multiplying both sides of this equation by $-\tilde{g}^{-1/2}$ results in the law

$$\frac{1}{\sqrt{\tilde{g}}}\frac{\partial}{\partial \tilde{x}^{\tilde{i}}}\left(\sqrt{\tilde{g}}\,\tilde{b}^{\tilde{i}}\right) = 0. \tag{15}$$

The differential operator on the left-hand side of this equation is the curvilinear divergence [14,18], so this equation means that the magnetic field is solenoidal as expected. Similarly, by taking $\left(\tilde{\alpha}, \tilde{\beta}, \tilde{\theta}\right) = (2, 3, 0), (3, 1, 0)$ and $(1, 2, 0)$ in turn and multiplying by $\tilde{g}^{-1/2}$ results in Faraday's law

$$\frac{\epsilon^{\tilde{i}\tilde{k}\tilde{\ell}}}{\sqrt{\tilde{g}}}\frac{\partial \tilde{e}_{\tilde{\ell}}}{\partial \tilde{x}^{\tilde{k}}} + \frac{1}{\sqrt{\tilde{g}}}\frac{\partial}{\partial \tilde{t}}\left(\sqrt{\tilde{g}}\,\tilde{b}^{\tilde{i}}\right) = 0, \tag{16}$$

where $\tilde{t} = \tilde{\mathsf{x}}^0/c$. The spatial operator on the left-hand side of this equation is the curvilinear curl. If $\tilde{g}$ is independent of $\tilde{t}$, the temporal derivative term is merely the negation of the time rate of change of the magnetic flux density as expected. If $\tilde{g}$ varies with time, however, the extra factors take this into account in the computation of the time variation of the flux.

The other two Maxwell equations depend on the antisymmetric displacement tensor which combines the displacement current density and magnetic field, and the four-current which combines the current and charge densities. The antisymmetric displacement tensor $\tilde{\mathsf{m}}^{\tilde{\alpha}\tilde{\beta}}$ has terms given by

$$\tilde{\mathsf{m}}^{\tilde{i}0} = \frac{c\tilde{d}^{\tilde{i}}}{\sqrt{\tilde{\mathsf{g}}_{00}}}$$

$$\tilde{\mathsf{m}}^{\tilde{i}\tilde{k}} = -\frac{\epsilon^{\tilde{i}\tilde{k}\tilde{\ell}}\tilde{h}_{\tilde{\ell}}}{\sqrt{\tilde{g}}}, \tag{17}$$

where $\tilde{d}^{\tilde{i}}$ is the displacement current density and $\tilde{h}_{\tilde{\ell}}$ is the magnetic field. The four-current $\tilde{\mathsf{j}}^{\alpha}$ has components

$$\tilde{\mathsf{j}}^{0} = \frac{\tilde{\rho} c}{\sqrt{\tilde{\mathsf{g}}_{00}}}, \qquad \tilde{\mathsf{j}}^{\tilde{i}} = \frac{\tilde{j}^{\tilde{i}}}{\sqrt{\tilde{\mathsf{g}}_{00}}}, \tag{18}$$

where $\tilde{\rho}$ is the free charge density and $\tilde{j}^{\tilde{i}}$ is the free current density. In terms of these quantities, the Ampère–Maxwell–Gauss law is

$$\frac{1}{\sqrt{-\tilde{\mathsf{g}}}} \frac{\partial}{\partial \tilde{\mathsf{x}}^{\tilde{\beta}}} \left(\sqrt{-\tilde{\mathsf{g}}}\, \tilde{\mathsf{m}}^{\tilde{\alpha}\tilde{\beta}} \right) = \tilde{\mathsf{j}}^{\tilde{\alpha}}. \tag{19}$$

The scalar Gauss law for the electric field is recovered by taking $\tilde{\alpha} = 0$ in the above equation. The vector Maxwell–Ampère law obtains by taking $\tilde{\alpha} = 1, 2,$ and 3 in turn. Ultimately, these equations become

$$\frac{1}{\sqrt{\tilde{g}}} \frac{\partial}{\partial \tilde{x}^{\tilde{i}}} \left(\sqrt{\tilde{g}}\, \tilde{d}^{\tilde{i}} \right) = \tilde{\rho}, \tag{20}$$

$$\frac{1}{\sqrt{-g}} \left[\epsilon^{\tilde{i}\tilde{k}\tilde{\ell}} \frac{\partial \tilde{h}_{\tilde{\ell}}}{\partial \tilde{x}^{\tilde{k}}} - \frac{\partial}{\partial \tilde{t}} \left(\sqrt{-g}\, \tilde{d}^{\tilde{i}} \right) \right] = \tilde{j}^{\tilde{i}}. \tag{21}$$

Moreover, just as in the more familiar, three-dimensional exposition of electrodynamics, these equations give rise to the continuity equation (i.e., the charge conservation law.) In particular, continuity is derived by taking the four-divergence of Eq. 19 (times the metric determinant) and noting the antisymmetry of $\tilde{\mathsf{m}}^{\tilde{\alpha}\tilde{\beta}}$:

$$\begin{aligned}
\frac{\partial}{\partial \tilde{\mathsf{x}}^{\tilde{\alpha}}} \left[\sqrt{-\tilde{\mathsf{g}}}\, \tilde{\mathsf{j}}^{\tilde{\alpha}} \right] &= \frac{\partial}{\partial \tilde{\mathsf{x}}^{\tilde{\alpha}}} \left[\sqrt{-\tilde{\mathsf{g}}} \frac{1}{\sqrt{-\tilde{\mathsf{g}}}} \frac{\partial}{\partial \tilde{\mathsf{x}}^{\tilde{\beta}}} \left(\sqrt{-\tilde{\mathsf{g}}}\, \tilde{\mathsf{m}}^{\tilde{\alpha}\tilde{\beta}} \right) \right] \\
&= \frac{\partial^2}{\partial \tilde{\mathsf{x}}^{\alpha} \partial \tilde{\mathsf{x}}^{\beta}} \left(\sqrt{-\tilde{\mathsf{g}}}\, \tilde{\mathsf{m}}^{\tilde{\alpha}\tilde{\beta}} \right) \\
&= -\frac{\partial^2}{\partial \tilde{\mathsf{x}}^{\alpha} \partial \tilde{\mathsf{x}}^{\beta}} \left(\sqrt{-\tilde{\mathsf{g}}}\, \tilde{\mathsf{m}}^{\tilde{\beta}\tilde{\alpha}} \right) \\
&= -\frac{\partial^2}{\partial \tilde{\mathsf{x}}^{\beta} \partial \tilde{\mathsf{x}}^{\alpha}} \left(\sqrt{-\tilde{\mathsf{g}}}\, \tilde{\mathsf{m}}^{\tilde{\alpha}\tilde{\beta}} \right) \\
&= 0.
\end{aligned} \tag{22}$$

Written in terms of the components of $\tilde{\mathsf{j}}^{\alpha}$, this becomes

$$\frac{1}{\sqrt{\tilde{g}}} \frac{\partial}{\partial \tilde{x}^{\tilde{i}}} \left(\sqrt{\tilde{g}}\, \tilde{j}^{\tilde{i}} \right) + \frac{1}{\sqrt{\tilde{g}}} \frac{\partial}{\partial \tilde{t}} \left(\sqrt{\tilde{g}}\, \tilde{\rho} \right) = 0, \tag{23}$$

as expected.

To complete the discussion of the relativistic formulation of electrodynamics, the general form of electromagnetic constitutive relations must be stipulated. Actually, this turns out to be the most important part of describing Maxwell's equations where continuum mechanics is concerned: Ultimately, the formulations proposed in [6,7,12,19] all differ on the formulation of these relations. Despite this confusion, the difficulty is already clear from the above exposition: The electromagnetic tensor that appears in the equations is covariant, and the displacement tensor that appears in the equations is contravariant. Formulations based on Cartesian tensor notation will categorically miss this issue.

