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Toward the Second-Order Discontinuities of the Simple and Double Layers

by Michael Grinfeld and Pavel Grinfeld

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14. ABSTRACT

Simple and double layers are the standard physical models and the keynote mathematical tools. They first appeared in electrostatics and found later various applications in mathematical physics. The discontinuities of these functions and their first derivatives have been under extensive study for about two centuries. In this report, we announce the jump conditions for the second spatial derivatives of those layers.

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Contents

1. Introduction

In 1987, Julius Weingarten (1836–1910) announced in his publication[1] a remarkable formula for the jumps of second derivatives of the Newtonian potential across the boundary of gravitating bodies. We remind the reader that the Newtonian potential $V(\vec{z})$ of a self-gravitating body $B,$ with mass distributed with the mass density $\rho(\vec{z})$, is defined as the 3-D integral

$$V\ \ \vec{z}\ \ = \int_B dB^* \frac{\rho(\vec{z}^{\,*})}{\left|\vec{z} - \vec{z}^{\,*}\right|} \tag{1}$$

(See, for instance, Sretensky's[2] monograph for analysis of Weingarten's result presented in a very clear form, close to Weingarten's original).

In recounting Weingarten's result, we must first explain our notation and some elementary facts from differential geometry of hypersurfaces in the Euclidean space. Weingarten[1] and Sretensky[2] were dealing with the classical theory of gravitation. Following their footsteps, in this report we assume that the space is Euclidean and it is referred to curvilinear coordinates z^i; all Latin (spatial) indexes run values 1, 2, 3. In Eq. 1, $\vec{z}(z^i)$ is the radius vector; in the following, we do not explicitly show indexes in the arguments of functions.

In the following, $\vec{z}_i(z) \equiv \partial\vec{z}(z) / \partial z^i$ is the covariant basis, $z_{ij}(z) \equiv \vec{z}_i(z)\cdot\vec{z}_j(z)$ is the covariant basis, and the contravariant basis $z^{ij}(z)$ is defined by the inverse of $z_{ij}(z)$. We are using standard operations of tensor calculus in the Euclidean space, including the direct and inner multiplication of tensors, the contraction of indexes, and the covariant differentiation denoted as ∇_i. The standard definition can be found in multiple textbooks on tensor calculus including McConnell[3] and Thomas.[4] We especially recommend the book and numerous papers by the outstanding geometer and physicist Tracy Thomas, one of the key figures of the theory of compatibility conditions for discontinuous tensorial fields. Also, we recommend our books,[5,6] from which we borrow our notation and some results on the compatibility conditions. In those books, we present relevant historical information.

The body B under study is bounded by the smooth surface $\Xi,$ referred to as the Gaussian coordinates ξ^α; the Greek (surface) indices run values 1, 2. In the following, $z^i = z^i(\xi^\alpha)$ is the equation of the surface Ξ; $\vec{\xi}\ \ \xi\ \ \equiv \vec{z}(z(\xi))$ is the radius-vector of Ξ; $z^{i\cdot}_{\cdot\alpha}(\xi) \equiv \partial z^i(\xi) / \partial\xi^\alpha$ is the shift tensor; $\vec{\xi}_\alpha(\xi) = \vec{z}_i z^{i\cdot}_{\cdot\alpha}$ is the covariant basis; $\xi_{\alpha\beta}(\xi) \equiv \vec{\xi}_\alpha(\xi)\cdot\vec{\xi}_\beta(\xi)$ is the surface covariant basis; and the contravariant basis $\xi^{\alpha\beta}(\xi)$ is defined by the inverse of $\xi_{\alpha\beta}(\xi)$. For the mixed spatio-surface tensors, we use standard operations of tensor calculus for hypersurfaces in the Euclidean space, including the direct and inner multiplication of tensors, the contraction of indexes, and the covariant differentiation denoted as ∇_{α}.

The important information, characterizing the surface, imbedded in the Euclidian space, contains the field of unit normal $N_i(\xi)$ and the tensor of the second quadratic form $B_{\alpha\beta}(\xi)$. The following relationships are particularly important

$$\nabla_{\beta} z^{i\cdot}_{\cdot\alpha} = B_{\alpha\beta} N^i, \; \nabla_{\beta} N^i = -z^{i\cdot}_{\cdot\alpha} B^{\alpha\cdot}_{\cdot\beta}\,, \; z^{i\cdot}_{\cdot\alpha} N_i = 0 \quad (2)$$

For the points $\vec{z}$ inside the body B, the integrand assume arbitrarily large values, and the exact mathematical meaning of the integration requires a vigilant interpretation. In this report, we do not dwell on such issues and refer interested readers to various brilliant monographs and textbooks on the potential theory.

