



ARL-TR-9254 • AUG 2021



Mechanical Characterization of Advanced Ultra-High Molecular Weight Polyethylene (UHMWPE) Composites by Ultrasonic Method

by Stephen L Alexander and Tusit Weerasooriya

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Mechanical Characterization of Advanced Ultra-High Molecular Weight Polyethylene (UHMWPE) Composites by Ultrasonic Method

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REPORT DOCUMENTATION PAGE			Form Approved OMB No. 0704-0188		
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1. REPORT DATE (DD-MM-YYYY) August 2021		2. REPORT TYPE Technical Report		3. DATES COVERED (From - To) October 2019 – March 2021	
4. TITLE AND SUBTITLE Mechanical Characterization of Advanced Ultra-High Molecular Weight Polyethylene (UHMWPE) Composites by Ultrasonic Method			5a. CONTRACT NUMBER W911QX-16-D-0014		
			5b. GRANT NUMBER		
			5c. PROGRAM ELEMENT NUMBER		
6. AUTHOR(S) Stephen L Alexander and Tusit Weerasooriya			5d. PROJECT NUMBER		
			5e. TASK NUMBER		
			5f. WORK UNIT NUMBER		
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) DEVCOM Army Research Laboratory ATTN: FCDD-RLW-TB Aberdeen Proving Ground, MD 21005			8. PERFORMING ORGANIZATION REPORT NUMBER ARL-TR-9254		
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)			10. SPONSOR/MONITOR'S ACRONYM(S)		
			11. SPONSOR/MONITOR'S REPORT NUMBER(S)		
12. DISTRIBUTION/AVAILABILITY STATEMENT Approved for public release: distribution unlimited.					
13. SUPPLEMENTARY NOTES ORCID ID: Tusit Weerasooriya, 0000-0003-3299-2166					
14. ABSTRACT Advanced ultra-high molecular weight polyethylene (UHMWPE) composites may offer breakthroughs in terms of increasing ballistic performance with less back-surface deformation, while decreasing cost. An example is Tensylon, which is a solid-state-extruded (SSE) UHMWPE film used to fabricate film-based composites. Researchers aiming to operationalize these advances into better warfighter protection through physics-based modeling and simulation of the deformation and failure of material in the target interaction zone require mesostructure-inspired and mechanism-based (MIMB) material (deformation and failure) models. Here, as a first step toward obtaining deformation and failure models, a direct contact ultrasonic method is used to provide quasi transversely isotropic elastic parameters of novel fiber- and film-based UHMWPE composite materials. The technique accounted for the anisotropy of the material, providing the full matrix of elastic constants, the anisotropic moduli, and Poisson's ratios, which are a challenge or almost impossible to obtain through traditional mechanical experimentation. The physical principles involved are thoroughly reviewed, discussed, and analyzed, thereby providing a reference report for future researchers who could apply the methodology to other complex classes of innovative anisotropic Army-relevant materials.					
15. SUBJECT TERMS UHMWPE composites; Tensylon; DSM Protective Materials; Dyneema; SSE films; ultrasonic measurement of elastic constants; Kelvin–Christoffel equation; elastic constants; orthotropic; tetragonal; UHMWPE panels; UHMWPE ply; UHMWPE ballistic performance					
16. SECURITY CLASSIFICATION OF:			17. LIMITATION UU	18. NUMBER OF PAGES 105	19a. NAME OF RESPONSIBLE PERSON Tusit Weerasooriya
a. REPORT Unclassified	b. ABSTRACT Unclassified	c. THIS PAGE Unclassified			19b. TELEPHONE NUMBER (Include area code) 410-652-9450

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Acknowledgments

The authors acknowledge the Weapons and Materials Research Directorate (WMRD) of the US Army Combat Capabilities Development Command Army Research Laboratory for funding as a WMRD 6.1 Research Project. Authors would also like to acknowledge Micah Gallagher, Michael Neblett, Fred Racine, and Michael Thompson from WMRD for laying up and pressing the ultra-high molecular weight polyethylene panels, water-jet cutting the specimens, and general assistance with the materials. Authors also thank Daniel O'Brien of WMRD, who graciously always made himself available to discuss and assist with background information, most notably with material processing. Finally, authors acknowledge Ross Rozansky of DSM Protective Materials and Bryce VanArsdalen of DuPont for providing materials and feedback as needed.

1. Introduction

Ultra-high molecular weight polyethylene (UHMWPE) is an advanced material used in composites designed for personnel protection as soft armor. These composites are formed by consolidating a set of plies. Traditionally, plies used to form these composites have consisted of gel-spun polyethylene fibers arranged in a single direction embedded in a matrix material. For example, a common UHMWPE composite material is the HB Dyneema series from DSM Protective Materials (Heerleen, Netherlands), such as HB26, HB50, HB80, HB212, etc. Of this list, HB26, HB50, and HB80 are all constructed with UHMWPE fibers called SK76, whereas HB212 uses newer SK99 fibers. Fibers are either embedded in a polyurethane matrix (in HB26 and HB80) or a softer rubber matrix (HB50 and HB212). Other common examples of UHMWPE fiber composites are the SpectraShield series from Honeywell International Inc (Colonial Heights, Virginia, USA).

A potential breakthrough in UHMWPE composites is the use of solid-state-extruded (SSE) films. These films do not have fibers, in contrast to the examples cited above. Commercial examples of SSE films include Tensylon from DuPont (Delaware, USA), Endumax from Teijin Aramid (Arnhem, Netherlands), and BT10 from DSM Protective Materials. These films may offer significant improvements in ballistic performance over fiber-based composites as measured by backface deflection (BFD) and velocity of penetration (V50) (Freitas et al. 2014). Moreover, Singletary and Lauke (2016) note that with optimized manufacturing, “SSE-PE film-reinforced composites can achieve 10%–15% higher V50 than the PE fiber-reinforced composite benchmark molded under similar conditions.” In addition to improved ballistic performance, SSE films also provide significant cost savings. SSE films can be manufactured without the use of solvent, which could provide a cost savings over fiber-based composites by a theoretical factor of 5 (Mrozek 2019). Reddy et al. (2020) constructed 20-mm panels made from either Tensylon film or Dyneema HB50 made from SK76 fibers and noted that the cost of the Tensylon panels was less than 60% the cost of the Dyneema panels.

Successfully exploiting the potential advantage of these state-of-the-art UHMWPE materials in Soldier protection requires an understanding of their deformation and failure mechanisms. The process of exploitation of these materials for protection is greatly aided by modeling and simulation, which relies on accurate material properties at the different length scales. In smaller-scale simulations, the composite may be able to be modeled at the scale of the plies or even at the microscale of individual fibers or films. However, it is often not feasible to include such detailed lower-scale refinements in larger simulations of protective equipment consisting of

multiple composites, which may also need to model the interaction with the biological components of the Soldier to quantify the relevant injuries. Larger system-level simulations require macroscale properties at the continuum length scale of the entire UHMWPE panel.

The macroscale properties of soft armor panels required for system-level simulations can be difficult to determine with individual experiments. As noted in a review by van der Werff and Heisserer (2016, p. 85), both affiliated with DSM, one challenge is the high strength and stiffness combined with low coefficient of friction. A particular example is tensile strength. Tension experiments of the panel need to apply load to the entire thickness of the panel (all plies). The DSM authors note that this is difficult in Dyneema panels due to the low interlaminar shear strength, which hinders transfer of load through the combined thickness and causes failure near the point of contact with the grips or slipping at the grips.

An alternative method to derive the elastic constants of anisotropic, heterogeneous materials such as composites is by use of ultrasonic characterization. Ultrasonic material testing is a branch of nondestructive evaluation that involves transmitting waves at higher-than-audible frequencies into a test specimen and then measuring the received pulse. These measurements can be used to calculate material properties because the speed of the ultrasonic wave depends on the properties of the material through which it passes, such as density and stiffness. For example, there is a large body of work on ultrasonic measurement of the elastic properties of biological tissues and materials such as the femur (e.g., Van Buskirk et al. 1981; Ashman 1982), skull (Peterson and Dechow 2002), teeth (Ghorayeb et al. 2013), wood (Gonçalves et al. 2014), and bioengineered scaffolds (Kohlhauser 2009).

Of particular interest to the present study is the use of ultrasonic velocity measurements to quantify the macroscopic properties of fiber-reinforced composites. For example, various ultrasonic techniques have been developed over half a century for unidirectional (UD) carbon fiber-reinforced plastics (CFRP), as reviewed by Paterson et al. (2018). However, ultrasonic measurement of the elastic constants of composites has not been limited to UD materials or to carbon fiber reinforcements. Ultrasonic methods have also been used to measure the elastic constants of composites reinforced with other types of fibers, such as glass (Zimmer and Cost 1970; Dandekar et al. 1998; Mistou and Karama 2000) and graphite (Kriz and Stinchcomb 1979; Pearson and Murri 1987; Lhermitte and Perrin 1991). Moreover, some of these studies have included composites constructed of 0°/90° cross plies and weaves (Lhermitte and Perrin 1991; Dandekar et al. 1998; Mistou and Karama 2000).

Measuring all of the elastic constants of anisotropic materials, such as composites, by ultrasonic method requires waves to be transmitted through the material in various directions. In their recent review, Paterson et al. (2018) noted that two different techniques have been used to transmit the waves in the necessary directions. For all cases, waves are transmitted and received using transducers, which transfer electrical signals to mechanical waves and back to electric signals. One method for measuring the wave speed involves immersing both the composite and the transducers in a liquid, such as water, and then rotating the specimen. The special transducers used in this method are known as immersion transducers. The second method requires coupons to be cut out from the composite at the predetermined required orientations. Then, the transducers are directly coupled to these coupons via a thin layer of coupling fluid, without immersion. Therefore, this method is known as the direct contact method, and the special transducers used are classified as contact transducers.

The present study used the direct contact method to measure the ultrasonic wavespeed through UHMWPE panels in order to calculate the effective elastic constants of the panels for use in future modeling and simulation efforts. Five different panels were constructed from different UHMWPE materials. The materials consisted of novel fiber-based composites and SSE films. In addition, a legacy fiber-based composite was also characterized to compare the mechanical properties measured by the present ultrasonic technique with previously published properties derived through other experimental methods.

2. Review of Physical Principles and Terminology

This section presents an overview of the physical principles used to measure the speed of sound in the UHMWPE panels and relate these speeds to the material properties of the panels. In addition, several key notations and terminology used in this study are explicitly defined here to avoid ambiguity, since it is assumed that readers are from various scientific backgrounds where terminology may have different connotations. Here, only those portions of the concepts are given that are necessary to provide the requisite background for the methodology used in the current study. References are supplied for more systematic presentations of the physical principles.

2.1 Construction of UHMWPE Composites: Introduction of Coordinate System and Vector Notation

A UD ply is defined here as a single ply consisting of UHMWPE fibers all aligned in the same orientation, referred to as the orientation of the ply. For the case of SSE

films that lack fibers, the UD ply is defined as a single ply consisting of the UHMWPE mechanical properties highly oriented in a single direction: the ply orientation.

UHMWPE material came from the manufacturer in rolls. The roll material will be referred to here as precursors. Usually, precursors consisted of a few UD plies stacked over each other in alternating, defined orientations and preconsolidated into a single sheet. Information regarding the orientation of the plies relative to each other within the precursor was supplied by the manufacturer. For example, one particular precursor configuration consisted of two plies stacked on top of each other with ply orientations at right angles. This type of configuration was referred to as a $[0^\circ/90^\circ]$ layup.

Panels were constructed by stacking many precursor sheets over each other. Therefore, the panel had three unique directions: the 0° direction, 90° direction, and the stacking direction. The coordinate system used in this study was defined with the 1-direction in the 0° direction, the 2-direction in the 90° direction, and the 3-direction in the stacking direction as shown in Fig. 1.* Vectors were written based on this coordinate system using the notation summarized in Table 1.

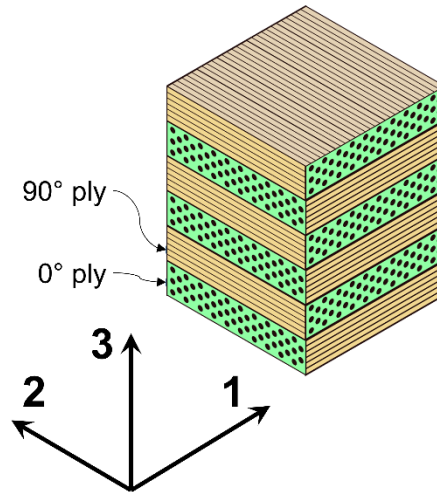


Fig. 1 Definition of coordinate system, with plies stacked in the 3-direction. Here, a ply is defined as a single UD layer.

* The reader should be aware that some of the literature on composites uses an alternative coordinate system where the stacking direction is defined as the 1-direction and the ply lies in the plane spanned by the 2- and 3-directions (e.g., Nguyen et al. 2016).

Table 1 Vector notation with representative examples

Vector Notation	Description
[1,0,0]	+1-direction (0° fiber direction)
[0,1,0]	+2-direction (90° fiber direction)
[0,0,1]	+3-direction (ply stacking direction). Panels had a nominal thickness of 1 inch in the 3-direction.
[1,1,0]	Referred to as the 1-2 direction. Lies in the 1-2 plane and is 45° to both the +1 and +2-directions.
[-1,1,0]	Referred to as the $\bar{1}$ -2 direction, where overbar indicates negative direction in the present study. Lies in the 1-2 plane and is 45° to both the -1 and +2-directions (180° in opposite direction of the 1-2 direction).
[0,1,1]	Referred to as the 2-3 direction. Lies in the 2-3 plane and is 45° to both the +2 and +3-directions.
[1,0,1]	Referred to as the 1-3 direction. Lies in the 1-3 plane and is 45° to both the +1 and +3-directions.

2.2 Material Symmetries

The elastic properties such as the stiffness of a material generally vary from point to point within the material. The values of these properties depend on how the material is oriented when it is loaded. However, many engineering materials have symmetry in their elastic properties. A plane of elastic symmetry exists when the elastic properties show reflective symmetry along the plane. For example, a slab of rebar-reinforced concrete has reflective symmetry along the axis of the rebar, whereas a UD fiber composite has reflective symmetry along the axis of the fibers.

