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Exact Solution for a Consistent Model of a Charged Spherical Conductor

by Michael Grinfeld and Pavel Grinfeld

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Exact Solution for a Consistent Model of a Charged Spherical Conductor

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14. ABSTRACT

Classical engineering models of electricity carry an essential physical drawback—namely, they explicitly assume that the electronic liquid has unlimited compressibility. This results in electric charges accumulating at the boundaries of the conductor, and 3-D density reaching infinitely large values. In our recent publications, we suggested to fix to this drawback by assuming that the compressibility is finite. In this report, we consider the special case of a spherical conductor permitting the exact analytical solution for our model of conductor. The result is to be used for validation and verification of Alegra code in the area of electromagnetic armor.

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1. Introduction

Theoretical and engineering support of US Army Combat Capabilities Development Command Army Research Laboratory projects dealing with armor includes a lot of electrical engineering modeling. These days, electrical engineering models are quite complex and multifaceted. Therefore, modeling is impossible to accomplish without the use of computer programs. Those models and programs include numerous effects and relevant parameters, even when the underlying physical effects are, taken separately, quite transparent. Because of the use of models with multiple parameters, there are various problems with validation and verification (V&V). When the amount of available data is limited but the number of relevant parameters is large, there are plenty of opportunities to choose seemingly appropriate parameters. This simplicity of choosing the appropriate parameters is misleading because multiple superficial V&V procedures are very unstable. This instability means that tiny changes in experimental or engineering data bring about deviations in the choice of the relevant parameters. How to avoid this instability? Decompose the complex program into smaller chunks and apply the V&V to those chunks.

The V&V procedure includes, among other instruments, comparison with exact analytical solutions of the appropriate problem. In this report, we establish a solution for the radially symmetric problem of electrostatic equilibrium of a charged sphere. The conducting sphere is modelled as suggested in our earlier reports.[1–4]

2. Formulation of the Model of Electric Current in Unbounded Space

Our approach relies on the electrostatics model of the conductor, which we suggested in the publications.[1–4] In the case of a spherically symmetric conductor, the electrostatic potential j and the density r of the electronic “liquid” depend on only the radial coordinate r.

The master system of electrostatic equilibrium includes

- the bulk equation of electrostatic equilibrium

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\varphi}{dr}\right)=-4\pi\left(\sigma_e\rho_e+\sigma_i\rho_i\right) \tag{1}$$

- and the chemo-electric equilibrium condition

$$\varphi\sigma_e + a_e^2\rho_e = P, \tag{2}$$

where P is an unknown constant, whereas the function $a_e^2(\rho_e)$ is a given positive function[1–4] that can be determined using the internal or free energy density of the electronic liquid; σ_e and σ_i are the respective charge densities per unit mass of the negative (mobile) charge and positive (immobile) charge components per unit mass, and ρ_e and ρ_i are the mass densities of the components; φ is the electric potential.

The total positive charge of the lattice ions Q_i and the total negative charge of the electron liquid Q_e are equal to

$$Q_i = \frac{4\pi R^3\rho_i\sigma_i}{3}, \; Q_e = 4\pi\sigma_e\int_0^R d\xi\xi^2\rho_e(\xi) \tag{3}$$

Let Q_{tot} be the given total charge of the charged conductor

$$Q_{tot} = \frac{4\pi R^3\rho_i\sigma_i}{3} + 4\pi\sigma_e\int_0^R d\xi\xi^2\rho_e(\xi) \tag{4}$$

Out of the conductor, the electrostatics potential is given by the relationship

$$\varphi_p(r,t) = \frac{Q_{tot}}{r} \tag{5}$$

The bulk electrostatics system (Eqs. 1–4) should be supplied with the standard electrostatics boundary condition of the electrostatics potential across the conductor boundary

$$\left[\varphi\right]_-^+ = 0, \quad \left[\frac{\partial\varphi}{\partial r}\right]_-^+ = 0 \tag{6}$$

Eliminating the electrostatics potential φ between the bulk (Eqs. 1 and 2), we arrive at the following equation, which is closed with respect to the unknown density ρ_e

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{1}{\sigma_e}\frac{d(a_e^2\rho_e)}{dr}\right) - 4\pi\left(\sigma_e\rho_e + \sigma_i\rho_i\right) = 0 \tag{7}$$

