ARL-MR-1030 • FEB 2021

An Examination of Connectivity across 3-D Cubic-Lattice Networks

by Steven B Segletes

NOTICES

Disclaimers

ARL-MR-1030 • FEB 2021

An Examination of Connectivity across 3-D Cubic-Lattice Networks

by Steven B Segletes
Weapons and Materials Research Directorate, DEVCOM Army Research Laboratory

REPORT DOCUMENTATION PAGE	*Form Approved* *OMB No. 0704-0188*

1. REPORT DATE *(DD-MM-YYYY)*	February 2021
2. REPORT TYPE	Memorandum Report
3. DATES COVERED (From - To)	October 2020–February 2021
4. TITLE AND SUBTITLE	An Examination of Connectivity across 3-D Cubic-Lattice Networks
5a. CONTRACT NUMBER	
5b. GRANT NUMBER	
5c. PROGRAM ELEMENT NUMBER	
6. AUTHOR(S)	Steven B Segletes
5d. PROJECT NUMBER	
5e. TASK NUMBER	
5f. WORK UNIT NUMBER	
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)	DEVCOM Army Research Laboratory ATTN: FCDD-RLW-PC Aberdeen Proving Ground, MD 21005-5066
8. PERFORMING ORGANIZATION REPORT NUMBER	ARL-MR-1030
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)	
10. SPONSOR/MONITOR’S ACRONYM(S)	
11. SPONSOR/MONITOR'S REPORT NUMBER(S)	
12. DISTRIBUTION/AVAILABILITY STATEMENT	Approved for public release; distribution is unlimited.
13. SUPPLEMENTARY NOTES	primary author’s email: <steven.b.segletes.civ@mail.mil>.

14. ABSTRACT

Square- and cubic-lattice networks are examined as a metaphor for a mixed computational cell. A probability-cascade approach is developed and systematized as a means to determine the aggregate probability of connectivity across various size 2-D and 3-D networks, as a function of the inter-node-linkage probability.

15. SUBJECT TERMS

connectivity, recombinant probability cascade, 3-D network, mixed cell

16. SECURITY CLASSIFICATION OF: a. REPORT	b. ABSTRACT	c. THIS PAGE	17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON / 19b. TELEPHONE NUMBER (Include area code)
Unclassified	Unclassified	Unclassified	UU	22	Steven B Segletes 410-278-6010

Standard Form 298 (Rev. 8/98)
Prescribed by ANSI Std. Z39.18

Contents

List of Figures

List of Tables

Acknowledgments

The author is grateful to his esteemed colleague, Dr Michael Grinfeld, for his helpful review of this manuscript. Likewise, the author appreciates and recognizes his editor Ms Carol Johnson, for her excellent work in bringing this manuscript to fruition.

1. Introduction

In ARL-TR-8899,[1] electrical connectivity across square 2-D lattice networks, composed of a mixture of conducting and insulating linkages, was examined as a metaphor for the connectivity across a mixed computational cell. The method utilized there developed its statistical results by exhaustively enumerating and categorizing all realizations (possible states) of the network, for the situation wherein each network link has a given probability of exhibiting inter-node connectivity. Such a brute force approach reaches its practical limit of utility quickly, since the number of network realizations increases as 2^n, where n signifies the number of linkages in the network. This limitation is especially onerous since the number of linkages grows as $n \sim m^2$ in 2-D and $n \sim m^3$ in 3-D, where m is the linear dimension of the network.

An alternative (approximate) method was also explored in Appendix C of that report, in which the likelihood of connectivity (given by F_P) between an origin O and a given node P could be calculated as a probability *cascade*[*] originating from O. The goal of this report is to revisit this alternative *unidirectional* approach to network connectivity in a more systematic fashion and extend the results to networks of three spatial dimensions.

2. Nodal Families

In three dimensions, we wish to consider the point-to-point connectivity across the diagonal of a cubic lattice, from the node O, at $(0, 0, 0)$, to X, at (m, m, m). Let us define notation that will be useful, moving forward in this analysis.

