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Weak Fronts of Maxwell Equations for Conducting Media

by Michael Grinfeld and Pavel Grinfeld

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Weak Fronts of Maxwell Equations for Conducting Media

Michael Grinfeld
Weapons and Materials Research Directorate,
DEVCOM Army Research Laboratory

Pavel Grinfeld
Drexel University

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We analyze propagation of electromagnetic fronts in unbounded electric conductors. Our model is based on the Maxwell model of electromagnetism, which includes the displacement current and the Ohm's law in their simplest forms. The weak electromagnetic front is a propagating surface at which the electric and magnetic fields remain continuous, whereas their first and higher-order derivatives experience finite jumps. Remarkably, full analysis of such fronts appears to be autonomous. This means that in order to analyze such fronts, there is no need to explore the solutions out of the fronts. This property opens the possibility of establishing analytically exact solution implications of the full Maxwell systems and establishing the laws of evolution of intensity of the jumps on such fronts.

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1. Introduction

James C Maxwell's model of electromagnetism, although not accepted with enthusiasm by the giants of the 19th century like Kelvin and Helmholtz, eventually proved to be instrumental in thousands of applications. This model is of outstanding beauty, though it is described by the system of four partial differential equations for the electric and magnetic fields. Multiple efforts of outstanding scholars resulted in many classes of exact analytical solutions of the Maxwell model. In addition, these efforts allowed the development of various methods to approximate solutions. One widespread shortcoming of those solutions is their complexity. Quite often, exact solutions in the form of infinite series and cumbersome integrals appear to be less transparent than the original Maxwell model equations. For example, great interest generated a solution describing wavefronts—the surfaces carrying different discontinuities of electromagnetic fields. Moreover, these discontinuities can be analyzed exactly (i.e., without any linearization or neglecting some terms).

The key instrument in analysis of wavefronts is the so-called compatibility conditions. In the 20th century, the biggest contributions in their usage belong to the outstanding geometers and mathematical physicists Hadamard (1903), Levi-Civita (1931), and Thomas (1958). The first usage of this approach in electromagnetism belongs to Luneburg (1964), as well as Born and Wolf (1999), and Keller (1962). In those publications, interested readers can find further references and historic discussions.

Compatibility conditions have been discussed in detail in the monographs of Michael Grinfeld (1990) and Pavel Grinfeld (2014). From these monographs we borrow required results and notation. We use standard tensorial notation and operations in the form, presented in Michael Grinfeld (1990) and Pavel Grinfeld (2014). The Latin (spatial) indices $i, j, k, \ldots$ run the values 1,2,3; the Greek (surface) indices $\alpha, \beta, \gamma \ldots$ run the values 1,2. The space is referred to the immobile coordinates z^i; δ_i^j is the spatial Kronecker delta; z_{ij} and z^{ij} are the co- and contravariant components of the metrics, which are used for raising and lowering ("juggling") spatial indices, and for defining covariant differentiation ∇_i; z_{ijk} is the Levi–Civita skew-symmetric tensor.

2. The Master System of Equations

We assume that there are no noncompensated mobile charges in the conductor; however, there are electric currents I^i inside the conductor under study. The bulk Maxwell equations read

$$\nabla_i E^i = 0 \tag{1.1}$$

$$\nabla_i H^i = 0 \tag{1.2}$$

$$\frac{1}{c}\frac{\partial H^i}{\partial t} = -z^{ijk}\nabla_j E_k \tag{1.3}$$

$$\frac{\partial E^i}{\partial t} + 4\pi I^i = cz^{ijk}\nabla_j H_k \tag{1.4}$$

where c is a speed of light, E^i is an electric field, H^i is a magnetic field, and I^i is an electric current.

We assume that the electric field E^i and electric current I^i are connected by the simplest Ohm's law

$$I^i = \sigma E^i \tag{1.5}$$

where σ is an electroconductity constant.