Engineering materials can be classified according to the number of planes of symmetry, partly based on the classification system developed for crystal structures. In crystallography, there are 32 different crystal classes divided into 7 crystal systems, which are summarized in Fig. 2.

The systems of relevance to the present study are shown in Table 2, which is ordered by an increasing amount of symmetry. The highest degree of symmetry is termed isotropic and corresponds to the case where the material has the same properties in every direction. Anisotropy refers to deviation from the isotropic ideal case, with the completely anisotropic case referring to a material that has different properties in each direction. For a more formal presentation, the reader is referred to a thorough tutorial by NASA (Nemeth 2011, p. 197) and the many publications on the symmetry classes and systems of elastic crystals (e.g., Mason 1958, p. 380; Neighbors and Schacher 1967; Schreiber et al. 1973, p. 15).

Seven Primitive Crystal Systems

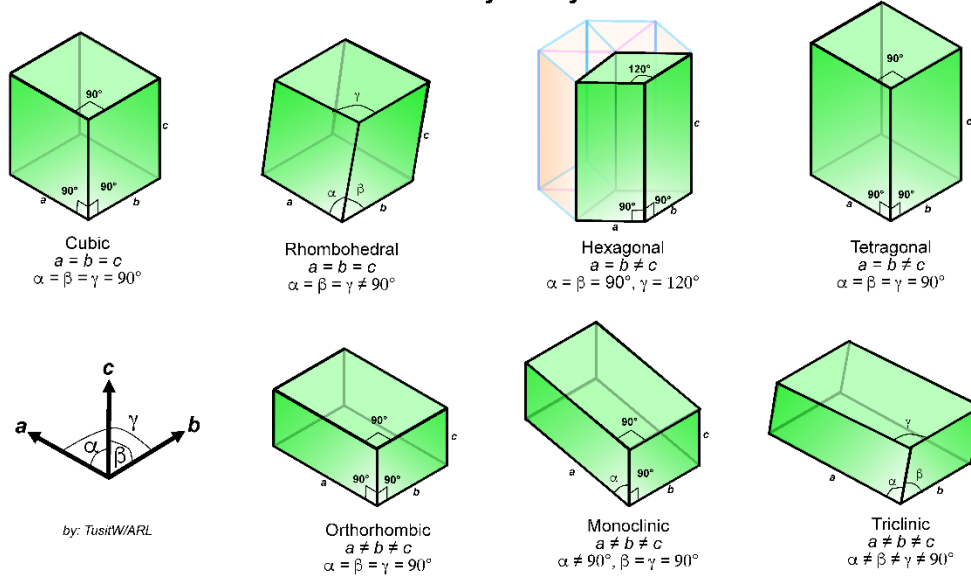


Fig. 2 Summary of crystal systems. The UHMWPE panels used in the current study follow the tetragonal primitive structure (quasi-transversely isotropic), with three mutually perpendicular planes of symmetry and with additional constraint that two of the three perpendicular planes of symmetry are also identical.

Table 2 Material symmetries

Symmetry classification	Description (informal)
Completely anisotropic (<i>triclinic</i>)	No symmetry. Different properties in every tested direction.
Orthotropic (<i>orthorhombic</i>)	Three mutually perpendicular planes of symmetry.
Quasi-transversely isotropic (<i>tetragonal</i>)	Orthotropic, with additional constraint that two of the three perpendicular planes of symmetry are also identical.
Transversely isotropic (<i>hexagonal</i>)	Orthotropic, with additional constraint that there is a plane of isotropy where the material is the same in all directions.
Completely isotropic	Completely symmetric. Same properties in every direction.

The UHMWPE panels used in the current study were classified as tetragonal (quasi-transversely isotropic) due to the following considerations. One plane of symmetry existed in the 1-direction (aligned in the 0° direction as defined in Table 1), another in the 2-direction, and another in the 3-direction. These three mutually perpendicular planes of symmetry indicate at least orthotropic symmetry. However, in addition, the 1-direction and 2-direction of the UHMWPE panels in this study were assumed to be identical because the same material was used in both directions. The classification of $[0^\circ/90^\circ]_n$ panels that have identical material in the 0° and 90° directions as tetragonal was in keeping with previous studies that used ultrasonic technique to measure the elastic constants of cross-ply composites reinforced with

glass fibers (Dandekar et al. 1998; Mistou and Karama 2000) or graphite fibers (Lhermitte and Perrin 1991).

The tetragonal crystal system has within it seven different crystal classes (Mason 1958, p. 381; Schreiber et al. 1973, p. 15; Clayton 2011, p. 545) and is divided into two Laue groups designated as TI and TII (Clayton 2011). In this study, the UHMWPE panels are considered as a tetragonal material in the TI Laue group. This designation of the particular Laue group within the tetragonal system is important because the TI and TII Laue groups have different implications for the elastic behavior of the material. Hereafter, the term tetragonal in this study, if it appears without additional qualifying statements, will refer to the TI Laue group.

2.3 Elasticity

Generalized Hooke's law relates the Cauchy stress tensor, $\boldsymbol{\sigma}$, to the infinitesimal engineering strain tensor, $\boldsymbol{\varepsilon}$, as

$$\boldsymbol{\sigma} = \mathbb{C} : \boldsymbol{\varepsilon} \quad (1)$$

The fourth-order tensor $\mathbb{C}$ will be referred to as the elasticity tensor^{*}. The inverse of Eq. 1 is written as

$$\boldsymbol{\varepsilon} = \mathbb{S} : \boldsymbol{\sigma} \quad (2)$$

The fourth-order tensor $\mathbb{S}$ will be referred to as the compliance tensor. The elasticity and compliance tensors are inversely related:

$$\mathbb{S} = \mathbb{C}^{-1} \quad (3)$$

The elasticity and compliance tensors can be written as 6×6 matrices (Appendix A). The matrix corresponding to the elasticity tensor will be referred to here as the C-Matrix. Analogously, the matrix corresponding to the compliance tensor will be referred to as the S-Matrix.

Material symmetry reduces the number of independent elements of the C-Matrix and S-Matrix (Mason 1958, p. 385; Neighbors and Schacher 1967; Schreiber et al. 1973, p. 17; Nemeth 2011). For a completely anisotropic material with no material symmetry, the C-Matrix and S-Matrix each have 21 independent elements. The UHMWPE panels were assumed as tetragonal materials of the TI Laue group. For these tetragonal materials, the C-Matrix and S-Matrix each have six independent

^{*} The tensor $\mathbb{C}$ is also referred to as the stiffness tensor. However, this term will be avoided in this present study to avoid confusion with the tensor $\mathbb{S}$.

elements (Appendix B) and can be written in a matrix format as Eqs. 4 and 5*. The corresponding expressions for anisotropic, orthotropic, and transversely isotropic materials are included in Appendix B for reference and comparison.

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix} \quad (4)$$

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{11} & S_{13} & 0 & 0 & 0 \\ S_{13} & S_{13} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} \quad (5)$$

Equations 4 and 5 are matrix representations of Hooke's law using mixed notation. More details regarding the matrix representation and subscript notation are provided in Appendix A. In these equations, the subscripts of strain and stress do not have the same meaning as the subscripts of the elements of the C-Matrix and S-Matrix. Stress and strain have subscripts 11, 22, 33, 23, 13, or 12. These indices match the 1,2,3 coordinate system of Fig. 1. However, the C-Matrix and S-Matrix in Eqs. 4 and 5 are in Voigt reduced notation. In the reduced notation, the subscript indices range from 1 to 6 and do not exactly match the 1,2,3 coordinate system but are related to it as described in Appendix A.

The elements of the C-Matrix are known as the elastic constants. They are also referred to in this study as the C_{ij} elements. Analogously, the elements of the S-Matrix are referred to in this study as the S_{ij} elements.

Young's moduli (E), shear moduli (G), and Poisson's ratios (ν) are collectively referred to as engineering constants. Materials have different numbers of independent engineering constants depending on their symmetry, as described in Appendix B. Tetragonal materials such as the UHMWPE panels have six independent engineering constants, as summarized in Table 3. There are two independent elastic moduli and two independent shear moduli. In addition, there are two sets of Poisson's ratios. The ratios from one set are independent from the

* For tetragonal materials of the TII Laue group, the C-Matrix and S-Matrix each have seven independent elements (Mason 1958, p. 387–388; Schreiber et al. 1973, p. 21–23). As noted previously, the term tetragonal in this report, when used without additional qualification, will generally refer to materials of the TI Laue group, with C- and S-Matrices as in Eqs. 4 and 5.

ratios of the other set but are not independent from the other ratio(s) within their own set. The first set will be referred to as the in-plane Poisson's ratios consisting of ν_{12} and its equivalent, ν_{21} . The second set will be referred to as the through-thickness Poisson's ratios and consists of ν_{13} , ν_{31} , ν_{23} , ν_{32} . The through-thickness ratios are related by the following:

$$\frac{\nu_{31}}{E_{33}} = \frac{\nu_{13}}{E_{11}} = \frac{\nu_{23}}{E_{22}} = \frac{\nu_{32}}{E_{33}} \quad (6)$$

The engineering constants are directly related to the elements of the S-Matrix. Since the S-Matrix is the inverse of the C-Matrix (Eq. 3), the engineering constants can also be directly related to the C-Matrix elements. These relationships are arithmetic closed-form expressions due to the symmetric simplified form of the C-Matrix for materials with orthotropic or higher symmetry. Appendix B provides the equations for calculating the engineering constants for different classes of material symmetries either from the S-Matrix elements or directly from the C-Matrix elements. Table 3 summarizes the equations whereby each constant can be calculated from the S-Matrix elements for the case of the tetragonal UHMWPE panels.

Table 3 Calculating engineering constants from S-Matrix elements in UHMWPE panels

Engineering constant	Description	Equation (tetragonal symmetry)	Eq. no.
$E_{11} = E_{22}$	Elastic modulus in the 0° direction (for E_{11}) or 90° direction (for E_{22})	$E_{11} = E_{22} = 1/S_{11}$	(1)
E_{33}	Elastic modulus in the stacking direction	$E_{33} = 1/S_{33}$	(2)
G_{12}	Shear modulus in the plane of the plies	$G_{12} = 1/S_{66}$	(3)
$G_{13} = G_{23}$	Shear modulus when the panel is sheared in the stacking direction	$G_{13} = G_{23} = 1/S_{44}$	(4)
$\nu_{12} = \nu_{21}$	Poisson's ratio in the plane of the plies	$\nu_{12} = \nu_{21} = -S_{12}E_{11}$	(5)
$\nu_{13} = \nu_{23}^a$	Through-thickness Poisson's ratios	$\nu_{13} = \nu_{23} = -S_{13}E_{11}$	(6)
$\nu_{31} = \nu_{32}^a$	Through-thickness Poisson's ratios	$\nu_{31} = \nu_{32} = -S_{13}E_{33}$	(7)

Note: See Appendix B for more detail, including the equations relating these engineering constants directly to the C-Matrix elements.

^a The four through-thickness Poisson's ratios (ν_{13} , ν_{23} , ν_{31} , ν_{32}) are all linearly related. See Eq. 6.

2.4 Generating Ultrasonic Waves for Purpose of Calculating Elastic Constants

2.4.1 Direct Contact Transducers

In ultrasonic characterization, waves are transmitted and received using ultrasonic transducers. Transducers are used in transmit mode to generate the ultrasonic wave

by converting an electrical signal into mechanical energy. In receiver mode, transducers are also used to measure the ultrasonic wave by converting the mechanical energy of an ultrasonic wave impinging upon the transducer back into an electrical signal for display to the experimentalist. The transducers used in this study were direct contact transducers. These transducers must be directly coupled to the material being characterized. A coupling liquid is used to couple the transducer to the material. This method is distinct from the many other ultrasonic studies using immersion transducers^{*}. In immersion studies, the transducer is not directly in physical contact with the material being characterized. Instead, both the transducer(s) and the material are submersed in a liquid. The liquid is amenable for propagating ultrasonic waves, thereby enabling the transducer to be physically separated from the material. Unless otherwise stated, the methods and discussion regarding ultrasonic characterization in this study will generally be using the direct contact method.

2.4.2 Propagation and Polarization, Pure and Quasi Waves

Plane waves traveling in a semi-infinite solid are characterized by their propagation vector $\vec{a} = [a_1, a_2, a_3]$ and polarization vector $\vec{p} = [p_1, p_2, p_3]$.[†] Waves can be classified as either longitudinal or transverse. Transverse waves will also be referred to as shear waves in this report. Pure longitudinal waves are classified by having parallel $\vec{a}$ and $\vec{p}$ such that

$$a_1 = p_1, a_2 = p_2, a_3 = p_3 \quad (14)$$

Pure transverse waves are classified by having perpendicular $\vec{a}$ and $\vec{p}$ such that

$$a_1 p_1 + a_2 p_2 + a_3 p_3 = 0 \quad (15)$$

Waves that are either pure longitudinal or pure transverse are referred to as pure waves. However, in general, $\vec{a}$ and $\vec{p}$ will not be completely parallel or perpendicular but will be close to satisfying these conditions (Neighbors and Schacher 1967, p. 5366). Such waves are referred to as quasi-longitudinal, quasi-transverse, or simply quasi waves. Pure waves only exist for specific propagation directions as detailed in Appendix C.

^{*} The reader is referred to a recent review by Paterson et al. (2018) documenting the dichotomy between the direct contact method and the immersion method in the context of measuring the elastic constants of unidirectional fiber composites. Other studies that used the immersion technique include Pearson and Murri (1987), Jeong (1990), and Aristégui and Baste (1997).

[†] Many authors also refer to the polarization vector as the direction of particle motion/displacement (e.g., Mason 1958, Kriz et al. 1979, van Buskirk et al. 1981, Dandekar et al. 1998).