Our presentation of constitutive relations is not based on any particular material, but simply describes how bound charges and currents are involved in relating the displacement tensor to the electromagnetic tensor. In any particular material, the bound charges and currents would be functions of an applied field, but this is not of current concern. Moreover, in general, we might be concerned about the movement of the bound charge relative to the frame in which the equations are being formulated, but this is of no concern in the convected frame, which moves with the material. Therefore, if we define the vector polarization per unit volume (measured in C/m^2) to be $\tilde{p}^{\tilde{i}}$, and the vector magnetization per unit volume (measured in A/m) to be $\tilde{m}_{\tilde{k}}$, we can define a contravariant antisymmetric second-order polarization tensor through the equations

$$\begin{aligned} \tilde{\mathsf{p}}^{\tilde{i}0} &= -\frac{c\tilde{p}^{\tilde{i}}}{\sqrt{\tilde{\mathsf{g}}_{00}}} \\ \tilde{\mathsf{p}}^{\tilde{i}\tilde{k}} &= -\frac{\epsilon^{\tilde{i}\tilde{k}\tilde{\ell}}\tilde{m}_{\tilde{\ell}}}{\sqrt{-\tilde{\mathsf{g}}}}. \end{aligned} \tag{24}$$

The bound four-current $\tilde{\mathsf{j}}_{\mathrm{b}}^{\tilde{\beta}}$ is just the four-divergence of this tensor:

$$\tilde{\mathsf{j}}_{\mathrm{b}}^{\tilde{\beta}} = \frac{1}{\sqrt{-\tilde{\mathsf{g}}}}\frac{\partial}{\partial\tilde{x}^{\tilde{\alpha}}}\left(\sqrt{-\tilde{\mathsf{g}}}\,\tilde{\mathsf{p}}^{\tilde{\alpha}\tilde{\beta}}\right). \tag{25}$$

In terms of these quantities, the general electromagnetic constitutive law in a medium is

$$\tilde{\mathsf{m}}^{\alpha\beta} = \frac{1}{\mu_0}\tilde{\mathsf{n}}^{\alpha\beta} - \tilde{\mathsf{p}}^{\alpha\beta}. \tag{26}$$

Of course, the contravariant electromagnetic tensor is related to its covariant form in the usual manner:

$$\tilde{\mathsf{n}}^{\tilde{\alpha}\tilde{\beta}} = \tilde{\mathsf{g}}^{\tilde{\alpha}\tilde{\theta}}\tilde{\mathsf{g}}^{\tilde{\beta}\tilde{\psi}}\tilde{\mathsf{n}}_{\tilde{\theta}\tilde{\psi}}. \tag{27}$$

All of these equations are covariant in the relativistic sense: their form is preserved in the face of parameterization changes in space-time. In particular, they hold equally with the tildes removed, i.e., in the spatial frame. While the details have not been expounded here, all of the vector differentiations used in the equations are covariant in form and the forms of these equations are identical upon coordinate changes effected using standard tensor transformation formulas [16,18]. The only cost associated with this reformulation is the need to view events, at least for the purpose of variable changes, in four-dimensional space-time. We now examine two useful coordinate changes for continuum mechanics.

4 Universal time

Perhaps the most straightforward way to derive a convective theory of electromagnetomechanics is to measure time in all systems in the same manner. This implies that the missing transformation in Eq. 5 should be supplied by defining

$$\tilde{\mathsf{x}}^0 = \mathsf{x}^0 = ct. \tag{28}$$

Note that this definition does not imply that time appears the same to all observers; the actual passage of time must be computed with the help of the metric tensor. The choice to use a universal time is merely a matter of accounting, not physics.

To apply this to the derivation of a convective theory, we must first compute the relativistic deformation gradient matrices, that is, the matrix of derivatives $\partial\mathsf{x}^{\alpha}/\partial\tilde{\mathsf{x}}^{\tilde{\alpha}}$ and its inverse. This is done in terms of the particle velocity (as measured in the laboratory)

$$v^i \doteq \frac{\mathrm{d}x^i}{\mathrm{d}t} = c\frac{\mathrm{d}\mathsf{x}^i}{\mathrm{d}\mathsf{x}^0}, \tag{29}$$

and the motion of the laboratory as seen from the particle

$$\tilde{v}^{\tilde{i}} \doteq c\frac{\mathrm{d}\tilde{\mathsf{x}}^{\tilde{i}}}{\mathrm{d}\tilde{\mathsf{x}}^0} = -\frac{\partial\tilde{x}^{\tilde{i}}}{\partial x^i}\frac{v^i}{c}. \tag{30}$$

In terms of these quantities, the deformation gradient matrix has terms

$$\frac{\partial \mathsf{x}^0}{\partial \tilde{\mathsf{x}}^0} = 1, \qquad \frac{\partial \mathsf{x}^0}{\partial \tilde{\mathsf{x}}^{\tilde{i}}} = 0, \qquad \frac{\partial \mathsf{x}^i}{\partial \tilde{\mathsf{x}}^0} = \frac{v^i}{c}, \qquad \frac{\partial \mathsf{x}^i}{\partial \tilde{\mathsf{x}}^{\tilde{i}}} = \delta^I_{\tilde{i}} \frac{\partial x^i}{\partial X^I}, \tag{31}$$

where the elements of the standard (three-dimensional) deformation gradient matrix are given by

$$\frac{\partial x^i}{\partial X^I} = \delta^{\tilde{i}}_I \frac{\partial x^i}{\partial \tilde{x}^{\tilde{i}}}. \tag{32}$$

This matrix can easily be inverted using its block structure. The elements of the inverse are given by

$$\frac{\partial \tilde{\mathsf{x}}^0}{\partial \mathsf{x}^0} = 1, \qquad \frac{\partial \tilde{\mathsf{x}}^0}{\partial \mathsf{x}^i} = 0, \qquad \frac{\partial \tilde{\mathsf{x}}^{\tilde{i}}}{\partial \mathsf{x}^0} = \frac{\tilde{v}^{\tilde{i}}}{c}, \qquad \frac{\partial \tilde{\mathsf{x}}^{\tilde{i}}}{\partial \mathsf{x}^i} = \frac{\partial \tilde{x}^{\tilde{i}}}{\partial x^i} = \delta^{\tilde{i}}_I \frac{\partial X^I}{\partial x^i}. \tag{33}$$

From these expressions, we can compute the elements of the metric tensor (to first order in v^i/c) using Eq. 1, giving

$$\tilde{\mathsf{g}}_{00} = 1, \qquad \tilde{\mathsf{g}}_{\tilde{i}0} = \tilde{\mathsf{g}}_{0\tilde{i}} = -\delta_{ik} \frac{\partial x^i}{\partial \tilde{x}^{\tilde{i}}} \frac{v^k}{c}, \qquad \tilde{\mathsf{g}}_{\tilde{i}\tilde{k}} = \delta_{ik} \frac{\partial x^i}{\partial \tilde{x}^{\tilde{i}}} \frac{\partial x^k}{\partial \tilde{x}^{\tilde{k}}}. \tag{34}$$

Note that from Eq. 31, the purely spatial part of the metric tensor is just the right Cauchy-Green tensor C_{IK} [13]:

$$\tilde{\mathsf{g}}_{\tilde{i}\tilde{k}} = \delta_{ik} \frac{\partial x^i}{\partial \tilde{x}^{\tilde{i}}} \frac{\partial x^k}{\partial \tilde{x}^{\tilde{k}}} = \delta_{\tilde{i}I} \delta_{\tilde{k}K} \delta_{ik} \frac{\partial x^i}{\partial X^I} \frac{\partial x^k}{\partial X^K} = \delta_{\tilde{i}I} \delta_{\tilde{k}K} C_{IK}. \tag{35}$$

Similarly, the elements of the inverse metric tensor are given to first order by

$$\tilde{\mathsf{g}}^{00} = 1, \qquad \tilde{\mathsf{g}}^{\tilde{i}0} = \tilde{\mathsf{g}}^{0\tilde{i}} = -\frac{\tilde{v}^{\tilde{i}}}{c}, \qquad \tilde{\mathsf{g}}^{\tilde{i}\tilde{k}} = \delta_{ik} \frac{\partial \tilde{x}^{\tilde{i}}}{\partial x^i} \frac{\partial \tilde{x}^{\tilde{k}}}{\partial x^k}. \tag{36}$$

With these relationships in hand, we can proceed to find the relationships between the various electromagnetic quantities in different systems. The convected electric field and magnetic flux densities can be related between systems using the standard tensor transformation formula

$$\tilde{\mathsf{n}}_{\tilde{\alpha}\tilde{\beta}} = \frac{\partial x^\alpha}{\partial \tilde{x}^{\tilde{\alpha}}} \frac{\partial x^\beta}{\partial \tilde{x}^{\tilde{\beta}}} \mathsf{n}_{\alpha\beta}. \tag{37}$$

This formula leads to the formulas

$$\tilde{e}_{\tilde{i}} = \frac{\partial x^i}{\partial \tilde{x}^{\tilde{i}}} \left(e_i + \epsilon_{ik\ell} v^k b^\ell \right), \tag{38}$$

$$\tilde{b}^{\tilde{k}} = \frac{\partial \tilde{x}^{\tilde{k}}}{\partial x^k} b^k, \tag{39}$$

for changing the field definition from one frame to another. The strange asymmetry is a consequence of the choice of a twice-covariant matrix for the electromagnetic tensor, and the consequent expression of the electric field in the covariant basis and the magnetic field in the contravariant basis. Indeed, the indices on these "vectors" cannot directly be "raised" or "lowered" because the vectors themselves have no independent existence; they are inextricably linked by Eq. 37. Moreover, an attempt to compute the relationship between contravariant electric fields and covariant magnetic fluxes by examining the contravariant version of Eq. 37 leads to the conclusion that the $\tilde{b}_{\tilde{k}}$ depends on both e^i and b_k, but that $\tilde{e}^{\tilde{i}}$ depends only on e^i!