Eliminating electric current I^i between Eqs. 1.4 and 1.5, we get

$$\frac{\partial E^i}{\partial t} + 4\pi\sigma E^i = cz^{ijk}\nabla_j H_k \tag{1.6}$$

Differentiating Eq. 1.6 with respect to t, we get

$$\frac{\partial^2 E^i}{\partial t^2} + 4\pi\sigma\frac{\partial E^i}{\partial t} - c^2\nabla_k\nabla^k E^i = 0 \tag{1.7}$$

Similarly, differentiating Eq. 1.3 with respect to t, we get

$$\frac{1}{c}\frac{\partial^2 H^i}{\partial t^2} = -z^{ijk}\nabla_j \frac{\partial E_k}{\partial t} \tag{1.8}$$

We then get, using Eq. 1.6,

$$\frac{\partial^2 H^i}{\partial t^2} + 4\pi\sigma \frac{\partial H^i}{\partial t} - c^2 \nabla^k \nabla_k H^i = 0 \tag{1.9}$$

as implied by the following chain

$$\begin{aligned}
&\frac{\partial E_k}{\partial t} + 4\pi\sigma E_k = c z_{k..}^{.mn} \nabla_m H_n \to \\
&\frac{\partial E_k}{\partial t} + 4\pi\sigma E_k = c z_{k..}^{.mn} \nabla_m H_n \to \\
&\frac{1}{c}\frac{\partial^2 H^i}{\partial t^2} = -z^{ijk}\nabla_j (c z_{k..}^{.mn} \nabla_m H_n - 4\pi\sigma E_k) \to \\
&\frac{1}{c}\frac{\partial^2 H^i}{\partial t^2} = -c z^{ijk} z_{k..}^{.mn} \nabla_j \nabla_m H_n + 4\pi\sigma z^{ijk}\nabla_j E_k \to \\
&\frac{\partial^2 H^i}{\partial t^2} = -c^2 (z^{im} z^{jn} - z^{in} z^{jm}) \nabla_j \nabla_m H_n - 4\pi\sigma \frac{\partial H^i}{\partial t} \to \\
&\frac{\partial^2 H^i}{\partial t^2} + 4\pi\sigma \frac{\partial H^i}{\partial t} = -c^2 (z^{im} z^{jn} \nabla_j \nabla_m H_n - z^{in} z^{jm} \nabla_j \nabla_m H_n) \to \\
&\frac{\partial^2 H^i}{\partial t^2} + 4\pi\sigma \frac{\partial H^i}{\partial t} = c^2 z^{jm} \nabla^k \nabla_k H^i
\end{aligned}$$

Equations 1.7 and 1.9 are sometimes called telegraphic equations.

3. Compatibility Conditions for Jumps

We assume that there is a weak front inside the conductor: The moving smooth surface S, at which the fields $E^i(z,t)$, $H^i(z,t)$, $I^i(z,t)$ remain continuous, whereas there first and second spatial and temporal derivatives experience finite jumps.

We assume that the weak fronts are referred to the Gaussian coordinates ξ^α. Thus, the fronts are described by the equation $z^i = z^i(\xi^\alpha, t)$. When possible, we will not explicitly show indices in the argument. The tensor $z_{.\alpha}^{i\cdot}(\xi, t)$, defined by the derivatives $z_{.\alpha}^{i\cdot}(\xi, t) \equiv \partial z^i(\xi^\alpha, t) / \partial \xi^a$ is called the shift tensor. It allows us to define the covariant surface metrics $\xi_{\alpha\beta}(\xi,t) = z_{ij}(x(\xi,t))z^{i\cdot}_{\cdot\alpha}(\xi,t)z^{i\cdot}_{\cdot\beta}(\xi,t)$; the contravariant surface metrics $\xi^{\alpha\beta}(\xi,t)$ are defined by the inverse of $\xi_{\alpha\beta}(\xi,t)$. We use the surface metrics for juggling surface (the Greek) indices and for defining the surface covariant differentiation, denoted as ∇_{α}. Let $N^{i}(\xi,t)$ be the field of unit normals to the front S, and $b_{\alpha\beta}(\xi,t)$ is the tensor of the second quadratic form of the surface. Let $C(\xi,t)$ be the velocity of the front**.**