In the following are some elementary facts from the theory of Newtonian potential. Following Sretensky,[2] we assume that we use physical units in which the gravitational constant γ is equal to 1. Out of boundaries, the potential $V(\vec{z})$ also has the second derivatives and satisfies the Poisson equation

$$\nabla_i \nabla^i V = -4\pi\rho \quad (3)$$

(The Latin indexes run the values 1, 2, 3, and summation over repeated indexes is implied.)

Within mass free space, the Poisson Eq. 3 reduces to the Laplace equation

$$\nabla_i \nabla^i V = 0 \quad (4)$$

Let the body be bounded with the sufficiently smooth boundary S. Despite the use of the singular kernel in the definition of the Newtonian potential, the potential $V(z)$ not only exists, but is also continuous and differentiable everywhere in and out of the body B, including also the boundary S of the self-gravitating body. Formally, these properties are characterized by the relationships

$$\left[V\right]_{-}^{+} = 0 \text{ and } \left[\nabla_i V\right]_{-}^{+} = 0 \quad (5)$$

That pair of continuity relationships can be replaced with the following equivalent pair:

$$\left[V\right]_{-}^{+} = 0 \text{ and } \left[\nabla_i V\right]_{-}^{+} N^i = 0 \tag{6}$$

At the same time, in view of the bulk Eqs. 3 and 4, at least, one of the second-order derivatives $\nabla_i \nabla_j V$ must be discontinuous. A more-detailed analysis of the discontinuities of the second derivative $\left[\nabla_i \nabla_j V\right]_{-}^{+}$ shows that these jumps are given by the relationship

$$\left[\nabla_i \nabla_j V\right]_{-}^{+} = -4\pi\rho N_i N_j \,, \tag{7}$$

where $N_i(\xi)$ is the unit normal to the interface S. Also, $\nabla_i \nabla_j V_+$ and $\nabla_i \nabla_j V_-$ are limit values of $\nabla_i \nabla_j V$ when approaching S from inside and outside the self-gravitating body B. Equation 7 is the main result of the Weingarten paper.[1]

In addition to volumetric distributions of the density ρ or charge q, there are also two popular distributions localized on the surfaces. They are used as convenient physical models and as convenient mathematical tools in the problems of mathematical physics, which are very far from gravity and electricity.

Consider a smooth, closed surface Ξ referred to the Gaussian coordinates ξ^α. The so-called simple layer is the 3-D function $S_\mu(\vec{z})$, associated with the 2-D function $\mu \ \xi$, and it is defined by the 2-D integral over the surface Ξ

$$S_\mu(\vec{z}) \equiv \int_\Xi d\Xi \frac{\mu(\xi)}{\left|\vec{z}(z) - \vec{\xi}(\xi)\right|} \,, \tag{8}$$

where $z^i = z^i(\xi^\alpha)$ is the equation of the surface Ξ, and the vector-function $\vec{\xi}(\xi)$ is defined by the relationship

$$\vec{\xi}(\xi) \equiv \vec{z}(z(\xi)) \tag{9}$$

The double layer is a 3-D function, associated with the 2-D function $\tau(\xi)$, and is defined by the 2-D integral over the surface Ξ

$$D_\tau \ \vec{z} \ \equiv -\int_\Xi d\Xi \tau(\xi) N^i(\xi) \nabla_i \frac{1}{\left|\vec{z}(z) - \vec{\xi}(\xi)\right|} \tag{10}$$

Similar to the 3-D Newtonian potential $V(z)$, the simple and double layers are twice differentiable out of the interface Ξ and satisfy the Laplace equation (Eq. 4). At the same time, at least one of the jump conditions (Eq. 6) is violated across the simple and double layer.

