ARL-TR-9150 ● FEB 2021

Boundary Charge Layer and Current for Spherical Inclusion in Conducting System

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ARL-TR-9150 ● FEB 2021

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REPORT DOCUMENTATION PAGE			*Form Approved* *OMB No. 0704-0188*

1. REPORT DATE *(DD-MM-YYYY)*	**2. REPORT TYPE**	**3. DATES COVERED (From - To)**
February 2021	Technical Report	1 July 2020–10 January 2021

4. TITLE AND SUBTITLE Boundary Charge Layer and Current for Spherical Inclusion in Conducting System	**5a. CONTRACT NUMBER**
	5b. GRANT NUMBER
	5c. PROGRAM ELEMENT NUMBER
6. AUTHOR(S) Michael Grinfeld and Pavel Grinfeld	**5d. PROJECT NUMBER**
	5e. TASK NUMBER
	5f. WORK UNIT NUMBER
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) DEVCOM Army Research Laboratory ATTN: FCDD-RLW-PA Aberdeen Proving Ground, MD 21005	**8. PERFORMING ORGANIZATION REPORT NUMBER** ARL-TR-9150
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)	**10. SPONSOR/MONITOR'S ACRONYM(S)**
	11. SPONSOR/MONITOR'S REPORT NUMBER(S)

12. DISTRIBUTION/AVAILABILITY STATEMENT

Approved for public release: distribution unlimited.

13. SUPPLEMENTARY NOTES

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14. ABSTRACT

The steady-state current in a heterogeneous conducting system is considered. One of the goals is to generate a battery of exact solutions for Validation and Verification (V&V) purposes. These solutions are to be used for the electromagnetic software (as of now, for the Sandia package ALEGRA). The system consists of an isotropic unbounded matrix containing spherical inclusion (the inclusion can be isotropic or anisotropic and even nonlinear). This solution was suggested and used for V&V in technical reports published in 2020. The solutions of these previous reports, however, miss analysis of the boundary charge layer and the current in this layer. In this report, we establish these elements of the exact solution.

15. SUBJECT TERMS

electric current in conductors, irreversible thermodynamics, ALEGRA verification, boundary value problems, exact solution

16. SECURITY CLASSIFICATION OF:			**17. LIMITATION OF ABSTRACT**	**18. NUMBER OF PAGES**	**19a. NAME OF RESPONSIBLE PERSON** Michael Grinfeld
a. REPORT	**b. ABSTRACT**	**c. THIS PAGE**			**19b. TELEPHONE NUMBER (Include area code)**
Unclassified	Unclassified	Unclassified	UU	19	(410) 278-7030

Standard Form 298 (Rev. 8/98)
Prescribed by ANSI Std. Z39.18

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1. Introduction

The steady-state current in a heterogeneous conducting system is considered. Fundamentals and early history of the analysis of steady-state electric current can be found in the classic monographs.[1,2]

One of the goals of this effort is to generate a battery of exact solutions for Validation and Verification (V&V) purposes. These solutions are to be used for the electromagnetic software (as of now, for the Sandia package ALEGRA Magnetohydrodynamic [or MHD] and Full Maxwell Hydrodynamics [or FMHD]). The system consists of an isotropic, unbounded matrix containing spherical inclusion (the inclusion can be either isotropic or anisotropic and even nonlinear). These solutions were suggested and used for V&V in previous reports.[3,4] The solutions of previous reports,[3,4] however, miss the analysis of the boundary charge layer and the current in this layer. In this report, we explore these elements of the exact solutions. After establishing the general steady-state solution we proceed with considering various asymptotic situations with high and low conductance of the matrix and inclusion and the high and low conductance of the boundary layer.

2. Mathematical Formulation of the Problem

We remind the reader of the exact solution of the report[3] from which we borrow our notation and also Fig. 1. Consider a spherical conductor embedded in an infinite conducting space, as shown in Fig. 1.

