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Instantaneous Point Explosion in Incompressible Fluid-like Media

by Michael A Grinfeld and Steven B Segletes

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14. ABSTRACT

This report covers different issues relating to point explosion in incompressible fluid-like media. Suggested analysis takes into account 1) a finite size of the domain of the instantaneous explosion and 2) a finite size of the liquid media. The solutions in those cases cease to be self-similar but still can be reduced to the analysis of a single ordinary differential equation. The equation is rather complex, and the usage of computer-based numerical analysis is required. Fortunately, in the asymptotic case of unbounded fluid, the equation permits exact solution, which is not self-similar.

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1. Introduction

The problem of a point explosion[*] is one of the most famous and extensively developed in the sense of corresponding physics, mechanics, and applied mathematics. There are many reasons for the continuing interest to this problem, which is based on its practical importance and theoretical beauty. The point explosion is an attractive notion because the physical properties of the explosive are expressed in terms of a single constant. This schematic model, however, allows us to concentrate on the mathematical peculiarities of the problem and explore those in more detail. Any expansion from the point of explosion produces an unphysical feature of infinite relative gas expansion and, thus, a net explosive pressure of zero. Therefore, the point explosion provides the convenient means to directly introduce a specified amount of kinetic energy into a system without regard to the explosive properties or the characteristics of the blast.

The problem of point explosion, even when based on the simplest models of compressible liquid, leads to analysis of the initial boundary value problems for systems of partial differential equations. Relevant solutions lead to substantial computations and rather cumbersome relationships. In some special cases of high geometrical and mathematical symmetries, the systems of partial differential equations can be reduced conceptually to much simpler systems of ordinary differential equations: we are talking about the so-called self-similar solutions. Another assumption, which simplifies the problem, concerns the assumption of incompressibility. Spherically symmetric blast in incompressible ideal liquid was presented in the early editions of Sedov's monograph *Similarity and Dimensional Methods in Mechanics* (1993).

The problem of spherically symmetric blast explosions allows the self-similar solutions only if the zone of explosion can be treated as a singular point and the ambient medium is unbounded (and it is not the whole set of constraints). Clearly these assumptions are not applicable in the vicinity of the center of the explosion and in the vicinity of the exterior boundary of the media.

In this report, we limit ourselves with the case in incompressible liquid and spherical symmetry. At the same time, we assume that the explosion happens initially with finite spatial domain and that the liquid has a varying exterior radius. The solutions in those situations are not self-similar.

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[*] The early history of point explosion is summarized in "Taylor–von Neuman–Sedov blast wave" (2021); further developments can be found in Taylor (1945); Sedov (1993); Landau and Lifshitz (1959); and Zeldovich and Raizer (2002).

2. Formulation of the Model

Consider an instantaneous explosion of a given energy E_c that is spherically symmetric. We assume that originally the "effective" fluid has radius R_{out}°; the big structure of the resulting geometry is presented in Fig. 1. A spherical empty hole with the variable radius $R_*(t)$ begins propagating from the center of the explosion. Different interpretations of this mathematical model are possible. For instance, one may consider that the total explosive energy E_c is *impulsively* applied to the liquid shell at the moment of detonation. During the subsequent cavity expansion, there is no residual pressure inside the cavity of the fluid shell, precisely because the *instantaneous* point explosion has converted the explosive energy into a purely kinetic form from the outset.

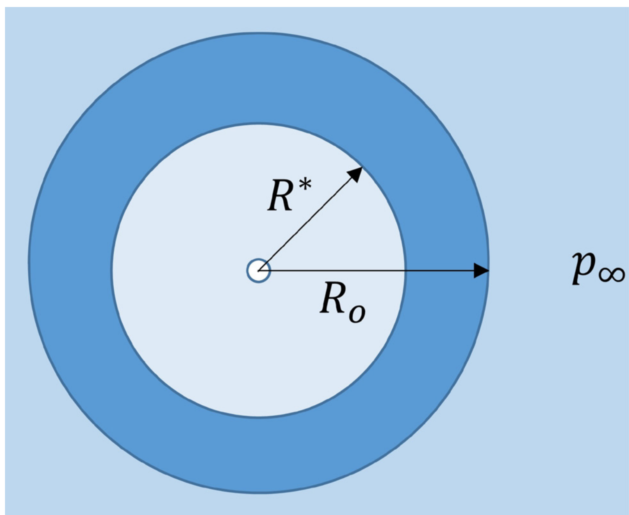

Fig. 1 Spherically symmetric point explosion geometry

Because of that, the external radius R_{out} of the fluid will become a function of time as well, $R_{out} = R_{out}(t)$. It is physically natural to define the initial geometry