While parentheses are used to denote an indexed node in the Cartesian network, for example $(1, 1, 1)$, square brackets are used here to denote a set of similar nodes that we call a *family*. Members of the family may be obtained by way of permuting the nodal indices. We use a subscript as a shorthand to remind the reader of the number of unique permutations. Thus, $[2, 1, 0]_6$ indicates a family that possesses six nodes,

[*]By cascade, we mean that network information flows across the lattice *in one direction*, from O to P. Nodes downstream from P (toward the endpoint X) play no role in affecting F_P (it is this *unidirectional* flow of information that constitutes the *approximation* to the general solution explored in ARL-TR-8899). This type of construction is also known as a recombinant "probability" or "decision" tree.

Another way to characterize this approximation is to say that only electrical pathways of minimal length are considered (pathlength $2(m-1)$ in an $m{\times}m$ network and $3(m-1)$ in an $m{\times}m{\times}m$ 3-D network). Longer meandering pathways (which, it should be noted, are less probable) are excluded from consideration as viable electrical pathways.

namely $\{(2,1,0),(1,2,0),(0,2,1),(0,1,2),(2,0,1),(1,0,2)\}$. If the family has repeated indices, the number of members is diminished—for example, $[2,1,1]_3$ has three members $\{(2,1,1),(1,2,1),(1,1,2)\}$. In the extreme case of triple-repeated indices, the family has a single member: $[1,1,1]_1 \equiv \{(1,1,1)\}$.

We have introduced the concept of nodal families to save on effort and notation, because the nodes within a family are topologically symmetric, relative to the network diagonal of interest (that which connects nodes O and X). This symmetry implies (for a set of randomly distributed conductors in the network) that the likelihood of connectivity between nodes O and P is the same for all nodes P of a given family. Likewise is true for the connectivity between nodes P and X.

To understand this concept of topological symmetry better, we begin with the analysis of 2-D networks (recall, in Appendix C of ARL-TR-8899, such a case was studied). In considering the connectivity between the diagonally extreme nodes of the lattice (which we denote O and X), we look to symmetry relative to the O–X diagonal. As an example, for the 3×3 square network depicted in Fig. 1, the points $(2,0)$ and $(0,2)$ possess such symmetry. Based on our newly introduced notation of family, we note that both nodes are part of the $[2,0]$ family. Likewise, there are also two nodes each in the $[1,0]$ and $[2,1]$ families.

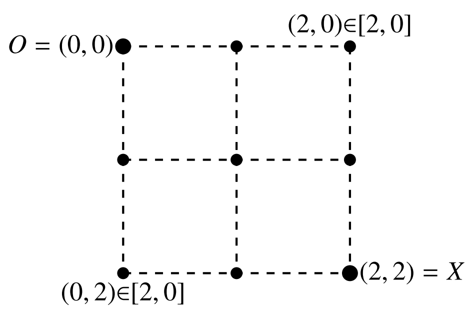

Fig. 1 A 3×3 network, depicting the symmetry of the $[2,0]$ node family, relative to the O–X diagonal

A convenient way to capture this notion of symmetric node families is given in Table 1, for the case of 2-D square networks. Columns depict the cumulative addition of nodal families, as the network size grows. Rows depict the unidirectional (*i.e.*, minimized) path length from O to any member of the given nodal family. The use of parentheses rather than brackets around diagonal nodes merely indicates a family of size one (*i.e.*, $(j,j) \equiv [j,j]_1$).

Table 1 Topological node family hierarchy for 2-D square networks

Path length	1×1	2×2	3×3	4×4	5×5
0	$(0,0)$				
1		$[1,0]_2$			
2		$(1,1)$	$[2,0]_2$		
3			$[2,1]_2$	$[3,0]_2$	
4			$(2,2)$	$[3,1]_2$	$[4,0]_2$
5				$[3,2]_2$	$[4,1]_2$
6				$(3,3)$	$[4,2]_2$
7					$[4,3]_2$
8					$(4,4)$
Node count:					
Marginal	1	3	5	7	9
Cumulative	1	4	9	16	25
=	1^2	2^2	3^2	4^2	5^2