The derivation of the transformation rule for the magnetic field and displacement flux density follows a similar approach, and similar conclusions apply to the transformations of the resulting fields. The transformation formulas are given by

$$\tilde{d}^{\tilde{i}} = \frac{\partial \tilde{x}^{\tilde{i}}}{\partial x^i} d^i, \tag{40}$$

$$\tilde{h}_{\tilde{i}} = \frac{\partial x^i}{\partial \tilde{x}^{\tilde{i}}} \left(h_i - \epsilon_{ik\ell} v^k d^\ell \right). \tag{41}$$

Currents and charges, including bound currents and charges, transform as contravariant world vectors. Separated into their component pieces, the laws read

$$\tilde{\rho} = \rho, \tag{42}$$

$$\tilde{j}^{\tilde{i}} = \frac{\partial \tilde{x}^{\tilde{i}}}{\partial x^i}\left(j^i - \rho v^i\right). \tag{43}$$

Finally, the transformation rule for the polarization tensor can be derived by comparison with the displacement tensor. This process yields the equations

$$\tilde{p}^{\tilde{i}} = \frac{\partial \tilde{x}^{\tilde{i}}}{\partial x^i} p^i, \tag{44}$$

$$\tilde{m}_{\tilde{i}} = \frac{\partial x^i}{\partial \tilde{x}^{\tilde{i}}}\left(m_i + \epsilon_{ik\ell} v^k p^\ell\right). \tag{45}$$

The inverse of this rule, given by

$$p^i = \frac{\partial x^i}{\partial \tilde{x}^{\tilde{i}}} \tilde{p}^{\tilde{i}}, \tag{46}$$

$$m_i = \frac{\partial \tilde{x}^{\tilde{i}}}{\partial x^i}\left(\sqrt{\tilde{g}}\,\epsilon_{\tilde{i}\tilde{j}\tilde{k}} \tilde{v}^{\tilde{j}} \tilde{p}^{\tilde{k}} + \tilde{m}_{\tilde{i}}\right), \tag{47}$$

is of particular interest, because the material properties are almost certainly known primarily in the convected frame, where the material is stationary, rather than the laboratory frame. Notice also that polarization and magnetization are inextricably linked; this appears to have been ignored in previous work on this topic. This is no small matter, either: the current formulation demonstrates that no measurement can distinguish between moving polarization and magnetization. On the other hand, the asymmetry present in this equation, like that present in the equations for the components of the electromagnetic and displacement tensors, is essentially illusory. It changes character if the covariant and contravariant natures of the tensors in question are switched and can be made to disappear entirely with the formulation given in the next section.

Given all of these transformation laws, it only remains to write the appropriate laws of physics in each of the systems. Maxwell's equations are particularly easy: They are of the forms shown in Eqs. 15, 16, 20, and 21 in general. In spatial coordinates, all of the tildes are removed for the field and current variables, and $-\mathsf{g} = g = \mathsf{g}_{00} = 1$. In the convected coordinates, the tildes remain and given the particulars of the transformation, $-\tilde{\mathsf{g}} = \tilde{g}$ and $\tilde{\mathsf{g}}_{00} = 1$.

Finally, the constitutive laws of Eq. 26 become

$$d^i = \epsilon_0 \delta^{ik} e_k + p^i, \tag{48}$$

$$h_i = \frac{1}{\mu_0}\delta_{ik} b^k - m_i, \tag{49}$$

and

$$\tilde{d}^{\tilde{i}} = \epsilon_0 \tilde{g}^{\tilde{i}\tilde{k}}\left(\tilde{e}_{\tilde{k}} + \sqrt{\tilde{g}}\,\epsilon_{\tilde{k}\tilde{\ell}\tilde{r}} \tilde{v}^{\tilde{\ell}} \tilde{b}^{\tilde{r}}\right) + \tilde{p}^{\tilde{i}}, \tag{50}$$

$$\tilde{h}_{\tilde{i}} = \frac{1}{\mu_0}\tilde{g}_{\tilde{i}\tilde{k}} \tilde{b}^{\tilde{k}} - \tilde{m}_{\tilde{i}} + \sqrt{\tilde{g}}\,\epsilon_0 \epsilon_{\tilde{i}\tilde{k}\tilde{\ell}} \tilde{v}^{\tilde{k}} \tilde{g}^{\tilde{\ell}\tilde{r}} \tilde{e}_{\tilde{r}}. \tag{51}$$

Non-relativistic formulations of electrodynamics often falter right here. For instance, many formulations of Eq. 49 include a term proportional to the cross-product of the polarization and the velocity, but in a manifestly covariant formulation, this term is provided solely by the transformation of the polarization tensor, i.e., Eq. 45.

The presentation of Maxwell's equations formulated in this section is correct and complete, except of course for the low-velocity approximation. That said, the result is still very strange and unnatural: the formulation hinges on the "unit vector in the direction of time," and intertwines electricity and magnetism to such a degree that the formula for transforming fields between systems takes a completely different form depending on whether covariant or contravariant bases are used. To some extent, such effects are unavoidable: after all, the electric and magnetic fields are not really vectors, but specific parts of an antisymmetric tensor. Nonetheless, some confusion may be avoided by replacing Eq. 28 with an equation that makes time and space orthogonal, so that the "unit vector in the direction of time" can be eliminated. We turn to such a formulation in the next section.

5 Time-orthogonal transformation

To formulate electromagnetic theory in a time-orthogonal manner, we replace the first equation of equation set 5 with

$$\tilde{\mathsf{x}}^0 = \tilde{a}\mathsf{x}^0 + \tilde{s}_{\tilde{i}}\frac{\tilde{v}^{\tilde{i}}}{c}, \tag{52}$$

where the constants $\tilde{a}$ and $\tilde{s}_{\tilde{i}}$ are to be determined. Of course, this equation is to be used on an instant-by-instant basis, that is, we describe what a spectator-physicist (see [16] and [18]) would see when convected with the material and measuring time in a specific way. Given this description of time, Eq. 33 becomes

$$\frac{\partial\tilde{\mathsf{x}}^0}{\partial\mathsf{x}^0} = \tilde{a} + \tilde{s}_{\tilde{i}}\frac{\tilde{v}^{\tilde{i}}}{c}, \qquad \frac{\partial\tilde{\mathsf{x}}^0}{\partial\mathsf{x}^i} = \frac{\partial\tilde{x}^{\tilde{i}}}{\partial x^i}\tilde{s}_{\tilde{i}}, \qquad \frac{\partial\tilde{\mathsf{x}}^{\tilde{i}}}{\partial\mathsf{x}^0} = \frac{\tilde{v}^{\tilde{i}}}{c}, \qquad \frac{\partial\tilde{\mathsf{x}}^{\tilde{i}}}{\partial\mathsf{x}^i} = \frac{\partial\tilde{x}^{\tilde{i}}}{\partial x^i} = \delta^{\tilde{i}}_I\frac{\partial X^I}{\partial x^i} \tag{53}$$

in the new system. This tensor is easily inverted, resulting in

$$\frac{\partial\mathsf{x}^0}{\partial\tilde{\mathsf{x}}^0} = \frac{1}{\tilde{a}}, \qquad \frac{\partial\mathsf{x}^0}{\partial\tilde{\mathsf{x}}^{\tilde{i}}} = \frac{\tilde{s}_{\tilde{i}}}{a}, \qquad \frac{\partial\mathsf{x}^i}{\partial\tilde{\mathsf{x}}^0} = \frac{1}{\tilde{a}}\frac{v^i}{c}, \qquad \frac{\partial\mathsf{x}^i}{\partial\tilde{\mathsf{x}}^{\tilde{i}}} = \delta^I_{\tilde{i}}\frac{\partial x^i}{\partial X^i} - \frac{\tilde{s}_{\tilde{i}}}{\tilde{a}}\frac{v^i}{c}. \tag{54}$$

To arrive at a time-orthogonal mapping, the condition $\tilde{\mathsf{g}}_{0\tilde{j}} = 0$ must be satisfied. Using Eqs. 1, 6, and 54 to flesh this condition out in the current circumstance leads to the equation

$$\tilde{s}_{\tilde{i}} = -\frac{\tilde{a}\delta_{ij}\frac{\partial x^j}{\partial\tilde{x}_{\tilde{i}}}\frac{v^i}{c}}{\sqrt{1-\delta_{k\ell}\frac{v^k v^\ell}{c^2}}}. \tag{55}$$

If we further specify that we would like purely temporal measurements to be instantaneously the same for both systems (that is, if we enforce $\tilde{g}_{00} = 1$), we find that

$$\tilde{a} = \sqrt{1-\delta_{ij}\frac{v^i v^j}{c^2}} \approx 1, \tag{56}$$

$$\tilde{s}_{\tilde{i}} = -\delta_{ij}\frac{\partial x^j}{\partial\tilde{x}^{\tilde{i}}}\frac{v^i}{c}. \tag{57}$$

