



**Modeling a High Speed Pin-on-Disc Experiment by  
Comparison of Numerical Solutions to a Moving  
Boundary Nonlinear Heat Equation**

THESIS

Aron R. Wing, Captain, USAF  
AFIT-ENC-MS-21-M-003

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MODELING A HIGH SPEED PIN-ON-DISC EXPERIMENT BY COMPARISON OF  
NUMERICAL SOLUTIONS TO A MOVING BOUNDARY NONLINEAR HEAT  
EQUATION

THESIS

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in Partial Fulfillment of the Requirements for the  
Degree of Master of Science in Applied Mathematics

Aron R. Wing, M.S.

Captain, USAF

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## **Abstract**

The purpose of this work is to analyze the heat transfer characteristics of Vascomax®C300 during high-speed sliding. This work extends previous research that is intended to help predict the wear-rate of connecting shoes for a hypersonic rail system at Holloman Air Force Base. Solutions were generated using finite element analysis, finite difference methods, and spectral methods. The frictional heat generated by the pin-on-disc is assumed to flow uniformly and normal to the face of the pin and the pin is assumed to be a perfect cylinder resulting in two-dimensional heat flow. Displacement data obtained from the experiment is used to define the moving boundary. The distribution of temperature resulting from transient finite element analysis is used to justify a one-dimensional model. The one dimensional heat equation is solved with variable parameters using a variable-step, variable-order, implicit finite-difference method. Spectral methods are then employed to calculate the spatial derivatives improving the approximation of the function which represents the data. It is concluded that a one-dimensional approach with constant heat transfer parameters sufficiently models the high-speed pin-on-disc experiment.

## **Acknowledgements**

The Air Force Research Laboratory, Aerospace Systems Directorate sponsored this project and facilitated the experimental testing.

To my best friend, my wife. Thank you. I love you.

Aron R. Wing

# Table of Contents

|                                                        | Page      |
|--------------------------------------------------------|-----------|
| Abstract .....                                         | iv        |
| Acknowledgements .....                                 | v         |
| List of Figures .....                                  | viii      |
| List of Tables .....                                   | xiii      |
| <b>I. Introduction .....</b>                           | <b>1</b>  |
| 1.1 Background .....                                   | 1         |
| 1.2 Brief History of Related Work .....                | 2         |
| 1.3 Experiment Design .....                            | 5         |
| 1.4 Prologue.....                                      | 7         |
| <b>II. Literature Review .....</b>                     | <b>9</b>  |
| 2.1 Heat Transfer.....                                 | 9         |
| 2.2 Tribology .....                                    | 10        |
| 2.2.1 Friction .....                                   | 10        |
| 2.2.2 Conduction .....                                 | 13        |
| 2.3 Assumptions Regarding Friction and Conduction..... | 15        |
| <b>III. Numerical Solutions .....</b>                  | <b>16</b> |
| 3.1 Finite Element Analysis .....                      | 16        |
| 3.1.1 FEA Solution Methodology .....                   | 16        |
| 3.1.2 Steady State- FEA .....                          | 25        |
| 3.1.3 Transient- FEA .....                             | 27        |
| 3.2 Finite Difference Methods .....                    | 33        |
| 3.2.1 Rescaled Partial Differential Equation.....      | 33        |
| 3.2.2 Finite Differences.....                          | 34        |
| 3.2.3 FDM- Constant Parameters .....                   | 44        |
| 3.2.4 pdepe- Non-constant Parameters .....             | 46        |
| 3.2.5 Optimizing Parameters .....                      | 48        |
| 3.3 Spectral Methods .....                             | 50        |
| 3.3.1 Investigating Stiffness .....                    | 57        |
| 3.4 Future Explorations .....                          | 61        |
| <b>IV. Conclusion .....</b>                            | <b>63</b> |

|                                             | Page |
|---------------------------------------------|------|
| Appendix A. Abaqus Model Tutorial .....     | 65   |
| 1.1 Creating a New File and Directory ..... | 65   |
| 1.2 The Model in Review .....               | 66   |
| 1.3 Build Part .....                        | 68   |
| 1.4 Define Material Properties .....        | 72   |
| 1.5 Model Assembly .....                    | 73   |
| 1.6 Create Steps .....                      | 73   |
| 1.7 Define Interactions .....               | 75   |
| 1.8 Applying Load .....                     | 79   |
| 1.9 Creating the Mesh .....                 | 80   |
| 1.10 Define Boundary Condition .....        | 81   |
| 1.11 Create Output Sets .....               | 82   |
| 1.12 Create Initial Condition .....         | 84   |
| 1.13 Create and Run Job .....               | 85   |
| 1.14 Viewing Results .....                  | 85   |
| 1.15 Create Outputs .....                   | 88   |
| Bibliography .....                          | 93   |

## List of Figures

| Figure |                                                                                                                                  | Page |
|--------|----------------------------------------------------------------------------------------------------------------------------------|------|
| 1      | Hypersonic Rail System Figures .....                                                                                             | 3    |
| 2      | Pin-on-Disc Figures [15] .....                                                                                                   | 4    |
| 3      | Experiment Rig .....                                                                                                             | 6    |
| 4      | Raw Experiment Data- Force and Displacement .....                                                                                | 6    |
| 5      | Raw Experiment Data- Temperature .....                                                                                           | 6    |
| 6      | Laraqi Partition [21] .....                                                                                                      | 11   |
| 7      | Lim and Ashby Coefficient of Friction [23] .....                                                                                 | 12   |
| 8      | Thermal Conductivity Approximation [15] .....                                                                                    | 14   |
| 9      | Specific Heat Approximation [17] .....                                                                                           | 15   |
| 10     | Thermal Diffusivity Approximation [17] [9] .....                                                                                 | 15   |
| 11     | FEA Flowchart [16] .....                                                                                                         | 17   |
| 12     | Planes Normal to, and through the center of, a Cylinders<br>Face.....                                                            | 18   |
| 13     | Finite Element Analysis Model .....                                                                                              | 18   |
| 14     | Finite Element Analysis Partitions .....                                                                                         | 23   |
| 15     | Force Data vs Force Average .....                                                                                                | 24   |
| 16     | Laraqi Partition with Estimate for 12 inch AISI 4340 Disc .....                                                                  | 25   |
| 17     | Steady State- FEA Model Maximum and Minimum<br>Temperatures at Source Boundary After Removal From<br>Step 8 - 117 nodes .....    | 26   |
| 18     | Steady State- FEA Model Convergence of Maximum and<br>Minimum Temperatures at Source Boundary After<br>Removal From Step 8 ..... | 26   |
| 19     | Steady State- FEA Model at initial and with 1 displacment .....                                                                  | 27   |

| Figure |                                                                                                        | Page |
|--------|--------------------------------------------------------------------------------------------------------|------|
| 20     | Steady State- FEA Model after 2 and 3 displacements .....                                              | 27   |
| 21     | Steady State- FEA Model after 7 and 8 displacements .....                                              | 28   |
| 22     | Steady State- FEA Thermocouple Temperatures vs Data .....                                              | 28   |
| 23     | Transient 2d- Abaqus Model at time 0 and 1 sec .....                                                   | 29   |
| 24     | Transient 2d- FEA Model at time 2 and 3 sec .....                                                      | 29   |
| 25     | Transient 2d- FEA Model at time 7 and 8 sec .....                                                      | 30   |
| 26     | Transient 2d- FEA Maximum Temperature Difference in<br>Direction Parallel to the Bottom Boundary ..... | 30   |
| 27     | Transient 2d- FEA Thermocouple Temperatures vs Data .....                                              | 30   |
| 28     | Transient 2d- FEA Thermocouple Temperatures vs<br>Transient 1d- FEA Thermocouple Temperatures .....    | 31   |
| 29     | Transient 1d- FEA Thermocouple Temperatures vs Data .....                                              | 31   |
| 30     | Transient 1d Alternate- FEA Thermocouple Temperatures<br>vs Data .....                                 | 32   |
| 31     | Finite Difference Method Model .....                                                                   | 34   |
| 32     | Finite Difference Method Model With Moving Boundary .....                                              | 42   |
| 33     | Removal by Displacement Data using pdepe with Constant<br>Parameters- Top View .....                   | 44   |
| 34     | Removal by Displacement Data using pdepe with Constant<br>Parameters- Isometric View .....             | 45   |
| 35     | Comparing Removal by Displacement with Constant<br>Parameters using pdepe with Abaqus .....            | 45   |
| 36     | Comparing Removal by Displacement with Constant<br>Parameters using pdepe with Data .....              | 46   |
| 37     | Variable Force Averaged Interpolant .....                                                              | 47   |
| 38     | 6th Order Polynomial Interpolant of Deleon's Specific Heat .....                                       | 47   |

| Figure |                                                                                                                | Page |
|--------|----------------------------------------------------------------------------------------------------------------|------|
| 39     | Comparing Removal by Displacement with Variable Parameters using pdepe with Constant Parameters and Data ..... | 48   |
| 40     | Optimized Parameters at Thermocouples using pdepe vs Data .....                                                | 49   |
| 41     | Optimized Parameters using pdepe Surface Plot .....                                                            | 49   |
| 42     | Convergence of a Fourth order FDM [27] .....                                                                   | 52   |
| 43     | Convergence of a Spectral Method using Chebyshev Points [27] .....                                             | 52   |
| 44     | Spectral Differentiation with Constant Parameters- Surface Temp and Thermocouples vs Data .....                | 54   |
| 45     | Spectral Differentiation with Constant Parameters- Scatter Plot Top View .....                                 | 54   |
| 46     | Spectral Differentiation with Variable Parameters- Surface Temp and Thermocouples vs Data .....                | 55   |
| 47     | Spectral Differentiation with Variable Parameters- Scatter Plot Top View .....                                 | 55   |
| 48     | Spectral Differentiation with Optimized Parameters- Surface Temp and Thermocouples vs Data .....               | 56   |
| 49     | Spectral Differentiation with Optimized Parameters- Scatter Plot Top View .....                                | 56   |
| 50     | Convergence of ODE45 Solving Equation 42 .....                                                                 | 58   |
| 51     | Convergence of ODE15s Solving Equation 42 .....                                                                | 58   |
| 52     | ODE45 Solving Equation 42 with Constant Parameters and N=18 .....                                              | 59   |
| 53     | ODE45 Solving Equation 42 with Variable Parameters and N=18 .....                                              | 59   |
| 54     | ODE45 with N=36 vs ODE15s with N=150 with Constant Parameters .....                                            | 60   |
| 55     | ODE45 with N=36 vs ODE15s with N=150 with Variable Parameters .....                                            | 61   |

| Figure |                                                          | Page |
|--------|----------------------------------------------------------|------|
| 56     | Start Session .....                                      | 65   |
| 57     | Save As .....                                            | 65   |
| 58     | Model .....                                              | 66   |
| 59     | Create Part .....                                        | 68   |
| 60     | Add Rectangle .....                                      | 69   |
| 61     | Create Datum Planes .....                                | 70   |
| 62     | Create Partitions .....                                  | 71   |
| 63     | Switch Module .....                                      | 72   |
| 64     | Create Material .....                                    | 72   |
| 65     | Create Section .....                                     | 73   |
| 66     | Create Step .....                                        | 74   |
| 67     | Create Increment .....                                   | 74   |
| 68     | Create Convection Interaction with Surrounding Air ..... | 75   |
| 69     | Select Conv-1 Surfaces .....                             | 76   |
| 70     | Define Convection-2 .....                                | 76   |
| 71     | Select and Define Convection-8 .....                     | 77   |
| 72     | Deactivating Propagated Interactions .....               | 78   |
| 73     | Create Removal Interaction .....                         | 78   |
| 74     | Select Region for Removal .....                          | 79   |
| 75     | Create Load .....                                        | 79   |
| 76     | Select Load-1 Surface .....                              | 79   |
| 77     | Select Load-2 Surface .....                              | 80   |
| 78     | Select Load-2 Surface Side .....                         | 80   |
| 79     | All 9 Loads Pointing "into" the Plate .....              | 80   |

| Figure |                                                                             | Page |
|--------|-----------------------------------------------------------------------------|------|
| 80     | Create Mesh .....                                                           | 81   |
| 81     | Create Boundary Condition .....                                             | 82   |
| 82     | Boundary Condition Complete .....                                           | 82   |
| 83     | Select Thermocouple Nodes .....                                             | 83   |
| 84     | Select Surface Nodes .....                                                  | 84   |
| 85     | Create Initial Condition .....                                              | 84   |
| 86     | Watch Movie .....                                                           | 85   |
| 87     | Create XY Plot of Thermocouple Temperatures with<br>Respect to Time 1 ..... | 86   |
| 88     | Create XY Plot of Thermocouple Temperatures with<br>Respect to Time 2 ..... | 87   |
| 89     | XY Plot of Thermocouple Temperatures vs Time .....                          | 87   |
| 90     | Create Display Groups .....                                                 | 88   |
| 91     | Create Thermocouple Temperature Report File .....                           | 89   |
| 92     | Selecting Step and Frame .....                                              | 89   |
| 93     | Excel Report File 1 .....                                                   | 90   |
| 94     | Excel Report File 2 .....                                                   | 91   |
| 95     | Excel Report File 3 .....                                                   | 92   |

## List of Tables

| Table | Page                                                                                                                                    |
|-------|-----------------------------------------------------------------------------------------------------------------------------------------|
| 1     | Symbols Definitions and Units ..... 2                                                                                                   |
| 2     | Vascomax® 300 Material Properties at Room Temperature<br>[1] ..... 10                                                                   |
| 3     | Vascomax® 300 Specific Heat [19] ..... 10                                                                                               |
| 4     | AISI 4340 Steel Material Properties ..... 10                                                                                            |
| 5     | Vascomax® 300 Mean Coefficient of Thermal Expansion<br>[1] ..... 14                                                                     |
| 6     | FEA Material Removal ..... 23                                                                                                           |
| 7     | FEA Material Properties ..... 23                                                                                                        |
| 8     | FEA Flux Parameters ..... 24                                                                                                            |
| 9     | FEA- Summary of Relative Differences at Thermocouples<br>1, 2, and 3 (Labeled Here As $T_i$ ) ..... 32                                  |
| 10    | Material Removal- Scaled for FDM ..... 35                                                                                               |
| 11    | FDM- Forward Difference Errors of $\sin'(\pi/6)$ With<br>Respect to Spacing ..... 37                                                    |
| 12    | FDM- Stiffness Ratios for Equation 31 ..... 41                                                                                          |
| 13    | FDM- Summary of Relative Differences at thermocouples<br>1, 2, and 3 (Labeled Here As $T_i$ ) ..... 49                                  |
| 14    | Relative Displacements- Re-scaled for Spectral Methods ..... 53                                                                         |
| 15    | Spectral Method- Summary of Relative Differences at<br>thermocouples 1, 2, and 3 (Labeled Here As $T_i$ ) ..... 56                      |
| 16    | Stiffness: Time Steps and Computation Time Required for<br>ODE45 and ODE15s to Generate 1 Second of Solution of<br>Equation 42 ..... 57 |
| 17    | FEA Material Removal ..... 67                                                                                                           |
| 18    | FEA Material Properties ..... 67                                                                                                        |

| Table |                           | Page |
|-------|---------------------------|------|
| 19    | FEA Flux Parameters ..... | 67   |

# MODELING A HIGH SPEED PIN-ON-DISC EXPERIMENT BY COMPARISON OF NUMERICAL SOLUTIONS TO A MOVING BOUNDARY NONLINEAR HEAT EQUATION

## I. Introduction

The purpose of this work is to analyze the heat transfer characteristics of Vascomax®C300 during high-speed sliding. This work extends previous research that was intended to help predict the wear-rate of connecting shoes for a hypersonic rail system at Holloman Air Force Base. Several techniques are employed to solve the non-linear heat equation with a moving boundary and empirical evidence from a high-speed pin-on-disc experiment is used to provide validation. Solutions were generated using finite element analysis, finite difference methods, and spectral methods. Throughout this thesis, the nomenclature in Table 1 will be used.

### 1.1 Background

Holloman Air Force Base is the home of the Holloman High Speed Test Track (HHSTT) which is designed to test various instruments and carry payloads into targets at speeds greater than mach 5. Four train cars are linked together, each carrying a rocket propulsion system. Each time the caboose runs out of fuel, it is jettisoned and the next rocket in line fires up to continue accelerating the remaining cars toward their target along the 16 km track. The front two cars, also called sleds, are attached via slippers made of Vascomax®C300 which wrap around the track made of AISI 4340 steel. Below is a chart representing the speeds corresponding with each rocket stage and a picture of the rocket sled with payload [9]. Many studies have been conducted to examine the dynamics of this

**Table 1. Symbols Definitions and Units**

| Symbol     | Definition                       | Units                  |
|------------|----------------------------------|------------------------|
| $T$        | Temperature                      | $K$                    |
| $T_s$      | Scaled Temperature               | None                   |
| $T_0$      | Ambient Temperature (296)        | $K$                    |
| $T_{melt}$ | Melting Temperature (1685)       | $K$                    |
| $t$        | Time                             | seconds, s             |
| $L$        | Length of The Pin                | meters, m              |
| $\rho$     | Density                          | $\frac{kg}{m^3}$       |
| $\mu$      | Coefficient of Friction          | None                   |
| $\alpha$   | Coefficient of Thermal Expansion | $1/K$                  |
| $\beta$    | Partition Function               | None                   |
| $k$        | Thermal Conductivity             | $\frac{W}{m \cdot K}$  |
| $\kappa$   | Thermal Diffusivity              | $\frac{m^2}{s}$        |
| $V$        | Velocity                         | $\frac{m}{s}$          |
| $c$        | Specific Heat                    | $\frac{J}{kg \cdot K}$ |
| $q$        | Heat Flux                        | $\frac{W}{m^2}$        |
| $q_s$      | Scaled Heat Flux                 | None                   |
| $Q$        | Heat Rate                        | $W$                    |
| $A$        | Contact Area                     | $m^2$                  |
| $r$        | Radius                           | $m$                    |
| $z$        | Length                           | $m$                    |
| $x$        | Scaled Length                    | None                   |
| $h$        | Heat transfer coefficient        | $W/m^2K$               |
| $d$        | Displacement                     | $m$                    |

hypersonic rail system and this chapter starts off by discussing a selection of these works. While each of these studies focused in on a particular topic, they all attempt to answer the same problem statement: predict temperature of the slipper shoes that connect the rocket sleds to the rail. In this thesis we continue with the same end goal by modeling temperature data through Vascomax®C300 during a high-speed pin-on-disc experiment.

## 1.2 Brief History of Related Work

In 2008 a special run of the HHSTT was conducted. The sled system was highly instrumented to record measurements regarding the extreme aerodynamic forces the sled system

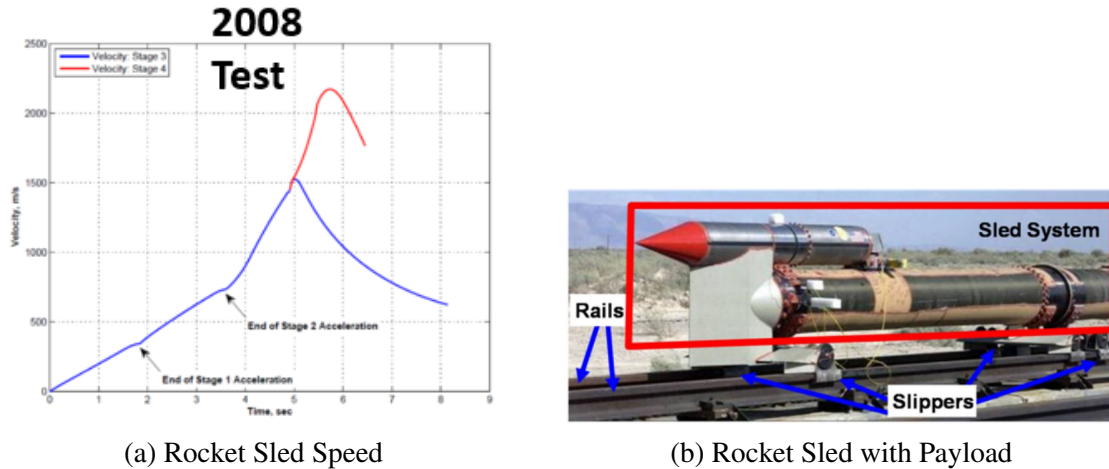


**Figure 1. Hypersonic Rail System Figures**

undergoes. A slipper from the third rail car was recovered after the test. Using the Dynamic Analysis and Design System, the speed profile of the third car was recorded as the blue line in Figure 1 showing a maximum speed of about Mach 4.5 [9]. The fourth rocket speed profile in Figure 1 is drawn in red showing the leading sled system reached about Mach 6.5 [17]. As an interesting side note, in 2003 the HHSTT set the world land speed record at Mach 8.5 [3].

Alban [9] and DeLeon [17] both used the 2008 test data as a baseline for the models in their research. Alban utilized a 2d Finite Elements Analysis (FEA) model for the connecting shoe. The FEA is fed by a 1d Finite Difference Method (FDM) model which calculated the heat flux and adjusted for melt. Alban's model predicted melt temperatures would be reached inside of the shoe (away from the leading edge). Alban suggests that this would result in additional material wear that the model was not capable of removing [9].

DeLeon looked to improve upon the wear removal modeling and abandoned the FEA approach, focusing instead on retooling the 1d FDM model. DeLeon improved Alban's model by considering material removal from the surface through the thickness of the shoe instead of only at the surface [17]. While DeLeon made significant progress toward predicting wear due to melt, the problem statement remained unsatisfied since melt wear is

only a portion of the total wear; DeLeon predicted that only about 40 percent of total wear was due to melt [17].

Boardman took a different approach than Alban and DeLeon and used data from a pin-on-disc experiment in the research [15]. While the pin-on-disc experiment does not reach nearly the same velocity profiles as the sled system, data collection and modeling processes are greatly simplified. Traditional pin-on-disc experiments, such as performed by Ashby [10], are conducted with velocities on the order of 1 to 10 meters per second. The pin-on-disc design used by Boardman has capability of reaching over 40 times that of the traditional test velocity, but experiments were conducted at a maximum of 240 meters per second [15].

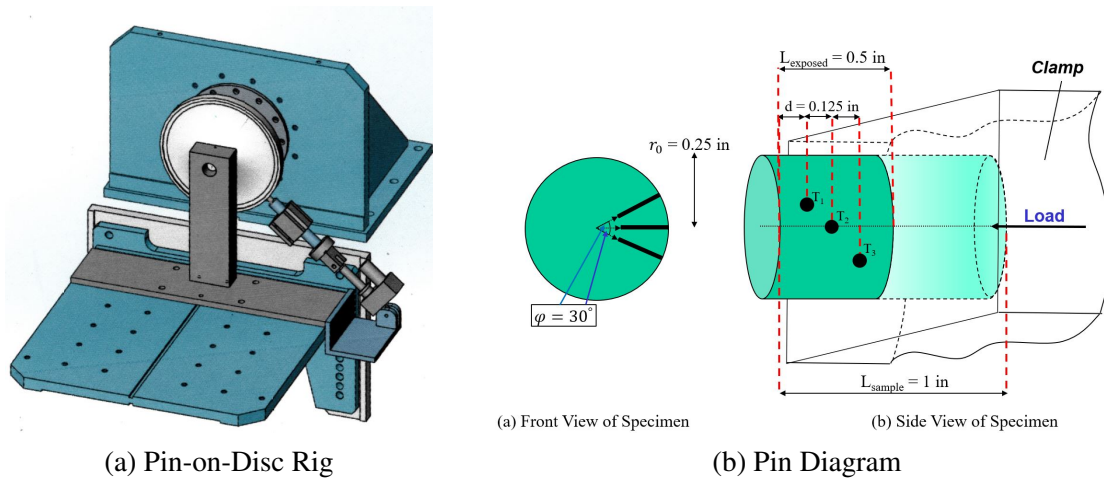


**Figure 2. Pin-on-Disc Figures [15]**

In order to try and account for total wear, Dr William Baker began to investigate if total wear could be described by a function dependent only on temperature. Under this guidance, the FDM model from DeLeon was extended to 2d by Boardman under assumption of radial axisymmetry. Boardman used a "Critical Temperature" of 1276 Kelvin as the removal criteria similar to DeLeon's methodology [15].

### 1.3 Experiment Design

The pin-on-disc experimentation configuration, seen in Figures 2 and 3, was set-up and conducted by the Air Force Research Laboratory at Wright-Patterson Air Force Base. The pin has a radius,  $r$ , of 0.00635 meters and a length of  $2r$  to the clamp. Displacement was captured by a LDI-119 linear variable inductive transducer position sensor. Force was measured using a LC501 steel cantilever beam. Force and displacement were recorded every  $5 \times 10^{-4}s$ ; the raw data is shown in Figure 4. Temperature was recorded using type-J thermocouples spaced out in 1/8 inch intervals, offset by 30 degrees, and placed in holes drilled 1/5 inch deep. Type-J thermocouples are rated to  $1030K \pm 0.75\%$ . Temperature was recorded every  $1 \times 10^{-2}s$ ; the raw data is shown in Figure 5. At 5.5 seconds thermocouple 1 exceeds 1100K and there is a noticeable change in the behavior of the curve. For this reason quantitative comparisons between the models and data are only made from 0 to 5.5 seconds. The locations of thermocouple 1, 2, and 3 are shown as  $T_1$ ,  $T_2$ , and  $T_3$  in Figure 2. The pin is made from a discarded slipper shoe (Vascomax®C300) and the 12-inch diameter disc is made of the same material as the rail at the HHSTT (AISI 4340 steel). The disc is cooled by a cold air gun during the test run to keep it from overheating.

The raw data from the experiment in Figures 4 and 5 exhibit noise and irregularities. The force was maintained manually by turning a knob on the end of the motor-actuated holder. As the pin was worn down, a technician applied more pressure by turning the knob. While the technician did their best to maintain a constant pressure on the pin, it is clear that vibration effects of the friction and the manual knob adjustments generated considerable distortions to the forcing data.

The temperature at the thermocouple closest to the disc (thermocouple 1) has an irregularity at 4 seconds that is unexplained. After 5.5 seconds this thermocouple begins to read temperatures that exceed the rated threshold. Therefore, the data for thermocouple 1 cannot be trusted to be within the error tolerance after 5.5 seconds. At 7 seconds there is an

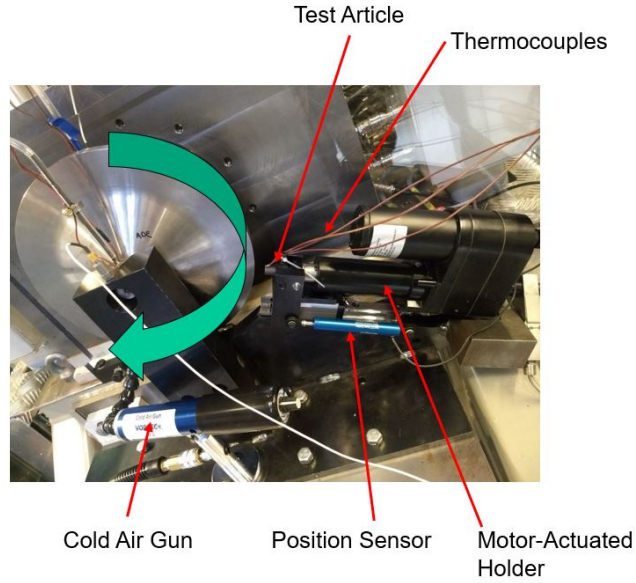


**Figure 3. Experiment Rig**

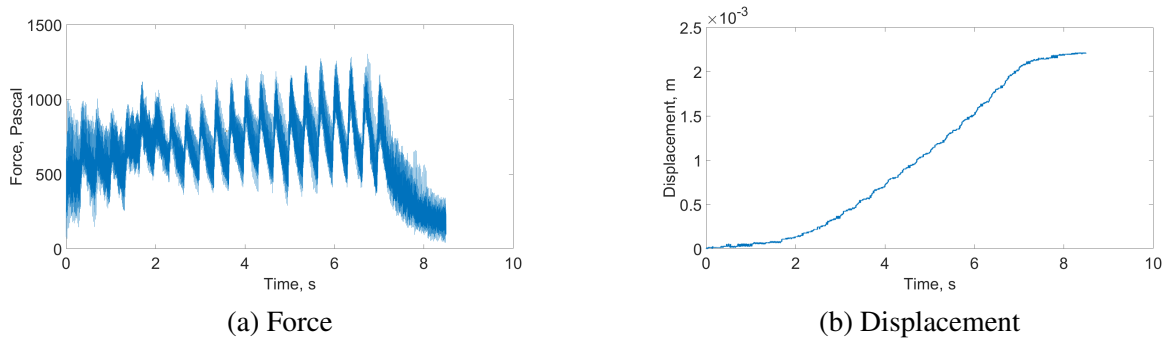


**Figure 4. Raw Experiment Data- Force and Displacement**

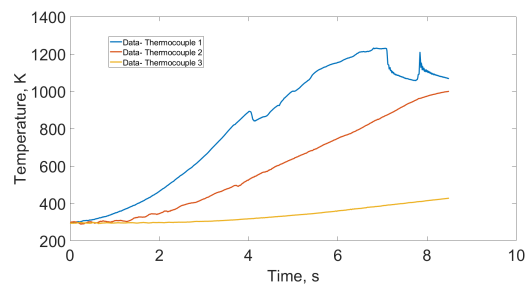


**Figure 5. Raw Experiment Data- Temperature**

irregularity to which some is explained by a drop in force, but most of which is assumed to be faulty readings by the thermocouple.

The displacement is the smoothest, most regular, raw data. The smoothness of the data

indicates that vibration effects did not play a large role in the collection of displacement data as it did in the forcing data. However, note in Figure 3 that the position sensor is not located in-line with the pin. There may have been some loss of information through the connecting system.