As demonstrated in different ways by outstanding mathematicians and physicists of the past, the simple potential remains to be a continuous function even across the points $\vec{\xi}(\xi)$ belonging to the surface Ξ. At the same time, the first derivatives of $S_{\mu}(\vec{z})$ experience irregularities at the points of the surface S. Concerning the double layers, even the function $D_{\tau}(\vec{z})$ appears to be discontinuous at S.

When characterizing the behavior of $S_{\mu}(\vec{z})$ and $D_{\tau}(\vec{z})$ across the surface S, we ought to distinguish between the one-sided limits of these functions and their direct values on S. The direct values are of particular importance in the theory of integral equations, though integral equations are out of scope of this report, so we will not discuss the problem of direct values. We deal only with one-sided limits of the values of $S_{\mu}(\vec{z})$ and $D_{\tau}(\vec{z})$ and their derivatives. We assume that the surface is orientable and has two sides—the "plus" side and the "minus" side. Consider a point P of the surface S with the surface coordinates ξ^{α}. Consider a sufficiently small spherical ball ω centered at M. The surface S splits this ball into two subdomains, ω_{+} and ω_{-}. Let the point $\vec{z}$ approach the point $\vec{z}(z^{m}(\xi^{\gamma}))$ staying all the time within the subdomain ω_{+} (ω_{-}). Let the function $F(z)$ under study have a one-sided finite limit $F_{+}(\xi)$ ($F_{-}(\xi)$) when the approaching point permanently stays within the subdomain ω_{+} (ω_{-}), respectively. These limits are called the one-sided limits of $F(z)$ at M.

Under rather general conditions, the functions $F_{\pm}(\xi)$ appear to be rather smooth functions of ξ^{α}. We can introduce, similarly, the one-sided limits of the spatial gradient of $\nabla_{i} F(z)$ for which we use notation $\nabla_{i} F_{\pm}$, which are the functions of the surface coordinates ξ^{α}. The derivatives of the functions $F_{\pm}(\xi)$ with respect to ξ^{α} can be expressed in terms of the functions $\nabla_{i} F_{\pm}$ as follows:

$$\nabla_{\alpha} F_{\pm}(\xi) = z^{i\cdot}_{\cdot\alpha} \nabla_{i} F_{\pm}, \tag{11}$$

where the shift-tensor $z^{i\cdot}_{\cdot\alpha}$ is defined as

$$z^{i\cdot}_{\cdot\alpha} \equiv \frac{\partial z^i(\xi)}{\partial \xi^\alpha} \tag{12}$$

The mathematically rigorous derivation of the relationship Eq. 11 is known as the Hadamard lemma and can be found, for instance, in Smirnov's textbook.[7] The meaning of this relationship is much more transparent in the following form:

$$\nabla_\alpha \lim_{z \to z(\xi) \pm 0} F(z) = z^{i\cdot}_{\cdot\alpha} \lim_{z \to z(\xi) \pm 0} \nabla_i F(z) \tag{13}$$

In fact, Eq. 12 is nothing else but the natural generalization of the formula of the derivative of composed function of elementary calculus; however, it now concerns the smooth functions inside the semi-ball rather than entire ball centered at $z^i(\xi^\alpha)$.

Generally speaking, the values $F_\pm$ are not equal to each other. The difference $F_+(\xi) - F_-(\xi)$ is called the jump of $F(z)$ at M, and it is denoted as $\left[F\right]_-^+ \equiv F_+(\xi) - F_-(\xi)$ Subtracting two Eqs. 3s with plus and minus signs, we get

$$\nabla_\alpha \left[F\right]_-^+ = z^{i\cdot}_{\cdot\alpha} \left[\nabla_i F\right]_-^+ \tag{14}$$

Contracting both sides of Eq. 14 with the tensor $\Xi^{\alpha\beta} z^{j\cdot}_{\cdot\beta}$, we then get

$$\Xi^{\alpha\beta} z^{j\cdot}_{\cdot\beta} \nabla_\alpha \left[F\right]_-^+ = \Xi^{\alpha\beta} z^{j\cdot}_{\cdot\beta} z^{i\cdot}_{\cdot\alpha} \left[\nabla_i F\right]_-^+, \tag{15}$$

where $\Xi^{\alpha\beta}(\xi)$ is the metrics of the surface S.