3. The Cascade Relations (2-D)

We can use Table 1 to systematically recreate an analysis similar to that found in Appendix C of ARL-TR-8899, but for any size 2-D network (for which the hierarchy has been specified). The probability of connectivity between O and any node P in a given family derives from the node families that lie immediately upstream from P. For nodes containing a value of "0" as an index (certain edge nodes), information derives from one upstream neighbor, according to the relation

$$F_{[j,0]} = f F_{[j-1,0]} \quad . \tag{1}$$

Here, f denotes the probability that any given link (connecting two adjacent nodes) will be conductive, while F denotes the probability of connectivity from O to *any* particular node in the family denoted by the index. For families containing nodes without an index equal to "0", information derives from two upstream neighbors, according to the relation

$$F_{[j,k]} = f(F_{[j-1,k]} + F_{[j,k-1]}) - f^2 F_{[j-1,k]} F_{[j,k-1]} \quad . \tag{2}$$

This relation is a statement of mutually non-exclusive event probability, in which $(0,0)$–$[j,k]$ connectivity can be established by way of $[j-1,k]$, $[j,k-1]$, or both. Note that, based on the definition of a node family, the index order is not important, so that $[j, k] \equiv [k, j]$. However, for consistency, we adopt the convention in Table 1 of ordering the indices in decreasing order.

With these rules established, we can now systematically define the cascade through which the probability of connectivity from O to X is calculated. Starting with the tautology that $F_{(0,0)} = 1$ (*i.e.*, O is connected to itself), simply proceed down successive columns of Table 1, applying the rules of Eqs. 1 and 2, until the desired 2-D network size has been reached. For example, in the case of the 3×3 network depicted in Fig. 1,

$$\begin{aligned}
F_{(0,0)} &= 1 \\
F_{[1,0]} &= f F_{(0,0)} \\
F_{(1,1)} &= 2 f F_{[1,0]} - f^2 F_{[1,0]}^2 \\
F_{[2,0]} &= f F_{(1,0)} \\
F_{[2,1]} &= f (F_{(1,1)} + F_{[2,0]}) - f^2 F_{(1,1)} F_{[2,0]} \\
F_{O \to X} \equiv F_{(2,2)} &= 2 f F_{[2,1]} - f^2 F_{[2,1]}^2 \quad .
\end{aligned}$$

One may compare this cascade to the result found in Appendix C of ARL-TR-8899 and find, by way of substitution, the result to be identical. A primary advantage of introducing this *family* notation is that symmetry can be systematically taken advantage of in reducing the calculation of nodal connectivity probabilities. For a 2-D $m \times m$ matrix, the reduction in nodal evaluations reduces by up to a factor of 2, from m^2 to $\frac{1}{2} m(m+1)$. For this 3×3 network of nine nodes, the method requires $\frac{1}{2} \cdot 3 \cdot 4 = 6$ equations, as shown above.

And while this cascade method is only an approximation to the former exhaustive approach of ARL-TR-8899 that we hope to replace, the difference in calculation is stark. With the exhaustive approach, one requires for the 3×3 case (with 12 linkages) the analysis of $2^{12} = 4096$ network realizations (*i.e.*, unique configurations of conducting/insulating linkages that might describe the state of the network), with each realization providing up to 12 unique connectivity pathways from O to X. As the scale of the network grows, the comparison only becomes that much more lopsidedly prohibitive for the exhaustive approach—a 3×3×3 cubic network has 54 linkages and thus $2^{54} = 18{,}014{,}398{,}509{,}481{,}984$ realizations.

4. The 3-D Cascade

The real power of this systematic approach is garnered for the more challenging problem of a 3-D network with additional planes of symmetry, wherein the reduction in nodal evaluations approaches a factor as large as 6, going from m^3 to $\frac{1}{3!}m(m+1)(m+2)$. With the addition of the third dimension, an extra consideration is that some node families (those without a zero index) will get their probability information from three upstream neighbors. Stating the rules for the 3-D cascade,

$$\begin{aligned}
F_{[j,0,0]} &= fF_{[j-1,0,0]} \\
F_{[j,k,0]} &= f(F_{[j-1,k,0]} + F_{[j,k-1,0]}) - f^2 F_{[j-1,k,0]}F_{[j,k-1,0]} \\
F_{[j,k,l]} &= f(F_{[j-1,k,l]} + F_{[j,k-1,l]} + F_{[j,k,l-1]}) - \\
&\quad f^2(F_{[j-1,k,l]}F_{[j,k-1,l]} + F_{[j,k-1,l]}F_{[j,k,l-1]} + F_{[j-1,k,l]}F_{[j,k,l-1]}) + \\
&\quad f^3 F_{[j-1,k,l]}F_{[j,k-1,l]}F_{[j,k,l-1]}
\end{aligned} \quad .(3)$$