## **1.4 Prologue**

Three separate numerical solutions will be generated and discussed for the non-linear heat equation with a moving boundary. Data from the high-speed pin-on-disc experiment is used to compare each model to observation. Finite element analysis, finite-difference methods, and spectral methods can each provide unique approximations of, and insight to the progression of, heat transfer through the pin.

Finite element analysis will be used to show that a steady state solution can not effectively represent the data, leading to transient solution methods. Finite element analysis had not been previously applied to this particular pin-on-disc experimental set-up, although finite elements had previously been applied to model slipper wear during a run of the hypersonic rail car [9]. Transient finite elements show negligible convection and reduces the model to one dimension. Using constant thermal properties, the transient finite element is shown to qualitatively represent the thermocouple data well. Investigation into if variable parameters can provide a better model is performed with finite difference and spectral methods.

Finite-difference method in one dimension provided a simplification to the two dimensional model formulated by Boardman [15]. Using variable definitions for the parameters in the thermal diffusivity relation to represent the heat transfer is shown to not improve the model. The temperature curves at the thermocouples produced herein have a concave down, while the data has a concave up profile. This inconsistency was also produced in previous work on this experimental set-up using finite differences to model wear of the

pin [15]. It is hypothesized that the mesh used is too coarse to accurately model the heat transfer and spectral methods are turned to for solution instead of refining the mesh due to stiffness that is observed.

Spectral methods were implemented to achieve high accuracy on a coarse mesh by virtue of spectral accuracy. Using variable parameters it is again shown that the model does not improve and it is concluded that constant parameters are sufficient to represent this pin-on-disc experiment. Spectral methods were not found to have been previously applied to any aspect of the hyper-sonic rail system or this particular pin-on-disc experiment. The spectral method produced the best model of the experimental data out of each of the three solution techniques.

## II. Literature Review

### 2.1 Heat Transfer

Heat transfer is thermal energy in transit due to a spatial temperature difference [12]. The three types of heat transfer are conduction, convection and radiation. Conduction refers to heat transfer through the body of a medium, convection refers to heat transfer from the surface of a medium into a fluid, and radiation refers to heat transfer from the surface of a medium via electromagnetic waves (including into a vacuum) [12]. Conduction in solids is attributed to the thermal energy inducing atomic motion that transmits lattice waves and to translational motion of free electrons [12]. Convection is comprised of diffusion and bulk motion of the fluid [12].

Heat energy transfer occurs in the direction of decreasing temperature and can be quantified by the appropriate rate equation [12]. The rate equation for conduction is given by Equation 1 [12]; the rate equation for convection is given by Equation 2 [12]; radiation is assumed to be negligible in this application. Thermal conductivity,  $k$ , is related to other material properties through the diffusivity relation given in Equation 10. The tables below provide a summary of thermal properties of the pin (Vascomax®C300) and the disc (AISI 4340) that are used in this study.

$$-k\nabla T = \text{Conduction Heat Flux} \quad (1)$$

$$h\Delta T = \text{Convection Heat Flux} \quad (2)$$

**Table 2. Vascomax® 300 Material Properties at Room Temperature [1]**

| Property             | Value                  |
|----------------------|------------------------|
| Density              | 8000 kg/m <sup>3</sup> |
| Melt Temperature     | 1685 K                 |
| Thermal Conductivity | 30.807 W/(m · K)       |

**Table 3. Vascomax® 300 Specific Heat [19]**

| Temperature (K) | Specific Heat, $\hat{c}_p$ (J/kg · K) |
|-----------------|---------------------------------------|
| 298             | 360                                   |
| 422             | 481                                   |
| 598             | 599                                   |
| 700             | 858                                   |

**Table 4. AISI 4340 Steel Material Properties**

| Property                  | Value                              |
|---------------------------|------------------------------------|
| Density                   | 7750 kg/m <sup>3</sup> [5]         |
| Heat Transfer Coefficient | 72.14 W/m <sup>2</sup> K [18] [15] |
| Specific Heat             | 502 J/(kg · K) [5]                 |
| Thermal Conductivity      | 41.8 W/(m · K) [5]                 |

## 2.2 Tribology

Tribology is the study of the interaction between surfaces in contact. Friction refers to the force that resists sliding. Despite having been studied since prehistoric times, friction has not received a full physical explanation, but there are two basic issues at play: the nature of the friction force and the energy dissipation mechanism [13]. In other words, we look to understand how much heat will enter the system due to friction, and how that heat will conduct through the system.

### 2.2.1 Friction

Flux density,  $q_{tot}$ , is the amount of heat that enters the system due to friction per unit area. Friction is treated as a heat source at the point of contact. Flux density is defined as the product of the partition function [21], the coefficient of friction, pressure, and velocity

divided by the contact area [10]:

$$q_{tot} = \beta \mu p V / A \quad (3)$$

- The Partition Function,  $\beta$ , defines the proportion of the heat generated that will enter the pin. The partition function becomes almost constant at high speeds where the Peclet number is greater than ten [21]. The Peclet Number is defined by Equation 4 [13]. The Peclet number is a dimensionless quantity that is used to characterize temperature evolution. When the Peclet number is low, the temperature distribution can be represented by a stationary source; when it is high, the heat across the plane of the source can be assumed to generate uniformly [21]. Using the material properties given in Tables 2 and 3, the Peclet number will be on the order of  $10^4$ , which is large and leads to the assumption of uniform heat generation across the face of the pin.

$$Pe = \frac{Vr}{2\kappa} \quad (4)$$

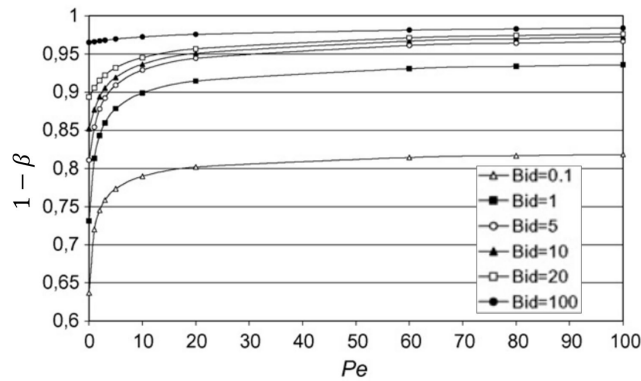


**Figure 6. Laraqi Partition [21]**

The value of the Partition coefficient can be shown to depend only on the Biot number of the disc [21]. For the disc, the properties of AISI 4340 are well studied. Using Table 4, the Biot number of the disc is approximately .526. The Biot number

is defined by Equation 5 [21].

$$Biot = \frac{hr}{k} \quad (5)$$

- The coefficient of friction,  $\mu$ , at high speeds, is inversely proportional to velocity [10]. The coefficient of friction in steel sliding conditions is approximated by Equation 6 [23]. Lim and Ashby plotted this approximation against several other published findings as seen in Figure 7.

$$\mu = 0.78 - 0.13 \log_{10} \tilde{V} \quad (6)$$

Where  $\tilde{V}$  is the normalized velocity. Normalized velocity is found by Equation 7 [10].

$$\tilde{V} = \frac{Vr}{.5(\kappa_{disc} + \kappa_{pin})} \quad (7)$$

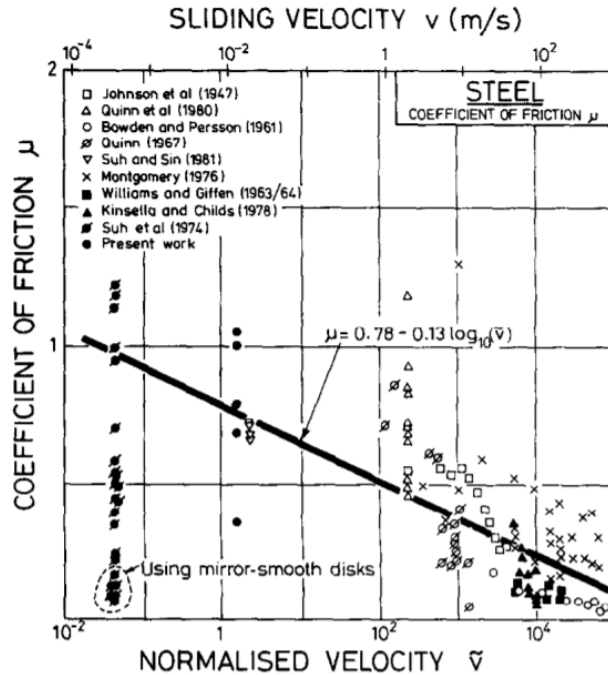


Figure 7. Lim and Ashby Coefficient of Friction [23]

The normalized velocity is dimensionless and can be calculated using the data from

Tables 2, 3, and 4 using the relationship in Equation 10.

### 2.2.2 Conduction

The conduction of heat through the pin is determined by density, thermal diffusivity, thermal conductivity, specific heat, and convective heat loss. Vascomax®C300 is a Maraging 300. Maraging describes types steels that are fully martensitic and strengthened in an aging process. An 18% nickel alloy, Vascomax®C300 is ductile and strong composed of copper, molybdenum, titanium, and aluminum. Published data for material properties at temperatures beyond 800 K is sparse, and it was found that around 800-1000 K Vascomax®C300 undergoes solid-solid phase change transition from martensite to austenite [19]. Phase transition to austenite would theoretically drastically change the thermal properties of the pin [19], but no data was found in, or above, the phase change temperature to support what the thermal property values may be.

- density,  $\rho$ , can be determined as a function of temperature,  $T$ , and coefficient of thermal expansion,  $\alpha$ , using Equation 8. The temperature dependence for  $\alpha$  is complex, but can be determined experimentally [28]. Table 5 provides some mean values of  $\alpha$ . It has been found that most metals that have a melting point above 900 K linearly expand approximately 2% when heating from  $T_0$  to  $T_{melt}$  [28]. Linear expansion,  $\Delta L$ , is defined by Equation 9. That said, density is assumed to be constant in this thesis.

$$\rho_T = \rho_{T_0} / (1 + \alpha T)^3 \quad (8)$$

$$\Delta L = \alpha T \quad (9)$$

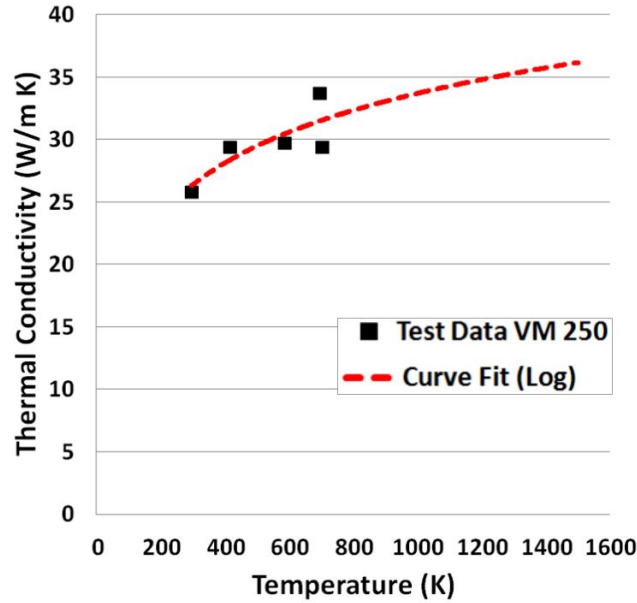
- Thermal diffusivity,  $\kappa$ , thermal conductivity,  $k$ , and specific heat,  $c$ , are related by Equation 10. Experimental data was not found for these material properties in the

**Table 5. Vascomax® 300 Mean Coefficient of Thermal Expansion [1]**

| Temperature Range (K) | Value ( $10^{-6}/K$ ) |
|-----------------------|-----------------------|
| 293.15-373.15         | 9.9                   |
| 293.15-473.15         | 10.2                  |
| 293.15-573.15         | 10.6                  |
| 293.15-673.15         | 11.0                  |
| 293.15-753.15         | 11.3                  |

temperature ranges experienced during the pin-on-disc experiment. Previous work [15] has been done using data from Vascomax®250 and approximating thermal conductivity using a log fit extrapolation seen in Figure 8. The specific heat function in Figure 9 was found by DeLeon [17]. A constant thermal diffusivity ( $4.4883 \times 10^{-6}$ ), was used prior [9] to DeLeon developing the function seen in Figure 10 [15]. Solving Equation 10 with the constants  $\kappa = 4.4883 \times 10^{-6}$ ,  $k = 30.806$ , and  $\rho = 8000$  results in  $c = 858$  which is plotted in Figure 9.

$$\kappa = \frac{k}{\rho c} \quad (10)$$

**Figure 8. Thermal Conductivity Approximation [15]**

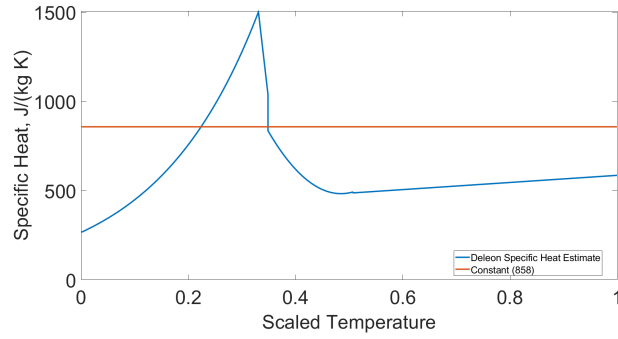


**Figure 9. Specific Heat Approximation [17]**

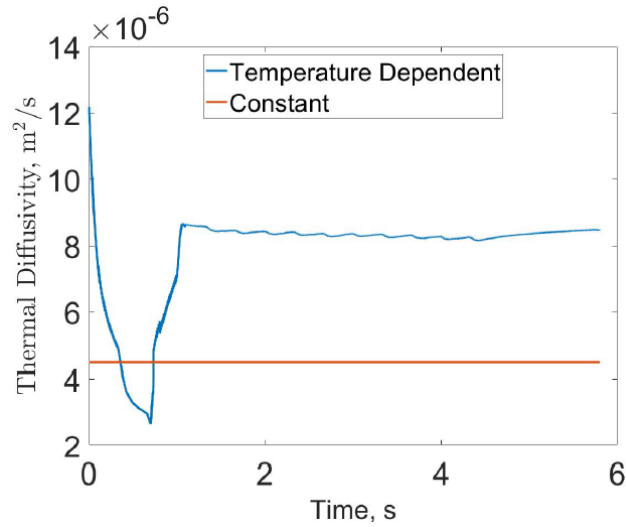


**Figure 10. Thermal Diffusivity Approximation [17] [9]**

### 2.3 Assumptions Regarding Friction and Conduction

The clamp is assumed to be a heat sink that maintains ambient temperature. Surface zones are assumed to be removed from the disc and pin nearly instantaneously and the materials to be Gaussian isotropic [13]. These homogeneity assumptions result in thermal properties that are independent of position. The heat flux density is assumed to act as a heat source that is distributed uniformly across the contact surface.

### III. Numerical Solutions

#### 3.1 Finite Element Analysis

Finite element analysis (FEA) is a method for numerical solution of field problems such as the distribution of temperature in a body. The individual finite elements can be visualized as small pieces of a larger structure such that the temperature is assumed to have simple variation through the element defined by a shape function, while the actual variations may be much more complicated. Therefore, the finite element analysis provides an approximation of the variation of temperature throughout a body such that the distribution is continuous across the element connections, or nodes. A particular arrangement of elements making up the overall shape is called the *mesh*. The mesh can be represented by a system of algebraic equations where the nodal quantities represent the temperature, so that solving the algebraic equations approximates the temperature distribution [16].

While developing the FEA model in the next section, the flowchart in Figure 11 was followed [16]. First, a mathematical model was developed and the empirical data was obtained for subsequent comparison to the FEA results. Initial discretization was chosen based upon thermocouple and displacement locations. The model was then built in Abaqus [6] and a tutorial for building this model can be found in the Appendix. Several outputs are displayed for examination after solutions were found. Finally, the mesh was refined until the refinement no longer significantly altered the results.

##### 3.1.1 FEA Solution Methodology

The assumption of uniform heat flux is sufficient to reduce the model to 2d, assuming perfect cylindrical geometry of the pin. However, the assumption of uniform heat flux is not a necessary condition. Any axisymmetric heat flux distribution would also be sufficient. Furthermore, at high enough velocities the problem itself, irrespective of the heat

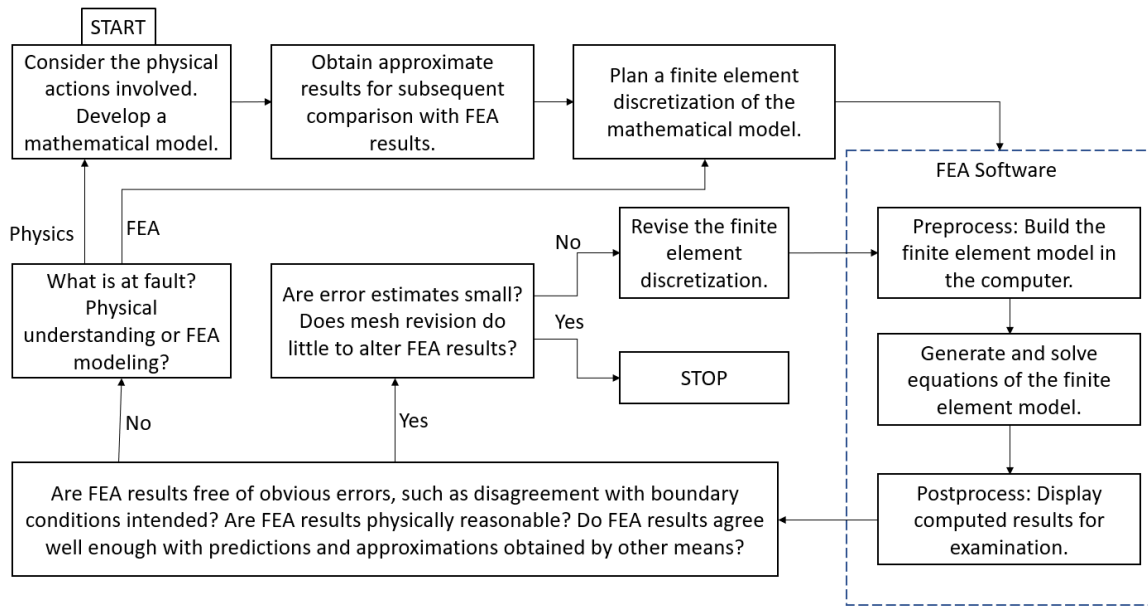


**Figure 11. FEA Flowchart [16]**

flux distribution, is axisymmetric. So that at high enough speeds the pin-on-disc problem only depends only on the radial and length dimensions, not angle [21]. The experiment being modeled was conducted at speeds that are considered very high for a pin-on-disc. Therefore, it is determined that reducing to 2d is reasonable and proceed by constructing a 2d planar finite elements solution as follows.

Every plane which is normal to the pin face and cuts through the center of the pin face, as shown in Figure 12, will have the same temperature profile. As a result, the pin-on-disc can be viewed as a 2d plane heat transfer problem. Redrawing the pin from Figure 2 under this assumption produces the schematic shown as the 2d plane object in Figure 13. All three thermocouples are projected onto the same plane.

Starting from conservation of energy the basic finite elements matrix equation is given in Equation 11. The temperature at each node is placed in the temperature vector,  $\mathbf{T}$ , and the thermal loads (amount of heat flux density) applied to each load is placed in the vector  $\mathbf{Q}$ . The heat rate (heat flow per unit time) is given in Equation 12 [20].

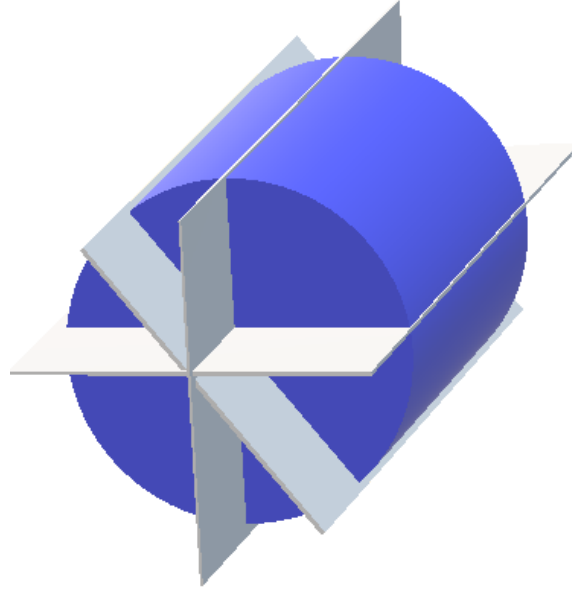


**Figure 12. Planes Normal to, and through the center of, a Cylinders Face**

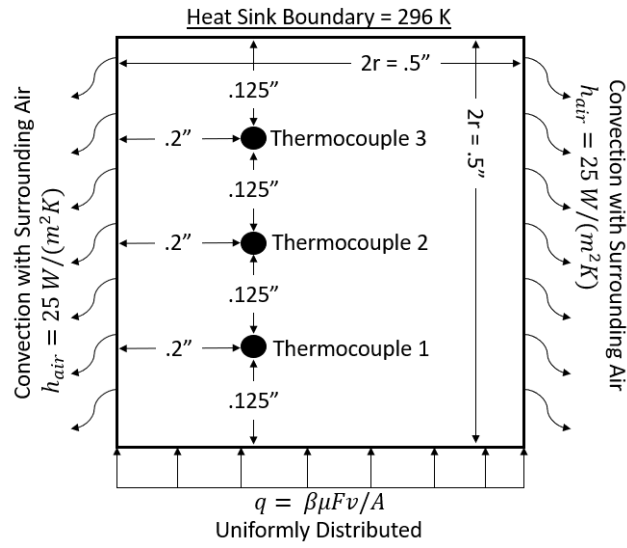


**Figure 13. Finite Element Analysis Model**

$$[\mathbf{K}_T]\{\mathbf{T}\} = \{\mathbf{Q}\} \quad (11)$$

$$Q_x = -kA \frac{dT}{dx} \quad (12)$$

For example, if we reduced Figure 13 to 1d and solved the steady-state problem using 4 elements the problem would reduce to Equation 13 containing 5 nodes. Let us assume that the lengths of the elements are equal, so that the interior nodes represent the three thermocouples; constant thermal conductivity,  $k_i = k_j = k$ ;  $T_4$  is held to ambient temperature  $T_\infty$ ;  $Q_0$  is a known source  $Q$ , and  $Q_1 = Q_2 = Q_3 = 0$  (no generation or loss); and, let the unknowns be  $T_0, T_1, T_2, T_3$  and  $Q_4$ . Then Equation 13 is completely defined, but the matrix in Equation 14 is singular. Since  $T_4$  is known, one method to remedy this is to impose the boundary condition and move the known quantities to the right hand side as in Equation 15 [16]. Now the system can be solved by performing Gaussian elimination on Equation 15 to find the temperatures given in Equation 16. With all the temperatures known,  $Q_4$  can be readily solved by going back to Equation 14.

$$\begin{bmatrix} Ak_1/L_1 & -Ak_1/L_1 & 0 & 0 & 0 \\ -Ak_1/L_1 & Ak_1/L_1 + Ak_2/L_2 & -Ak_2/L_2 & 0 & 0 \\ 0 & -Ak_2/L_2 & Ak_2/L_2 + Ak_3/L_3 & -Ak_3/L_3 & 0 \\ 0 & 0 & -Ak_3/L_3 & Ak_3/L_3 + Ak_4/L_4 & -Ak_4/L_4 \\ 0 & 0 & 0 & -Ak_4/L_4 & Ak_4/L_4 \end{bmatrix} \begin{Bmatrix} T_0 \\ T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = - \begin{Bmatrix} Q_0 \\ Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix} \quad (13)$$

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} T_0 \\ T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = - \begin{Bmatrix} Q \\ 0 \\ 0 \\ 0 \\ Q_4 \end{Bmatrix} \frac{L/4}{Ak} \quad (14)$$

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} T_0 \\ T_1 \\ T_2 \\ T_3 \\ T_4 \end{pmatrix} = \begin{pmatrix} -QL/4Ak \\ 0 \\ 0 \\ T_4 \\ T_\infty \end{pmatrix} \quad (15)$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} T_0 \\ T_1 \\ T_2 \\ T_3 \\ T_4 \end{pmatrix} = \begin{pmatrix} T_\infty - 4QL/4Ak \\ T_\infty - 3QL/4Ak \\ T_\infty - 2QL/4Ak \\ T_\infty - QL/4Ak \\ T_\infty \end{pmatrix} \quad (16)$$

The example above may be applied to the model in Figure 13 assuming  $h = 0$ :  $L/4 = .125'' = .003175\text{m}$ ,  $k = 30.806\text{W/m} \cdot \text{K}$ ,  $T_\infty = 296\text{K}$ . The heat rate is given by the heat flux times the area to which the flux is applied [12]. With the risk of jumping ahead a bit, Equation 24 provides the value to use for  $q = QA$ . With the temperatures solved as thus in Equation 17, the remaining unknown,  $Q_4$ , can be solved by the equation given in the last row of Equation 14, as shown in Equation 18. Noting, again with risk of jumping ahead, that A is provided by Equation 25.

$$\begin{pmatrix} T_0 \\ T_1 \\ T_2 \\ T_3 \\ T_4 \end{pmatrix} = \begin{pmatrix} T_\infty - 4qL/4k \\ T_\infty - 3qL/4k \\ T_\infty - 2qL/4k \\ T_\infty - qL/4k \\ T_\infty \end{pmatrix} \approx \begin{pmatrix} 2750 \\ 2135 \\ 1520 \\ 910 \\ 296 \end{pmatrix} \text{ Kelvin} \quad (17)$$

$$Q_4 = -\frac{Ak}{L/4}(T_4 - T_3) \approx 755 \text{ Watts} \quad (18)$$

This approximate solution to the model in Figure 13 serves the purpose of the initial

cycle of the first three boxes in Figure 11. Developing an initial approximation can be used to verify outputs received from FEA software and help detect errors that may occur when building the computer model [16]. While leading to construction of a more robust transient model, we shall first develop the analogous steady state model to corroborate the results of Equation 17.

The heat conduction problem is given in Equation 19 [24]. Semi-discretization of the heat equation (Equation 19) produces the initial-value problem in Equation 20 [24]. For the model in Figure 13, the initial condition, and boundary condition along the top, are set to an ambient temperature of  $296K$ .

$$\rho c \dot{T} = \nabla(k \nabla T) \quad (19)$$

$$\mathbf{C}(t, \mathbf{T}) \dot{\mathbf{T}} = \mathbf{f}(t, \mathbf{T}) - \mathbf{K}(t, \mathbf{T}) \mathbf{T}(t) ; t > 0 \quad (20)$$

To generate  $\mathbf{K}$  in two-dimensions, bilinear quadrilateral elements were used to interpolate across a discretized mesh of Figure 13. The shape functions for the bilinear quadrilateral are given in Equation 21 [16]. The conductivity matrix,  $\mathbf{K}$ , is defined element-wise in Equation 22 [24]. The capacitance matrix,  $\mathbf{C}$ , is defined element-wise by Equation 23 [24]. The nodal flux vector,  $\mathbf{f}$ , contains the heat flux and boundary terms for each element, including convection with the surrounding air [16]. Convective heat flow can be found by  $q = hA(T_0 - T)$  [20]. Note that in the model provided in Figure 13,  $h = 25W/(m^2K)$ . Equation 20 is then fully defined and can be solved.

$$\begin{aligned}
N_1 &= \frac{1}{4}(1 - \xi)(1 - \eta) \\
N_2 &= \frac{1}{4}(1 + \xi)(1 - \eta) \\
N_3 &= \frac{1}{4}(1 + \xi)(1 + \eta) \\
N_4 &= \frac{1}{4}(1 - \xi)(1 + \eta)
\end{aligned} \tag{21}$$

$$K_{i,j} = \int_{-1}^1 \int_{-1}^1 k \nabla N_i^T \nabla N_j d\xi d\eta \tag{22}$$

$$C_{i,j} = \int_{-1}^1 \int_{-1}^1 \rho c N_i N_j d\xi d\eta \tag{23}$$

The version of the commercial software Abaqus [6] that was used to generate the mesh and perform the time integration limits the mesh to 1000 nodes. 4-node linear heat transfer bilinear quadrilaterals partition the part seen in Figure 13 by displacement data and thermocouple locations producing the partitions in Figure 14. Partitioning at thermocouple locations (as shown in Figure 14) ensures that when the part is meshed by Abaqus [6] there will be a node located at the same geometric location as the thermocouples. Therefore, the values at these nodes represent the interpolated approximation of data from the experiment. The values for the displacement is from experimental data obtained in 1 second intervals and are given in Table 6. Partitioning at displacement locations accounts for the wear: eight steps were created and after each step the model is changed so that that the elements in the lowest partition are removed from the model to represent the wear.