We introduce the following notation of the jump vectors $f^i_E(\xi,t), f^i_H(\xi,t), F^i_E(\xi,t), F^i_H(\xi,t)$, defined as follows:

$$\begin{aligned} &f^i_E(\xi,t) \equiv \left[\nabla_m E^i\right]^+_- N^m,\ \ f^i_H(\xi,t) \equiv \left[\nabla_m H^i\right]^+_- N^m, \\ &F^i_E(\xi,t) = \left[\nabla_n \nabla_m E^i\right]^+_- N^m N^n, F^i_H(\xi,t) = \left[\nabla_n \nabla_m H^i\right]^+_- N^m N^n \end{aligned} \tag{2.1}$$

Remarkably, the jumps vectors $f^i_E(\xi,t), f^i_H(\xi,t), F^i_E(\xi,t), F^i_H(\xi,t)$ permit us to express all the first and second temporal and spatial derivatives in terms of the temporal and surface derivatives of the jump vectors as well as the geometrical and kinematical characteristics of the front. In particular, for the first derivatives of the electric and magnetic fields, we get relationships called the first-order compatibility conditions:

$$\begin{aligned} &\left[\nabla_k E^i\right]^+_- = N_k f^i_E, \left[\frac{\partial E^i(z,t)}{\partial t}\right]^+_- = -C f^i_E, \\ &\left[\nabla_k H^i\right]^+_- = N_k f^i_H, \left[\frac{\partial H^i(z,t)}{\partial t}\right]^+_- = -C f^i_H \end{aligned} \tag{2.2}$$

For the jumps of the second derivatives, we get the following compatibility relationships:

$$\begin{aligned} &\left[\nabla_k \nabla_l E^i\right]^+_- = F^i_E N_k N_l + 2N_{(k} z^{\cdot\alpha}_{l.)} \nabla_\alpha f^i_E - f^i_E b^{\alpha\beta} z^{\cdot}_{k\alpha} z^{\cdot}_{l\beta}, \\ &\left[\nabla_k \frac{\partial E^i(z,t)}{\partial t}\right]^+_- = -F^i_E N_k C - C f^i_E + \frac{\delta f^i_E}{\delta t} N_k - z^{\cdot\alpha}_{k.} \nabla_\alpha (C f^i_E), \\ &\left[\frac{\partial^2 E^i(z,t)}{\partial t^2}\right]^+_- = F^i_E C^2 - 2C\frac{\delta f^i_E}{\delta t} - f^i_E \frac{\delta C}{\delta t} \end{aligned} \tag{2.3}$$

and

$$
\begin{aligned}
&\left[\nabla_k \nabla_l H^i\right]_-^+ = F_H^i N_k N_l + 2N_{(k} z_{l.)}^{.\alpha} \nabla_\alpha f_H^i - f_H^i b^{\alpha\beta} z_{k\alpha}^{.} z_{l\beta}^{.}, \\
&\left[\nabla_k \frac{\partial H^i(z,t)}{\partial t}\right]_-^+ = -F_H^i N_k C - C f_H^i + \frac{\delta f_H^i}{\delta t} N_k - z_{k.}^{.\alpha} \nabla_\alpha (C f_H^i), \\
&\left[\frac{\partial^2 H^i(z,t)}{\partial t^2}\right]_-^+ = F_H^i C^2 - 2C \frac{\delta f_H^i}{\delta t} - f_H^i \frac{\delta C}{\delta t}
\end{aligned}
\quad (2.4)
$$

In Eqs. 2.3 and 2.4, $\delta / \delta t$ is the symbol of the covariant time-derivative on the moving surfaces (Thomas 1958; Grinfeld 1990; Grinfeld 2014).

4. Weak Electromagnetics Fronts: Their Velocity and Jump Vectors

Let us calculate the jumps of Eqs. 1.1–1.3 and 1.6 across the weak front:

$$\left[\nabla_i E^i\right]_-^+ = 0 \quad (3.1)$$

$$\left[\nabla_i H^i\right]_-^+ = 0 \quad (3.2)$$

$$\frac{1}{c}\left[\frac{\partial H^i}{\partial t}\right]_-^+ = -z^{ijk}\left[\nabla_j E_k\right]_-^+ \quad (3.3)$$

$$\frac{1}{c}\left[\frac{\partial E^i}{\partial t}\right]_-^+ = z^{ijk}\left[\nabla_j H_k\right]_-^+ \quad (3.4)$$