The propagation direction is always normal to the face of the transducer. Therefore, in order to change $\vec{a}$, the transducer must be physically repositioned at different orientations to the material being characterized. For example, considering the specimen shown in Fig. 1, placing the transducer under the specimen would transmit a wave with $\vec{a}$ in the 3-direction. Placing the transducer on one of the side lateral faces would transmit a wave with $\vec{a}$ in either the 1-direction or the 2-direction. However, this method of repositioning the transducer only allows for certain propagation vectors to be generated, based on the external faces of the specimen available for placement of the transducer. For example, in Fig. 1, it would not be possible to generate a wave propagating at 45° between two axes, such as $\vec{a} = [1,1,0]$. An alternative method to change $\vec{a}$ is to cut coupons out from the panel at different orientations. This second method allows for any propagation vector, limited only by what can be cut from the panel.

The polarization of the ultrasonic wave is dependent on the type of transducer. Transducers are commercially available as either longitudinal or shear transducers. For longitudinal transducers, the polarization direction is always normal to the face of the transducer, and therefore also parallel with $\vec{a}$. Shear transducers, specifically what are known as normal incidence shear wave (NISW) transducers, generate waves with $\vec{p}$ in the plane of the transducer face, orthogonal to $\vec{a}$. For the example specimen in Fig. 1, placing a NISW transducer under the specimen would transmit a wave with $\vec{p}$ oriented in the plane spanned by the 1-direction and 2-direction (the 1-2 plane). The particular direction of $\vec{p}$ in the orthogonal plane is dictated by the rotation of the transducer, indicated by a marker on the transducer. For example, for the NISW transducers from Olympus used in the present study, the direction of $\vec{p}$ was parallel to the BNC connector of the transducer.

3. Methods

Figure 3 schematically represents the method whereby the elastic and engineering constants of the panels were calculated by measuring the ultrasonic velocity. The method is summarized as a five-step process. This section first describes the UHMWPE materials and the construction of the panels (Section 3.1). Next, Section 3.2 describes the first two steps outlined in Fig. 3: measurement of time of flight (TOF) and velocity. The remaining steps in Fig. 3 consist of data analysis and are described in the Results section (4).

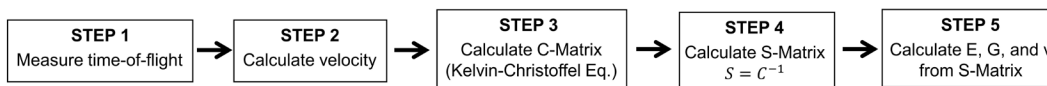


Fig. 3 Overview of method to calculate elastic and engineering constants by measuring time of flight in the UHMWPE panels

3.1 Precursor Materials and UHMWPE Panels

Five UHMWPE panels were constructed, each from different UHMWPE precursors. These precursors were supplied in rolls by the manufacturer and are listed in Table 4. Three of the panels were constructed from precursors from DSM Protective Materials. These panels will be referred to by the manufacturer-specified names of the precursor materials: HB25, HB311, and HB460. In addition, two panels were made from Tensylon products from DuPont. One panel was constructed using proprietary processes and material specifications, and will be referred to as Tensylon X. Another panel was made from commercially available Tensylon HSBD 30A and will be referred to here as Tensylon 30A.

Table 4 UHMWPE precursors (as received from manufacturers)

	HB25	HB311	HB460	Tensylon X	Tensylon 30A
Manufacturer	DSM	DSM	DSM	DuPont	DuPont
Fiber Type	SK76	SK99	PI	N/A (film)	N/A (film)
Matrix Type	PADP	PI	PI	PI	PU ^b
# of plies in Precursor^a	2	4	PI	PI	2 ^b
Precursor Layup	$[0^\circ/90^\circ]_1$	$[0^\circ/90^\circ]_2$	PI	PI	$[0^\circ/90^\circ]_1^b$

^a A ply is a single UD layer, while the precursor is the sheet as received from the manufacturer.

PADP: polyether diol-aliphatic diisocyanate polyurethane

^b Reference: White and Sherwood (2018)

Notes: PI = proprietary information; PU = polyurethane; N/A = not applicable

Table 5 UHMWPE panel processing

	HB25	HB311	HB460	Tensylon X	Tensylon 30A
Temperature (°F)	257	PI	PI	PI	230 ^d
Pressure (psi)	2400	3000	PI	3000	3000 ^d
Time (min)	45	60	PI	45	20
# of precursor sheets^a	189	194	PI	PI	221
# of single plies^b	378	776	PI	PI	442
Panel thickness, avg. (mm)	24.95	25.54	PI	26.75	25.99
Ply thickness after consolidation^c (μm)	66	33	PI	PI	59

^a As received from the manufacturer.

^b A ply is a single UD layer. Number of plies in the panel is calculated by multiplying the number of precursor sheets by the number of plies in the precursor (Table 4).

^c Calculated by dividing the panel thickness by the number of single plies.

^d References: Cline et al. 2017, Cline and Love 2020.

PI: Proprietary information

The panels were constructed by first laying up sheets of precursor material in a mold. The mold was custom-fabricated for this study to accommodate a 9- × 9-inch panel that was nominally 1 inch thick in the ply stacking direction. Precursor materials were cut from the manufacturer-supplied rolls (nominally 63 inches wide for HB25, HB311, and Tensylon 30A) to the dimensions of the mold using a table cutter (Gerber Technology, Tolland, Connecticut). Mold dimensions were derived from the specimen geometries and configurations that were necessary for ultrasonic measurements, as described below. Table 5 documents the panel processing parameters, which were derived from manufacturer recommendations. Panels were consolidated in a uniaxial press (Wabash MPI, Wabash, Indiana) and had a nominal areal density of 5 psf.

3.2 Ultrasonic Velocity Measurements for Purpose of Calculating Elastic Constants

3.2.1 Elasticity-Velocity Relationships

The UHMWPE panels had six independent elastic constants (elements of the C-Matrix, described in Section 2.3). The constants were calculated by measuring the phase velocity of ultrasonic waves propagated through the panel. The phase velocity, V , of an ultrasonic wave passing through the UHMWPE panel with propagation direction $\vec{a}$ and polarization direction $\vec{p}$ is dependent on the material properties of the panel. In particular, V is related to the mass density, ρ , and some of the elastic constants of the panel. The governing relationship is referred to in this report as the Kelvin–Christoffel equation, which is presented and analyzed in Appendix C.

As detailed in Appendix C, it was necessary to measure the phase velocity of waves generated with different combinations of $\vec{a}$ and $\vec{p}$ in order to calculate all six independent elastic constants of the panels. Here, six different propagation directions were used, listed in the $\vec{a}$ column of Table 6. Three different polarizations were used for each propagation direction. Therefore, 18 unique ultrasonic waves were generated.

Table 6 lists the elasticity-velocity relationships for the 18 waves measured in each panel in this study. The velocity in each equation is annotated with two sets of subscripts separated by a comma. The first set of subscripts denotes $\vec{a}$. The second set of subscripts, after the comma, refers to $\vec{p}$. The relationships are derived in Appendix C in the form of V as a function of C-Matrix elements (C_{ij}). The relationships in Table 6 are simply the relationships derived in Appendix C, but rearranged into the form of C_{ij} as a function of V , since this was the form used to calculate the elastic constants. The square root term in the quasi waves could be either positive or negative but was assumed to be positive in the current study.

Table 6 Relationships specifying elastic constants of UHMWPE tetragonal panels as a function of velocity of ultrasonic wave

Spec. ^a	Equation ^b	Wave type ^c	$\vec{a}$	$\vec{p}$	Eq. no.
A	$C_{11} = \rho(V_{1,1})^2$	L	[1,0,0]	[1,0,0]	(16)
A	$C_{66} = \rho(V_{1,2})^2$	T	[1,0,0]	[0,1,0]	(17)
A	$C_{44} = \rho(V_{1,3})^2$	T	[1,0,0]	[0,0,1]	(18)
B	$C_{66} = \rho(V_{2,1})^2$	T	[0,1,0]	[1,0,0]	(19)
B	$C_{11} = \rho(V_{2,2})^2$	L	[0,1,0]	[0,1,0]	(20)
B	$C_{44} = \rho(V_{2,3})^2$	T	[0,1,0]	[0,0,1]	(21)
C	$C_{44} = \rho(V_{3,1})^2$	T	[0,0,1]	[1,0,0]	(22)
C	$C_{44} = \rho(V_{3,2})^2$	T	[0,0,1]	[0,1,0]	(23)
C	$C_{33} = \rho(V_{3,3})^2$	L	[0,0,1]	[0,0,1]	(24)
D	$C_{12} = 2\rho(V_{12,12})^2 - 2C_{66} - C_{11}$	L	[1,1,0]	[1,1,0]	(25)
D	$C_{12} = C_{11} - 2\rho(V_{12,\bar{1}2})^2$	T	[1,1,0]	[-1,1,0]	(26)
D	$C_{44} = \rho(V_{12,3})^2$	T	[1,1,0]	[0,0,1]	(27)
E	$C_{13} = \pm \sqrt{(C_{11} + C_{44} - 2\rho V_{13,13}^Q)^2 \cdot (C_{33} + C_{44} - 2\rho V_{13,\bar{1}3}^Q)^2} - C_{44}$	QL	[1,0,1]	[1 ^d ,0,1]	(28)
E	$C_{13} = \pm \sqrt{(C_{11} + C_{44} - 2\rho V_{13,\bar{1}3}^Q)^2 \cdot (C_{33} + C_{44} - 2\rho V_{13,13}^Q)^2} - C_{44}$	QT	[1,0,1]	[-1 ^d ,0,1]	(29)
E	$C_{66} = 2\rho(V_{13,2})^2 - C_{44}$	T	[1,0,1]	[0,1,0]	(30)
F	$C_{13} = \pm \sqrt{(C_{11} + C_{44} - 2\rho V_{23,23}^Q)^2 \cdot (C_{33} + C_{44} - 2\rho V_{23,\bar{2}3}^Q)^2} - C_{44}$	QL	[0,1,1]	[0,1 ^d ,1]	(31)
F	$C_{13} = \pm \sqrt{(C_{11} + C_{44} - 2\rho V_{23,\bar{2}3}^Q)^2 \cdot (C_{33} + C_{44} - 2\rho V_{23,23}^Q)^2} - C_{44}$	QT	[0,1,1]	[0,-1 ^d ,1]	(32)
F	$C_{66} = 2\rho(V_{23,1})^2 - C_{44}$	T	[0,1,1]	[1,0,0]	(33)

Notes:

See Table 1 for an explanation of vector notation.

^a Specimen designation corresponding to Fig. 4.

^b Velocities are written with two sets of subscripts separated by a comma. The first set refers to the propagation direction of the wave; the second subscript refers to the polarization. Superscript Q indicates a quasi mode (wave not pure, see Section 2.4.2). Overbar indicates negative direction (as described in Table 1).

^c L or T alone indicates a pure wave, L: longitudinal (Eq. 14), T: transverse (Eq. 15). Q indicates quasi (see Section 2.4.2).

^d These were the polarization directions assumed when making the measurements. Actual polarization directions could not be exactly known a priori since they were dependent on the C-Matrix values (see Appendix C, Table C-3).

3.2.2 Extraction of Coupons from UHMWPE Panels

The six different propagation directions shown in Table 6 required coupons to be cut from the UHMWPE panels in six different orientations, one orientation for each propagation direction. The coupons corresponding to each orientation were identified in Table 6 as Specimens A–F. The representative orientation of each specimen is shown graphically in Fig. 4 (top). Figure 4 (bottom) depicts the actual photograph of specimens extracted from panels.

Coupons were extracted from the UHMWPE panels using a water-jet cutter. Figure 4 includes a schematic of the 3-D geometry of the coupons. The two surfaces of the coupon on which the transducer could be placed are referred to here as the cross-sectional faces, with characteristic dimension L . The dimensions of the cross-sectional face varied for the various coupons based on the cutting process, but were at least 20×20 mm ($L \geq 20$ mm) in order to place a 1/2-inch-diameter transducer. The depth, d , of the specimen was the dimension normal to these two surfaces and connecting them. The depth corresponded with the distance travelled by the wave propagating in direction $\vec{a}$ from one cross-sectional face to the other. All coupons had a nominal depth of $d = 10$ mm. The actual depth was measured with digital calipers to 0.01-mm accuracy, and this value was used in computing the velocity of the ultrasonic wave.

3.2.3 Density Measurements

Density was calculated by independently measuring the mass and volume of coupons from each panel. Mass was measured with a digital microgram scale. Volume was measured either using digital calipers (for Tensylon 30A) or using a pycnometer (for the other panels). The densities were 994 kg/m³ for HB25, 986 kg/m³ for HB311, 982 kg/m³ for HB460, 978 kg/m³ for Tensylon X, and 927 kg/m³ for Tensylon 30A.

3.2.4 Transducers and Waveform Generator

Most of the transducers used in these experiments were manufactured by either Panametrics or Olympus, as listed in Table 7. These transducers had a diameter of 1/2 inch. Apart from these transducers, a single additional longitudinal 5.0-MHz transducer was used, manufactured by Matec and with a diameter of 3/8 inch. Longitudinal transducers were coupled to the coupon by applying gel couplant (Couplant D, Olympus) directly on the specimen. Shear transducers were coupled to the coupon by applying a small amount of specialized, highly viscous shear couplant (SWC-2, Olympus) onto the face of the transducer and then using a straight edge to smooth the couplant into a thin layer. The particular transducers used varied between the different coupons and panels, depending on the challenges of each particular measurement. One variable affecting transducer selection was attenuation. For example, the wave generated with both propagation and polarization in the stacking direction ($\vec{a} = \vec{p} = [0,0,1]$) showed a particularly high attenuation: the received signal was much weaker than for most other velocity measurements. Therefore, it was generally necessary to use a transducer with 1-MHz frequency to measure this wave, since lower frequencies have increased penetration into attenuating materials such as plastics. However, for most other

velocity measurements, it was preferable to use a higher frequency to ensure that bulk velocity, rather than bar velocity, was being measured (Section 5.2).