Properties were held as constants. Abaqus requires manual entry to define piece-wise function for most properties. This process can become restrictive when performing analyses for properties which are not well-defined. While we show in Section 2.2 that friction is fairly well defined for the temperature range of the high-speed pin-on-disc experiment, the conductivity of Vascomax®300 is not. Therefore the model is generated using material

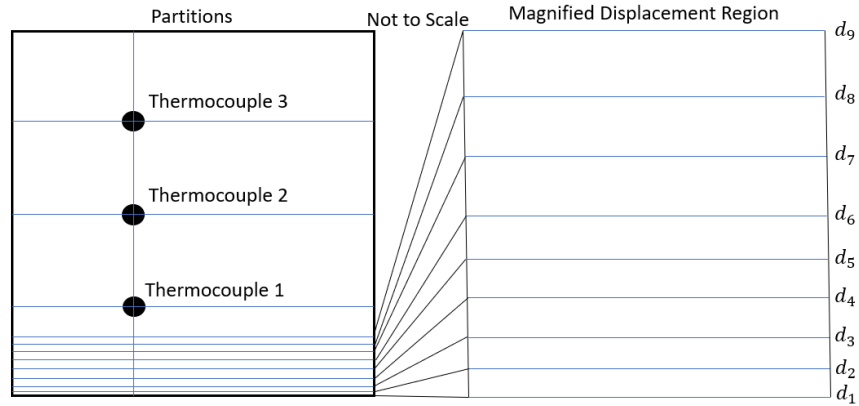


**Figure 14. Finite Element Analysis Partitions**

| Step    | Total Displacement at step $i$ , $d_i$ | Total Time |
|---------|----------------------------------------|------------|
| Initial | .00 mm                                 | 0          |
| 1       | .0000 m                                | 0-1 sec    |
| 2       | .00005 m                               | 1-2 sec    |
| 3       | .000125 m                              | 2-3 sec    |
| 4       | .00035 m                               | 3-4 sec    |
| 5       | .0007 m                                | 4-5 sec    |
| 6       | .0011 m                                | 5-6 sec    |
| 7       | .0015 m                                | 6-7 sec    |
| 8       | .0020 m                                | 7-8 sec    |
| 9       | .0022 m                                | 8-9 sec    |

**Table 6. FEA Material Removal**

properties that were defined using Table 2, except for specific heat which was chosen such that the the thermal diffusivity would be  $4.4883 \times 10^{-6}$  (recall Figure 10). Conductivity parameters used for FEA are summarized in Table 7.

| Property      | Value      | Temperature |
|---------------|------------|-------------|
| Conductivity  | 30.807 [1] | all         |
| Density       | 8000       | all         |
| Specific Heat | 857.98     | all         |

**Table 7. FEA Material Properties**

The friction parameters defined in Table 8 were used to calculate the surface heat flux that is applied to the bottom surface during each step. The amount of the heat entering the

system, from Equation 3, is:

$$q(t) = \frac{\beta(t)\mu(t)F(t)v(t)}{A(t)} = 5.95 \times 10^6 \quad (24)$$

The flux parameters were approximated either from experiment data or previous literature and are summarized in Table 8. Force,  $F$ , represents an average taken from data obtained during the entire experiment which is plotted against force data was reduced to  $1 \times 10^{-1}s$  intervals in Figure 15 to remove some of the noise. Velocity,  $v$ , was maintained constant by the experiment rig. The partition function,  $\beta$ , is estimated from Figure 6 using Equation 5 and data from Table 4 (see Figure 16). The coefficient of friction,  $\mu$ , was calculated using Equation 6.

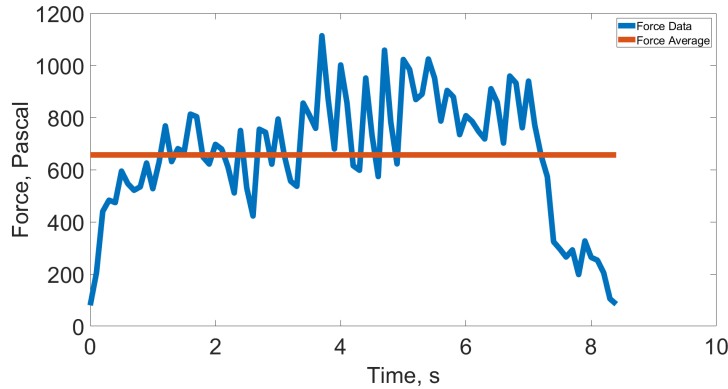


**Figure 15. Force Data vs Force Average**

|            |                         |
|------------|-------------------------|
| $\beta(t)$ | 0.12                    |
| $\mu(t)$   | 0.20                    |
| $F(t)$     | 656.76 N                |
| $v(t)$     | 47.83 m/s               |
| $A(t)$     | .0001267 m <sup>2</sup> |

**Table 8. FEA Flux Parameters**

The plate is given a thickness of .00997455m so that the surface area acted upon by  $q$  is the same as the pin face (see Equation 25). This creates an issue for the convection boundary with the air though. The surface area of the sides of the cylinder is  $2r(2\pi r) =$

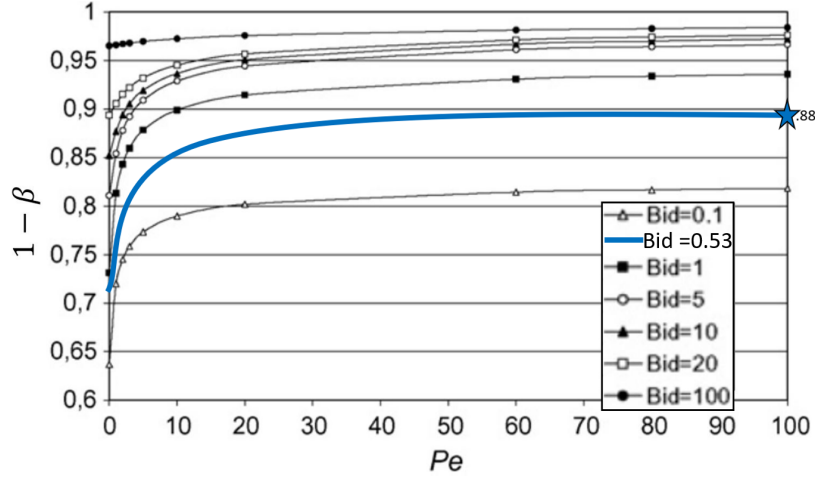


Figure 16. Laraqi Partition with Estimate for 12 inch AISI 4340 Disc

$5.067 \times 10^{-4}$ , but the area on the sides of the plate is  $2(2r(.00997455)) = 2.534 \times 10^{-4}$  which is smaller by a factor of 2. Therefore, a surface film condition of  $50W/(m^2K)$ , instead of 25 as shown in Figure 13, is applied to the sides of the plane to correct for the total area of the cylinder walls since  $q = hA(T_0 - T)$  [16].

$$\text{Area of Pin Face} = \pi r^2 = 1.26768 \times 10^{-4} \quad (25)$$

$$\text{Surface Flux Applied Area} = 2r * .00997455 = 1.26768 \times 10^{-4}$$

### 3.1.2 Steady State- FEA

Steady state solutions were obtained following removal of the lowest partition at the end of each step. A coarse mesh containing 96 elements and 117 nodes, a fine mesh containing 728 elements with 783 nodes, and a control mesh of 899 elements with 960 nodes was used to check convergence of the solution. Convergence of the mesh is achieved for the fine mesh of 728 elements. The maximum and minimum values along the bottom boundary were used to determined convergence. The difference in maximum and minimum temperatures between the control and fine mesh are within .0001% of each other, as shown in Figures 17 and 18. The fine mesh is used for all subsequent analysis.

Figures 19 to 21 depict nearly horizontal isotherms and visually illustrate the temper-

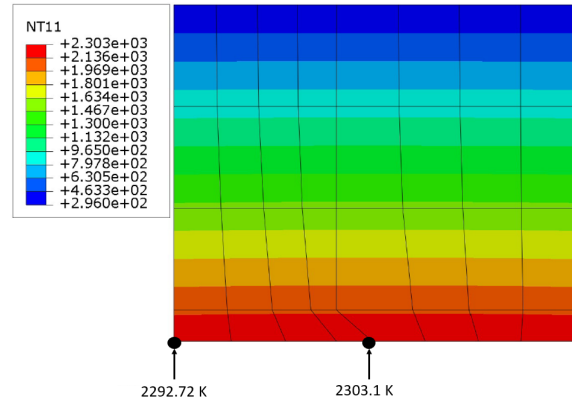


**Figure 17. Steady State- FEA Model Maximum and Minimum Temperatures at Source Boundary After Removal From Step 8 - 117 nodes**

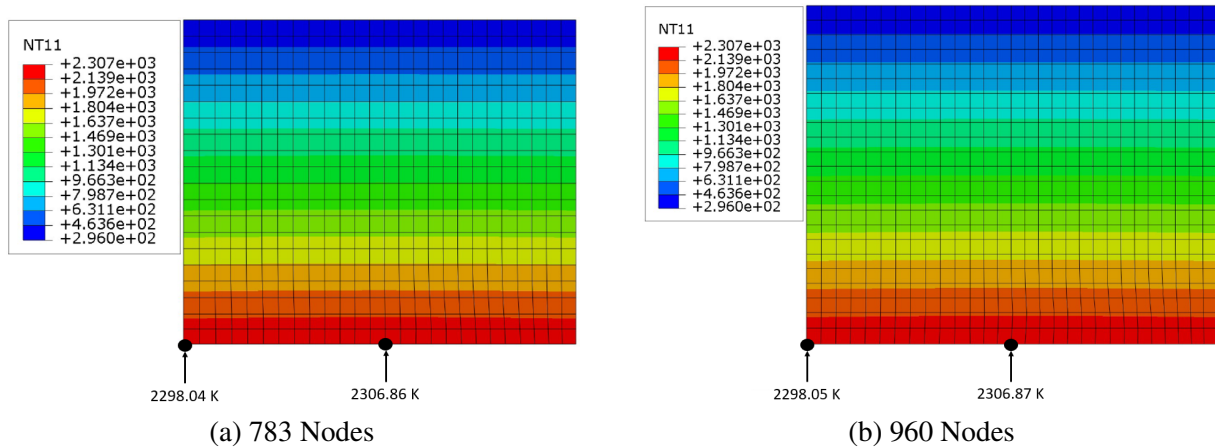


**Figure 18. Steady State- FEA Model Convergence of Maximum and Minimum Temperatures at Source Boundary After Removal From Step 8**

ature profile slowly decaying as the heat source at the bottom boundary moves closer to the heat sink at the top boundary. Figure 22 plots a comparison of the FEA data with data from the experiment. The step axis and time axis is equivalent in this case since each step corresponds to one second.

To generate a more representative model it was decided, from looking at Figure 22 that a transient model is required. Recall though that the need for a transient solution was expected and that the main purpose of this steady state model was to corroborate the computer model results with that of the solution found in Equation 17. The FEA software model generates a solution of about 2108, 1501, and 898 K at thermocouples 1, 2, and 3,

respectively while Equation 17 generated 2135, 1520, and 910 K. This appears satisfactory since the computer model is expected to produce a smaller value due to the convective terms included, and recognizing there is some error due to the rough mesh used for Equation 17. It is concluded that the computer model has been instantiated correctly to represent Figure 13.

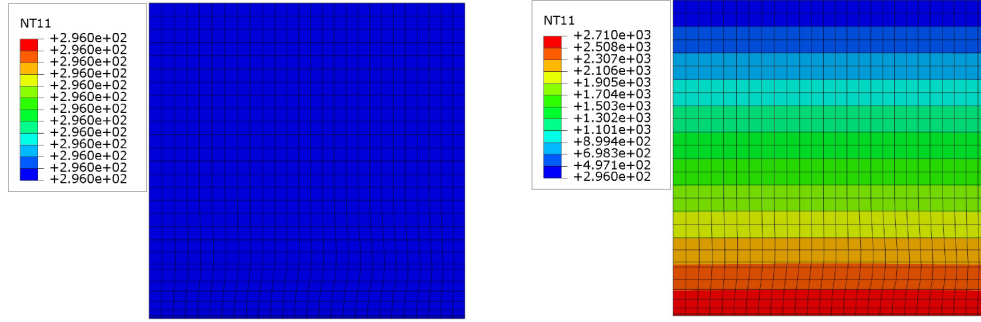


**Figure 19. Steady State- FEA Model at initial and with 1 displacment**

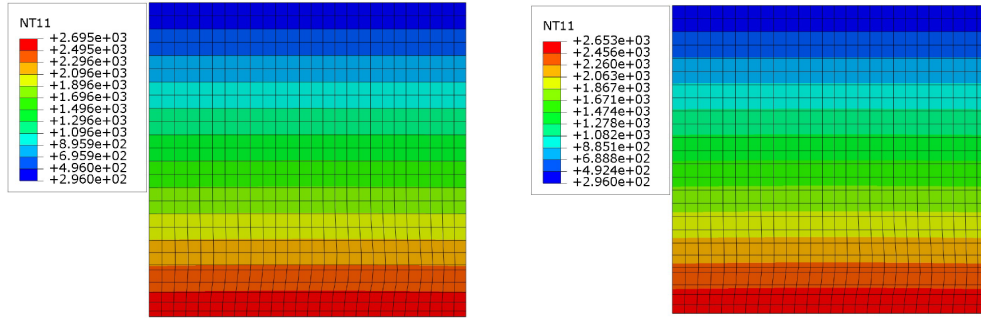


**Figure 20. Steady State- FEA Model after 2 and 3 displacements**

### 3.1.3 Transient- FEA

Transient steps were implemented using 1 second increments per step, corresponding to the displacement data in Table 6. All other conditions were kept the same as the steady state case. A transient solution is one that incorporates the time-dependent nature of the problem by discretizing and integrating over the time-domain. Abaqus standard solves for the next time increment of Equation 20 using the BFGS (Broyden, Fletcher, Goldfarb,

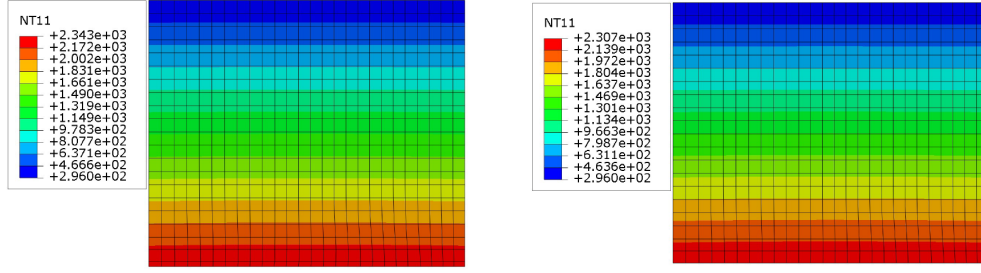


Figure 21. Steady State- FEA Model after 7 and 8 displacements

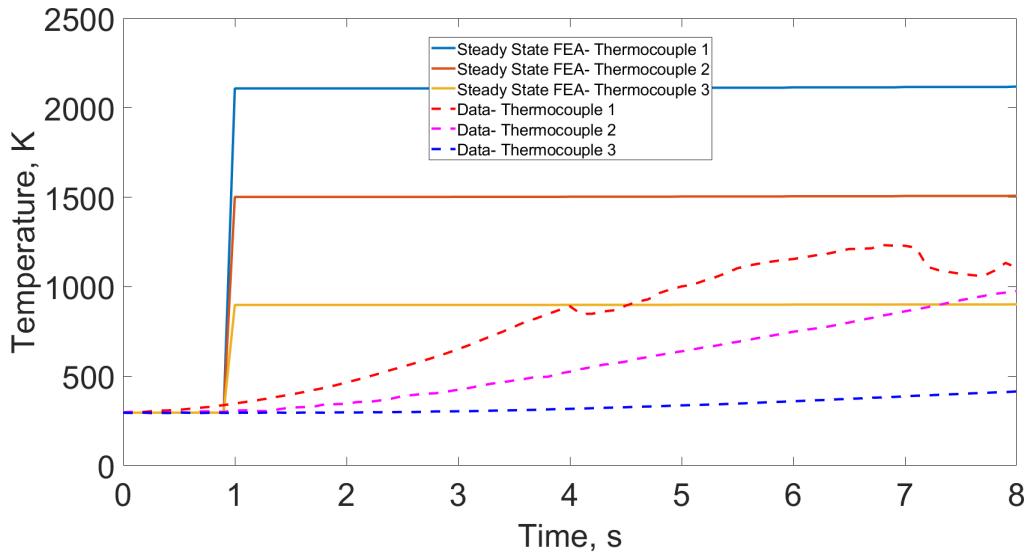


Figure 22. Steady State- FEA Thermocouple Temperatures vs Data

Shanno) quasi-Newton technique [7]. The temperature is assumed to vary linearly over small time increments [8]; increment size is automatically controlled by using such methods as presented by Bergan [11] or Hibbit and Karlson [2] to ensure accuracy.

Data obtained at the thermocouples was recorded every tenth of a second and plotted against the data from the experiment in Figure 27. Type-J thermocouples are rated to  $1030K \pm 0.75\%$ . At 5.5 seconds thermocouple 1 exceeds  $1100K$  and there is a noticeable change in the behavior of the curve. For this reason comparisons between the model and data are made from 0 to 5.5 seconds. To calculate the difference between solutions and data, a relative difference is found at one-tenth of a second increments from 0 to 5.5 seconds for

a total of 56 data points henceforth so that the sizes can be fairly compared. The relative difference is defined as the mean of the absolute value of the model divided by the data minus 1. The relative difference between this Transient 2d FEA solution and the data is 11.7%, 1.0%, and 8.3% for thermocouples 1, 2, and 3 respectively for an average relative difference of 7%. A summary of FEA relative differences is given in Table 16.

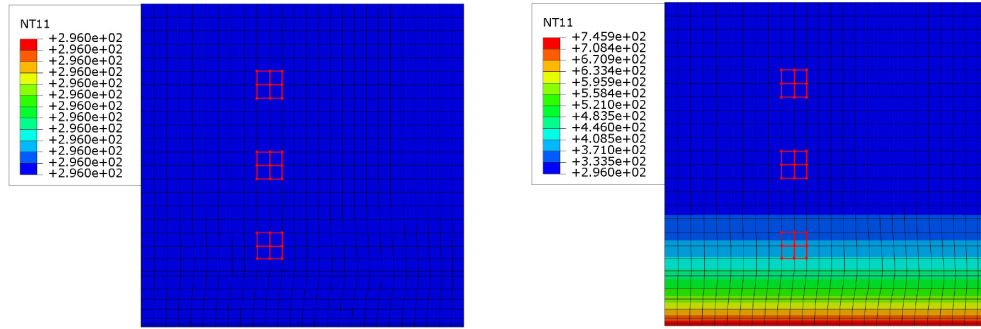


**Figure 23. Transient 2d- Abaqus Model at time 0 and 1 sec**

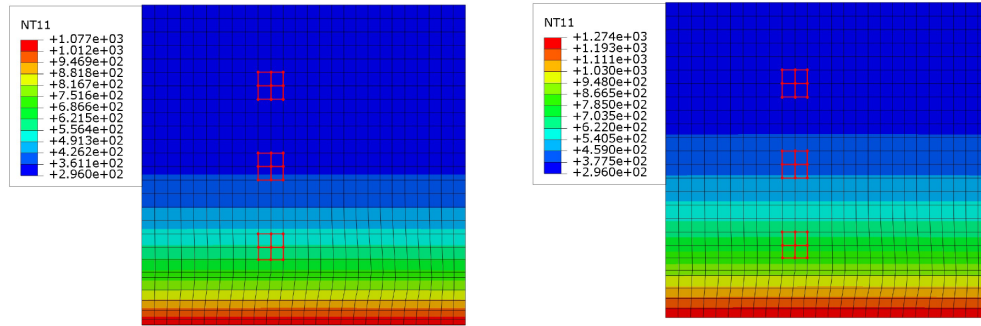


**Figure 24. Transient 2d- FEA Model at time 2 and 3 sec**

For visual reference to thermocouple locations, the elements surrounding each thermocouple are highlighted. The isotherms in Figures 23 through 25 are nearly horizontal again, like in the steady state case, leading to consideration of whether or not convection of the sides plays a significant role in the results. The largest difference in temperatures observed in a direction parallel to the bottom boundary is less than half-a-percent as seen in Figure 26.

To investigate the effects of convection on the system, the surface film on the left and

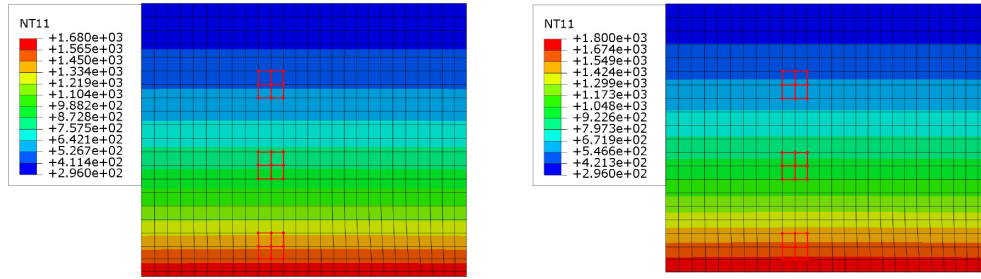


**Figure 25. Transient 2d- FEA Model at time 7 and 8 sec**

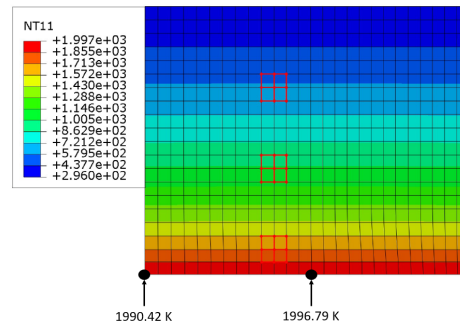


**Figure 26. Transient 2d- FEA Maximum Temperature Difference in Direction Parallel to the Bottom Boundary**

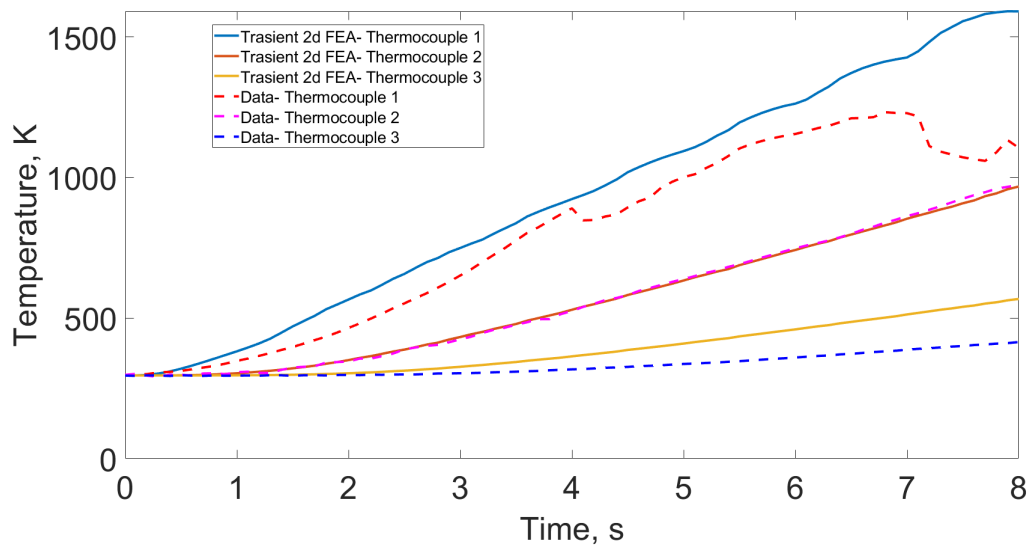


**Figure 27. Transient 2d- FEA Thermocouple Temperatures vs Data**

right boundaries were removed. This results in a model with no flux boundaries and effectively reduces the model to 1d. The 1d and 2d FEA models are plotted against each other in

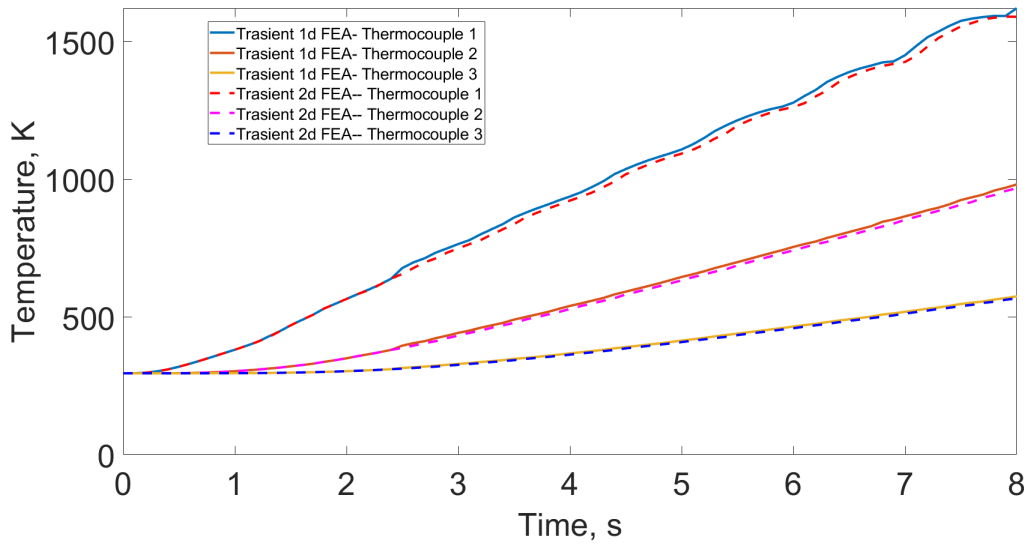


**Figure 28. Transient 2d- FEA Thermocouple Temperatures vs Transient 1d- FEA Thermocouple Temperatures**

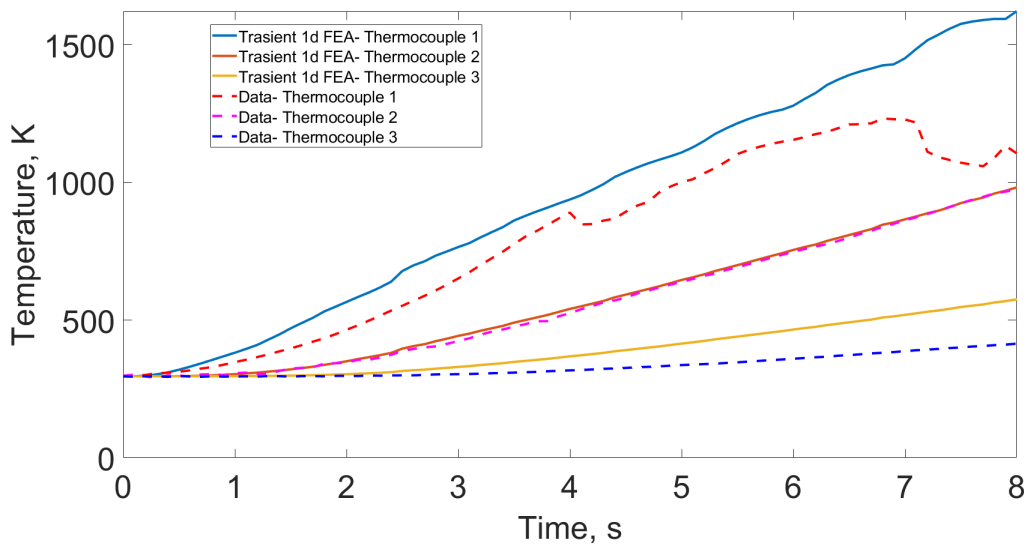


**Figure 29. Transient 1d- FEA Thermocouple Temperatures vs Data**

Figure 28; the average difference is 0.9% which will be considered negligible. The relative difference between the 1d FEA and the data is 13.0%, 1.8%, and 9.0% at thermocouples 1, 2 and 3 respectively for an average relative difference of 7.9%.

