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Verifying the Outer Boundary Condition of the Point Explosion

by Steven B Segletes

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14. ABSTRACT
In this technical note, an explicit verification is offered for the outer boundary pressure condition in an incompressible fluid shell that is subjected to an inner point-explosion.

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1. Introduction

In the recent report, ARL-TR-9247,[1] the classical concept of point explosion in an incompressible fluid was revisited, but generalized to include a nonzero-sized initial crater ($r = R_*^\circ$) and a finite-sized outer dimension to the incompressible fluid shell ($r = R_{\mathrm{out}}^\circ$). The boundary conditions associated with the problem are that the pressure on the inner crater wall, for time $t > 0$, will be zero, as the point explosion converts its internal energy into a purely kinetic response at $t = 0$, and the crater expands under its own inertia, subject to the outer boundary condition.

The outer boundary condition is that of a reservoir at fixed pressure, p_∞. The reservoir is assumed to have no inertial characteristics. The resulting kinematic motion is therefore one in which the purely radial inertial flow of the incompressible fluid shell is monotonically diminishing under the work applied by the reservoir, until such time that the expansion halts. The sample result of ARL-TR-9247 is presented again in Fig. 1, showing the pressure distribution in the fluid shell, at different levels of crater expansion, ranging from the initial state to that of maximum expansion.

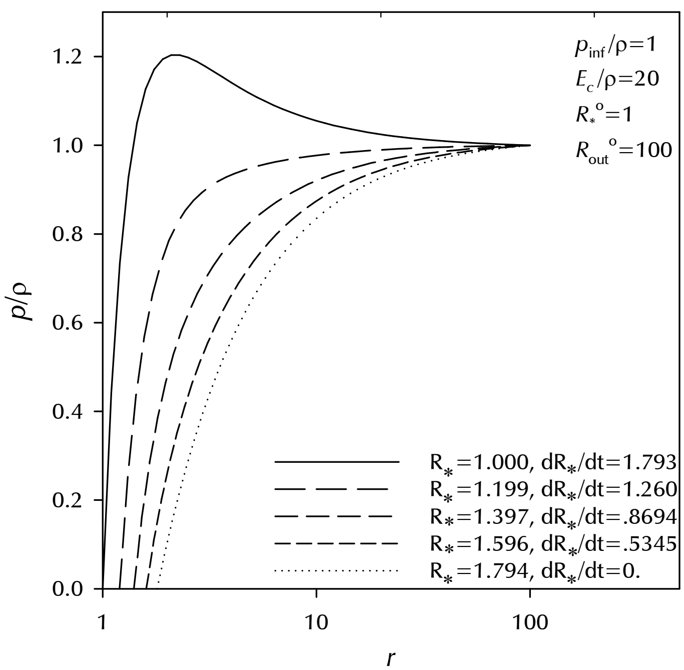

Fig. 1 Pressure as a function of radius for the sample problem described in ARL-TR-9247

As the model advertised, the pressure at the crater boundary was indeed ever $p(R_*(t), t) = 0$ and at the outer extent of the fluid shell, $p(R_{\mathrm{out}}(t), t) = p_\infty$.

However, while the zero crater-wall pressure was explicitly built in to the integration constant of the model, the outer-boundary pressure constraint at $R_{\text{out}}(t)$ was never analytically verified, even though Fig. 1 clearly shows that it is indeed assured.

In ARL-TR-9247, it was argued (but not proven) that the energy equation that was built in to the model could "only hold true when the outer boundary is subjected to a pressure of p_∞". The argument was tacit but direct: taking the derivative of the energy equation (Eq. 3 in ARL-TR-9247) with respect to the outer shell radius R_{out}, the change in kinetic energy per unit change of outer shell radius, dK/dR_{out} is $-4\pi R_{\text{out}}^2\, p_\infty$. This loss of kinetic energy arises from the work accomplished by the reservoir pressure—the derivative represents a force acting on the outer surface of the shell, whose area is at any instant $4\pi R_{\text{out}}^2(t)$. Thus, the force magnitude, being delivered as a pressure per unit area, *implies* that the pressure delivering the force must be p_∞. This argument is valid—however, it is an *implicit* argument.

To confirm, for example, that the subsequent derivations of ARL-TR-9247 were, in fact, correct, one would desire to show *explicitly* that the pressure in the shell, at $r = R_{\text{out}}(t)$ exactly matches the p_∞ value implied by the energy balance. So as not to divert the reader of the original work with a sidetracking derivation, ARL-TR-9247 left the assertion of the p_∞ boundary condition at $R_{\text{out}}(t)$ implicit.