Table 7 Transducers

Frequency (MHz)	Wave type	Part number ^a
1.0	Longitudinal	V103
1.0	Shear	V153
5.0	Longitudinal	V109
5.0	Shear	V155
10.0	Longitudinal	V111

^a These transducers were manufactured either by Olympus or Panametrics, but the part numbers were the same for both manufacturers.

Transducers were connected to a digital flaw detector (Epoch 650, Olympus, Waltham, Massachusetts). The Epoch 650 was used to control the transmitted wave (e.g., peak voltage) and to measure the received signal. This particular model was chosen because it enabled quantitative visualization of the received waveform, which was highly important in discriminating between signal and noise. Figure 5 shows a photograph of the experimental setup.



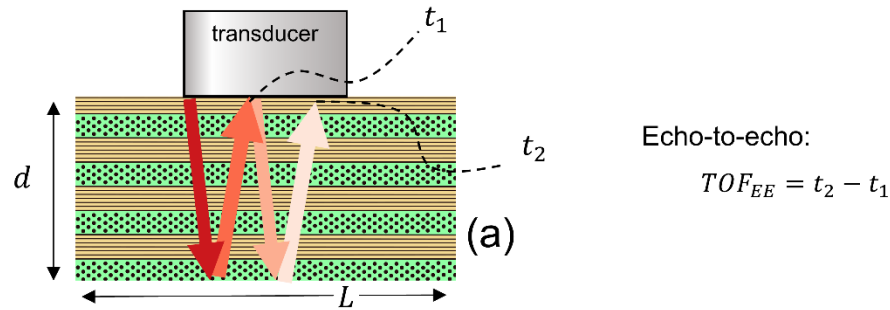
Fig. 5 Experimental setup showing an example of a measurement on a specimen from one of the panels. Measurement is in through-thickness mode (defined in Section 3.2.5) showing both the transmitting and receiving transducers. The transducers and specimen are held in place with a custom clamping device. Transducers are connected to the digital flaw detector used for generating and measuring waveforms. The waveform shown on the flaw detector is one of the best examples, showing clearly the first and second reflected pulses.

3.2.5 Techniques for Measuring Time of Flight of Ultrasonic Waves through UHMWPE Composites

Two different transmission modes were used: Pulse-Echo or Through-Transmission. Pulse-Echo mode involved a single ultrasonic transducer that acted both to send the ultrasonic wave (the pulse) and measure the return signal (echo). Through-Transmission mode involved using two ultrasonic transducers, one on each of the cross-sectional faces of the coupon. One transducer sent the ultrasonic wave while the other acted solely as the receiver to measure the wave.

With either measurement mode, the digital readout generally displayed multiple echos. This occurred because each time the wave reached one of the cross-sectional faces, part of the wave passed through the boundary while part of the wave was reflected, as schematically represented in Fig. 6.

Pulse-Echo Mode



Through-Transmission Mode

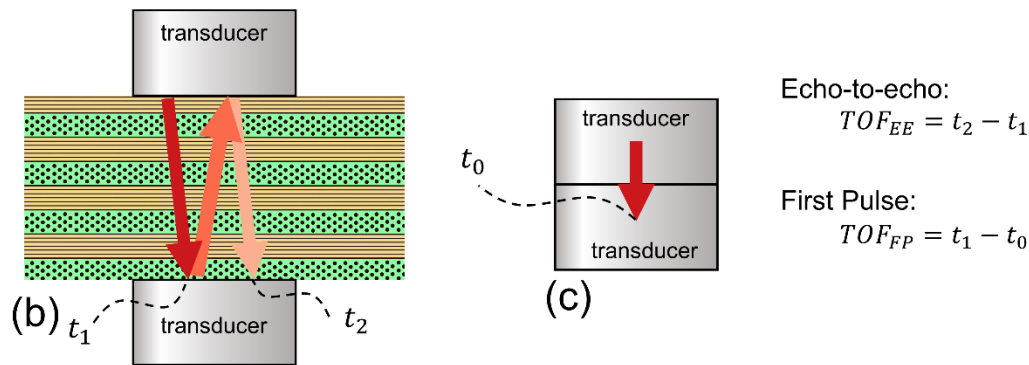


Fig. 6 Schematic representation of techniques for measuring TOF in a coupon cut out from the UHMWPE panel. Ultrasonic wave is represented by a red arrow, originating from the transmitting transducer. Part of the wave is reflected when it hits a specimen boundary while part of the wave passes through the boundary. Waves are shown in (a) and (b) as separated across the horizontal axis (left to right) for ease of visualization. In reality, the wave should propagate orthogonal to the face of the transducer, as in (c), and the reflected wave would also travel orthogonal to the face of the transducer. Schematic (c) represents the setup used for measuring the calibration pulse.

In addition to the two measurement modes, two different methods were also used to measure TOF. The first method of calculating TOF, referred to here as the First Pulse (FP) method, was only used with the Through-Transmission mode as it required two transducers, one sending and one receiving the pulse. The transducers were coupled to opposite sides of the coupon, and the time of arrival of the first echo was recorded as t_1 . Then, the coupon was removed and the transducers were brought into direct contact with each other, with only the couplant between them. The time of arrival of this calibration pulse (also known as the zero-delay pulse) was then measured as t_0 . The TOF for the FP method was then calculated as

$$TOF_{FP} = t_1 - t_0 \quad (34)$$

The second method for calculating TOF was used with both the Pulse-Echo and Through-Transmission modes. In this method, the transducer(s) remained coupled to the specimen. The time of arrival of one pulse was recorded as t_i , and the time of arrival of the next subsequent pulse was recorded as t_{i+1} . This method will be referred to as the Echo-to-Echo (EE) method. The TOF for the EE method was calculated as

$$TOF_{EE} = t_{i+1} - t_i \quad (35)$$

The different transmission modes and TOF methods are shown schematically in Fig. 6.

The sound velocity, V , was calculated by dividing the soundpath with the associated TOF method. The sound-path was defined as the distance that the sound wave traveled during the TOF measurement. The sound-path for the FP method was simply the depth of the coupon (Fig. 6b). Therefore, the velocity was calculated as

$$V = \frac{d}{TOF_{FP}} \quad (36)$$

The sound-path for the EE method was double the depth (Fig. 6a,b). In these examples, the wave is shown to travel a distance of twice the depth between t_1 and t_2 . Therefore, the velocity for TOF measured by the EE method was calculated as

$$V = \frac{2d}{TOF_{EE}} \quad (37)$$

Multiple measurements were made for the 18 velocities listed in Table 6. Different measurement modes (Pulse-Echo and Through-Transmission) and TOF methods (FP and EE) were used based on various challenges arising from diffraction, scattering, attenuation, etc. For example, for highly attenuated waves, such as when both propagation and polarization were in the stacking direction, the Through-

Transmission mode and FP method were necessary since the signal was not strong enough to measure the multiple echoes needed for the EE method.

3.2.6 Techniques for Identifying Waveforms for Challenging Measurements

Waveforms were sometimes difficult to identify due to artifacts, poor signal, etc. For these cases, iterative and inverse procedures were used. For example, for the HB460 material, the four velocities corresponding to C_{66} were difficult to measure. Therefore, the value of C_{66} found for the other panels was used as a starting point. Then, the approximate values of the four velocities were calculated by inverting Eqs. 17, 19, 30, and 33 (Table 6). An expected TOF was calculated from these approximate velocity values. These expected TOF values were used as starting points around which to try to identify the waveforms and the actual TOF for HB460. When identifying the actual TOF, potential TOF values were evaluated by calculating the resulting C_{66} value and then following the procedure described in Fig. 3 to calculate the engineering constants. These potential results for C_{66} and the engineering constants were then evaluated against literature results and the values from the other panels to select the possible number.

4. Results

The velocity measurements are listed in Section 4.1. Elastic constants are listed in Section 4.2, along with a description of calculation techniques. The engineering constants calculated from the elasticity tensors are listed in Section 4.3.

4.1 Velocity Measurements

Table 8 lists the velocity measurements for the 18 ultrasonic waves described in Table 6. The panel that was easiest to measure was Tensylon X. For the other panels, some of the velocities were not able to be recorded for all 18 waves due to difficulties in distinguishing the waveform from noise.

For Tensylon 30A, the longitudinal velocity in the stacking direction, $V_{3,3}$ was not able to be measured from Specimen C (defined in Fig. 4) extracted from the thick panel. The transmitted waveform was not able to be detected by the receiving transducer, even when using Through-Transmission mode. The problem was assumed to be the thickness of the specimen ($d = 10.10$ mm) combined with the low density of the Tensylon 30A panel. Since this velocity was critical for the calculation of elastic constants, $V_{3,3}$ was measured from a thinner panel of Tensylon HSBD 30A ($d = 7.82$ mm). This thinner panel had been processed following a similar procedure to the Tensylon 30A panel, as listed in Table 5.

Table 8 Velocity measurements in meters per second

Velocity	Wave type	HB25	HB311	HB460	Tensylon X	Tensylon 30A
$V_{1,1}$	L	7697	8469	8842	7906	8294
$V_{1,2}$	T	2344	~	1133	1426	1341
$V_{1,3}$	T	2405	1148	2657	1301	~
$V_{2,1}$	T	2425	~	1128	1386	1309
$V_{2,2}$	L	8452	8272	9007	7858	7992
$V_{2,3}$	T	2448	1128	2764	1290	~
$V_{3,1}$	T	~	~	~	1296	895
$V_{3,2}$	T	~	~	~	1296	885
$V_{3,3}$	L	2083	2436	2320	2651	2074
$V_{12,12}$	L	6299	6213	6601	6046	~
$V_{12,\bar{1}2}$	T	5684	5762	~	5278	5751
$V_{12,3}$	T	2549	~	2565	1271	~
$V_{13,13}^Q$	QL	~	~	~	5656	~
$V_{13,\bar{1}3}^Q$	QT	~	1877	~	2046	~
$V_{13,2}$	T	2419	1184	~	1283	1193
$V_{23,23}^Q$	QL	~	1867	~	5657	~
$V_{23,\bar{2}3}^Q$	QT	~	~	~	2086	~
$V_{23,1}$	T	2450	1139	~	1346	1146

Notes: Refer to Table 6 for definition of velocities and wave types. Velocity measurements marked by ~ were either not able to be made due to attenuation or artifacts or were considered as outliers and not used in the calculation of elastic constants.

4.2 Elastic Constants

The elastic constants were calculated from the velocity measurements using the equations shown in Table 6 and the procedure described below. Elastic constants were calculated in the following order: C_{11} , C_{33} , C_{44} , C_{66} , C_{12} and then C_{13} . This order was important because some of the equations required other elastic constants to have already been calculated. For example, Eq. 25 (in Table 6) for C_{12} required both C_{11} and C_{66} to have already been determined. Generally, there were multiple equations for each elastic constant, and the final value of the elastic constant was taken as the average of the values calculated from the different equations. For example, both Eq. 16 and from Eq. 20 gave C_{11} (see Table 6). Therefore, C_{11} was taken as the average of these two equations. The procedure for calculating the elastic constants is shown in Table 9, which lists the different equations from Table 6 used to calculate each elastic constant. Table 10 lists the elastic constants calculated for each panel.

Several exceptions were made to the general methodology for calculating elastic constants summarized in Table 9. First, many velocities were unable to be measured. In these cases, the corresponding equation was simply omitted from the calculation procedure. Second, for each elastic constant, the different values calculated from the equations were checked prior to averaging and clear outliers were omitted. These procedural exceptions were possible because almost all of the elastic constants could be calculated from multiple different velocity measurements. Only C_{33} depended on a single velocity measurement ($V_{3,3}$, see Eq. 24 in Table 6), making the measurement of this velocity critical to the procedure.

Table 9 Procedure for calculating elastic constants

Elastic constant (listed in order of evaluation)	Equation(s) ^a
C_{11}	avg(Eq. 16, Eq. 20)
C_{33}	Eq. 24
C_{44}	avg(Eq. 18, Eq. 21, Eq. 22, Eq. 23, Eq. 27)
C_{66}	avg(Eq. 17, Eq. 19, Eq. 30, Eq. 33)
C_{12}	avg(Eq. 25, Eq. 26)
C_{13}	avg(Eq. 28, Eq. 29, Eq. 31, Eq. 32)

^a These equations are from Table 6.

Table 10 Elastic constants for UHMWPE panels in gigapascals

Velocity	HB25	HB311	HB460	Tensylon X	Tensylon 30A
C_{11}	64.95	69.09	78.21	60.76	61.49
C_{33}	4.31	5.85	5.29	6.87	3.99
C_{44}	6.06	1.28	6.96	1.63	0.73
C_{66}	5.69	1.39	1.26	1.84	1.72
C_{12}	1.64	3.93	4.85	6.66	0.16
C_{13}	0.05 ^a	2.07	0.85 ^a	1.86	0.05 ^a

^a The C_{13} values listed for the HB25, HB460, and Tensylon 30A panels are approximations due to difficulties in measuring the corresponding velocities (see explanation below).

The other procedural exception in calculating elastic constants was in determining C_{13} . There were four velocities used to calculate C_{13} : $V_{13,13}^Q$, $V_{13,\bar{1}3}^Q$, $V_{23,23}^Q$, and $V_{23,\bar{2}3}^Q$ as indicated in Table 6 (see the table for the definition of velocities). The equations to calculate C_{13} were of the same form for all of these velocities (Table 6). This equation is rewritten below using V^Q to refer to any of the four velocities:

$$C_{13} = \sqrt{(C_{11} + C_{44} - 2\rho(V^Q)^2) \cdot (C_{33} + C_{44} - 2\rho(V^Q)^2) - C_{44}} \quad (38)$$

The velocities referred to as V^Q were very challenging to measure. As an example, Fig. 7 plots C_{13} as a function of V^Q for HB25. One challenge was that the wave transmission modes were not pure but were quasi (defined in Section 2.4.2). Therefore, even though a single waveform was transmitted, multiple different waves could be received and displayed by the measurement device. In addition, the equation to calculate these waves (Eq. 38) included a square root term. The square root term limited the range of velocities that lead to real (not complex) C_{13} values. Moreover, the form of Eq. 38 caused C_{13} to be extremely sensitive to small variations in the TOF measurements, even on the order of nanoseconds. This sensitivity is illustrated for the HB25 panel in Fig. 7 as an example. For this panel, the Poisson's ratio increased from $0.00 < \nu_{13} < 0.50$ when the velocity increased from 6001.4 m/s to 6022.2 m/s. Specimen F for HB25 had a depth of 9.90 mm. Therefore, the velocity range corresponded to a TOF range of 1.6496 to 1.6439 μ s for the FP method (Eq. 36). In other words, a difference in TOF measurement of only 6 ns would change the calculated Poisson's ratio between $\nu_{13} = 0.00$ and $\nu_{13} = 0.50$.