$$R_*(0) = R_*^{\circ} \tag{1}$$

and

$$R_{out}(0) = R_{out}^{\circ} \tag{2}$$

We will ignore the inertial and internal energy of the surrounding atmosphere but assume that it maintains the fixed pressure p_∞.

Since we ignore any energy dissipation in the fluid and atmosphere, the total energy E_c will be used to transfer to the fluid the kinetic energy K and produce the work against external pressure. For the total energy we accept the following formula:

$$E_c = \frac{4\pi}{3}(R_{out}^3 - R_{out}^{\circ 3})p_\infty + K \tag{3}$$

Because of the incompressibility, we have the following mass balance condition:

$$R_{out}^3(t) - R_*^3(t) = R_{out}^{\circ 3} - R_*^{\circ 3} \tag{4}$$

Combining Eqs. 3 and 4, we get

$$E_c = \frac{4\pi}{3}(R_*^3(t) - R_*^{\circ 3})p_\infty + K \tag{5}$$

Because we consider a point explosion, any expanded gas pressure on the inner wall of the crater becomes zero as soon as $R_*(t) > 0$, due to the infinite relative expansion of the explosive gas. Thus, we get Eq. 5 in which no work is applied to the cavity interior for $t > 0$.

Consider a radially symmetric flow of incompressible fluid with the radial velocity $v(r,t)$ and pressure distribution $p(r,t)$. These two functions obey the following system of partial differential equations:

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} \tag{6}$$

and

$$\frac{\partial v}{\partial r} + \frac{2}{r}v = 0 \tag{7}$$

which express momentum and mass conservation, respectively.

This pair allows the following solution:

$$v(r,t) = \frac{A(t)}{r^2} \tag{8}$$

and

$$\frac{p(r,t)}{\rho} = \frac{1}{r}\frac{dA}{dt} - \frac{1}{2}\frac{A^2}{r^4} + B(t) \tag{9}$$

where $A(t)$ and $B(t)$ are functions of time that should be determined from the initial and boundary conditions. Equation 8 follows immediately from the continuity of Eq. 7. To get Eq. 9, expression (7) of the velocity $v(r,t)$ should be inserted into Eq. 6 and integrated once, subject to the boundary condition

$$p(R_*) = 0 \tag{10}$$

Combining Eqs. 9 and 10, we arrive at the relationship

$$B(t) = -\frac{1}{R_*}\frac{dA}{dt} + \frac{1}{2}\frac{A^2}{R_*^4} \tag{11}$$

Combining Eqs. 9 and 11, we arrive at the relationship

$$\frac{p(r,t)}{\rho} = -\frac{r - R_*}{rR_*}\frac{dA}{dt} + \frac{A^2}{2}\frac{r^4 - R_*^4}{r^4 R_*^4} \tag{12}$$

The total kinetic energy K of the hollow sphere of incompressible fluid,

(R_*, R_{out}), is equal to

$$K = \frac{\rho}{2}\int_{R_*(t)}^{R_{out}(t)} dr\, 4\pi r^2 v^2(r,t) \tag{13}$$

Inserting the solution, delivered by Eq. 8, into Eq. 13, we get

$$K = \frac{\rho A^2}{2}\int_{R_*(t)}^{R_{out}(t)} dr\, 4\pi r^{-2} = 2\pi\rho\frac{R_{out} - R_*}{R_{out}R_*}A^2 \tag{14}$$