The purpose of this technical note is to provide one such *explicit* derivation, showing that the pressure condition on the external boundary of the fluid shell can, indeed, be analytically verified as $p(R_{\text{out}}(t), t) = p_\infty$.

Henceforth, equations that point directly to those in ARL-TR-9247 are labeled with a preceding "PE" (for "point explosion"), to distinguish the equation from those derived here and specifically identify their origin in the original work. Thus, Eq. PE24 specifically refers, for example, to an equation that appeared as Eq. 24 in ARL-TR-9247.

2. Re-expressing dA/dt in terms of A^2

We already know the following equation from ARL-TR-9247[1]:

$$A(t) = R_*^2 \frac{dR_*}{dt} \quad , \tag{PE24}$$

where $R_* = R_*(t)$ is the time-dependent radius of the crater that is acted upon by the point explosion. The $A(t)$ term is the time-dependent portion of the velocity solution v to the point explosion, namely,

$$v(r,t) = \frac{A(t)}{r^2} \quad . \tag{PE8}$$

Apply the chain rule to the time-derivative of A^2 in order to express it in terms of $R_*(t)$, rather than in terms of t directly. Doing so leads to

$$\frac{d}{dt}(A^2) = \frac{d}{dR_*}(A^2)\,\frac{dR_*}{dt} \quad . \tag{1}$$

Equation 1 may be further expanded through the following chain, making use of Eq. PE24:

$$\begin{aligned} \frac{d}{dt}(A^2) &= \frac{d}{dR_*}(A^2)\,\frac{dR_*}{dt} \\ 2A\frac{dA}{dt} &= \\ 2\left(R_*^2\frac{dR_*}{dt}\right)\frac{dA}{dt} &= \\ \frac{dA}{dt} &= \frac{1}{2R_*^2}\frac{d}{dR_*}(A^2) \quad . \end{aligned}$$

Thus, we accomplish our goal of expressing dA/dt in terms of A^2 (and $R_*(t)$):

$$\frac{dA}{dt} = \frac{1}{2R_*^2}\frac{d}{dR_*}(A^2) \quad . \tag{2}$$

3. Pressure Equation in terms of A^2

Equation 2 is particularly useful because the pressure, given in the point-explosion report as

$$\frac{p(r,t)}{\rho} = -\frac{r - R_*}{rR_*}\frac{dA}{dt} + \frac{A^2}{2}\frac{r^4 - R_*^4}{r^4 R_*^4} \quad , \tag{PE12}$$

can now be expressed in terms that always appear as A^2, which becomes our new working variable in time (we see later that A^2 contains no square roots, which is another fortuitous development in expediting the calculation). That the pressure is always expressible in terms of A^2 and its derivative may be seen by substituting Eq. 2 into Eq. PE12, obtaining

$$\frac{p(r,t)}{\rho} = -\frac{1}{2}\frac{r - R_*}{rR_*^3}\frac{d}{dR_*}(A^2) + \frac{A^2}{2}\frac{r^4 - R_*^4}{r^4 R_*^4} \quad . \tag{3}$$

Through Eq. 3, the inner $p(R_*, t) = 0$ boundary condition may be immediately confirmed. However, it is also through Eq. 3 that we prove the outer boundary condition $p(R_{\text{out}}, t) = p_\infty$.

4. Expressing A^2 in terms of R_*, while Retaining $W(R_*)$ Terms

In the point-explosion report, ARL-TR-9247,

$$A = Q(R_*) = \sqrt{\frac{E_c - p_\infty(R_*^3 - R_*^{\circ 3})4\pi/3}{2\pi\rho} \cdot \frac{R_* W}{W - R_*}} \quad , \tag{PE25}$$

where W is the outer radius of the fluid expressed as a function of the crater radius ($W = W(R_*)$) and thus corresponding exactly to the outer fluid radius R_{out}, which is expressed as a function of time t ($\therefore\ W(R_*) \equiv R_{\text{out}}(t)$). From Eq. PE25, we may thus express

$$A^2 = Q^2(R_*) = MN \quad , \tag{4}$$

where

$$M(R_*) = \frac{E_c - p_\infty(R_*^3 - R_*^{\circ 3})4\pi/3}{2\pi\rho} \tag{5}$$

and

$$N(R_*) = \frac{R_* W}{W - R_*} \quad . \tag{6}$$

5. Derivatives of M and N with Respect to R_*

The derivative of M may be directly obtained as

$$\frac{dM}{dR_*} = -2\frac{p_\infty}{\rho}R_*^2 \quad . \tag{7}$$

Let us recite Eq. PE19 from ARL-TR-9247:

$$\frac{dW}{dR_*} = \left(\frac{R_*}{W}\right)^2 \quad . \tag{PE19}$$

Now, let us calculate the derivative of N, making use of Eq. PE19:

$$\begin{aligned}
\frac{dN}{dR_*} &= \frac{(W - R_*)\left(W + R_*\dfrac{dW}{dR_*}\right) - R_*W\left(\dfrac{dW}{dR_*} - 1\right)}{(W - R_*)^2} \\
&= \frac{(W - R_*)\left(\dfrac{W^3 + R_*^3}{W^2}\right) - R_*W\dfrac{R_*^2 - W^2}{W^2}}{(W - R_*)^2} \\
&= \frac{W^4 + WR_*^3 - W^3R_* - R_*^4 - WR_*^3 + W^3R_*}{W^2(W - R_*)^2} \\
\therefore \frac{dN}{dR_*} &= \frac{W^4 - R_*^4}{W^2(W - R_*)^2} \quad .
\end{aligned} \tag{8}$$

6. Verifying the Outer Boundary Condition, $p(R_{\text{out}}, t) = p_\infty$

We wish to evaluate the term $d(A^2)/dR_*$ that appears in the pressure Eq. 3. Therefore, let us take the derivative of Eq. 4 with respect to R_*:

$$\frac{d}{dR_*}(A^2) = M\frac{dN}{dR_*} + N\frac{dM}{dR_*} \quad . \tag{9}$$

Now, substitute Eqs. 7 and 8 into Eq. 9 to obtain

$$\frac{d}{dR_*}(A^2) = M\frac{W^4 - R_*^4}{W^2(W - R_*)^2} - 2N\frac{p_\infty}{\rho}R_*^2 \quad . \tag{10}$$

Substitute Eqs. 10 and 4 into the pressure Eq. 3 and evaluate at the outer fluid boundary $r = W(R_*) \equiv R_{\text{out}}(t)$ to obtain

$$\begin{aligned}\frac{p(R_{\text{out}}, t)}{\rho} &= -\frac{1}{2}\frac{W - R_*}{WR_*^3}\left(M\frac{W^4 - R_*^4}{W^2(W - R_*)^2} - 2N\frac{p_\infty}{\rho}R_*^2\right) + \frac{MN}{2}\frac{W^4 - R_*^4}{W^4R_*^4} \\ &= -\frac{M}{2}\frac{(W^4 - R_*^4)}{W^3R_*^3(W - R_*)} + N\frac{W - R_*}{WR_*}\frac{p_\infty}{\rho} + \frac{M}{2}\left(\frac{R_*W}{W - R_*}\right)\frac{W^4 - R_*^4}{W^4R_*^4} \quad .\end{aligned}$$

The last term, into which the definition of N, Eq. 6, was substituted, may be further reduced:

$$\frac{p(R_{\text{out}}, t)}{\rho} = -\frac{M}{2}\frac{W^4 - R_*^4}{W^3R_*^3(W - R_*)} + N\frac{W - R_*}{WR_*}\frac{p_\infty}{\rho} + \frac{M}{2}\frac{W^4 - R_*^4}{W^3R_*^3(W - R_*)} \quad .$$

The first and third terms on the right are seen to cancel. Further, the second term may also be reduced, through substitution of Eq. 6:

$$\frac{p(R_{\text{out}}, t)}{\rho} = N\frac{1}{N}\frac{p_\infty}{\rho} \quad .$$

Thus, we finally arrive at the result that

$$\frac{p(R_{\text{out}}, t)}{\rho} = \frac{p_\infty}{\rho} \quad , \tag{11}$$

thereby proving that the moving outer boundary of the incompressible fluid shell, located at $r = W(R_*) \equiv R_{\text{out}}(t)$, is always at the boundary-value pressure of $p(R_{\text{out}}(t), t) = p_\infty$. The value of this boundary pressure is independent of R_* and thus independent of time t.

7. Conclusion

In this report, an explicit derivation is offered to support an implicit assertion present in ARL-TR-9247—namely, that the pressure at the external boundary of the fluid shell that is subject to an internal point explosion remains fixed at a value equal to the pressure of the outer reservoir, $p(R_{\text{out}}(t), t) = p_\infty$. Whereas arguments concerning energy conservation were used to make the original implicit assertion, the current derivations explicitly evaluate the fluid pressure at the time-dependent radius of $R_{\text{out}}(t)$, corresponding to the moving outer boundary of the fluid shell.

8. References

1. Grinfeld MA, Segletes SB. Instantaneous point explosion in incompressible fluid-like media. DEVCOM Army Research Laboratory (US); 2021 Aug. Report No.: ARL-TR-9247.

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