Choosing the special model for which

$$a_e^2 = const\,, \tag{8}$$

we can rewrite Eq. 7 as

$$\frac{1}{r^2}\frac{d}{dr}(r^2\frac{d\rho_e}{dr}) - \frac{4\pi\sigma_e}{a_e^2}(\sigma_e\rho_e + \sigma_i\rho_i) = 0 \tag{9}$$

Using the identity

$$r^2\frac{d\rho_e}{dr} = r\frac{d(r\rho_e)}{dr} - r\rho_e\,, \tag{10}$$

we can rewrite Eq. 9 as follows:

$$\frac{1}{r^2}\frac{d}{dr}(r\frac{d(r\rho_e)}{dr} - r\rho_e) - \frac{4\pi\sigma_e}{a_e^2}(\sigma_e\rho_e + \sigma_i\rho_i) = 0 \tag{11}$$

and then

$$\frac{d^2(r\rho_e)}{dr^2} - \frac{4\pi\sigma_e^2}{a_e^2}r\rho_e = \frac{4\pi\sigma_e\sigma_i\rho_i}{a_e^2}r \tag{12}$$

The general solution of the linear inhomogeneous Eq. 12 can be presented as a sum of any particular solution $\tilde{\rho}_e(z)$ and the general solution $\rho_e^{\mathrm{hom}}(z)$ of its homogeneous part.[5] The simplest partial solution reads

$$\tilde{\rho}_e = -\frac{\sigma_i}{\sigma_e}\rho_i \tag{13}$$

The general solution of the homogenous part of Eq. 12 reads

$$\rho_e^{\mathrm{hom}}(r) = C_s\frac{1}{r}\sinh\sqrt{\frac{4\pi\sigma_e^2r^2}{a_e^2}} + C_c\frac{1}{r}\cosh\sqrt{\frac{4\pi\sigma_e^2r^2}{a_e^2}} \tag{14}$$

Thus, we arrive at the following general solution of Eq. 9:

$$\rho_e(r) = C_s\frac{1}{r}\sinh\sqrt{\frac{4\pi\sigma_e^2r^2}{a_e^2}} + C_c\frac{1}{r}\cosh\sqrt{\frac{4\pi\sigma_e^2r^2}{a_e^2}} - \frac{\sigma_i}{\sigma_e}\rho_i \tag{15}$$

where C_s and C_c are certain constants to be determined.

For the solution of Eq. 15, to be finite at $r = 0$, the constant C_c must be equal to zero; thus, the solution of Eq. 15 can be rewritten as

$$\rho_e(r) = C_s \frac{1}{r} \sinh\sqrt{\frac{4\pi\sigma_e^2 r^2}{a_e^2}} - \frac{\sigma_i}{\sigma_e}\rho_i \tag{16}$$

Inserting Eq. 16 into the charge balance of Eq. 4, we get

$$Q_{tot} = Q_e + Q_i = C_s 4\pi\sigma_e \int_0^R d\xi \xi \sinh\sqrt{\frac{4\pi\sigma_e^2 \xi^2}{a_e^2}} \tag{17}$$

as implied by the following chain:

$$\begin{aligned} Q_e &= 4\pi\sigma_e \int_0^R d\xi \xi^2 \rho_e(\xi) = \\ &4\pi\sigma_e \int_0^R d\xi \xi^2 \left\{ C_s \frac{1}{\xi} \sinh\sqrt{\frac{4\pi\sigma_e^2 \xi^2}{a_e^2}} - \frac{\sigma_i}{\sigma_e}\rho_i \right\} = \\ &4\pi\sigma_e \int_0^R d\xi \left\{ C_s \xi \sinh\sqrt{\frac{4\pi\sigma_e^2 \xi^2}{a_e^2}} - \xi^2 \frac{\sigma_i}{\sigma_e}\rho_i \right\} = \\ &C_s 4\pi\sigma_e \int_0^R d\xi \xi \sinh\sqrt{\frac{4\pi\sigma_e^2 \xi^2}{a_e^2}} - \frac{4\pi R^3}{3}\sigma_i \rho_i \end{aligned} \tag{18}$$

The integral in Eq. 18 can be integrated by changing the independent variation:

$$\begin{aligned} &\int_0^R d\xi \xi \sinh\sqrt{\frac{4\pi\sigma_e^2 \xi^2}{a_e^2}} = \frac{a_e^2}{4\pi\sigma_e^2} \int_0^{\sqrt{\frac{4\pi\sigma_e^2 R^2}{a_e^2}}} d\eta\, \eta \sinh\eta = \\ &\frac{a_e^2}{4\pi\sigma_e^2}\left(\sqrt{\frac{4\pi\sigma_e^2 R^2}{a_e^2}} \cosh\sqrt{\frac{4\pi\sigma_e^2 R^2}{a_e^2}} - \sinh\sqrt{\frac{4\pi\sigma_e^2 R^2}{a_e^2}} \right) \end{aligned} \tag{19}$$