If it is not clear to the reader from whence these rules arise, they can be reckoned from the combined probability of one, two, and three (mutually non-exclusive) events A, B, and C (representing connectivity by way of distinct upstream nodes, respectively), where $P_A = fF_{[j-1,k,l]}$, $P_B = fF_{[j,k-1,l]}$, and $P_C = fF_{[j,k,l-1]}$. Equation 3, restated in these terms, would be much more recognizable:

$$\begin{aligned}
P(A) &= P_A \\
P(A \cup B) &= P_A + P_B - P_{A\cap B} \\
P(A \cup B \cup C) &= P_A + P_B + P_C - (P_{A\cap B} + P_{B\cap C} + P_{A\cap C}) + P_{A\cap B\cap C}
\end{aligned} \quad .$$

As in 2-D, the node families for 3-D can be systematically laid out, as is done in Table 2. Here, because of increased possibilities for symmetry, nodal families can possess one, three, or six members, depending on whether they have repeated indices (and if so, whether two or three). For the 5×5×5 case, there are 35 node families represented, as compared to 125 individual nodes, representing a savings of more than 3.5 times over an equivalent brute-force calculation analyzing each node separately.

Using the node families delineated in Table 2, with the rules of Eq. 3, the cascade for the 3-D cases may be constructed by proceeding through each successive column of the table, as in the 2-D case. In the Appendix, the *f77* code for doing so is provided (as well as the numerical results). The logic of Eq. 3 is laid out in the functions

Table 2 Topological node family hierarchy for 3-D cubic networks

Path length	1×1×1	2×2×2	3×3×3	4×4×4	5×5×5
0	$(0,0,0)$				
1		$[1,0,0]_3$			
2		$[1,1,0]_3$	$[2,0,0]_3$		
3		$(1,1,1)$	$[2,1,0]_6$	$[3,0,0]_3$	
4			$[2,1,1]_3$ $[2,2,0]_3$	$[3,1,0]_6$	$[4,0,0]_3$
5			$[2,2,1]_3$	$[3,1,1]_3$ $[3,2,0]_6$	$[4,1,0]_6$
6			$(2,2,2)$	$[3,2,1]_6$ $[3,3,0]_3$	$[4,1,1]_3$ $[4,2,0]_6$
7				$[3,2,2]_3$ $[3,3,1]_3$	$[4,2,1]_6$ $[4,3,0]_6$
8				$[3,3,2]_3$	$[4,2,2]_3$ $[4,3,1]_6$ $[4,4,0]_3$
9				$(3,3,3)$	$[4,3,2]_6$ $[4,4,1]_3$
10					$[4,3,3]_3$ $[4,4,2]_3$
11					$[4,4,3]_3$
12					$(4,4,4)$
Node count:					
Marginal	1	7	19	37	61
Cumulative	1	8	27	64	125
=	1^3	2^3	3^3	4^3	5^3

`From1`, `From2`, and `From3`. The cascade itself is provided in the main program block, which is cycled through for successive increments on the value of f. The output is graphically presented in Fig. 2.

While the results presented in Fig. 2 are approximate, because they derive from a unidirectional cascade rather than from an exhaustive tabulation of all possible network realizations, a comparison to the 2-D results presented in ARL-TR-8899 is nonetheless illustrative. We have reproduced those earlier results in Fig. 3.

While both results exhibit the characteristic S-curve associated with threshold behavior, we note, comparing Figs. 2 and 3, several significant differences:

- The threshold for transitioning to a conducting network occurs sooner in 3-D, at $f \approx 0.45$ rather than at $f \approx 0.65$ for 2-D.

