

NPS-MA-20-001



**NAVAL
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MONTEREY, CALIFORNIA

**A NEW TRIGONOMETRICALLY-FITTED METHOD FOR
SECOND ORDER INITIAL VALUE PROBLEMS**

by

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December 2020

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Prepared for: Naval Postgraduate School

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REPORT DOCUMENTATION PAGE				<i>Form Approved</i> OMB No. 0704-0188	
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1. REPORT DATE (DD-MM-YYYY) 12-09-2020		2. REPORT TYPE Technical Report		3. DATES COVERED (From-To) 10-01-2019 - 09-30-2020	
4. TITLE AND SUBTITLE A New Trigonometrically-Fitted Method for Second Order Initial Value Problems				5a. CONTRACT NUMBER	
				5b. GRANT NUMBER	
				5c. PROGRAM ELEMENT NUMBER	
6. AUTHOR(S) Beny Neta and Changbum Chun				5d. PROJECT NUMBER	
				5e. TASK NUMBER	
				5f. WORK UNIT NUMBER	
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) AND ADDRESS(ES) Naval Postgraduate School				8. PERFORMING ORGANIZATION REPORT NUMBER NPS-MA-20-001	
9. SPONSORING / MONITORING AGENCY NAME(S) AND ADDRESS(ES)				10. SPONSOR/MONITOR'S ACRONYM(S)	
				11. SPONSOR/MONITOR'S REPORT NUMBER(S)	
12. DISTRIBUTION / AVAILABILITY STATEMENT Approved for public release; distribution is unlimited					
13. SUPPLEMENTARY NOTES					
14. ABSTRACT Numerical schemes approximating the solution of ordinary initial value problems interpolate polynomial up to a certain degree. Brock and Murray have suggested in 1952 to interpolate exponential functions when the solution is of exponential type. In 1961 Gautschi has suggested the use of complex exponential functions of the frequency and multiples of it. Later others developed methods based on combination of both. The basis of all these methods are the well-known linear multistep (including Obrechhoff schemes) and Runge-Kutta (RKN) schemes. We develop methods for the solution of first and second order systems having periodic solution with approximately known period. Our methods based on Obrechhoff schemes. We compare our new methods to existing ones.					
15. SUBJECT TERMS Initial value problems, Obrechhoff-tpe methods, periodic and almost-periodic solutions.					
16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT Unlimited	18. NUMBER OF PAGES 23	19a. NAME OF RESPONSIBLE PERSON Beny Neta
a. REPORT Unclassified	b. ABSTRACT Unclassified	c. THIS PAGE Unclassified			

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A new trigonometrically-fitted method for second order initial value problems

Changbum Chun and Beny Neta

Abstract. Numerical schemes approximating the solution of ordinary initial value problems interpolate polynomial up to a certain degree. Brock and Murray have suggested in 1952 to interpolate exponential functions when the solution is of exponential type. In 1961 Gautschi has suggested the use of complex exponential functions of the frequency and multiples of it. Later others developed methods based on combination of both. The basis of all these methods are the well known linear multistep (including Obrechhoff schemes) and Runge-Kutta (RKN) schemes. We develop methods for the solution of first and second order systems having periodic solution with approximately known period. Our methods based on Obrechhoff schemes. We compare our new methods to existing ones.

Mathematics Subject Classification (2010). 65L05, 65L06.

Keywords. Obrechhoff schemes; Initial value problems; Trigonometrically-fitted methods.

1. Introduction

In this paper we develop new schemes for approximating the solution of ordinary initial value problems

$$\begin{aligned}y''(x) &= f(x, y(x)), \quad x \in [a, b] \\y(a) &= y_0, \\y'(a) &= y'_0.\end{aligned}\tag{1.1}$$

If the right hand side depends on $y'(x)$ then the equation is replaced by a system

$$\mathbf{y}' = \mathbf{f}(x, \mathbf{y}), \quad \mathbf{y}(a) = \mathbf{y}_0,\tag{1.2}$$

where $\mathbf{y} = [y, y']^T$, $\mathbf{f} = [y', f(x, y, y')]^T$, and \mathbf{y}_0 is a vector of the initial values.

There are basically two classes of methods, i.e. linear multistep methods [12] which include Obrechhoff methods [19] and Runge-Kutta methods [1]. In this paper, we develop trigonometrically-fitted Obrechhoff-type schemes for (1.1) and (1.2) whose solution is periodic or almost periodic with approximately known period, see review article [4].

