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Target-Centric Probabilistic Multi-Hypothesis Tracking

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PREFACE

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14. ABSTRACT

This report examines an alternative assignment model for Probabilistic Multi-Hypothesis Tracking (PMHT) in which targets are assigned to measurements, in contrast to the original PMHT model in which measurements are assigned to targets. A new "target-centric" PMHT algorithm is derived and compared to its original "measurement-centric" counterpart. The relationship between target-centric PMHT and the probabilistic data association (PDA) filter for single-target tracking in clutter was examined, and a PDA-style approximation to the target state covariance matrices for target-centric PMHT is proposed. A target-centric/measurement-centric PMHT hybrid is also proposed to address algorithm performance in the case of closely spaced targets.

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LIST OF ABBREVIATIONS AND ACRONYMS

E-Step	Expectation Step
EM	Expectation-Maximization
JPDA	Joint Probabilistic Data Association
M-Step	Maximization Step
MAP	Maximum *a posteriori*
NUWCDIVNPT	Naval Undersea Warfare Center Division Newport
PDA	Probabilistic Data Association
PDF	Probability Density Function
PMHT	Probabilistic Multi-Hypothesis Tracking

TARGET-CENTRIC PROBABILISTIC MULTI-HYPOTHESIS TRACKING

1. INTRODUCTION

1.1 BACKGROUND AND EXISTING WORK

The Probabilistic Multi-Hypotheses Tracking (PMHT) method (references 1 and 2) is a multitarget tracking method for situations where the assignments between measurements and targets are unknown. Such situations are typical of tracking problems where targets are only partially observed and closely spaced in the observable dimensions. For example, in bearings-only tracking the ranges to targets are not observed, and hence measured bearings can be associated with any number of possible targets along, or near to, the lines of bearing. Even for problems where target range is observable, or where there is only a single target present, non-target related clutter measurements (due to environmental effects for example) create an assignment problem between measurements and targets.

Models for this assignment problem fall into two categories—those that treat the assignments as mutually exclusive and exhaustive, and those that treat them as statistically independent. Tracking approaches based on mutually exclusive and exhaustive assignments include Joint Probabilistic Data Association (JPDA) tracking (reference 3), in which all target and measurement combinations in the current scan are considered, and Multi-Hypothesis Tracking (MHT) (reference 3), in which all target and measurement combinations in all available scans are considered. MHT is consequently exponential in computational complexity, and practical applications require judicious pruning of the hypothesis tree. Due to enumeration in JPDA tracking being confined to the current scan, it is much less computationally intensive than MHT, but is nevertheless exponential in complexity within the current scan and is thus limited to sequential tracking applications.

Approaches based on statistically independent assignments include PMHT, which allows multiple measurements within a scan to be assigned to the same target. PMHT's greatest strength is its computational efficiency—it is linear in the number of targets under track, the number of scans, and the numbers of measurements with a scan. However, this efficiency comes at a cost; PMHT's assignment model yields a likelihood function with many local maxima, making the algorithm difficult to initialize and prone to track loss in situations of heavy clutter, low signal-to-noise ratios, and closely spaced tracks.

As discussed in reference 4, PMHT's admission of multiple measurements per target leads to "over-hospitality" with respect to clutter—as well as measurements from interfering targets—near each target under track. A potential solution to this problem is to change the PMHT assignment model to admit at most one measurement per target. One way to do this is to swap the roles of measurements and targets in the PMHT assignment model with respect to the assumption on independence and the direction of assignment; that is, to assume the targets are independent conditioned on the measurements, and to introduce missing target-to-measurement assignments (versus measurement-to-target assignments) at each scan. Unlike in PMHT, where multiple measurements within a scan may be assigned to the same target, in this alternative PMHT, multiple targets may be assigned to the same measurement within a scan. This consequence would seem to be reasonable for targets closely spaced in the measurement domain or with respect to the sensor resolution. The basic idea of swapping the roles of measurements and targets in the PMHT assignment model is explored in reference 5, although many details in this reference are either omitted or unclear, and the statistical model for the missing target-to-measurement assignments does not appear to be entirely justified. Nevertheless, reference 5 is the first open-literature reference to swapping the roles of measurements and targets in the PMHT assignment model. This report thoroughly examines the implications of this role-reversal and the new PMHT algorithm that results. In the sequel, this new algorithm will be referred to as target-centric PMHT to emphasize its target-to-measurement assignment model;* the original PMHT, as derived in references 1 and 2, which may be thought of as measurement-centric in light of its measurement-to-target assignment model, will be referred to interchangeably as measurement-centric PMHT, original PMHT, or simply PMHT.

1.2 NOTATION

The existence of $M \geq 1$ targets is assumed throughout this report, observed via a measurement process consisting of a batch of $T \geq 1$ discrete scans. Let $X = \{X_t\}$ denote the set of target states at scans $t = 1, \ldots, T$, where the set $X_t = \{x_{tm}\}, m = 1, \ldots, M$, denotes the states of the targets at scan, t. Similarly, let $Z = \{Z_t\}$ denote the set of measurements collected over the T scans, where $Z_t = \{z_{tr}\}, r = 1, \ldots, n_t$, denotes the set of n_t measurements collected at scan, t. The target states X (i.e., the target tracks) are the unknown quantities in PMHT, to be estimated from the observed data Z. It is customary in PMHT to include a known clutter model for measurements that are not target originated. (Such measurements may arise, for example, from random noise in the environment.) Consequently, a clutter target with states $\{x_{t0}\} = \{\emptyset\}, t = 1, \ldots, T$, are introduced where appropriate to avoid burdensome notation, though these states are not part of the estimation process. Throughout the report, $p(\cdot)$ and $P\{\cdot\}$ will be used generically to denote probability density functions (PDFs) and probability distributions, respectively.

*The version of target-centric PMHT presented in 5 is referred to as target-oriented PMHT.

2. PMHT REVIEW

2.1 COMPLETE, INCOMPLETE, AND MISSING DATA PDFS

In PMHT, associated with each measurement, z_{tr}, is an assumed missing (unobserved) measurement, k_{tr}, of the discrete measurement-to-target assignment random variable such that $k_{tr} \in \{0, 1, \ldots, M\}$ indicates the target to which the measurement z_{tr} is assigned. Let $K = \{K_t\}$ denote the set of missing measurement-to-target assignments at scans $t = 1, \ldots, T$, where the set $K_t = \{k_{tr}\}$ denotes the missing assignments associated with the n_t measurements Z_t collected at scan, t.

The goal of PMHT is to estimate the target states, X, from the observed (incomplete) data, Z, by finding the value of X that maximizes the incomplete-data PDF $p(Z, X)$; it does so by using the expectation-maximization (EM) method (reference 6) to find values of X that maximize a sequence of simpler estimation problems involving the complete-data PDF $p(Z, K, X)$. Development of the PMHT complete-data PDF follows from the following set of assumptions.

Assumption 1. The target states, X, and measurement-to-target assignments, K, are independent.

Under this assumption, the complete-data PDF can be written as

$$p(Z, K, X) = p(Z|K, X)P\{K\}p(X), \tag{2-1}$$

where $p(X)$ is the target state prior PDF, $P\{K\}$ is the prior probability distribution for the measurement-to-target assignments, and $p(Z|K, X)$ is the PDF of the measurements conditioned on the target states and assignments, also referred to as the complete-data likelihood function in the sequel.

Assumption 2. The target state sequences are independent, and each sequence follows a Markov process.

Under this assumption, the target state prior PDF is given by

$$p(X) = p(X_1) \prod_{t=2}^{T} p(X_t|X_{t-1}), \tag{2-2}$$

with

$$p(X_1) = \prod_{m=1}^{M} p_m(x_{1m}), \tag{2-3}$$

and

$$p(X_t|X_{t-1}) = \prod_{m=1}^{M} p_m(x_{tm}|x_{t-1,m}). \tag{2-4}$$

Assumption 3. The measurements, Z, are independent across and within scans when conditioned on the target states, X, and measurement-to-target assignments, K.

Under this assumption, the complete-data likelihood function is given by

$$p(Z|X,K) = \prod_{t=1}^{T} p(Z_t|X_t,K_t), \tag{2-5}$$

with

$$p(Z_t|X_t,K_t) = \prod_{r=1}^{n_t} p(z_{tr}|x_{tm}, k_{tr}=m) = \prod_{r=1}^{n_t} p(z_{tr}|x_{tm})\big|_{m=k_{tr}}. \tag{2-6}$$

Assumption 4. The measurement-to-target assignments, K, are independent within and across scans.

Under this simplifying (but unconventional) assumption, the prior probability distribution for the measurement-to-target assignments is given by

$$P\{K\} = \prod_{t=1}^{T} P\{K_t\}, \tag{2-7}$$

with

$$P\{K_t\} = \prod_{r=1}^{n_t} P\{k_{tr}=m\} = \prod_{r=1}^{n_t} \pi_{tm}\big|_{m=k_{tr}}, \tag{2-8}$$

where π_{tm} denotes the *a priori* probability of assigning a measurement to target, m, at scan, t. Assumption 4 greatly simplifies the assignment problem and leads to a state estimation procedure that grows linearly with the numbers of measurements and targets. However, this simplification comes at the cost of an incomplete-data PDF that is littered with local extrema, due to the fact this assignment model includes assignment hypotheses in the incomplete-data PDF that are excluded in conventional methods such as JPDA and MHT—namely, tracks with more than one assigned measurement per scan.

Combining the results of Assumptions 1 through 4, the PMHT complete-data PDF becomes

$$p(Z,K,X) = \left[\prod_{m=1}^{M} p_m(x_{1m}) \prod_{t=2}^{T} p_m(x_{tm}|x_{t-1,m})\right]\left[\prod_{t=1}^{T}\prod_{r=1}^{n_t} \pi_{tm}\, p_m(z_{tr}|x_{tm})\big|_{m=k_{tr}}\right]. \tag{2-9}$$

The PMHT incomplete-data PDF is obtained from the complete-data PDF by marginalizing over the missing measurement-to-target assignments:

$$P(Z,X) = \sum_{K} p(Z,K,X) = \sum_{t=1}^{T}\sum_{r=1}^{n_t}\sum_{k_{tr}=0}^{M} p(Z,K,X). \tag{2-10}$$

Substituting equation (2-9) into equation (2-10) and simplifying the resulting expression yields

$$p(Z,X) = \left[\prod_{m=1}^{M} p_m(x_{1m}) \prod_{t=2}^{T} p_m(x_{tm}|x_{t-1,m})\right] \left[\prod_{t=1}^{T}\prod_{r=1}^{n_t}\sum_{m=0}^{M} \pi_{tm}\, p_m(z_{tr}|x_{tm})\right]. \qquad \text{(2-11)}$$

Integral to the expectation step (E-step) of the EM method is the conditional distribution of the missing data with respect to the observed data. The conditional missing-data probability distribution for PMHT is obtained from the ratio of equations (2-9) and (2-11):

$$P\{K|Z,X\} = \frac{p(Z,K,X)}{p(Z,X)} = \prod_{t=1}^{T}\prod_{r=1}^{n_t} \omega_{trm}\big|_{m=k_{tr}}, \qquad \text{(2-12)}$$

where ω_{trm} denotes the conditional probability of assigning measurement, z_{tr}, to target, x_{tm}, and is given by

$$\omega_{trm} = P\{k_{tr} = m|z_{tr}, x_{tm}\} = \begin{cases} \dfrac{\pi_{t0}\, p_0(z_{tr}|x_{t0})}{\pi_{t0}\, p_0(z_{tr}|x_{t0}) + \sum_{l=1}^{M} \pi_{tl}\, p_l(z_{tr}|x_{tl})}, & m = 0 \\ \dfrac{\pi_{tm}\, p_m(z_{tr}|x_{tm})}{\pi_{t0}\, p_0(z_{tr}|x_{t0}) + \sum_{l=1}^{M} \pi_{tl}\, p_l(z_{tr}|x_{tl})}, & m = 1,\ldots,M. \end{cases} \qquad \text{(2-13)}$$

The PMHT conditional measurement-to-target assignment probability, ω_{trm}, for a given measurement depends on all M targets states at scan, t. Interestingly, the roles of measurements and targets are reversed for the target-centric PMHT conditional target-to-measurement assignment probabilities to be derived in section 3.