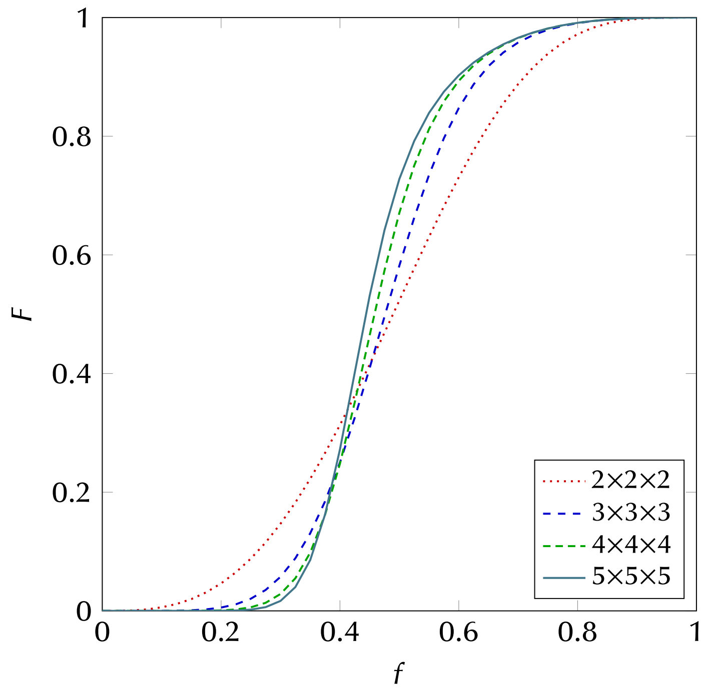

Fig. 2 Probability F of establishing connectivity across the diagonal of various 3-D cubic network *cascades*, as a function of f, the likelihood that any given network link is conductive

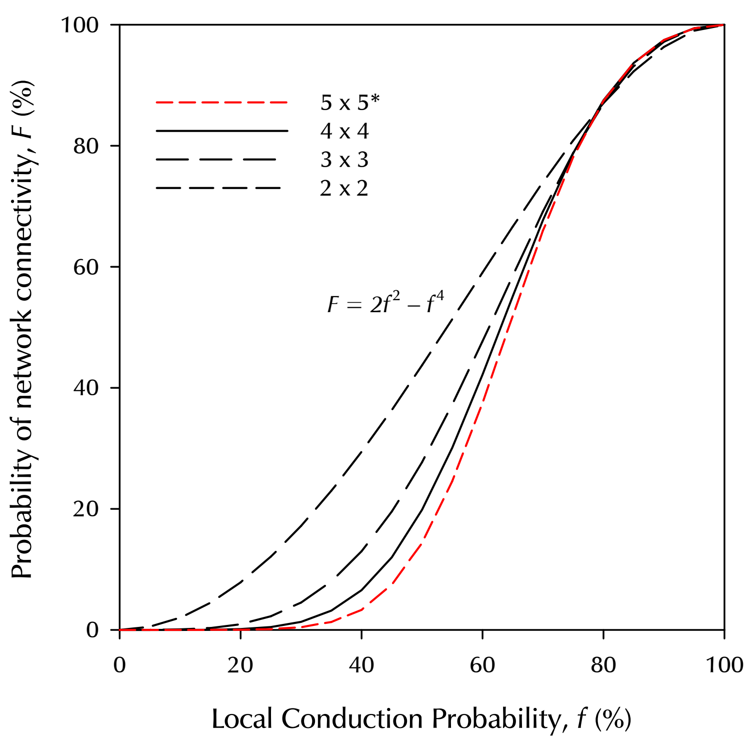

Fig. 3 Probability F of diagonal 2-D network connectivity, as a function of the likelihood f that any given network link is conductive (figure originally appeared as Fig. B-3a in ARL-TR-8899)

- The threshold transition for a 3-D network is steeper (and thus narrower) than the corresponding 2-D threshold.

While these distinctions are important, they are not necessarily surprising. In 3-D, interior nodes of the network have three mutually non-exclusive pathways to provide viable continuity, as compared to two such pathways for the 2-D case. Therefore, the overall probability of network connectivity increases for a given value of f, shifting the curves leftward in Fig. 2 *vis-à-vis* Fig. 3. The addition of the extra pathway in 3-D also means that the order of the polynomial (in f) that governs the $O \rightarrow X$ continuity is higher than in 2-D—thus, a narrower threshold window (*i.e.*, steeper gradient) becomes possible, as witnessed in Fig. 2.