2. Obrechhoff-type methods

Simos [20] has developed a trigonometrically-fitted Obrechhoff P-stable tenth order method for (1.1).

$$y_{n+1} + y_{n-1} - 2y_n = \sum_{j=1}^{\ell} h^{2j} \left[\beta_{j0} y_{n+1}^{(2j)} + 2\beta_{j1} y_n^{(2j)} + \beta_{j0} y_{n-1}^{(2j)} \right], \quad (2.1)$$

where $\ell = 3$ and

$$\begin{aligned} \beta_{10} &= \frac{89}{1878} - \frac{15120}{313} \beta_{31}, \\ \beta_{11} &= \frac{425}{939} + \frac{15120}{313} \beta_{31}, \\ \beta_{20} &= -\frac{1907}{1577520} + \frac{660}{313} \beta_{31}, \\ \beta_{21} &= \frac{30257}{1577520} + \frac{690}{313} \beta_{31}, \\ \beta_{30} &= \frac{59}{3155040} - \frac{13}{313} \beta_{31}. \end{aligned} \quad (2.2)$$

The free coefficient β_{31} is obtained from the P-stability condition

$$\begin{aligned} \beta_{31} &= (190816819200[1 - \cos(v)] - 95408409600v^2 + 7950700800v^4 \\ &\quad - 265023360v^6 + 4732560v^8 - 52584v^{10} + 1727v^{12}) / (3568320v^{12}), \end{aligned} \quad (2.3)$$

where $v = \omega h$. The approximation of the first derivative is given by (36) in Chun and Neta [4]. Wang et al. [24] have modified slightly β_{31} to read

$$\beta_{31} = \frac{3155040 - 1428000v^2 + 60514v^4 - \alpha_1 \cos(v)}{5040v^2(-15120 + 6900v^2 - 313v^4 + \alpha_2 \cos(v))}, \quad (2.4)$$

where $\alpha_1 = 3155040 + 149520v^2 + 3814v^4 + 59v^6$ and $\alpha_2 = 15120 + 660v^2 + 13v^4$.

Wang et al. [24] have developed a new P-stable Obrechhoff-type twelfth order method

$$\begin{aligned} y_{n+1} - 2y_n + y_{n-1} &= h^2 (A_1 (y''_{n+1} + y''_{n-1}) + A_2 y''_n) \\ &+ h^4 (B_1 (y^{(4)}_{n+1} + y^{(4)}_{n-1}) + B_2 y^{(4)}_n) \\ &+ h^6 (C_1 (y^{(6)}_{n+1} + y^{(6)}_{n-1}) + C_2 y^{(6)}_n), \end{aligned} \quad (2.5)$$

where

$$A_1 = \frac{229}{7788}, \quad B_1 = -\frac{1}{2360}, \quad B_2 = \frac{711}{12980},$$

$$C_1 = \frac{127}{39251520}, \quad C_2 = \frac{2923}{3925152},$$

and A_2 is chosen to satisfy P-stability,

$$A_2 = 2v^{-2} + v^2 B_2 - v^4 C_2 + 2 \cos(v) (-v^{-2} - A_1 + v^2 B_1 - v^4 C_1).$$

The first order derivative approximation is given by (3.9) in [24].

Remark 2.1. The typographical errors in the coefficients given in [24] are corrected here.

Vanden Berghe and Van Daele [23] have suggested fitting the functions

$$\{x^0, x^1, \dots, x^K, e^{\pm \lambda x}, x e^{\pm \lambda x}, \dots, x^P e^{\pm \lambda x}\}.$$

When $P = -1$ we get the well known Obrechhoff method and for $K = -1$, we get no monomials. We list here two of their eighth order methods in the form (2.6) with $K = 5$, $P = 1$ and $K = 7$, $P = 0$.

The methods are symmetric and given by

$$y_{n+2} - 2y_{n+1} + y_n = \sum_{i=1}^{\ell} h^{2i} (b_{i0} y_{n+2}^{(2i)} + b_{i1} y_{n+1}^{(2i)} + b_{i0} y_n^{(2i)}), \quad \ell \geq 2. \quad (2.6)$$

The first method, denoted K5p1, is given by

$$\begin{aligned} \beta_{10} &= \frac{1}{12} - 2\beta_{20} - 2\beta_{21}, \\ \beta_{11} &= \frac{5}{12} + 2\beta_{20} + 2\beta_{21}, \\ \beta_{20} &= \frac{v^5 \sin(v) + 2v^4 \cos(v) + 10v^4 + 48Q(\cos(v) - 1)}{C}, \\ \beta_{21} &= \frac{5v^5 \sin(v) - 2v^4 \cos(v)(\cos(v) + 5) - 48R(\cos(v) - 1)}{C}, \end{aligned} \quad (2.7)$$

where $Q = v^2 - 1 + \cos(v)$, $R = (v^2 + 1) \cos(v) - 1$ and $C = 12v^7 \sin(v) - 48v^4(1 - 2 \cos(v) + \cos(v)^2)$.