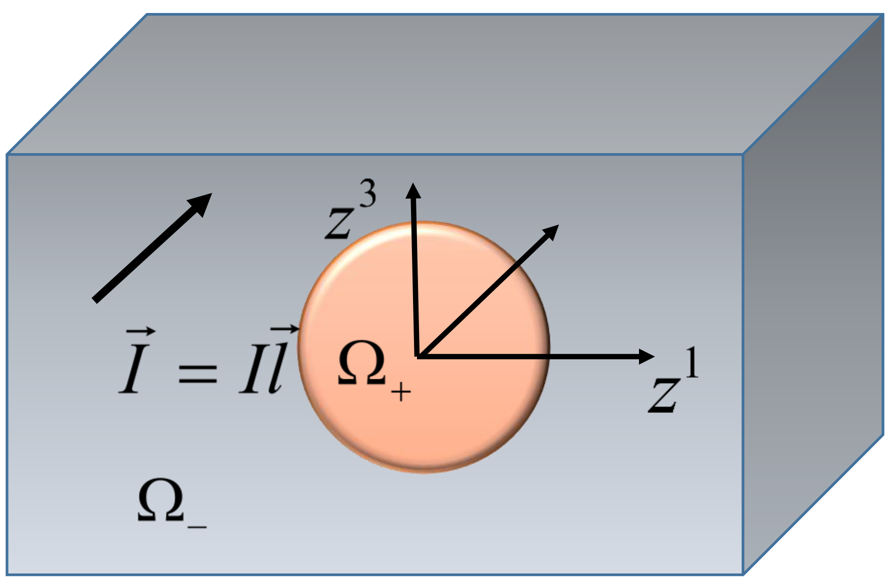

Fig. 1 Conducting inclusion within conducting matrix

Let Ω_+ be the domain inside the inclusion, and Ω_- be the domain outside the inclusion. The electrostatic potential φ satisfies the Laplace equation everywhere (inside and outside the inclusion)

$$\nabla_i \nabla^i \varphi = 0 \tag{1}$$

At the boundary S the following conditions should be satisfied

$$[\varphi]_-^+ = 0 \tag{2}$$

and

$$[I_i]_-^+ N^i = 0 \tag{3}$$

where I_i are the components of the electric current and N^i are the components of the normal to the boundary.

At infinity, we use the following condition

$$I_i(z) \to Il_i = I_i^\infty \text{ at } |z^i| \to \infty \tag{4}$$

It is assumed the matrix is made of a linear isotropic conductor; this means the current I^i is connected with the potential gradient $\nabla_i \varphi$ via the classical Ohm's law

$$I_i = -\Sigma_{mat} \nabla_i \varphi \tag{5}$$

where Σ_{mat} is a positive constant called the conductivity. We assume a much more general conductivity law inside the inclusion

$$I_i = F_i(\nabla_m \varphi) \tag{6}$$

where F_i is an arbitrary vector-function of the gradient of the electrostatic potential.

When F_i is a linear vector-function, we get

$$F_i(\nabla_m \varphi) = -\Sigma_{ij} \nabla^j \varphi \;\to\; I_i = -\Sigma_{ij} \nabla^j \varphi \tag{7}$$

where Σ_{ij} is the conductivity tensor. It is usually assumed (using different arguments) that this tensor is positive and symmetric. When the matrix is isotropic we get, by definition:

$$\Sigma_{ij} = \Sigma_{inc} z_{ij} \tag{8}$$

and Eqs. 6 and 7 lead to the standard Ohm's law

$$I_i = -\Sigma_{inc} \nabla_i \varphi \tag{9}$$

In this case, we can rewrite the current continuity Eq. 3 in the form

$$\Sigma_{mat} \nabla_i \varphi N^i = \Sigma_{inc} \nabla_i \varphi N^i \tag{10}$$

However, in general, we neither need the assumption of isotropy or linearity of the inclusion. In this general situation, the boundary condition of the current continuity reads

$$\sigma_{mat} \nabla_i \varphi N^i = F_i(\nabla_m \varphi) N^i \tag{11}$$

where $\Phi(|\nabla \varphi|)$ is a certain function of the module of the potential gradient.