This yields the final form of the transformation matrices to first order; they become

$$\frac{\partial\tilde{\mathsf{x}}^0}{\partial\mathsf{x}^0} = 1, \qquad \frac{\partial\tilde{\mathsf{x}}^0}{\partial\mathsf{x}^i} = \delta_{ij}\frac{v^j}{c}, \qquad \frac{\partial\tilde{\mathsf{x}}^{\tilde{i}}}{\partial\mathsf{x}^0} = \frac{\tilde{v}^{\tilde{i}}}{c}, \qquad \frac{\partial\tilde{\mathsf{x}}^{\tilde{i}}}{\partial\mathsf{x}^i} = \frac{\partial\tilde{x}^{\tilde{i}}}{\partial x^i}, \tag{58}$$

and

$$\frac{\partial\mathsf{x}^0}{\partial\tilde{\mathsf{x}}^0} = 1, \qquad \frac{\partial\mathsf{x}^0}{\partial\tilde{\mathsf{x}}^{\tilde{i}}} = \delta_{ij}\frac{\partial x^j}{\partial\tilde{x}^{\tilde{i}}}\frac{v^i}{c}, \qquad \frac{\partial\mathsf{x}^i}{\partial\tilde{\mathsf{x}}^0} = \frac{v^i}{c}, \qquad \frac{\partial\mathsf{x}^i}{\partial\tilde{\mathsf{x}}^{\tilde{i}}} = \frac{\partial x^i}{\partial\tilde{x}^{\tilde{i}}}. \tag{59}$$

Of course, by design, the metric tensors are even simpler, with elements given (as usual, to first order in v/c) by

$$\tilde{\mathsf{g}}_{00} = 1, \qquad \tilde{\mathsf{g}}_{\tilde{i}0} = \tilde{\mathsf{g}}_{0\tilde{i}} \equiv 0, \qquad \tilde{\mathsf{g}}_{\tilde{i}\tilde{j}} = -\tilde{g}_{\tilde{i}\tilde{j}} \doteq -\delta_{ij}\frac{\partial x^i}{\partial\tilde{x}^{\tilde{j}}}\frac{\partial x^i}{\partial\tilde{x}^{\tilde{j}}}, \tag{60}$$

and, obviously,

$$\tilde{\mathsf{g}}^{00} = 1, \qquad \tilde{\mathsf{g}}^{\tilde{i}0} = \tilde{\mathsf{g}}^{0\tilde{i}} \equiv 0, \qquad \tilde{\mathsf{g}}^{\tilde{i}\tilde{j}} = -\tilde{g}_{\tilde{i}\tilde{j}} = -\delta^{ij}\frac{\partial\tilde{x}^{\tilde{i}}}{\partial x^i}\frac{\partial\tilde{x}^{\tilde{j}}}{\partial x^j}. \tag{61}$$

Before turning to the relationship between the convected and spatial fields, we first create definitions for contravariant and covariant versions of three-vectors that was avoided in the last section. Four-vectors and four-tensors, of course, have definitions fixed by our tensor scheme from the start. We may consistently define

$$\tilde{e}^{\tilde{i}} \doteq -\tilde{g}^{\tilde{i}\tilde{j}}\tilde{e}_{\tilde{j}}, \qquad \tilde{d}_{\tilde{i}} \doteq -\tilde{g}_{\tilde{i}\tilde{j}}\tilde{d}^{\tilde{j}}, \tag{62}$$

$$\tilde{h}^{\tilde{i}} \doteq +\tilde{g}^{\tilde{i}\tilde{j}}\tilde{h}_{\tilde{j}}, \qquad \tilde{b}_{\tilde{i}} \doteq +\tilde{g}_{\tilde{i}\tilde{j}}\tilde{b}^{\tilde{j}}, \tag{63}$$

and

$$\tilde{p}_{\tilde{i}} \doteq -\tilde{g}_{\tilde{i}\tilde{j}}\tilde{p}^{\tilde{j}}, \quad \tilde{m}^{\tilde{i}} \doteq \tilde{g}^{\tilde{i}\tilde{j}}\tilde{m}_{\tilde{j}}, \tag{64}$$

The negative signs in Eqs. 62 and 64 arise from the sign convention used in our metric definition and are unavoidable. Switching to a metric definition where the negative sign is associated with time rather than space would simply alter the signs of the magnetic fields rather than the electric ones. In any case, the choice of sign above is telling: Because our metric inverts space on transferring between covariant and contravariant bases, Eqs. 62 and 63 can be remembered by recalling that **e** is a polar vector and **h** is an axial vector.

With these definitions in place, and the rules for raising and lowering indices of four-vectors fixed, the elements of the twice-contravariant electromagnetic tensor and the twice-covariant displacement tensor become

$$\tilde{n}^{\tilde{i}0} = -\frac{\tilde{e}^{\tilde{i}}}{c}, \qquad \tilde{n}^{\tilde{i}\tilde{j}} = -\frac{\epsilon^{\tilde{i}\tilde{j}\tilde{k}}}{\sqrt{\tilde{g}}}\tilde{b}_{\tilde{k}}, \tag{65}$$

$$\tilde{m}_{\tilde{i}0} = c\tilde{d}_{\tilde{i}}, \qquad \tilde{m}_{\tilde{i}\tilde{j}} = -\sqrt{\tilde{g}}\,\epsilon_{\tilde{i}\tilde{j}\tilde{k}}\tilde{h}^{\tilde{k}}, \tag{66}$$

$$\tilde{p}_{\tilde{i}0} = -c\tilde{p}_{\tilde{i}}, \qquad \tilde{p}_{\tilde{i}\tilde{j}} = -\sqrt{\tilde{g}}\,\epsilon_{\tilde{i}\tilde{j}\tilde{k}}\tilde{m}^{\tilde{k}}. \tag{67}$$

These new variables satisfy Maxwell's equations in both tensor form (i.e., Eqs. 13 and 19) and the more familiar "vector" form (i.e., Eqs. 15, 16, 20, and 21), as expected. All that remains is to describe the relationship between fields in the spatial and convected coordinates.

By examining the components of the change of basis formula for the electromagnetic tensor n_{ij}, the spatial-to-convected coordinate transformation for the electric field and the magnetic flux density become

$$\tilde{e}_{\tilde{i}} = \frac{\partial x^i}{\partial \tilde{x}^{\tilde{i}}}\left(e_i + \epsilon_{ijk}v^j b^k\right), \qquad \tilde{b}^{\tilde{i}} = \frac{\partial \tilde{x}^{\tilde{i}}}{\partial x^i}\left(b^i + \epsilon^{ijk}\frac{v_j e_k}{c^2}\right). \tag{68}$$

Because of the simple orthogonal form of the metric, the contravariant electric transformation and the covariant magnetic flux transformation are symmetric with the above:

$$\tilde{e}^{\tilde{i}} = \frac{\partial \tilde{x}^{\tilde{i}}}{\partial x^i}\left(e^i + \epsilon^{ijk}v_j b_k\right), \qquad \tilde{b}_{\tilde{i}} = \frac{\partial x^i}{\partial \tilde{x}^{\tilde{i}}}\left(b_i + \epsilon_{ijk}\frac{v^j e^k}{c^2}\right). \tag{69}$$

These equations also eliminate the asymmetry in the way electric and magnetic fields intertwine with one another in the former formulation. The change of variable formulas for the displacement tensor quantities becomes

$$\tilde{d}^{\tilde{i}} = \frac{\partial \tilde{x}^{\tilde{i}}}{\partial x^i}\left(d^i - \epsilon^{ijk}\frac{v_j h_k}{c^2}\right), \qquad \tilde{h}_{\tilde{i}} = \frac{\partial x^i}{\partial \tilde{x}^{\tilde{i}}}\left(h_i - \epsilon_{ijk}v^j d^k\right), \tag{70}$$

and those for the polarization tensor becomes

$$\tilde{p}^{\tilde{i}} = \frac{\partial \tilde{x}^{\tilde{i}}}{\partial x^i}\left(p^i + \epsilon^{ijk}\frac{v_j m_k}{c^2}\right), \qquad \tilde{m}_{\tilde{i}} = \frac{\partial x^i}{\partial \tilde{x}^{\tilde{i}}}\left(m_i + \epsilon_{ijk}v^j p^k\right). \tag{71}$$

The inverse relation of this equation is of interest, since the material parameters are probably known when the material is at rest. These equations are simply the tensor inverse of the above, that is,

$$p^i = \frac{\partial x^i}{\partial \tilde{x}^{\tilde{i}}}\left(\tilde{p}^{\tilde{i}} + \frac{\epsilon^{\tilde{i}\tilde{j}\tilde{k}}}{\sqrt{\tilde{g}}}\frac{\tilde{v}_{\tilde{j}}\tilde{m}_{\tilde{k}}}{c^2}\right), \qquad m_i = \frac{\partial \tilde{x}^{\tilde{i}}}{\partial x^i}\left(\tilde{m}_{\tilde{i}} + \sqrt{\tilde{g}}\,\epsilon_{\tilde{i}\tilde{j}\tilde{k}}\tilde{v}^{\tilde{j}}\tilde{p}^{\tilde{k}}\right). \tag{72}$$

Finally, the charges and currents transform according to the formulas

$$\tilde{\rho} = \rho + \frac{v_i j^i}{c^2}, \qquad \tilde{j}^{\tilde{i}} = \frac{\partial \tilde{x}^{\tilde{i}}}{\partial x^i}\left(j^i - \rho v^i\right). \tag{73}$$

6 A comparison with the literature

Given these two formulations of Maxwell's equations in convective coordinate systems, we now turn to a comparison with the work of earlier authors on this topic. This task is gravely complicated by the fact that all of these works seek to write their relations not in the convective coordinates advocated here, but in the reference coordinate system. We therefore need a way to discuss the relationship between these systems and relate parameters in one system to those in the other.