5. Conclusion

In this report, we study networks comprising 2-D square and 3-D cubic lattices, in search of how the probability of linkage connectivity affects the overall likelihood of cross-network connectivity. A probability-cascade approach has been developed and studied as an approximation to a more exhaustive network-connectivity analysis. A downside of the cascade approach is that the more circuitous pathways across the network are excluded from consideration in establishing connectivity. This exclusion is mitigated somewhat by the fact that circuitous pathways will occur with a lower probability than direct pathways. More importantly, the cascade approximation allows for much larger networks to be considered, without running into an insurmountable calculational wall.

While an example of the cascade approach was originally presented in an appendix of ARL-TR-8899, here we systematize the approach and expand its application to that of 3-D cubic networks. The systematization not only provides calculation savings, by exploiting network symmetries relative to the diagonal, but also defines the topological hierarchy of nodal families (*e.g.*, Tables 1 and 2) through which the network connectivity must be established.

The establishment of such a hierarchy is necessary to facilitate automation of the connectivity cascade for any arbitrarily sized network, whether 2-D or 3-D—one merely proceeds downward through each successive column of the tabulated hierarchy until the desired matrix size has been reached. In this way, the probability inputs for any given nodal family in the network have already been established through prior calculations in the sequence. And while the hierarchies (Tables 1 and 2) provided in this report were deduced from inspection, such constructions can, likewise, be deduced via algorithm.

Results indicate a threshold behavior that is even more pronounced than that for 2-D networks. As the probability of linkage connectivity remains small (*e.g.*, below 40%), the overall likelihood of cross-network connectivity remains close to zero. However, if the linkage-connectivity likelihood rises to 50% or more, the likelihood of cross-network connectivity rises rapidly, becoming near certain by the time the linkage probability reaches a mere 60%.

It is hoped that these results may help guide an understanding of electrical conductivity and joule heating in mixed computational cells, for which the lattice networks of this report serve as a possible metaphor. The problem is challenging, especially for cases in which one species in the mixed cell occurs in small proportion to the overall cell composition.

6. References

1. Segletes S. On the electrical connectivity of a 2-D, randomly distributed, two-component (conducting/insulating) mixture. DEVCOM Army Research Laboratory; 2020 Jan. Report No.: ARL-TR-8899.

Appendix. Unidirectional Probability Cascade to Evaluate the Cross-Network Connectivity of 3-D Cubic Networks

The *f77* code on the following page demonstrates how the probability of O–X (diagonal, cross-network) connectivity can be calculated for 3-D lattices. In the code, the probability is calculated, using the cascade method, for $m \times m \times m$ cubic networks, for values of m up to 5.

The code loops through a range of per-linkage probabilities, f, between 0 and 1. The functions `From1`, `From2`, and `From3` calculate the combined probability for one, two, and three mutually non-exclusive events, respectively (connectivity by way of upstream nodes). The output of the code is given below.