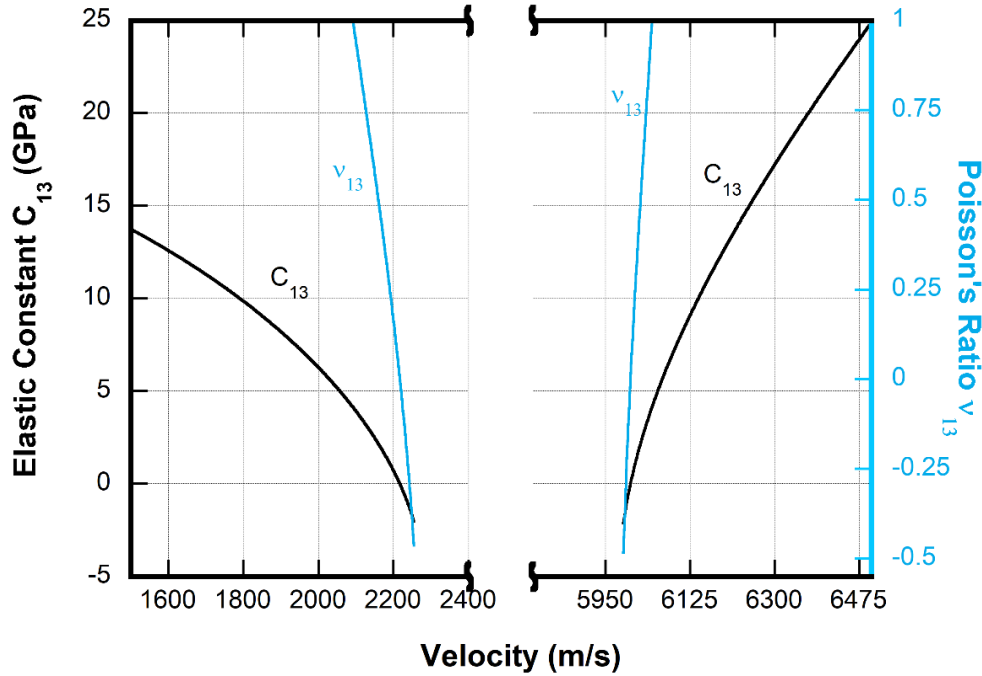


Fig. 7 Example of the sensitivity of C_{13} and ν_{13} to the measured velocity of the quasi waves for HB25. The elastic constant C_{13} was calculated using Eq. 38, with the values of C_{11} , C_{44} , and C_{33} taken from Table 10 for HB25. The velocity, V^Q , refers to the velocity of any of the four quasi waves: $V_{13,13}^Q$, $V_{13,\bar{1}3}^Q$, $V_{23,23}^Q$, and $V_{23,\bar{2}3}^Q$. The Poisson's ratio ν_{13} was calculated using the procedure outlined in Section 4.3. Due to the square root term in Eq. 38, C_{13} is undefined for $2283 < V^Q < 5977$ m/s.

To overcome these challenges, an inverse procedure similar to that described in Section 3.2.6 was used. An initial range of C_{13} values was approximated by determining the C_{13} values, which resulted in ν_{13} values between 0.00 and 0.50 (this starting constraint could be relaxed for anisotropic material [Ting and Chen 2005]). For this range of C_{13} values, the corresponding velocities and TOFs were then determined using the inverse of Steps 3, 2, and 1 in Fig. 3. Waveforms were examined to identify TOF values near these approximations. Even so, the correct waveforms for C_{13} could not be identified for three of the panels: HB460, HB25, and Tensylon 30A. For Tensylon 30A as an example, the velocity range yielding values of ν_{13} between 0.00 and 0.50, given the values in Table 10 for the five elastic constants other than C_{13} , was between $1574 < V^Q < 1595$ m/s and between $5793 < V^Q < 5800$ m/s. For these three panels, an approximate value of C_{13} was used for further calculations and listed in Table 10, based on matching the through-thickness Poisson's ratios with literature.

4.3 Engineering Constants

For each panel, the C-Matrix was constructed from the C_{ij} elements listed in Table 10. The S-Matrix was then calculated by inverting the C-Matrix (Eq. 3). Then, the engineering constants were calculated from the elements of the S-Matrix using the equations in Table 3. The UHMWPE panels had six independent engineering constants. The calculated values for the six constants are shown in Table 11.

Table 11 Engineering constants for UHMWPE panels

Engineering constant	Description	HB25	HB311	HB460	Tensylon X	Tensylon 30A
E_{11} (GPa)	Elastic modulus in the 0° or 90° directions (equivalent to E_{22})	64.90	68.21	77.79	59.63	61.49
E_{33} (GPa)	Elastic modulus in the stacking direction	4.31	5.73	5.27	6.77	3.99
G_{12} (GPa)	In-plane shear modulus	5.69	1.39	1.26	1.84	1.72
G_{13} (GPa)	Through-thickness shear modulus (equivalent to G_{23})	6.06	1.28	6.96	1.63	0.73
ν_{12}	In-plane Poisson's ratio (equivalent to ν_{21})	0.025	0.047	0.060	0.102	0.003
ν_{31}	Through-thickness Poisson's ratio (equivalent to ν_{32} and linearly related to ν_{13} and ν_{23} by Eq. 6)	0.001	0.028	0.010	0.028	0.001

5. Discussion

5.1 Comparison with Literature

Table 12 compares the velocities measured in the current study with other common engineering materials. Previous experimental and computational studies have reported on HB26, which is also a UHMWPE material produced by DSM Protective Materials. HB26 is almost identical to the HB25 material characterized in the present study. The only difference is that while HB25 is commercially available as a two-ply precursor material with the plies in a $0^\circ/90^\circ$ configuration, the HB26 precursor material has four plies arranged in a $0^\circ/90^\circ/0^\circ/90^\circ$ configuration. Therefore, the two materials were considered to be comparable for macroscopic properties of panels several millimeters thick.

Table 12 Comparison of sound velocities (m/s)

Material	Longitudinal	Shear
HB25	In ply plane ^a : 8075 Stacking direction ^b : 2083	2841 ^c
HB311	In ply plane ^a : 8371 Stacking direction ^b : 2436	2072 ^c
HB460	In ply plane ^a : 8925 Stacking direction ^b : 2320	2049 ^c
Tensylon X	In ply plane ^a : 7882 Stacking direction ^b : 2651	1717 ^c
Tensylon 30A	In ply plane ^a : 8143 Stacking direction ^b : 2074	1789 ^c
Aluminum	6320 ^d , 6420 ^e	3130 ^d , 3040 ^e
Beryllium	12890 ^e , 12900 ^d	8880 ^{d, e}
Copper	4660 ^d , 5010 ^e	2330 ^d , 2270 ^e
Lead	1960 ^e , 2160 ^d	690 ^e , 700 ^d
Polyethylene	1950 ^e , 2670 ^d	540 ^e
Polyurethane	1398 ^f , 1780 ^g	
Steel 347	5740 ^d , 5790 ^e	3090 ^d , 3100 ^e
Tungsten	5180 ^d , 5410 ^e	2640 ^e , 2870 ^d

^a Average of $V_{1,1}$ and $V_{2,2}$.

^b Taken from $V_{3,3}$.

^c Average velocities from pure transverse waves (marked as T in Table 8 under Wave Type).

^d NDE Resource Center, Materials properties tables, n.d.

^e American Institute of Physics Handbook (1957)

^f Cafarelli et al. (2016)

^g Elcometer Inspection Equipment (2021).

The previous studies used for comparison are overviewed in Section 5.1.1. The engineering constants and mass densities are tabulated in Section 5.1.2. Finally, differences between Poisson's ratios and moduli are discussed in Sections 5.1.3 and 5.1.4, respectively.

5.1.1 Background on Previous Studies

A sample of studies, mostly involving HB26, are used for comparison and are described in the following paragraphs. These studies were primarily chosen because they reported properties applicable for the whole panel produced by orthogonally stacking individual plies ($0^\circ/90^\circ$ orientation), rather than for values for individual plies or for panels produced by stacking UD plies all in the same direction (referred to as unidirectional composites). The fact that the properties were reported at the macroscopic scale of the orthogonally stacked panel was important because the ultrasonic technique used in the present study also characterized the properties of the UHMWPE panels at the macroscopic scale.

The study by Levi-Sasson et al. (2014) was chiefly experimental, reporting innovative experimental methods for measuring tensile and shear properties of HB26. They also included two micromechanical models. In Model 1, parameters were optimized to match HB26 properties derived from the manufacturer. In Model 2, parameters were optimized to match HB26 properties derived from the experiments of the authors.

Authors Lässig and Nguyen et al. (2015) presented a nonlinear constitutive model for HB26 and implemented it in ANSYS AUTODYN. The model was validated against different experiments including ballistic impact. The elastic portion of the model considered HB26 as an orthotropic material. Most of the engineering constants for the elastic portion were directly obtained from experiments, which were also documented in the study. The only exception was the Poisson's ratios, which were taken from a previous study by Grujicic et al. (2009).

Later, the authors Nguyen and Lässig published an altered version of the HB26 model (Nguyen et al. 2016). Some of the elastic constants for the orthotropic elastic model were different compared to the previous study. This model was also validated against ballistic experiments (Nguyen et al. 2016, 2020).*

Hazzard et al. (2018) modeled HB26 panels using an available model in LS-DYNA and inputting custom parameter values. The parameter values were mostly based on previous experimental literature, which was also used to validate the model.

Finally, the 2009 study by Grujicic et al. presented a model for cross-ply UHMWPE composite material. The specific type of composite was not explicitly mentioned. However, the Grujicic study is included in this discussion because it is widely cited; several of the other studies considered here derived some of their engineering constants from it.

* Additional information for this model was also presented in the thesis of Nguyen (2015).

5.1.2 Table of Parameter Comparisons

Table 13 relists the engineering constants calculated for the five panels in the current study (from Table 11) and compares these to the engineering constants listed in the previous publications. Many of the previous studies modeled HB26 as orthotropic. Treating the panels as orthotropic assumed less symmetry than the approach taken in the current study of characterizing the panels as tetragonal. The independent engineering constants for orthotropic materials are presented in Appendix B. Importantly, as described in the appendix, orthotropic materials have three independent Poisson's ratios, whereas tetragonal materials have only two. Therefore, to avoid confusion, Table 13 lists all of the Poisson's ratios for the materials. The Poisson's ratios from the current study were either taken directly from Table 11 or were calculated in Eq. 6. Poisson's ratios from the literature were either directly reported in the corresponding study or were calculated from the reported ratios using the equations presented in Appendix B.

Table 13 Comparison of engineering constants

Material/Source	E_{11}^a (GPa)	E_{33} (GPa)	G_{12} (GPa)	G_{13}^b (GPa)	ν_{12}^c	ν_{31}	ν_{13}	ν_{32}	ν_{23}	ρ (kg/m ³)
HB25	64.90	4.31	5.69	6.06	0.025	0.001	0.012	0.001	0.012	994
HB311	68.21	5.73	1.39	1.28	0.047	0.028	0.337	0.028	0.337	986
HB460	77.79	5.27	1.26	6.96	0.060	0.010	0.150	0.010	0.150	982
Tensylon X	59.63	6.77	1.84	1.63	0.102	0.028	0.243	0.028	0.243	978
Tensylon 30A	61.49	3.99	1.72	0.73	0.003	0.001	0.013	0.001	0.013	927
Tensylon Experiments at ARL	55 ^e		1.50 ^f		0.15 ^e					
Grujicic 2009	93.00	11.50	4.60	5.00	0.000	0.012 ^d	0.100	0.062 ^d	0.500	~
Levi-Sasson 2014 (Model 1)	43.90	3.65	1.12	0.88	0.024	0.002 ^d	0.019	0.002 ^d	0.019	~
Levi-Sasson 2014 (Model 2)	22.31	1.12	0.40	0.37	0.013	0.001 ^d	0.014	0.001 ^d	0.014	~
Lässig 2015	26.90	3.62	0.04	0.03	0.000	0.013 ^d	0.100	0.067 ^d	0.500	980
Nguyen 2016	51.10	3.62	0.19	2.00	0.000	0.013	0.184 ^d	0.035 ^d	0.500	980
Hazzard 2018	34.26	3.26	0.17	0.55	0.000	0.013	0.137 ^d	0.013	0.137 ^d	970

Notes: Almost all previous studies shown here were explicitly based on HB26. The exception was Grujicic et al. (2009), in which the specific type of UHMWPE composite was not specified.

^a E_{22} was equivalent to E_{11} in all cases.

^b G_{23} was equivalent to G_{13} in all cases.

^c ν_{21} was equivalent to ν_{12} in all cases.

^d Indicates Poisson's ratios, which were not directly reported in the study but are here calculated from the reported ratios using the equations for orthotropic materials presented in Appendix B.

^e From single plies of UD Tensylon (Gunnarsson, personal communication 2020).

^f Derived from analysis of experiments on Tensylon HSB 30A (cross-ply panels) performed by Cline et al. (2017), see Fig. 8.

5.1.3 Comparison of Poisson's Ratios

Table 13 indicates that the Poisson's ratios showed significant variation. It is important to note that both Nguyen et al. (2016) and Hazzard et al. (2018) cite Lässig et al. (2015) for the values of Poisson's ratios. In turn, the Lässig study cites the previous study by Grujicic et al. (2009). The 2009 study by Grujicic et al. uses engineering constants derived from a previous computational model (Grujicic et al. 2008). The 2008 study modeled the UHMWPE composite at the level of the unit cell. The original source of the parameters used for the unit cell is unclear to us.