Observing the solution curves of the of the 1d FEA can provide intuition for tuning parameters that would fit the data better. For example, observing that thermocouples 1 and

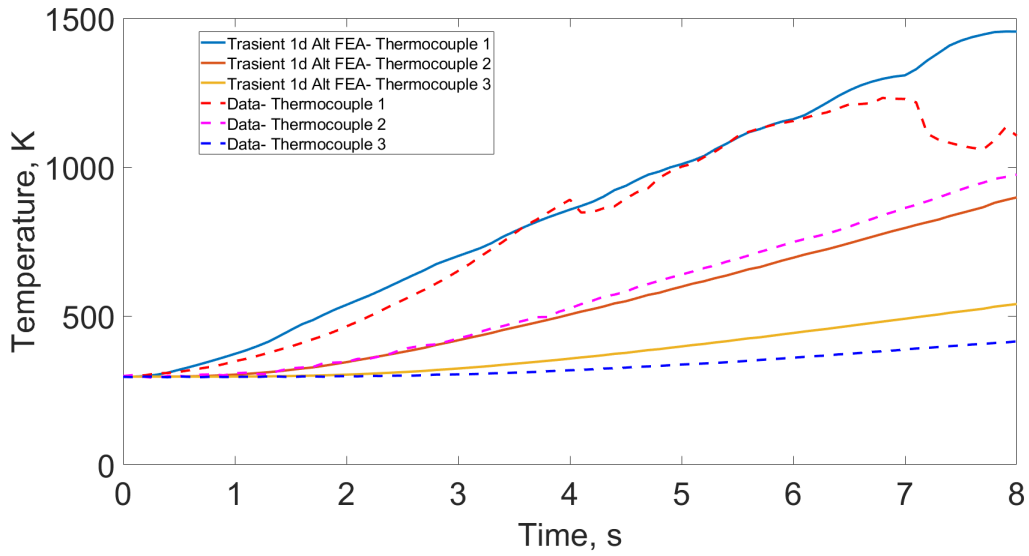


**Figure 30. Transient 1d Alternate- FEA Thermocouple Temperatures vs Data**

3 are reporting values too large, it may be that too much flux has entered the system. If the estimated value of  $\beta$  or  $\mu$  was taken too large, then  $q$  would be too large. Adjusting  $\beta$  to .11 yields  $q = 5.45 \times 10^6$ . The result is plotted as an "alternate" transient 1d FEA against the data in Figure 30. The relative difference is 6.3%, 2.7%, and 7.0% at thermocouples 1, 2, and 3 respectively for an average relative difference of 5.3%, which is a much improved approximation.

| Set 1            | Set 2        | $T_1$ rel diff | $T_2$ rel diff | $T_3$ rel diff | Avg rel diff |
|------------------|--------------|----------------|----------------|----------------|--------------|
| Steady State     | Data         | 200.2%         | 202.3%         | 155.0%         | 185.8%       |
| Transient 2d     | Data         | 11.7%          | 1.0%           | 8.3%           | 7.0%         |
| Transient 1d     | Data         | 13.0%          | 1.8%           | 9.0%           | 7.9%         |
| Transient 1d Alt | Data         | 6.3%           | 2.7%           | 7.0%           | 5.3%         |
| Transient 2d     | Transient 1d | 1.2%           | 1.1%           | 0.6%           | 0.9%         |

**Table 9. FEA- Summary of Relative Differences at Thermocouples 1, 2, and 3 (Labeled Here As  $T_i$ )**

### 3.2 Finite Difference Methods

The general form of the heat diffusion equation is given by [12]:

$$\begin{aligned}
\rho c(T) \frac{\partial T}{\partial t} &= \nabla \cdot (k \nabla T) \\
&= k(\nabla \cdot \nabla T) + \nabla k \nabla T \\
&= k \nabla^2 T + (T) \frac{dk}{dT} |\nabla T|^2
\end{aligned} \tag{26}$$

In cylindrical coordinates,

$$\nabla^2 T = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{1}{r^2} \frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2} \quad \& \quad |\nabla T|^2 = \left( \frac{\partial T}{\partial r} \right)^2 + \frac{1}{r^2} \left( \frac{\partial T}{\partial \phi} \right)^2 + \left( \frac{\partial T}{\partial z} \right)^2$$

Substituting into 26, and dividing both sides by  $\kappa$  yields:

$$\begin{aligned}
\frac{1}{\kappa(T)} \frac{\partial T}{\partial t} &= \left( \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{1}{r^2} \frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2} \right) \\
&+ \frac{T}{k(T)} \frac{dk}{dT} \left( \left( \frac{\partial T}{\partial r} \right)^2 + \frac{1}{r^2} \left( \frac{\partial T}{\partial \phi} \right)^2 + \left( \frac{\partial T}{\partial z} \right)^2 \right)
\end{aligned} \tag{27}$$

FEA results suggest, under assumption that the friction produces uniform heat flux, the temperature gradient in the radial direction is negligible for this experiment. Removing dependence on  $\theta$  and  $r$  results in the 1d heat diffusion equation:

$$\frac{1}{\kappa(T)} \frac{\partial T}{\partial t} = \left( \frac{\partial^2 T}{\partial z^2} \right) + \frac{T}{k(T)} \frac{dk}{dT} \left[ \left( \frac{\partial T}{\partial z} \right)^2 \right] \tag{28}$$

#### 3.2.1 Rescaled Partial Differential Equation

Scaling the parameters to the unit-less dimensions of the same magnitude can stabilize a numerical solver [26] and assist in analyzing the influence each parameter has on the solution so that values can be compared based on their magnitude without consideration

of the units involved [13]. Equation 27 is reduced to 1d and rescaled using the following definitions:

1. Length:  $x = \frac{z}{2r}; \quad 0 \leq x \leq 1$
2. Temperature:  $T_s = \frac{T - T_0}{T_{crit} - T_0}; \quad 0 \leq T_s \leq 1$

Where  $T_{crit}$  is some maximum value that the material is able to take on during the experiment; for this work,  $T_{crit} = T_{melt}$ . Equation 27 then becomes:

$$\frac{r^2}{\kappa(T_s)} \frac{\partial T_s}{\partial t} = \frac{\partial^2 T_s}{\partial x^2} + \frac{T_s}{k(T_s)} \frac{dk}{dT_s} \left[ \left( \frac{\partial T_s}{\partial x} \right)^2 \right] \quad (29)$$

Figure 31 shows the resulting 1d pin model after re-scaling. The system is solved for 1 second increments. At the end of each increment, the left boundary is moved according to the scaled displacement data given in Table 10 to reflect material removal. The solution is actually only solved for the first 7 seconds since the data for thermocouple 1 is assumed to be completely irrelevant thereafter.

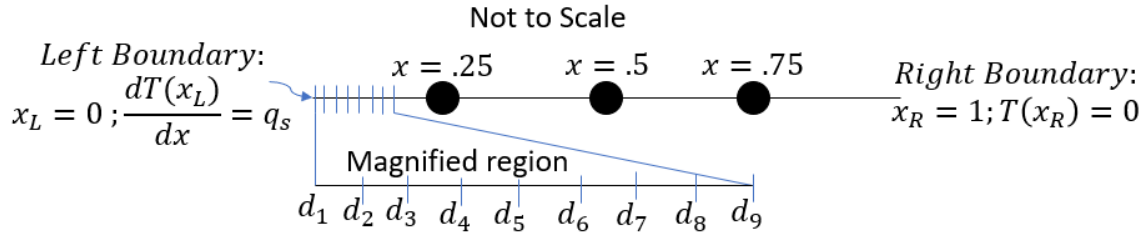


**Figure 31. Finite Difference Method Model**

### 3.2.2 Finite Differences

Finite difference methods may be employed to approximate a partial differential equation when an analytic formula cannot be found [22]. Finite difference methods work by approximating the derivative using the value of the function at nearby points. There is a wide array of approximation methods that provide various degrees of accuracy. Approximations may also be tailored to account for the location, for example by using forward, or

| Step    | Total Displacement | Total Time |
|---------|--------------------|------------|
| Initial | .00                | 0          |
| 1       | .0000              | 0-1 sec    |
| 2       | .0039              | 1-2 sec    |
| 3       | .0098              | 2-3 sec    |
| 4       | .0276              | 3-4 sec    |
| 5       | .0551              | 4-5 sec    |
| 6       | .0866              | 5-6 sec    |
| 7       | .1181              | 6-7 sec    |
| 8       | .1575              | 7-8 sec    |
| 9       | .1732              | 8-9 sec    |

**Table 10. Material Removal- Scaled for FDM**

backward, finite differences at the edges of a domain. Since the functions  $\frac{dk}{dT_s}$ ,  $\frac{1}{\kappa(T_s)}$  and  $\frac{1}{k(T_s)}$  will be assumed to be known based on extrapolations of data, the only unknowns in Equation 29 is the value of the time derivative and the spatial derivatives.

The classic way to develop finite difference equations is to start with the Taylor series expansion definition of a function,  $f$  [22]:

$$f(x_0 + \Delta x) = f(x_0) + f'(x_0)\Delta x/1! + f''(x_0)\Delta x^2/2! + f'''(x_0)\Delta x^3/3! + \dots + f^{(n)}(x_0)\Delta x^n/n! + \dots$$

Where  $\Delta x$  references the distance from the point  $x_0$  to the next point on the discretization. The discretization error, or so-called truncation error, is introduced by truncating the Taylor expansion somewhere to create an approximation. If the series is truncated such that a remainder  $R$  is left behind, then the error in  $R$  can be calculated. Let an approximation be:

$$f(x_0 + \Delta x) = f(x_0) + f'(x_0)\Delta x/1! + R(x)$$

$$\therefore f'(x_0) = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} + \frac{R(x)}{\Delta x}$$

$$\text{where } \frac{R(x)}{\Delta x} = f''(x_0)\Delta x/2! + f'''(x_0)\Delta x^2/3! + \dots + f^{(n)}(x_0)\Delta x^{n-1}/n! + \dots$$

Therefore, the approximation is said to be on the order of  $\Delta x$ . That is, the size of the error

is linearly associated with the size of the discretization chosen. Typically, Big O notation is used to associate the rate of decay of the dominant error term. So, for this example

$$f'(x_0) = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} + \mathcal{O}(\Delta x)$$

Removing the error term produces the standard forward-difference equation

$$f'(x_0) \approx \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

Similarly, the Taylor series can be made to approximate the derivative using the previous point on the discretization:

$$f(x_0 - h) = f(x_0) - f'(x_0)\Delta x/1! + f''(x_0)\Delta x^2/2! - f'''(x_0)\Delta x^3/3! + \dots$$

$$f'(x_0) = \frac{f(x_0) - f(x_0 - \Delta x)}{\Delta x} + \mathcal{O}(\Delta x)$$

Producing the standard backward-difference equation:

$$f'(x_0) \approx \frac{f(x_0) - f(x_0 - h)}{\Delta x}$$

The forward and backward difference equations can be combined to generate the standard central-difference equation. If  $x_0$  is centered between the points ahead and behind, then

$$f(x_0 + \Delta x) - f(x_0 - \Delta x) = 2f'(x_0)\Delta x/1! + 2f'''(x_0)\Delta x^3/3! + \dots$$

$$\therefore f'(x_0) = \frac{f(x_0 + \Delta x) - f(x_0 - \Delta x)}{2\Delta x} + \mathcal{O}(\Delta x^2)$$

$$f'(x_0) \approx \frac{f(x_0 + \Delta x) - f(x_0 - \Delta x)}{2\Delta x}$$

The central-difference finite difference equation shows how cancellations can occur by

incorporating additional points resulting in higher order accuracy. By extending these concepts, a large number of finite difference equations exist so that any order of truncation error desired can be theoretically achieved by including more points in the approximation. The lowest number that IEEE standard double format stores at full precision is  $1 \times 10^{-16}$ , so the spacing between the points could be chosen as low as  $\Delta x = 1 \times 10^{-16}$  in type double. To illustrate the concept of balancing computational errors with truncation errors, approximate the derivative of  $\sin(x)$  at  $\pi/6$  using forward-difference. Table 11 shows the approximation and the errors for this calculation given spacing between the points  $\Delta x = 10^{-1}, 10^{-2}, \dots, 10^{-16}$ . This table shows that the total error reduces until  $\Delta x = 10^{-8}$ . Continuing to reduce the truncation error, by using smaller values of  $\Delta x$ , creates computational errors that start to dominate the total error. So while truncation errors continue to decrease with  $\Delta x \rightarrow 0$ , the total error is seen to increase when  $\Delta x < 10^{-8}$ .

$$\sin'(\pi/6) \approx \frac{\sin(\pi/6+\Delta x) - \sin(\pi/6)}{\Delta x}$$

| $\Delta x$ | Approximation     | Error                |
|------------|-------------------|----------------------|
| $10^{-1}$  | 0.839603576017623 | 0.0264218277668158   |
| $10^{-2}$  | 0.863510990933142 | 0.00251441285129694  |
| $10^{-3}$  | 0.865775259467683 | 0.000250144316756074 |
| $10^{-4}$  | 0.866000402341416 | 2.50014430228651e-05 |
| $10^{-5}$  | 0.866022903761588 | 2.50002285118178e-06 |
| $10^{-6}$  | 0.866025153867245 | 2.49917193939631e-07 |
| $10^{-7}$  | 0.866025378631896 | 2.51525426042676e-08 |
| $10^{-8}$  | 0.866025401391468 | 2.39297059945187e-09 |
| $10^{-9}$  | 0.866025418044813 | 1.42603747699255e-08 |
| $10^{-10}$ | 0.866025584578267 | 1.80793828463699e-07 |
| $10^{-11}$ | 0.86602391924373  | 1.48454070858506e-06 |
| $10^{-12}$ | 0.866029470358853 | 4.06657441465175e-06 |
| $10^{-13}$ | 0.866529070719935 | 0.000503666935495972 |
| $10^{-14}$ | 0.860422844084496 | 0.00560255969994239  |
| $10^{-15}$ | 0.832667268468867 | 0.0333581353155714   |
| $10^{-16}$ | 0.555111512312578 | 0.31091389147186     |

**Table 11. FDM- Forward Difference Errors of  $\sin'(\pi/6)$  With Respect to Spacing**

The size of the computational error depends on storage type and the calculations per-

formed during the numerical approximation. Label the error created by storing a number as  $\varepsilon$  and the total computational error created by the approximation formula,  $E_c$ . The original representation of the number prior to any calculation results in an error of  $\mathcal{O}(\varepsilon)$  and every operation thereafter results in additional errors that can be computed. For forward-difference we have

$$f'(x) \approx [f(x_0 + \Delta x) - f(x_0)]/\Delta x + E$$

$$E_c = [\varepsilon f(x_0 + \Delta x) - \varepsilon f(x_0)]/\varepsilon \Delta x = [2\varepsilon + \varepsilon(f(x_0 + \Delta x) - f(x_0))]/\varepsilon \Delta x$$

From the original Taylor expansion, the storage error of  $f(x_0 + \Delta x) - f(x_0) = \Delta x f'(x_0)$ , is  $\Delta x \varepsilon$

$$\therefore E_c = [2\varepsilon + \varepsilon^2 \Delta x]/\varepsilon \Delta x = \varepsilon(2/\Delta x + 1) + \varepsilon = 2\varepsilon/\Delta x + 2\varepsilon$$

Note that while minimizing the truncation error, more and more points were created so that  $\Delta x \rightarrow 0$ . However, it is now apparent that as  $\Delta x$  tends to zero the computational error will blow up. To find the optimal spacing,  $\Delta x$ , let the total error of the approximation,  $E$ , be the sum of truncation error and computational error, then  $E$  can be approximated by

$$E = \Delta x + 2\varepsilon/\Delta x + 2\varepsilon$$

$$\text{and the optimal } \Delta x \text{ occurs when } \frac{dE}{d\Delta x} = 1 - 2\varepsilon/\Delta x^2 = 0$$

$$\Delta x = \sqrt{2\varepsilon}$$

For standard type double  $\varepsilon = 10^{-16}$ , which shows now why the error begins to increase for  $\Delta x < 10^{-8}$  in Table 11.

In solving equations using FDM that include a time derivative the terms implicit and explicit are introduced. Explicit methods solve for forward points in time using only previously assumed or computed values in backward points of time. An example would be a

differential equation of the form  $f'(t) = G$ , where  $G$  is any known function. Then take the forward-difference approximation of the time derivative. Since this is the time derivative it is common notation to let  $t_0 = 0$  ;  $t_1 - t_0 = \Delta t = t_{n+1} - t_n$  which yields

$$[f(t_1) - f(t_0)]/\Delta t \approx G$$

$$\therefore f(t_1) = (\Delta t)G + f(t_0)$$

Which can be solved for any initial value of  $f$ . Since the solution at  $f(t_{n+1})$  only relies on values of  $f$  at previous time steps, this is an explicit method. If instead,  $f(t_0)$  was solved for then the implicit method  $f(t_0) = f(t_1) - \Delta t G$  would have been generated. For implicit methods, the value of the function in the future must be known or estimated. Additional computations per step and treatments to find the value at future time-steps are required to solve implicit methods, so they are not generally the preferred method of solution. However, implicit methods can be more numerically stable and, since implicit methods solve with knowledge about the future, can solve stiff equations with less overall computations than explicit methods.

Stiff equations are ones such that the solution progresses very slowly, but there exists nearby solutions that progress very quickly [22]. If the solution requires very small time-steps to maintain desired accuracy, then the equation might be stiff. Stiff equations arise when  $G$  contains spatial derivatives. For an example, let  $T_t = T_{xx} = G$ , taking the forward-difference in time and central difference in space. Let the space be finite and represented by  $N$  points so that  $-\infty < x_1 < x_2 \cdots < x_N < \infty$ , then central-difference cannot be taken at  $x_1$  or  $x_N$  since these points are not central. Take forward-difference at  $x_1$  and backward-difference at  $x_N$ . Let the notation  $T_i^j$  denote the  $i$ th point in space and  $j$ th point in time, then

$$(T(x, t + \Delta t) - T(x, t))/\Delta t = (T(x + h, t) - T(x, t))/\Delta x ; x = x_1$$

$$(T(x, t + \Delta t) - T(x, t)) / \Delta t = (T(x + \Delta x, t) - 2T(x, t) + T(x - \Delta x, t)) / 2\Delta x ; x \in (x_1, x_N)$$

$$\& (T(x, t + \Delta t) - T(x, t)) / \Delta t = (T(x, t) - T(x - \Delta x, t)) / \Delta x ; x = x_N$$

The finite difference equations are

$$(T_n^{j+1} - T_n^j) / \Delta t = (T_{n+1}^j - T_n^j) / \Delta x ; n = 1$$

$$(T_n^{j+1} - T_n^j) / \Delta t = (T_{n+1}^j - 2T_n^j + T_{n-1}^j) / 2\Delta x ; 1 < n < N$$

$$(T_n^{j+1} - T_n^j) / \Delta t = (T_n^j - T_{n-1}^j) / \Delta x ; n = N$$

The spatial derivative can be rewritten in matrix notation

$$(T_n^{j+1} - T_n^j) / \Delta t = \frac{1}{2\Delta x} \begin{bmatrix} -2 & 2 & 0 & \dots & \dots & \dots & 0 \\ 1 & -2 & 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & -2 & 1 & 0 & \dots & 0 \\ \vdots & 0 & 0 & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & 1 & -2 & 1 \\ 0 & \dots & \dots & \dots & \dots & 2 & -2 \end{bmatrix} \vec{T}^j \quad (30)$$

A quasi-formal method of determining stiffness is to find the stiffness-ratio by calculating the eigenvalues of the matrix generated by the spatial derivatives. The stiffness ratio is said to be equal to the largest magnitude eigenvalue divided by the smallest magnitude eigenvalue [22]. A problem may be considered stiff if the stiffness ratio is large, but this is not a necessary condition [22]. The matrix in Equation 30 is actually singular, and therefore this quasi-formal method breaks down since there exists a zero eigenvalue. However, from Figure 31,  $T_N$  is held constant. This can be achieved by rewriting the matrix in Equation 30 as Equation 31. The matrix in Equation 31 is not singular, and the stiffness ratio rapidly

increases as more points are introduced as shown in Table 12.

$$(T_n^{j+1} - T_n^j)/\Delta t = \frac{1}{2\Delta x} \begin{bmatrix} -2 & 2 & 0 & \dots & \dots & \dots & 0 \\ 1 & -2 & 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & -2 & 1 & 0 & \dots & 0 \\ \vdots & 0 & 0 & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & 1 & -2 & 1 \\ 0 & \dots & \dots & \dots & \dots & 0 & 1 \end{bmatrix} \vec{T}^j \quad (31)$$

| Number of Points, N | Stiffness Ratio |
|---------------------|-----------------|
| 10                  | 130             |
| 20                  | 584             |
| 26                  | 1012            |
| 255                 | 104588          |

**Table 12. FDM- Stiffness Ratios for Equation 31**

Using the same methodology for solution as in FEA, the left boundary is moved after each one second interval to represent wear (as shown if Figure 32). Including displacement locations and thermocouples on a uniform grid determines the smallest  $\Delta x$  that can be used. The temperature distribution at the end of each one second interval is recast as the initial values of temperature during the next one second time interval. The smallest displacement after any step is the first displacement taken after 1 second, by Table 6, which is  $d_2 = .00005m$ . Since the domain is  $.0127m$  long, it takes 255 points to cover the domain uniformly. The heat equation is known to be able to exhibit stiffness [22]. The stiffness ratio for the second derivative using central differences and a constant boundary using 255 points is 104588. So Equation 29 may be stiff.

When solving with implicit and explicit methods, numerical stability of the method is also a concern. For any finite difference equation the maximum step size must be constrained so that the equation can maintain the desired order of accuracy. For  $T_t = T_{xx}$  using

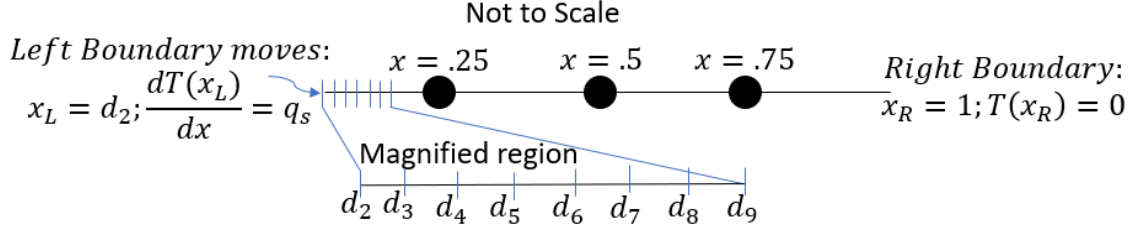


Figure 32. Finite Difference Method Model With Moving Boundary

central-differences in space and forward differences in time,  $\Delta t$  must be less than or equal to  $\Delta x^2/2$ . The stability requirements would then constrain  $t$  to be less than  $(5/2) \times 10^{-10}$ . Since solution is requested for 7 seconds it would take about  $3 \times 10^{10}$  steps to complete solution so that it may be beneficial to switch to an implicit method. .

Generating finite differences for  $T_t = T_{xx}$  using central-differences in space and backward differences in time, yields

$$\frac{T_n^{j+1} - T_n^j}{\Delta t} = \frac{T_{n+1}^{j+1} - 2T_n^{j+1} + T_{n-1}^{j+1}}{\Delta x^2}. \quad (32)$$

Where  $T_n^{j+1}$  can be solved by the linear system

$$(1 + 2\frac{\Delta t}{\Delta x^2})T_n^{j+1} - \frac{\Delta t}{\Delta x^2}T_{n-1}^{j+1} - \frac{\Delta t}{\Delta x^2}T_{n+1}^{j+1} = T_n^j. \quad (33)$$

Solving this system for each time step creates additional computations per step, but it is always numerically stable and convergent for all step sizes with accuracy  $\mathcal{O}(\Delta t) + \mathcal{O}(\Delta x^2)$ . Including additional points in the time difference calculation can increase the accuracy of the method in the same way as in the space dimension. For example, the Crank-Nicolson method takes central difference at time  $t_{n+1/2}$  and second order central differences in space to yield an accuracy of  $\mathcal{O}(\Delta t^2) + \mathcal{O}(\Delta x^2)$  [22].

Searching for fast and accurate result, we look to solve Equation 29 using an implicit method. The pdepe built-in function provided by MATLAB calls ODE15s which

is a variable-step, variable-order finite difference method (FDM) solver. ODE15s is a stiff solver, which means that it uses implicit solution methods. The ODE15s routine maintains the desired level of accuracy while applying first through fifth order finite difference methods. The pdepe solver accepts 1d parabolic and elliptic partial differential equations which are of the form:

$$c\left(x, t, u, \frac{\partial u}{\partial x}\right) \frac{\partial u}{\partial t} = x^{-m} \frac{\partial}{\partial x} \left( x^m f\left(x, t, u, \frac{\partial u}{\partial x}\right) \right) + s\left(x, t, u, \frac{\partial u}{\partial x}\right). \quad (34)$$

With boundary conditions:

$$p^*(x, t, u) + q^*(x, t) f\left(x, t, u, \frac{\partial u}{\partial x}\right) = 0 \quad (35)$$

Comparing Equations 34 and 29, note that

$$m = 0 ; c = \frac{r^2}{\kappa(T_s)} ; f = \frac{\partial T_s}{\partial x} ; s = \frac{T_s}{k(T_s)} \frac{dk}{dT_s} \left[ \left( \frac{\partial T_s}{\partial x} \right)^2 \right]$$

$$p_L^* = q_s ; q_L^* = 1 ; p_R^* = T_{sR} ; q_R^* = 0$$

$$\implies \frac{\partial T_s}{\partial x} = q_s = \frac{rq}{k(T_{crit} - T_0)} [15] ; T_{sR} = 0$$

Where the subscript  $L$  and  $R$  refer to the left and right boundary, respectively in Figure 31.

Optimization routines are then applied to fit the FDM model to the data. In FDM solution form, changing and optimizing the parameters to fit the data proved easier than in FEA form. Although there are many concerns regarding the accuracy of the data recovered by the thermocouples in this experiment there is ongoing effort to perform the experiment again with better instrumentation and collection methods using lessons learned. Once a trusted data set has been collected the framework below may serve as a good reference point to determine the material properties of Vascomax®300.

### 3.2.3 FDM- Constant Parameters

For a baseline analysis the solver was set-up with the same constant properties as the FEA model (Tables 7 & 8). The boundary was moved after each 1 second transient solution. The solution at the end of each second is saved and used as the initial condition for the next 1 second period of solution to reflect the boundary location in Table 10. Time-steps of .1 second were requested so that the solution trajectory can be compared to the FEA results. Space domain results were requested at uniform spaced grid of 21 points and the displacement locations. This provides interpolation points for the thermocouples and for the initial condition of the next step and is reflective of the formulation used by Boardman's 2d FDM [14]. The solution grid can be seen in Figure 33; note that thermocouples 1, 2, and 3 are located at  $x = .25, .5$  &  $.75$ , respectively. Notice the moving boundary on the left end representing the friction wear in Figure 33. The solution was generated only for the first 7 seconds due to irregularity of the data for thermocouple 1 thereafter. For comparison between solution methods, the relative difference is taken the same way as with FEA methods.

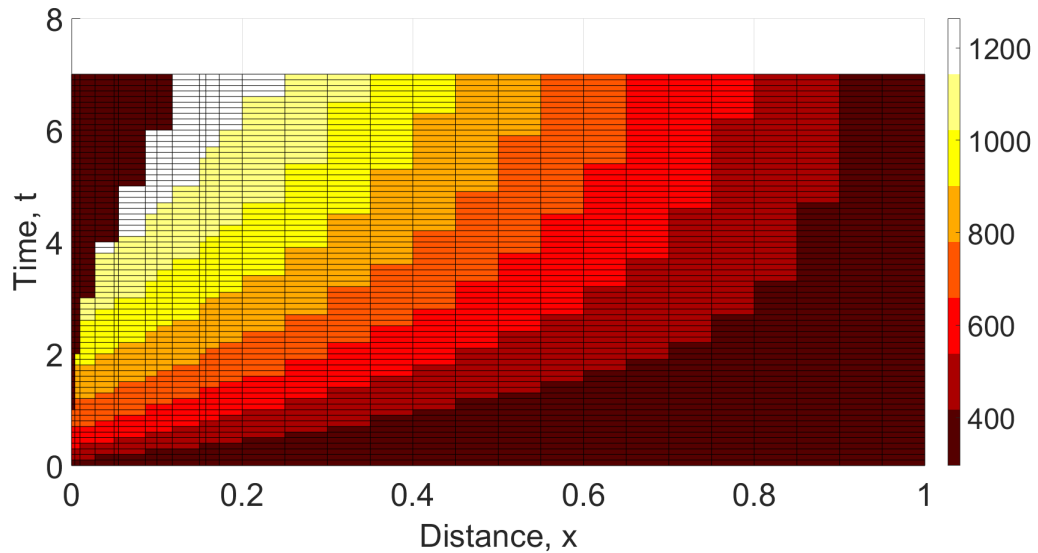


Figure 33. Removal by Displacement Data using pdepe with Constant Parameters- Top View

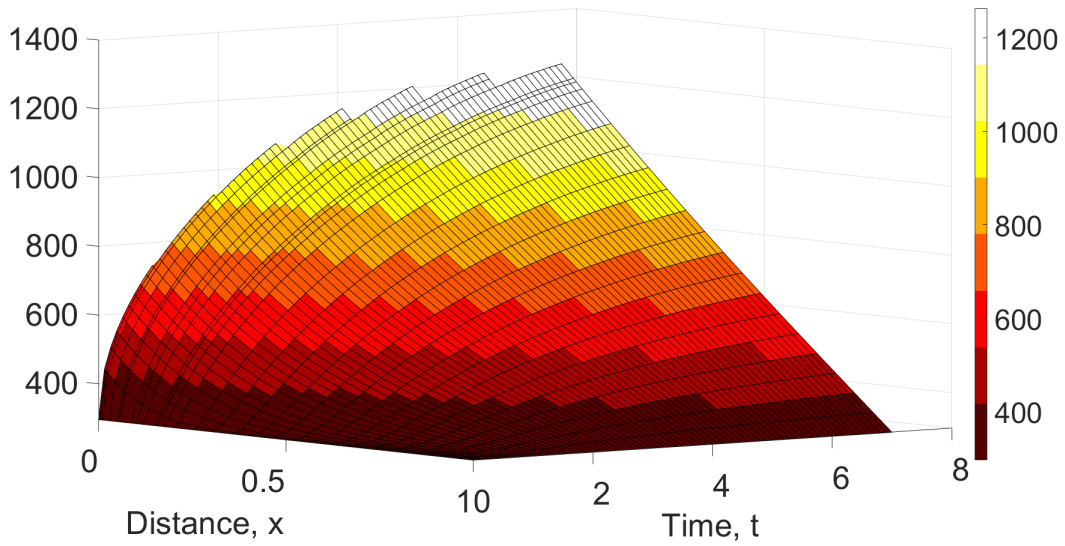


**Figure 34. Removal by Displacement Data using pdepe with Constant Parameters- Isometric View**

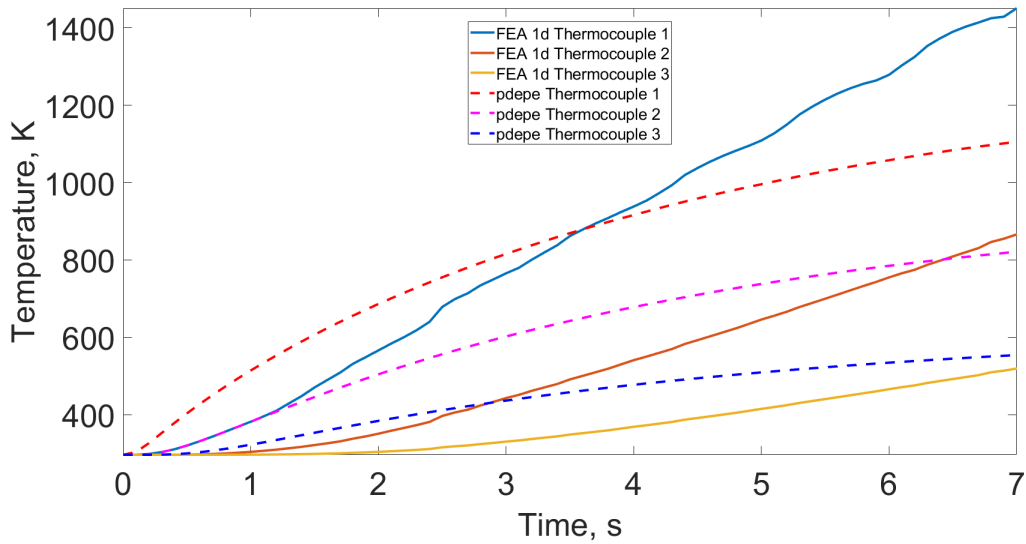


**Figure 35. Comparing Removal by Displacement with Constant Parameters using pdepe with Abaqus**

The entire program runs in less than 7 seconds. The result qualitatively and quantitatively fits the data much more poorly than that of the Abaqus model. Qualitatively, Figure 36 shows that the model does not trend well with the data. Quantitatively, the relative differences at thermocouples 1, 2, and 3 are 24.4%, 28.1%, and 32.9% respectively for an average relative difference of 28.5%. While considerably less accurate, the pdepe method

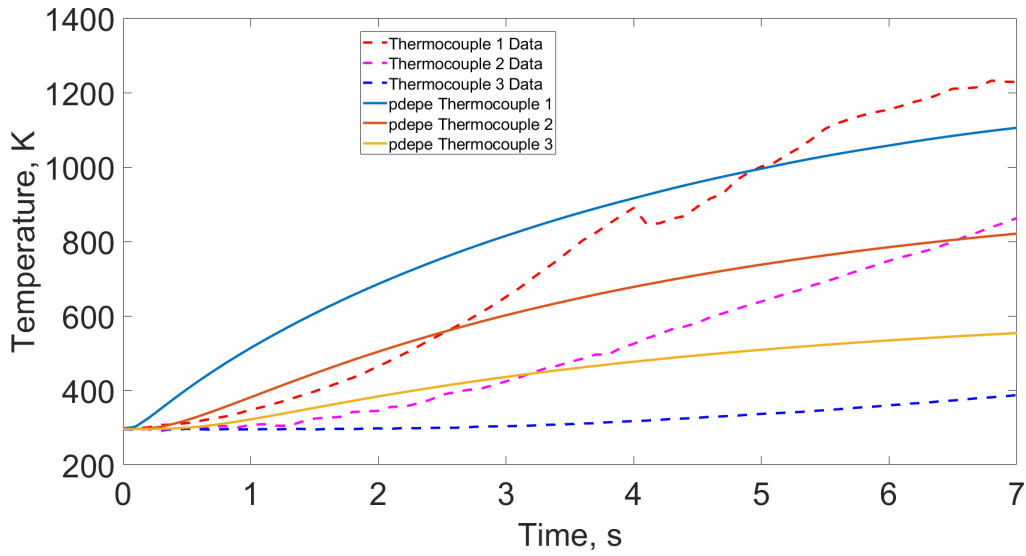


**Figure 36. Comparing Removal by Displacement with Constant Parameters using pdepe with Data**

of solution makes it much easier to determine the effects of non-constant parameters on the solution, which is performed in the next section. Variable functions can be used in the problem definition whereas Abaqus required manual entry of step-wise functions be defined.