When the inclusion is isotropic but still nonlinear, we get

$$F_i(\nabla_m \varphi) = \Phi(|\nabla \varphi|) \nabla_i \varphi \tag{12}$$

When the inclusion is isotropic but still nonlinear, Eq. 11 should be replaced with the following one:

$$\Sigma_{mat} \nabla_i \varphi N^i = \Phi(|\nabla \varphi|) \nabla_i \varphi N^i \tag{13}$$

In the case of a linear isotropic inclusion, we get

$$\Phi(|\nabla \varphi|) = \Sigma_{inc} \tag{14}$$

whereas Eq. 13 should be replaced with the following one

$$\Sigma^{mat} \nabla_i \varphi N^i = \Sigma_{ij}^{inc} \nabla^j \varphi N^i \tag{15}$$

If the inclusion is isotropic (i.e., when $\Sigma_{ij}^{inc} = \Sigma^{inc} z_{ij}$), Eq. 15 can be rewritten as

$$\Sigma^{mat} \nabla_i \varphi N^i = \Sigma^{inc} \nabla_i \varphi N^i \tag{16}$$

3. Exact Solution for Spherical Inclusion (Vanishing Boundary Conductivity)

In the absence of boundary current, the master system (Eqs. 1–16) has been analyzed in Grinfeld et al.[3] For the case of unbounded linear isotropic matrix and the spherical isotropic inclusion, the solution is the following:

$$\varphi_{mat}\left(z\right)=\left[A_i\frac{1}{r^3}-\frac{1}{\Sigma_{mat}}I_i^{\infty}\right]z^i \text{ outside the sphere} \tag{17}$$

and

$$\varphi_{inc}\left(z\right)=K_i z^i \text{ inside the sphere} \tag{18}$$

The constant vectors A_i and K_i appeared to be the following:

$$A_i\frac{1}{R^3}=\frac{\Sigma_{inc}\Sigma_{mat}^{-1}-1}{\Sigma_{inc}+2\Sigma_{mat}}I_i^{\infty},\quad K_i=-\frac{3}{\Sigma_{inc}+2\Sigma_{mat}}I_i^{\infty} \tag{19}$$

Combining Eqs. 17–19 we arrive at the following solution for the field potential

a) outside the sphere

$$\varphi_{mat}(z)=\frac{1}{\Sigma_{mat}}\left[\frac{\Sigma_{inc}-\Sigma_{mat}}{\Sigma_{inc}+2\Sigma_{mat}}\frac{R^3}{r^3}-1\right]I_i^{\infty}z^i$$

b) inside the sphere

$$\varphi_{inc}(z)=-\frac{3}{\Sigma_{inc}+2\Sigma_{mat}}I_i^{\infty}z^i \tag{20}$$

4. Exact Solution for Spherical Inclusion (Finite Boundary Conductivity)

According to the classical theory of electric conductors by Landau and Lifshitz,[3] at equilibrium there are no free electric charges inside conductors. All of the charges are distributed at the boundaries with the finite surface density τ. Generally speaking, the density τ is not uniform (i.e., it changes from point to point).

When there are stationary currents in the conductors, the free charges do exist in the bulk of the conductor. This does not mean, though, there are no charges and currents concentrated at the interfaces. In this section, we calculated those distributed charges and currents.

The electric field E_i experiences finite jump at the boundary, and the limit values $E_{i\pm}$ of the density at the interface is connected with the charge density τ by means of the classical relationship

$$4\pi\tau = \left[\nabla_i \varphi\right]_-^+ N^i \tag{21}$$

Using the solution Eq. 20, we arrive at the following relationship for the surface charge density:

$$\tau = \frac{3}{4\pi\Sigma_{mat}} \frac{\Sigma_{inc} - \Sigma_{mat}}{\Sigma_{inc} + 2\Sigma_{mat}} I_i^\infty N^i \tag{22}$$

It is important to realize that Eq. 22 relies on the assumption there are no surface currents. Let us get rid of this assumption. Let $\vec{i}$ be the vector of the surface current. It is a vector parallel to the tangent plane of the interface matrix/inclusion. For the steady-state currents, we postulate the following surface balance equation for the interface:

$$\left[I^i\right]_-^+ N_i = div_\Sigma \vec{i}\ , \tag{23}$$

where div_Σ is a symbol of surface divergence of a surface vector field.