2.2 *A PRIORI* ASSIGNMENT PROBABILITIES

Expressions for the *a priori* measurement-to-target assignment probabilities,

$$\pi_{tm}\big|_{m=k_{tr}} = P\{k_{tr} = m\}, \qquad \text{(2-14)}$$

in terms of the target probability of detection, P_D, and clutter density, λ, are derived in appendix B of reference 7. Specifically, under the assumption there is at most one measurement per target, the *a priori* assignment probabilities, π_{tm}, are given by

$$\pi_{tm} = \begin{cases} 1 - \frac{1}{n_t}\alpha_t, & m = 0, \\ \frac{1}{n_t M}\alpha_t, & m = 1,\ldots,M, \end{cases} \qquad \text{(2-15)}$$

where α_t is a constant that depends on P_D and λ and is defined, for $\tilde{n}_t = \min\{n_t, M\}$, by

$$\alpha_t = \frac{\sum_{s=0}^{\tilde{n}_t} s\binom{M}{s} P_D^s (1-P_D)^{M-s} \mu(n_t - s; \lambda V)}{\sum_{s=0}^{\tilde{n}_t} \binom{M}{s} P_D^s (1-P_D)^{M-s} \mu(n_t - s; \lambda V)}, \qquad \text{(2-16)}$$

where $\mu(\cdot)$ denotes the clutter count probability distribution, and V denotes the measure of the surveillance region.

2.3 STATE ESTIMATION

PMHT uses the EM method (reference 6) to find the target states, $\hat{X}$, that maximize the incomplete-data PDF $p(Z, X)$ or, equivalently, the logarithm of this PDF (log-PDF). Rather than maximizing this log-PDF directly, the method proceeds by iteratively maximizing the conditional expectation of the complete-data log-PDF over X, where the expectation is with respect to the missing measurement-to-target assignments, K, conditioned on the observed data, Z, and the values for the states obtained from the previous iteration. Let $X^{(\ell)}, \ell = 1, 2, \ldots$ denote the estimate for the target states obtained from the ℓth EM iteration, where $X^{(0)}$ denotes the values for the states used in the initial evaluation of the E-step. Under the mild regularity conditions discussed in references 6 and 8, the sequence $p(Z, X^{(\ell)}), \ell = 1, 2, \ldots$ obtained by the EM iterations converges to at least a local maximum of $p(Z, X)$. Whether the sequence $X^{(\ell)}, \ell = 1, 2, \ldots$ converges to $\hat{X}$ depends on the complexity of $p(Z, X)$, the values for the states, $X^{(0)}$, used in the initial evaluation of the E-step, and possibly the method used for the maximization step (M-step) when this step cannot be evaluated in closed form.

2.3.1 E- and M-Steps

In PMHT, the E-step of the EM method consists of evaluating the following conditional expectation with respect to the missing measurement-to-target assignments, K:

$$\Psi(X, X^{(\ell-1)}) = \mathrm{E}\left[\log p(Z, K, X) | \, Z, X^{(\ell-1)}\right] = \sum_{K} p(K|Z, X^{(\ell-1)}) \log p(Z, K, X), \tag{2-17}$$

where $X^{(\ell-1)}$ denotes the state estimates obtained from the previous iteration. The M-step for the current iteration, (ℓ), is written explicitly as

$$X^{(\ell)} = \arg\max_{X} \Psi(X, X^{(\ell-1)}). \tag{2-18}$$

EM iterations are performed for $\ell = 1, 2, \ldots$ until prescribed convergence criteria are achieved. For the first iteration ($\ell = 1$), an initial "guess" for the target states, $X^{(0)}$, is required; depending on the clutter density and the separation of the targets, PMHT can be sensitive to this guess.

When evaluating the conditional expectation (2-17), it is convenient to partition the target states, X, by target index, m, rather than scan index, t. To this end, let $X = \{X^m\}$ denote the set of M target state sequences, where $X^m = \{x_{tm}\}$ denotes the state sequence of the mth target for scans $t = 1, \ldots, T$. Then, substituting equation (2-9) and equation (2-12) into equation (2-17), evaluating the summation and simplifying the resulting expression yields

$$\Psi(X, X^{(\ell-1)}) = \sum_{m=1}^{M} \Psi_m(X^m, (X^m)^{(\ell-1)}), \tag{2-19}$$

with

$$\Psi_m(X^m, (X^m)^{(\ell-1)}) = \log p_m(x_{1m}) + \left[\sum_{t=2}^{T} \log p_m(x_{tm}|x_{t-1,m})\right] + \left[\sum_{t=1}^{T}\sum_{r=1}^{n_t} \omega_{trm}^{(\ell-1)} \log p_m(z_{tr}|x_{tm})\right], \tag{2-20}$$

where the conditional measurement-to-target assignment probabilities for the previous iteration, $\omega_{trm}^{(\ell-1)}$, are given by

$$\omega_{trm}^{(\ell-1)} = \frac{\pi_{tm}\, p_m\left(z_{tr}|x_{tm}^{(\ell-1)}\right)}{\pi_{t0}\, p_0(z_{tr}|x_{t0}) + \sum_{l=1}^{M} \pi_{tl}\, p_l\left(z_{tr}|x_{tl}^{(\ell-1)}\right)}. \tag{2-21}$$

Consequently, because the conditional expectation as expressed in equation (2-17) is additively separable with respect to the state sequences, X^m, the M-step expressed in equation (2-18) is realized as M independent maximizations:

$$(X^m)^{(\ell)} = \arg\max_{X^m} \Psi_m(X^m, (X^m)^{(\ell-1)}). \tag{2-22}$$

At this point, explicit state estimates for PMHT cannot be derived without specifying functional forms for the target state prior PDFs and measurement likelihood functions in equation (2-20). However, as shown in the following section, explicit state estimates are easily obtained for the standard linear state-space model with Gaussian statistics.

2.3.2 Linear-Gaussian Case

In the case of the standard linear-Gaussian state-space model, the target state prior PDFs and measurement likelihood functions in equation (2-20) are given by

$$p_m(x_{1m}) = \mathcal{N}(x_{1m}|\hat{x}_{1|0,m}, P_{1|0,m}), \tag{2-23}$$
$$p_m(x_{tm}|x_{t-1,m}) = \mathcal{N}(x_{tm}|F_{tm}x_{t-1,m}, Q_{tm}), \tag{2-24}$$
$$p_m(z_{tr}|x_{tm}) = \mathcal{N}(z_{tr}|H_{tm}x_{tm}, R_{trm}), \tag{2-25}$$

where F_{tm} and Q_{tm} are known state transition and process noise covariance matrices, H_{tm} and R_{trm} are known measurement and measurement noise covariance matrices, and $\hat{x}_{1|0,m}$ and $P_{1|0,m}$ are known mean vectors and covariance matrices for the target states at the first scan. Let $\hat{x}_{0|0,m}$ and $P_{0|0,m}$ denote the state estimate and covariance matrix for the mth target at scan $t = 0$, defined as the scan just prior to the first scan in the batch. Then $\hat{x}_{1|0,m}$ and $P_{1|0,m}$ are given by

$$\begin{aligned} \hat{x}_{1|0,m} &= F_{1m}\hat{x}_{0|0,m}, \\ P_{1|0,m} &= F_{1m}P_{0|0,m}F_{1m}^T + Q_{1m}. \end{aligned} \tag{2-26}$$

Furthermore, when conditioning on the clutter target ($m = 0$), it is typically assumed the measurements are uniformly distributed in the surveillance region; in this case, if this region has measure V, the likelihood function $p_0(z_{tr}|x_{t0})$ in equation (2-25) is given by

$$p_0(z_{tr}|x_{t0}) = p_0(z_{tr}) = \frac{1}{V} \tag{2-27}$$

for all scans and measurements. Substituting equations (2-23) through (2-25) into equation (2-20), simplifying the resulting expression and dropping terms not dependent on the target states yields the following form for the conditional expectation Ψ_m:

$$\begin{aligned}\Psi_m(X^m, (X^m)^{(\ell-1)}) = &- \left(x_{1m} - \hat{x}_{1|0,m}\right)^T P_{1|0,m}^{-1} \left(x_{1m} - \hat{x}_{1|0,m}\right) \\ &- \left[\sum_{t=2}^{T} \left(x_{tm} - F_{tm}x_{t-1,m}\right)^T Q_{tm}^{-1} \left(x_{tm} - F_{tm}x_{t-1,m}\right)\right] \\ &- \left[\sum_{t=1}^{T}\sum_{r=1}^{n_t} \omega_{trm}^{(\ell-1)} \left(z_{tr} - H_{tm}x_{tm}\right)^T R_{trm}^{-1} \left(z_{tr} - H_{tm}x_{tm}\right)\right],\end{aligned} \tag{2-28}$$

where the measurement-to-target assignment probabilities for the previous iteration are given by

$$\omega_{trm}^{(\ell-1)} = \frac{\pi_{tm}\,\mathcal{N}\Big(z_{tr}|H_{tm}x_{tm}^{(\ell-1)}, R_{trm}\Big)}{\pi_{t0}/V + \sum_{l=1}^{M}\pi_{tl}\,\mathcal{N}\Big(z_{tr}|H_{tl}x_{tl}^{(\ell-1)}, R_{trl}\Big)}. \tag{2-29}$$

The double sum in equation (2-28) can be simplified by defining a composite measurement, $\tilde{z}_{tm}$, and corresponding measurement covariance matrix, $\tilde{R}_{tm}$, for each target at each scan as follows:

$$\tilde{R}_{tm}^{(\ell-1)} = \left(\sum_{r=1}^{n_t} \omega_{trm}^{(\ell-1)} R_{trm}^{-1}\right)^{-1}, \tag{2-30}$$

$$\tilde{z}_{tm}^{(\ell-1)} = \tilde{R}_{tm}^{(\ell-1)} \sum_{r=1}^{n_t} \omega_{trm}^{(\ell-1)} R_{trm}^{-1}\, z_{tr}. \tag{2-31}$$

Expanding the quadratic form in the double sum in equation (2-28), distributing the inner sum over r, substituting the definitions given by equations (2-30) and (2-31), completing the square and dropping terms not dependent on the target states yields the simplified form for the conditional expectation Ψ_m:

$$\begin{aligned}\Psi_m(X^m, (X^m)^{(\ell-1)}) = &- \left(x_{1m} - \hat{x}_{1|0,m}\right)^T P_{1|0,m}^{-1} \left(x_{1m} - \hat{x}_{1|0,m}\right) \\ &- \left[\sum_{t=2}^{T} \left(x_{tm} - F_{tm}x_{t-1,m}\right)^T Q_{tm}^{-1} \left(x_{tm} - F_{tm}x_{t-1,m}\right)\right] \\ &- \left[\sum_{t=1}^{T} \left(\tilde{z}_{tm}^{(\ell-1)} - H_{tm}x_{tm}\right)^T \left(\tilde{R}_{tm}^{(\ell-1)}\right)^{-1} \left(\tilde{z}_{tm}^{(\ell-1)} - H_{tm}x_{tm}\right)\right].\end{aligned} \tag{2-32}$$

The simplified form of the conditional expectation, Ψ_m, in equation (2-32) is functionally equivalent to the posterior log-PDF for the standard Kalman smoothing problem with measurements and measurement covariance matrices given by the composite measurements and composite measurement covariance matrices defined by equations (2-30) and (2-31). Hence, the estimated target state sequence, $(X^m)^{(\ell)}$, for the mth target at the ℓth EM iteration may be obtained by standard Kalman smoothing techniques. This connection with Kalman filtering theory is not generally shared by target-centric PMHT; however, as shown in section 3.3, state estimation for target-centric PMHT is nevertheless a linear estimation problem.