f	2x2x2	3x3x3	4x4x4	5x5x5
0.0250	0.0001	0.0000	0.0000	0.0000
0.0500	0.0007	0.0000	0.0000	0.0000
0.0750	0.0025	0.0000	0.0000	0.0000
0.1000	0.0060	0.0001	0.0000	0.0000
0.1250	0.0116	0.0003	0.0000	0.0000
0.1500	0.0199	0.0010	0.0001	0.0000
0.1750	0.0313	0.0025	0.0003	0.0000
0.2000	0.0463	0.0056	0.0008	0.0002
0.2250	0.0651	0.0111	0.0024	0.0006
0.2500	0.0881	0.0205	0.0060	0.0021
0.2750	0.1153	0.0353	0.0136	0.0063
0.3000	0.1469	0.0574	0.0285	0.0168
0.3250	0.1827	0.0887	0.0550	0.0401
0.3500	0.2226	0.1308	0.0982	0.0857
0.3750	0.2663	0.1847	0.1622	0.1626
0.4000	0.3133	0.2502	0.2481	0.2721
0.4250	0.3632	0.3260	0.3519	0.4020
0.4500	0.4153	0.4092	0.4643	0.5311
0.4750	0.4689	0.4960	0.5738	0.6418
0.5000	0.5232	0.5819	0.6709	0.7280
0.5250	0.5773	0.6626	0.7505	0.7920
0.5500	0.6305	0.7349	0.8123	0.8394
0.5750	0.6819	0.7967	0.8588	0.8751
0.6000	0.7307	0.8474	0.8934	0.9026
0.6250	0.7763	0.8874	0.9193	0.9242
0.6500	0.8179	0.9181	0.9389	0.9413
0.6750	0.8553	0.9411	0.9539	0.9550
0.7000	0.8880	0.9580	0.9655	0.9660
0.7250	0.9159	0.9704	0.9745	0.9748
0.7500	0.9390	0.9794	0.9816	0.9816
0.7750	0.9576	0.9859	0.9870	0.9870
0.8000	0.9720	0.9906	0.9911	0.9911
0.8250	0.9826	0.9940	0.9942	0.9942
0.8500	0.9900	0.9963	0.9964	0.9964
0.8750	0.9948	0.9979	0.9980	0.9980
0.9000	0.9977	0.9990	0.9990	0.9990
0.9250	0.9992	0.9996	0.9996	0.9996
0.9500	0.9998	0.9999	0.9999	0.9999
0.9750	1.0000	1.0000	1.0000	1.0000
1.0000	1.0000	1.0000	1.0000	1.0000

```
      implicit none
      double precision f,g, F000,F100,F110,F111,F200,F210,F211,F220,
     &                 F221,F222,F300,F310,F311,F320,F321,F330,F322,
     &                 F331,F332,F333,F400,F410,F411,F420,F421,F430,
     &                 F422,F431,F440,F432,F441,F433,F442,F443,F444,
     &                 From1, From2, From3
      write(*,'(5a10)') 'f', '2x2x2', '3x3x3', '4x4x4', '5x5x5'
      f = 0.d0
      do while (f .le. 1.d0)
        f = f + 0.025d0
        F000 = 1.d0                     ! (0,0,0)     +1 = 1 = 1^3

        F100 = From1(f,F000)            ! [1,0,0]_3
        F110 = From2(f,F100,F100)       ! [1,1,0]_3
        F111 = From3(f,F110,F110,F110)! (1,1,1)     +7 = 8 = 2^3

        F200 = From1(f,F100)            ! [2,0,0]_3
        F210 = From2(f,F110,F200)       ! [2,1,0]_6
        F211 = From3(f,F111,F210,F210)! [2,1,1]_3
        F220 = From2(f,F210,F210)       ! [2,2,0]_3
        F221 = From3(f,F211,F211,F220)! [2,2,1]_3
        F222 = From3(f,F221,F221,F221)! (2,2,2)     +19 = 27 = 3^3

        F300 = From1(f,F200)            ! [3,0,0]_3
        F310 = From2(f,F210,F300)       ! [3,1,0]_6
        F311 = From3(f,F211,F310,F310)! [3,1,1]_3
        F320 = From2(f,F220,F310)       ! [3,2,0]_6
        F321 = From3(f,F221,F311,F320)! [3,2,1]_6
        F330 = From2(f,F320,F320)       ! [3,3,0]_3
        F322 = From3(f,F222,F321,F321)! [3,2,2]_3
        F331 = From3(f,F321,F321,F330)! [3,3,1]_3
        F332 = From3(f,F322,F322,F331)! [3,3,2]_3
        F333 = From3(f,F332,F332,F332)! (3,3,3)     +37 = 64 = 4^3

        F400 = From1(f,F300)            ! [4,0,0]_3
        F410 = From2(f,F310,F400)       ! [4,1,0]_6
        F411 = From3(f,F311,F410,F410)! [4,1,1]_3
        F420 = From2(f,F320,F410)       ! [4,2,0]_6
        F421 = From3(f,F321,F311,F420)! [4,2,1]_6
        F430 = From2(f,F330,F420)       ! [4,3,0]_6
        F422 = From3(f,F322,F421,F421)! [4,2,2]_3
        F431 = From3(f,F331,F421,F430)! [4,3,1]_6
        F440 = From2(f,F430,F430)       ! [4,4,0]_3
        F432 = From3(f,F332,F422,F431)! [4,3,2]_6
        F441 = From3(f,F431,F431,F440)! [4,4,1]_3
        F433 = From3(f,F333,F432,F432)! [4,3,3]_3
        F442 = From3(f,F432,F432,F441)! [4,4,2]_3
        F443 = From3(f,F433,F433,F442)! [4,4,3]_3
        F444 = From3(f,F443,F443,F443)! (4,4,4)     +61 = 125 = 5^3

        write (*,'(5f10.4)') f, F111, F222, F333, F444
      end do
      stop
      end
C**************************************************************
      double precision function From1(f, F1)
      implicit none
      double precision f, F1
      From1 = f *F1
      return
      end
C**************************************************************
      double precision function From2(f, F1, F2)
      implicit none
      double precision f, F1, F2
      From2 = f *(F1 + F2) - f**2 *F1*F2
      return
      end
C**************************************************************
      double precision function From3(f, F1, F2, F3)
      implicit none
      double precision f, F1, F2, F3
      From3 = f *(F1 + F2 + F3) - f**2 *(F1*F2 + F2*F3 + F1*F3)
     &     + f**3 *F1*F2*F3
      return
      end
C**************************************************************
```