### 3.2.4 pdepe- Non-constant Parameters

In this section investigation into the effects of varying the parameters will be performed by treating the force and thermal diffusivity relation parameters as functions. The empirical data shows the forcing data having considerable noise. The noise was removed by assuming force starts at zero and taking the average over each .1 s increment to create an averaged force per .1s. Then fitting the averaged points to a 5th order polynomial produced the function seen in Figure 37.

Very limited data was found regarding specific heat and thermal conductivity. For specific heat, the data in Table 3 was approximated and extrapolated by the function in Figure 9 by Deleon [17]. Lacking any other data, Deleon's extrapolated function was used. Deleon defined the function piece-wise which is not acceptable in the pdepe function format so a

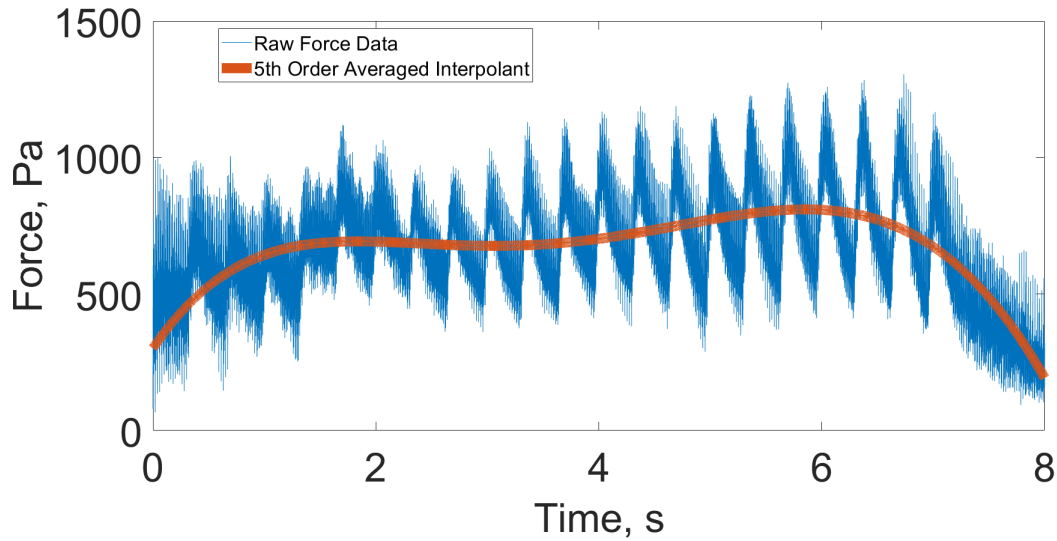


**Figure 37. Variable Force Averaged Interpolant**

6th order polynomial interpolant was used instead and is shown in Figure 38. Data concerning thermal conductivity is similarly sparse, so that the log fit suggested by Deleon in Figure 8 was used.

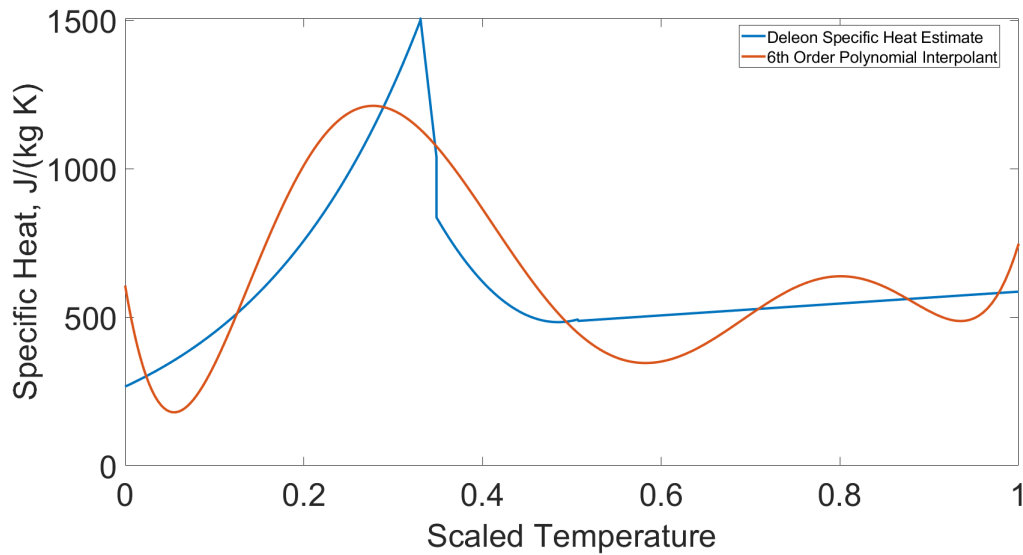


**Figure 38. 6th Order Polynomial Interpolant of Deleon's Specific Heat**

Figure 39 shows qualitatively little difference in the solution profile of the constant parameter and variable parameter solutions. The relative difference between the constant and

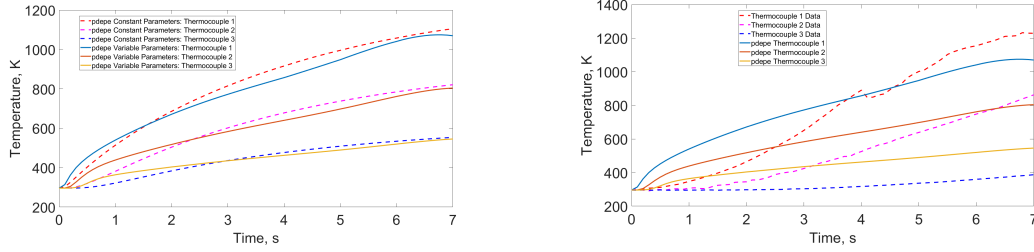


**Figure 39. Comparing Removal by Displacement with Variable Parameters using pdepe with Constant Parameters and Data**

non-constant solutions at thermocouples 1, 2 and 3 are 5.2%, 5.9% and 4.3%, respectively, for an average relative difference of 5.1%. The relative difference between the non-constant solution and the data at thermocouples 1, 2 and 3 are 24.4%, 29.6%, and 34.7%, respectively, for an average relative difference of 29.6%. While the constant parameter model performs slightly better quantitatively, the finite-differences model performs unsatisfactorily in both cases. In order to increase the accuracy of the model and hopefully generate better results, spectral methods are turned to in the next section. It was decided to avoid continuing to refine the mesh using finite-differences since taking smaller time-steps using spectral methods is more affordable than generating finer grid-spacing for finite-difference methods [27].

### 3.2.5 Optimizing Parameters

A fundamental question of the problem statement is to estimate the temperature at the boundary to determine if significant melt wear is occurring, or if it is primarily mechanical wear. Extrapolation from the thermocouple solutions demands a good fit to the empirical data. In order to fit the solution to the thermocouple data, the constant parameters used previously were employed. Thermal diffusivity,  $\kappa$ , and the product of the coefficient of friction and the partition function,  $\mu\beta$ , were optimized using the `fmincon` function of MATLAB. The average of the three thermocouple's 2-norm difference between the solution and the data taken every tenth of a second was used to determine the optimal results.

The relative difference at thermocouple 1, 2, and 3 , respectively, was 4.7%, 5.8% and 4.8% for an average relative difference of 5.1%, the best fit so far. Figure 41 shows the predicted temperature at the contacting surface reaching melt temperature. The values for the two optimization parameters found were:  $\kappa = 1.0510 \times 10^{-6}$  and  $\mu\beta = 0.0506$ .

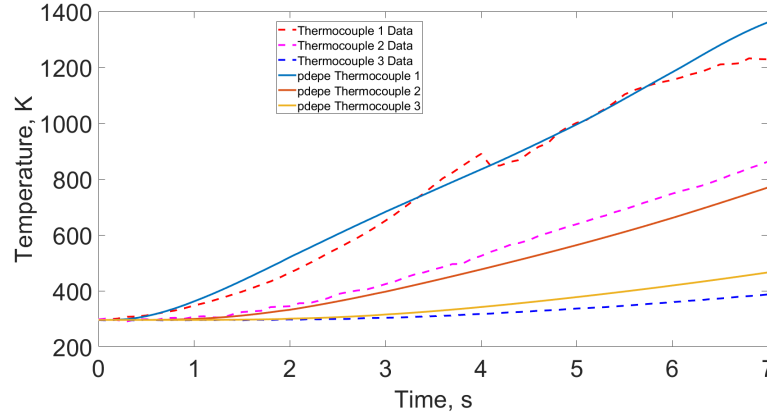


**Figure 40. Optimized Parameters at Thermocouples using pdepe vs Data**

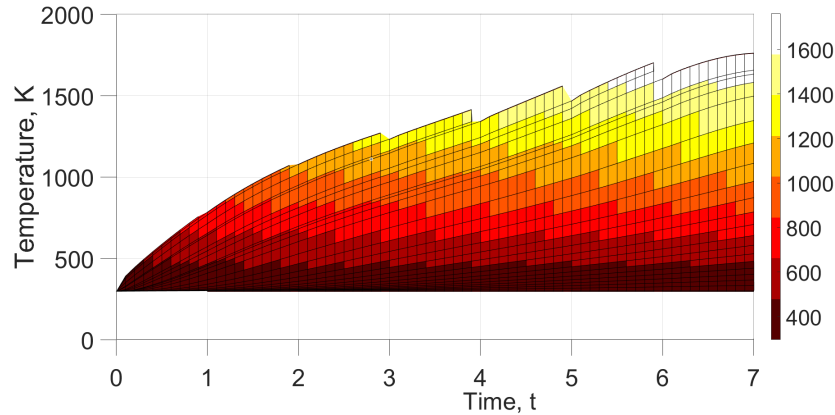


**Figure 41. Optimized Parameters using pdepe Surface Plot**

| Set 1                | Set 2               | $T_1$ rel diff | $T_2$ rel diff | $T_3$ rel diff | Avg rel diff |
|----------------------|---------------------|----------------|----------------|----------------|--------------|
| Constant Parameters  | Data                | 24.4%          | 28.1%          | 32.9%          | 28.5%        |
| Variable Parameters  | Data                | 24.4%          | 29.6%          | 34.7%          | 29.6%        |
| Optimized Parameters | Data                | 4.7%           | 5.8%           | 4.8%           | 5.1%         |
| Constant Parameters  | Variable Parameters | 5.2%           | 5.9%           | 4.3%           | 5.1%         |

**Table 13. FDM- Summary of Relative Differences at thermocouples 1, 2, and 3 (Labeled Here As  $T_i$ )**

### 3.3 Spectral Methods

Diffusion of heat through an initially constant temperature cylinder is a problem that results in smooth data on a simple domain. For problems on simple domains, with smooth data, spectral methods provide good approximations [27]. Spectral methods became famous in the 1970's for their effectiveness in solving partial differential equations. The foundation of a spectral method is to implement approximation methods on a global scale by incorporating all of the grid points in the approximation of the derivative [27]. Finite-difference methods generate sparse differentiation matrices that increase in accuracy for each additional neighboring point used to approximate the derivative. A method which uses all of the points in the discretization in the approximation of the derivative, resulting in a dense differentiation matrix, may be called a "spectral method".

For bounded, non-periodic, domains the use of equispaced points produces bad derivative approximations. When smooth functions are interpolated on equispaced points by a polynomial, the approximation error grows at a rate of  $2^N$  and will generally fail to converge as  $N \rightarrow \infty$  [27]. The Chebyshev points provide the preferred uneven spacing to effectively interpolate bounded, non-periodic, domains [27]:

$$x_j = \cos(j\pi/N), \quad j = 0, 1, \dots, N \quad (36)$$

Interpolating at the Chebyshev points through  $N + 1$  data points and taking the derivative results in the differentiation matrix,  $D_N$ . Consider  $N = 1$ , then by Equation 36  $x_0 = 1$  and  $x_1 = -1$ . For arbitrary data points  $v_0$  and  $v_1$ , the Lagrange form of the interpolant is [27]:

$$p(x) = \frac{1}{2}(1+x)v_0 + \frac{1}{2}(1-x)v_1 \quad (37)$$

$$\therefore p'(x) = \frac{1}{2}v_0 - \frac{1}{2}v_1 \quad (38)$$

Since  $x_0$  and  $x_1$  are not present in Equation 38, the differentiation matrix,  $D$ , immediately follows to be [27]:

$$D_1 = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \ni D_1 \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} = p'(x) \quad (39)$$

This process can be carried out for any  $N \in \mathbb{N}$ , and results in a general formula for the matrix  $D_N$  [27]:

$$(D_N)_{ij} = \begin{cases} \frac{2N^2+1}{6} & \text{for } i = j = 0 \\ -\frac{2N^2+1}{6} & \text{for } i = j = N \\ \frac{-x_j}{2(1-x_j^2)} & \text{for } i = j = 1, \dots, N-1 \\ \frac{c_i}{c_j} \frac{(-1)^{i+j}}{(x_i-x_j)} & \text{for } i \neq j, \quad i, j = 0, \dots, N \end{cases} \quad (40)$$

$$\text{where } c_i = \begin{cases} 2 & \text{for } i = 0 \text{ or } N, \\ 1 & \text{otherwise} \end{cases} \quad (41)$$

The MATLAB cheb command [4] generates the  $D$  matrix using identities which have better stability properties in the presence of rounding errors, but is functionally the same as the above definition [27]. Using the spectral differentiation matrix,  $D_N$ , Equation 29 can be written as:

$$\frac{\partial T_s}{\partial t} = \frac{\kappa(T_s)}{r^2} \left[ (D_N)^2 T_s + \frac{T_s}{k(T_s)} \frac{dk}{dT_s} \left[ (D_N T_s)^2 \right] \right] \quad (42)$$

With boundary conditions:

$$(T_s)_1 = 0 \quad \text{for } x = -1 \quad (43)$$

$$(T_s)_{N+1} = \frac{1}{(D_N)_{N+1,N+1}} [-Q - (D_N)_{N+1,1:N} T_{s1:N}] \quad \text{for } x = 1 \quad (44)$$

The derivative in time then becomes the only unknown, which is solved for using finite differences. While, in principle, spectral accuracy is sacrificed by taking the finite-difference in time, in practice time steps of order 2 or higher will maintain satisfactory

global accuracy [27]. Taking smaller time-steps using spectral methods is more affordable than generating finer grid-spacing for finite-difference methods [27]. Therefore, the spectral method is more efficient in generating high accuracy solutions using small time-steps than finite-difference methods using small grid-spacing due to spectral accuracy. The advantage comes from what is called "spectral accuracy" of the derivative approximation. Trefethen illustrates this spectral accuracy phenomenon by showing the difference between convergence of a fourth order finite-difference scheme (Figure 42) and a spectral method using Chebyshev points (Figure 43) to solve for the derivative of  $e^{\sin x} \cos x$  [27]. The spectral method achieves an error of  $10^{-10}$  with  $N = 20$ , where the FDM requires  $N = 20,000$ .

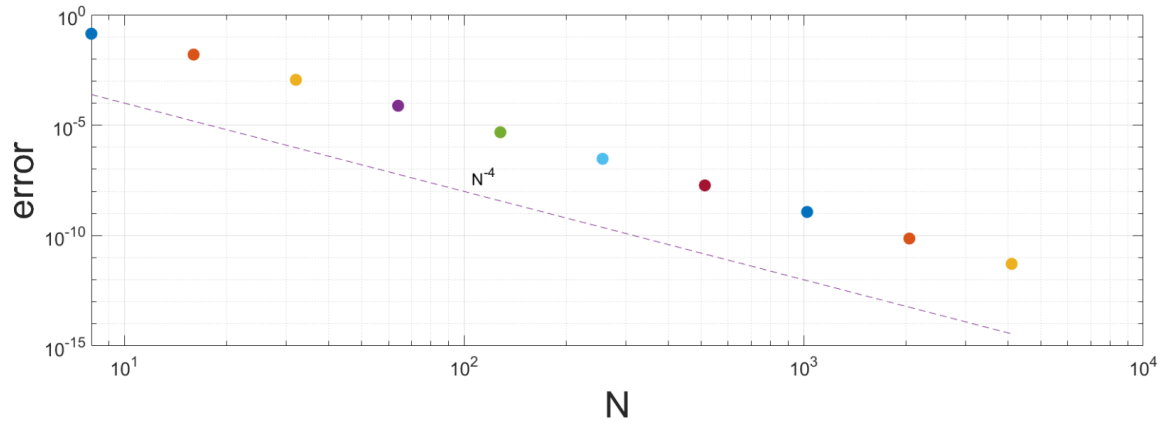
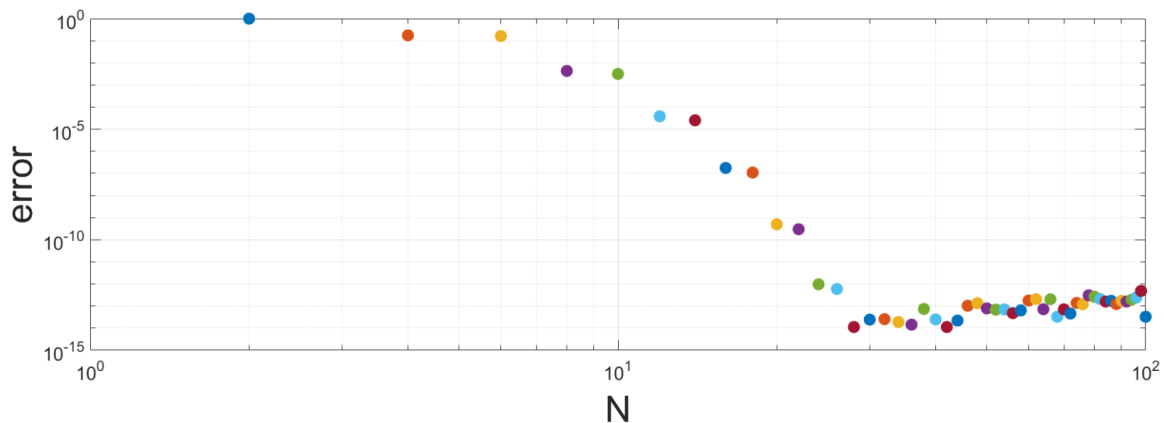


**Figure 42. Convergence of a Fourth order FDM [27]**



finite-differences in time, which is the same solver called by pdepe in the FDM section. After each one second run the solution is interpolated using the chebfun MATLAB command [4] and rescaled back onto a Chebyshev grid of  $N$  points on  $[-1, 1]$ . This requires first completing a change of coordinates after each run since the  $[-1, 1]$  Chebyshev grid of the  $i$ th run represents the  $[d_i, 1]$  scaled grid from Table 10. This results in the rescaled coordinates in Table 14. This table gives the rescaled relative displacement,  $r_i$  so that the interpolating on the Chebyshev points  $x(2 - r_i)/2 - r_i/2$  provides the initial values for the next run where  $x \in [-1, 1]$ . This is done by first finding the relative displacement,  $d_i^* = d_i - d_{i-1}$ , then finding the re-scaled relative displacement,  $r_i = 2d_i^*/(1 - d_i)$ . So that, although the solutions are computed on the  $[-1, 1]$  Chebyshev grid, they are interpolated and re-scaled back onto the same  $[d_i, 1]$  grid that was used for FDM (see Figure 32).

| Step    | Re-scaled Relative Displacement | Total Time |
|---------|---------------------------------|------------|
| Initial | .00                             | 0          |
| 1       | .0000                           | 0-1 sec    |
| 2       | .0078                           | 1-2 sec    |
| 3       | .012                            | 2-3 sec    |
| 4       | .0366                           | 3-4 sec    |
| 5       | .0582                           | 4-5 sec    |
| 6       | .069                            | 5-6 sec    |
| 7       | .0714                           | 6-7 sec    |
| 8       | .0936                           | 7-8 sec    |
| 9       | .038                            | 8-9 sec    |

**Table 14. Relative Displacements- Re-scaled for Spectral Methods**

As with FDM, constant parameters were considered first using the same properties (Tables 7 & 8). The relative difference between the constant parameter model and the data at thermocouples 1, 2 and 3 are 8.4%, 5.8%, and 5.7%, respectively, for an average relative difference of 6.6%. Therefore, spectral methods provides the best model to represent the data when using constant parameters. Recall that the FEA and FDM models produced 7.9% and 28.5% relative differences, respectively. Figure 44 shows the thermocouple temperature data plotted with the modelled temperature at the thermocouples and at the surface.

Figure 45 shows a scatter plot of the model output viewed from the top.

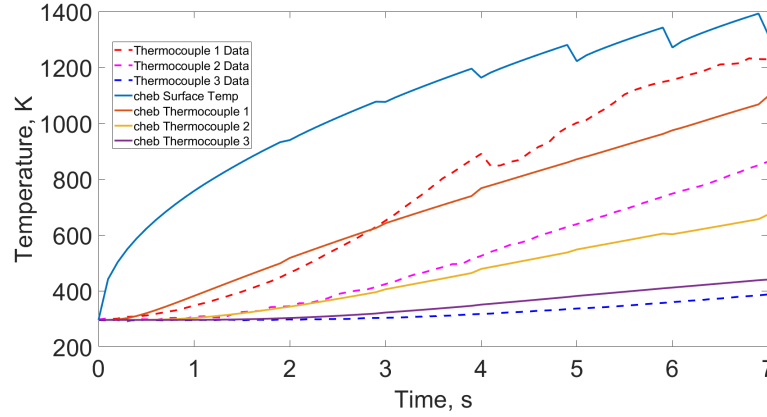


**Figure 44. Spectral Differentiation with Constant Parameters- Surface Temp and Thermocouples vs Data**

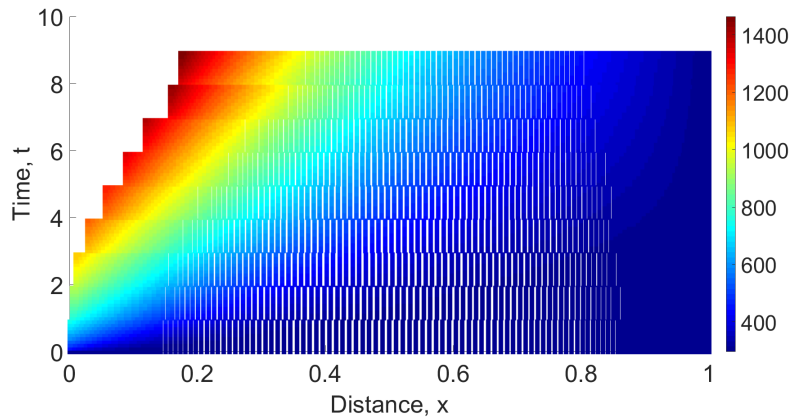


**Figure 45. Spectral Differentiation with Constant Parameters- Scatter Plot Top View**

Variable parameters were also considered using the 5th order averaged interpolant for the forcing data and Deleon's estimate of the thermal conductivity. Thermal diffusivity was held constant at  $4.4883 \times 10^{-6}$ . The relative difference between the variable parameter model and the data at thermocouples 1, 2 and 3 are 7.8%, 8.6%, and 4.3%, respectively, for an average relative difference of 6.9%. Therefore, spectral methods provides the best model to represent the data when using variable parameters as well. Recall that the FDM models produced a 29.6% relative difference. The variable parameter model quantitatively fits the data slightly worse compared the constant parameter model. Figure 46 shows the

thermocouple temperature data plotted with the modelled temperature at the thermocouples and at the surface. Figure 47 shows a scatter plot of the model output viewed from the top.

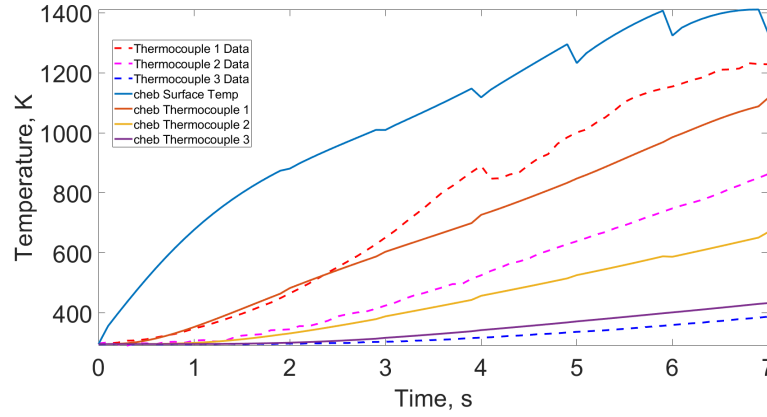


**Figure 46. Spectral Differentiation with Variable Parameters- Surface Temp and Thermocouples vs Data**

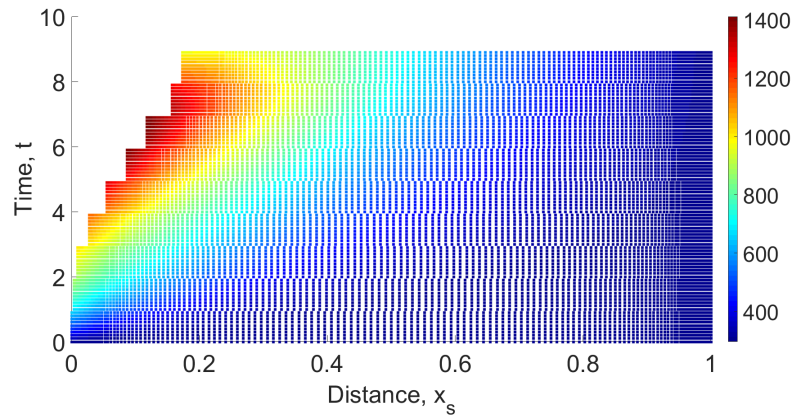


**Figure 47. Spectral Differentiation with Variable Parameters- Scatter Plot Top View**

Optimization of the parameters  $\kappa$  and  $\mu\beta$  was then interrogated. The relative difference between the optimized parameter model and the data at thermocouples 1, 2 and 3 are 2.7%, 12.7%, and 0.3%, respectively, for an average relative difference of 5.2%. Figure 48 shows the thermocouple temperature data plotted with the modelled temperature at the thermocouples and at the surface. This optimization model quantitatively improves the fit of the data compared to the constant and variable parameter models. Figure 49 shows a scatter

plot of the model output viewed from the top. The optimized parameters determined were  $\kappa = 2.5877 \times 10^{-6}$  and  $\mu\beta = 0.0541$ .