The second method, denoted K7p0, is

$$\begin{aligned}
 \beta_{10} &= \frac{1}{30} - 12\beta_{20}, \\
 \beta_{11} &= \frac{7}{15} + 12\beta_{20}, \\
 \beta_{20} &= \frac{4v^2 \cos(v) + S}{D}, \\
 \beta_{21} &= \frac{1}{40} + 5\beta_{20},
 \end{aligned} \tag{2.8}$$

where $S = -3v^4 + 56v^2 + 120(\cos(v) - 1)$ and $D = 120v^2(12(\cos(v) - 1) + (\cos(v) + 5)v^2)$.

Cash [2] considered hybrid two-step methods, i.e. using off-step points. Neta [13], [14] has constructed high order hybrid Störmer-Cowell implicit and explicit methods. See also Ibrahim and Ikhile [8], Felix and Okuonghae [5] and Kovalnogov et al. [11] for other hybrid methods.

Fukushima [6] has introduced the super-implicit methods, i.e. using also future value. Clearly, in this case one has to design schemes for initial and terminal points in each block of points and then solve for all the unknown values in the block. The initial value for the next block is the terminal value at the previous one. Neta and Fukushima [16] and Neta [17], [18] have developed various P-stable symmetric super-implicit methods.

Ibrahim and Ikhile [9] have combined the ideas of hybrid and super-implicit methods to construct high order schemes with minimal phase-lag. Here we list the twelfth order method

$$\begin{aligned}
 y_{n+1} + y_{n-1} - 2y_n &= h^2 \left(\frac{7201}{1596672} f_n + \frac{2662343}{79833600} (f_{n+1} + f_{n-1}) \right. \\
 &\quad - \frac{9307}{19958400} (f_{n+2} + f_{n-2}) + \frac{547}{22809600} (f_{n+3} + f_{n-3}) \\
 &\quad \left. - \frac{10499}{11416204800} (f_{n+4} + f_{n-4}) + \frac{18718533}{40268800} (f_{n+\frac{1}{3}} + f_{n-\frac{1}{3}}) \right), \tag{2.9}
 \end{aligned}$$

where the off-step values are computed by

$$\begin{aligned}
 y_{n+\frac{1}{3}} &= \frac{21790}{59049} y_{n+1} + \frac{35152}{59049} y_n + \frac{2107}{59049} y_{n-1} \\
 &+ h^2 \left(-\frac{151099}{1771470} f_n - \frac{69352}{885735} f_{n+1} + \frac{19843}{885735} f_{n+2} \right. \\
 &\quad \left. - \frac{1924}{295245} f_{n+3} + \frac{1621}{1771470} f_{n+4} \right), \tag{2.10}
 \end{aligned}$$

and

$$\begin{aligned}
 y_{n-\frac{1}{3}} &= \frac{2107}{59049}y_{n+1} + \frac{35152}{59049}y_n + \frac{21790}{59049}y_{n-1} \\
 + h^2 &\left(-\frac{151099}{1771470}f_n - \frac{69352}{885735}f_{n-1} + \frac{19843}{885735}f_{n-2} \right. \\
 &\quad \left. - \frac{1924}{295245}f_{n-3} + \frac{1621}{1771470}f_{n-4} \right). \tag{2.11}
 \end{aligned}$$

Khalsaraei and Shokri [10] have developed an explicit six-step P-stable method of order 10. It is given by (note the typographical errors in [10]).

$$\begin{aligned}
 \sum_{i=1}^k a_i(y_{n+i} + y_{n-i}) + a_0 y_n &= h^2 \left[\sum_{i=1}^{k-1} b_i(y''_{n+i} + y''_{n-i}) + b_0 y''_n \right] \\
 &+ h^4 \left[\sum_{i=1}^{k-1} d_i(y_{n+i}^{(4)} + y_{n-i}^{(4)}) + d_0 y_n^{(4)} \right] \\
 &+ h^6 \left[\sum_{i=1}^{k-1} e_i(y_{n+i}^{(6)} + y_{n-i}^{(6)}) + e_0 y_n^{(6)} \right] \tag{2.12}
 \end{aligned}$$