The reason for this inter-citation is due to the inherent difficulty in measuring Poisson's ratios for UHMWPE panels.

5.1.4 Moduli Comparisons

The elastic and shear moduli ultrasonically determined for HB25 in the current study fell within the range of values previously used in models for HB26, though the HB25 values were generally slightly higher. However, the density of HB25 was also higher than the density of HB26, consistent with higher moduli values. The higher density could indicate differences in processing technique, which led to the panel in the current study being more consolidated, and thus stiffer.

The biggest discrepancies were in the shear moduli. The ultrasonically determined shear moduli were 1 to 2 orders of magnitude higher than some of the previously reported HB26 values. The higher ultrasonically determined values could be due to strain rate sensitivity of the shear response. For example, Russell et al. (2013) characterized the in-plane shear response of HB26 by applying tension to $\pm 45^\circ$ dogbone specimens at strain rates between 10^{-4} s^{-1} and 10^{-2} s^{-1} . The shear response was shown to be significantly strain rate dependent.

In addition, differences between the experimental and ultrasonically determined shear moduli could also be due to challenges inherent in characterizing the shear response with both methods, as shear waves are of lower magnitude and seemed to get damped out sooner than the longitudinal waves. In mechanical characterization of materials, shear characterization is far more difficult than tensile or compressive characterization. Also, gripping difficulties inherent with UHMWPE further amplify during shear characterization. The ultrasonically determined shear moduli could also be affected from the limitations and challenges discussed in Section 5.2. Specifically, the shear moduli were directly related to C_{44} and C_{66} . All of the velocity measurements used to calculate C_{44} and C_{66} came from shear waves (see Table 6), which were generally more challenging to measure than longitudinal waves, as discussed in Section 5.2.5.

Despite these caveats, the ultrasonically determined shear moduli were still close to the values used by Grujicic et al. (2009). This is noteworthy since the Grujicic et al. model was used as a source for some of the other parameters in the other studies included in Table 13, such as Poisson's ratios (Section 5.1.3).

The ultrasonically determined in-plane shear modulus for the panels from Tensylon also approximately matched the value found from experiments performed by Cline et al. (2017). The initial portion of the shear response of Tensylon HSBD 30A from the Cline et al. study is shown in Fig. 8, with an initial in-plane shear modulus of approximately $G_{12} = 1.50$ GPa. Cline et al. also measured G_{12} for a fiber-based composite. The G_{12} for Tensylon HSBD 30A was found to be more than 20 times larger than for the fiber-based composite. This finding was not confirmed in the current study. Here, the ultrasonically determined G_{12} of the Tensylon panels was not significantly greater than the other fiber-based composites. The large difference found by Cline et al. may have been due in part to the particular fiber-based composite they characterized, Spectra Shield SR3136.

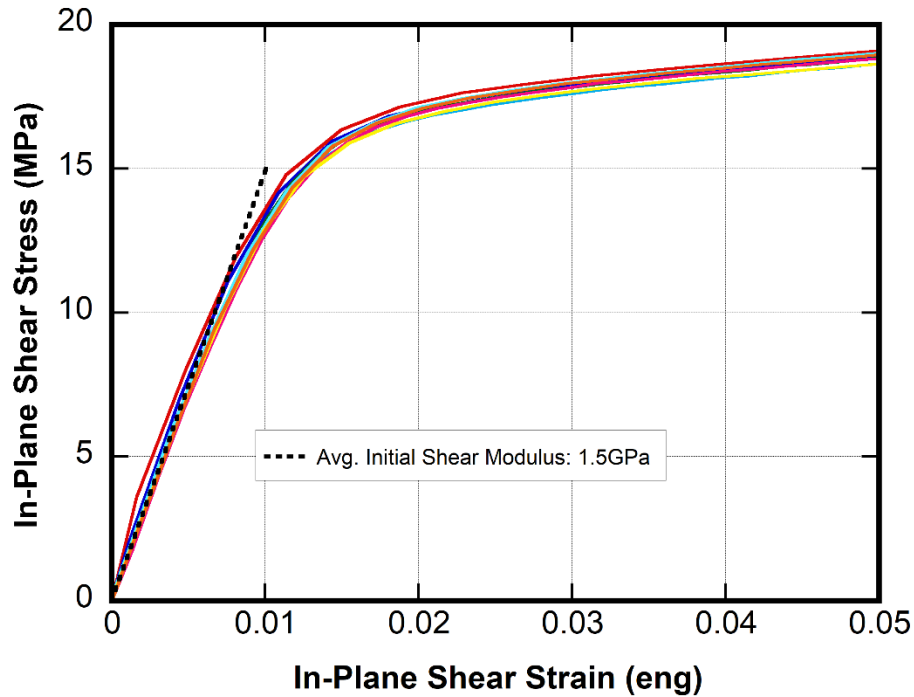


Fig. 8 Shear stress-strain response of Tensylon HSBD 30A, from experiments performed by Cline et al. (2017). Response of specimens cut out from the panel are shown in solid lines. Dotted line shows approximate average initial shear modulus.

5.2 Challenges and Limitations

The underlying component of the present methodology was the Kelvin–Christoffel equation, which enabled calculation of elastic constants from wave velocities. The equation is valid under a certain set of conditions, as described in Appendix C. Here, three of the conditions are considered, and the extent to which they were met in the current study is evaluated. First, Section 5.2.1 analyzes whether the ultrasonic waves were plane waves. Section 5.2.2 analyzes whether the waves propagated as bulk waves rather than bar waves. Finally, Section 5.2.3 discusses whether the measured velocities were the phase velocities rather than the group velocities.

Measurement of TOF from the waveforms was often challenged by difficulties in identifying the waveform from artifacts. These artifacts in the waveforms are discussed in Section 5.2.4. Finally, additional sources of imprecision in the TOF measurements are considered in Section 5.2.5.

5.2.1 Plane Waves

The Kelvin–Christoffel equation assumed that the waves were plane waves. Ultrasonic waves generated from a circular transducer, such as those used in the current study, travel from the transducer first as plane waves for some distance before becoming spherical waves, as shown in Fig. 9. The space over which the wave is planar is known as the Fresnel region or zone (Mason 1958, p. 96). After traveling inside the Fresnel zone a distance $d = d_{crit}$, the ultrasonic wave transitions into the Fraunhofer zone (Hendee and Ritenour 2002, p. 322). The dichotomy between the Fresnel/Fraunhofer zones is also referred to as the near/far fields or zones (NDE Resource Center, Radiated fields of ultrasonic transducers, n.d.).

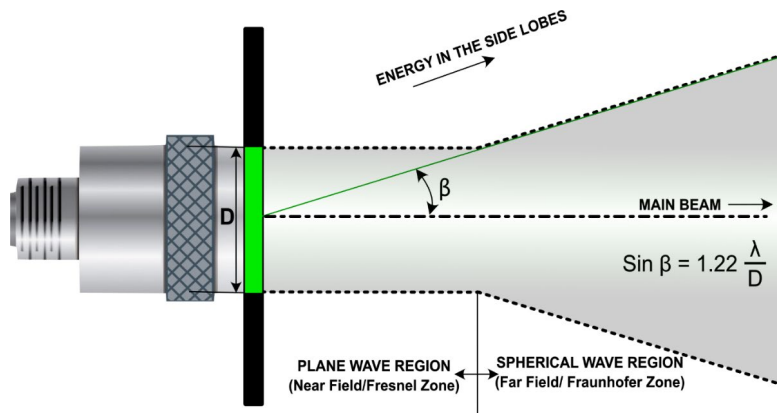


Fig. 9 Schematic showing Fresnel and Fraunhofer zones. An ultrasonic beam generated by a circular transducer with diameter D travels first as a plane wave within the Fresnel zone, also referred to as the Near Field. At a critical travel distance of d_{crit} (Eq. 41), the wave transitions into the Fraunhofer zone, also referred to as the Far Field, and the beam energy starts to diverge from the center line.

The transition distances d_{crit} were calculated from the diameter of the transducer, D , as (Hendee and Ritenour 2002)

$$d_{crit} = \frac{D^2}{4\lambda} \quad (39)$$

The wavelength of the wave is given by λ . The wavelength is related to the frequency, f , and velocity of the wave by

$$\lambda = V/f \quad (40)$$

Therefore, the transition distances were calculated as

$$d_{crit} = \frac{f \cdot D^2}{4V} \quad (41)$$

Different types of transducers were used in the current study (Section 3.2.4). In order to assess the assumption of plane waves, the transducers were categorized into four different cases, Case A through Case D, as shown in Table 14. Records were not kept of which transducers were used to measure which specific velocity. However, the general approach was to preferentially use the highest frequency possible, resorting to lower frequencies when unable to measure the TOF with higher-frequency transducers due to attenuation. Therefore, to evaluate the assumption of plane waves, d_{crit} was calculated for each case of transducers using Eq. 41 and the velocity measurements listed in Table 8.

Table 14 Transition distances (d_{crit}) between near and far fields for the different transducers used in current study

Case	Freq. (MHz)	Diam. (inch)	Transition Distances (d_{crit} , mm)					
			Longitudinal ^a			Shear ^b		
			Min.	Max.	Avg.	Min.	Max.	Avg.
A	1	0.5	4.5	19.4	8.5	7.0	45.6	25.6
B	5	0.375	12.6	54.7	24.0	19.7	128.1	72.1
C	5	0.5	22.4	97.2	42.7	35.0	227.8	128.2
D	10	0.5	44.8	194.4	85.5	~	~	~

Note: Transition distances were calculated using Eq. 41. The 10-MHz transducers were only used for longitudinal waves, not for shear waves.

^a Calculated using the velocities from pure longitudinal waves (marked as “L” Wave Type in Table 8).

^b Calculated using the velocities from pure transverse waves (marked as “T” Wave Type in Table 8).

The assumption that the ultrasonic waves in the current study were plane waves required the travel distance of the wave to be less than d_{crit} . For the FP method

(Section 3.2.5), the travel distance was the depth of the specimen, d (Fig. 6), which was nominally $d = 10$ mm. Therefore, for the wave to travel as a plane wave,

$$\text{For plane wave: } d_{crit} > 10\text{mm} \quad (42)$$

Table 14 indicates that transition distances were generally greater than 10 mm. Therefore, Eq. 42 suggests that the waves were mostly planar. The exception would be the 1-MHz transducers. For these transducers, the transition distances for longitudinal waves were on average less than 10 mm. The shear waves generated by 1 MHz were generally greater than 10 mm, with the exception of the fastest waves. Therefore, the 1-MHz transducers may have not generated planar longitudinal waves. They may have generated planar shear waves, except for the fastest directions.

The previous analysis was for the FP method. The findings may be different for the EE method, which relies on multiple reflections of the wave from the cross-sectional surfaces of the specimen. These reflections may change the way in which the wave transitions from the near to the far field.

5.2.2 Bulk Waves

The Kelvin–Christoffel equation assumes that the ultrasonic wave travels as a bulk wave rather than a bar wave. The dichotomy between bulk and bar waves is based on the size of the wavelength of the wave, λ , relative to the dimensions of the material through which the wave is passing. The cross-sectional dimensions of the coupons in the present study varied but were at least 20×20 mm. The cross-sectional dimension was depicted as L in Fig. 4.

Waves travel as bulk waves when $\lambda \ll L$. In this limit, the specimen is considered as a semi-infinite solid. Since the wavelength is much smaller than cross-sectional dimensions, the wave does not interact with the sides of the specimen as it travels. In the opposite limit, $\lambda > L$, and the wave is referred to as a bar or extensional wave (Kohlhauser 2009). In this case, the wave is expected to interact with the sides of the specimen as it travels, producing a different type of motion than bulk waves (Kohlhauser 2009, p. 11). The motion has been described as the “entire cross section [being] excited by the passing wave” (Ashman et al. 1984, p. 355). Waves traveling as bulk waves, without interacting with the lateral boundaries of the specimen, are faster than waves traveling as bar waves (Kohlhauser 2009, p. 15). The two types of velocities are referred to as bulk velocity and bar (or extensional) velocity, respectively.

Shear waves do not exhibit this dichotomy: the velocity of the shear wave is not affected by the cross-sectional dimension (Tu et al. 1955; Schreiber et al. 1973, p.

44; Kohlhauser 2009). For longitudinal waves, the transition between bulk and bar motion is not well-defined. For example, Tu et al. (1955) measured the velocity of the ultrasonic pulse in metallic cylindrical specimens with radius r and various values of r/λ . The measured velocity corresponded to the bulk velocity when $r > 2.5\lambda$. Writing this requirement in terms of the cross-sectional length, L , would correspond to a ratio of $L/\lambda > 5$.

However, a single critical value of L/λ was not found to be repeatable in other literature. Ashman et al. (1984, p. 355–356, Fig. 6) ultrasonically measured the velocity of different materials while systematically reducing L in order to characterize the transition between bulk and bar velocities. First, the bulk velocity was measured when the specimens had large L . The bulk velocity continued to be measured as L was reduced until reaching a critical value of L/λ . Then, as L was further decremented, the velocity measurements also decreased in steps. The bar velocity, as a lower limit, was only reached in some of the tested materials at sufficiently low values of L . The critical value of L/λ was approximately 10 for aluminum and carbon-reinforced epoxy, 20 for bovine femur, and 100 for plexiglass.