The pressures at $r = R^*$ and $r = R_{out}$ should be equal to 0 and p_∞, respectively. Thus, Eq. 9 implies the equations

$$\frac{1}{R_*}\frac{dA}{dt} - \frac{1}{2}\frac{A^2}{R_*^4} + B(t) = 0 \tag{15}$$

$$\frac{1}{R_{out}}\frac{dA}{dt}-\frac{1}{2}\frac{A^2}{R_{out}^4}+B(t)=p_\infty \tag{16}$$

The energy relationship in Eq. 5 now reads

$$E_c=2\pi\rho\frac{R_{out}-R_*}{R_*R_{out}}A^2+\frac{4\pi}{3}p_\infty(R_*^3(t)-R_*^{\circ 3}) \tag{17}$$

Using Eq. 4, let us introduce function $W(R_*)$ such that

$$W(R_*)\equiv(R_*^3+R_{out}^{\circ 3}-R_*^{\circ 3})^{\frac{1}{3}}=R_{out}(R_*) \tag{18}$$

Differentiating Eq. 18, we get

$$W_{R^*}(R_*)=\frac{\partial W(R_*)}{\partial R_*}=\frac{R_*^2}{W^2(R_*)} \tag{19}$$

as implied by the following chain:

$$W_{R^*}(R_*)=\frac{\partial W(R_*)}{\partial R_*}=\frac{1}{3}\frac{1}{(R_*^3+R_{out}^{\circ 3}-R_*^{\circ 3})^{\frac{2}{3}}}\frac{\partial R_*^3}{\partial R_*}=$$
$$\frac{R_*^2}{(R_*^3+R_{out}^{\circ 3}-R_*^{\circ 3})^{\frac{2}{3}}}=\frac{R_*^2}{W^2(R_*)}$$

Equation 19 appears to be useful in our future calculations.

The boundary “explosive/fluid” moves with a velocity equal to that of the fluid particles located at $r=R_*$. Therefore, for the velocity C_* of this boundary, we get the following kinematic equation:

$$C_*(t)=v(R_*,t) \tag{20}$$

Because of the radial symmetry of the solution under study, we get

$$C_*=\frac{dR_*(t)}{dt} \tag{21}$$

Combining Eqs. 20 and 21, we get the equation

$$\frac{dR_*(t)}{dt} = v(R_*, t) \tag{22}$$

Inserting the solution described by Eq. 8 into Eq. 22, we get

$$\frac{dR_*(t)}{dt} = \frac{A(t)}{R_*^2} \tag{23}$$

or

$$A(t) = R_*^2 \frac{dR_*(t)}{dt} \tag{24}$$

Resolving Eq. 17 to isolate A, we get

$$\begin{gathered} A = Q(R_*), \text{ where} \\ Q(R_*) \equiv \sqrt{\frac{E_c - p_\infty (R_*^3 - R_*^{\circ 3}) 4\pi / 3}{2\pi\rho} \frac{R_* W(R_*)}{W(R_*) - R_*}} \end{gathered} \tag{25}$$

Inserting Eq. 25 into Eq. 23, we arrive at the following strongly nonlinear ordinary differential equation:

$$\frac{dR_*(t)}{dt} = \frac{1}{R_*^2} Q(R_*) \tag{26}$$

Or in full:

$$\frac{dR_*(t)}{dt} = \sqrt{\frac{E_c - p_\infty (R_*^3 - R_*^{\circ 3}) 4\pi / 3}{2\pi\rho} \frac{R_*^{-3} R_{out}}{R_{out} - R_*}} \tag{27}$$

Using the function $W(R_*)$, defined by Eq. 18, we can rewrite Eq. 27 in the following form:

$$\frac{dR_*(t)}{dt} = \sqrt{\frac{E_c - p_\infty (R_*^3 - R_*^{\circ 3}) 4\pi / 3}{2\pi\rho} \frac{R_*^{-3} W(R_*)}{W(R_*) - R_*}} \tag{28}$$

or in the full explicit form as

$$\frac{dR_*(t)}{dt} = \sqrt{\frac{E_c - p_\infty(R_*^3 - R_*^{\circ 3})4\pi / 3}{2\pi\rho} \frac{R_*^{-3}\sqrt[3]{R_*^3 + R_{out}^{\circ 3} - R_*^{\circ 3}}}{\sqrt[3]{R_*^3 + R_{out}^{\circ 3} - R_*^{\circ 3}} - R_*}} \quad (29)$$