where $k = 3$, $a_3 = 1$, and

$$\begin{aligned}
 d_2 &= -\frac{1}{250}, \quad e_0 = -\frac{1}{125}, \quad e_1 = \frac{1}{125}, \quad e_2 = -\frac{1}{250}, \\
 a_0 &= -2 + \frac{7483599}{128800}v^2 - \frac{1931802681}{65172800}v^4 + \frac{3515150550753}{545626681600}v^6 - \dots, \\
 b_0 &= -\frac{6324399}{128800} - \frac{223989351}{325864000}v^2 + \frac{19329982344467}{2728133408000}v^4 - \frac{1664204965051393}{828261302668800}v^6 + \dots, \\
 d_0 &= -\frac{7592313}{322000} + \frac{170261477}{14812000}v^2 - \frac{80242849913}{32737600896}v^4 + \frac{58346712470407}{188241205152000}v^6 - \dots, \\
 a_1 &= -\frac{100467}{4025}v^2 + \frac{45020631}{3258640}v^4 - \frac{161999074637}{51152501400}v^6 + \dots, \\
 b_1 &= \frac{100467}{4025} - \frac{786506407}{40733000}v^2 + \frac{61809044239}{11625568500}v^4 - \frac{8889665876381}{12941582854200}v^6 + \dots, \\
 d_1 &= -\frac{1768763}{322000} + \frac{87888151}{40733000}v^2 - \frac{31066652473}{102305002800}v^4 + \frac{456219337129}{21569304757000}v^6 - \dots, \\
 a_2 &= -\frac{1053711}{257600}v^2 + \frac{18711063}{18620800}v^4 - \frac{177510875491}{3273760089600}v^6 + \dots, \\
 b_2 &= \frac{1053711}{257600} - \frac{93927731}{93104000}v^2 + \frac{953029579247}{16368800448000}v^4 - \frac{9966542939}{25484963159040}v^6 + \dots.
 \end{aligned}$$

3. New schemes

In this section, we develop new Obrechhoff-type explicit and implicit schemes for first order and second order systems of ordinary initial value problems whose solution is periodic or almost periodic. The methods fit a combination of monomials and complex exponentials, i.e. the set

$$\{x^i\}_{i=0}^K \cup \{x^n \sin(r\omega x), x^n \cos(r\omega x)\}_{r=1,2,\dots,q}^{n=0,1}.$$

For example, the method (2.7) using monomials to power 5 with $q = n = 1$. The method (2.8) uses monomials to power 7 and $q = 1$ and $n = 0$. Both of these methods are implicit with $\ell = 2$ in (2.6). We only developed schemes for which $n = 0$. Therefore, from now on we will not mention the index n .

For first order systems, we developed Obrechhoff-type two-step methods

$$y_{n+2} - (1+a)y_{n+1} + ay_n = \sum_{i=1}^{\ell} h^i (b_{i0}y_{n+2}^{(i)} + b_{i1}y_{n+1}^{(i)} + b_{i2}y_n^{(i)}), \quad \ell \geq 2. \quad (3.1)$$

The method is not stable unless $|a| < 1$.

For $\ell = 2$, the explicit method of trigonometric order 1, ($K = q = 1$) is given by

$$y_{n+2} - (1+a)y_{n+1} + ay_n = \sum_{i=1}^{\ell} h^i (b_{i1}y_{n+1}^{(i)} + b_{i2}y_n^{(i)}), \quad (3.2)$$

where a and b_{11} are free parameters and

$$\begin{aligned} b_{12} &= -a - b_{11} + 1, \\ b_{21} &= \frac{a(\sin(v) - v) + b_{11}v(\cos(v) - 1) + v + \sin(v) - 2\sin(v)\cos(v)}{v^2 \sin(v)}, \\ b_{22} &= \frac{(a+1)(v\cos(v) - \sin(v)) + b_{11}v(\cos(v) - 1)}{v^2 \sin(v)}. \end{aligned} \quad (3.3)$$

There is no way of choosing the free parameters to increase the trigonometric order.

The implicit method (3.1) for $\ell = 2$ led to $a = 1$ which is **not** zero stable. So we have taken

$$y_{n+1} - y_n = \sum_{i=1}^{\ell} h^i (b_{i1}y_{n+1}^{(i)} + b_{i2}y_n^{(i)}). \quad (3.4)$$

For $\ell = 2$ we have an implicit method of trigonometric order 1 ($K = 2$ and $q = 1$)

$$\begin{aligned}
 b_{11} &= \frac{1}{2}, \\
 b_{12} &= \frac{1}{2}, \\
 b_{21} &= \frac{1}{2} \frac{v \cos(v) - 2 \sin(v) + v}{v^2 \sin(v)}, \\
 b_{22} &= -\frac{1}{2} \frac{v \cos(v) - 2 \sin(v) + v}{v^2 \sin(v)}.
 \end{aligned} \tag{3.5}$$

For $\ell = 3$ we have the implicit method of trigonometric order 2 ($K = q = 2$)

$$\begin{aligned}
 b_{11} &= \frac{1}{2}, \\
 b_{12} &= \frac{1}{2}, \\
 b_{21} &= -\frac{1}{4} \frac{-3v \cos(v)^2 + 7 \sin(v) \cos(v) - 3v \cos(v) - \sin(v)}{v^2 \sin(v)(\cos(v) - 1)}, \\
 b_{22} &= \frac{1}{4} \frac{-3v \cos(v)^2 + 7 \sin(v) \cos(v) - 3v \cos(v) - \sin(v)}{v^2 \sin(v)(\cos(v) - 1)}, \\
 b_{31} &= \frac{1}{4} \frac{-v \cos(v) + 3 \sin(v) - 2v}{v^3(\cos(v) - 1)}, \\
 b_{32} &= \frac{1}{4} \frac{-v \cos(v) + 3 \sin(v) - 2v}{v^3(\cos(v) - 1)}.
 \end{aligned} \tag{3.6}$$

For second order ordinary initial value problems, we have developed a new Obrechhoff-type scheme with trigonometric order 2. The method is fitting complex exponentials and monomials with $K = 5$ and $q = 2$.