3. TARGET-CENTRIC PMHT

3.1 COMPLETE, INCOMPLETE, AND MISSING DATA PDFS

Target-centric PMHT reverses the direction of the original (measurement-centric) PMHT assignment model; that is, target-centric PMHT considers target-to-measurement assignments instead of measurement-to-target assignments. Specifically, in target-centric PMHT, associated with each target state, x_{tm}, is an assumed missing (unobserved) discrete state, $j_{tm} \in \mathcal{J}_t = \{0, 1, \ldots, n_t\}$, that indicates the measurement in Z_t to which x_{tm} is assigned. The element 0 is included in the assignment set, $\mathcal{J}_t$, to account for the possibility that a target is assigned to no measurement in the scan. The non-assignment event $j_{tm} = 0$ in target-centric PMHT is analogous to the measurement-centric PMHT event in which no measurement at scan, t, is assigned to target, m; that is, the event $\{k_{tr} \neq m : r = 1, \ldots, n_t\}$. In the sequel, let $J = \{J_t\}$ denote the set of missing target-to-measurement assignments at scans $t = 1, \ldots, T$, where $J_t = \{j_{tm}\}$ denotes the missing assignments associated with the M targets at scan, t.

Like PMHT, the goal of target-centric PMHT is to estimate the target states, X, from the observed (incomplete) data, Z, by finding the value of X that maximizes the incomplete-data PDF $p(X, Z)$. It does so by using the EM method to find values of X that maximize a sequence of simpler estimation problems involving the complete-data PDF $p(X, J, Z)$. Development of the target-centric complete-data PDF, which can be factored in terms of the conditional prior probability distribution for the target-to-measurement assignments, $P\{J|Z\}$, and the complete-data target state posterior PDF, $p(X|J, Z)$, as

$$p(X, J, Z) = p(X|J, Z)P\{J|Z\}p(Z), \tag{3-1}$$

follows from the following set of assumptions:

Assumption 5. The targets are independent and their states follow a Markov process.

Under this assumption, the complete-data target state posterior PDF is given by

$$p(X|J, Z) = p(X_1|J_1, Z_1) \prod_{t=2}^{T} p(X_t|X_{t-1}, J_t, Z_t), \tag{3-2}$$

with

$$p(X_1|J_1, Z_1) = \prod_{m=1}^{M} p_m(x_{1m}|j_{1m} = r, z_{1r}) = \prod_{m=1}^{M} p_m(x_{1m}|z_{1r})\big|_{r=j_{1m}}, \tag{3-3}$$

and

$$p(X_t|X_{t-1}, J_t, Z_t) = \prod_{m=1}^{M} p_m(x_{tm}|x_{t-1,m}, j_{tm} = r, z_{tr}) = \prod_{m=1}^{M} p_m(x_{tm}|x_{t-1,m}, z_{tr})\big|_{r=j_{tm}}. \tag{3-4}$$

Assumption 6. The target-to-measurement assignments, J, are independent within and across scans, conditioned on the measurements, Z.

Under this simplifying assumption, the conditional prior probability distribution for the target-to-measurement assignments is given by

$$P\{J|Z\} = \prod_{t=1}^{T} P\{J_t|Z_t\}, \tag{3-5}$$

with

$$P\{J_t|Z_t\} = \prod_{m=1}^{M} P\{j_{tm} = r|Z_t\} = \prod_{m=1}^{M} \gamma_{tmr}\big|_{r=j_{tm}}, \tag{3-6}$$

where γ_{tmr} denotes the conditional *a priori* probability of assigning target, m, to measurement, r, at scan, t, conditioned on all measurements at scan, t. As is the case for the analogous assumption in the original PMHT assignment model, this assumption greatly simplifies the assignment problem and leads to a state estimation procedure that grows linearly with the numbers of measurements and targets. This assumption also explicitly allows more than one target to be assigned to a given measurement within a given scan—a reasonable assumption for closely-spaced targets.

Combining the results of Assumptions 5 and 6, the target-centric PMHT complete-data PDF becomes

$$p(X, J, Z) = \left[\prod_{m=1}^{M} \gamma_{1mr}\, p_m(x_{1m}|z_{1r})\big|_{r=j_{1m}}\right] \left[\prod_{t=2}^{T}\prod_{m=1}^{M} \gamma_{tmr}\, p_m(x_{tm}|x_{t-1,m}, z_{tr})\big|_{r=j_{tm}}\right]. \tag{3-7}$$

For $r = j_{tm} \neq 0$, the target state posterior densities in equation (3-7) may be written in terms of target state prior PDFs and measurement likelihood functions using the following factorizations:

$$p_m(x_{1m}|z_{1r}) = \frac{p_m(x_{1m}, z_{1r})}{p_m(z_{1r})}, = \frac{p_m(z_{1r}|x_{1m})p_m(x_{1m})}{p_m(z_{1r})}, \tag{3-8}$$

$$p_m(x_{tm}|x_{t-1,m}, z_{tr}) = \frac{p_m(x_{tm}, x_{t-1,m}, z_{tr})}{p_m(x_{t-1,m}, z_{tr})}, = \frac{p_m(z_{tr}|x_{tm})p_m(x_{tm}|x_{t-1,m})}{p_m(z_{tr}|x_{t-1,m})}, \tag{3-9}$$

where the denominators in equation (3-8) and equation (3-9) are given by the integrals

$$p_m(z_{1r}) = \int p_m(z_{1r}|x_{1m})\, p_m(x_{1m})\, dx_{1m}, \tag{3-10}$$

$$p_m(z_{tr}|x_{t-1,m}) = \int p_m(z_{tr}|x_{tm})\, p_m(x_{tm}|x_{t-1,m})\, dx_{tm}, \tag{3-11}$$

respectively. For the case $r = j_{tm} = 0$, the "measurement," z_{t0}, is interpreted as the empty set, and the target state posterior densities in equation (3-7) are defined as

$$\begin{aligned} p_m(x_{1m}|z_{10}) &\equiv p_m(x_{1m}), \\ p_m(x_{tm}|x_{t-1,m}, z_{t0}) &\equiv p_m(x_{tm}|x_{t-1,m}). \end{aligned} \tag{3-12}$$

Substituting equations (3-8) and (3-9) into equation (3-7) and rearranging terms gives the following form for the target-centric PMHT complete-data PDF:

$$\begin{aligned} p(X, J, Z) = &\left[\prod_{m=1}^{M} \gamma_{1mr} \left. \frac{p_m(z_{1r}|x_{1m}) p_m(x_{1m})}{p_m(z_{1r})} \right|_{r=j_{1m}} \right] \\ &\times \left[\prod_{t=2}^{T} \prod_{m=1}^{M} \gamma_{tmr} \left. \frac{p_m(z_{tr}|x_{tm}) p_m(x_{tm}|x_{t-1,m})}{p_m(z_{tr}|x_{t-1,m})} \right|_{r=j_{tm}} \right], \end{aligned} \tag{3-13}$$

where, in light of equation (3-12), to simplify notation the following interpretations for the case $r = j_{tm} = 0$ is adopted:

$$\left. \frac{p_m(z_{1r}|x_{1m})}{p_m(z_{1r})} \right|_{r=j_{tm}=0} \equiv 1, \tag{3-14}$$

$$\left. \frac{p_m(z_{tr}|x_{tm})}{p_m(z_{tr}|x_{t-1,m})} \right|_{r=j_{tm}=0} \equiv 1. \tag{3-15}$$

Finally, rearranging terms in equation (3-13) and adopting the convention

$$p_m(z_{1r}|x_{0m}) \equiv p_m(z_{1r}), \tag{3-16}$$

yields the final form for the target-centric complete-data PDF:

$$p(X, J, Z) = \left[\prod_{m=1}^{M} p_m(x_{1m}) \prod_{t=2}^{T} p_m(x_{tm}|x_{t-1,m}) \right] \left[\prod_{m=1}^{M} \prod_{t=1}^{T} \gamma_{tmr} \left. \frac{p_m(z_{tr}|x_{tm})}{p_m(z_{tr}|x_{t-1,m})} \right|_{r=j_{tm}} \right]. \tag{3-17}$$

In this form, it is clear that when target, m, is assigned to no measurement at scan, t, that is when $j_{tm} = 0$, the contribution of target, m, to the complete-data PDF, $p(X, J, Z)$, at that scan is due entirely to its prior PDF, namely $p_m(x_{tm}|x_{t-1,m})$, by virtue of equation (3-15).

The target-centric incomplete-data PDF is obtained from the complete-data PDF by marginalizing over the missing target-to-measurement assignments:

$$P(X, Z) = \sum_{J} p(X, J, Z) = \sum_{t=1}^{T} \sum_{m=1}^{M} \sum_{j_{tm}=0}^{n_t} p(X, J, Z). \tag{3-18}$$

Substituting equation (3-17) into equation (3-18), performing the summations and simplifying the resulting expression yields

$$p(X,Z) = \left[\prod_{m=1}^{M} p_m(x_{1m}) \prod_{t=2}^{T} p_m(x_{tm}|x_{t-1,m})\right] \left[\prod_{m=1}^{M} \prod_{t=1}^{T} \sum_{r=0}^{n_t} \gamma_{tmr} \frac{p_m(z_{tr}|x_{tm})}{p_m(z_{tr}|x_{t-1,m})}\right]. \tag{3-19}$$

Finally, the conditional missing-data PDF for target-centric PMHT is obtained from the ratio of equations (3-17) and (3-19):

$$P\{J|X,Z\} = \frac{p(X,J,Z)}{p(X,Z)} = \prod_{m=1}^{M} \prod_{t=1}^{T} \xi_{trm}|_{r=j_{tm}}, \tag{3-20}$$

where ξ_{trm} denotes the conditional probability of assigning target, x_{tm}, to measurement, z_{tr} (or no measurement at all in the case of $r = 0$), given by

$$\xi_{trm} = P\{j_{tm} = r|x_{tm}, x_{t-1,m}, z_{tr}\} = \begin{cases} \dfrac{\gamma_{tm0}}{\gamma_{tm0} + \sum_{\rho=1}^{n_t} \gamma_{tm\rho} \frac{p_m(z_{t\rho}|x_{tm})}{p_m(z_{t\rho}|x_{t-1,m})}}, & r = 0, \\ \dfrac{\gamma_{tmr} \frac{p_m(z_{tr}|x_{tm})}{p_m(z_{tr}|x_{t-1,m})}}{\gamma_{tm0} + \sum_{\rho=1}^{n_t} \gamma_{tm\rho} \frac{p_m(z_{t\rho}|x_{tm})}{p_m(z_{t\rho}|x_{t-1,m})}}, & r = 1, \ldots, n_t. \end{cases} \tag{3-21}$$

The target-centric PMHT conditional target-to-measurement assignment probability, ξ_{trm}, for a given target depends on all n_t measurements at scan, t. In contrast, the PMHT conditional measurement-to-target assignment probabilities, ω_{trm}, depend on all M targets; the assignment probabilities given by equation (3-21) may be interpreted as the duals to those given by equation (2-13).

3.2 *A PRIORI* ASSIGNMENT PROBABILITIES

Expressions for the conditional *a priori* target-to-measurement assignment probabilities

$$\gamma_{tmr}|_{r=j_{tm}} = P\{j_{tm} = r|Z_t\}, \tag{3-22}$$

in terms of the target probability of detection, P_D, and expected clutter density, λ, are derived in appendix A. Specifically, under the assumption there is at most one measurement per target, the conditional *a priori* assignment probabilities, γ_{tmr}, for target-centric PMHT are given by

$$\gamma_{tmr} = \begin{cases} 1 - \frac{1}{M}\alpha_t, & r = 0, \\ \frac{1}{M} \frac{p_m(z_{tr})}{\sum_{\rho=1}^{n_t} p_m(z_{t\rho})} \alpha_t, & r = 1, \ldots, n_t, \end{cases} \tag{3-23}$$

where α_t, a function of P_D and λ, is defined by equation (2-16).