The left-hand side of Eq. 23 expresses resulting influx of electric charges from both sides of the interface. The right-hand side of Eq. 23 expresses the local divergence of surface fluxes due to surface fluxes. In the steady-state flows these two effects should be in balance. When interface currents are equal to zero, the right-hand side of Eq. 23 vanishes and this equation reduces to Eq. 3.

Also, we have to add the analogy of the Ohm's law for the surface flux. For isotropy conductors we can postulate the Ohm's law in the following form:

$$\vec{i} = \sigma \vec{E}_\parallel \tag{24}$$

where σ is the surface conductivity constant and $\vec{E}_\parallel$ is a tangent component of the steady electric field.

The steady-state electric field can still be presented in the form of Eqs. 17 and 18, but the values of the vectors A_i and K_i will be different as compared with the case of nonconducting interface; namely, we get now

$$A_i = \frac{R^3}{\Sigma_{mat}} \frac{\Sigma_{inc} - \Sigma_{mat} + 2\sigma R^{-1}}{\Sigma_{inc} + 2\Sigma_{mat} + 2\sigma R^{-1}} I_i^\infty \tag{25}$$

and

$$K_i = -\frac{3}{\Sigma_{inc} + 2\Sigma_{mat} + 2\sigma R^{-1}} I_i^\infty \tag{26}$$

Using Eqs. 25 and 26, we arrive at the following expressions of electrostatic field:

a) $r > R$:

$$E_k^{mat}(z) = \frac{1}{\Sigma_{mat}} \left\{ \delta_k^i - \frac{\Sigma_{inc} - \Sigma_{mat} + 2\sigma R^{-1}}{\Sigma_{inc} + 2\Sigma_{mat} + 2\sigma R^{-1}} \frac{R^3}{r^3} \left[\delta_k^i - 3\frac{1}{r^2} z_{kj} z^j z^i \right] \right\} I_i^\infty \tag{27}$$

and

b) $r < R$

$$E_i^{inc}(z) = \frac{1}{\Sigma_{mat}} \frac{3\Sigma_{mat}}{\Sigma_{inc} + 2\Sigma_{mat} + 2\sigma R^{-1}} I_i^\infty \tag{28}$$

Using Eq. 27, we get for E_k^{mat} at the interface

$$E_k^{mat} = \frac{I_i^\infty}{\Sigma_{mat}} \left\{ \delta_k^i - \frac{\Sigma_{inc} - \Sigma_{mat} + 2\sigma R^{-1}}{\Sigma_{inc} + 2\Sigma_{mat} + 2\sigma R^{-1}} \left(\delta_k^i - 3N_k N^i \right) \right\} \tag{29}$$

Using Eqs. 28 and 29, we get

$$E_k^{inc} N^k = \frac{I_k^\infty}{\Sigma_{mat}} N^k \frac{3\Sigma_{mat}}{\Sigma_{inc} + 2\Sigma_{mat} + 2\sigma R^{-1}} \tag{30}$$

and

$$E_k^{mat} N^k = \frac{3I_k^\infty}{\Sigma_{mat}} N^k \frac{\Sigma_{inc} + 2\sigma R^{-1}}{\Sigma_{inc} + 2\Sigma_{mat} + 2\sigma R^{-1}} \tag{31}$$

Inserting Eqs. 30 and 31 in Eq. 21, we arrive at the following relationship of the interface charge density τ:

$$\tau = \frac{3}{4\pi} \frac{I_k^\infty}{\Sigma_{mat}} N^k \frac{\Sigma_{inc} - \Sigma_{mat} + 2\sigma R^{-1}}{\Sigma_{inc} + 2\Sigma_{mat} + 2\sigma R^{-1}} \tag{32}$$

