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Singularity-Free Great-Circle Sailing

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Traditional approaches to solving great-circle navigation problems (geodesics on a spherical-Earth approximation) involve spherical trigonometry and can have singularities at the poles. We present a singularity-free approach to solving both the direct and the indirect great-circle problems: rotate the coordinate system so that the trajectory is on the equator, compute the trajectory and rotate back. This is done, because great-circle navigation problems are simple on the equator. Such great-circle navigation algorithms can be useful in a number of techniques for ship self-localization.

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SINGULARITY-FREE GREAT-CIRCLE SAILING

1. INTRODUCTION

When using a spherical-Earth approximation, great-circles are geodesic paths (the straightest path along a curved surface) and form the shortest distances between two points. Geodesic navigation stands in contrast to loxodromic (a.k.a. rhumb) navigation [1–3] in that the heading with respect to true North is constantly changing, except on trajectories along the equator or those going directly North-South. Though ellipsoidal Earth approximations are more accurate and modern approaches such as [4–6] avoid problems associated with more classical techniques such as Vicenty's algorithm [7], great-circle navigation has been noted to be a sufficient approximation for most navigation problems [8, Sec. 104] and such techniques have been used in navigation for a very long time.

Spherical Earth approximations play a role in the time-delay-of-arrival-only surface-wave passive localization [9] for the now-defunct Loran navigation system. They also allow ships to localize themselves via direction-of-arrival-only surface-wave measurements [10] exist. Such algorithms require the ability to solve the direct and indirect great circle navigation problems. The ability to utilize such algorithms without worrying about singularities near the geographic poles will be increasingly important in the years to come due to the melting of polar ice. The melting of the polar ice is expected to make the Northwest Passage a popular trading route by the end of the 21st century due to global warming [11]

Navigation using a spherical-Earth approximation is often called "great-circle sailing." A summary of algorithms for great-circle sailing from 1899 is given in [12]. Computational algorithms for great-circle navigation such as [8, Sec. 1208], [13, 14] typically utilize spherical trigonometry and can have singularities at the geographic poles. This paper presents singularity-free algorithms for the direct great-circle problem (given a starting point, heading, and distance, find the endpoint) and the indirect great-circle problem (given two points on the sphere, find the great-circle trajectory between them). The solutions are based on coordinate-system transformations and the fact that great-circle trajectories that are known to go along the (spherical) Earth's equator are simple to compute.

The solutions in this paper effectively get Cartesian vector locations and headings, which can then be re-expressed in terms of latitude and longitude and in terms of a heading angle East of North. This stands in contrast to the solution of [15], which while also performing a type of rotation to simplify the navigation problems, expresses things in trigonometric, not vector terms. Additionally, though a solution utilizing vectors is derived in [16], it is inherently based in spherical trigonometry and singularities are present at the poles in some vector definitions.

In [17, Ch. 2.4.4], the notion of a "wander frame" is defined for targets moving over a curved Earth. As noted in [18], such a coordinate frame can be used for mapping tagrte dynamic models to a curved Earth, and thus also for solving the direct geodetic problem. However, the wander frame is defined in terms of a set of nonlinear differential equations. The navigation solutions in this paper *do not* require the integration of any differential equations. The solutions are exact on a sphericla Earth.

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The algorithm presented here is based on rotating the coordinate system so that the trajectory runs along the equator. Sections 2 and 3 present two necessary transformations. Section 2 presents expressions for rotating two vectors to be along the equator (z components zero). Section 3 then describes the transformation of a point and a heading such that the point is on the equator and the heading is along the equator. The actual algorithms are then described in Section 4, with the solution to the indirect geodetic problem presented in Subsection 4.1, and the solution to the direct geodetic problem presented in Subsection 4.2. The results are summarized in Section 5.