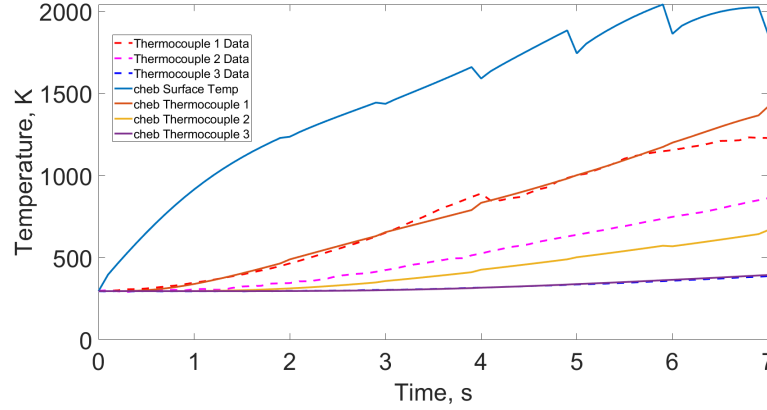


**Figure 48. Spectral Differentiation with Optimized Parameters- Surface Temp and Thermocouples vs Data**

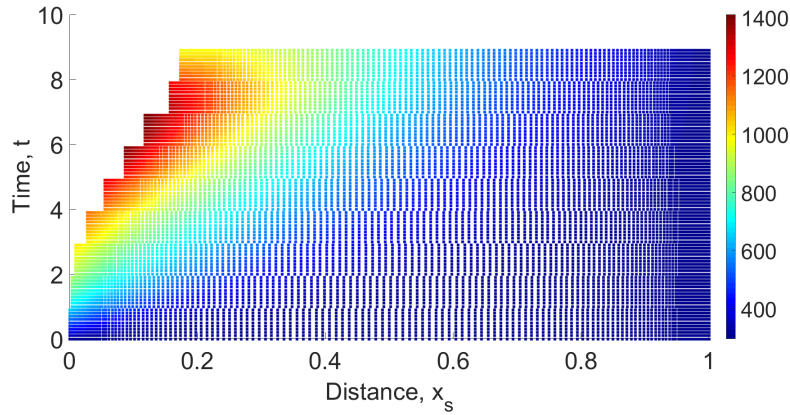


**Figure 49. Spectral Differentiation with Optimized Parameters- Scatter Plot Top View**

| Set 1                | Set 2               | $T_1$ rel diff | $T_2$ rel diff | $T_3$ rel diff | Avg rel diff |
|----------------------|---------------------|----------------|----------------|----------------|--------------|
| Constant Parameters  | Data                | 8.4%           | 5.8%           | 5.7%           | 6.6%         |
| Variable Parameters  | Data                | 7.8%           | 8.6%           | 4.3%           | 6.9%         |
| Optimized Parameters | Data                | 2.7%           | 12.7%          | 0.3%           | 5.2%         |
| Constant Parameters  | Variable Parameters | 5.5%           | 3.2%           | 1.4%           | 3.4%         |

**Table 15. Spectral Method- Summary of Relative Differences at thermocouples 1, 2, and 3 (Labeled Here As  $T_i$ )**

The stiffness in the solution can be seen by changing the implicit ODE15s solver to

the explicit ODE45 solver. Equation 42 is solved for the first second using both ODE15s and ODE45 recording the number of time steps and the computation time required. The number of time steps and computation time required using ODE45 grows exponentially as  $N$  is increased linearly. The number of time steps and computation time required to generate solutions using ODE15s grows linearly as  $N$  is increased linearly.

| N   | ODE15s Steps | ODE15s Time (s) | ODE45 Steps | ODE45 Time (s) |
|-----|--------------|-----------------|-------------|----------------|
| 5   | 50           | 0.01            | 53          | 0.01           |
| 10  | 59           | 0.01            | 93          | 0.01           |
| 20  | 69           | 0.02            | 1041        | 0.04           |
| 40  | 91           | 0.04            | 16317       | 0.75           |
| 80  | 106          | 0.08            | 260437      | 9.38           |
| 160 | 114          | 0.15            | N/A*        | N/A*           |

\* MATLAB Error: Out of memory

**Table 16. Stiffness: Time Steps and Computation Time Required for ODE45 and ODE15s to Generate 1 Second of Solution of Equation 42**

### 3.3.1 Investigating Stiffness

While the equation is stiff, if  $N$  is small enough the ODE45 solver may be reasonable to use. Convergence is tested by determining the relative differences with  $N = 150$  taken to be the finest mesh available. For ODE45, the full solution is obtained for  $N = 6, 12, 18, \dots, 150$ . Solution is obtained in multiples of six due to time constraints in finding the solution resulting from stiffness; generating Figure 50 took 8601 seconds. The relative difference is found for each run relative to  $N = 150$  and plotted in Figure 50. For ODE15s the full solution is obtained for  $N = 1, 2, 3, \dots, 150$  using ODE15s and the relative difference with respect to  $N = 150$  is plotted in Figure 51.

The solutions in the previous sections were all obtained using ODE15s with  $N = 150$  in order to avoid the stiffness. But, Figure 50 shows that convergence of the solution occurs fairly rapidly with the relative difference dropping below  $10^{-6}$  by  $N = 18$ . A  $10^{-6}$  level of convergence is commensurate with the level of convergence attained for the FEA model.

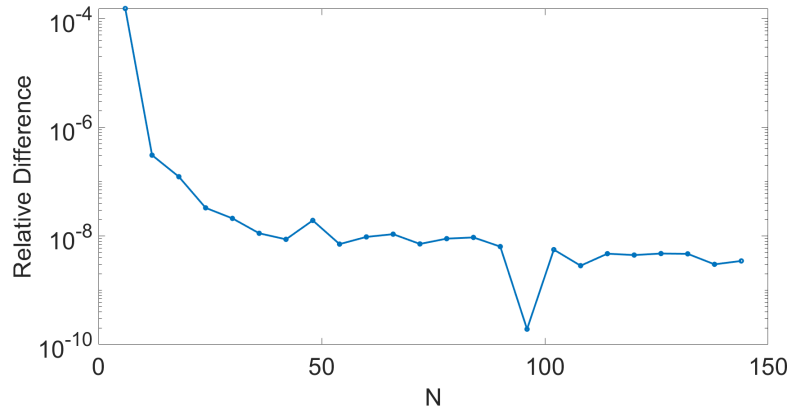


**Figure 50. Convergence of ODE45 Solving Equation 42**

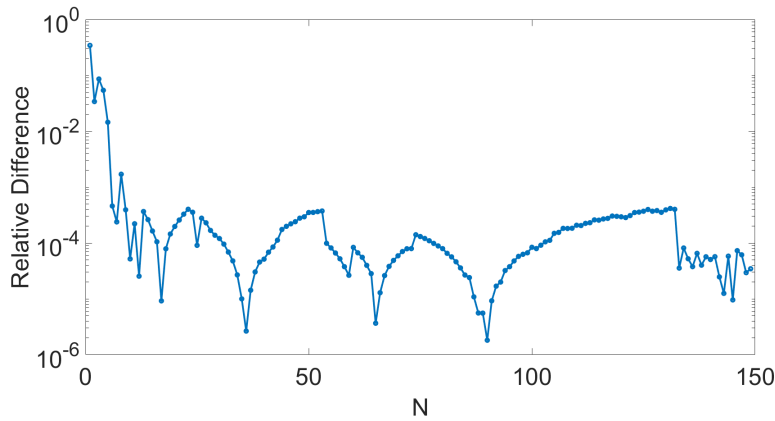


**Figure 51. Convergence of ODE15s Solving Equation 42**

The full solution using ODE45 takes less than 1 second, so in practice there is no restriction due to the stiffness. Furthermore, Figure 51 shows no gain to be had by using larger  $N$  with ODE15s since the ODE15s solution converges by  $N = 10$  to a relative difference on the order of  $10^{-4}$ . For this reason the solutions for the spectral methods are re-ran for the constant and variable parameter cases using ODE45 with  $N = 18$ .

The result of using a much courser mesh are quantitatively similar to the previous fine mesh results. For the constant parameters, solving with ODE45 produces relative differences of 7.8%, 6.4% and 5.0% to thermocouples 1, 2, and 3, respectively, for an average relative difference of 6.4%. Using variable parameters produces relative differences of 7.5%, 9.1% and 3.6% to thermocouples 1, 2 and 3, respectively, for an average relative

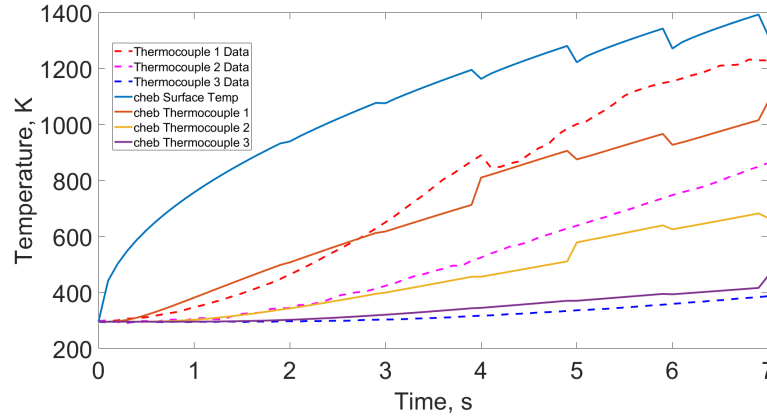


**Figure 52. ODE45 Solving Equation 42 with Constant Parameters and N=18**

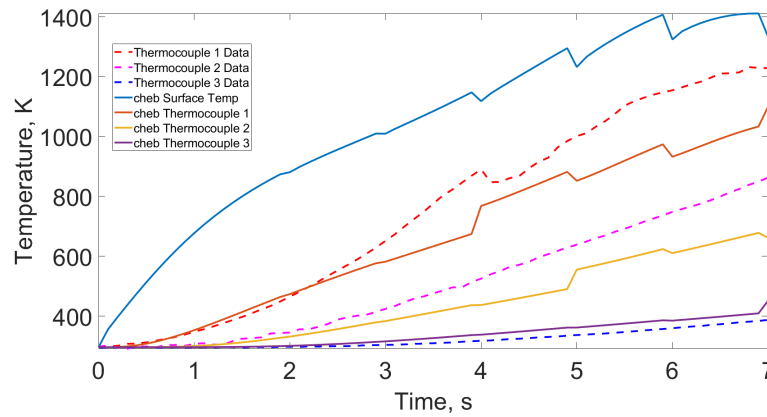


**Figure 53. ODE45 Solving Equation 42 with Variable Parameters and N=18**

difference of 6.7%. Qualitatively, Figures 52 and 53 appear very similar. The relative difference between the constant parameter and variable parameter models are 1.3%, 3.2% and 5.4% at thermocouples 1, 2, and 3, respectively, for an average relative difference of 3.3%.

Therefore, while Figures 52 and 53 show data which appears less smooth and therefore qualitatively less desirable, using ODE45 with N=18 provides data points which are very similar to using ODE15s with N=150. Using constant parameters, the relative difference between ODE45 with N=18 to ODE15s with N=18 is 2.1%, 1.6%, and 0.7% at thermocouples 1, 2, and 3, respectively, for an average relative difference of 1.5%. Using variable parameters, the relative difference between ODE45 with N=18 to ODE15s with N=18 is 2.1%, 1.6%, and 0.6% at thermocouples 1, 2, and 3, respectively, for an average relative

difference of 1.4%.

The time required to generate the full solution using ODE45 with N=18 and constant parameters was .07 s, and with variable parameters was .14 s. The time required to generate the full solution using ODE15s with N=18 and constant parameters was 0.3 s and with variable parameters was 0.6 s. So, that the explicit ODE45 solver using forward differences is faster prior to full convergence. Going back to Figure 50, note that ODE45 nearly reaches full convergence at  $10^{-8}$  by N=36. The relative difference between the solutions with constant parameters using ODE45 with N=36 and ODE15s with N=150 drops to 1.4%, .7% and .5%, at thermocouples 1, 2, and 3, respectively, for an average relative difference of 0.9%. Similarly, for variable parameters the relative difference between ODE45 with N=150 and ODE15s with N=36 drops to 1.4%, .7% and .5%.

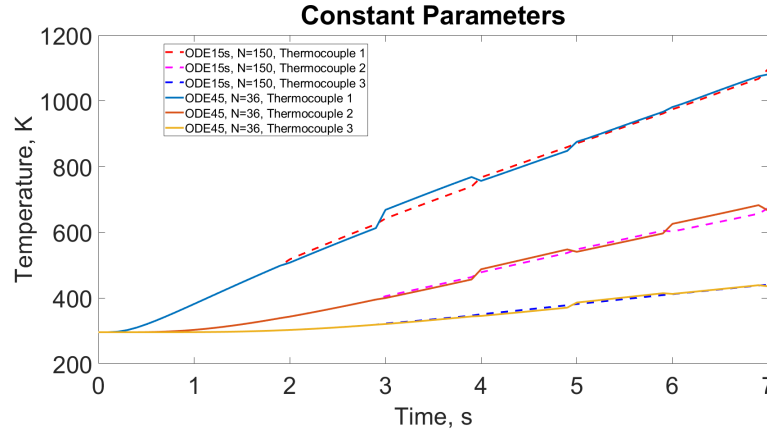


**Figure 54. ODE45 with N=36 vs ODE15s with N=150 with Constant Parameters**

Now that the relative differences in solutions has fallen below 1%, we say that the differences in solutions are negligible and are shown in Figures 54 and 55. So that, ODE45 with N=36 provides solutions which are about as good as can be expected using any method presented herein. The time to compute the full solution for constant parameters was 0.5 s and for variable parameters was 1.7 s. So that, the stiffness has made the ODE45 solver slower than the backward difference solver ODE15s. However, the time scale makes the added cost negligible in this case, adding only 1 second in computation time. So, while

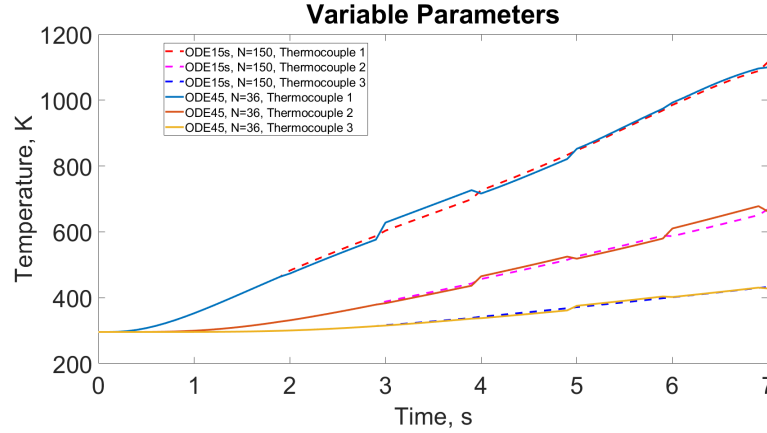


**Figure 55. ODE45 with N=36 vs ODE15s with N=150 with Variable Parameters**

Equation 42 is shown to be stiff, the spectral accuracy obtained by using the derivative matrix in Equation 40, allows the use of a coarse enough grid that the stiffness can be virtually avoided.

### 3.4 Future Explorations

The assumptions that were made to reduce the cylindrical pin to a two dimensional plane model were uniform heat generation across the face of the pin and uniform radial convection with the surrounding air. Extending the finite element model to three dimensions to explore if any aerodynamic effects of the spinning wheel significantly effect the model; and, if non-uniform frictional heat generation [13] plays a significant role.

The finding that reduced the two dimensional plane model to one dimension was that radial convection was negligible in the finite elements analysis. Radial basis functions may be used to generate a derivative matrix with high accuracy and can be easily extended to higher dimensions [25] so that Equation 27 can be solved in two or three dimensions using derivative matrices similar to Equation 42 in one dimension. Radial basis function- finite-difference (RBF-FD) methods can reduce the derivative matrix to a sparse matrix [25]. While it was shown that standard finite-difference methods proved to inaccurately represent the experimental data (Sections 3.2.3 and 3.2.4), RBF-FD is versatile since any number

of nearest neighbors can be used to compute the derivative at any location [25]. Since the size of the matrix will grow exponentially in higher dimensions, it may be of benefit to explore if the computational cost can be reduced by finding an appropriate number of nearest neighbors while adding negligible errors to the solution.

The RBF system is defined by equation 46. There are many examples of functions that the RBF,  $\phi$ , can take. What is important for now is that they measure the distance from a given point to a center-point, the derivatives of  $\phi$  can be easily computed [25], and that choosing an infinitely smooth RBF will avoid any singularity issues that may arise in this formulation [25]. Since  $f(x_i)$  is the value of the data at the given point, the system is completely defined and  $c_i$  can be solved by row reduction [25].

$$\Phi \vec{c} = \vec{f} \quad (45)$$

$$\begin{bmatrix} \phi(|x_1 - x_1|) & \phi(|x_1 - x_2|) & \dots & \phi(|x_1 - x_N|) \\ \phi(|x_2 - x_1|) & \phi(|x_2 - x_2|) & \dots & \phi(|x_2 - x_N|) \\ \vdots & \vdots & \ddots & \vdots \\ \phi(|x_N - x_1|) & \phi(|x_N - x_2|) & \dots & \phi(|x_N - x_N|) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{bmatrix} = \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_N) \end{bmatrix} \quad (46)$$

Once the  $c_j$  are known, one can easily find the derivative of  $\Phi \vec{c}$  so that an approximate derivative of  $f$  can be found. The two attributes of the RBF that will lend well to this problem are: 1) there is no need to redefine Equation 46 in higher dimensions as would be required in FDM and spectral methods since Equation 46 only depends on the distance between two points regardless of the dimension; and, 2) there is no dependency on uniform or pre-formulated grid-spacing.

## IV. Conclusion

Three separate numerical solutions were generated for to solve the non-linear heat equation with a moving boundary and empirical evidence from a high-speed pin-on-disc experiment was used to provide validation. Finite element analysis, finite-difference methods, and spectral methods each provided unique opportunities to inspect the progression of heat transfer through the pin. The frictional heat source was assumed to be generated uniformly across the pin face and the opposite end of the pin was assumed to maintain the temperature of the ambient air. Finally, assuming uniform convective heat loss off the sides of the pin, the model reduced to two dimensions.

Finite element analysis, in Abaqus [6], was used to first show that the steady state solution could not effectively represent the empirical evidence (Figure 22) so that a transient solution would be required. The transient finite element analysis model then provided insight that the pin may be radially convecting negligible heat loss with the surrounding air (Figure 28). Negligible convection effectively reduces the model to one dimension. The one dimensional finite element model qualitatively represents the thermocouple data well, as exhibited by the tracking of thermocouple 2 in Figure 29. However, the overall model over-represents the temperature profile along thermocouples 1 and 3. The finite element analysis was completed using only constant parameters, and analysis into the effect of variable parameters was desired. With the reduction to 1 dimension, it was believed that switching to a different solution method would provide an easier form to interrogate variable parameters for two reasons: Abaqus [6] requires piece-wise defined functions for several parameters, and the time required to run the model is around 2 minutes. The time required is not by any means overly restrictive, but can certainly be improved.

Finite difference methods, via pdepe in MATLAB, were employed to easily input functional forms of variable parameters. Solution time was reduced to about 3 seconds, but the temperature profiles were qualitatively poor (Figures 35 and 39). The temperature curves

at the thermocouples have a concave down profile. This concave down profile was also exhibited in previous work that used finite differences with removal based on displacement data (see Figure 14 of [14]). The introduction of variable parameters does not improve the model qualitatively or quantitatively. However, the model can be made to represent the empirical data well by optimizing the  $\kappa$  and  $\mu\beta$  terms. It is concluded that standard finite-difference models may not be accurate enough with the given grid spacing.

Spectral methods, formed via the differentiation matrix in Equation 40, were implemented to form a more accurate model by Equation 42. As an added benefit, solution times using spectral methods were further reduced to about 1 second. The solution profiles of the temperature at the thermocouples using spectral methods qualitatively represents the empirical data well, as exhibited by Figures 44 and 46. While thermocouple 2 was virtually traced by the finite elements methods, the the overall representation of the three thermocouples is done best by the spectral methods. The introduction of variable parameters does not improve the model qualitatively or quantitatively when using spectral methods either, so it is concluded that variable parameters are negligible in this pin-on-disc experiment. Optimizing the parameters provided a  $\kappa$  that is about half of the assumed constant and a  $\mu\beta$  that is about twice that of assumed value from Table 8.

Finite elements analysis and spectral methods both provided good qualitative temperature profiles for the thermocouple temperatures using one dimension and constant parameters. Spectral methods provided the best quantitative measure of fit by producing an average relative error of 6.6% vs 7.9% using finite elements. Implementing spectral methods for the space derivative also provided the ability to avoid the stiffness presented by the mathematical model; achieving spectral accuracy with a course mesh does not present an issue for the computational efficiency or accuracy of explicit methods for the time derivative.

## Appendix A. Abaqus Model Tutorial

### 1.1 Creating a New File and Directory

Launch Abaqus and click "With Standard/Explicit Model". Or, if Abaqus is already open, click File → New Model Database → With Standard/Explicit Model.

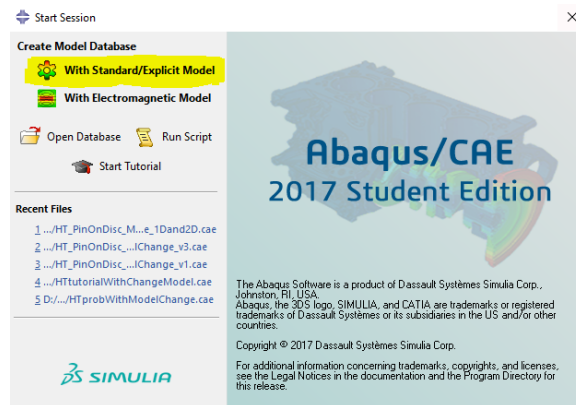


Figure 56. Start Session

Click File → Save As...

Navigate to where you would like to save your model and click the "Create new Directory" icon. Enter a name and click "OK". Double click the directory to open it. Enter a File Name and click OK.

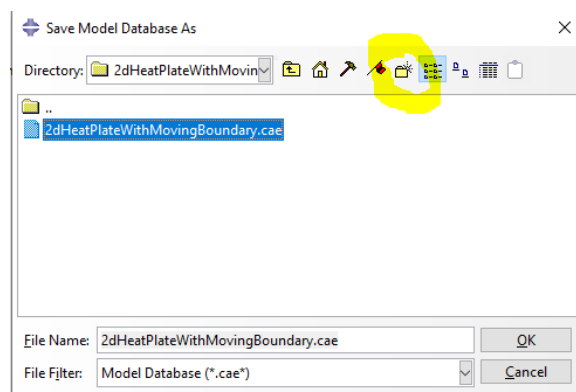


Figure 57. Save As

Click File → Set Work Directory. Click the "Select" icon to open the "Select a Work

Directory” window. Navigate to and select the Directory that was just created. Click OK in the ”Select a Work Directory” window. Click ”OK” in the ”Set Work Directory” window.

Setting a new work directory for every project keeps the output files separated and easier to manage.

## 1.2 The Model in Review

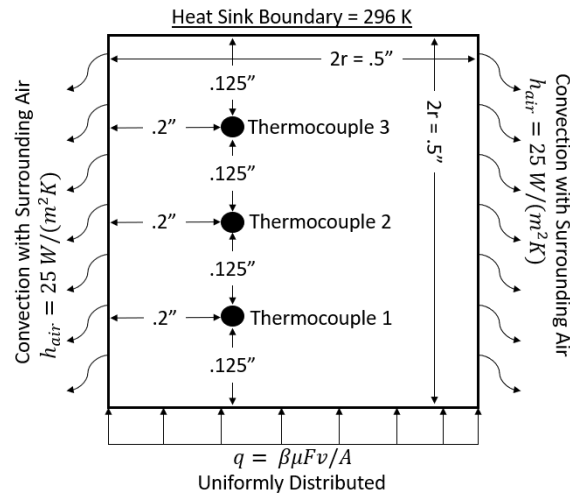


**Figure 58. Model**

Note that  $.5'' = .0127m$ ,  $.125'' = .003175m$ ,  $.2'' = .00508m$ . Abaqus does not have a place to specify units. One must ensure all values entered are in corresponding units throughout a model.

$$q(t) = \frac{\beta(t)\mu(t)F(t)v(t)}{A(t)} = 5.95 \times 10^6$$

| Step    | Total Displacement at step $i$ , $d_i$ | Total Time |
|---------|----------------------------------------|------------|
| Initial | .00 mm                                 | 0          |
| 1       | .0000 m                                | 0-1 sec    |
| 2       | .00005 m                               | 1-2 sec    |
| 3       | .000125 m                              | 2-3 sec    |
| 4       | .00035 m                               | 3-4 sec    |
| 5       | .0007 m                                | 4-5 sec    |
| 6       | .0011 m                                | 5-6 sec    |
| 7       | .0015 m                                | 6-7 sec    |
| 8       | .0020 m                                | 7-8 sec    |
| 9       | .0022 m                                | 8-9 sec    |

**Table 17. FEA Material Removal**

| Property      | Value      | Temperature |
|---------------|------------|-------------|
| Conductivity  | 30.807 [1] | all         |
| Density       | 8000       | all         |
| Specific Heat | 857.98     | all         |

**Table 18. FEA Material Properties**

|            |                |
|------------|----------------|
| $\beta(t)$ | 0.12           |
| $\mu(t)$   | 0.20           |
| $F(t)$     | 656.76 $N$     |
| $v(t)$     | 47.83 $m/s$    |
| $A(t)$     | .0001267 $m^2$ |

**Table 19. FEA Flux Parameters**

### 1.3 Build Part

Click Part → Create...

Enter "Plate" in the "Name" field. Choose 2D Planar, Deformable, Shell. Enter .1 is the "Approximate size" field. Click "Continue...".

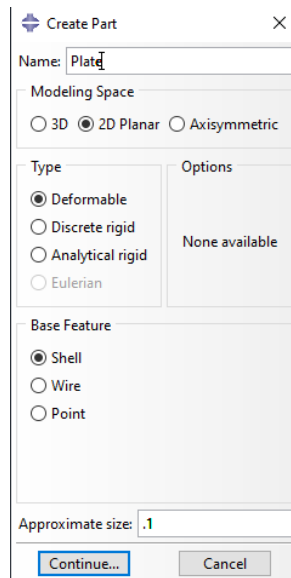


**Figure 59. Create Part**

Click Add → Line → Rectangle. In the field enter "0,0" for the starting corner. Enter ".0127,.0127" for the opposite corner. Click "Done".

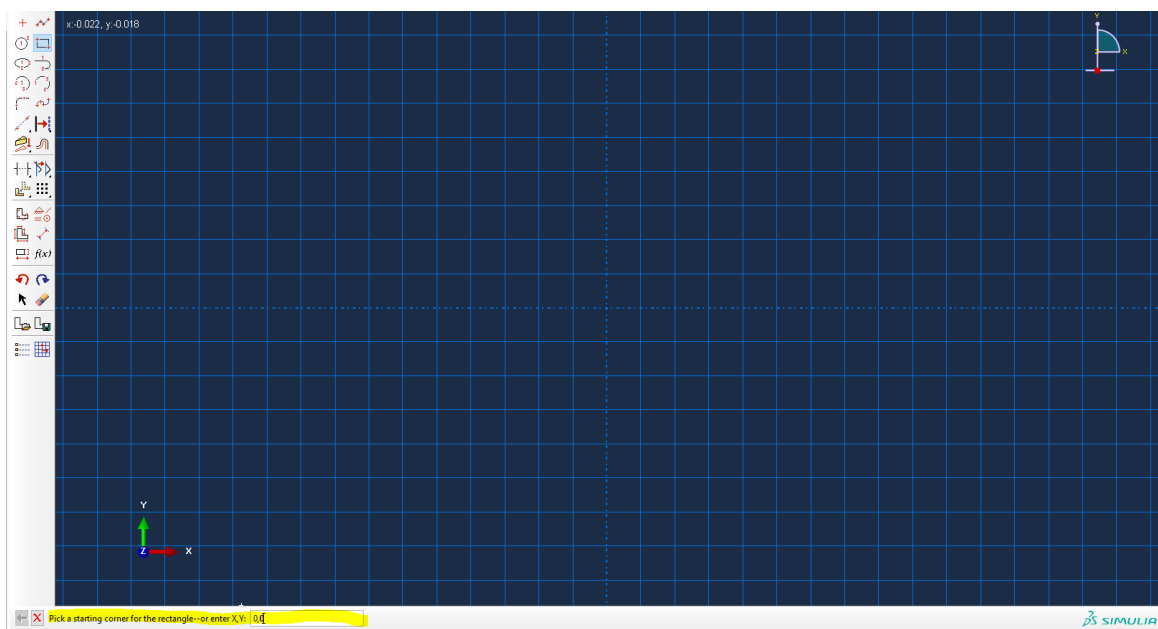


**Figure 60. Add Rectangle**

Click "Create Datum Plane: Offset From Principal Plane". Click "XZ Plane"; enter .00005 in the field. Repeat to enter all the non-zero values in the "FEA Material Removal" table (.000125, .00035, .0007, .0011, .0015, .0020, .0022).

Repeat 3 more times to create planes through the thermocouple positions (.003175, .00635, .009525).

Click "YZ Plane"; enter .00508 in the field to create a plane through the thermocouple positions.

Click the "Cancel Procedure" icon.

Turning the perspective view off may help visualizing the planes.