This task is significantly eased by imagining the reference coordinates to refer to the continuum in a condition of initial rest. Under the action of forces, the continuum is deformed to a different shape at a later time. The reference coordinate system X^I can be envisioned as a set of Cartesian coordinate planes drawn on the continuum in this original configuration. As it deforms, these lines describe the convective system. By tracking this deformation continuously, we could in principle relate the current configuration of these planes to their initial configuration and thus relate the convective coordinates to the reference coordinates by preserving the mapping the coefficients of vectors in the convective system onto the reference system.

The problem with this prescription is that the convective system is necessarily oblique in general, and two possible sets of basis vectors can be chosen at each point in the field. The first type of basis that can be chosen can be used with contravariant coefficients and consists of vectors tangential to the coordinate curves. In vector notation, we can call the basis vector tangent to coordinate curve $\tilde{i}$, "$\tilde{\mathbf{u}}_{\tilde{i}}$" and define it by the formula

$$\tilde{\mathbf{u}}_{\tilde{i}} = \frac{\partial \mathbf{r}}{\partial \tilde{x}^{\tilde{i}}}. \tag{74}$$

This basis is most useful when thinking about fields as acting along a line. For instance, this is the definition of current density that views it as charge density times velocity, a linear motion of electricity.

The second basis comes from looking not at the directions tangent to the coordinate lines, but those normal to the coordinate planes. The basis vector chosen normal to plane $\tilde{i}$ is given by

$$\tilde{\mathbf{u}}^{\tilde{i}} = \nabla \tilde{x}^{\tilde{i}}. \tag{75}$$

This basis is to be used with the covariant coefficients and appears most often when a vector field describes a flux crossing a surface. In the case of current density, it is the viewpoint embraced by defining the total current as its flux through a surface.

Because the original reference system is invariably Cartesian (and hence orthogonal), there is no unique choice of which set of coefficients should be preserved in the transformation. The cacophony of different interpretations in the literature is testimony to this: The different results arise from different interpretations of each field. If the electric field is imagined to be the source of a force (as both the Lorentz force equation and Faraday's law seem to imply), the author transforms it as a set of contravariant coefficients; if it is seen as a flux, the author treats it as a set of covariant ones. Because ultimately some expression requires a relationship between one set for which the author has chosen to use covariant coefficient transformations and another in which he has chosen contravariant transformations, each of these previous theories is inconsistent. For example, Lax and Nelson [12] choose to force the macroscopic Maxwell's equations (i.e., those involving **D** and **H** to model the behavior of fields on materially bound charges) to transform consistently, and then find an inconsistent relationship in the constitutive relationship. Yang and Batra [19] (implicitly) imply that they would choose both the electric field (**E**) and the electric displacement density (**D**) to transform as vectors with contravariant coefficients. This choice preserves the consistency of the constitutive relationship at the expense of that of Gauss's law for the electric field. Dorfmann and Ogden [7] make the same choice as Lax and Nelson [12], but muse that

$$P_I = \frac{\partial x^i}{\partial X^I} p_i \tag{76}$$

Table 1 The transformation of the elements of the electromagnetic tensor in different formulations

	e	**b**
Universal time, $\tilde{n}_{\alpha\beta}$	$\tilde{e}_{\tilde{i}} = \frac{\partial x^i}{\partial \tilde{x}^{\tilde{i}}}\left(e_i + \epsilon_{ik\ell} v^k b^\ell\right)$	$\tilde{b}^{\tilde{i}} = \frac{\partial \tilde{x}^{\tilde{i}}}{\partial x^i} b^i$
Universal time, $\tilde{n}^{\alpha\beta}$	$\tilde{e}^{\tilde{i}} = \frac{\partial \tilde{x}^{\tilde{i}}}{\partial x^i} e^i$	$\tilde{b}_{\tilde{i}} = \frac{\partial x^i}{\partial \tilde{x}^{\tilde{i}}}\left(b_i - \epsilon_{ijk}\frac{v^j e^k}{c^2}\right)$
Time orthogonal, $\tilde{n}_{\alpha\beta}$	$\tilde{e}_{\tilde{i}} = \frac{\partial x^i}{\partial \tilde{x}^{\tilde{i}}}\left(e_i + \epsilon_{ijk} v^j b^k\right)$	$\tilde{b}^{\tilde{i}} = \frac{\partial \tilde{x}^{\tilde{i}}}{\partial x^i}\left(b^i + \epsilon^{ijk}\frac{v_j e_k}{c^2}\right)$
Lax and Nelson [12]	$E_I = \frac{\partial x_i}{\partial X_I}\left(e_i + \epsilon_{ijk} v_j b_k\right)$	$B_I = \sqrt{C}\frac{\partial X_I}{\partial x_i} b_i$
Yang and Batra [19]	$E_I = \sqrt{C}\frac{\partial X_I}{\partial x_i} e_i$	(Electrostatic)
Dorfmann and Ogden [7]	$E_I = \frac{\partial x_i}{\partial X_I} e_i$	(Electrostatic)
Clayton [6]	$E_I = \frac{\partial x_i}{\partial X_I} e_i$	(Electrostatic)

Table 2 The transformation of the elements of the displacement tensor in different formulations

	d	**h**
Universal time, $\tilde{m}^{\alpha\beta}$	$\tilde{d}^{\tilde{i}} = \frac{\partial \tilde{x}^{\tilde{i}}}{\partial x^i} d^i$	$\tilde{h}_{\tilde{i}} = \frac{\partial x^i}{\partial \tilde{x}^{\tilde{i}}}\left(h_i - \epsilon_{ik\ell} v^k d^\ell\right)$
Universal time, $\tilde{m}_{\alpha\beta}$	$\tilde{d}_{\tilde{i}} = \frac{\partial x^i}{\partial \tilde{x}^{\tilde{i}}}\left(d_i - \epsilon_{ijk}\frac{v^j h^k}{c^2}\right)$	$\tilde{h}^{\tilde{i}} = \frac{\partial \tilde{x}^{\tilde{i}}}{\partial x^i} h^i$
Time orthogonal, $\tilde{m}^{\alpha\beta}$	$\tilde{d}^{\tilde{i}} = \frac{\partial \tilde{x}^{\tilde{i}}}{\partial x^i}\left(d^i - \epsilon^{ijk}\frac{v_i h_k}{c^2}\right)$	$\tilde{h}_{\tilde{i}} = \frac{\partial x^i}{\partial \tilde{x}^{\tilde{i}}}\left(h_i - \epsilon^{ijk} v_j d_k\right)$
Lax and Nelson [12]	$D_I = \sqrt{C}\frac{\partial X_I}{\partial x_i} d_i$	$H_I = \frac{\partial x_i}{\partial X_I}\left(h_i - \epsilon_{ik\ell} v_k d_\ell\right)$
Yang and Batra [19]	$D_I = \sqrt{C}\frac{\partial X_I}{\partial x_i} d_i$	(Electrostatic)
Dorfmann and Ogden [7]	$D_I = \sqrt{C}\frac{\partial X_I}{\partial x_i} d_i$	(Electrostatic)
Clayton [6]	$D_I = \sqrt{C}\frac{\partial X_I}{\partial x_i} d_i$	(Electrostatic)

would be equally acceptable. Clayton [6] chooses this relationship for polarization density after claiming that there is no natural choice for polarization.

Fortunately, the source of these inconsistencies is well known: they come from ignoring the impact of an observer's motion on his perception of time. In a four-dimensional formulation, the standard electromagnetic fields described above become entries in four-dimensional tensors, and the transformation of these tensors properly and consistently describes their relationship in any physical coordinate system as we have shown. Of course, looking at results in the reference system is now impossible: different points in the continuum have experienced the passage of time differently. Besides, there is no physical entity corresponding to, say, the "reference electric field," so an unbiased choice is impossible. Furthermore, this is not merely a matter of notation: we have written our results in the notation used by Ricci, Levi-Civita, and Einstein, but the same sorts of problems would show up if we used, for instance, the geometric algebra advocated by Arthur [2]. In geometric algebra notation, the distinction is not between covariant and contravariant, but between vectors and bivectors. Still, even when this issue is sorted out, the result can only be made consistent with an appeal to the theory of special relativity.