Using the tensor identity (see, for instance, any of the following: McConnell,[3] Thomas,[4] Grinfeld,[5,6] or Smirnov[7])

$$\Xi^{\alpha\beta} z^{j\cdot}_{\cdot\beta} z^{i\cdot}_{\cdot\alpha} = z^{ij} - N^i N^j, \tag{16}$$

we can rewrite Eq. 15 as follows:

$$\left[\nabla_i F\right]_-^+ (z^{ij} - N^i N^j) = \Xi^{\alpha\beta} z^{j\cdot}_{\cdot\beta} \nabla_\alpha \left[F\right]_-^+ \tag{17}$$

It is instructive to rewrite Eq. 17as follows:

$$\left[\nabla_i F\right]_-^+ = \Xi^{\alpha\beta} z_{ik} z^{k\cdot}_{\cdot\beta} \nabla_\alpha \left[F\right]_-^+ + N^k N_i \left[\nabla_k F\right]_-^+ \tag{18}$$

or, using the standard tensorial operation of the "indexes juggling", in the following shorter form:

$$\left[\nabla_i F\right]_-^+ = z_{i.}^{\cdot\alpha} \nabla_\alpha H_0 + N_i H_1 \tag{19}$$

In Eq. 18, we use the following jump amplitudes $H_0(\xi)$, $H_1(\xi)$,... defined as follows:

$$H_0(\xi) \equiv \left[F\right]_-^+ ,\ H_1(\xi) \equiv \left[\nabla_k F\right]_-^+ N^k,\ H_2(\xi) \equiv \left[\nabla_k \nabla_l F\right]_-^+ N^k N^l, \ldots \tag{20}$$

According to Eqs. 19 and 20, two surface functions, $H_0(\xi)$ and $H_1(\xi)$, allow three jumps $\left[\nabla_i F\right]_-^+$ of the spatial derivatives of $F(z)$. When the spatial function $F(z)$ is continuous across Ξ, the function $H_0(\xi)$ vanishes identically, and the jumps $\left[\nabla_i F\right]_-^+$ can be expressed in terms of only one spatial function, $H_1(\xi)$:

$$\left[\nabla_i F\right]_-^+ = N_i H_1 \tag{21}$$

Equations 19 and 21 have purely geometric origins: Any particular physical nature of F plays no role in the validity of these relationships. On the other hand, Eq. 6 has, basically, a physical origin, mostly based on the Newton's law of gravitation (Eq. 1).

Per Eq. 6, for the Newtonian potential, both $V(z)$ and $\nabla_i V$ remain continuous across S. For such functions F, instead of Eq. 21 we get the following relationship for the jumps of the second derivatives:

$$\left[\nabla_i \nabla_j F\right]_-^+ = H_2 N_i N_j \tag{22}$$

According to Eq. 22, nine jumps of second derivatives can be expressed with the help of a single surface function, $H_2(\xi)$. The purely geometric Eq. 22 reminds us of Eq. 7 for the second derivatives of the Newtonian potential, but it is not equivalent to it. We see that the pure geometry allows a lot, but it does not allow the relationship

$$H_2(\xi) = -4\pi\rho(z(\xi)) \ , \tag{23}$$

which is valid only for the Newtonian potential $V(z)$.

To make this point more transparent, let us raise the index i in Eq. 21 and contract the resulting equation with respect to the indices i and j. We then get

$$\left[\nabla_i \nabla^i F\right]_-^+ = H_2 \tag{24}$$

Let us apply Eq. 24 to the Newtonian potential $V(z)$, and use Eqs. 2 and 3. We then get

$$-4\pi(\rho_{+} - \rho_{-}) = H_{2} \tag{25}$$

If the domain Ω_{+} is filled with the self-gravitating substance and the domain Ω_{-} is vacuum, then Eq. 25 takes the form of Eq. 6, and hence we arrive at the beautiful Weingarten relationship, Eq. 7 (see Weingarten[1] and Sretensky[2]).

The goal of this report is to find the analogies of the Weingarten formulae for the simple and double potentials. Our plan is the following. First, we find the analogy of Eq. 22. This is applicable not only to the volumetric distributions of mass or charge, but also applicable to the simple and double layers and even to abstract discontinuous functions with regular one-sided limits on the smooth hypersurfaces (those general relationships include not only the jump functions $H_{2}(\xi)$ and the unit normal $N^{i}(\xi)$, but also the jumps $H_{0}(\xi)$ and $H_{1}(\xi)$ and the geometric tensors $\Xi_{\alpha\beta}$, $B_{\alpha\beta}$). Then we establish the expressions of the jumps $H_{0}(\xi)$ and $H_{1}(\xi)$, and $H_{2}(\xi)$ in terms of the densities $\mu(\xi)$ and $\tau(\xi)$ of the simple and double layers.