The ordinary differential Eq. 29 should be amended with the initial data of Eq. 1. We notice that the quantities E_c, p_∞, ρ, R_*°, R_{out}° are fixed boundary or initial conditions, fully specified in advance, and only R_* is a function of time to be determined. In this sense, it is expedient to use a shortcut

$$\frac{dR_*(t)}{dt} = G(R_*) \quad (30)$$

where

$$G(R_*) \equiv \sqrt{\frac{E_c - p_\infty(R_*^3 - R_*^{\circ 3})4\pi / 3}{2\pi\rho} \frac{R_*^{-3}\sqrt[3]{R_*^3 + R_{out}^{\circ 3} - R_*^{\circ 3}}}{\sqrt[3]{R_*^3 + R_{out}^{\circ 3} - R_*^{\circ 3}} - R_*}} \quad (31)$$

By setting Eq. 31 to zero and solving for R_*, the maximum expansion of the crater, $R_{*\max}$, may be determined as $R_{*\max} = (R_*^{\circ 3} - 3E_c / 4\pi p_\infty)^{1/3}$. This represents the expansion at which the exterior pressure, p_∞, performs an amount of work sufficient to overcome the energy E_c that was dumped into the system at $t = 0$. Using Eqs. 24, 30, and 31, we present the function $A(t)$ in the following form:

$$A = R_*^2 G(R_*) \quad (32)$$

Combining Eqs. 12 and 32, we arrive at the following formula for the pressure distribution:

$$\frac{p(r,t)}{\rho} = \frac{R_* - r}{rR_*} \frac{d(R_*^2 G(R_*))}{dR_*} G(R_*) - \frac{1}{2} \frac{R_*^4 - r^4}{r^4} G^2(R_*) \quad (33)$$

An application of Eq. 33 was performed to verify the expected result. For this example, the fluid was taken to occupy a range of r, from $R_*^\circ = 1$ to $R_{out}^\circ = 100$. The exterior pressure, p_∞ / ρ, was set at a value of unity, while the initial energy dump into the system, E_c / ρ, was set to a value of 20. The result is shown in Fig.

2. There, each of the five curves represents the pressure profile in the fluid at different levels of expansion (i.e., different values of R_*), ranging from the initial to the final state of expansion. The legend notes the value of dR_* / dt associated with each level of expansion.