3.3 STATE ESTIMATION

As with PMHT, target-centric PMHT uses the EM method to find the target states, $\hat{X}$, that maximize the incomplete-data PDF, $p(X, Z)$, or, equivalently, the log-PDF, $\log p(X, Z)$. The method proceeds by iteratively maximizing the conditional expectation of the complete-data log-PDF over X, where the expectation is with respect to the missing target-to-measurement assignments, J, conditioned on the observed data, Z, and the values for the states obtained from the previous iteration.

3.3.1 E- and M-Steps

The E-step for target-centric PMHT consists of evaluating the following conditional expectation with respect to the missing target-to-measurement assignments, J:

$$\Theta(X, X^{(\ell-1)}) = \mathrm{E}\left[\log p(X, J, Z) |\, X^{(\ell-1)}, Z\right] = \sum_{J} p(J|X^{(\ell-1)}, Z) \log p(X, J, Z). \tag{3-24}$$

Substituting equations (3-17) and (3-20) for the complete-data and missing-data PDFs, respectively, into equation (3-24), evaluating the summations, using the identities in equations (3-14) and (3-15), dropping terms not dependent on X, and simplifying the resulting expression yields

$$\Theta(X, X^{(\ell-1)}) = \sum_{m=1}^{M} \Theta_m(X^m, (X^m)^{(\ell-1)}), \tag{3-25}$$

with

$$\begin{aligned}\Theta_m(X^m, (X^m)^{(\ell-1)} = \log p_m(x_{1m}) &+ \left[\sum_{t=2}^{T} \log p_m(x_{tm}|x_{t-1,m})\right] \\ &+ \left[\sum_{t=1}^{T}\sum_{r=1}^{n_t} \xi_{trm}^{(\ell-1)} \log p_m(z_{tr}|x_{tm})\right] \\ &- \left[\sum_{t=2}^{T}\sum_{r=1}^{n_t} \xi_{trm}^{(\ell-1)} \log p_m(z_{tr}|x_{t-1,m})\right],\end{aligned} \tag{3-26}$$

where the conditional target-to-measurement assignment probabilities for the previous iteration are given by

$$\xi_{trm}^{(\ell-1)} = \frac{\gamma_{tmr} \frac{p_m\left(z_{tr}|x_{tm}^{(\ell-1)}\right)}{p_m\left(z_{tr}|x_{t-1,m}^{(\ell-1)}\right)}}{\gamma_{tm0} + \sum_{\rho=1}^{n_t} \gamma_{tm\rho} \frac{p_m\left(z_{t\rho}|x_{tm}^{(\ell-1)}\right)}{p_m\left(z_{t\rho}|x_{t-1,m}^{(\ell-1)}\right)}}. \tag{3-27}$$

As is the case for the conditional expectation $\Psi(X, X^{(\ell-1)})$ in the original PMHT, the target-centric conditional expectation (3-24) is additively separable with respect to the state sequences, X^m. Hence, the M-step for target-centric PMHT is realized as M independent maximizations:

$$(X^m)^{(\ell)} = \arg\max_{X^m} \Theta_m(X^m, (X^m)^{(\ell-1)}). \tag{3-28}$$

Likewise, explicit state estimates for target-centric PMHT are easily obtained for the standard linear state-space model with Gaussian statistics, as shown in the following section.

3.3.2 Linear-Gaussian Case

The target state prior PDFs, $p_m(x_{1m})$ and $p_m(x_{tm}|x_{t-1,m})$, and measurement likelihood function, $p_m(z_{tr}|x_{tm})$, for the standard linear-Gaussian state-space model are given by equations (2-23) through (2-25). The target-centric PMHT conditional missing-data PDFs in equation (3-21) and conditional expectations in equation (3-26) require evaluation of the additional PDFs in equations (3-10) and (3-11). For the linear-Gaussian model PDFs given by equations (2-23) through (2-25), the PDFs in equations (3-10) and (3-11) are easily evaluated using the Gaussian Refactorization Lemma derived in reference 9 and provided in appendix B for convenient reference. Substituting equations (2-23) through (2-25) into in equations (3-10) and (3-11) and applying this lemma (and its corollary, equation (B-8)) yields

$$p_m(z_{1r}) = \mathcal{N}(z_{1r}|H_{1m}\hat{x}_{1|0,m}, H_{1m}P_{1|0,m}H_{1m}^T + R_{1rm}) \tag{3-29}$$

for $t = 1$ and, for $t > 1$,

$$p_m(z_{tr}|x_{t-1,m}) = \mathcal{N}(z_{tr}|\Phi_{tm}x_{t-1,m}, \Lambda_{trm}), \tag{3-30}$$

with

$$\begin{aligned} \Phi_{tm} &= H_{tm}F_{tm}, \\ \Lambda_{trm} &= H_{tm}Q_{tm}H_{tm}^T + R_{trm}. \end{aligned} \tag{3-31}$$

Substituting equations (2-23) through (2-25) and equation (3-30) into equation (3-26), simplifying the resulting expression and dropping terms not dependent on the target states yields the following form for the conditional expectation, Θ_m:

$$\begin{aligned} \Theta_m(X^m, (X^m)^{(\ell-1)}) = &- \left(x_{1m} - \hat{x}_{1|0,m}\right)^T P_{1|0,m}^{-1} \left(x_{1m} - \hat{x}_{1|0,m}\right) \\ &- \left[\sum_{t=2}^{T} \left(x_{tm} - F_{tm}x_{t-1,m}\right)^T Q_{tm}^{-1} \left(x_{tm} - F_{tm}x_{t-1,m}\right)\right] \\ &- \left[\sum_{t=1}^{T}\sum_{r=1}^{n_t} \xi_{trm}^{(\ell-1)} \left(z_{tr} - H_{tm}x_{tm}\right)^T R_{trm}^{-1} \left(z_{tr} - H_{tm}x_{tm}\right)\right] \\ &+ \left[\sum_{t=2}^{T}\sum_{r=1}^{n_t} \xi_{trm}^{(\ell-1)} \left(z_{tr} - \Phi_{tm}x_{t-1,m}\right)^T \Lambda_{trm}^{-1} \left(z_{tr} - \Phi_{tm}x_{t-1,m}\right)\right], \end{aligned} \tag{3-32}$$

where the conditional target-to-measurement assignment probabilities for the previous iteration are given by

$$\xi_{trm}^{(\ell-1)} = \frac{\frac{\gamma_{tmr}}{\eta_{trm}^{(\ell-1)}} \mathcal{N}\left(z_{tr}|H_{tm}x_{tm}^{(\ell-1)}, R_{trm}\right)}{\gamma_{tm0} + \sum_{\rho=1}^{n_t} \frac{\gamma_{tm\rho}}{\eta_{t\rho m}^{(\ell-1)}} \mathcal{N}\left(z_{t\rho}|H_{tm}x_{tm}^{(\ell-1)}, R_{t\rho m}\right)}, \tag{3-33}$$

with

$$\eta_{trm}^{(\ell-1)} = \begin{cases} \mathcal{N}(z_{1r}|H_{1m}\hat{x}_{1|0,m}, H_{1m}P_{1|0,m}H_{1m}^T + R_{1rm}), & t = 1, \\ \mathcal{N}(z_{tr}|\Phi_{tm}x_{t-1,m}^{(\ell-1)}, \Lambda_{trm}), & t > 1. \end{cases} \tag{3-34}$$

As in the original PMHT, the double sums in this expression can be simplified by defining composite measurement, $\tilde{z}_{tm}$ and $\tilde{\zeta}_{tm}$, and corresponding composite measurement covariance matrices, $\tilde{R}_{tm}$ and $\tilde{\Lambda}_{tm}$, for each target at each scan, as follows:

$$\tilde{R}_{tm}^{(\ell-1)} = \left(\sum_{r=1}^{n_t} \xi_{trm}^{(\ell-1)} R_{trm}^{-1}\right)^{-1}, \tag{3-35}$$

$$\tilde{z}_{tm}^{(\ell-1)} = \tilde{R}_{tm}^{(\ell-1)} \sum_{r=1}^{n_t} \xi_{trm}^{(\ell-1)} R_{trm}^{-1} z_{tr}, \tag{3-36}$$

and, similarly,

$$\tilde{\Lambda}_{tm}^{(\ell-1)} = \left(\sum_{r=1}^{n_t} \xi_{trm}^{(\ell-1)} \Lambda_{trm}^{-1}\right)^{-1}, \tag{3-37}$$

$$\tilde{\zeta}_{tm}^{(\ell-1)} = \tilde{\Lambda}_{tm}^{(\ell-1)} \sum_{r=1}^{n_t} \xi_{trm}^{(\ell-1)} \Lambda_{trm}^{-1} z_{tr}. \tag{3-38}$$

Expanding the quadratic forms in the double sums in equation (3-32), distributing the inner sums over r, substituting the definitions given by equations (3-35) through (3-38), completing the squares and dropping terms not dependent on the target states yields the simplified form for the conditional expectation Θ_m:

$$\begin{aligned} \Theta_m(X^m, (X^m)^{(\ell-1)}) = &- \left(x_{1m} - \hat{x}_{1|0,m}\right)^T P_{1|0,m}^{-1} \left(x_{1m} - \hat{x}_{1|0,m}\right) \\ &- \left[\sum_{t=2}^{T} \left(x_{tm} - F_{tm}x_{t-1,m}\right)^T Q_{tm}^{-1} \left(x_{tm} - F_{tm}x_{t-1,m}\right)\right] \\ &- \left[\sum_{t=1}^{T} \left(\tilde{z}_{tm}^{(\ell-1)} - H_{tm}x_{tm}\right)^T \left(\tilde{R}_{tm}^{(\ell-1)}\right)^{-1} \left(\tilde{z}_{tm}^{(\ell-1)} - H_{tm}x_{tm}\right)\right] \\ &+ \sum_{t=2}^{T} \left(\tilde{\zeta}_{tm}^{(\ell-1)} - \Phi_{tm}x_{t-1,m}\right)^T \left(\tilde{\Lambda}_{tm}^{(\ell-1)}\right)^{-1} \left(\tilde{\zeta}_{tm}^{(\ell-1)} - \Phi_{tm}x_{t-1,m}\right). \end{aligned} \tag{3-39}$$