2. The General Geometric Compatibility Conditions for the Second Derivatives

We begin by establishing the following relationship for the jumps of the second derivatives:

$$\begin{aligned}\left[\nabla_{i}\nabla_{j}F\right]_{-}^{+} = H_{2}N_{j}N_{i} + 2N_{(i}z_{j).}^{\cdot\alpha}\nabla_{\alpha}H_{1} - H_{1}B_{ij} + \\ 2N_{(i}z_{j).}^{\cdot\alpha}B_{\alpha}^{\beta}\nabla_{\beta}H_{0} + z_{i.}^{\cdot\alpha}z_{j.}^{\cdot\beta}\nabla_{\alpha}\nabla_{\beta}H_{0}\end{aligned} \tag{26}$$

First, using Eq. 16, we can decompose the identity (Kronekker) tensor into two orthogonal projectors:

$$\delta_{n}^{m} \equiv z_{n.}^{\cdot\alpha}z_{\cdot\alpha}^{m\cdot} + N^{m}N_{n} \tag{27}$$

Using Eq. 27, we get

$$\left[\nabla_{i}\nabla_{j}F\right]_{-}^{+} = \left[\nabla_{k}\nabla_{j}F\right]_{-}^{+} N^{k}N_{i} + z_{i.}^{\cdot\alpha}\nabla_{\alpha}\left[\nabla_{j}F\right]_{-}^{+}, \tag{28}$$

as implied by the directly verifiable chain:

$$
\begin{aligned}
&\left[\nabla_i \nabla_j F\right]_-^+ = \left[\nabla_k \nabla_j F\right]_-^+ \delta_i^k = \\
&\left[\nabla_k \nabla_j F\right]_-^+ (z_{i.}^{.\alpha} z_{.\alpha}^{k.} + N^k N_i) = \\
&\left[\nabla_k \nabla_j F\right]_-^+ N^k N_i + z_{i.}^{.\alpha} \nabla_\alpha \left[\nabla_j F\right]_-^+
\end{aligned}
\tag{29}
$$

and the chain rule $\nabla_\alpha = z_{.\alpha}^{k.} \nabla_k$.

Let us transform separately each of the two terms in the right-hand side of Eq. 28. First, we get

$$
\left[\nabla_k \nabla_j F\right]_-^+ N^k N_i = H_2 N_i N_j + z_{j.}^{.\beta} \nabla_\beta \left[\nabla_k F\right]_-^+ N^k N_i \tag{30}
$$

as implied by the directly verifiable chain:

$$
\begin{aligned}
&\left[\nabla_k \nabla_j F\right]_-^+ N^k N_i = \delta_j^l \left[\nabla_k \nabla_l F\right]_-^+ N^k N_i = \\
&(z_{j.}^{.\beta} z_{.\beta}^{l.} + N^l N_j) \left[\nabla_k \nabla_l F\right]_-^+ N^k N_i = \\
&\left[\nabla_k \nabla_l F\right]_-^+ N^k N_i N^l N_j + z_{j.}^{.\beta} \nabla_\beta \left[\nabla_k F\right]_-^+ N^k N_i = \\
&H_2 N_i N_j + z_{j.}^{.\beta} \nabla_\beta \left[\nabla_k F\right]_-^+ N^k N_i
\end{aligned}
\tag{31}
$$

For the last term in Eq. 30, we get

$$
z_{j.}^{.\beta} \nabla_\beta \left[\nabla_k F\right]_-^+ N^k N_i = N_i z_{j.}^{.\beta} \nabla_\beta H_1 + N_i z_{j.}^{.\beta} B_\beta^\alpha \nabla_\alpha H_0 \tag{32}
$$

as implied by the directly verifiable chain:

$$
\begin{aligned}
&z_{j.}^{.\beta} \nabla_\beta \left[\nabla_k F\right]_-^+ N^k N_i = \\
&N_i z_{j.}^{.\beta} \quad \nabla_\beta (\left[\nabla_k F\right]_-^+ N^k) - \left[\nabla_k F\right]_-^+ \nabla_\beta N^k \quad = \\
&N_i z_{j.}^{.\beta} \nabla_\beta H_1 + N_i z_{j.}^{.\beta} \left[\nabla_k F\right]_-^+ B_\beta^\alpha z_\alpha^{k.}
\end{aligned}
\tag{33}
$$