The method is

$$y_{n+1} - 2y_n + y_{n-1} = \sum_{j=1}^2 h^{2j} \left[\beta_{j0} y_{n+1}^{(2j)} + 2\beta_{j1} y_n^{(2j)} + \beta_{j0} y_{n-1}^{(2j)} \right]. \tag{3.7}$$

The set of equations to solve is:

$$\begin{aligned}
& -4\beta_{10} - 4\beta_{11} + 2 = 0, \\
& (-\cos(wh)\beta_{20} - \beta_{21})h^4w^4 + (\cos(wh)\beta_{10} + \beta_{11})h^2w^2 + \cos(wh) - 1 = 0, \\
& -(\beta_{20}\cos(wh) + \beta_{21})h^4w^4 + (\beta_{10}\cos(wh) + \beta_{11})h^2w^2 + \cos(wh) - 1 = 0, \\
& -24\beta_{10} - 48\beta_{20} - 48\beta_{21} + 2 = 0, \\
& (-16\cos(wh)^2\beta_{20} + 8\beta_{20} - 8\beta_{21})h^4w^4 + (4\cos(wh)^2\beta_{10} - 2\beta_{10} + 2\beta_{11})h^2w^2 \\
& \quad + \cos(wh)^2 - 1 = 0, \\
& (4\cos(wh)^2 - 4) + h^2w^2(16\beta_{10}\cos(wh)^2 - 8\beta_{10} + 8\beta_{11}) \\
& \quad + h^4w^4(-64\beta_{20}\cos(wh)^2 + 32\beta_{20} - 32\beta_{21}) = 0.
\end{aligned} \tag{3.8}$$

We solved this system for the coefficients $\beta_{10}, \beta_{11}, \beta_{20}, \beta_{21}$ using Maple software [3] to get

$$\begin{aligned}
\beta_{10} &= -\frac{1}{12} \frac{-(2\cos(v) + 1)v^4 + 3(8\cos(v) + 7)v^2 + 45(\cos(v)^2 - 1)}{D_1}, \\
\beta_{11} &= \frac{1}{12} \frac{5(2\cos(v) + 1)v^4 + 3(6\cos(v)^2 + 8\cos(v) + 1)v^2 + 45(\cos(v)^2 - 1)}{D_1}, \\
\beta_{20} &= -\frac{1}{24} \frac{(-\cos(v)^2 + 2\cos(v) + 8)v^4 + 9(\cos(v)^2 + 2\cos(v) - 3)v^2 + 18N_2}{D_2}, \\
\beta_{21} &= \frac{1}{24} \frac{N_3 + 3(15\cos(v)^3 - 12\cos(v)^2 - 9\cos(v) + 6)v^2 + N_2}{D_2},
\end{aligned} \tag{3.9}$$

where

$$\begin{aligned}
D_1 &= (2\cos(v)v^2 + v^2 + 3\cos(v)^2 - 3)v^2, \\
N_2 &= \cos(v)^3 - \cos(v)^2 - \cos(v) + 1, \\
N_3 &= (3\cos(v)^3 + 20\cos(v)^2 - 4\cos(v) - 10)v^4, \\
D_2 &= [(2\cos(v)^2 - \cos(v) - 1)v^2 + 3N_2]v^4.
\end{aligned}$$

Taylor series expansion up to degree 8 is as follows:

$$\begin{aligned}
\beta_{10} &= \frac{11}{252} + \frac{59}{63504}v^2 + \frac{2297}{88016544}v^4 + \frac{3600761}{2883421981440}v^6 \\
&\quad + \frac{25113727}{363311169661440}v^8, \\
\beta_{11} &= \frac{115}{252} - \frac{59}{63504}v^2 - \frac{2297}{88016544}v^4 - \frac{3600761}{2883421981440}v^6 \\
&\quad - \frac{25113727}{363311169661440}v^8, \\
\beta_{20} &= -\frac{13}{15120} - \frac{59}{762048}v^2 - \frac{139199}{26404963200}v^4 - \frac{11233841}{34601063777280}v^6 \\
&\quad - \frac{2072169709}{108993350898432000}v^8, \\
\beta_{21} &= \frac{313}{15120} - \frac{295}{762048}v^2 - \frac{205351}{26404963200}v^4 - \frac{2074145}{6920212755456}v^6 \\
&\quad - \frac{1694889341}{108993350898432000}v^8.
\end{aligned} \tag{3.10}$$

4. Comments on order

Note that a method for first or second order ODEs is of order p if it fits monomials up to degree $p + 1$ or $p + 2$, respectively. Therefore, the definition of order for trigonometrically-fitted methods is now based on the number of equations satisfied. Method, such as (2.7), for second order IVPs is of order 8, because it fits the set

$$\{1, x, \dots, x^5\} \cup \{\sin(\omega x), \cos(\omega x), x \sin(\omega x), x \cos(\omega x)\}.$$

In Table 1, we list methods used here along with their order.