Using Eqs. 27 and 28, we get for the bulk currents

a) inside matrix

$$I_k^{mat}(z) = I_i^{\infty}\left\{\delta_k^i - \frac{\Sigma_{inc} - \Sigma_{mat} + 2\sigma R^{-1}}{\Sigma_{inc} + 2\Sigma_{mat} + 2\sigma R^{-1}}\frac{R^3}{r^3}\left[\delta_k^i - 3\frac{1}{r^2}z_{kl}z^j z^l\right]\right\} \quad (33)$$

b) inside inclusion

$$I_k^{inc}(z) = I_k^{\infty}\frac{3\Sigma_{inc}}{\Sigma_{inc} + 2\Sigma_{mat} + 2\sigma R^{-1}} \quad (34)$$

For the interface current we get the relationship

$$i_j = I_k^{\infty}(\delta_j^k - N^k N_j)\frac{3\sigma}{\Sigma_{inc} + 2\Sigma_{mat} + 2\sigma R^{-1}} \quad (35)$$

5. Special Cases

5.1 Vanishing Surface Conductance $\sigma = 0$

Electrostatic field outside inclusion:

$$E_k^{mat}(z) = \frac{I_i^{\infty}}{\Sigma_{mat}}\left\{\delta_k^i - \frac{\Sigma_{inc} - \Sigma_{mat}}{\Sigma_{inc} + 2\Sigma_{mat}}\frac{R^3}{r^3}\left[\delta_k^i - 3\frac{1}{r^2}z_{kj}z^j z^i\right]\right\} \quad (36)$$

Electrostatic field inside inclusion:

$$E_k^{inc}(z) = \frac{I_k^{\infty}}{\Sigma_{inc}}\frac{3\Sigma_{mat}}{\Sigma_{inc} + 2\Sigma_{mat}} \quad (37)$$

Electric current outside inclusion:

$$I_k^{mat}(z) = I_i^{\infty}\left\{\delta_k^i - \frac{\Sigma_{inc} - \Sigma_{mat}}{\Sigma_{inc} + 2\Sigma_{mat}}\frac{R^3}{r^3}\left[\delta_k^i - 3\frac{1}{r^2}z_{kj}z^j z^i\right]\right\} \quad (38)$$

Electric current inside inclusion:

$$I_k^{inc}(z) = I_k^{\infty}\frac{3\Sigma_{mat}}{\Sigma_{inc} + 2\Sigma_{mat}} \quad (39)$$

Interface current:

$$i_k = 0 \tag{40}$$

Interface charge density:

$$\tau = \frac{3}{4\pi}\frac{I_k^\infty}{\Sigma_{mat}} N^k \frac{\Sigma_{inc} - \Sigma_{mat}}{\Sigma_{inc} + 2\Sigma_{mat}} \tag{41}$$

5.2 Nonconducting Inclusion $\Sigma_{inc} = 0$

Electrostatic field outside inclusion:

$$E_k^{mat}(z) = \frac{I_i^\infty}{\Sigma_{mat}}\left\{\delta_k^i + \frac{\Sigma_{mat} - 2\sigma R^{-1}}{2\Sigma_{mat} + 2\sigma R^{-1}}\frac{R^3}{r^3}\left[\delta_k^i - 3\frac{1}{r^2} z_{kj} z^j z^i\right]\right\} \tag{42}$$

Electrostatic field inside inclusion:

$$E_k^{inc}(z) = I_k^\infty \frac{3}{2\Sigma_{mat} + 2\sigma R^{-1}} \tag{43}$$

Electric current outside inclusion:

$$I_k^{mat}(z) = I_i^\infty\left\{\delta_k^i + \frac{\Sigma_{mat} - 2\sigma R^{-1}}{2\Sigma_{mat} + 2\sigma R^{-1}}\frac{R^3}{r^3}\left[\delta_k^i - 3\frac{1}{r^2} z_{kl} z^j z^l\right]\right\} \tag{44}$$