Inserting Eq. 32 in Eq. 30, we get

$$
\left[\nabla_k \nabla_j F\right]_-^+ N^k N_i = H_2 N_i N_j + N_i z_{j.}^{.\beta} \nabla_\beta H_1 + N_i z_{j.}^{.\beta} B_\beta^\alpha \nabla_\alpha H_0 \tag{34}
$$

Let us now transform the term $z_{i.}^{.\alpha} \nabla_\alpha \left[\nabla_j F\right]_-^+$ in Eq. 28. First, we get

$$
z_{i.}^{.\alpha} \nabla_\alpha \left[\nabla_j F\right]_-^+ = z_{i.}^{.\alpha} \nabla_\alpha (z_{j.}^{.\beta} \nabla_\beta H_0 + H_1 N_j) \tag{35}
$$

as implied by the directly verifiable chain:

$$
\begin{aligned}
& z_{i.}^{\cdot\alpha}\nabla_{\alpha}\left[\nabla_{j}F\right]_{-}^{+} = z_{i.}^{\cdot\alpha}\nabla_{\alpha}\ \left[\nabla_{k}F\right]_{-}^{+}\delta_{j}^{k} \ = \\
& z_{i.}^{\cdot\alpha}\nabla_{\alpha}\ \left[\nabla_{k}F\right]_{-}^{+}(z_{j.}^{\cdot\beta}z_{\cdot\beta}^{k\cdot} + N^{k}N_{j}) \ = \\
& z_{i.}^{\cdot\alpha}\nabla_{\alpha}\ \left[\nabla_{k}F\right]_{-}^{+}z_{j.}^{\cdot\beta}z_{\cdot\beta}^{k\cdot} + \left[\nabla_{k}F\right]_{-}^{+}N^{k}N_{j} \ = \\
& z_{i.}^{\cdot\alpha}\nabla_{\alpha}\ \ \nabla_{\beta}\left[F\right]_{-}^{+}z_{j.}^{\cdot\beta} + \left[\nabla_{k}F\right]_{-}^{+}N^{k}N_{j} \ = \\
& z_{i.}^{\cdot\alpha}\nabla_{\alpha}(z_{j.}^{\cdot\beta}\nabla_{\beta}H_{0} + H_{1}N_{j})
\end{aligned}
\tag{36}
$$

We can further transform Eq. 35 to get

$$
\begin{aligned}
z_{i.}^{\cdot\alpha}\nabla_{\alpha}(z_{j.}^{\cdot\beta}\nabla_{\beta}H_{0} + H_{1}N_{j}) = z_{i.}^{\cdot\alpha}z_{j.}^{\cdot\beta}\nabla_{\alpha}\nabla_{\beta}H_{0} + \\
z_{i.}^{\cdot\alpha}N_{j}B_{\alpha}^{\beta}\nabla_{\beta}H_{0} + N_{j}z_{i.}^{\cdot\alpha}\nabla_{\alpha}H_{1} - B_{ij}H_{1}
\end{aligned}
\tag{37}
$$

where we use notation

$$
B_{ij} \equiv z_{i.}^{\cdot\alpha}B_{\alpha\beta}^{\cdot\cdot}z_{j.}^{\cdot\beta}
\tag{38}
$$

Using Eq. 37, we can rewrite Eq. 35 as

$$
\begin{aligned}
& z_{i.}^{\cdot\alpha}\nabla_{\alpha}\left[\nabla_{j}F\right]_{-}^{+} = z_{i.}^{\cdot\alpha}z_{j.}^{\cdot\beta}\nabla_{\alpha}\nabla_{\beta}H_{0} + z_{i.}^{\cdot\alpha}N_{j}B_{\alpha}^{\beta}\nabla_{\beta}H_{0} + \\
& \quad N_{j}z_{i.}^{\cdot\alpha}\nabla_{\alpha}H_{1} - B_{ij}H_{1}
\end{aligned}
\tag{39}
$$