The variation that Ashman et al. recorded in the critical value of L/λ may indicate other confounding factors. A more recent study by Kohlhauser and Hellmich (2013) found the transition between bar and bulk velocities in the through-transmission experimental setup to not only depend on L and λ , but also on the travel distance of the wave from one transducer to the other. In the current study, this travel distance was denoted by d , the depth of the specimen (Fig. 6). Kohlhauser and Hellmich developed an empirical relationship from a detailed set of experiments to indicate when bulk velocity, rather than the bar velocity, was measured. Written using the d notation of the current study, the relationship indicated that the bulk velocity was measured when

$$-1.426 \log\left(\frac{L}{d}\right) - 0.530 \log\left(\frac{d}{\lambda}\right) \leq 1 \quad (43)$$

The assumption that the longitudinal velocities measured in the current study were bulk velocities was evaluated by two methods: comparing L/λ to the critical values reported in Tu et al. (1955) and Ashman et al. (1984), and by evaluating whether Eq. 43 was satisfied. First, the ratio L/λ was calculated using Eq. 40 as

$$\frac{L}{\lambda} = \frac{fL}{v} \quad (44)$$

The most conservative case was for the lowest-frequency transducers, 1 MHz. For $f = 1$ MHz, the longitudinal velocity measurements listed in Table 8 gave $2.2 \leq$

$L/\lambda \leq 9.6$, with an average ratio of 4.2. On average, these measurements fell below the critical value found in Tu et al. of $L/\lambda > 5$, and almost all of the range fell below the critical values in Ashman et al. For the higher-frequency transducers, the velocity measurements in Table 8 gave $11.1 \leq L/\lambda \leq 48.2$ for $f = 5$ MHz and $22.2 \leq L/\lambda \leq 96.4$ for $f = 10$ MHz. These ratios were well above the critical value of 5 found by Tu et al. and were above some of the critical values found by Ashman et al.

Equation 43 was evaluated by calculating the left-hand side, which will be referred to as the LHS. In the current study, d was 10 mm when using the Through-Transmission mode and FP method of measuring TOF (Fig. 6). With $d = 10$ mm, the LHS was found to be negative for all three transducer frequencies and for all measured velocities. Therefore, Eq. 43 was satisfied, indicating that bulk velocity was measured in all cases for the FP method. Equation 43 was developed from Through-Transmission experiments using the FP method. The EE method of measuring TOF involves reflection of the wave from the cross-sectional surfaces. The effect of these reflections may alter the physics of the wave, and so no attempt is made here to apply Eq. 43 to the EE method.

In conclusion, the uncertainty in the literature of defining the transition between the bulk and bar velocities precludes a definitive evaluation regarding whether the assumption of bulk wave propagation was satisfied. If the transition is only dependent on L/λ , then the assumption may have been satisfied with the higher-frequency transducers but was likely not satisfied for the 1-MHz transducer. On the other hand, if the transition is also dependent on the travel distance of the wave as described by Kohlhauser and Hellmich, then the present analysis indicates that the assumption was likely satisfied for all frequencies when the FP method was used to measure the TOF.

5.2.3 Phase Velocities

The Kelvin–Christoffel equation relates the elastic constants specifically to the phase velocity. For anisotropic materials, the phase velocity can be distinct from another velocity, known as the group velocity. Therefore, it was important to verify that the phase, rather than the group, velocity was measured with the ultrasonic techniques in the present study.

Further information regarding the definition of phase and group velocities is detailed in the presentation provided by Kohlhauser and Hellmich (2012, p. 96–97). Briefly, the phase velocity direction, V_p , is associated with the direction of wave propagation, $\vec{a}$, while the group velocity is associated with the direction of energy propagation, V_e (Kohlhauser 2009, p. 2). The direction of energy

propagation is also referred to as the energy-flux vector (Kriz and Stinchcomb 1979). For anisotropic materials, V_e can deviate from $\vec{a}$. This deviation leads to differences between the group and phase velocities, as shown in Fig. 10. Here, a transmitting transducer is coupled to the bottom of an off-axis specimen. A wave is generated with wave propagation direction $\vec{a}$ toward the top of the specimen, to the point marked as A. The depth of the specimen, d , is the travel distance for $\vec{a}$ between the bottom and top surfaces and was measured by digital calipers in the current study. The direction of energy propagation, V_e , deviates from $\vec{a}$ and reaches the top surface at the point marked by B. The travel distance for V_e is denoted by ℓ_e .

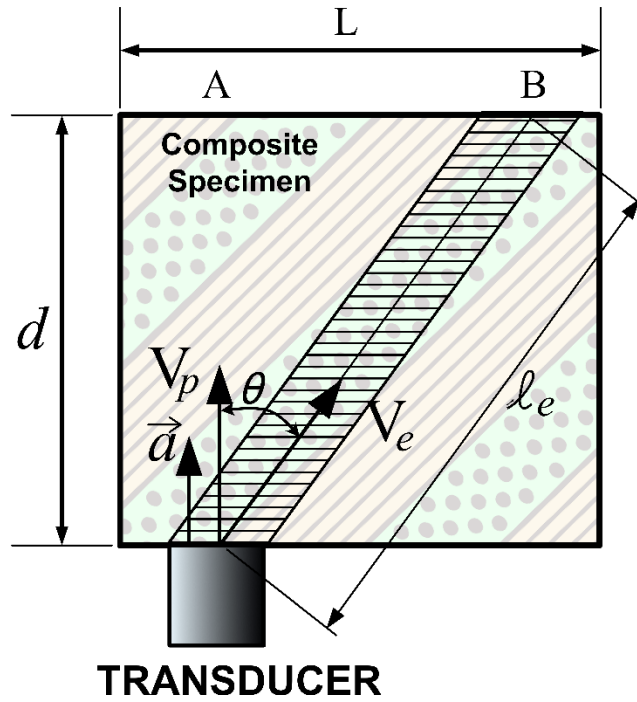


Fig. 10 Schematic of deviation between the phase and group velocities (recreated based on Kohlhauser and Hellmich 2012)

The question of whether phase or group velocities were measured depends on experimental setup and technique. In the current study, two different transmission modes were used to measure TOF: Pulse-Echo and Through-Transmission (Section 3.2.5). The Pulse-Echo mode will always give the phase velocity rather than the group velocity, provided that the travel distance is taken as a multiple of d rather than ℓ_e (Sahay et al. 1992, Every 1994, Kohlhauser and Hellmich 2012). In other words, Pulse-Echo gives the phase velocity if velocity is calculated as $(2d)/\text{TOF}$, as it was in the present study.

On the other hand, correctly measuring the phase velocity with Through-Transmission mode required certain necessary conditions, as described by Pearson and Murri (1987), Sahay et al. (1992), and Kohlhauser and Hellmich (2012). First, to correctly measure the TOF, the receiving transducer must be in contact with the specimen at the point where the energy flux reaches the top of the specimen, marked as B in Fig. 10. For the example shown in Fig. 10, a receiving transducer with a small diameter similar to the transmitting transducer would have to be deliberately placed at point B rather than at point A. However, if the transducers had sufficiently wide diameters, or if the depth, d , was sufficiently small, the transmitting and receiving transducers would not have to be laterally displaced from each other, and no a-priori knowledge of the point B would be needed (Sahay et al. 1992, Kohlhauser and Hellmich 2012). After measurement of TOF, phase velocity is calculated using a travel distance of d rather than ℓ_e (velocity = d/TOF).

However, the potential difference between group and phase velocities may not have been an issue for the majority of the specimen orientations in the current study (shown in Fig. 4). Kriz and Stinchcomb (1979) calculated the angle of deviation (θ in Fig. 10) between V_e and $\vec{a}$ in UD CFRP specimens. Nonzero deviation was recorded only for the two off-axis specimens in which quasi waves were generated. Using the notation from the Kriz study, these two specimens were in the 1–3 and 1–2 planes. In the current study, the two corresponding specimens associated with quasi waves were Specimen E and Specimen F (see Table 6). Kriz and Stinchcomb found zero deviation ($\theta = 0$) for the other specimens in which pure waves were generated. Similarly, Pearson and Murri (1987) experimentally determined the deviation between phase and group velocities in unidirectional CFRP specimens by propagating waves at different angles to the axis of the fibers. There was insignificant deviation between the phase and group velocities when waves were propagated either parallel or orthogonal to the fibers. Deviation occurred when waves were propagated in directions not strictly parallel or orthogonal to the fibers.

The considerations provided here indicate that divergence between phase and group velocities in the current study were likely limited to only two of the six different specimen orientations (Specimens E and F). Furthermore, here, the velocity was always calculated using d or a multiple of d , as required. In addition, the Pulse-Echo technique, which always gives the phase velocity, was used for several measurements, although records were not kept regarding which transmission modes were used for each velocity measurement. For Through-Transmission, the receiving transducer was generally kept in-line with the transmitting transducer and was not displaced to try to find the energy flux vector. This may have contributed to the difficulty in measuring the quasi waves in Specimens E and F, and indicates potential error in the TOF measurement of the quasi waves.

5.2.4 Artifacts in the Measured Pulses

A frequent problem encountered in the current study was the muddling of the measured wave with artifacts. For example, sometimes the measured wave was contaminated with what are referred to as trailing pulses. In addition, sometimes multiple different pulses with different amplitudes were visible in the oscilloscope.

These artifacts were likely due to multiple effects on the ultrasonic waveform imposed by the viscoelastic, heterogeneous, anisotropic UHMWPE material in the current study. Several terms are used in the field to describe these effects, some of which are considered here. Mode conversion describes the phenomenon of a wave changing type (longitudinal or transverse). For example, a longitudinal wave reflecting from a surface at an oblique angle can produce both a longitudinal and a shear wave (Mason 1958, p. 23–27). Scattering can be defined as the “random reflection of ultrasonic waves by small discontinuities or surface irregularities” (NDE Resource Center, NDE Glossary, n.d.) and is frequency dependent. Attenuation can be described as a loss of wave energy, as seen, for example, in a decrease in amplitude of the waveform. Attenuation can be due to scattering or to the viscoelasticity of the material and is frequency dependent (Kim and Park 1987). Finally, dispersion refers to the frequency dependence of velocity. A formal definition of dispersion is provided in the theses by Kohlhauser (2009, p. 6–8) and Ashman (1982, p. 34–35). Briefly, dispersion is associated with the inequality between the group and phase velocities (Section 5.2.3). For nondispersive conditions, the phase and group velocities are equal, whereas they deviate under dispersive conditions. Plastics and their composites, such as CFRP, are known to be dispersive (Mason 1958, Kim and Park 1987, Jeong 1990, Raišutis et al. 2008, Paterson et al. 2018).

The frequency-dependent effects, such as dispersion and attenuation, could have been responsible for the distortion of some of the waveforms in the current study. Here, the frequency of the ultrasonic transducers has been referenced by a single number (1, 5, or 10 MHz). However, this number only refers to the center frequency of the transducer frequency spectrum. The waves generated by these transducers included frequencies across the spectrum of each transducer. Since the UHMWPE specimens were likely dispersive, the different frequencies generated by a single transducer traveled at different velocities through the specimen, thus distorting the waveform (Mason 1958, p. 45–47, 100; Schreiber et al. 1973, p. 43).

The extent to which attenuation effects could have been responsible for the artifacts in the current study was directly assessed. Generally, interaction effects are minimized if the ultrasonic wavelength is sufficiently larger than the characteristic length scale of the inclusions (Jeong 1990; Lhermitte and Perrin 1991; Ourak et al.

1993; Rho 1996; Kohlhauser and Hellmich 2013; Paterson et al. 2018). The case when the wavelength is much larger than the characteristic length scale has been referred to as the long wavelength limit. The microstructural details possibly creating interaction effects in cross-plyed, fiber-based composites, such as the UHMWPE panels in the current study, include the fibers and the individual plies themselves. To minimize interaction with the fibers, the wavelength needed to be much larger than D_{fiber} , the fiber diameter:

$$\lambda \gg D_{fiber} \quad (45)$$

To evaluate whether Eq. 45 was satisfied in the current study, the limiting case was identified by the smallest wavelengths. The smallest wavelengths corresponded to the slowest waves generated with the highest-frequency transducers (Eq. 40). The highest-frequency transducers, 10 MHz, were only used for longitudinal waves. The slowest longitudinal waves corresponded to $V_{3,3}$. The average wavelength for these waves, if generated by 10-MHz transducers, was $\lambda = 231 \mu\text{m}$. The next highest-frequency transducers, 5 MHz, were used for both longitudinal and shear waves. The lowest velocity recorded in Table 8 was 885 m/s. Generated by a 5-MHz transducer, this wave would have a wavelength of $\lambda = 177 \mu\text{m}$. Therefore, the shortest wavelengths possibly generated in this study were $\lambda = 177 \mu\text{m}$. The fibers in the UHMWPE panels were either SK76 or SK99 (Table 4), which have been reported to have diameters of approximately $D_{fiber} = 18$ and $10 \mu\text{m}$, respectively (Sanborn et al. 2015; O'Masta et al. 2016). Given that the shortest possible wavelengths were more than 10 times larger than the fibers, Eq. 45 indicates that the waves were not altered by interaction with the fibers.

To minimize interaction with the plies, the wavelength needed to be much larger than the characteristic length scale associated with the plies. Lhermitte and Perrin (1991) considered this characteristic length scale to be two times the ply thickness for the case of plies arranged in a $0^\circ/90^\circ$ configuration. Therefore, in the long wavelength limit, the wavelength is much larger than twice the effective thickness of the individual ply:

$$\lambda \gg 2d_{ply} \quad (46)$$

In Eq. 46, d_{ply} is the effective thickness of the ply, which is the thickness through which the ultrasonic wave must pass. In Specimens A, B, and C, the effective ply thickness was equal to the thickness of individual plies within the panel after consolidation. The post-consolidation thicknesses are listed in Table 5 and give values of $2d_{ply} = 132$, 66 , and $118 \mu\text{m}$ for HB25, HB311, and Tensylon 30A, respectively. The lowest velocity recorded in Table 8 for Specimens A, B, or C was 885 m/s. This measurement could have corresponded to a wavelength of $\lambda = 177$

μm if a 5-MHz transducer was used, as described previously. This limiting wavelength value was greater than $2d_{ply}$ for the panels mentioned above and was even greater than $4d_{ply}$ for HB311. Therefore, Eq. 46 was likely satisfied for almost all of the waves generated in the present study in Specimens A, B, and C. The exceptions could have been some of the slowest waves generated with the higher frequencies, and even these were likely in the long wavelength limit except when characterizing the HB25 or Tensylon panels.