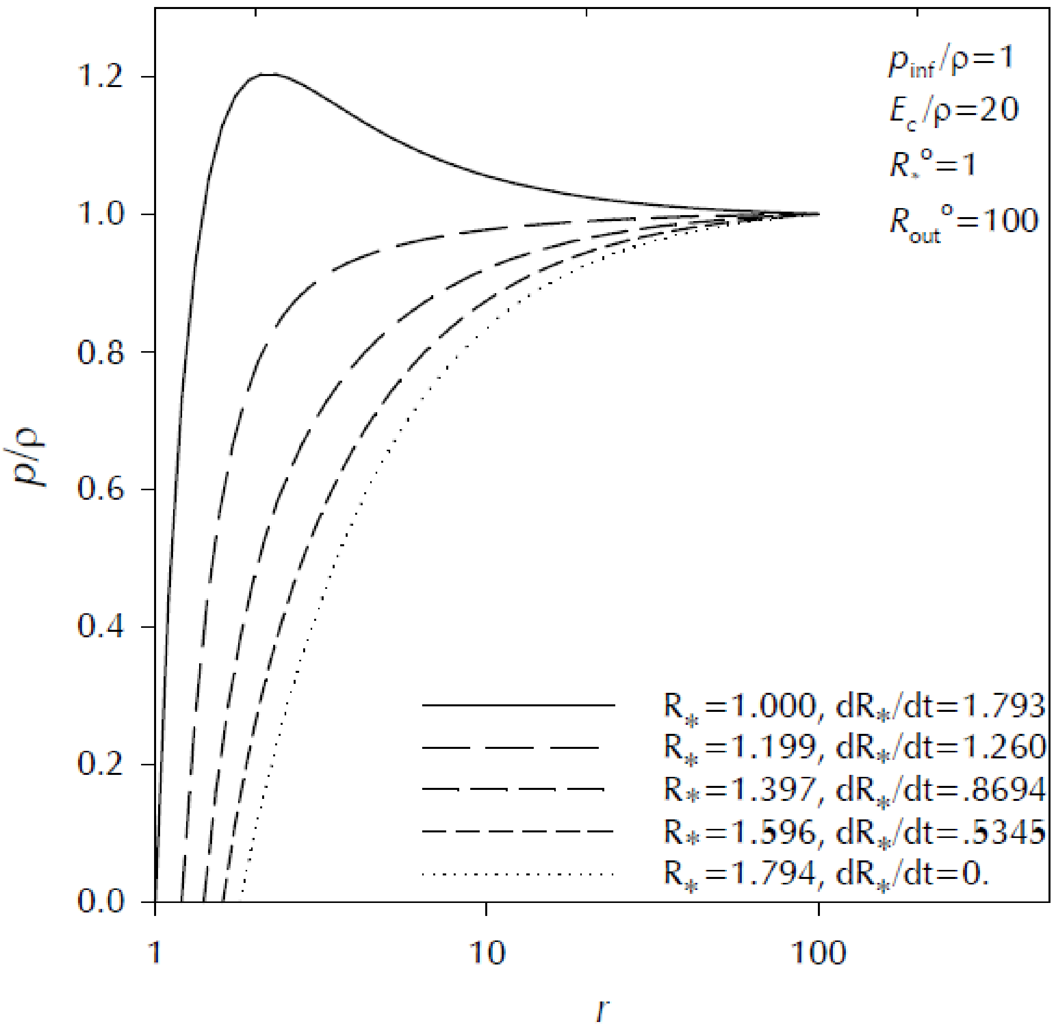

Fig. 2 Pressure as function of radius

We see that, in all cases, the outer boundary pressure matches the p_∞ condition. Note that while $p(R_{out}, t) = p_\infty$ was never guaranteed by way of an explicit pressure boundary condition in the form similar to Eq. 10 leading to Eq. 11, it was nonetheless assured by way of the energy equation, Eq. 3, which can only hold true when the outer boundary is subjected to a pressure of p_∞.

3. Explosion in Unbounded Fluid

When the exterior pressure vanishes, $p_\infty = 0$, Eq. 31 implies

$$G(R_*) \equiv \sqrt{\frac{K}{2\pi\rho} \frac{R_*^{-3} \sqrt[3]{R_*^3 + R_{out}^{\circ 3} - R_*^{\circ 3}}}{\sqrt[3]{R_*^3 + R_{out}^{\circ 3} - R_*^{\circ 3}} - R_*}} \tag{34}$$

In Eq. 34 we replaced the full energy E_c with the kinetic energy K, because no work is being done on the fluid for $t > 0$, and so all the energy remains kinetic.

When the fluid is unbounded, we get $R_{out}^{\circ} = \infty$ and Eq. 34 formally implies

$$G(R_*) \equiv \sqrt{\frac{K}{2\pi\rho}} R_*^{-3/2} \tag{35}$$

Using Eq. 35, we can rewrite Eq. 30 as follows:

$$\frac{dR_*(t)}{dt} = \sqrt{\frac{K}{2\pi\rho}} R_*^{-3/2} \tag{36}$$

Integrating Eq. 36 with the initial data $R_*(0) = R_*^{\circ}$, we arrive at the relationship

$$R_*(t) = \left[R_*^{\circ \frac{5}{2}} + \left(\frac{25}{8} \frac{K}{\pi\rho} \right)^{\frac{1}{2}} t \right]^{\frac{2}{5}} \tag{37}$$