Using Eq. 19, we can rewrite Eq. 17 as follows

$$Q_{tot} = C_s \frac{a_e^2}{\sigma_e}\left(\sqrt{\frac{4\pi\sigma_e^2 R^2}{a_e^2}} \cosh\sqrt{\frac{4\pi\sigma_e^2 R^2}{a_e^2}} - \sinh\sqrt{\frac{4\pi\sigma_e^2 R^2}{a_e^2}} \right) \tag{20}$$

and it allows us to find the constant C_s

$$C_s = Q_{tot}\,\frac{\sigma_e}{a_e^2}\,\frac{1}{\sqrt{\dfrac{4\pi\sigma_e^2 R^2}{a_e^2}}\,\cosh\sqrt{\dfrac{4\pi\sigma_e^2 R^2}{a_e^2}} - \sinh\sqrt{\dfrac{4\pi\sigma_e^2 R^2}{a_e^2}}} \tag{21}$$

Inserting Eq. 21 into Eq. 16, we arrive at the following result:

$$\rho_e(r) = Q_{tot}\,\frac{\sigma_e}{a_e^2}\,\frac{\dfrac{1}{r}\sinh\sqrt{\dfrac{4\pi\sigma_e^2 r^2}{a_e^2}}}{\sqrt{\dfrac{4\pi\sigma_e^2 R^2}{a_e^2}}\,\cosh\sqrt{\dfrac{4\pi\sigma_e^2 R^2}{a_e^2}} - \sinh\sqrt{\dfrac{4\pi\sigma_e^2 R^2}{a_e^2}}} - \frac{\sigma_i}{\sigma_e}\rho_i \tag{22}$$

In particular, we get the following relationship for the density at the outer boundary of the sphere and in its center:

$$\rho_e(R) = Q_{tot}\,\frac{\sigma_e}{Ra_e^2}\,\frac{\tanh\sqrt{4\pi\sigma_e^2 a_e^{-2} R^2}}{\sqrt{4\pi\sigma_e^2 a_e^{-2} R^2} - \tanh\sqrt{4\pi\sigma_e^2 a_e^{-2} R^2}} - \frac{\sigma_i}{\sigma_e}\rho_i \tag{23}$$

and

$$\rho_e(0) = Q_{tot}\,\frac{\sigma_e}{Ra_e^2}\,\frac{\dfrac{\sqrt{4\pi\sigma_e^2 a_e^{-2} R^2}}{\sinh\sqrt{4\pi\sigma_e^2 a_e^{-2} R^2}}\tanh\sqrt{4\pi\sigma_e^2 a_e^{-2} R^2}}{\sqrt{4\pi\sigma_e^2 a_e^{-2} R^2} - \tanh\sqrt{4\pi\sigma_e^2 a_e^{-2} R^2}} - \frac{\sigma_i}{\sigma_e}\rho_i \tag{24}$$

Using Eqs. 23 and 24, we get the following formula for the density gap:

$$\rho_e(R) - \rho_e(0) = \frac{Q_{tot}\sigma_e}{Ra_e^2}\,\frac{\sinh\sqrt{4\pi\sigma_e^2 a_e^{-2} R^2} - \sqrt{4\pi\sigma_e^2 a_e^{-2} R^2}}{\cosh\sqrt{4\pi\sigma_e^2 a_e^{-2} R^2}\left(\sqrt{4\pi\sigma_e^2 a_e^{-2} R^2} - \tanh\sqrt{4\pi\sigma_e^2 a_e^{-2} R^2}\right)}$$

(25)