5. Numerical examples

In this section we compare our new schemes for first order, denoted E1 and I2, and second order systems, denoted OM2, to the following known methods for the solution of five examples:

1. K7p0, Vanden Berghe and Van Daele [23] method (2.8)
2. K5p1, Vanden Berghe and Van Daele [23] method (2.7)
3. Wang, Wang et al. [24] method of order 12, given here by (2.5)
4. RKNS, Runge-Kutta trigonometrically-fitted scheme of Simos [21]

Method	first order ODEs	second order ODEs
K7p0 (2.8)		8
K5p1 (2.7)		8
E1 (3.3)	3	
I2 (3.4), (3.6)	6	
OM2 (3.7)		8
Wang (2.5)		12
RKNS [21]	4	
KS20 (2.12)		10

TABLE 1. Methods order.

5. KS20, Explicit P-stable Obrechhoff-type method of Khalsaraei and Shokri [10], given by (2.12)

We did not include Adams implicit [7] and Neta and Ford [15] generalized Milne-Simpson methods because of their poor performance in the examples in our previous paper [4]. The RKNS was corrected for typographical errors in [21], therefore we list the method and the expansion in Taylor series of the coefficients up to sixth order here.

$$\begin{aligned}
 a_0 &= 0, \\
 a_1 &= \frac{1}{4}, \\
 a_2 &= \frac{7}{10}, \\
 a_3 &= 1,
 \end{aligned}
 \tag{5.1}$$

$$\begin{aligned}
 c_0 &= \frac{1}{14} + \frac{89}{13440000}v^4 + \frac{314477}{290304000000}v^6 + \dots, \\
 c_1 &= \frac{27}{25}, \\
 c_2 &= \frac{189}{433}, \\
 c_3 &= \frac{40320000}{41472000000}v^4 + \frac{49729}{41472000000}v^6 + \dots, \\
 \dot{c}_0 &= \frac{1}{14}, \\
 \dot{c}_1 &= \frac{32}{81}, \\
 \dot{c}_2 &= \frac{550}{567} + \frac{1}{86400}v^4 - \frac{17}{53760000}v^6 + \dots, \\
 \dot{c}_3 &= \frac{5}{54} - \frac{1}{86400}v^4 - \frac{43}{161280000}v^6 + \dots,
 \end{aligned}
 \tag{5.2}$$

$$\begin{aligned}
 g_1 &= 1 - \frac{1}{96}v^2 + \frac{1}{30720}v^4 - \frac{1}{20643840}v^6 + \dots, \\
 g_2 &= 1 + \frac{31}{120}v^2 + \frac{2206271}{25200000}v^4 + \frac{28425955949}{952560000000}v^6 + \dots, \\
 g_3 &= 1 + \frac{11}{42}v^2 + \frac{1663}{17010}v^4 + \frac{659}{17010}v^6 + \dots, \\
 \gamma_{10} &= \frac{1}{32} - \frac{1}{6144}v^2 + \frac{1}{2949120}v^4 - \frac{1}{2642411520}v^6 + \dots, \\
 \gamma_{20} &= \frac{1000}{119}, \\
 \gamma_{21} &= \frac{119}{500} + \frac{167431}{2160000}v^2 + \frac{10803521483}{40824000000}v^4 + \frac{557091047587691}{61725888000000000}v^6 + \dots, \\
 \gamma_{30} &= \frac{1}{14}, \\
 \gamma_{31} &= \frac{8}{27}, \\
 \gamma_{32} &= \frac{25}{189} + \frac{5447}{40824}v^2 + \frac{509}{8505}v^4 + \frac{560971381}{22861440000}v^6 + \dots.
 \end{aligned}
 \tag{5.3}$$

Our methods for first and second order systems used for the comparison are:

1. E1, Explicit method (3.3) for first order systems with $\ell = 2$ and trigonometric order 1 with $a = b_{11} = 1/3$
2. I2, Implicit method (3.4), (3.6) for first order system with $\ell = 3$ and trigonometric order 2
3. OM2, Our method (3.7) for second order systems with trigonometric order 2

Example 5.1. In our first example, we use the almost periodic problem

$$y''(x) + \left(100 + \frac{1}{4x^2}\right)y(x) = 0, \quad 1 \leq x \leq 90. \quad (5.4)$$

We choose the initial condition so that

$$y_{exact}(x) = \sqrt{x}J_0(10x),$$

and pick $\omega = 10$.