To appreciate the differences between the formulation presented here and the earlier approaches, we present Tables 1, 2, 3 listing the transformations of the most important variables according to this work and other authors. Unfortunately, this comparison is complicated by the fact that the papers in the literature use different notation from that used here and from each other. Furthermore, formulations in the literature refer not to our curvilinear convected variables $\tilde{x}^{\tilde{i}}$, but to the Cartesian reference variables X^I. This allows them to use Cartesian tensor notation, which eschews superscripts and ignores the difference between covariant and contravariant variables since they coincide for orthonormal coordinate systems. To be as thorough as possible, we present results pertaining to the universal time technique of Sect. 4 in both covariant and contravariant forms. Since our field definitions are elements of second-order, four-dimensional tensors, indices are raised or lowered by raising or lowering the indices of the enveloping tensor and taking the appropriate field definition. Thus, Sect. 4 defines $\tilde{e}_{\tilde{k}} = c\tilde{n}_{0\tilde{k}}$; we therefore define $\tilde{e}^{\tilde{k}} = c\tilde{n}^{0\tilde{k}}$ to facilitate comparisons. We also include the results presented in Sect. 5.

Table 3 The transformation of the elements of the polarization tensor in different formulations

	p	**m**
Universal time, $\tilde{p}^{\alpha\beta}$	$\tilde{p}^{\tilde{i}} = \frac{\partial \tilde{x}^{\tilde{i}}}{\partial x^i} p^i$	$\tilde{m}_{\tilde{i}} = \frac{\partial x^i}{\partial \tilde{x}^{\tilde{i}}} \left(m_i - \epsilon_{ik\ell} v^k p^\ell\right)$
Universal time, $\tilde{p}_{\alpha\beta}$	$\tilde{p}_{\tilde{i}} = \frac{\partial x^i}{\partial \tilde{x}^{\tilde{i}}} \left(p_i + \epsilon_{ijk} \frac{v^j m^k}{c^2}\right)$	$\tilde{m}^{\tilde{i}} = \frac{\partial \tilde{x}^{\tilde{i}}}{\partial x^i} m^i$
Time orthogonal, $\tilde{p}^{\alpha\beta}$	$\tilde{p}^{\tilde{i}} = \frac{\partial \tilde{x}^{\tilde{i}}}{\partial x^i} \left(p^i + \epsilon^{ijk} \frac{v_i m_k}{c^2}\right)$	$\tilde{m}_{\tilde{i}} = \frac{\partial x^i}{\partial \tilde{x}^{\tilde{i}}} \left(m_i + \epsilon^{ijk} v_j p_k\right)$
Lax and Nelson [12]	$P_I = \sqrt{C} \frac{\partial X_I}{\partial x_i} p_i$	$M_I = \frac{\partial x^i}{\partial X^I} m_i$
Yang and Batra [19]	$P_I = \sqrt{C} \frac{\partial X_I}{\partial x_i} p_i$	(Electrostatic)
Dorfmann and Ogden [7]	$P_I = \sqrt{C} \frac{\partial X_I}{\partial x_i} p_i$	(Electrostatic)
Clayton [6]	$P_I = \frac{\partial x^i}{\partial X^I} p_i$	(Electrostatic)

Finally, even with all of the caveats and explanations provided above, a few additional things about the tables should be further clarified. Lax and Nelson [12] is the only paper to mention magnetic effects, even though these exist to first order in v/c in Maxwell's equations, however formulated. Also, the results attributed to Yang and Batra [19] are not explicitly given in their paper (except for the electric field translation) but can be inferred from their work.

To effectuate the comparison, we must introduce the right Cauchy-Green tensor,

$$C_{IJ} = \delta_{ij} \frac{\partial x^i}{\partial X^I} \frac{\partial x^j}{\partial X^J}, \tag{77}$$

and its determinant C. Assuming that the reference coordinates refer to an initial, undeformed continuum, at any given time, the elements of C_{IJ} are the same as those of $\tilde{g}_{\tilde{i}\tilde{j}}$, though their meanings are different: The former is a measure of the deformation of the medium relative to reference coordinates, whereas the latter is a metric tensor in the current configuration. Thus, one clear difference between our formulations and the previous formulations from the literature is the presence of factors of $\sqrt{C}$ in the latter to account for the volumetric deformation of the continuum.

Of all of the previous presentations related here, Lax and Nelson [12] comes closest to a theory that agrees with ours, but with some caveats. First, their theory is presented in the reference coordinates, rendering it immune from experiment. If, however, we interpret all of their results in the manner most agreeable to our theory, the only formula on which there is any major substantive difference is the transformation of the magnetization. Interestingly, even with this success, [12] laments its inability to render electromagnetics in a tensorially consistent manner. The work here demonstrates, however, that in some cases (i.e., in the case of the definition $\mathbf{d} = \epsilon_0 \mathbf{e} + \mathbf{p}$), the failure of frame invariance was only apparent: the presented formula is entirely correct, but only frame invariant when viewed through the space-time of relativity.

7 Conclusions

This paper has clarified the expression of Maxwell's equations of electrodynamics in the language of continuum mechanics, resulting in two systems for the solution of non-relativistic electromagnetomechanical problems. The first of these is closer to currently available schemes, but clarifies the nature of difficulties faced by previous authors, and corrects earlier mistakes including the relationship between polarization and magnetization. The second method is based on the relativistic idea of time orthogonality, and thus results in more symmetric formulas.

This work has also illuminated the causes of the confusion that currently pervades the literature. First, there is an insistence on formulating the equations in the reference domain, a concept with no meaning in the realm of electromagnetics and possibly no physical meaning to boot. This led to the independent assignment of "reference domain" values to polarization and magnetization (which are connected in reality), and to an indeterminate method of evaluating the correctness of the derived transformations. Second, even when low velocities are considered, Maxwell's equations are simply inconsistent with Galilean relativity, so the Lorentz transformation must be employed. Indeed, that this is necessarily the case is hinted at in every previous derivation: Upon changing coordinates to a moving frame (even one with constant velocity), the electric field in the new frame depends on the magnetic flux density in the old. This demonstrates conclusively that these two physical quantities must be part of a single tensor. Furthermore, this is no trifling matter: the connection between the two fields is of first order in the quantity v/c.

This work clears up the confusion evident in the forgoing explanation of the state of the literature. By insisting on convective rather than reference coordinates, the work presented here has removed the ambiguity of previous formulations by allowing the theory to be associated with quantities that can be measured. By further weaving in the theory of relativity, contradictions in earlier formulations can be overcome.

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ERRATUM

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Erratum to: On the proper formulation of Maxwellian electrodynamics for continuum mechanics

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In the original publication of the article, Maxwell's equations for continuum mechanics contain a few errors of sign and indicial notation. These corrections do little to alter the conclusions of the paper except that the resultant equations are more symmetric.

Equation 8 has a small typographical error and should be

$$\tilde{g}_{\tilde{i}\tilde{k}} \doteq \tilde{\mathsf{g}}_{\tilde{i}}\tilde{\mathsf{g}}_{\tilde{k}} - \tilde{\mathsf{g}}_{\tilde{i}\tilde{k}}. \tag{1}$$

Equation 17(b) has a slightly more significant mistake in its treatment of the metric element, and should read

$$\tilde{\mathsf{m}}^{\tilde{i}\tilde{k}} = -\frac{\epsilon^{\tilde{i}\tilde{k}\tilde{\ell}}\tilde{h}_{\tilde{\ell}}}{\sqrt{-\tilde{\mathsf{g}}}}. \tag{2}$$

In Eq. 19, the order of $\tilde{\alpha}$ and $\tilde{\beta}$ are reversed, and the argument should be $m^{\tilde{\beta}\tilde{\alpha}}$. In Eq. 21, there is a spurious minus sign in $\sqrt{\tilde{g}}$. The corrected expression is

$$\frac{1}{\sqrt{\tilde{g}}}\left[\epsilon^{\tilde{i}\tilde{k}\tilde{\ell}}\frac{\partial \tilde{h}_{\tilde{\ell}}}{\partial \tilde{x}^{\tilde{k}}} - \frac{\partial}{\partial \tilde{t}}\left(\sqrt{\tilde{g}}\,\tilde{d}^{\tilde{i}}\right)\right] = \tilde{j}^{\tilde{i}}. \tag{3}$$

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In Eq. 22, tildes are missing on some of the α and β indices. The order should also be reversed to agree with the revised Eq. 19. A tilde on α is missing on $\tilde{\mathsf{j}}^{\tilde{\alpha}}$ in the line after Eq. 22. A tilde is missing, and a spurious factor of c appears in Eq. 30. The corrected version is

$$\tilde{v}^{\tilde{i}} \doteq c\frac{\mathrm{d}\tilde{\mathsf{x}}^{\tilde{i}}}{\mathrm{d}\tilde{\mathsf{x}}^{0}} = -\frac{\partial \tilde{x}^{\tilde{i}}}{\partial x^{i}} v^{i}. \tag{4}$$