2. ROTATING TWO VECTORS ONTO THE $x-y$ PLANE

Given two 3D vectors **x**, and **y**, the goal is to find a rotation matrix that rotates the vectors such that their z-components are zero. An infinite number of solutions to this problem exist. However, the number of solutions is narrowed down by requiring that the rotation matrix be a product of a rotation about the y-axis followed by a rotation about the x-axis. That is, the rotation matrix **R** is the product

$$\mathbf{R} = \mathbf{R}_x\mathbf{R}_y. \tag{1}$$

Consequently, a rotated vector $\mathbf{v}_{\text{rot}}$ and an unrotated vector **v** are related by

$$\mathbf{v}_{\text{rot}} = \mathbf{R}\mathbf{v}. \tag{2}$$

Right-handed rotations about the x and y axes are given by the matrices

$$\mathbf{R}_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_x) & -\sin(\theta_x) \\ 0 & \sin(\theta_x) & \cos(\theta_x) \end{bmatrix} \qquad \mathbf{R}_y = \begin{bmatrix} \cos(\theta_y) & 0 & \sin(\theta_y) \\ 0 & 1 & 0 \\ -\sin(\theta_y) & 0 & \cos(\theta_y) \end{bmatrix}. \tag{3}$$

Let $\mathbf{x} = [x_1, x_2, x_3]'$ and $\mathbf{y} = [y_1, y_2, y_3]'$. The goal is to find an **R** as in (1) such that the z components of the rotated vectors are 0. Using (3), this means that

$$x_3\cos(\theta_x)\cos(\theta_y) + x_2\sin(\theta_x) - x_1\cos(\theta_x)\sin(\theta_y) = 0 \tag{4}$$

$$y_3\cos(\theta_x)\cos(\theta_y) + y_2\sin(\theta_x) - y_1\cos(\theta_x)\sin(\theta_y) = 0. \tag{5}$$

Four solutions (unique within factors of 2π) come from explicitly solving for θ_x and θ_y using the four-quadrant inverse-tangent function. However, one must explicitly address the case where $\mathbf{x} = \mathbf{y}$ and $\mathbf{x} = -\mathbf{y}$. Define:

$$n_1 \triangleq \sqrt{\left(x_1^2 + x_3^2\right) y_2^2 - 2x_2y_2\left(x_1y_1 + x_3y_3\right) + x_2^2\left(y_1^2 + y_3^2\right)} \tag{6}$$

$$n_2 \triangleq x_1y_3 - x_3y_1 \tag{7}$$

$$n_3 \triangleq x_1y_2 - x_2y_1 \tag{8}$$

$$n_4 \triangleq x_3y_2 - x_2y_3. \tag{9}$$

One solution is:

$$\theta_x = \arctan(n_2, -n_1) \qquad\qquad \theta_y = \arctan(n_4, n_3) \tag{10}$$

where if the solution is in the right half-plane, then the four-quadrant inverse tangent is equivalent to the standard arctangent of $\arctan(a, b) = \arctan\left(\frac{a}{b}\right)$ and equals zero if $b = 0$. This is how the four-quadrant inverse-tangent is defined in Matlab. In Mathematica, the ordering of the arguments a and b is reversed. Only the first solution should be needed for the algorithm developed in this paper. Note that in (6), though the argument of the square root should be positive, one might want to take the maximum of the argument and 0 to avoid finite-precision issues.

For the case where $|n_1| < \epsilon$, $|n_2| < \epsilon$, $|n_3| < \epsilon$, and $|n_4| < \epsilon$, where ϵ is a value related to machine finite-precision limits (for example, one might use `eps()` in Matlab), that designates the special cases on $\mathbf{x} = \mathbf{y}$ and $\mathbf{x} = -\mathbf{y}$. In such an instance, we could choose a single rotation matrix using the technique of [19]. There, a technique is given to rotate a coordinate axis to point in a particular direction. However, the use of a Householder transformation, as in [20, Ch. 5.1.3], is numerically more stable. The rotation matrix from a Householder transformation has the form

$$\mathbf{R} = \mathbf{I} - 2\frac{\mathbf{v}\mathbf{v}'}{\|\mathbf{v}\|}. \tag{11}$$