The methods require the approximation of the first derivative in computing the higher derivatives of $y(x)$ and we used the fourth order method in Simos [21], see also [23] and (36) in [4]. For Wang (2.5) we use (3.9) in [24] to approximate the first derivative.

Remark 5.1. The approximation (3.9) in [24] for the first derivative requires higher order derivative that is used in (2.5) but not in the other methods.

The results at $x = 9.0$ are given in Table 2. It is clear that KS20 and Wang gave superior results as expected by the highest order method used. Our method OM2 (3.7) and Van Daele methods K7p0 (2.8) were second best followed by I2, Vanden Berghe K5p1 (2.7) and RKNS. The worst method is E1. We ran Wang's method with the fourth order predictor as all others and the results are about the same as OM2, i.e. the error was 0.156719(-13).

Note that even though K5p1 is of the same order as OM2, our method performed much better with the fourth order predictor.

Method	L_2 Error
K7p0	0.764761(-13)
K5p1	0.174977(-8)
E1	0.496207(-7)
I2	0.100363(-8)
OM2	0.764769(-13)
Wang	0.350947(-17)
RKNS	0.182420(-8)
KS20	0.582199(-18)

TABLE 2. The L_2 error of various methods at $x = 90$ using $h = 0.002$ where the exact solution is 0.1898627894

Example 5.2. A second example is

$$y^{(4)} + 2y'' + y = \sin(x), \quad (5.5)$$

subject to

$$y(0) = 1, y'(0) = 1, y''(0) = 1, y'''(0) = 1. \tag{5.6}$$

$$y_{exact} = \cos(x) + \frac{19}{8} \sin(x) + x \left(\sin(x) - \frac{11}{8} \cos(x) \right) - \frac{1}{8} x^2 \sin(x). \tag{5.7}$$

We can rewrite this as a system of 2 second order initial value problems

$$\begin{aligned} y_1'' &= y_2, \\ y_2'' &= -2y_2 - y_1 + \sin(x), \end{aligned} \tag{5.8}$$

$$y_1(0) = 1, y_2(0) = 1. \tag{5.9}$$

We can also present it as a first order system

$$\begin{aligned} y_1' &= y_2, \\ y_2' &= y_3, \\ y_3' &= y_4, \\ y_4' &= -2y_3 - y_1 + \sin(x), \end{aligned} \tag{5.10}$$

$$y_1(0) = 1, y_2(0) = 1, y_3(0) = 1, y_4(0) = 1. \tag{5.11}$$

The results at $x = 40\pi$ using two values of h are given in Table 3.

Method	$h = \frac{\pi}{250}$	$h = \frac{\pi}{500}$
K7p0	0.184810(-7)	0.571794(-9)
K5p1	0.565016(-4)	0.132882(-3)
E1	0.268008(-2)	0.335158(-3)
I2	0.448942(-4)	0.280728(-5)
OM2	0.184809(-7)	0.577588(-9)
Wang	0.681180(-8)	0.265514(-10)
RKNS	0.534170(-3)	0.667603(-4)
KS20	0.732229(-15)	0.118648(-14)

TABLE 3. The L_2 error of various methods at $x = 40\pi$ using two values of time steps

Now KS20 is best even though Wang is of higher order. Wang came second and K7p0 and OM2 were closely following. I2, surprisingly performed better than K5p1. The worst method is again E1.

Example 5.3. In our third example we took

$$z''(x) + z(x) = 0.001e^{ix}, \tag{5.12}$$

and initially

$$\begin{aligned} z(0) &= 1.0, \\ z'(0) &= 0.9995i. \end{aligned} \tag{5.13}$$

This example should give a leg up to the method K5p1 using $x \sin(\omega x)$ and $x \cos(\omega x)$, as can be seen in the exact solution below.

The exact solution is

$$\begin{aligned} z(x) &= u(x) + iv(x), \\ u(x) &= \cos(x) + 0.0005x \sin(x), \\ v(x) &= \sin(x) - 0.0005x \cos(x). \end{aligned} \tag{5.14}$$

We solved the equation on the interval $0 \leq x \leq 40\pi$ using $\omega = 1$ and compared the exact value of γ defined as $\gamma = \sqrt{u^2 + v^2} = \sqrt{1 + (0.0005x)^2}$ to the approximate value at the end of the interval using two values of $h = \pi/25, \pi/50$. The results for various methods are given in Table 4.

Method	$h = \frac{\pi}{25}$	$h = \frac{\pi}{50}$
K7p0	0.214602(-7)	0.671245(-9)
K5p1	0.214617(-7)	0.671258(-9)
E1	0.3914441	0.374640
I2	0.721347(-5)	0.458647(-6)
OM2	0.214568(-7)	0.679291(-9)
AI2	0.599316(-4)	0.686719(-5)
Wang	0.569522(-11)	0.222028(-13)
RKNS	0.292407(-7)	0.167719(-8)
KS20	0.637693(-10)	0.611745(-13)

TABLE 4. The L_2 error of various methods at $x = 40\pi$ using two values of time steps

In this case KS20 and Wang performed best followed by the methods K5p1, K7p0 and OM2. The method E1 is worst.