1 (PDF) DEFENSE TECHNICAL
INFORMATION CTR
DTIC OCA

1 (PDF) DEVCOM ARL
FCDD RLD DCI
TECH LIB

1 (PDF) COMMANDER
US ARMY AVN & MISSILE CMD
S CORNELIUS

1 (PDF) NSWC
INDIAN HEAD DIVISION
T P MCGRATH II

1 (PDF) NAVAIR
E SIEVEKA

1 (PDF) US ARMY ERDEC
J Q EHRGOTT JR

3 (PDF) SANDIA NATIONAL LABS
J NIEDERHAUS
C SIEFERT
A E RODRIGUEZ

1 (PDF) ATR CORP
A WARDLAW

1 (PDF) SOUTHWEST RESEARCH INST
T HOLMQUIST

3 (PDF) DE TECHNOLOGIES
R CICCARELLI
W FLIS
W CLARK

1 (PDF) APPLIED RESEARCH ASSOC INC
R T BOCCHIERI

1 (PDF) DREXEL UNIVERSITY
B FAROUK

ABERDEEN PROVING GROUND

78 (PDF) DEVCOM ARL
FCDD RLD
P BAKER
FCDD RLD P
T ROSENBERGER
FCDD RLW
S SCHOENFELD
FCDD RLW C
D LYON
FCDD RLW LA
W OBERLE
FCDD RLW LC
K MCNESBY
B ROOS
FCDD RLW LF
J CONDON
FCDD RLW LH
T EHLERS
E KENNEDY
L MAGNESS
C MEYER
B SORENSEN
R SUMMERS
FCDD RLW MB
B LOVE
G GAZONAS
FCDD RLW MC
R JENSEN
FCDD RLW MD
S WALSH
FCDD RLW ME
P PATEL
FCDD RLW P
R FRANCART
J HOGGE
T VONG
R YEAGER
FCDD RLW PA
S R BILYK
P BERNING
M COPPINGER
J FLENIKEN
M GREENFIELD
W C UHLIG
C WOLFE
FCDD RLW PB
C HOPPEL
M J GRAHAM
S SATAPATHY
FCDD RLW PC
J CAZAMIAS
M FERMEN-COKER
R BECKER
D CASEM
J CLAYTON
B LEAVY
J LLOYD
S SEGLETES
L SHANNAHAN
A TONGE
C WILLIAMS

FCDD RLW PD
A BARD
N BRUCHEY
R DONEY
S HALSEY
M KEELE
D KLEPONIS
F MURPHY
D PETTY
C RANDOW
S SCHRAML
K STOFFEL
G VUNNI
V WAGONER
M ZELLNER
FCDD RLW PE
P SWOBODA
P BARTKOWSKI
D GALLARDY
D HORNBAKER
E HORWATH
J HOUSKAMP
E KLIER
C KRAUTHAUSER
M LOVE
K MCNAB
D SHOWALTER
FCDD RLW PF
N GNIAZDOWSKI
C CUMMINS
E FIORAVANTE
D FOX
R GUPTA
S HUG
FCDD RLW PG
S KUKUCK
C PECORA
J STEWART

12 (PDF) DEVCOM DAC
FCDD DAS LBA
D FORDYCE
FCDD DAS LBD
R GROTE
J POLESNE
FCDD DAS LBE
C BARKER
D HOWLE
FCDD DAS LBG
P MERGLER
J ABELL
P HORTON
FCDD DAS LBS
M PERRY
D LYNCH
J SHINDELL
FCDD DAS LBW
R BOWERS