A Householder vector can be found as in [20, Ch. 5.1.3]. For a 3D problem where one is specifically zeroing the x and z components, one can obtain a simple explicit solution. Define $\mathbf{u}$ as the normalized $\mathbf{x}$ vector

$$\mathbf{u} = \frac{\mathbf{x}}{\|\mathbf{x}\|}. \tag{12}$$

Designate the elements of $\mathbf{u} = [u_x, u_y, u_z]'$. The Householder vector $\mathbf{v}$ is

$$\mathbf{v} = \begin{bmatrix} \dfrac{u_x}{\sqrt{2\left(1 - u_y\right)}} \\ \dfrac{u_y - 1}{\sqrt{2\left(1 - u_y\right)}} \\ \dfrac{u_z}{\sqrt{2\left(1 - u_y\right)}} \end{bmatrix} \tag{13}$$

where if $\mathbf{u}_z = 1$, one uses $\mathbf{v} = \mathbf{0}$.

3. ROTATING A VECTOR AND A HEADING TO THE EQUATOR

Whereas the rotation of Section 2 can be used for solving the direct great-circle problem, we now consider solving the indirect great-circle problem. In this instance, we are given an initial location and a heading. We wish to rotate the point to the equator and perform an additional rotation so that the heading also points along the equator.

Let the location on the surface of the reference sphere be specified in terms of an azimuth λ counterclockwise from the x-axis in the $x-y$ plane and an elevation ϕ above the $x-y$ plane. Though we are working with a reference sphere in this report, if the Earth were an ellipse, then λ would be East longitude and ϕ would be North latitude. Additionally, suppose that we are given a heading θ in radians East of North in a local East-North-Up coordinate system. We wish to obtain a rotation matrix such that:

1. The Cartesian location of the point on the sphere rotates to the x-axis (the y and z components are zero).
2. A unit vector in the heading direction is $[0,1,0]'$ (East) in the rotated coordinate system.

A 3D unit vector for the unrotated heading direction can be obtained as

$$\mathbf{u}_h=\sin(\theta)\mathbf{u}_{\text{East}}+\cos(\theta)\mathbf{u}_{\text{North}} \tag{14}$$

where $\mathbf{u}_{\text{East}}$ and $\mathbf{u}_{\text{North}}$ are unit vectors in the direction of the local East and North vectors on the unit sphere. Expressions for these vectors in ellipsoidal coordinates are given in [21]. Spherical coordinates are the same as ellipsoidal coordinates with the flattening term $f=0$. Thus, these vectors are:

$$c_E\mathbf{u}_{\text{East}}=\begin{bmatrix}-\sin(\lambda)\cos(\phi)\\ \cos(\lambda)\cos(\phi)\\ 0\end{bmatrix} \qquad c_N\mathbf{u}_{\text{North}}=\begin{bmatrix}-\cos(\lambda)\sin(\phi)\\ -\sin(\lambda)\sin(\phi)\\ \cos(\phi)\end{bmatrix}. \tag{15}$$

whereby c_E and c_N reflect the fact that the expressions on the right are not necessarily unit vectors and thus need to be normalized. These derivatives correspond to a coordinate transformation on a reference sphere of radius r of

$$x=r\cos(\lambda)\cos(\phi) \tag{16}$$
$$y=r\sin(\lambda)\cos(\phi) \tag{17}$$
$$z=r\sin(\phi). \tag{18}$$

The inverse transformation is similarly

$$r=\sqrt{x^2+y^2+z^2} \tag{19}$$
$$\lambda=\arctan\left(y,x\right) \tag{20}$$
$$\phi=\arcsin\left(\frac{z}{r}\right). \tag{21}$$