Unlike for the original PMHT, except for the special case $T = 1$, the simplified form of the conditional expectation, Θ_m, in equation (3-39) is not in general functionally equivalent to the posterior log-PDF for the standard Kalman smoothing problem with respect to the composite measurements and measurement covariance matrices defined by equations (3-35) through (3-38). Nevertheless, the estimated target state sequence, $(X^m)^{(\ell)}$, for the mth target at the ℓth EM iteration are obtained as the solution to the linear system of equations

$$(\mathbf{A}^m)^{(\ell-1)} (\mathbf{X}^m)^{(\ell)} = (\mathbf{b}^m)^{(\ell-1)}, \tag{3-40}$$

where, if s denotes the dimension of the target state, $\mathbf{X}^m$ is an $sT \times 1$ block vector formed by concatenating the states in $X^m = \{x_{tm}\}$ and, in general, $\mathbf{A}^m$ is an $sT \times sT$ block-tridiagonal matrix, and $\mathbf{b}^m$ is an $sT \times 1$ block vector. Equation (3-40) is obtained by differentiating equation (3-39) with respect to X^m, setting the resulting expression equal to zero, and simplifying the result; that is, equation (3-40) results from the necessary condition for X^m to maximize $\Theta_m(X^m, (X^m)^{(\ell-1)})$. For the special case $T = 1$, the matrix $\mathbf{A}^m$ and vector $\mathbf{b}^m$ are given by

$$\begin{aligned} (\mathbf{A}^m)^{(\ell-1)} &= P_{1|0,m}^{-1} + H_{1m}^T \left(\tilde{R}_{1m}^{(\ell-1)}\right)^{-1} H_{1m}, \\ (\mathbf{b}^m)^{(\ell-1)} &= P_{1|0,m}^{-1}\,\hat{x}_{1|0,m} + H_{1m}^T \left(\tilde{R}_{1m}^{(\ell-1)}\right)^{-1} \tilde{z}_{1m}^{(\ell-1)}, \end{aligned} \tag{3-41}$$

respectively. For the general case $T > 1$, the block entries of $\mathbf{A}^m$ and $\mathbf{b}^m$ are given by

$$\begin{aligned} (\mathbf{A}_{tt}^m)^{(\ell-1)} &= \begin{cases} P_{1|0,m}^{-1} + H_{1m}^T \left(\tilde{R}_{1m}^{(\ell-1)}\right)^{-1} H_{1m} + F_{2m}^T Q_{2m}^{-1} F_{2m} \\ \quad - \Phi_{2m}^T \left(\tilde{\Lambda}_{2m}^{(\ell-1)}\right)^{-1} \Phi_{2m}, & t = 1, \\ Q_{tm}^{-1} + H_{tm}^T \left(\tilde{R}_{tm}^{(\ell-1)}\right)^{-1} H_{tm} + F_{t+1,m}^T Q_{t+1,m}^{-1} F_{t+1,m} \\ \quad - \Phi_{t+1,m}^T \left(\tilde{\Lambda}_{t+1,m}^{(\ell-1)}\right)^{-1} \Phi_{t+1,m}, & 1 < t < T, \\ Q_{Tm}^{-1} + H_{Tm}^T \left(\tilde{R}_{Tm}^{(\ell-1)}\right)^{-1} H_{Tm}, & t = T. \end{cases} \\ \left(\mathbf{A}_{t,t+1}^m\right)^{(\ell-1)} &= -F_{t+1,m}^T Q_{t+1,m}^{-1}, \quad 1 \le t < T, \\ \left(\mathbf{A}_{t,t-1}^m\right)^{(\ell-1)} &= -Q_{tm}^{-1} F_{tm}, \quad 1 < t \le T, \end{aligned} \tag{3-42}$$

and

$$(\mathbf{b}_t^m)^{(\ell-1)} = \begin{cases} P_{1|0,m}^{-1}\hat{x}_{1|0,m} + H_{1m}^T \left(\tilde{R}_{1m}^{(\ell-1)}\right)^{-1} \tilde{z}_{1m}^{(\ell-1)} - \Phi_{2m}^T \left(\tilde{\Lambda}_{2m}^{(\ell-1)}\right)^{-1} \tilde{\zeta}_{2m}^{(\ell-1)}, & t = 1, \\ H_{tm}^T \left(\tilde{R}_{tm}^{(\ell-1)}\right)^{-1} \tilde{z}_{tm}^{(\ell-1)} - \Phi_{t+1,m}^T \left(\tilde{\Lambda}_{t+1,m}^{(\ell-1)}\right)^{-1} \tilde{\zeta}_{t+1,m}^{(\ell-1)}, & 1 < t < T, \\ H_{Tm}^T \left(\tilde{R}_{Tm}^{(\ell-1)}\right)^{-1} \tilde{z}_{Tm}^{(\ell-1)}, & t = T, \end{cases} \tag{3-43}$$

respectively.

4. COMPARISON OF INCOMPLETE-DATA PDFS FOR PMHT AND TARGET-CENTRIC PMHT

The objectives of PMHT and target-centric PMHT are to find the target states, $\hat{X}$, that maximize the incomplete-data PDFs given by equations (2-11) and (3-19), respectively. In this section, a qualitative comparison of these PDFs for a simple one-dimensional tracking problem is provided. The problem consists of tracking two constant-velocity targets, in clutter, from measurements of their positions in the surveillance region, denoted by $\mathcal{S}$; practical examples include tracking in time-bearing and time-frequency. To simplify the problem even further, only a single measurement scan is considered, so that $t = T = 1$. In this case, the PMHT incomplete-data PDF, equation (2-11), reduces to

$$p(Z_1, X_1) = \left[\prod_{m=1}^{2} p_m(x_{1m})\right] \prod_{r=1}^{n_1} \left[\pi_{10} + \sum_{m=1}^{2} \pi_{1m}\, p_m(z_{1r}|x_{1m})\right]. \tag{4-1}$$

Likewise, the target-centric PMHT incomplete-data PDF, equation (3-19), reduces to

$$p(X_1, Z_1) = \left[\prod_{m=1}^{2} p_m(x_{1m})\right] \prod_{m=1}^{2} \left[\gamma_{1m0} + \sum_{r=1}^{n_1} \gamma_{1mr} \frac{p_m(z_{1r}|x_{1m})}{p_m(z_{1r})}\right]. \tag{4-2}$$

Let the state of target, m, at scan, t, be defined as $x_{tm} = [u_{tm}, \dot{u}_{tm}]^T$, where u_{tm} and $\dot{u}_{tm}$ denote the target position and velocity, respectively. For a given scan of measurements, $Z_1 = \{z_{1r}\}, r = 1, 2, \ldots, n_1$, the PDFs in equations (4-1) and (4-2) are functions of the four kinematic parameters $\{u_{11}, \dot{u}_{11}, u_{12}, \dot{u}_{12}\}$, and are hence difficult to visualize without some form of dimensional reduction. To address this difficulty, equations (4-1) and (4-2) are integrated over the target velocities, $\dot{u}_{11}$ and $\dot{u}_{12}$, to obtain the marginal incomplete-data PDFs

$$p(Z_1, \{u_{11}, u_{12}\}) = \iint p(Z_1, X_1)\, d\dot{u}_{11} d\dot{u}_{12}, \tag{4-3}$$

and

$$p(\{u_{11}, u_{12}\}, Z_1) = \iint p(X_1, Z_1)\, d\dot{u}_{11} d\dot{u}_{12} \tag{4-4}$$

for PMHT and target-centric PMHT, respectively. When the densities, $p_m(x_{1m})$, $p_m(z_{1r}|x_{1m})$, and $p_m(z_{1r})$, are given by the Gaussian densities in equations (2-23), (2-25), and (3-29), respectively, the integrals in equations (4-3) and (4-4) are easy to compute; the results are

$$p(Z_1, \{u_{11}, u_{12}\}) = \left[\prod_{m=1}^{2} \mathcal{N}(u_{1m}|\hat{u}_{1|0,m}, \xi^2_{1|0,m})\right] \times \prod_{r=1}^{n_1} \left[\frac{\pi_{10}}{V} + \sum_{m=1}^{2} \pi_{1m}\, \mathcal{N}(z_{1r}|u_{1m}, \sigma^2_{1rm})\right], \tag{4-5}$$

and

$$p(\{u_{11},u_{12}\},Z_1)=\left[\prod_{m=1}^{2}\mathcal{N}(u_{1m}|\hat{u}_{1|0,m},\xi^2_{1|0,m})\right]\times\prod_{m=1}^{2}\left[\gamma_{1m0}+\sum_{r=1}^{n_1}\gamma_{1mr}\,\frac{\mathcal{N}(z_{1r}|u_{1m},\sigma^2_{1rm})}{\mathcal{N}(z_{1r}|\hat{u}_{1|0,m},\xi^2_{1|0,m}+\sigma^2_{1rm})}\right], \tag{4-6}$$

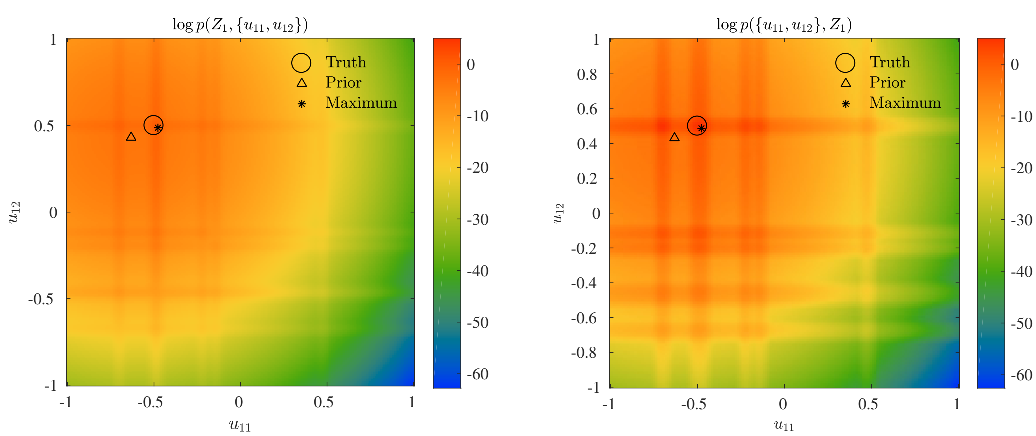
where $\hat{u}_{1|0,m}$ and $\xi^2_{1|0,m}$ denote the *a priori* mean and variance of the mth target's position at scan $t=1$, and σ^2_{1rm} denotes the variance of the rth measurement at $t=1$ under the mth target model.

For example, consider the one-dimensional surveillance region $\mathcal{S}=[-1,1]$, with targets 1 and 2 positioned at $u_{11}=-0.5$ and $u_{12}=0.5$, respectively, at scan $t=1$. Furthermore, assume the *a priori* expected target positions at $t=1$ are given by $\hat{u}_{1|0,1}=-0.6322$ and $\hat{u}_{1|0,2}=0.4314$ with standard deviations for both target positions given by $\xi_{1|0,m}=0.2$. For this example, a target detection probability, P_D, of 0.9, and an expected number of clutter points, λV, of 4 are assumed, where the number of clutter points is Poisson distributed, with a uniform spatial distribution in $\mathcal{S}$, as discussed in appendix A. Finally, target-originated measurements are assumed Gaussian distributed with an error standard deviation, σ_{1rm}, of 0.02 for all measurements and each target.

Plots of the marginal incomplete-data log-PDFs for PMHT and target-centric PMHT, obtained by taking the logarithms of equations (4-3) and (4-6), are shown in figures 1(a) and 1(b), respectively, for the following random draw of target and clutter measurements:

$$Z_1=\{-0.4899,0.4883,-0.6896,-0.4624,-0.1322,-0.2082\}, \tag{4-7}$$

where the first two measurements are target-originated, and the last four are clutter points.



(a) PMHT ***(b) Target-Centric PMHT***

***Figure 1. Marginal Incomplete-Data Log-PDF for (a) PMHT and (b) Target-Centric PMHT for Two Constant-Rate Targets in Clutter, and Measurements Listed in Equation* (4-7)**

The PMHT measurement-to-target *a priori* assignment probabilities, $\{\pi_{1m}\}, m = 0, 1, 2$, and the target-centric target-to-measurement *a priori* conditional assignment probabilities, $\{\gamma_{1mr}\}, m = 1, 2, r = 0, 1, \ldots, 6$, are computed using equations (2-15), (2-16), and (3-23), and are listed in table 1.

Table 1. A Priori Assignment Probabilities for PMHT (Second Column) and Target-Centric PMHT (Last Two Columns) for Example Shown in Figure 1

m	π_{1m}	r	γ_{11r}	γ_{12r}
0	0.8343	0	0.0806	0.0806
1	0.0829	1	0.2762	0.0000
2	0.0829	2	0.0000	0.8951
		3	0.3405	0.0000
		4	0.2483	0.0000
		5	0.0161	0.0183
		6	0.0383	0.0059

The true target positions, $u_{11} = -0.5$ and $u_{12} = 0.5$, are plotted as circles in figure 1, while the *a priori* expected positions, $\hat{u}_{1|0,1}$ and $\hat{u}_{1|0,2}$, are plotted as triangles. The maximum values of the two marginal incomplete-data log-PDFs are marked with asterisks in the plots; the values of target position that maximize these log-PDFs are identical (to within the resolution of the grid used for the plots in figure 1) for both PMHT and target-centric PMHT in this case. In fact, the two log-PDFs are remarkably similar, with the same numbers of local maxima at the same locations in the position state subspace. The results of this example would seem to undermine the claim in reference 5 that the target-centric assignment model should reduce the number of local maxima in the incomplete-data PDF due to clutter, though further study is clearly warranted.