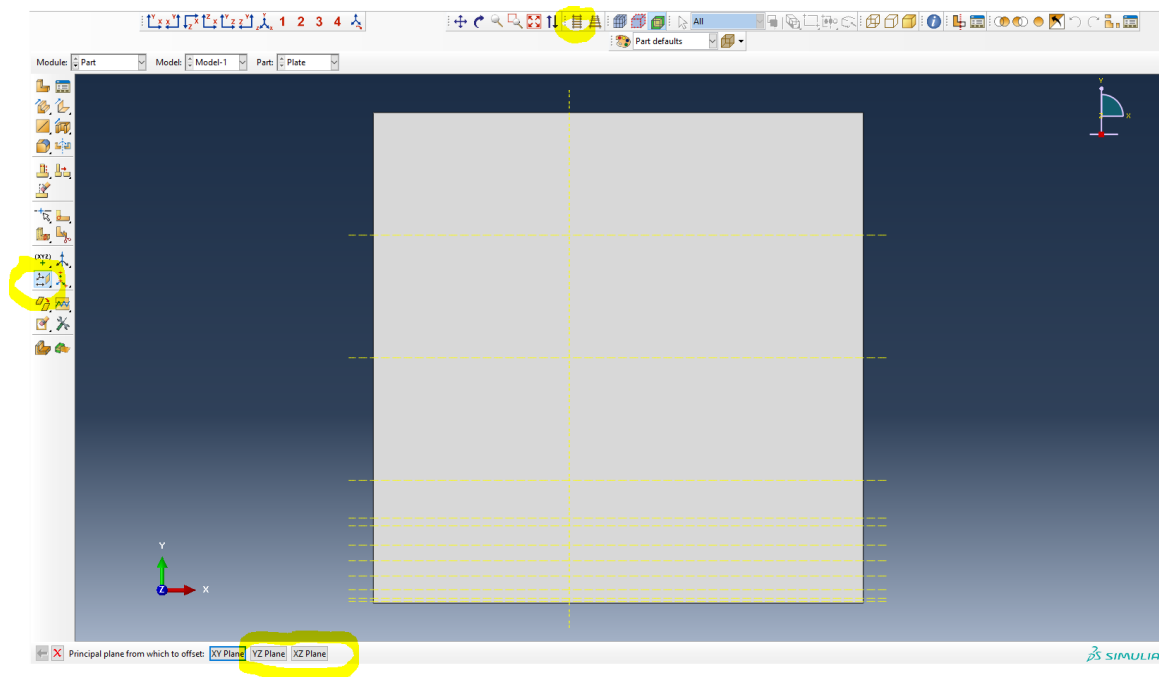


**Figure 61. Create Datum Planes**

Click and hold the "Partition Face" icon to expand it. Select the "Partition Face: Use Datum Plane" icon. Select the plate; click "Done". Select a datum plane; click "Create Partition". Repeat until the plate is partitioned by all datum planes. Note that more than one face can be partitioned at a time by holding shift when selecting faces to partition.

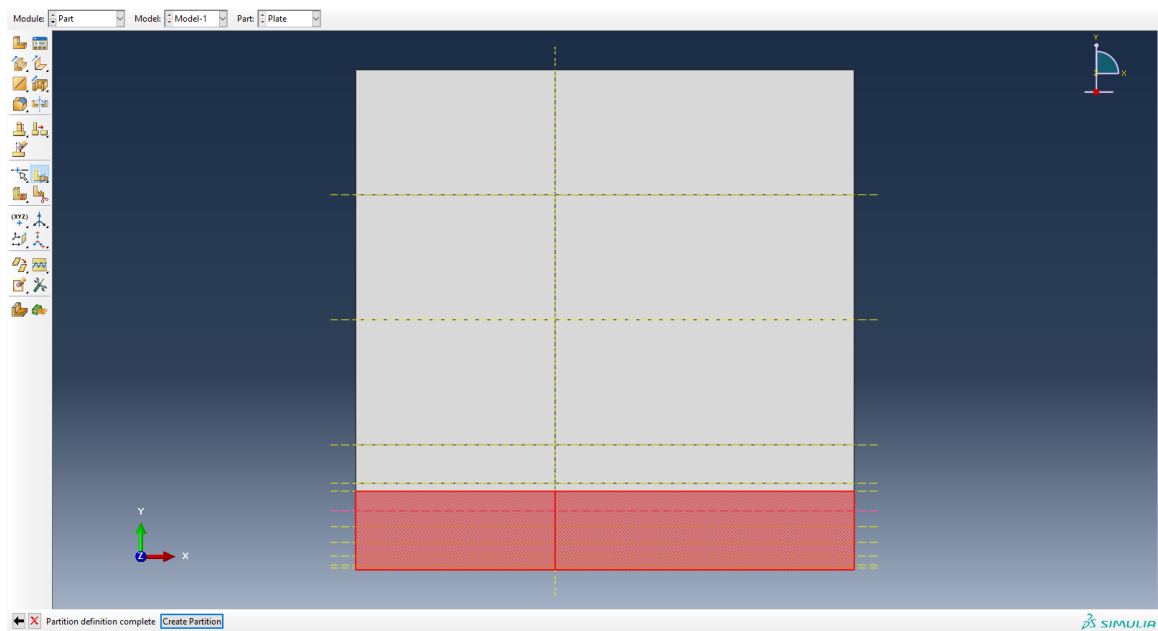


**Figure 62. Create Partitions**

## 1.4 Define Material Properties

Switch to the "Property" module by clicking on, and selecting from, the "Module" drop down list.



**Figure 63. Switch Module**

Click Material → Create...

Enter "Vascomax300-Constant" in the "Name" field.

Click General → Density. Enter 8000 in the "Mass Density" field.

Click Thermal → Conductivity. Enter 30.807 in the "Conductivity" field.

Click Thermal → Specific Heat. Enter 857.98 in the "Specific Heat" field.

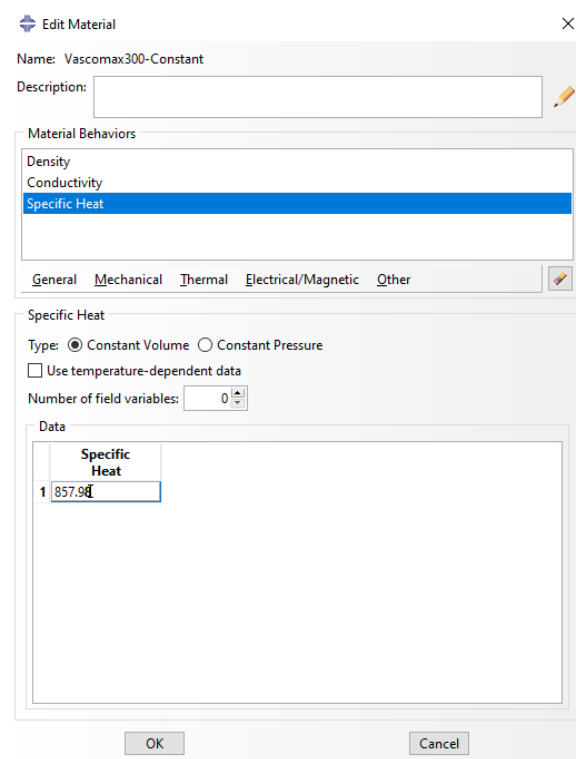


**Figure 64. Create Material**

Click OK to exit the "Edit Material" window.

Click Section → Create...

Enter "Plate" in the "Name" field. Select "Solid" and "Homogeneous".

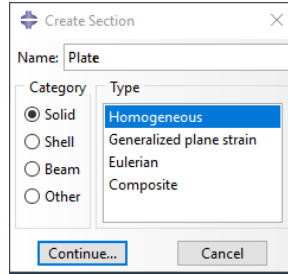


**Figure 65. Create Section**

Click "Continue..."

Ensure that the "Material" field has "Vascomax300-Constant" in it. Click the box next to "Plane stress/strain thickness:"; enter 0.00997455 in the field. Click "OK".

Note that the area of the face of the pin is  $\pi r^2 \approx .00012668m$ , and that  $2r = .0127m$ . Defining the thickness as .00997455 makes the area of the bottom of the plate approximately the same as the area of the face of the pin. This ensures that the heat rate (flux times area)[12] will be equivalent.

Click Assign → Section

Click and drag to select the entire plate. Click "Done".

Accept the defaults in the "Edit Section Assignment" window. Click "OK".

## 1.5 Model Assembly

Switch to the "Assembly" module.

Click Instance → Create

Select "Independent" for the "Instance Type"; accept the other defaults and click "OK".

## 1.6 Create Steps

Switch to the "Step" module.

Click Step → Create...

Select "Heat Transfer" for the "General" procedure type. Click "Continue..."

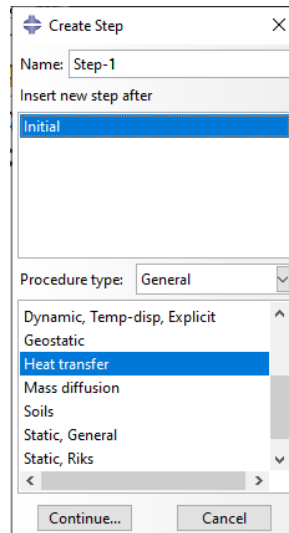


**Figure 66. Create Step**

For steady-state, select "Steady State"; click "OK". Repeat 9 times to create 9 steps after initial.

For transient, select the "Incrementation" tab. Select "Type: Fixed". Change the "Increment size:" field to .1; click "OK". Repeat 9 times to create 9 steps after initial.

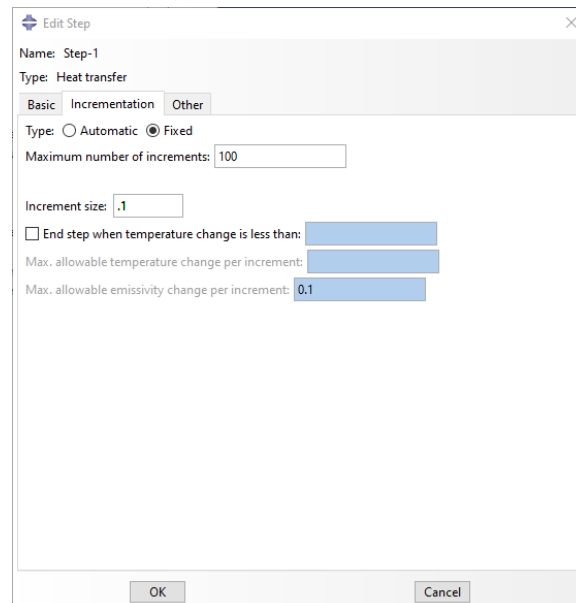


**Figure 67. Create Increment**

## 1.7 Define Interactions

Switch to the "Interaction" module.

### Define Convection with Surrounding Air:

**Note:** If building for 1d there will be no convection with surrounding air. Skip ahead to **Define Model Changes (Removal):**.

Click Interaction → Create...

Enter "Convection-1" in the "Name" field; use "Step" drop-down to select "Step-1"; select "Surface film condition"; click "Continue..."

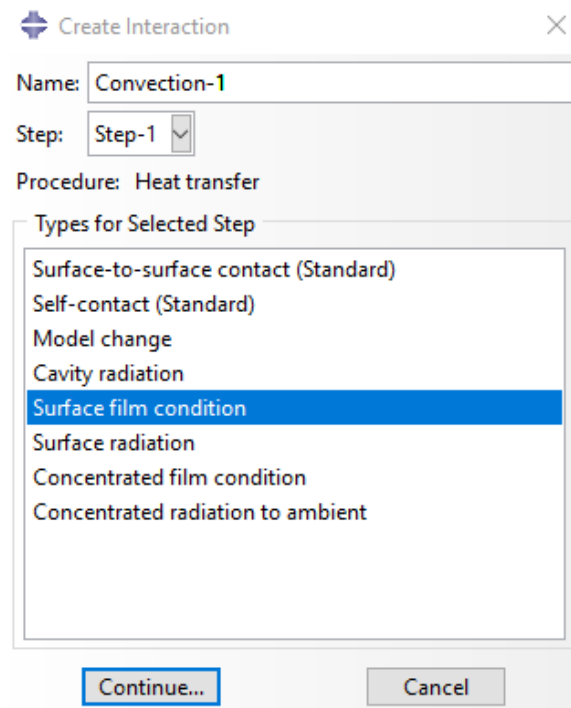


Figure 68. Create Convection Interaction with Surrounding Air

Select all 24 of the vertical edges. Note, clicking and dragging to select all 28 edges, then holding control to deselect the 4 horizontal edges may be faster.

Enter "Conv-1" in the "Create surface" field; click "Done".

Enter 50 in the "Film coefficient" field and 296 in the "Sink temperature" field; click "OK".

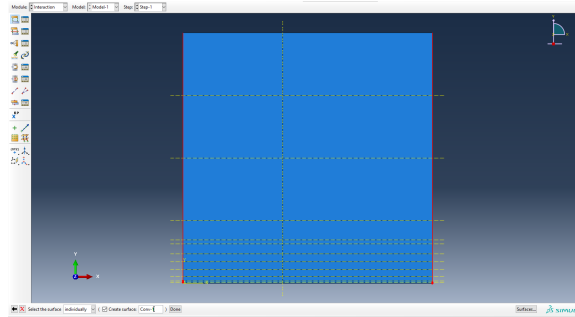


**Figure 69. Select Conv-1 Surfaces**

Note on why 50 instead of 25 for the "Film coefficient" as shown in the model diagram:  
 Since the thickness of the plate was defined as 0.00997455, the combined area of the sides of the plate is  $2(2r \times 0.00997455) \approx .00025336m = A_p$ , but the area of the side of the cylinder is  $2\pi rL \approx .00050671 = A_c$ . Since the flux is acting on an area that is a factor of 2 smaller than the pin, multiplying the flux by 2 will make the heat rate (flux times area [12]) equivalent.

$$q_p = hA_p(T - T_0); \text{ multiply by 2: } 2q_p = 2hA_p(T - T_0) = hA_c(T - T_0) = q_c$$

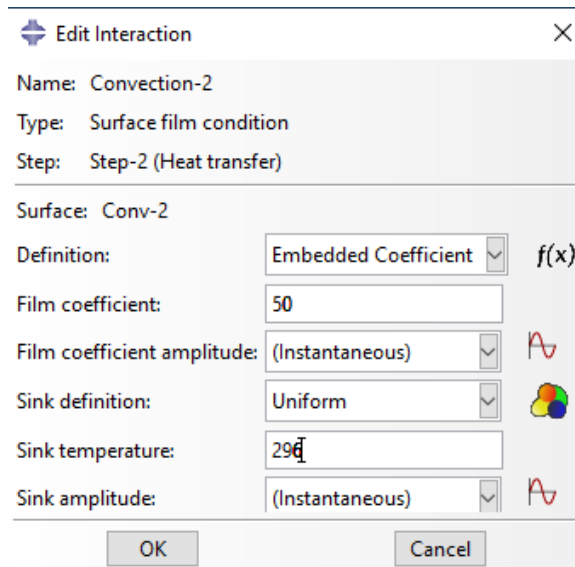


**Figure 70. Define Convection-2**

Repeat creating surface film conditions for Steps 2 through 9 named Convection-2 through Convection-9. When selecting the vertical faces for Step 2, do not select the bottom most vertical faces on either side as this section will be removed in this step. When selecting the vertical faces for Step 3, do not select the bottom 2 vertical faces as the bottom 2 section will be removed during this step.

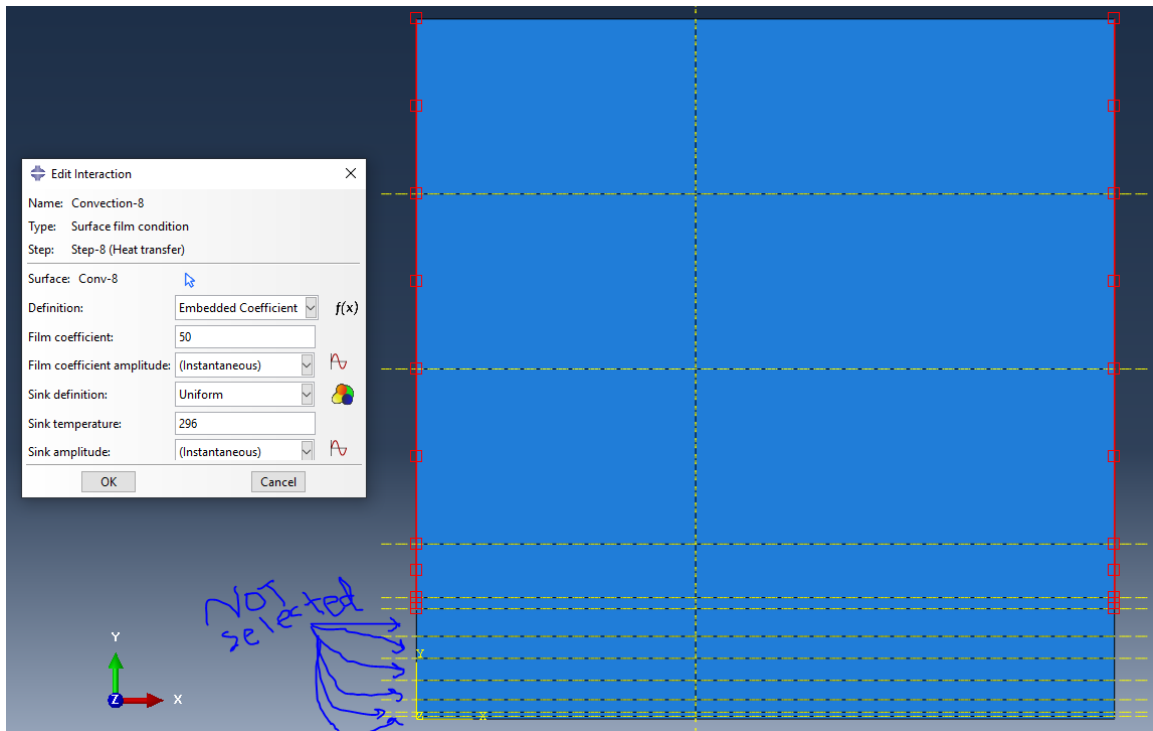


**Figure 71. Select and Define Convection-8**

Deactivate the interactions for future steps. By default, interactions created will propagate forward to the next steps. Since areas where the interactions were defined will be removed in future steps this will create an error if they are not deactivated.

In the model tree expand your model (named "Model-1" unless you changed it) by clicking the "plus" icon next to the name; expand "Interactions (9)"; expand "Convection-1"; expand "States". Right click on "Step-2"; select "Deactivate".

Repeat for "Convection-2" through "Convection-8" (Convection-9 only has the created step, so nothing needs to be deactivated) by right clicking the first step after "Created".

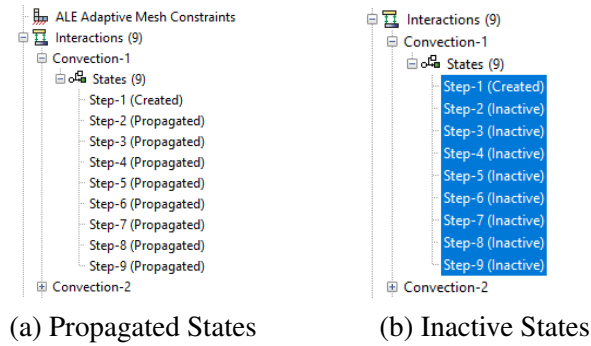


**Figure 72. Deactivating Propagated Interactions**

### Define Model Changes (Removal):

Click Interaction → Create...

Enter Removal-2 in the "Name" field. The number representing the step in which the interaction occurs. Use the "Step" drop-down to select "Step-2"; select the "Model change" type; click "Continue..."

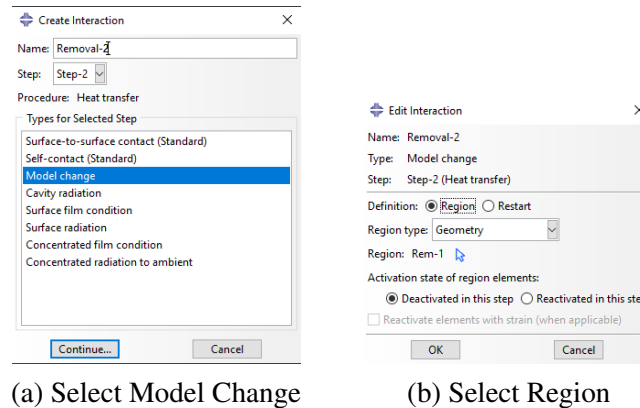


**Figure 73. Create Removal Interaction**

Ensure "Definition: Region" and "Region type: Geometry", and "Deactivated in this step" are defaulted. Click the blue cursor icon to select the region.

Hold shift and select the two bottom most partitions for removal; enter "Rem-2" in the "Create set" field; click "Done". Click "OK".

Repeat to create model change interactions named "Removal-3" through "Removal-9" for "Step-3" through "Step-9".

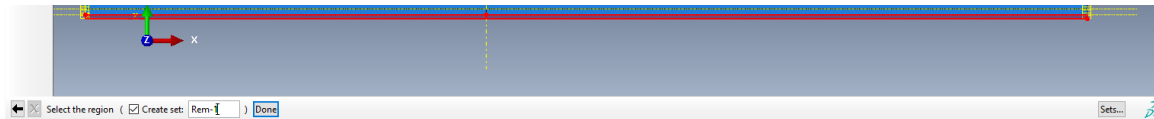


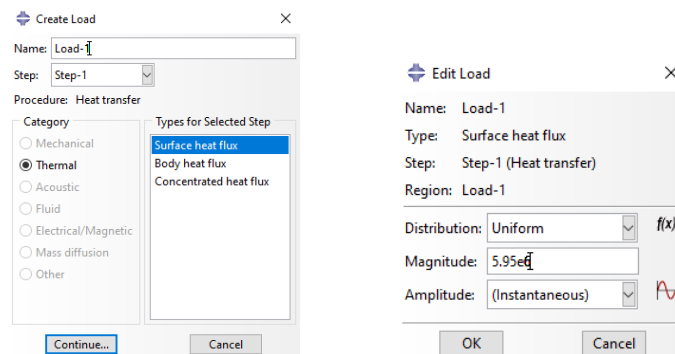
Figure 74. Select Region for Removal

## 1.8 Applying Load

Switch to the "Load" module.

Click Load → Create...

Ensure the defaults are "Name: Load-1", "Step: Step-1", "Category: Thermal", "Type: Surface heat flux". Click "Continue..."



(a) Name Load and Select Step

(b) Edit Load

Figure 75. Create Load

Hold Shift and select the two bottom edges; enter "Load-1" in the "Create surface field"; click "Done".

Enter 5.96e6 in the "Magnitude" field; click "OK".

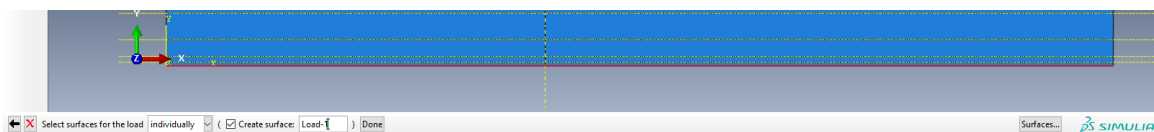


Figure 76. Select Load-1 Surface

The remaining surfaces may not be available to be selected in this fashion since they are inside the body of the plate. To manually create the surfaces, hold shift and select the two lines that represent the displacement at step 2.

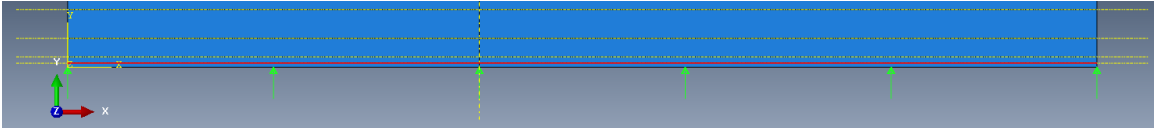


**Figure 77. Select Load-2 Surface**

In the model tree expand "Model-1", expand "Assembly", expand "Surfaces"

With the surface still selected, right click Surfaces → Create...

Enter "Load-2" in the "Name" field; click "Continue..."; click "Done".

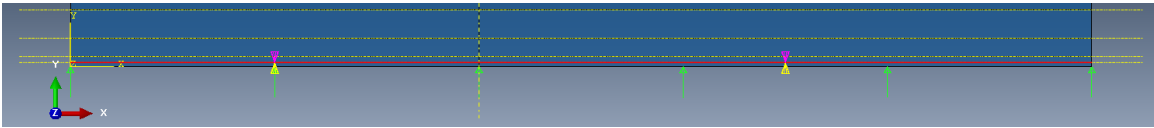


**Figure 78. Select Load-2 Surface Side**

Select whichever color is on top. In this picture, "Magenta".

Repeat to create surfaces named "Load-3" through "Load-9".

With the surfaces created, repeat creating loads "Load-3" through "Load-9" assigning them to their corresponding surfaces and steps. All load arrows should be pointing the same direction: "into" the plate.

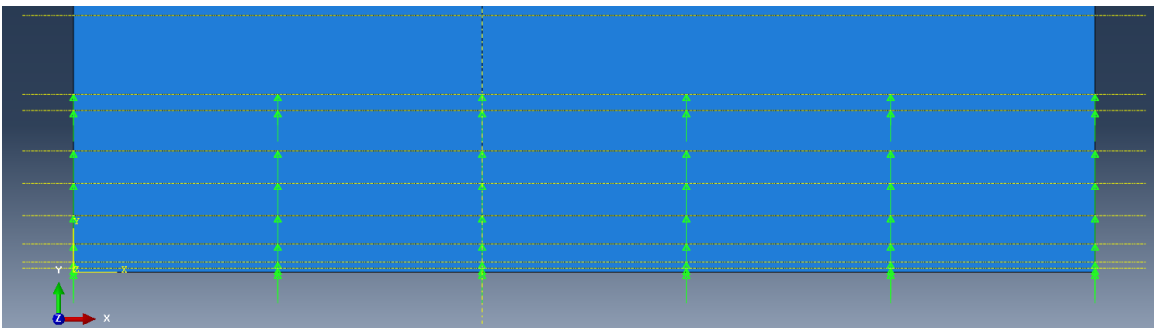


**Figure 79. All 9 Loads Pointing "into" the Plate**

## 1.9 Creating the Mesh

Switch to the "Mesh" module.

Click Mesh → Controls...

Click and drag to select the entire plate; click "Done".

Select "Element shape: Quad", "Technique: Free", "Algorithm: Medial axis"; click "OK".

Click Mesh → Element Type...

Select the entire plate.

Choose "Element: Standard", "Geometric Order: Linear", "Family: Heat Transfer", "Convection/Diffusion" to select DCC2D4, a 4-node convection/diffusion quadrilateral, element. Click "OK".

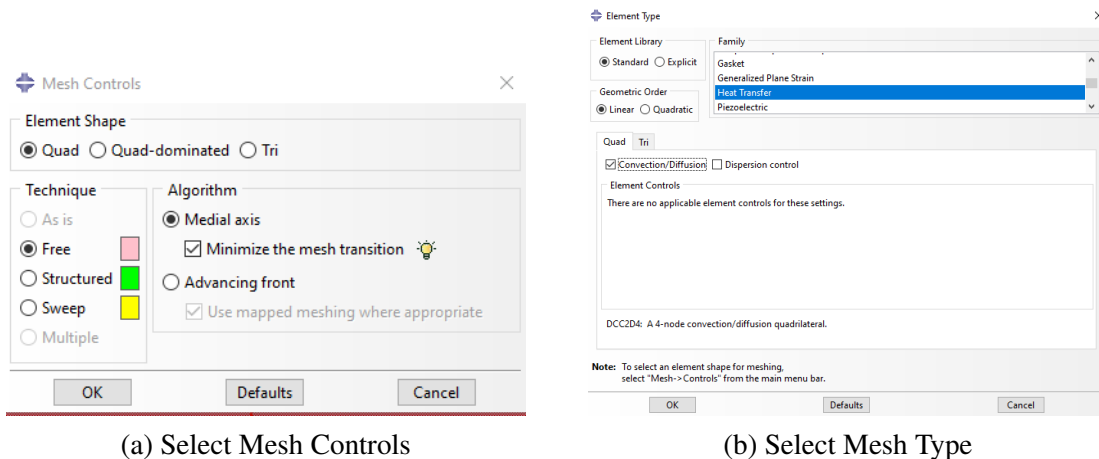


**Figure 80. Create Mesh**

Click Seed → Instance

Enter any number (try 0.0005) in the "Approximate Global Size" field; click "OK".

Note that the Student version of Abaqus only permits 1000 nodes to be executed.

Click Mesh → Instance...

Click "Yes"

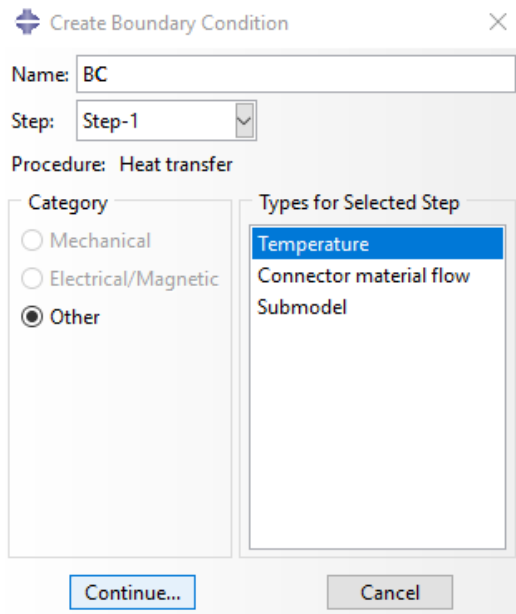
## 1.10 Define Boundary Condition

In the model tree right click BCs → Create...

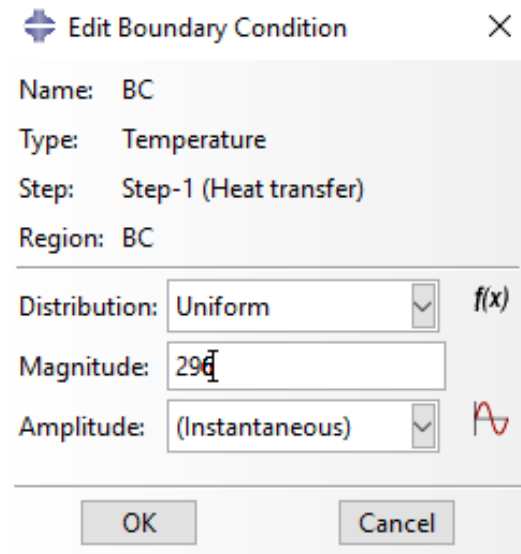
Enter "BC" in the name field. Select "Step: Step-1", "Category: Other", "Type: Temperature"; click "Continue..."

Hold shift and select the top two surface sections. Enter "BC" in the "Create set" field; click "Done".

Enter "296" in the "Magnitude" field; click "OK".



(a) Select Boundary Condition



(b) Define Boundary Condition

**Figure 81. Create Boundary Condition**



**Figure 82. Boundary Condition Complete**

## 1.11 Create Output Sets

Return to the Mesh Module

In the model tree expand model, "Assembly", "Sets".