A minus sign is missing in the $\tilde{\mathsf{g}}_{\tilde{i}\tilde{k}}$ term in Eq. 34. The equation should be

$$\tilde{\mathsf{g}}_{\tilde{i}\tilde{k}} = -\delta_{ik}\frac{\partial x^{i}}{\partial \tilde{x}^{\tilde{i}}}\frac{\partial x^{k}}{\partial \tilde{x}^{\tilde{k}}}. \tag{5}$$

Unlike the previous errors, this one carries into the remainder of the paper, causing errors in later expressions. The right Cauchy–Green tensor C_{IK} expression, Eq. 35, should then be

$$\tilde{\mathsf{g}}_{\tilde{i}\tilde{k}} = -\delta_{ik}\frac{\partial x^{i}}{\partial \tilde{x}^{\tilde{i}}}\frac{\partial x^{k}}{\partial \tilde{x}^{\tilde{k}}} = -\delta^{I}_{\tilde{i}}\delta^{K}_{\tilde{k}}\delta_{ik}\frac{\partial x^{i}}{\partial X^{I}}\frac{\partial x^{k}}{\partial X^{K}} = -\delta^{I}_{\tilde{i}}\delta^{K}_{\tilde{k}}C_{IK}, \tag{6}$$

and Eq. 36 is therefore

$$\tilde{\mathsf{g}}^{00} = 1, \qquad \tilde{\mathsf{g}}^{\tilde{i}0} = \tilde{\mathsf{g}}^{0\tilde{i}} = \frac{\tilde{v}^{\tilde{i}}}{c}, \qquad \tilde{\mathsf{g}}^{\tilde{i}\tilde{k}} = -\delta^{ik}\frac{\partial \tilde{x}^{\tilde{i}}}{\partial x^{i}}\frac{\partial \tilde{x}^{\tilde{k}}}{\partial x^{k}}. \tag{7}$$

In Eq. 37, the derivatives are with respect to the four-vector $\tilde{\mathsf{x}}^{\tilde{\alpha}}$, not the purely spatial components $\tilde{x}^{i}$ and should therefore be

$$\tilde{\mathsf{n}}_{\tilde{\alpha}\tilde{\beta}} = \frac{\partial x^{\alpha}}{\partial \tilde{\mathsf{x}}^{\tilde{\alpha}}}\frac{\partial x^{\beta}}{\partial \tilde{\mathsf{x}}^{\tilde{\beta}}}\mathsf{n}_{\alpha\beta}. \tag{8}$$

Equation 52 is obviously incorrect as the units do not match. It should read

$$\tilde{\mathsf{x}}^{0} = \tilde{a}\mathsf{x}^{0} + \tilde{s}_{\tilde{i}}\tilde{\mathsf{x}}^{\tilde{i}}. \tag{9}$$

A sign error appears in Eq. 54, and is carried to other equations. The corrected expressions are

$$\frac{\partial \mathsf{x}^{0}}{\partial \tilde{\mathsf{x}}^{0}} = \frac{1}{\tilde{a}}, \qquad \frac{\partial \mathsf{x}^{0}}{\partial \tilde{\mathsf{x}}^{\tilde{i}}} = -\frac{\tilde{s}_{\tilde{i}}}{\tilde{a}}, \qquad \frac{\partial \mathsf{x}^{i}}{\partial \tilde{\mathsf{x}}^{0}} = \frac{1}{\tilde{a}}\frac{v^{i}}{c}, \qquad \frac{\partial \mathsf{x}^{i}}{\partial \tilde{\mathsf{x}}^{\tilde{i}}} = \delta^{I}_{\tilde{i}}\frac{\partial x^{i}}{\partial X^{I}} - \frac{\tilde{s}_{\tilde{i}}}{\tilde{a}}\frac{v^{i}}{c}. \tag{10}$$

Incorporating these changes and correcting for an erroneous square root, Eq. 55 should read

$$\tilde{s}_{\tilde{i}} = -\frac{\tilde{a}\delta_{ij}\frac{\partial x^{j}}{\partial \tilde{x}^{\tilde{i}}}\frac{v^{i}}{c}}{1 - \delta_{k\ell}\frac{v^{k}v^{\ell}}{c^{2}}}. \tag{11}$$

The sign error propagates further to Eq. 58 which should now be

$$\frac{\partial \tilde{\mathsf{x}}^{0}}{\partial \mathsf{x}^{0}} = 1, \qquad \frac{\partial \tilde{\mathsf{x}}^{0}}{\partial \mathsf{x}^{i}} = -\delta_{ij}\frac{v^{j}}{c}, \qquad \frac{\partial \tilde{\mathsf{x}}^{\tilde{i}}}{\partial \mathsf{x}^{0}} = \frac{\tilde{v}^{\tilde{i}}}{c}, \qquad \frac{\partial \tilde{\mathsf{x}}^{\tilde{i}}}{\partial \mathsf{x}^{i}} = \frac{\partial \tilde{x}^{\tilde{i}}}{\partial x^{i}}. \tag{12}$$

In Eq. 60, the $\tilde{\mathsf{g}}_{\tilde{i}\tilde{j}}$ term should be

$$\tilde{\mathsf{g}}_{\tilde{i}\tilde{j}} = -\tilde{g}_{\tilde{i}\tilde{j}} = -\delta_{ij}\frac{\partial x^{i}}{\partial \tilde{x}^{\tilde{i}}}\frac{\partial x^{j}}{\partial \tilde{x}^{\tilde{j}}}, \tag{13}$$

as the expression presented in the paper is obviously not a proper indicial expression. In Eq. 61, the indices on $\tilde{g}^{\tilde{i}\tilde{j}}$ should be raised. The sign corrections lead to changes in the definitions given by Eqs. 62–64. The corrected definitions are

$$\tilde{e}^{\tilde{i}} \doteq \tilde{g}^{\tilde{i}\tilde{j}}\tilde{e}_{\tilde{j}}, \qquad \tilde{d}_{\tilde{i}} \doteq \tilde{g}_{\tilde{i}\tilde{j}}\tilde{d}^{\tilde{j}}, \tag{14}$$

$$\tilde{h}^{\tilde{i}} \doteq \tilde{g}^{\tilde{i}\tilde{j}}\tilde{h}_{\tilde{j}}, \qquad \tilde{b}_{\tilde{i}} \doteq \tilde{g}_{\tilde{i}\tilde{j}}\tilde{b}^{\tilde{j}}, \tag{15}$$

$$\tilde{p}_{\tilde{i}} \doteq \tilde{g}_{\tilde{i}\tilde{j}}\tilde{p}^{\tilde{j}}, \qquad \tilde{m}^{\tilde{i}} \doteq \tilde{g}^{\tilde{i}\tilde{j}}\tilde{m}_{\tilde{j}}. \tag{16}$$

Equations 66–67 now become

$$\tilde{m}_{\tilde{i}0} = -c\tilde{d}_{\tilde{i}}, \qquad \tilde{m}_{\tilde{i}\tilde{j}} = -\sqrt{\tilde{g}}\,\epsilon_{\tilde{i}\tilde{j}\tilde{k}}\tilde{h}^{\tilde{k}}, \tag{17}$$

$$\tilde{p}_{\tilde{i}0} = c\tilde{p}_{\tilde{i}}, \qquad \tilde{p}_{\tilde{i}\tilde{j}} = -\sqrt{\tilde{g}}\,\epsilon_{\tilde{i}\tilde{j}\tilde{k}}\tilde{m}^{\tilde{k}}. \tag{18}$$