In the spherical coordinate system, one can rotate the point in the direction specified by λ and ϕ to be along the x-axis $\mathbf{u}_x=[1,0,0]'$ by first performing a right-handed rotation of $-\phi$ about the y-axis and then performing a right-handed rotation of λ about the rotated z-axis. Consequently, a rotation matrix to rotate a vector with direction given by λ and ϕ to align with the x-axis is

$$\mathbf{R}_1 = \begin{bmatrix} \cos(\lambda)\cos(\phi) & \cos(\lambda)\sin(\phi) & \sin(\lambda) \\ -\sin(\phi) & \cos(\phi) & 0 \\ -\cos(\phi)\sin(\lambda) & -\sin(\lambda)\sin(\phi) & \cos(\lambda) \end{bmatrix}. \tag{22}$$

The unit vector of (14) in the direction of the heading in this rotated coordinate system is thus

$$\tilde{\mathbf{u}}_h = \mathbf{R}_1\mathbf{u}_h \tag{23}$$

and in general, this will not be equal to $[0, 1, 0]'$. That is, it will not be pointing East (nor West). An additional rotation about the local "up" axis to rotate the heading in the plane to point East must be performed to guarantee that the trajectory remains on the equator of the rotated coordinate system.

To do this, first extract the local heading North of East in the rotated local coordinate system as

$$\tilde{\theta}_{NE} = \arctan\left([\tilde{\mathbf{u}}_h]_3, [\tilde{\mathbf{u}}_h]_2\right) \tag{24}$$

where the subscripts indicate the element of the vector being selected. Note that $\tilde{\theta}_{NE}$ relates to the local heading East of North as $\tilde{\theta} = \frac{\pi}{2} - \tilde{\theta}_{NE}$. To undo the rotation, one thus needs to rotate $-\tilde{\theta}_{NE}$ about the local "up" axis, which is the x-axis in this case. Thus, the rotation matrix is

$$\mathbf{R}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\tilde{\theta}_{NE}) & \sin(\tilde{\theta}_{NE}) \\ 0 & -\sin(\tilde{\theta}_{NE}) & \cos(\tilde{\theta}_{NE}) \end{bmatrix}. \tag{25}$$

All together, the total rotation matrix to rotate a point to be on the equator and a collocated heading to simultaneously point East is

$$\mathbf{R} = \mathbf{R}_2\mathbf{R}_1. \tag{26}$$

4. THE SAILING ALGORITHMS

When traveling along the equator, one is simply going around a circle. Thus, to travel a distance of d along the equator, one travels an angle of

$$\frac{d}{r} = \Delta_\theta \tag{27}$$

where r is the radius if the Earth and Δ_θ is given in radians. The sailing algorithms simply relate distances traveled to such angles.

Note that both algorithms return the headings as unit vectors in the local tangent place $\mathbf{u}_h$. The unit vector can be transformed back to a heading in radians East of North using the formula:

$$\theta = \arctan\left(\mathbf{u}_h'\mathbf{u}_{\text{East}}, \mathbf{u}_h'\mathbf{u}_{\text{North}}\right) \tag{28}$$

where $\mathbf{u}_{\text{East}}$ and $\mathbf{u}_{\text{North}}$ are evaluated at the point where the heading is taken. Note that the headng angle assigned at the geographic poles depends on the choice of spherical longitude λ used at the pole.

4.1 The Indirect Great Circle Problem

In the indirect great-circle problem, one is given a starting point and ending point. The goal is to obtain the distance d traveled on the surface of the Earth as well as the initial and final headings. Given the starting point and an initial heading, one can then find waypoints along the trajectory by solving the direct great-circle problem of Section 4.2.

1. Given two points on the surface of the reference sphere of radius r. If these are specified in spherical coordinates, convert them to Cartesian points, $\mathbf{x}_1$ and $\mathbf{x}_2$, using (16), (17), and (18).
2. Rotate both vectors to a common axis using the technique of Section 2. Let the total rotation matrix be $\mathbf{R}$ and the rotated points be $\tilde{\mathbf{x}}_1$ and $\tilde{\mathbf{x}}_2$.
3. One must determine which way around the sphere the trajectory goes on the equator.
   - Get the angles of the rotated vectors in the x-y plane. That is:

     $$\theta_1 = \arctan\left([\tilde{\mathbf{x}}_1]_2, [\tilde{\mathbf{x}}_1]_1\right) \qquad \theta_2 = \arctan\left([\tilde{\mathbf{x}}_2]_2, [\tilde{\mathbf{x}}_2]_1\right) \tag{29}$$

     where the subscripts indicate the elements of the vectors to select.
   - Evaluate $\theta_2 - \theta_1$ and wrap the difference to the range of $-\pi$ to π. That is:

     $$\Delta_\theta = \text{modulo}\left(\theta_2 - \theta_1 + \pi, 2\pi\right) - \pi. \tag{30}$$