5. TARGET-CENTRIC PMHT RELATIONSHIP TO PDA

For the special case of a single linear-Gaussian target in clutter and a batch length of one scan, target-centric PMHT, a maximum *a posteriori* (MAP) estimator of target state, is closely related to the Probabilistic Data Association (PDA) filter (references 3, 10, and 11), a minimum mean-squared error (MMSE) estimator of target state, though there are differences in the way the two approaches combine measurements using target-to-measurement association probabilities. To simplify notation in this section, the subscript, m, is dropped on all variables indexed by target. Let $\hat{x}_{t-1|t-1}$ and $P_{t-1|t-1}$ denote the state estimate and state estimate covariance matrix, respectively, at scan $t-1$ given all measurements up to this scan, and let $\hat{x}_{t|t-1}$ and $P_{t|t-1}$ denote the predicted state estimate and corresponding covariance matrix at scan, t, given by

$$\hat{x}_{t|t-1} = F_t\,\hat{x}_{t-1|t-1} \tag{5-1}$$

and

$$P_{t|t-1} = F_t P_{t-1|t-1} F_t^T + Q_t, \tag{5-2}$$

respectively. Similarly, let $\hat{z}_{t|t-1}$ denote the predicted measurement at scan, t, given by

$$\hat{z}_{t|t-1} = H_t\,\hat{x}_{t|t-1}. \tag{5-3}$$

Finally, let ν_{tr} and S_{tr} denote the innovation (i.e., measurement residual) and innovation covariance matrix corresponding to the rth measurement at scan, t, given by

$$\nu_{tr} = z_{tr} - \hat{z}_{t|t-1} \tag{5-4}$$

and

$$S_{tr} = H_t P_{t|t-1} H_t^T + R_{tr}, \tag{5-5}$$

respectively.

5.1 PDA STATE UPDATE EQUATION AND ASSOCIATION PROBABILITIES

As shown in reference 11,* the PDA filter update for the target state at scan, t, is given by the predicted target state plus the weighted sum of the innovations at scan, t:

$$\hat{x}_{t|t} = \hat{x}_{t|t-1} + \sum_{r=1}^{n_t} \beta_{tr} W_{tr} \nu_{tr}, \tag{5-6}$$

where β_{tr} is the probability that z_{tr} is the target-originated measurement, and W_{tr} is the Kalman gain matrix conditioned on this measurement, that is,

$$W_{tr} = P_{t|t-1} H_t^T S_{tr}^{-1}. \tag{5-7}$$

*For this report, the modified form of the PDA filter derived in reference 11, which allows for each measurement in a scan to have a different covariance matrix, is used.

The PDA target-to-measurement association probabilities, β_{tr}, for this case are derived by combining the results of references 3 and 11 and take the following form:

$$\beta_{tr} = \begin{cases} \dfrac{c_{t0}}{c_{t0} + \sum_{\rho=1}^{n_t} \mathcal{N}\left(z_{t\rho} | \hat{z}_{t|t-1}, S_{t\rho}\right)}, & r = 0, \\ \dfrac{\mathcal{N}\left(z_{tr} | \hat{z}_{t|t-1}, S_{tr}\right)}{c_{t0} + \sum_{\rho=1}^{n_t} \mathcal{N}\left(z_{t\rho} | \hat{z}_{t|t-1}, S_{t\rho}\right)}, & r = 1, \ldots, n_t, \end{cases} \tag{5-8}$$

where the scan constants, c_{t0}, are functions of the target detection probability, P_D, and clutter density, λ.

Let $\hat{x}_{t|t,r}$ denote the target state update conditioned on measurement z_{tr}, that is,

$$\hat{x}_{t|t,r} = \hat{x}_{t|t-1} + W_{tr}\nu_{tr}, \tag{5-9}$$

and define $\hat{x}_{t|t,0} \equiv \hat{x}_{t|t-1}$. Then, because the association probabilities in equation (5-8) sum to 1 over the measurement index r, equation (5-6) may be written equivalently as

$$\hat{x}_{t|t} = \sum_{r=0}^{n_t} \beta_{tr}\, \hat{x}_{t|t,r}. \tag{5-10}$$

5.2 TARGET-CENTRIC PMHT STATE UPDATE EQUATION AND ASSOCIATION PROBABILITIES

For the case where $M = 1$ and $T = 1$, the conditional expectation of the E-step for target-centric PMHT takes the form

$$\begin{aligned} \Theta(x_t, \hat{x}_{t|t}^{(\ell-1)}) = &- \left(x_t - \hat{x}_{t|t-1}\right)^T P_{t|t-1}^{-1} \left(x_t - \hat{x}_{t|t-1}\right) \\ &- \left(\tilde{z}_t^{(\ell-1)} - H_t x_t\right)^T \left(\tilde{R}_t^{(\ell-1)}\right)^{-1} \left(\tilde{z}_t^{(\ell-1)} - H_t x_t\right), \end{aligned} \tag{5-11}$$

where the composite measurements and corresponding composite measurement covariance matrices defined by equations (3-35) and (3-36) written in the notation of this section are given by

$$\tilde{R}_t^{(\ell-1)} = \left(\sum_{r=1}^{n_t} \xi_{tr}^{(\ell-1)} R_{tr}^{-1}\right)^{-1}, \tag{5-12}$$

$$\tilde{z}_t^{(\ell-1)} = \tilde{R}_t^{(\ell-1)} \sum_{r=1}^{n_t} \xi_{tr}^{(\ell-1)} R_{tr}^{-1} z_{tr}. \tag{5-13}$$

The target-centric PMHT conditional target-to-measurement assignment probabilities probabilities given by equation (3-21) are written in the notation of this section as

$$\xi_{tr}^{(\ell-1)} = \begin{cases} \dfrac{\gamma_{tm0}}{\gamma_{tm0} + \sum_{\rho=1}^{n_t} \frac{\gamma_{tm\rho}}{\eta_{t\rho}} \mathcal{N}(z_{t\rho}|H_t \hat{x}_{t|t}^{(\ell-1)}, R_{t\rho})}, & r = 0, \\ \dfrac{\frac{\gamma_{tmr}}{\eta_{tr}} \mathcal{N}(z_{tr}|H_t \hat{x}_{t|t}^{(\ell-1)}, R_{tr})}{\gamma_{tm0} + \sum_{\rho=1}^{n_t} \frac{\gamma_{tm\rho}}{\eta_{t\rho}} \mathcal{N}(z_{t\rho}|H_t \hat{x}_{t|t}^{(\ell-1)}, R_{t\rho})}, & r = 1, \ldots, n_t, \end{cases} \tag{5-14}$$

with

$$\eta_{tr} = \mathcal{N}(z_{tr}|\hat{z}_{t|t-1}, S_{tr}), \tag{5-15}$$

following from equation (3-29). The form of the conditional expectation in equation (5-11) is functionally equivalent to the posterior log-PDF for the standard Kalman filtering problem with measurements and measurement covariance matrices given by the composite measurements and composite measurement covariance matrices defined by equations (5-12) and (5-13), respectively. Hence, the target state update, $\hat{x}_{t|t}^{(\ell)}$, at the ℓth EM iteration may be obtained by standard Kalman filtering techniques in this special case. Specifically, the target state update at the ℓth EM iteration is given by

$$\hat{x}_{t|t}^{(\ell)} = \hat{x}_{t|t-1} + W_t^{(\ell-1)} \nu_t^{(\ell-1)}, \tag{5-16}$$

where the innovation at scan, t,

$$\nu_t^{(\ell-1)} = \tilde{z}_t^{(\ell-1)} - \hat{z}_{t|t-1}, \tag{5-17}$$

is defined with respect to the composite measurement given by equation (5-13), and the Kalman gain matrix at scan t,

$$W_t^{(\ell-1)} = P_{t|t-1} H_t^T \left(S_t^{(\ell-1)} \right)^{-1}, \tag{5-18}$$

is defined with respect to the composite measurement covariance matrix given by equation (5-12) and the corresponding innovation covariance matrix given by

$$S_t^{(\ell-1)} = H_t P_{t|t-1} H_t^T + \tilde{R}_t^{(\ell-1)}. \tag{5-19}$$

5.3 DISCUSSION

The developments in sections 5.1 and 5.2 reveal a close relationship between the PDA and target-centric PMHT approaches for the case of a single scan of a single target in clutter. In particular, both approaches reduce to Kalman filtering problems for some weighted combination of the target information available in the current scan. In particular, equation (5-10) shows the PDA filter is a weighted combination of Kalman filters—one for each measurement z_{tr}—with weights given by the association probabilities, β_{tr}. In comparison, the target-centric PMHT approach yields a single Kalman filter applied to a weighted combination of the measurements, z_{tr}, with weights given by the target-to-measurement assignment probabilities, $\xi_{tr}^{(\ell-1)}$. Moreover, target-centric PMHT results in an iterated Kalman filter, where the assignment probabilities, $\xi_{tr}^{(\ell)}$, are updated at each iteration, ℓ, according to the target state update from the previous iteration—namely, $\hat{x}_{t|t}^{(\ell-1)}$; by contrast, the PDA association probabilities, β_{tr}, are fixed by the target state update from the previous scan—namely, $\hat{x}_{t|t-1}$.

6. STATE COVARIANCE MATRIX ESTIMATION FOR TARGET-CENTRIC PMHT

Unlike other multitarget tracking approaches, PMHT does not automatically produce covariance matrices for the target state estimates as byproducts of the estimation procedure. Nevertheless, good approximations to these matrices have been proposed and derived (see references 12–14). The PDA-style approximation examined in references 13 and 14 is particularly attractive due to its straightforward implementation and consistency performance. This approximation is also particularly well suited to target-centric PMHT, in light of the close relationship between PDA and target-centric PMHT discussed in section 5.

The PDA-style approximation to the PMHT state covariance matrices derived in references 13 and 14 takes advantage of the fact that the M-step reduces to a bank of Kalman smoothing filters, one for each target state. The target-centric PMHT M-step does not, in general, share this attribute. While it is possible to derive a PDA-style approximation to the target state covariance matrices for target-centric PMHT in the general case, the analysis here is confined to the special case of a batch of length one scan (i.e., $T = 1$); in this case, the M-step for target-centric PMHT reduces to a bank of Kalman filters, as discussed in section 5.2.

6.1 PDA STATE COVARIANCE MATRIX UPDATE

The PDA state covariance matrix update equations for the case where each measurement has its own measurement covariance matrix is derived in reference 11. These equations are summarized as follows.