Example 5.4. In our fourth example, we consider the nonlinear undamped Duffing's equation, see e.g. Van Dooren [22] and Vanden Berghe and Van Daele [23]

$$y'' + y + y^3 = B \cos(\Omega t),$$

with $B = .002$ and $\Omega = 1.01$. The exact solution (see Vanden Berghe and Van Daele [23]) is given by

$$y(t) = A_1 \cos(\Omega t) + A_3 \cos(3\Omega t) + A_5 \cos(5\Omega t) + A_7 \cos(7\Omega t) + A_9 \cos(9\Omega t),$$

where

$$\begin{aligned} A_1 &= 0.2001794775361502, \\ A_3 &= 2.46946143255559(-4), \\ A_5 &= 3.0401498519692437(-7), \\ A_7 &= 3.743490701609247(-10), \\ A_9 &= 4.609682949622697(-13). \end{aligned}$$

The initial conditions are

$$y(0) = A_1 + A_3 + A_5 + A_7 + A_9,$$

$$y'(0) = 0.$$

The results are displayed in Table 5. The best methods are KS20 and Wang which are also of the highest order. Clearly, the methods based on Obrechhoff scheme (K7p0, K5p1 and OM2) are better, if we take under consideration the order of the schemes. As was shown in [23], the error can be reduced by using a more accurate approximation for the first derivative.

Method	$h = \frac{\pi}{50}$
K7p0	0.668752(-6)
K5p1	0.668764(-6)
E1	0.287577(-4)
I2	0.122594(-2)
OM2	0.667467(-6)
Wang	0.134252(-11)
RKNS	0.145211(-4)
KS20	0.115172(-11)

TABLE 5. The L_2 error of various methods at $x = 40\pi$.

Example 5.5. In our last example, we consider the Mathieu differential equation

$$y''(x) + 100(1 - \alpha \cos(2x))y(x) = 0,$$

subject to

$$\begin{aligned} y(0) &= 1, \\ y'(0) &= 0, \end{aligned}$$

and $\alpha = 0.1$. The frequency is $\pi/5$ (see Gautschi [7]). Mathieu equation appears in physical problems involving elliptical shapes or periodic potentials. We will solve the equation for $0 \leq x \leq 50$ and tabulate at several points. The results are compared to the values obtained from Maple using the function $\text{MathieuC}(100,5,x)$. The absolute error is computed at the end point of integration. The best method is again Wang, followed by OM2. The methods K7p0 and K5p1 did not perform as well as OM2. The worst is again E1. The explicit method KS20 diverged and that is why the results are not listed in the table.

Method	$x = 5.0$	$x = 10.0$	$x = 20.0$	$x = 30.0$	$x = 40.0$	$x = 50.0$	L_2 error
K7p0	0.941764	0.666764	0.186253	-0.128919	-0.511972	-0.951685	0.276245(-3)
K5p1	0.941764	0.666772	0.186275	-0.128924	-0.512066	-0.951940	0.255555(-3)
E1	0.907675	0.641881	0.221878	-0.0122481	-0.271998	-0.586723	0.369962
I2	0.934728	0.643377	0.140281	-0.187190	-0.560786	-0.927932	0.237532(-1)
OM2	0.941737	0.666732	0.186244	-0.128919	-0.511972	-0.951684	0.490900(-6)
Wang	0.941737	0.666732	0.186244	-0.128919	-0.511972	-0.951685	0.2808498(-9)
RKNS	0.944174	0.670037	0.187888	-0.131346	-0.523116	-0.976630	0.249453(-1)
KS20							
Exact	0.941737	0.666732	0.186244	-0.128919	-0.511972	-0.951685	

TABLE 6. The approximate solution using various methods (with $h = 0.02$) at various values of the independent variable. The exact value at each point was found using the Maple function MathieuC.

6. Conclusions

We have developed several trigonometrically-fitted methods of orders 3 to 8 and compared them to existing trigonometrically-fitted methods of order 8 and P-stable methods of order 10 and 12. We have used 5 linear and nonlinear second and fourth order initial value problems. The explicit method of order 10, KS20, did not converge for the last example. This shows the limitation of the explicit method as a standalone. Our eighth order method gave similar or better results than the other eighth order methods but could not compete with the twelfth order method. In the future we will develop a twelfth order trigonometrically-fitted method to see if it can compete with the P-stable method due to Wang et al. [24].

Funding

The work of the first author was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MSIT) (No.2020R1F1A1A101058502).

Conflicts of interest

The authors declare no conflict of interest.

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