Right click Sets → Create...

Enter "Thermo-Nodes" in the "Name" field; Select "Node"; click "Continue..."

Hold shift and select the nodes at the datum plane intersections where the thermocouples are located. Click "Done".

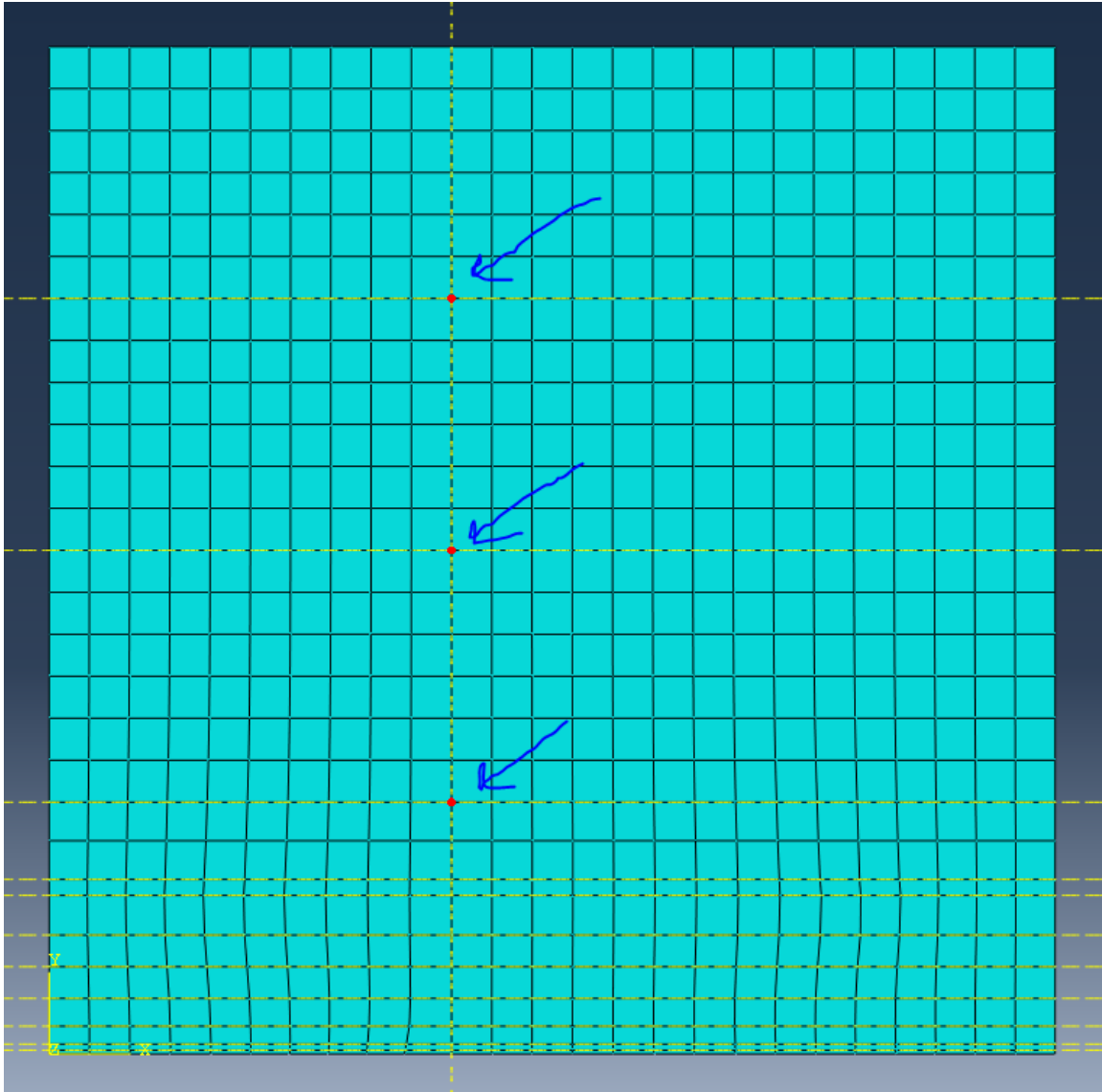


**Figure 83. Select Thermocouple Nodes**

Right click Sets → Create...

Enter "Surf1-Nodes" in the "Name" field; Select "Node"; click "Continue..."

Hold shift or click and drag to select all the nodes along the bottom edge.

Repeat for sets named "Surf2-Nodes" through "Surf9-Nodes". Using the datum planes

as a reference may help choosing the nodes corresponding to displacement.

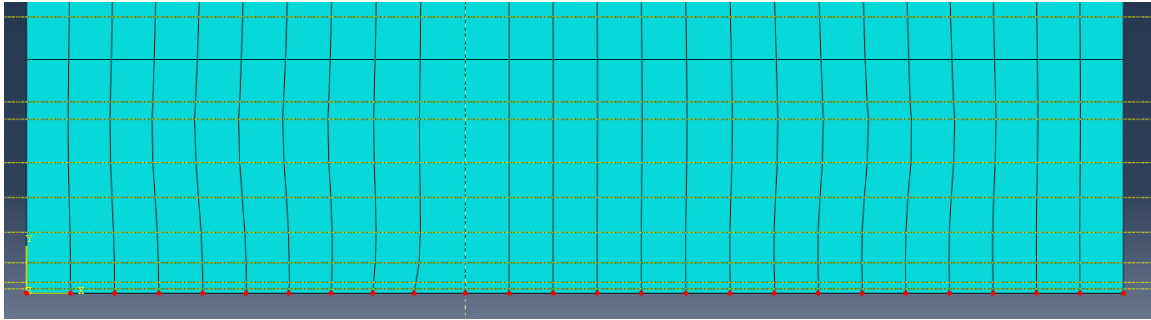


Figure 84. Select Surface Nodes

## 1.12 Create Initial Condition

Create one more set called "AllNodes" by selecting all the nodes in the mesh.

In the model tree right click Predefined Fields → Create...

Enter "InitialCondition" in the "Name" field; use the step drop down to select "Initial";  
select "Category: Other"; select "Type: Temperature"; click "Continue..."

Select by set "AllNodes"; click "Continue..."

Enter 296 in the "Magnitude" field; click "OK"

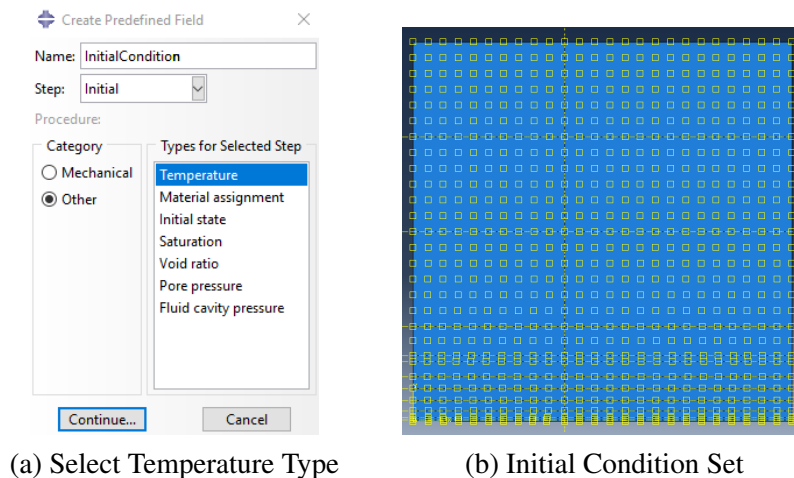


Figure 85. Create Initial Condition

### 1.13 Create and Run Job

Switch to the Job Module

Click Job → Create...

Enter "Model-1\_Job"; click "Continue..."; click "OK"

in the model tree expand "Analysis", expand "Jobs".

Right click Model-1\_Job → Submit. While it is running it is sometimes useful to watch for errors and progressions: Right click Model-1\_Job → Monitor.

### 1.14 Viewing Results

When the job is done running (Completed), right click Model-1\_Job → Results.

A quick way to see if the model was built correctly is to watch it run as a movie.

Use the drop down at the top to change from "HFL" to "NT11" (Nodal Temperature).

In visualization tools, click "Plot Contours on Deformed Shapes" then "Animate: Time History".

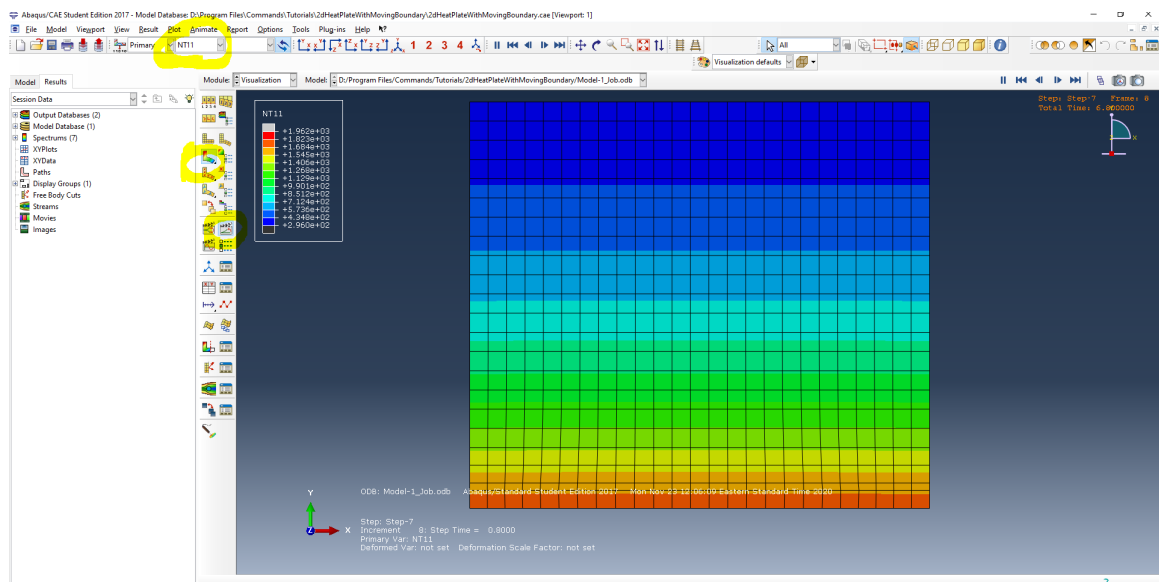


Figure 86. Watch Movie

The animation should show a gradient "flowing" upward as the heat comes in from the

bottom edge. The bottom should creep upward as the sections are removed after each step. The entire plate should start at 296 and top edge should maintain the sink temperature of 296.

To generate a chart to visualize the temperature of the nodes with respect to time, click the "XY" visualization tool icon. Select "ODB field output"; click "Continue..."

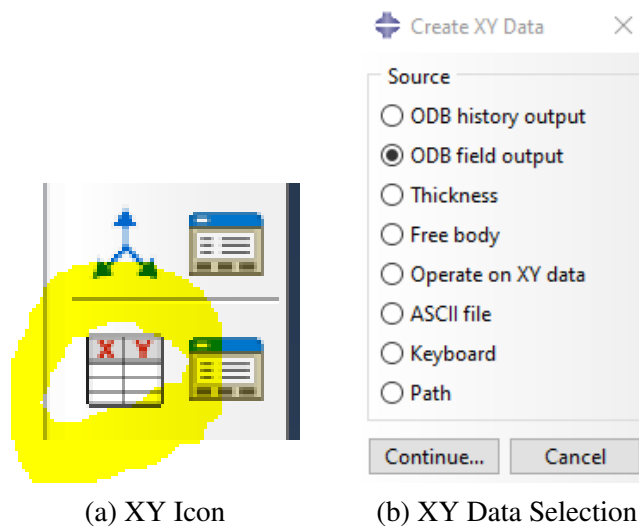
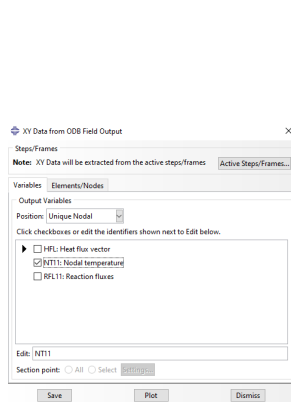


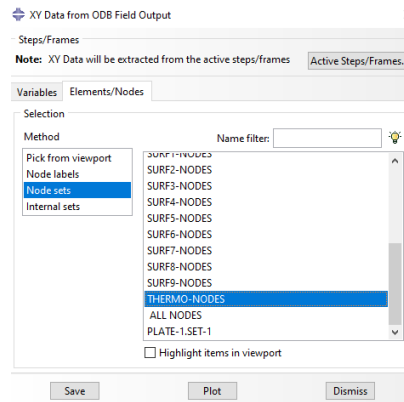
**Figure 87. Create XY Plot of Thermocouple Temperatures with Respect to Time 1**

Use the "Position" drop down to select "Unique Nodal"; click on the box to select "NT11: Nodal temperature".

Select the "Elements/Nodes" tab. Click "Method: Node sets"; select "THERMO-NODES"; click "Plot".



(a) XY Output



(b) XY Nodes

**Figure 88. Create XY Plot of Thermocouple Temperatures with Respect to Time 2**

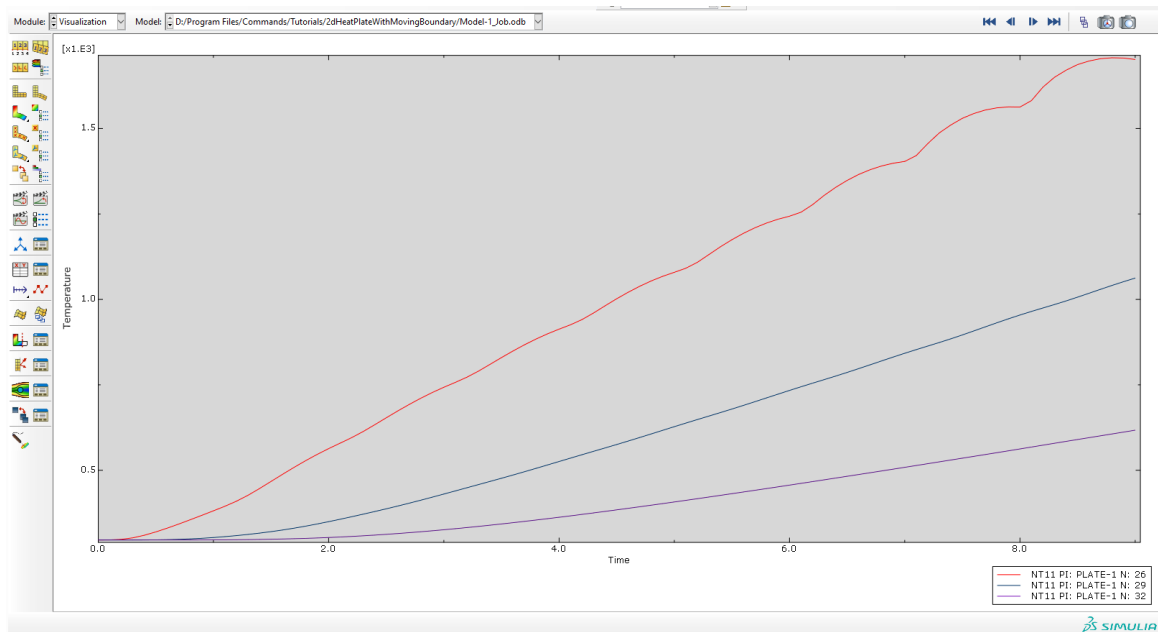


**Figure 89. XY Plot of Thermocouple Temperatures vs Time**

## 1.15 Create Outputs

First, we will create display groups. Activating the different display groups will generate outputs only for the items in the group.

In the Results tree right click Display Groups → Create...

Reset the display group to nothing by selecting "Item: All" then clicking "Remove".

Select "Item: Nodes"; select "THERMO-NODES"; click "Add"; click "Save As..."

Enter "Thermo-Display" in the "Name" field; click "OK".

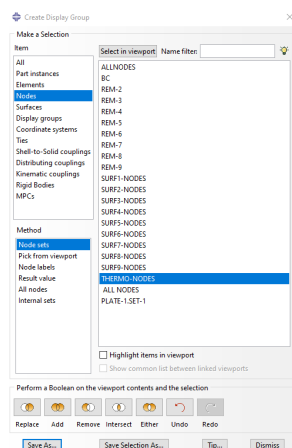
Reset the display group to nothing by selecting "Item: All" then clicking "Remove".

Select "Item: Nodes"; select "SURF1-NODES"; click "Add"; click "Save As..."

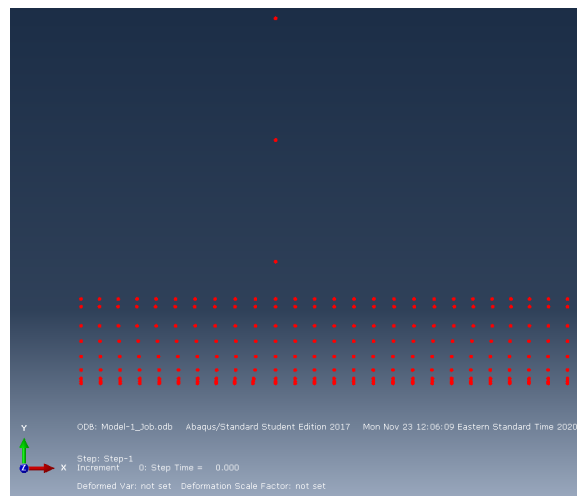
Enter "Surf1-Display" in the "Name" field; click "OK".

Repeat creating display groups "Surf2-Display" through "Surf9-Display"

Note that nodes alone cannot be displayed on the deformed shape. To visualize these display groups click on the "Plot of Undeformed Shape" tool icon.



(a) Selecting Display Groups



(b) Visual of Created Display Groups

**Figure 90. Create Display Groups**

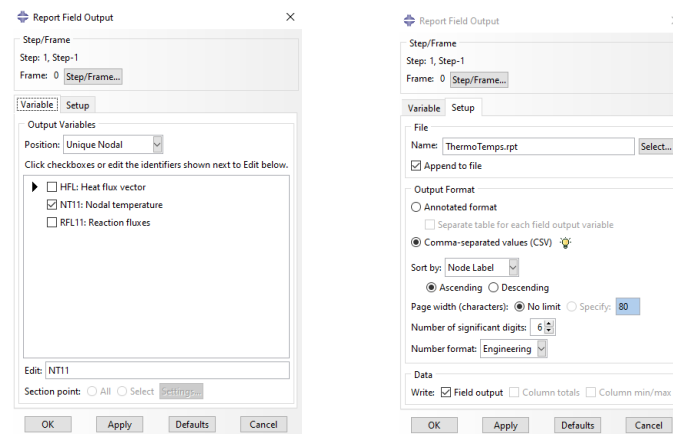
With the display groups created a report can be generated that only contains data from that group. There is a probably a better way to do this, but this is the only way I know.

In the Results tree "expand Display groups"; click on Thermo-Display. Only the three nodes representing the thermocouple locations should be visible.

Click Report → Field Output

Use the "Position" drop-down to select "Unique Nodal"; check the box for "NT11".

Go to the "Setup" tab. Enter "ThermoTemps.rpt" in the "Name" field; Select "Comma-separated values (CSV)"; uncheck "Column Totals" and "Column min/max".



(a) Selecting Temperature

(b) Setting up the Report File

**Figure 91. Create Thermocouple Temperature Report File**

Click "Step/Frame..."

Ensure "Step-1" and "Index: 0" are selected. If not select it and click "Apply" in the "Step/Frame" window". Index and Frame are synonymous.

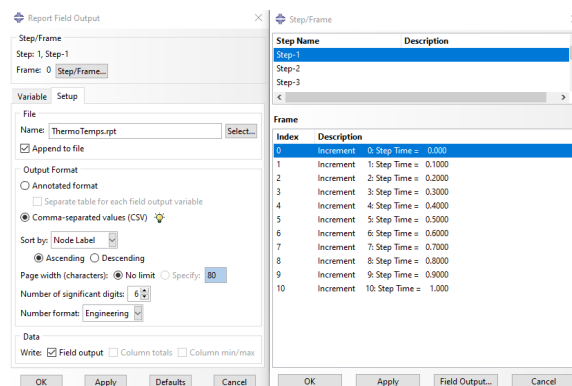


**Figure 92. Selecting Step and Frame**

Ensuring that Step-1, Frame: 0 is selected, click "Apply in the "Report Field Output" window. This will generate an excel file in your working directory called "ThermoTemps". Ensure that the Report is set up correctly by opening this excel file now.

|   | A                 | B         | C       | D             | E           | F        | G        | H | I            | J             | K                | L    | M | N | O | P |
|---|-------------------|-----------|---------|---------------|-------------|----------|----------|---|--------------|---------------|------------------|------|---|---|---|---|
| 1 | ODB Name          | Step      | Frame   | Part Instance | Node Labels | X        | Y        | Z | Section Name | Material Name | Section Property | NT11 |   |   |   |   |
| 2 | D:/Program/Step-1 | Increment | PLATE-1 |               | 26          | 5.08E-03 | 3.18E-03 | 0 | NA           | NA            |                  | 296  |   |   |   |   |
| 3 | D:/Program/Step-1 | Increment | PLATE-1 |               | 29          | 5.08E-03 | 6.35E-03 | 0 | NA           | NA            |                  | 296  |   |   |   |   |
| 4 | D:/Program/Step-1 | Increment | PLATE-1 |               | 32          | 5.08E-03 | 9.53E-03 | 0 | NA           | NA            |                  | 296  |   |   |   |   |

Figure 93. Excel Report File 1

Close the excel file and continue to generate the report by double clicking the next frame in the "Step/Frame" window to select it, then clicking "Apply" in the "Report Field Output" window to write to the report.

Write to the report Frame/Index 0 through 9. **DO NOT** write to the report Frame/Index 10. Note that Step-1 index 10 is the same as Step-2 index 1. So, writing to the report Index 10 will duplicate the data. For Step-9, Index 10 can be included in the report since this is the last Frame.

Be sure to go in order and not to skip. If a mistake is made the report file can be manually opened, edited and closed. Then start again writing to report from where appropriate. The Report Field Output will simply append to the bottom of whatever is already in the file.

Once complete open the excel file. In row 1, columns M, N and O enter Thermo1, Thermo2 and Thermo3, respectively. Abaqus may have generated your nodes differently. Look at column Y; Thermocouple 1 is located closest to the bottom so should be the smallest number. Thermocouple 2 is in the middle and Thermocouple 3 should be largest since

it is closest to the top. Adjust below as appropriate, if needed.

In row 2, column M, enter " =IF(E2=26,L2,") ". In row 2, column N, enter " =IF(E3=29,L3,") ". In row 2, column O, enter " =IF(E4=32,L4,") ".

click and drag to select row 2 columns M through O. Double click the little box at the bottom right of the highlighted region.

|   | A          | B      | C         | D             | E          | F        | G        | H | I            | J             | K                | L    | M       | N       | O       |
|---|------------|--------|-----------|---------------|------------|----------|----------|---|--------------|---------------|------------------|------|---------|---------|---------|
| 1 | ODB Name   | Step   | Frame     | Part Instance | Node Label | X        | Y        | Z | Section Name | Material Name | Section Property | NT11 | Thermo1 | Thermo2 | Thermo3 |
| 2 | D:/Program | Step-1 | Increment | PLATE-1       | 26         | 5.08E-03 | 3.18E-03 | 0 | NA           | NA            |                  | 296  | 296     | 296     | 296     |
| 3 | D:/Program | Step-1 | Increment | PLATE-1       | 29         | 5.08E-03 | 6.35E-03 | 0 | NA           | NA            |                  | 296  |         |         |         |
| 4 | D:/Program | Step-1 | Increment | PLATE-1       | 32         | 5.08E-03 | 9.53E-03 | 0 | NA           | NA            |                  | 296  |         |         |         |
| 5 | ODB Name   | Step   | Frame     | Part Instance | Node Label | X        | Y        | Z | Section Name | Material Name | Section Property | NT11 |         |         |         |

**Figure 94. Excel Report File 2**

Select all of the thermocouple temperatures then copy and paste them into MATLAB. Adding brackets around the copy and pasted data will generate a matrix with the columns 1, 2, and 3 representing the temperatures at thermocouples 1, 2 and 3, respectively.

generating surface temperature report data can be done by selecting "Surf1-Display" and repeating above to write Frames 0-9 to a new file named "SurfaceTemps.rpt". After writing Index 9 to the report file exit the "Report Field Output" window to select "Surf2-Display" and write Frames 0-9 to the report. Repeat...



## Bibliography

1. “18 per cent nickel maraging steels: Engineering properties,” Publication No. 4419, Nickel Development Institute, 1976.
2. “Analysis of pipe whip,” EPRI, Report NP-1208, 1979.
3. “Test sets world land speed record,” <https://www.af.mil/News/Article-Display/Article/139307/test-sets-world-land-speed-record>, 2003.
4. “Chebfun,” <https://www.chebfun.org/>, 2019.
5. “Material thermal properties database,” <https://thermtest.com/materials-database#4340>, 2019.
6. “Abaqus, dessault systems,” <https://edu.3ds.com/en/software/abaqus-student-edition>, 2020.
7. “Quasi-newton solution technique,” <https://abaqus-docs.mit.edu/2017/English/SIMACAETHERefMap/simathe-c-quasinewtsol.htm>, 2020.
8. “Quasi-newton solution technique,” <https://abaqus-docs.mit.edu/2017/English/SIMACAETHERefMap/simathe-c-nonlinearsol.htm>, 2020.
9. C. Alban, A. Palazotto, and J. Rutledge, “Thermal considerations with respect to sliding contact at high speed,” *Journal of Thermophysics and Heat Transfer*, vol. 30, no. 1, pp. 54–61, 2016.
10. M. Ashby and J. Abulawi, “Temperature maps for frictional heating in dry sliding,” *Tribology Transactions*, vol. 34, pp. 577–587, 1991.

11. P. Bergan, G. Horrigmoe, B. Krakeland, and T. Soreide, "Solution techniques for nonlinear finite element problems," *International Journal for Numerical Methods in Engineering*, vol. 12, p. 1677–1696, 1978.
12. T. Bergman, A. Lavine, F. Incopera, and D. Dewitt, *Fundamentals of Heat and Mass Transfer*. John Wiley and Sons, Inc, 2011.
13. B. Bhushan, *et. al.*, *Modern Tribology Handbook*. CRC Press, 2001.
14. B. Boardman, R. Uber, W. Baker, and A. Palazotto, "Modeling nonlinear heat transfer for a pin-on-disc sliding system," SciTech Orlando, Florida, January 2020.
15. B. Boardman, R. Uber, W. Baker, and A. Palazotto, "Modeling nonlinear heat transfer for a pin-on-disc sliding system," Wright Patterson AFB, OH, 2020.
16. R. Cook, D. Malkus, M. Plesha, and R. Witt, *Concepts and Applications of Finite Element Analysis*. John Wiley and Sons, Inc, 2001.
17. A. DeLeon, W. Baker, and A. Palazotto, "Modeling a nonlinear melt region as a result of high-speed sliding," *Journal of Thermophysics and Heat Transfer*, vol. 33, no. 3, pp. 808–816, 2019.
18. R. Fakir, N. Barka, and J. Brousseau, "Case study of laser hardening process applied to 4340 steel cylindrical specimens using simulation and experimental validation," *Case Studies in Thermal Engineering*, vol. 11, pp. 15–25, 2018.
19. A. Hall and C. Slunder, "The metallurgy, behavior, and application of the 18-percent nickel maraging steels," NASA SP-5051, Washington D.C., Tech. Rep., 1968.
20. N. Kim and B. Sankar, *Introduction to Finite Element Analysis and Design*. John Wiley & Sons, Inc, 2009.

21. N. Laraq, N. Alitat, J. Garcia de Maria, and A. Bairi, "Temperature and division of heat in a pin-on-disc fricitonal device-exact analytical solution," *Wear*, vol. 266, pp. 765–770, 2009.
22. R. LeVeque, *Finite Difference Methods for Ordinary and Partial Differential Equations*. Society for Industrial and Applied Mathematics, 2007.
23. S. Lim and M. Ashby, "Wear-mechanism maps," *Acta Metallurgica*, vol. 35, pp. 1–24, 1987.
24. J. Myllymaki and D. Baroudi, "A method to determine thermal conductivity using boundary temperature measurements," *Fire Safety Science*, vol. Proceedings of the Sixth International Symposium, pp. 349–360, 2018.
25. J. Ng, "Radial basis function generated finite differences for the nonlinear schrodinger equation," Wright Patterson AFB, OH, 2018.
26. K. A. Novak, *Numerical Methods for Scientific Computing*. Dover Publications, Inc, 1988.
27. L. Trefethen, *Spectral Methods in Matlab*. Society for Industrial and Applied Mathematics, 2000.
28. J. Valencia and P. Quested, *ASM Handbook Volume 15: Casting*. ASM International, 2008.

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| <b>14. ABSTRACT</b><br>The purpose of this work is to analyze the heat transfer characteristics of Vascomax®C300 during high-speed sliding. This work extends previous research that is intended to help predict the wear-rate of connecting shoes for a hypersonic rail system at Holloman Air Force Base. Solutions were generated using finite element analysis, finite difference methods, and spectral methods. The frictional heat generated by the pin-on-disc is assumed to flow uniformly and normal to the face of the pin and the pin is assumed to be a perfect cylinder resulting in two-dimensional heat flow. Displacement data obtained from the experiment is used to define the moving boundary. The distribution of temperature resulting from transient finite element analysis is used to justify a one-dimensional model. The one dimensional heat equation is solved with variable parameters using a variable-step, variable-order, implicit finite-difference method. Spectral methods are then employed to calculate the spatial derivatives improving the approximation of the function which represents the data. It is concluded that a one-dimensional approach with constant heat transfer parameters sufficiently models the high-speed pin-on-disc experiment. |                    |                     |                                          |                            |                                                                                             |  |
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