Electric current inside inclusion:

$$I_k^{inc}(z) = 0 \tag{45}$$

Interface current:

$$i_j = I_k^\infty(\delta_j^k - N^k N_j)\frac{3\sigma}{2(\Sigma_{mat} + \sigma R^{-1})} \tag{46}$$

Interface charge density:

$$\tau = \frac{3}{8\pi}\frac{I_k^\infty}{\Sigma_{mat}} N^k \frac{2\sigma R^{-1} - \Sigma_{mat}}{\sigma R^{-1} + \Sigma_{mat}} \tag{47}$$

5.3 Superconducting Inclusion $\Sigma_{inc} = \infty$

Electrostatic field outside inclusion:

$$E_k^{mat}(z) = \frac{I_i^\infty}{\Sigma_{mat}}\left\{\delta_k^i - \frac{R^3}{r^3}\left[\delta_k^i - 3\frac{1}{r^2}z_{kj}z^j z^i\right]\right\} \tag{48}$$

Electrostatic field inside inclusion:

$$E_k^{inc}(z) = 0 \tag{49}$$

Electric current outside inclusion:

$$I_k^{mat}(z) = I_i^\infty\left\{\delta_k^i - \frac{R^3}{r^3}\left[\delta_k^i - 3\frac{1}{r^2}z_{kl}z^j z^l\right]\right\} \tag{50}$$

Electric current inside inclusion:

$$I_k^{inc}(z) = 3I_k^\infty \tag{51}$$

Interface current:

$$i_k = 0 \tag{52}$$

Interface charge density:

$$\tau = \frac{3I_k^\infty}{4\pi\Sigma_{mat}}N^k \tag{53}$$

5.4 Superconducting Interface $\sigma = \infty$

Electrostatic field outside inclusion:

$$E_k^{mat}(z) = \frac{1}{\Sigma_{mat}}\left\{\delta_k^i - \frac{R^3}{r^3}\left[\delta_k^i - 3\frac{1}{r^2}z_{kj}z^j z^i\right]\right\}I_i^\infty \tag{54}$$

Electrostatic field inside inclusion:

$$E_i^{inc}(z) = 0 \quad \left(\cong \frac{3R}{2\sigma}I_i^\infty\right) \tag{55}$$

Electric current outside inclusion:

$$I_k^{mat}(z) = \left\{ \delta_k^i - \frac{R^3}{r^3} \left[\delta_k^i - 3\frac{1}{r^2} z_{kj} z^j z^i \right] \right\} I_i^{\infty} \tag{56}$$

Electric current inside inclusion:

$$I_k^{inc}(z) = 0 \quad \left[\approx I_k^{\infty} \frac{3R\Sigma_{inc}}{2\sigma} \right] \tag{57}$$

Interface current:

$$i_j = I_k^{\infty} (\delta_j^k - N^k N_j) \frac{3R}{2} \tag{58}$$

Interface charge density:

$$\tau = \frac{3}{4\pi} \frac{I_k^{\infty}}{\Sigma_{mat}} N^k \tag{59}$$

6. Conclusion

We explored the steady-state current in the heterogeneous system consisting of conducting unbounded matrix and conducting spherical inclusion. For simplicity, both are assumed isotropic. The effects of magnetic field are excluded from this study. At the same time, we explore the roles of the distributed charges at the interface. Also, we calculated explicitly the interface current.

The general solution is described by Eqs. 27, 28, and 32–35 in Section 4: Exact Solution for Spherical Inclusion (Finite Boundary Conductivity). Then, in Section 5, Special Cases, we consider different asymptotic cases: nonconducting and superconducting interface and nonconducting and superconducting spherical inclusions.

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List of Symbols, Abbreviations, and Acronyms

FMHD	Full Maxwell Hydrodynamics
MHD	Magnetohydrodynamic
V&V	Validation and Verification

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