Using the nondimensional parameter Γ,

$$\Gamma \equiv \sqrt{\frac{4\pi\sigma_e^2 R^2}{a_e^2}} \tag{26}$$

we can rewrite Eqs. 21–25 as

$$\rho_e(r) = Q_{tot}\frac{\sigma_e}{a_e^2}\frac{\frac{1}{r}\sinh\frac{\Gamma r}{R}}{\Gamma\cosh\Gamma - \sinh\Gamma} - \frac{\sigma_i}{\sigma_e}\rho_i \tag{27}$$

$$\rho_e(R) = Q_{tot}\frac{\sigma_e}{Ra_e^2}\frac{\tanh\Gamma}{\Gamma - \tanh\Gamma} - \frac{\sigma_i}{\sigma_e}\rho_i \tag{28}$$

$$\rho_e(0) = Q_{tot}\frac{\sigma_e}{Ra_e^2}\frac{\frac{\Gamma}{\sinh\Gamma}\tanh\Gamma}{\Gamma - \tanh\Gamma} - \frac{\sigma_i}{\sigma_e}\rho_i \tag{29}$$

$$\rho_e(R) - \rho_e(0) = \frac{Q_{tot}\sigma_e}{Ra_e^2}\frac{\sinh\Gamma - \Gamma}{\cosh\Gamma(\Gamma - \tanh\Gamma)} \tag{30}$$

When establishing the solution of Eq. 22, we assumed that the mass density is positive everywhere. It is not guaranteed. Let us calculated the radius $r = R_c$ at which the local density vanishes: $r_e(R_c) = 0$. Using Eq. 22, we arrive at the following nonlinear equation with respect to R_c:

$$\frac{\sinh(\Gamma Z)}{\Gamma Z} = -\frac{\Sigma}{\Theta_{tot}}\frac{\Gamma\cosh\Gamma - \sinh\Gamma}{\Gamma}, \tag{31}$$

or

$$G(\Gamma, Z_c, \Upsilon) = 0, \tag{32}$$

where $G(\Gamma, Z_c, \Upsilon)$ is defined as

$$G(\Gamma, Z, \Upsilon) \equiv \Upsilon\frac{\Gamma\cosh\Gamma - \sinh\Gamma}{\Gamma} - \frac{\sinh(\Gamma Z)}{\Gamma Z}\ . \tag{33}$$

In Eqs. 31–33, we use the following nondimensional parameters:

$$\begin{gathered}\Sigma \equiv -\frac{\sigma_i}{\sigma_e},\ Z \equiv \frac{r}{R},\ Z_c \equiv \frac{R_c}{R},\\ \Gamma \equiv \sqrt{\frac{4\pi\sigma_e^2R^2}{a_e^2}},\ \Theta_{tot} \equiv \frac{Q_{tot}\sigma_e}{\rho_i a_e^2 R},\\ \Pi \equiv \frac{\Theta_{tot}}{\Sigma} = -\frac{Q_{tot}}{\sigma_i}\frac{\sigma_e^2}{\rho_i a_e^2 R},\ \Upsilon = \frac{\sigma_i}{Q_{tot}}\frac{\rho_i a_e^2 R}{\sigma_e^2}\end{gathered} \tag{34}$$

Using Eq. 34, we can rewrite Eq. 27 as

$$\frac{\rho_e(Z)}{\rho_i} = \Theta_{tot} \frac{\Gamma}{\Gamma \cosh \Gamma - \sinh \Gamma} \frac{\sinh(\Gamma Z)}{\Gamma Z} + \Sigma \equiv \Sigma \Lambda(\Gamma, Z, \Pi) \quad (35)$$

where

$$\Lambda(\Gamma, Z, \Pi) \equiv \Pi \frac{\Gamma}{\Gamma \cosh \Gamma - \sinh \Gamma} \frac{\sinh(\Gamma Z)}{\Gamma Z} + 1 \quad (36)$$

Figure 1 presents Λ as functions of Z for G = 5 and several values of the parameter P = 1, 0, –1, –3, and –5. Let us analyze Fig. 1 and Eqs. 34–36.