   - If $\Delta_\theta < 0$, then the trajectory goes West in the transformed coordinate system.
4. From (27), the distance between the two points on the sphere is

   $$d = |\Delta_\theta|\, r. \tag{31}$$

5. Let $\mathbf{u}_{\text{East},1}$ and $\mathbf{u}_{\text{East},2}$ be the East unit vectors from (15) evaluated at the starting and stopping points in the *transformed* coordinate system.
6. If the trajectory is going West, then

   $$\mathbf{u}_{h,1} = -\mathbf{R}'\mathbf{u}_{\text{East},1} \qquad \mathbf{u}_{h,2} = -\mathbf{R}'\mathbf{u}_{\text{East},2} \tag{32}$$

otherwise

$$\mathbf{u}_{h,1} = \mathbf{R}'\mathbf{u}_{\text{East},1} \qquad \mathbf{u}_{h,2} = \mathbf{R}'\mathbf{u}_{\text{East},2} \tag{33}$$

where $\mathbf{u}_{h,1}$ and $\mathbf{u}_{h,2}$ are the initial and final headings of the target in global coordinates (the inverse rotation by $\mathbf{R}'$ transformed everything back), then (28) can be used to get the heading. If one wants waypoints between the beginning and end, one can use the direct great-circle solver of Section 4.2.

4.2 The Direct Great Circle Problem

The direct great-circle problem seeks to find the final location of a trajectory given an initial heading and a distance to travel. Additionally, one might desire waypoints along with headings at those points. These can be obtained as follows:

1. Given a distance to travel d, a location on the surface of a spherical-Earth of radius r in spherical coordinates of azimuth λ counterclockwise (East) from the x-axis in the $x-y$ plane and an elevation ϕ (North) above the $x-y$ plane as well as an initial heading θ East of North, obtain the rotation matrix $\mathbf{R}$ of (26).
2. The 3D location of the final point in the equatorially-rotated coordinate system is

$$\tilde{\mathbf{x}} = \begin{bmatrix}\cos(\Delta_\theta)\\ \sin(\Delta_\theta)\\ 0\end{bmatrix} \tag{34}$$

$$\Delta_\theta = \frac{d}{r} \tag{35}$$

where r is the radius of the Earth. Similarly, the heading of all points along the transformed trajectory is

$$\tilde{\mathbf{u}}_h \begin{bmatrix}1\\0\\0\end{bmatrix}. \tag{36}$$

3. The final point and unit-vector heading in 3D Cartesian coordinates are

$$\mathbf{x} = \mathbf{R}\tilde{\mathbf{x}} \tag{37}$$

$$\mathbf{u}_h = \mathbf{R}\tilde{\mathbf{u}}_h. \tag{38}$$

These can be converted back to spherical coordinates (λ, ϕ) as

$$\lambda = \arctan(y, x) \qquad \phi = \arcsin\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) \tag{39}$$

and a final heading in radians East of North can be obtained (28).

5. CONCLUSIONS

This paper presented singularity-free algorithms to solve the indirect and direct great-circle navigation problems. This can be useful for modern navigation and surface-wave location problems anywhere in the world, including near the geographic poles. The algorithms simply transform the trajectory being estimated to be along the equator; solve the problem, and then transform the results back. Open-source implementations of the algorithms presented in this paper are available online in the Tracker Component Library [22, 23] as options in the `indirectGreatCircleProb` and `directGreatCircleProb` functions.

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