The PDA state covariance matrix update, $P_{t|t,m}$, decomposes into two terms:

$$P_{t|t,m} = \bar{P}_{t|t,m} + \tilde{P}_{t|t,m}. \tag{6-1}$$

The first term in equation (6-1) is comprised of a combination of the standard Kalman filter state covariance matrix updates for each measurement in the scan processed independently, weighted by the PDA association probabilities given in equation (5-8):

$$\bar{P}_{t|t,m} = \beta_{t0m} P_{t|t-1,m} + \sum_{r=1}^{n_t} \beta_{trm} \bar{P}_{t|t,rm}, \tag{6-2}$$

where

$$\bar{P}_{t|t,rm} = P_{t|t-1,m} - W_{trm} S_{trm}^{-1} W_{trm}^{T} \tag{6-3}$$

is the state covariance matrix update conditioned on the rth measurement in the scan with the innovation covariance matrices, S_{trm}, and Kalman gain matrices, W_{trm}, given by equations (5-5) and (5-7), respectively. The second term in equation (6-1) quantifies the combined spread of the scaled innovations, $W_{trm}\nu_{trm}$, with the innovations given by

$$\nu_{trm} = z_{tr} - \hat{z}_{t|t-1,m}, \tag{6-4}$$

and weighted by the PDA association probabilities given in equation (5-8):

$$\tilde{P}_{t|t,m} = \left[\sum_{r=1}^{n_t} \beta_{trm} W_{trm} \nu_{trm} \nu_{trm}^T W_{trm}^T\right] - \left[\sum_{r=1}^{n_t} \beta_{trm} W_{trm} \nu_{trm}\right] \left[\sum_{r=1}^{n_t} \beta_{trm} W_{trm} \nu_{trm}\right]^T . \tag{6-5}$$

6.2 PDA-STYLE STATE COVARIANCE MATRIX APPROXIMATION FOR PMHT

As described in reference 13, equations (6-1) through (6-5) may be used to approximate the PMHT target state covariance matrices at the leading edge of the batch,* provided appropriate definitions for the association probabilities, β_{trm}, are specified. In reference 13, these association probabilities are defined in terms of the PMHT measurement-to-target assignment probabilities, ω_{trm}, given by equation (2-13), as follows. First, let $\hat{\omega}_{trm}$ denote the value of $\omega_{trm}^{(\ell)}$ upon convergence:

$$\hat{\omega}_{trm} = \lim_{\ell \to \infty} \omega_{trm}^{(\ell)} . \tag{6-6}$$

The probability target, m, associates to no measurement at scan, t, is equal to the joint probability that none of the n_t measurements associates to this target; that is,

$$\beta_{t0m} = \prod_{r=1}^{n_t} \left(1 - \hat{\omega}_{trm}\right) . \tag{6-7}$$

Next, assume the probability target, m, associates to measurement, r, at scan, t, is proportional to the PMHT measurement-to-target assignment, $\hat{\omega}_{trm}$; specifically,

$$\beta_{trm} = c_{tm} \hat{\omega}_{trm}, \quad r = 1, \ldots, n_t, \tag{6-8}$$

for some proportionality constant, c_{tm}, for all values of r. This constant is determined by observing the PDA association probabilities, β_{trm}, are target-to-measurement assignment probabilities and, hence, satisfy the constraint

$$\sum_{r=0}^{n_t} \beta_{trm} = 1. \tag{6-9}$$

Substituting equations (6-7) and (6-8) into equation (6-9) and solving for c_{tm} yields

$$c_{tm} = \frac{1 - \beta_{t0m}}{\sum_{r=1}^{n_t} \hat{\omega}_{trm}} . \tag{6-10}$$

*Extensions to equations (6-1) through (6-5) to include approximations for the target state covariance matrices for the smoothing problem are provided in reference 14.

6.3 PDA-STYLE STATE COVARIANCE MATRIX APPROXIMATION FOR TARGET-CENTRIC PMHT

Equations (6-1) through (6-5) may also be used to approximate target state covariance matrices in the target-centric PMHT approach for the special case of a batch of length one, where the M-step reduces to a bank of Kalman filters, one for each target state. In this approach, the target-centric PMHT target-to-measurement assignment probabilities, ξ_{trm}, given by equation (3-21) can be substituted directly for the PDA association probabilities, β_{trm}, in these expressions because both sets of probabilities describe the same set of target-to-measurement assignments, as discussed in section 5.1. Formally, let $\hat{\xi}_{trm}$ denote the value of $\xi_{trm}^{(\ell)}$ upon convergence:

$$\hat{\xi}_{trm} = \lim_{\ell \to \infty} \xi_{trm}^{(\ell)}. \tag{6-11}$$

Then, setting

$$\beta_{trm} = \hat{\xi}_{trm}, \quad r = 0, 1, \ldots, n_t, \tag{6-12}$$

in equations (6-1) through (6-5) will yield PDA-style approximations to the target-centric PMHT target state estimates for this special case. However, for the case of a batch length of $T > 1$, the M-step for target-centric PMHT does not reduce to a Kalman smoothing problem, hence the analogous approach to equations (6-1) through (6-5) derived in reference 14 does not directly apply. Nevertheless, the observed-information based approach derived in reference 12 would be applicable in this case.

7. A TARGET-CENTRIC/MEASUREMENT-CENTRIC PMHT HYBRID

Equation (3-21) of the target-centric PMHT derivation in section 3, repeated here for convenience,

$$\xi_{trm} = P\{j_{tm} = r | x_{tm}, z_{tr}\} = \begin{cases} \dfrac{\gamma_{tm0}}{\gamma_{tm0} + \sum_{\rho=1}^{n_t} \gamma_{tm\rho} \frac{p_m(z_{t\rho}|x_{tm})}{p_m(z_{t\rho}|x_{t-1,m})}}, & r = 0, \\ \dfrac{\gamma_{tmr} \frac{p_m(z_{tr}|x_{tm})}{p_m(z_{tr}|x_{t-1,m})}}{\gamma_{tm0} + \sum_{\rho=1}^{n_t} \gamma_{tm\rho} \frac{p_m(z_{t\rho}|x_{tm})}{p_m(z_{t\rho}|x_{t-1,m})}}, & r = 1, \ldots, n_t, \end{cases} \quad (7\text{-}1)$$

reveals the conditional probability target, m, associates to measurement, r, at scan, t, is independent of the other $M-1$ target states. This results in an estimation problem for the target states, X^m, given by equations (3-26) through (3-28), which is also independent of the other targets. Consequently, it is reasonable to expect the target-centric PMHT approach to, at a minimum, have difficulty resolving closely spaced targets with perhaps similar prior PDFs. To address this potential deficiency, two alternatives for blending the target-centric and measurement-centric approaches are proposed. The first alternative is to alternate between the two approaches between EM iterations. The motivation behind this alternative is to (almost) simultaneously take advantage of PMHT's inherent one-target-per-measurement model with its focus on sorting measurements between targets and clutter, and target-centric PMHT's inherent one-measurement-per-target model with its focus on sorting targets between measurements. The procedure would amount to finding the target states, $(X^m)^{(\ell)}$, that maximize the PMHT auxiliary function (equation (2-32)) at, for example, odd-numbered iterations, and those that maximize the target-centric PMHT auxiliary function (equation (3-39)) at even-numbered iterations. The procedure would also no longer constitute an EM algorithm and no longer guarantee a monotonic increase in the incomplete-data PDF at each iteration.

The second alternative is to form a new set of target-to-measurement assignment probabilities for target-centric PMHT that couple the target interactions across all measurements in a scan. This may be accomplished by the same argument used in section 6.2 to create association probabilities for the PDA-style state covariance matrix approximation for PMHT. Equation (2-13) of the original (measurement-centric) PMHT derivation in section 2, also repeated here for convenience,

$$\omega_{trm} = P\{k_{tr} = m | z_{tr}, x_{tm}\} = \begin{cases} \dfrac{\pi_{t0}\, p_0(z_{tr}|x_{t0})}{\pi_{t0}\, p_0(z_{tr}|x_{t0}) + \sum_{l=1}^{M} \pi_{tl}\, p_l(z_{tr}|x_{tl})}, & m = 0 \\ \dfrac{\pi_{tm}\, p_m(z_{tr}|x_{tm})}{\pi_{t0}\, p_0(z_{tr}|x_{t0}) + \sum_{l=1}^{M} \pi_{tl}\, p_l(z_{tr}|x_{tl})}, & m = 1, \ldots, M, \end{cases} \quad (7\text{-}2)$$

reveals the conditional probability measurement, r, associates to target, m, at scan, t, depends on all M target states and the clutter distribution. Using these probabilities and Assumption 4, the probability no measurement associates to target, m, at scan, t, is given by

$$\omega_{t0m} \equiv \prod_{\rho=1}^{n_t} \left(1 - \omega_{t\rho m}\right). \tag{7-3}$$

In light of equation (7-3), define a new conditional probability target, m, associates to no measurement at scan, t, denoted $\breve{\xi}_{t0m}$, as

$$\breve{\xi}_{t0m} \equiv \omega_{t0m}. \tag{7-4}$$

Furthermore, define a new conditional probability target, m, associates to measurement, r, at scan, t, $r = 1, \ldots, n_t$, denoted $\breve{\xi}_{trm}$, that is proportional to the conditional probability measurement, r, at scan, t, associates to target, m:

$$\breve{\xi}_{trm} \equiv u_{tm}\omega_{trm}, \quad r = 1, \ldots, n_t, \tag{7-5}$$

where u_{tm} is the proportionality constant. This constant is determined by observing the target-to-measurement assignment probabilities, $\breve{\xi}_{trm}$, satisfy the constraint

$$\sum_{r=0}^{n_t} \breve{\xi}_{trm} = 1. \tag{7-6}$$

Substituting equations (7-4) and (7-5) into equation (7-6) and solving for u_{tm} yields

$$u_{tm} = \frac{1 - \breve{\xi}_{t0m}}{\sum_{r=1}^{n_t} \omega_{trm}}. \tag{7-7}$$

Substituting the new conditional probabilities, $\breve{\xi}_{trm}$, for the original conditional probabilities, ξ_{trm}, in equation (3-26) leads to an estimation problem for the target states, X^m, at EM iteration, ℓ, that now depends on the estimates off all of the target states, $\{X^m : m = 1, \ldots, M\}$, obtained at iteration $\ell - 1$.

Both approaches to a hybrid assignment model for PMHT are easily implemented and warrant further investigation.

8. CONCLUDING REMARKS AND FUTURE INVESTIGATIONS

In this report an alternative, target-centric assignment model for PMHT is investigated, in which targets are assigned to measurements; this model is the dual of the original PMHT, measurement-centric assignment model, in which measurements are assigned to targets. An explicit state estimation algorithm for target-centric PMHT is derived for the linear-Gaussian state-space model. Unlike for measurement-centric PMHT, this algorithm does not, in general, reduce to an iterative Kalman smoothing problem for each target (except for a batch length of one). Nevertheless, target-centric PMHT remains an iterative, linear estimation problem.

The relationship between target-centric PMHT and PDA filtering, to which it is closely related for a batch length of one and a single target in clutter, is also investigated in this report. Both approaches are shown to reduce to Kalman filtering problems for some weighted combination of the target information available in the current scan, with weights given by the familiar association probabilities for PDA, β_{tr}, and by the conditional target-to-measurement assignment probabilities for target-centric PMHT, ξ_{tr}, both of which share remarkably similar forms (compare equations (5-8) and (5-14)). Additionally, an approximation to the target state estimate covariance matrix is derived for single-scan target-centric PMHT based on the PDA-style approximation proposed for PMHT in reference 13. Given this close relationship between the association and assignment probabilities, β_{tr} and ξ_{tr}, respectively, this approximation is particularly well-suited to target-centric PMHT. Extension of this approximation to the multiscan (i.e., batch length greater than one) case is left for future investigations.

The target-centric assignment model for PMHT is not new; to the authors' knowledge, reference 5 is the first open-literature report of the approach. Nevertheless, there are many details in reference 5 that are either omitted, unclear, or not entirely justified. This report attempts to address each of these issues. For instance, the authors in reference 5 claim, without proof or demonstration, that the target-centric assignment model should reduce the number of local maxima in the incomplete-data PDF compared to its measurement-centric counterpart. However, the example provided in section 4 of this report would seem to undermine this claim, as the target-centric and measurement-centric incomplete-data PDFs for the example are nearly identical with respect to the numbers and locations of local maxima. Further study of this behavior is clearly warranted.

Lastly, a hybrid target-centric/measurement-centric approach is investigated to address a potential deficiency of the target-centric model wherein the conditional probability a target associates to a measurement at a particular scan is independent of the other targets, effectively leading to independent estimation problems for each target. To address this issue, two alternatives are proposed for blending the two approaches: one based on swapping between the target-centric and measurement-centric assignment models between EM iterations, and one based on re-normalizing the target-to-measurement assignment probabilities for target-centric PMHT in a way—using ideas similar to those in reference 13—that couples the target interactions across all measurements in a scan. Both approaches are easily implemented, but are left for future investigation.