Using the first-order compatibility conditions (Eq. 2.2) we can transform Eqs. 3.1–3.4 as follows:

$$f_E^i N_i = 0 \quad (3.5)$$

$$f_H^i N_i = 0 \quad (3.6)$$

$$\frac{C}{c} f_H^i - z^{ijk} N_j f_{Ek} = 0 \tag{3.7}$$

$$\frac{C}{c} f_E^i + z^{ijk} N_j f_{Ek} = 0 \tag{3.8}$$

Equations 3.5–3.8 directly imply that three vectors, f_E^i, f_H^i, N^i, are mutually orthogonal. Eliminating the magnetic jump vector f_H^i between Eqs. 3.7 and 3.8 and using Eq. 3.6, we arrive at the linear uniform equation with respect to the electric jump vector:

$$(C^2 - c^2) f_E^i = 0 \tag{3.9}$$

Equation 3.9 implies that the nonvanishing jump vector f_E^i can exist only if

$$C = \pm c \tag{3.10}$$

Equation 3.10 in concert with Eqs. 3.5–3.8 implies the relationship

$$f_E^i f_{Ei} = f_H^i f_{Hi} \tag{3.11}$$

Let us present vectors f_E^i, f_H^i in the form

$$f_E^i = \Delta_E^i(\xi,t) A_E(\xi,t),\ f_H^i = \Delta_H^i(\xi,t) A_H(\xi,t), \tag{3.12}$$

where Δ_E^i, Δ_H^i are unit vectors called directors, and A_E, A_H are the magnitudes of the jump vectors.

Using the decomposition Eq. 3.12, we can rewrite Eq. 3.11 in the following form:

$$A_E^2 = A_H^2 \tag{3.14}$$

We can choose arbitrarily any orientation of the vector Δ_E^i in the plane orthogonal the unit vector N_i. The associated vector Δ_H^i should satisfy the following relationship:

$$\Delta_{Hi} = z_{ijk} N^j \Delta_E^k \tag{3.15}$$

The following relationship is valid:

$$N^i = z^{ijk} \Delta_{Ej} \Delta_{Hk} \tag{3.16}$$

We will see later that the director Δ_E^i can be chosen arbitrarily at $t = 0$ only. This choice automatically entails this vector $t > 0$ as long the front remains smooth.

5. Fronts and Rays

It is obvious, intuitively, that, if the position of the weak front is known at $t = 0$, then it can be determined for $t > 0$ since we know its velocity, $C = \pm c$. It is true for the initially smooth front, at least, within a finite interval $0 \le t \le t_c$, where t_c is the moment of appearance of caustics at the front (Born and Wolf 1999). Let $S : z^i = z^i(\xi, t)$ be the Gaussian equation of the front at time t, and let $S^\circ : z^i = z^i(\xi, 0) \equiv z^{i\circ}(\xi)$ at $t = 0$. Let $n^{i\circ}$ ξ be the field of unit normal vectors of the surface $S^\circ(\xi)$. Then, obviously, the surface S is defined by the Gaussian equation:

$$S : z^i(\xi, t) = z^{i\circ}(\xi) \pm c n^{i\circ}(\xi) t \tag{4.1}$$

where we assume that the spatial coordinate system is affine.

If we fix the moment t, then we treat the function $z^i = z^i(\xi, t)$ in Eq. 4.1 as the corresponding position of the front at moment t. At the same time, if we fix a pair of the Gaussian coordinates ξ^α, then by with changing t we get a straight line, called ray, as shown in Fig. 1.

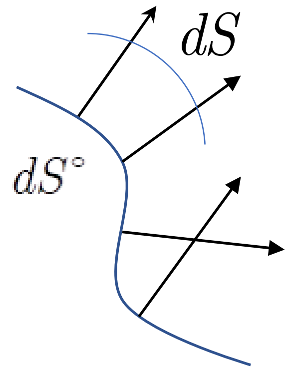

Fig. 1 Geometry of rays and front

If we choose certain area $dS^{\circ}(\xi)$ on the initial front and consider all the rays starting from $dS^{\circ}(\xi)$, we will get something that is called a tube of rays. The area dS crossed by all those rays from the current surface S differs from dS°, if the initial surface is not flat. When the tube is infinitesimally small, the ratio $J(\xi,t) \equiv dS \,/\, dS^{\circ}$ is called a rays divergence. The ray divergence satisfies the relationship

$$\frac{1}{J}\frac{\partial J}{\partial t} = -Cb_{\alpha}^{\alpha} \tag{4.2}$$

Equations. 4.1 and 4.2 are valid in the case of uniform, isotropic media. Otherwise, they should be replaced with more-general relationships (Grinfeld 1990 Grinfeld 2014).