Using Eqs. 34 and 39, we can rewrite Eq. 28 as the required Eq. 26. The intermediate calculations are shown in the following chain:

$$
\begin{aligned}
& \left[\nabla_{i}\nabla_{j}F\right]_{-}^{+} = \left[\nabla_{k}\nabla_{j}F\right]_{-}^{+}N^{k}N_{i} + z_{i.}^{\cdot\alpha}\nabla_{\alpha}\left[\nabla_{j}F\right]_{-}^{+} = \\
& H_{2}N_{i}N_{j} + N_{i}z_{j.}^{\cdot\beta}\nabla_{\beta}H_{1} + N_{i}z_{j.}^{\cdot\beta}B_{\beta}^{\alpha}\nabla_{\alpha}H_{0} + \\
& z_{i.}^{\cdot\alpha}z_{j.}^{\cdot\beta}\nabla_{\alpha}\nabla_{\beta}H_{0} + z_{i.}^{\cdot\alpha}N_{j}B_{\alpha}^{\beta}\nabla_{\beta}H_{0} + \\
& N_{j}z_{i.}^{\cdot\alpha}\nabla_{\alpha}H_{1} - z_{i.}^{\cdot\alpha}B_{\beta\alpha}^{\cdot\cdot}z_{j.}^{\cdot\beta}H_{1} = \\
& H_{2}N_{i}N_{j} + (N_{i}z_{j.}^{\cdot\alpha} + N_{j}z_{i.}^{\cdot\alpha})\nabla_{\alpha}H_{1} - B_{ij}H_{1} + \\
& z_{i.}^{\cdot\alpha}z_{j.}^{\cdot\beta}\nabla_{\alpha}\nabla_{\beta}H_{0} + (N_{i}z_{j.}^{\cdot\beta} + z_{i.}^{\cdot\beta}N_{j})B_{\beta}^{\alpha}\nabla_{\alpha}H_{0}
\end{aligned}
\tag{40}
$$

At $H_{0} = 0,$ the general relationship of Eq. 39 reduces to

$$
\left[\nabla_{i}\nabla_{j}F\right]_{-}^{+} = H_{2}N_{j}N_{i} + 2N_{(i}z_{j).}^{\cdot\alpha}\nabla_{\alpha}H_{1} - H_{1}B_{ij}\,,
\tag{41}
$$

which can be found in Thomas' monograph.[4] The case of $H_{0} \neq 0,$ however, is important for the analysis of double layers.

At $H_1 = 0$, Eq. 41 reduces to Eq. 22, as it should.

3. Simple and Double Layers

3.1 Jumps of the Second Derivatives of the Simple Layer

Now, we specify the surface functions $H_0,\ H_1,\ H_2$ in the general compatibility relationship, Eq. 26, for the simple and double layers.

The simple layer $S_\mu(\vec{z})$ with the surface density $\mu(\xi)$ is defined by Eq. 8. Out of the surface Ξ, it satisfies the Laplace equation:

$$\nabla^i \nabla_i S_\mu = 0 \tag{42}$$

At the smooth boundaries Ξ, the functions $S_\mu(\vec{z})$ and $\nabla_i S_\mu(\vec{z})$ have one-sided limits, $S_\mu^{\pm}(\vec{z})$ and $(\nabla_i S_\mu)^{\pm}$. Moreover, the following jump relationships are valid:

$$\left[S_\mu\right]_-^+ = 0 \tag{43}$$

and

$$\left[\nabla_i S_\mu\right]_-^+ N^i = -4\pi\mu \tag{44}$$

Mathematical proofs of Eqs. 43 and 44 for the simple layers can be found in any decent textbooks of mathematical physics, including Sretensky,[2] Smirnov,[7] and Vladimirov.[8]

In terms of the jump functions, Eqs. 43 and 44 imply

$$H_0(\xi) = 0,\ H_1(\xi) = -4\pi\mu \tag{45}$$

It remains to determine the function $H_2(\xi)$ in Eq. 25. To that end, let us "raise" the index i in Eq. 41 and make contraction with respect to the indexes i and j. As a result, we get

$$\left[\nabla^i \nabla_i S_\mu\right]_-^+ = H_2 - H_1 B_\alpha^\alpha \tag{46}$$