The sign error then propagates to Eqs. 68–73, which become

$$\tilde{e}_{\tilde{i}} = \frac{\partial x^i}{\partial \tilde{x}^{\tilde{i}}}\left(e_i + \epsilon_{ijk}v^j b^k\right), \qquad \tilde{b}^{\tilde{i}} = \frac{\partial \tilde{x}^{\tilde{i}}}{\partial x^i}\left(b^i - \epsilon^{ijk}\frac{v_j e_k}{c^2}\right), \tag{19}$$

$$\tilde{e}^{\tilde{i}} = \frac{\partial \tilde{x}^{\tilde{i}}}{\partial x^i}\left(e^i + \epsilon^{ijk}v_j b_k\right), \qquad \tilde{b}_{\tilde{i}} = \frac{\partial x^i}{\partial \tilde{x}^{\tilde{i}}}\left(b_i - \epsilon_{ijk}\frac{v^j e^k}{c^2}\right), \tag{20}$$

$$\tilde{d}^{\tilde{i}} = \frac{\partial \tilde{x}^{\tilde{i}}}{\partial x^i}\left(d^i + \epsilon^{ijk}\frac{v_j h_k}{c^2}\right), \qquad \tilde{h}_{\tilde{i}} = \frac{\partial x^i}{\partial \tilde{x}^{\tilde{i}}}\left(h_i - \epsilon_{ijk}v^j d^k\right), \tag{21}$$

$$\tilde{p}^{\tilde{i}} = \frac{\partial \tilde{x}^{\tilde{i}}}{\partial x^i}\left(p^i - \epsilon^{ijk}\frac{v_j m_k}{c^2}\right), \qquad \tilde{m}_{\tilde{i}} = \frac{\partial x^i}{\partial \tilde{x}^{\tilde{i}}}\left(m_i + \epsilon_{ijk}v^j p^k\right), \tag{22}$$

$$p^i = \frac{\partial x^i}{\partial \tilde{x}^{\tilde{i}}}\left(\tilde{p}^{\tilde{i}} - \frac{\epsilon^{\tilde{i}\tilde{j}\tilde{k}}}{\sqrt{\tilde{g}}}\frac{\tilde{v}_{\tilde{j}}\tilde{m}_{\tilde{k}}}{c^2}\right), \qquad m_i = \frac{\partial \tilde{x}^{\tilde{i}}}{\partial x^i}\left(\tilde{m}_{\tilde{i}} + \sqrt{\tilde{g}}\,\epsilon_{\tilde{i}\tilde{j}\tilde{k}}\tilde{v}^{\tilde{j}}\tilde{p}^{\tilde{k}}\right), \tag{23}$$

$$\tilde{\rho} = \rho - \frac{v_i j^i}{c^2}, \qquad \tilde{j}^{\tilde{i}} = \frac{\partial \tilde{x}^{\tilde{i}}}{\partial x^i}\left(j^i - \rho v^i\right). \tag{24}$$

Finally, the corrected versions of Tables 1, 2 and 3 are shown. As can be seen, after these corrections for sign errors, the time orthogonal transformation equations exhibit a completely symmetric relationship. They are also in complete agreement with the well-known frame transformation laws for electromagnetic field quantities in the limit of rigid motion. In contrast, even with the corrections, the transformation rules commonly used in continuum mechanics [1–4] remain asymmetric. As mentioned in the paper, this asymmetry is a source of confusion and possible errors in formulations that rely on them.

Table 1 Transformation of the elements of the electromagnetic tensor in different formulations

	e	**b**
Universal time, $\tilde{n}_{\alpha\beta}$	$\tilde{e}_{\tilde{i}} = \frac{\partial x^i}{\partial \tilde{x}^{\tilde{i}}}\left(e_i + \epsilon_{ijk}v^j b^k\right)$	$\tilde{b}^{\tilde{i}} = \frac{\partial \tilde{x}^{\tilde{i}}}{\partial x^i} b^i$
Universal time, $\tilde{n}^{\alpha\beta}$	$\tilde{e}^{\tilde{i}} = \frac{\partial \tilde{x}^{\tilde{i}}}{\partial x^i} e^i$	$\tilde{b}_{\tilde{i}} = \frac{\partial x^i}{\partial \tilde{x}^{\tilde{i}}}\left(b_i - \epsilon_{ijk}\frac{v^j e^k}{c^2}\right)$
Time orthogonal, $\tilde{n}_{\alpha\beta}$	$\tilde{e}_{\tilde{i}} = \frac{\partial x^i}{\partial \tilde{x}^{\tilde{i}}}\left(e_i + \epsilon_{ijk}v^j b^k\right)$	$\tilde{b}^{\tilde{i}} = \frac{\partial \tilde{x}^{\tilde{i}}}{\partial x^i}\left(b^i - \epsilon^{ijk}\frac{v_j e_k}{c^2}\right)$
Lax and Nelson [3]	$E_I = \frac{\partial x_i}{\partial X_I}\left(e_i + \epsilon_{ijk}v_j b_k\right)$	$B_I = \sqrt{C}\frac{\partial X_I}{\partial x_i} b_i$
Yang and Batra [4]	$E_I = \sqrt{C}\frac{\partial X_I}{\partial x_i} e_i$	(Electrostatic)
Dorfmann and Ogden [2]	$E_I = \frac{\partial x_i}{\partial X_I} e_i$	(Electrostatic)
Clayton [1]	$E_I = \frac{\partial x_i}{\partial X_I} e_i$	(Electrostatic)

Table 2 Transformation of the elements of the displacement tensor in different formulations

	d	**h**
Universal time, $\tilde{m}^{\alpha\beta}$	$\tilde{d}^{\tilde{i}} = \frac{\partial \tilde{x}^{\tilde{i}}}{\partial x^i} d^i$	$\tilde{h}_{\tilde{i}} = \frac{\partial x^i}{\partial \tilde{x}^{\tilde{i}}}\left(h_i - \epsilon_{ijk}v^j d^k\right)$
Universal time, $\tilde{m}_{\alpha\beta}$	$\tilde{d}_{\tilde{i}} = \frac{\partial x^i}{\partial \tilde{x}^{\tilde{i}}}\left(d_i + \epsilon_{ijk}\frac{v^j h^k}{c^2}\right)$	$\tilde{h}^{\tilde{i}} = \frac{\partial \tilde{x}^{\tilde{i}}}{\partial x^i} h^i$
Time orthogonal, $\tilde{m}^{\alpha\beta}$	$\tilde{d}^{\tilde{i}} = \frac{\partial \tilde{x}^{\tilde{i}}}{\partial x^i}\left(d^i + \epsilon^{ijk}\frac{v_j h_k}{c^2}\right)$	$\tilde{h}_{\tilde{i}} = \frac{\partial x^i}{\partial \tilde{x}^{\tilde{i}}}\left(h_i - \epsilon_{ijk}v^j d^k\right)$
Lax and Nelson [3]	$D_I = \sqrt{C}\frac{\partial X_I}{\partial x_i} d_i$	$H_I = \frac{\partial x_i}{\partial X_I}\left(h_i - \epsilon_{ijk}v_j d_k\right)$
Yang and Batra [4]	$D_I = \sqrt{C}\frac{\partial X_I}{\partial x_i} d_i$	(Electrostatic)
Dorfmann and Ogden [2]	$D_I = \sqrt{C}\frac{\partial X_I}{\partial x_i} d_i$	(Electrostatic)
Clayton [1]	$D_I = \sqrt{C}\frac{\partial X_I}{\partial x_i} d_i$	(Electrostatic)

Table 3 Transformation of the elements of the polarization tensor in different formulations

	p	**m**
Universal time, $\tilde{p}^{\alpha\beta}$	$\tilde{p}^{\tilde{i}} = \frac{\partial \tilde{x}^{\tilde{i}}}{\partial x^i} p^i$	$\tilde{m}_{\tilde{i}} = \frac{\partial x^i}{\partial \tilde{x}^{\tilde{i}}} \left(m_i + \epsilon_{ijk} v^j p^k\right)$
Universal time, $\tilde{p}_{\alpha\beta}$	$\tilde{p}_{\tilde{i}} = \frac{\partial x^i}{\partial \tilde{x}^{\tilde{i}}} \left(p_i - \epsilon_{ijk} \frac{v^j m^k}{c^2}\right)$	$\tilde{m}^{\tilde{i}} = \frac{\partial \tilde{x}^{\tilde{i}}}{\partial x^i} m^i$
Time orthogonal, $\tilde{p}^{\alpha\beta}$	$\tilde{p}^{\tilde{i}} = \frac{\partial \tilde{x}^{\tilde{i}}}{\partial x^i} \left(p^i - \epsilon^{ijk} \frac{v_j m_k}{c^2}\right)$	$\tilde{m}_{\tilde{i}} = \frac{\partial x^i}{\partial \tilde{x}^{\tilde{i}}} \left(m_i + \epsilon_{ijk} v^j p^k\right)$
Lax and Nelson [3]	$P_I = \sqrt{C} \frac{\partial X_I}{\partial x_i} p_i$	$M_I = \frac{\partial x^i}{\partial X^I} m_i$
Yang and Batra [4]	$P_I = \sqrt{C} \frac{\partial X_I}{\partial x_i} p_i$	(Electrostatic)
Dorfmann and Ogden [2]	$P_I = \sqrt{C} \frac{\partial X_I}{\partial x_i} p_i$	(Electrostatic)
Clayton [1]	$P_I = \frac{\partial x^i}{\partial X^I} p_i$	(Electrostatic)

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J LASALVIA
RDRL WMM F
M TSCHOPP
RDRL WML G
J ANDZELM
T CHANTAWANSRI
C RINDERSPACHER
T SIRK
Y SLIOZBERG
RDRL WMP
S SCHOENFELD
RDRL WMP B
S SATAPATHY
A SOKOLOW
T WEERASOORIYA
RDRL WMP C
R BECKER
S BILYK
T BJERKE
D CASEM
J CLAYTON
D DANDEKAR
M GREENFIELD
B LEAVY
J LLOYD
S SEGLETES
A TONGE
C WILLIAMS
RDRL WMP D
R DONEY
C RANDOW