Figure 1 clearly demonstrates the main difficulty of our analysis. It is connected with the fact that the rays can intersect at $t > 0$. Starting at this moment, this analysis fails and additional effects or models should be taken into account.

6. Evolution of the Jump-Vectors along the Rays

We proceed with calculating jumps of all terms in Eq. (1.7) we get

$$\left[\frac{\partial^2 E^i}{\partial t^2}\right]_-^+ + 4\pi\sigma\left[\frac{\partial E^i}{\partial t}\right]_-^+ - c^2\left[\nabla_k \nabla^k E^i\right]_-^+ = 0 \tag{5.1}$$

Using the compatibility Eqs. 2.2 and 2.3, we get the following:

$$F_E^i C^2 - 2C\frac{\delta f_E^i}{\delta t} - f_E^i\frac{\delta C}{\delta t} - 4\pi\sigma C f_E^i - c^2(F_E^i - f_E^i b_\alpha^\alpha) = 0 \qquad (5.2)$$

If we choose Eq. 3.1, then the $\delta / \delta t$ –derivative coincides with the partial $\partial / \partial t$ -derivative. Using Eq. 3.10, we can rewrite Eq. 5.2 as

$$2C\frac{\partial f_E^i}{\partial t} + f_E^i\frac{\partial C}{\partial t} + 4\pi\sigma C f_E^i - c^2 f_E^i b_\alpha^\alpha = 0 \qquad (5.3)$$

Using Eq. 3.10, we can rewrite Eq. 5.3 as

$$2c\frac{\partial f_E^i}{\partial t} + 4\pi\sigma c f_E^i - c^2 f_E^i b_\alpha^\alpha = 0 \qquad (5.4)$$

Using the decomposition Eq. 3.12, we can rewrite Eq. 5.4 as

$$\frac{\partial A_E}{\partial t}\Delta_E^i + \frac{\partial \Delta_E^i}{\partial t}A_E + 2\pi\sigma\Delta_E^i A_E - \frac{1}{2}cb_\alpha^\alpha\Delta_E^i A_E = 0 \qquad (5.5)$$

Contracting Eq. 5.5 with the unit vector $\Delta_{Ei}^{\cdot}$, we get

$$\frac{\partial A_E}{\partial t} + 2\pi\sigma A_E - \frac{1}{2}cb_\alpha^\alpha A_E = 0 \qquad (5.6)$$

Using Eq. 4.2, we can rewrite Eq. 5.6 as

$$\frac{\partial}{\partial t}(e^{4\pi\sigma t}JA_E^2) = 0 \qquad (5.7)$$

as implied by the chain Eq. 5.7:

$$\begin{aligned} &2\frac{\partial A_E}{\partial t} + 4\pi\sigma A_E + \frac{1}{J}\frac{\partial J}{\partial t}A_E = 0 \rightarrow \\ &\frac{\partial \ln(JA_E^2)}{\partial t} + 4\pi\sigma = 0 \rightarrow \\ &\frac{\partial JA_E^2}{\partial t} + 4\pi\sigma JA_E^2 = 0 \rightarrow \\ &\frac{\partial}{\partial t}(e^{4\pi\sigma t}JA_E^2) = 0 \end{aligned} \qquad (5.8)$$

Contracting Eq. 4.5 with unit vector $\Delta^{\cdot}_{Hi}$, we get

$$\frac{\partial \Delta^{i}_{E}}{\partial t}\Delta^{\cdot}_{Hi} = 0 \tag{5.9}$$

Also, differentiating relationships

$$\Delta^{i}_{E} N_{i} = 0\,, \quad \Delta^{i}_{E}\Delta_{Ei} = 1 \tag{5.10}$$

we get

$$\frac{\partial \Delta^{i}_{E}}{\partial t}\Delta^{\cdot}_{Hi} = 0, \frac{\partial \Delta^{i}_{E}}{\partial t}\Delta^{\cdot}_{Ei} = 0 \tag{5.11}$$

Summarizing, we arrive at the relationship

$$\frac{\partial \Delta^{i}_{E}}{\partial t} = 0 \tag{5.12}$$

In view of Eq. 5.8, the directors Δ^{i}_{E} and Δ^{i}_{H} remain the same along each of the rays.