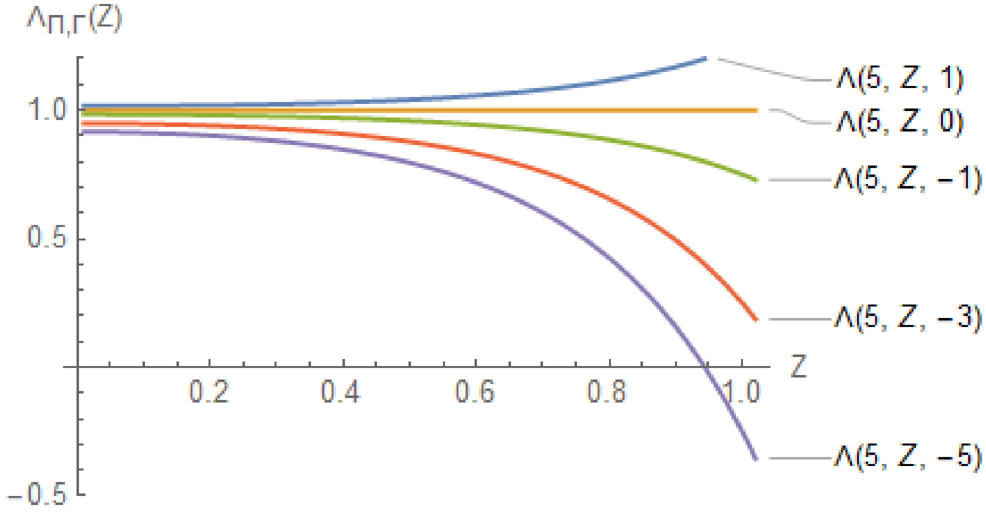

Fig. 1 Graph of the function L(G,*Z*,P) as a function of *Z* with parameters P and G

Per Eq. 34, S > 0, and therefore the behavior of $\rho_e(r)$ is the same as the behavior of L as a function of *r*.

Per the obvious relationship $Q_{tot} = Q_e + Q_c$; when $Q_e = -Q_c$, the total charge of conductor Q_{tot} vanishes. When the sign of the total charge Q_{tot} coincides with the sign of σ_e and opposite to the sign of s_i, the nondimensional parameter Π in this case appears to be positive. And vice versa when the sign of the total charge Q_{tot} coincides with the sign of s_i, the nondimensional parameter Π appears to be negative.

Per Fig. 1, when the conductor is positively charged (deficit of electrons), the parameter P is positive and the mass density of electrons $\rho_e(r)$ is growing with distance from the center. This behavior is illustrated by the blue curve for the case P = 1. This curve is also instructive to analyze the case when Q_{tot} approaches zero.

In this situation the density distribution curve cannot cross the Z axis within the interval $0 < Z < 1$.

The yellow (horizontal) line in Fig. 1 corresponds to the case when the conductor is electrically neutral and the density ratio is constant everywhere, as expected.

Three remaining curves correspond to the cases (P = –1, –3, –5) when there is a surfeit of electrons with respect to the neutral amount. In those cases, the parameter P assumes negative values and the functions $r_e(r)$ decay monotonically. Moreover, for some values of P, the functions $r_e(r)$ vanish inside the conductor at $r = r_c$, such that $0 < r_c < R$.

The solution with $\rho_e(r) < 0$ is physically unacceptable and should be modified. At the threshold value P_c, the density vanishes at the conductor's boundary $r_c = R$. Equation 35 implies the following values of P_c:

$$\Pi_c(\Gamma) = 1 - \frac{1}{\tanh \Gamma} \tag{37}$$

The graph of $P_c(G)$ is shown in Fig. 2.

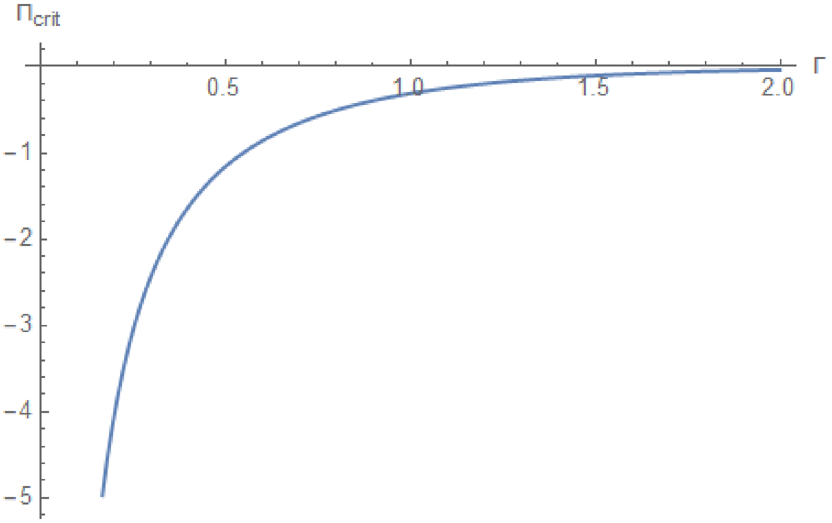

Fig. 2 Graph of P_c (G)

3. Conclusion

In previous publications, we suggested models of electric conductors that do not lead to the appearance of domains with infinitely high densities of electric current.[1–4] In this report, we establish an exact analytical solution of the earlier suggested model for the case of spherical conductors.

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