Now, in view of Eq. 42, Eq. 46 implies

$$H_2 = H_1 B_\alpha^\alpha = -4\pi\mu B_\alpha^\alpha \tag{47}$$

Summarizing, the jumps of second derivatives on the simple layer Eq. 26 take the form

$$\left[\nabla_i \nabla_j S_\mu\right]_-^+ = 4\pi(\mu B_{ij} - \mu B_\alpha^\alpha N_j N_i - 2N_{(i} z_{j).}^{\cdot\alpha} \nabla_\alpha \mu) \tag{48}$$

For the flat interface, Eq. 48 reads

$$\left[\nabla_i \nabla_j S_\mu\right]_-^+ = -8\pi N_{(i} z_{j).}^{\cdot\alpha} \nabla_\alpha \mu \tag{49}$$

3.2 Jumps of the Second Derivatives of the Double Layer

As with the simple layer, the double-layer function $D_\tau(\vec{z})$ with the surface density $\tau(\xi)$, out of the surface Ξ, satisfies the Laplace equation:

$$\nabla^i \nabla_i D_\tau = 0 \tag{50}$$

Also, at the smooth boundary Ξ, the functions $D_\tau(\vec{z})$ and $\nabla_i D_\tau(\vec{z})$ have one-sided limits, $D_\tau^\pm(\vec{z})$ and $(\nabla_i D_\tau)^\pm$. The jumps of the double-layer function, however, satisfy different jump relationships[4,8]:

$$\left[D_\tau\right]_-^+ = -4\pi\tau \tag{51}$$

and

$$\left[\nabla_i D_\tau\right]_-^+ N^i = 0 \tag{52}$$

Eqs. 51 and 52 imply the following relationships for the functions $H_0(\xi)$ and $H_1(\xi)$:

$$H_0 = -4\pi\tau, \quad H_1 = 0 \tag{53}$$

Inserting both relationships of Eq. 53 in the general compatibility relation of Eq. 25, we get

$$\left[\nabla_i \nabla_j D_\tau\right]_-^+ = H_2 N_j N_i - 8\pi N_{(i} z_{j).}^{\cdot\alpha} B_\alpha^\beta \nabla_\beta \tau - 4\pi z_{i.}^{\cdot\alpha} z_{j.}^{\cdot\beta} \nabla_\alpha \nabla_\beta \tau \tag{54}$$

Raising the index i and contracting with the indexes i and j, we transform Eq. 54 to the form

$$\left[\nabla^i \nabla_i D_\tau\right]_-^+ = H_2 - 4\pi \nabla_\alpha \nabla^\alpha \tau \tag{55}$$

With the help of Eqs. 50 and 55, we get the formula of H_2

$$H_2 = 4\pi\nabla_\alpha\nabla^\alpha\tau \tag{56}$$

Inserting Eqs. 53 and 56 in Eq. 54, we arrive at the required formula

$$\left[\nabla_i\nabla_j D_\tau\right]_-^+ = 4\pi(\nabla_\alpha\nabla^\alpha\tau N_j N_i - 2N_{(i} z_{j)\alpha}^{\cdot\cdot} B^{\alpha\beta}\nabla_\beta\tau - z_{i.}^{\cdot\alpha} z_{j.}^{\cdot\beta}\nabla_\alpha\nabla_\beta\tau) \tag{57}$$

For the flat double layer, Eq. 57 implies

$$\left[\nabla_i\nabla_j D_\tau\right]_-^+ = 4\pi\ \ \Xi^{\alpha\beta} N_j N_i - z_{i.}^{\cdot\alpha} z_{j.}^{\cdot\beta}\ \ \nabla_\alpha\nabla_\beta\tau \tag{58}$$

4. Conclusion

Three of the most popular distributions in classical potential theory are 1) the volumetric distribution, described by the singular 3-D integral Eq. 1; 2) the simple layer distribution, described by the 2-D singular surface integral Eq. 8; and 3) the double-layer distribution, described by the 2-D singular integral Eq 10. The second derivatives of the potential V experience finite jumps when crossing corresponding interfaces. For the 3-D distributions, the formula for those jumps, Eq. 7, was established by Julius Weingarten in 1887.[1] We established Eqs. 48 and 57, playing the same role for the simple layer and double layers, respectively.

5. References

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