Using Eq. 35, we can rewrite Eq. 33 as

$$p(r,t) = \frac{K}{4\pi} \frac{r^3 - R_*^3(t)}{r^4 R_*^2(t)} \tag{38}$$

as implied by the following chain:

$$\frac{p(r,t)}{\rho}=\frac{R_*-r}{rR_*}\frac{\partial(R_*^2G(R_*))}{\partial R_*}G(R_*)-\frac{1}{2}\frac{R_*^4-r^4}{r^4}G^2(R_*)\rightarrow$$

$$\frac{p(r,t)}{\rho}=\frac{R_*-r}{rR_*}\sqrt{\frac{K}{2\pi\rho}}\frac{1}{\sqrt{R_*^3}}\frac{\partial}{\partial R_*}\left(\sqrt{\frac{K}{2\pi\rho}}\sqrt{R_*}\right)-\frac{1}{2}\frac{R_*^4-r^4}{r^4}\frac{K}{2\pi\rho}\frac{1}{R_*^3}\rightarrow$$

$$\frac{4\pi\rho}{K}\frac{p(r,t)}{\rho}=\frac{R_*-r}{r}\frac{1}{R_*^3}-\frac{R_*^4-r^4}{r^4}\frac{1}{R_*^3}\rightarrow$$

$$p(r,t)=\frac{K}{4\pi}\left(\frac{R_*}{r}-\frac{R_*^4}{r^4}\right)\frac{1}{R_*^3}\rightarrow p(r,t)=\frac{K}{4\pi}\frac{r^3-R_*^3(t)}{r^4R_*^2(t)}$$

Substituting Eq. 37 in 38, we find that the pressure in the liquid is described by the relationship

$$p(r,t)=\frac{K}{4\pi}\frac{r^3-\left[\sqrt{R_*^{\circ 5}}+\sqrt{\frac{25}{8}\frac{K}{\pi\rho}}\,t\right]^{\frac{6}{5}}}{r^4\left[\sqrt{R_*^{\circ 5}}+\sqrt{\frac{25}{8}\frac{K}{\pi\rho}}\,t\right]^{\frac{4}{5}}} \tag{39}$$

At $R_*^\circ=0$, the relationship (37) reduces to the classical relationship (Sedov 1993):

$$R_*(t)=\left(\frac{25}{8}\frac{K}{\pi\rho}\right)^{\frac{1}{5}}t^{\frac{2}{5}} \tag{40}$$

At $R_*^\circ=0$, the formula (38) of pressure distribution becomes

$$p(r,t)=\frac{K}{4\pi}\frac{r^3-\left(\frac{25}{8}\frac{K}{\pi\rho}\right)^{\frac{3}{5}}t^{\frac{6}{5}}}{r^4\left(\frac{25}{8}\frac{K}{\pi\rho}\right)^{\frac{2}{5}}t^{\frac{4}{5}}} \tag{41}$$

4. Conclusion

We studied the spherically symmetric problem of a point explosion in the center of a spherical incompressible liquid. The explosion is caused by the instantaneous release of energy E_c. The solution to the problem reduces to solving the system of algebraic and ordinary differential equations. The current radius of the cavity $R_*(t)$ is defined by the ordinary differential Eq. 30, in which the right-hand side G is defined by Eq. 31. After determining the function $R_*(t)$, the pressure distribution function $p(r,t)$ can be determined with the help of Eq. 33.

When the outer pressure p_∞ vanishes, Eq. 30 reduces to a much simpler equation, which can be solved explicitly in the case of an unbounded liquid. In this case, the radius $R_*(t)$ and the pressure distributions $p(r,t)$ are given by Eqs. 37 and 39, respectively. These relationships generalize the classical self-similar problem of a point instantaneous explosion inside unbounded incompressible liquid for the case of a finite size of the initial cavity. At last, when the initial radius of the blast cavity R_*° approaches zero, the relationships for the radius $R_*(t)$ and pressure distribution $p(r,t)$ are given by Eqs. 40 and 41, respectively. Equation 40 agrees with the result for the self-similar solution. At $R_*^\circ \neq 0$, the solutions (37) and (39) cease to be self-similar solutions. In general, rather complex Eqs. 30 and 31 require the use of computer-based numerical analysis. The exact solutions (37) and (39) provide tools for the purpose of code verification.

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