7. Conclusion

We analyzed propagation of electromagnetic fronts in an unbounded electric conductor with isotropic electric conductivity constant σ. The weak electromagnetic front is a propagating surface at which the electric and magnetic fields remain continuous, whereas their first and higher-order derivatives experience finite jumps. All the jumps at the front can be expressed in terms of the jump vectors, defined on the front only and their derivatives with respect to time and the surface coordinates. In particular, the jumps of all or the first and second derivatives can be expressed in terms of the jump vectors $f^{i}_{E}(\xi,t), f^{i}_{H}(\xi,t), F^{i}_{E}(\xi,t), F^{i}_{H}(\xi,t)$, defined by Eq. 2.1. In particular, the jumps of the first-order derivatives are given by Eq. 2.2, whereas the jumps of the second-order derivatives are given by the second-order compatibility conditions in Eq. 2.3.

Remarkably, the evolution of the fields $f_E^i(\xi,t), f_H^i(\xi,t), F_E^i(\xi,t), F_H^i(\xi,t)$ can be separated from the analysis of the electric field E^i and magnetic field H^i in the domains of smoothness. In this sense, the analysis of the weak fronts appears to be autonomous. This autonomous character of the weak fronts appears to be quite convenient since the bulk equation contain four independent variables, whereas the autonomous system has only three.

Applying the first-order compatibility of Eq. 2.2 to the Maxwell system, Eqs. 1.1–1.4, and Ohm's law, Eq. 1.5, we arrive at the linear homogeneous system Eqs. 3.5–3.8. Analysis of Eqs. 3.5–3.8 directly implies 1) the velocity C of the weak front is equal to $\pm c$, where c is the speed of light in vacuum; 2) the three vectors f_E^i, f_H^i, N^i are mutually orthogonal, and they satisfy the relationship $f_E^i f_{Ei} = f_H^i f_{Hi}$ Eq. 3.11; and 3) the velocity C of the weak front does not depend upon the electroconductivity σ.

The position of the weak front $S : z^i = z^i(\xi,t)$ is defined in the affine coordinate system by Eq. 4.1: $z^i(\xi,t) = z^{i\circ}(\xi) \pm cn^{i\circ}(\xi)t$, where $S^\circ : z^i = z^i(\xi,0) \equiv z^{i\circ}(\xi)$ at $t = 0$ and $n^{i\circ}\ \xi$ is the field of unit normal vectors of the surface $S^\circ(\xi)$. If we fix the moment *t*, then we treat the function $z^i = z^i(\xi,t)$ in Eq. (4.1) as the corresponding position of the front at moment t. At the same time, if we fix a pair of the Gaussian coordinates ξ^α, then with changing t we get a straight line, called ray, as shown in Fig. 1. Thus, $z^i(\xi,t) = z^{i\circ}(\xi) \pm cn^{i\circ}(\xi)t$ delivers the Gaussian equation of the weak front in the special "ray" coordinates.

Applying the second-order compatibility equations 2.4 to the transport Eqs. 1.7 and 1.9, we establish our main result $\partial(e^{4\pi\sigma t} J A_E^2) / \partial t = 0$ (Eq. 5.9). This equation describes how the jump amplitude changes along the ray. It shows that the jump intensity grows when the ray divergence decays and vice versa. Also, there is an additional mechanism of exponential decay due to Ohm's resistance dissipation. In addition, Eq. 5.12 shows that the director vector Δ_E^i remains constant along the ray.

Ray constructions also appear when considering high-frequency asymptotics (Luneburg 1964; Born and Wolf 1999). However, the method of weak fronts is equally applicable in nonlinear problems, and it is exact, not asymptotic.

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