APPENDIX A
A PRIORI ASSIGNMENT PROBABILITIES FOR TARGET-CENTRIC PMHT

Expressions for the data-dependent *a priori* target-to-measurement assignment probabilities, γ_{tmr}, consistent with the target detection probability, P_D, (assumed equal for all targets) and the clutter density, λ, (i.e., the expected number of false alarms per unit volume, V, of the surveillance region) are derived in this appendix. In the sequel, let μ denote the distribution of the number of false alarms, η, in the surveillance region. For example, if η is assumed Poisson distributed with mean, λV, then

$$\mu(\eta) = \frac{(\lambda V)^\eta}{\eta!} \, \mathrm{e}^{-\lambda V}, \quad \eta = 0, 1, \ldots . \tag{A-1}$$

Recall, from equation (3-6), for each target $m \in \{1, \ldots, M\}$,

$$\gamma_{tmr} = P\{j_{tm} = r | Z_t\}, \quad r = 0, 1, \ldots, n_t. \tag{A-2}$$

The following derivation for γ_{tmr}, conditioned on the scan measurements, Z_t, follows that of the *a priori* PDA event probabilities presented in section 3.4.10, p. 136 of reference 3 and assumes:

1. There is at most one measurement per target;
2. There are at most M target measurements;
3. Missed detections are evenly distributed among the M targets;
4. The probability of assigning the mth target to the rth measurement at scan, t, is proportional to $p_m(z_{tr})$, where

$$p_m(z_{tr}) = \int p_m(z_{tr}|x_{tm}) p_m(x_{tm}) dx_{tm}. \tag{A-3}$$

First, an expression for γ_{tm0}, the *a priori* probability target, m, is assigned to no measurement at scan, t, is derived, followed by expressions for γ_{tmr}, $r = 1, 2, \ldots, n_t$, in terms of γ_{tm0}. In the sequel, let n_t^* and n_t^c denote the numbers of target measurements and clutter points observed, respectively, so that

$$n_t = n_t^* + n_t^c, \tag{A-4}$$

and let $\tilde{n}_t = \min\{n_t, M\}$, where M is the number of targets. Then, for $r = 0$,

$$\begin{aligned} P\{j_{tm} = 0|Z_t\} &= P\{j_{tm} = 0|n_t\} \\ &= P\{j_{tm} = 0|n_t, n_t^* = 0\} P\{n_t^c = n_t|n_t\} \\ &\quad + P\{j_{tm} = 0|n_t, n_t^* = 1\} P\{n_t^c = n_t - 1|n_t\} \\ &\quad \vdots \\ &\quad + P\{j_{tm} = 0|n_t, n_t^* = \tilde{n}_t\} P\{n_t^c = n_t - \tilde{n}_t|n_t\}. \end{aligned} \tag{A-5}$$

By assumptions 2 through 3, the preceding expression becomes

$$\begin{aligned} P\{j_{tm} = 0|n_t\} = P\{n_t^c = n_t|n_t\} \\ &+ \frac{M-1}{M} P\{n_t^c = n_t - 1|n_t\} \\ &\vdots \\ &+ \frac{M-\tilde{n}_t}{M} P\{n_t^c = n_t - \tilde{n}_t|n_t\}. \end{aligned} \tag{A-6}$$

Now, for $l = 0, 1, \ldots, \tilde{n}_t$,

$$\begin{aligned} P\{n_t^c = n_t - l|n_t\} &= \frac{P\{n_t|n_t^c = n_t - l\} P\{n_t^c = n_t - l\}}{P\{n_t\}} \\ &= \frac{P\{n_t^* = l\} P\{n_t^c = n_t - l\}}{P\{n_t\}} \\ &= \frac{\binom{M}{l} P_D^l (1 - P_D)^{M-l} \mu(n_t - l)}{P\{n_t\}}. \end{aligned} \tag{A-7}$$

To determine $P\{n_t\}$, observe

$$1 = \sum_{l=0}^{\tilde{n}_t} P\{n_t^c = n_t - l|n_t\} = \frac{\sum_{l=0}^{\tilde{n}_t} \binom{M}{l} P_D^l (1 - P_D)^{M-l} \mu(n_t - l)}{P\{n_t\}}, \tag{A-8}$$

from which it follows

$$P\{n_t\} = \sum_{l=0}^{\tilde{n}_t} \binom{M}{l} P_D^l (1 - P_D)^{M-l} \mu(n_t - l). \tag{A-9}$$

Combining equations (A-6), (A-7), and (A-9) yields, after some simplification,

$$\gamma_{tm0} = P\{j_{tm} = 0|n_t\} = 1 - \frac{1}{M} \frac{\sum_{l=0}^{\tilde{n}_t} l \binom{M}{l} P_D^l (1 - P_D)^{M-l} \mu(n_t - l)}{\sum_{l=0}^{\tilde{n}_t} \binom{M}{l} P_D^l (1 - P_D)^{M-l} \mu(n_t - l)}. \tag{A-10}$$

By assumption 4, the *a priori* assignment probabilities γ_{tmr} for $r = 1, 2, \ldots, n_t$ are given by

$$\gamma_{tmr} = c_{tm} p_m(z_{tr}), \quad r = 1, 2, \ldots, n_t, \tag{A-11}$$

where c_{tm} is the proportionality constant for the mth target at the tth scan. This proportionality constant is determined from the identity

$$\sum_{r=0}^{n_t} \gamma_{trm} = \sum_{r=0}^{n_t} P\{j_{tm} = r|Z_t\} = 1. \tag{A-12}$$

Substituting equation (A-11) into equation (A-12) and solving for c_{tm} yields

$$c_{tm} = \frac{1 - \gamma_{tm0}}{\sum_{r=1}^{n_t} p_m(z_{tr})}. \tag{A-13}$$

Finally, substituting equation (A-13) into equation (A-11) gives the data-dependent target-to-measurement assignment probabilities, γ_{trm}, for $r = 1, 2, \ldots, n_t$ in terms of the "miss" probability, γ_{tm0}, as

$$\gamma_{tmr} = \frac{p_m(z_{tr})}{\sum_{\rho=1}^{n_t} p_m(z_{t\rho})}(1 - \gamma_{tm0}), \quad r = 1, 2, \ldots, n_t, \tag{A-14}$$

with γ_{tm0} given in terms of the target detection probability, P_D, and expected number of clutter points, λV, by equation (A-10).

From equations (A-10) and (A-14), the following special cases are easily verified:

$$\gamma_{tmr} = \begin{cases} 0, & r = 0, \\ \dfrac{p_m(z_{tr})}{\sum_{\rho=1}^{n_t} p_m(z_{t\rho})}, & r = 1, 2, \ldots, n_t, \end{cases}, \quad P_D = 1, \tag{A-15}$$

$$\gamma_{tmr} = \begin{cases} 1, & r = 0, \\ 0, & r = 1, \ldots, n_t, \end{cases}, \quad P_D = 0. \tag{A-16}$$

The measurement PDFs, $p_m(z_{tr})$, in equation (A-14) are given, in general, by the integrals

$$p_m(z_{tr}) = \int p_m(z_{tr}|x_{tm})\, p_m(x_{tm})\, dx_{tm}. \tag{A-17}$$

For the linear-Gaussian case, these integrals are easily evaluated in closed-form using the PDFs given by equations (2-23) through (2-25) and the Gaussian Refactorization Lemma and its correlate (equation (B-8)) given in appendix B. For $t = 1$, closed-form expressions for these integrals (the same integrals given by equation (3-10)) are given by equation (3-29). Analogous results are obtained for $t = 2, 3, \ldots, n_t$ by an induction argument. Indeed, for $t = 2$, the likelihood functions, $p_m(z_{2r}|x_{2m})$, in equation (A-17) are given by

$$p_m(z_{2r}|x_{2m}) = \mathcal{N}(z_{2r}|H_{2m}x_{2m}, R_{2rm}), \tag{A-18}$$

and the target state PDFs, $p_m(x_{2m})$, in equation (A-17) are evaluated as follows:

$$\begin{aligned} p_m(x_{2m}) &= \int p_m(x_{2m}|x_{1m})\, p_m(x_{1m})\, dx_{1m}, \\ &= \int \mathcal{N}(x_{2m}|F_{2m}x_{1m}, Q_{2m})\, \mathcal{N}(x_{1m}|\hat{x}_{1|0,m}, P_{1|0,m})\, dx_{1m}, \\ &= \mathcal{N}(x_{2m}|F_{2m}\hat{x}_{1|0,m}, F_{2m}P_{1|0,m}F_{2m}^T + Q_{2m}), \\ &= \mathcal{N}(x_{2m}|\hat{x}_{2|0,m}, P_{2|0,m}), \end{aligned} \tag{A-19}$$

where the predicted target states, $\hat{x}_{2|0,m}$, and corresponding covariance matrices, $P_{2|0,m}$, are defined analogously to the predictions at $t = 1$, as in equation (2-26):

$$\begin{aligned} \hat{x}_{2|0,m} &= F_{2m}\hat{x}_{1|0,m}, \\ P_{2|0,m} &= F_{2m}P_{1|0,m}F_{2m}^T + Q_{2m}. \end{aligned} \tag{A-20}$$

Substituting equations (A-18) and (A-19) into equation (A-17), using the refactorization lemma, and evaluating the resulting integrals using equation (B-8) yields

$$p_m(z_{2r}) = \mathcal{N}(z_{2r}|H_{2m}\hat{x}_{2|0,m}, H_{2m}P_{2|0,m}H_{2m}^T + R_{2rm}), \tag{A-21}$$

which is identical in form to the result in equation (3-29) for $t = 1$. Analogous results are obtained for $t = 3, 4, \ldots, n_t$ by induction. In general, for $t = 1, 2, \ldots, n_t$,

$$p_m(z_{tr}) = \mathcal{N}(z_{tr}|H_{tm}\hat{x}_{t|0,m}, H_{tm}P_{t|0,m}H_{tm}^T + R_{trm}) \tag{A-22}$$

with

$$\begin{aligned} \hat{x}_{t|0,m} &= F_{tm}\hat{x}_{t-1|0,m}, \\ P_{t|0,m} &= F_{tm}P_{t-1|0,m}F_{tm}^T + Q_{tm}. \end{aligned} \tag{A-23}$$

APPENDIX B
GAUSSIAN REFACTORIZATION LEMMA

The Gaussian Refactorization Lemma stated and proved in reference 9 is useful for obtaining several marginal PDFs required in the derivation of target-centric PMHT. This lemma is stated below for convenient reference.

Lemma. *Given the L-dimensional vector x, the M-dimensional vector y, appropriately sized nonsingular covariance matrices S and P, and the $M \times L$ matrix F, the joint PDF*

$$\eta(y, x) = \mathcal{N}(y|Fx, S)\,\mathcal{N}(x|\mu, P) \tag{B-1}$$

can be refactored as

$$\eta(y, x) = \mathcal{N}(y|\omega, \Omega)\,\mathcal{N}(x; \lambda, \Lambda), \tag{B-2}$$

where the mean vectors and covariance matrices in the resulting product of Gaussian PDFs are given by

$$\omega = F\mu, \tag{B-3}$$

$$\Omega = S + FPF^T, \tag{B-4}$$

$$\lambda = (I - HF)\mu + Hy, \tag{B-5}$$

$$\Lambda = (I - HF)P, \tag{B-6}$$

with the matrix H given by

$$H = PF^T\Omega^{-1}. \tag{B-7}$$

This lemma is extremely useful when the marginal PDF of equation (B-1) over the random vector, x, is required. In this case, integrating equation (B-2) instead with respect to x easily yields

$$\int_{\mathbb{R}^L} \eta(y, x)\, dx = \mathcal{N}(y|\omega, \Omega), \tag{B-8}$$

because

$$\int_{\mathbb{R}^L} \mathcal{N}(x|\mu, P)\, dx = 1 \tag{B-9}$$

by properties of PDFs.

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