The off-axis specimens (Specimens D, E, and F in Fig. 4) were extracted at 45° relative to the panel principal directions. As a result, the effective ply thickness through which the ultrasonic wave passed was $\sqrt{2}$ times the post-consolidation thicknesses listed in Table 5. Therefore, for the off-axis specimens, the right-hand side of Eq. 46 was $2d_{ply} = 187, 93, \text{ and } 167$ for HB25, HB311, and Tensylon 30A, respectively. The lowest velocity recorded in Table 8 for Specimens D, E, or F was 1139 m/s . This measurement could have corresponded to a limiting case wavelength of $228 \mu\text{m}$ if measured with a 5-MHz transducer. Therefore, Eq. 46 was likely satisfied for most of the waves generated in the off-axis specimens with the possible exceptions being the slowest waves generated with the higher frequencies, as was found for Specimens A, B, and C.

The slowest waves were generally the transverse waves. Therefore, the finding that some of the slowest waves may have been scattered by interaction with the plies presents a possible explanation as to why shear waves were generally harder to measure than longitudinal waves.

In addition to the wave interacting with the microstructure, artifacts could have also arisen from the wave interacting with the boundaries of the specimen. Specifically, the wave can reflect from the lateral boundaries of the specimen if the cross-sectional dimension of the specimen is too narrow (Mason 1958; Schreiber et al. 1973). Reflection can result in mode conversion leading to the generation of slower waves. These slower waves can then arrive at the receiving transducer after the initial pulse, causing the phenomenon referred to as trailing pulses, and thus could contaminate the signal.

Mason (1958, p. 44) reported that, from practical experience, trailing pulses were mitigated only when the cross-sectional dimension was at least 20 times larger than the wavelength. The relative size of the cross section to the wavelength was already discussed earlier when evaluating whether the bulk or bar velocities were measured (Section 5.2.2) by analyzing the ratio of L/λ . The observation of Mason can be written in terms of this ratio as $L/\lambda \geq 20$. The previous analysis (Section 5.2.2) indicated that the 1-MHz transducers generated waves with $2.2 \leq L/\lambda \leq 10.7$, which fell far below the requirement. On the other hand, the 5-MHz and 10-MHz

transducers both generated waves with, on average, $L/\lambda > 20$. Therefore, the waves generated with 1-MHz transducers were likely contaminated with trailing pulses, which were less likely to be an issue with the higher-frequency transducers. This finding could explain why the artifacts in the present study were more pronounced with the 1-MHz transducers. In fact, many times the waveform generated by these transducers could not be interpreted, whereas the higher-frequency transducers produced noticeably cleaner waveforms.

However, the size of the cross section relative to the depth in the propagation direction may have also contributed to the extent to which the signal was contaminated by trailing pulses. For the Through-Transmission method, Schreiber et al. (1973, p. 43) recommended that the axial dimension of a cylindrical specimen (the dimension corresponding to the direction of wave propagation) be less than 10 times the diameter and “preferably less than 5”. This recommendation can be applied to the rectangular specimens of the present study. The cross-sectional length, L , and the travel distance, d , of the rectangular specimens in the current study would correspond to the diameter and the axial dimension, respectively, of the cylindrical specimen. Therefore, the recommendation of Schreiber et al. can be written in the notation of the present study as $d/L < 10$. In the present study, the cross-sectional dimension was nominally $L \geq 20$ mm, and for the through-transmission technique, $d = 10$ mm. Therefore, d/L was less than 1, thereby indicating that the recommendation was satisfied.

Finally, an additional cause of wave distortion was the quasi (not pure) nature of some of the waves. The distinction between quasi and pure waves was presented in Section 2.4.2. Kriz and Stinchcomb (1979) noted that when a quasi-wave is generated, the wave actually produces two waves: a quasi-transverse wave and a quasi-longitudinal wave. Each wave is displaced from the intended propagation direction by a different angle. If the receiving transducer is sufficiently wide, both of these waves will be measured. Thus, two waveforms could be displayed on the oscilloscope and must be differentiated. Kriz and Stinchcomb experimentally showed that by laterally displacing the receiving transducer relative to transmitting transducer, the relative amplitude of the two waves changes. In the current study, the only quasi waves generated were the off-axis waves in the 1–3 and 2–3 planes (Table 6). For some of these measurements, two waveforms were indeed visible on the oscilloscope. These were differentiated based on expected TOF, with quasi-longitudinal waves expected to be faster than quasi-transverse waves. However, in other measurements, the waveforms appeared to be partially overlapping, which further challenged TOF measurement.

5.2.5 Additional Sources of Error

Shear waves were generally more challenging to measure than longitudinal waves. Almost all of the longitudinal waves were successfully measured in the UHMWPE panels of the present study, whereas many of the shear waves were unable to be measured (Table 8). This may have been partly due to the requisite use of the shear couplant, which was much harder to use than the corresponding couplant for longitudinal waves. The shear couplant was much more viscous. It had to be directly applied to the transducer and then scraped with a straightedge to produce a thin film, whereas the couplant used for longitudinal waves was directly applied to the specimen and did not require a thin film (Olympus Industrial Resources, n.d.).

The specimens may have absorbed liquid during ultrasonic measurement. Each specimen was used for three different velocity measurements: two longitudinal and one shear (Table 6). Therefore, different types of couplant had to be applied to each specimen. In addition, for most cases, repeat measurements of the same wave were made, thereby increasing the potential amount of liquid that could be absorbed. Finally, the shear couplant was extremely viscous. In order to remove the shear couplant from the specimen after ultrasonic measurement, the specimen was submerged in a cup of water. If the specimens absorbed liquid, the density would have changed, leading to errors in the calculation of the elastic constants since the equations were dependent on the density (Table 6). Kohlhauser et al. (2009) tried to avoid absorption by using cellophane film when measuring porous wood specimens. Future studies could use this technique, wrapping the specimens in plastic film for ultrasonic measurements. This would not only avoid contact of couplant with the specimen, but would also obviate the need for submerging the specimen in water.

The current study was limited by uncertainties in measuring TOF from the waveforms, as well as in density measurements, thickness measurements, and the specimen preparation process. The difficulties and limitations of the waveform measurement techniques were analyzed previously. The density measurements were limited by the resolution of the digital scale and by the pycnometer. Furthermore, density was only determined from a subset of the specimens and then applied to all of the specimens, but there could have been intra-panel deviation due to processing conditions. The depth of the specimens, used to calculate the velocity, was measured for each specimen. However, after water-jet cutting, specimens were generally not further machined. Therefore, the depth varied slightly across the specimen; the two cross-sectional surfaces were not perfectly parallel. Finally, during the manufacturing of each panel, ply orientation within the layup may have deviated from the principal 1 and 2 directions. The water-jet cutting process likely also involved some angle of deviation from what is shown in Fig. 4.

These imprecisions could have led to errors in the calculation of elastic constants. Some of the equations for calculating elastic constants were extremely sensitive to small changes in the parameters. For example, in Section 4.2, the sensitivity of C_{13} and ν_{13} to velocity measurements in the HB25 panel was illustrated. A difference in TOF measurement of only 6 ns was shown to change the calculated ν_{13} between $\nu_{13} = 0.00$ and $\nu_{13} = 0.50$. However, the imprecisions identified above almost certainly precluded this level of resolution and precision in velocity measurements in the current study. The probable inability of the current method to measure the velocity with sufficient resolution for the quasi waves likely contributed to the difficulty in measuring the velocities associated with C_{13} and why they were not able to be measured or estimated in some of the panels.

6. Conclusion

A capability was adapted for calculating elastic and engineering constants for anisotropic materials by measuring the velocity of ultrasonic waves generated with contact transducers. The method was applied to characterize novel UHMWPE materials, including fiber-based and SSE film-based composites. The material properties listed in Tables 10 and 11 correspond to the macroscopic response of cross-ply panels manufactured from the UHMWPE materials, rather than the response of the individual plies. Therefore, these material properties from the current study can be directly used in macro-scale computer simulations, where the global response of the panel, or several cross-ply, is modeled.

In addition, a legacy UHMWPE fiber-based material (HB25) was also characterized for direct comparison of the present method with previously published material properties derived either through mechanical loading or through modeling and simulation. The moduli and Poisson's ratios calculated from ultrasonic measurements generally fell within the range from previous publications. The biggest deviations were in the shear moduli and Poisson's ratios. However, these constants have been notoriously difficult to measure experimentally through mechanical loading and were also challenging to measure with the current ultrasonic method.

The physical principles underlying the methods used to calculate elastic constants from ultrasonic velocity measurements were assessed. Several requirements were identified. The requirements that could be assessed via equations are summarized in Fig. 11. Specifically, a competing interest was found regarding the wavelength ($\lambda = V/f$). The wavelength needed to be minimized to ensure that the wave propagated as a plane and bulk wave. The wavelength also needed to be minimized to mitigate the waveform artifacts arising from the wave interacting with the lateral

boundaries of the specimen. On the other hand, the wavelength needed to be sufficiently larger than the characteristic length-scale of the microstructure in order to avoid contamination of the waveform by interaction with the microstructure.

In this study, waves generated by higher-frequency transducers (5 and 10 MHz) were generally found to satisfy the requirements. Waves generated by 1-MHz transducers may not have satisfied the requirements, especially for the slowest waves. This likely explains the experimental observation that the 1-MHz transducers often produced noisier waveforms.

Physical Requirement	Relationship
Planar wave propagation (req'd by Kelvin-Christoffel Eq.)	Maximize $\frac{f \cdot D^2}{4V}$ (must be $> d$ for FP method)
Bulk rather than bar wave propagation (req'd by Kelvin-Christoffel Eq.)	- Maximize $\frac{f \cdot L}{V}$ (at least > 5) - Bulk wave with FP method when: $-1.426 \log\left(\frac{L}{d}\right) - 0.530 \log\left(\frac{d \cdot f}{V}\right) \leq 1$
Long wavelength limit (avoid scattering effects of wave interacting with microstructure)	$(V/f) \gg$ fiber diameter $(V/f) \gg 2 \cdot$ (ply thickness)
Avoid artifacts from wave interacting with lateral boundaries of specimen	- Maximize $\frac{f \cdot L}{V}$ (recommended ≥ 20) - Minimize $\frac{d}{L}$ (recommended < 5)

Fig. 11 Summary of requirements evaluated in the current study. Relationships are in terms of transducer parameters (f : frequency, D : diameter), geometrical parameters of the specimen (as in Fig. 4, L : cross-sectional length, d : specimen depth in direction of wave propagation), and the velocity (V) of the ultrasonic wave.

The current method offers a relatively less expensive alternative to characterize the multiaxial elastic response of anisotropic advanced materials, which may be challenging (in some cases almost impossible) to grip and load with traditional mechanical techniques. Also, we would like to note that this method provides an estimation of the initial elastic constants at the start of the deformation through small deformation.

7. References

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Appendix A. Elasticity and Compliance Tensors: Matrix Representation in Mixed Notation

As a tutorial as well as for completeness of the work reported in this publication, this appendix provides more detail on the matrix format of Hooke's law used throughout the present document. Reference is frequently made to an excellent tutorial published by Nemeth.¹ An additional reference is the textbook from Malvern.²

Section A.1 defines the elasticity and compliance tensors. In Section A.2, the matrix representation of the elasticity tensor (C-Matrix) is presented as an example. The process is analogous for the compliance tensor (S-Matrix), as detailed in Section A.3.

No assumptions about material symmetry are made in this appendix. Instead, the C- and S-Matrices are presented for general anisotropic materials. Appendix B describes the simplifications to these matrices, which arise from various material symmetries such as orthotropy, transverse isotropy, or complete isotropy.

A.1 Generalized Hooke's Law

Generalized Hooke's law relates the Cauchy stress tensor, $\boldsymbol{\sigma}$, to the infinitesimal engineering strain tensor, $\boldsymbol{\varepsilon}$, as

$$\boldsymbol{\sigma} = \mathbb{C} : \boldsymbol{\varepsilon} \quad (\text{A-1})$$

The fourth-order tensor $\mathbb{C}$ will be referred to as the elasticity tensor. The inverted form of Eq. A-1 relates the strain to the stress:

$$\boldsymbol{\varepsilon} = \mathbb{S} : \boldsymbol{\sigma} \quad (\text{A-2})$$

The fourth-order tensor $\mathbb{S}$ is known as the compliance tensor.

A.2 Matrix Representation of Elasticity Tensor in Mixed Notation

In this section, the matrix representation of the elasticity tensor in mixed notation will be presented. First, Hooke's law is rewritten in matrix format without any simplifications of subscripts (Section A.2.1). Then, the subscripts for stress, strain, and the elasticity tensor in Hooke's law are simplified using reduced notation (Section A.2.2). Finally, Section A.2.3 presents the convenient mixed notation.

¹ Nemeth MP. An In-depth tutorial on constitutive equations for elastic anisotropic materials. National Aeronautics and Space Administration; 2011. Report No: NASA/TM-2011-217314.

² Malvern LE. Introduction to the mechanics of a continuous medium. Prentice-Hall; 1969.

A.2.1 Hooke's Law in Full Matrix Representation

Equation A-1 can be written in matrix format as in Nemeth (p. 41)¹:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1113} & C_{1112} \\ C_{2211} & C_{2222} & C_{2233} & C_{2223} & C_{2213} & C_{2212} \\ C_{3311} & C_{3322} & C_{3333} & C_{3323} & C_{3313} & C_{3312} \\ C_{2311} & C_{2322} & C_{2333} & C_{2323} & C_{2313} & C_{2312} \\ C_{1311} & C_{1322} & C_{1333} & C_{1323} & C_{1313} & C_{1312} \\ C_{1211} & C_{1222} & C_{1233} & C_{1223} & C_{1213} & C_{1212} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix} \quad (\text{A-3})$$

The elasticity tensor is represented as a 6×6 matrix, referred to in this document as the C-Matrix. Here, subscripts for all matrices, including stress, C-Matrix, and strain, consistently refer to the 1,2,3 coordinate system defined in Section 2.1 (Fig. 1), where the 1-direction coincides with the 0° ply direction, the 2-direction with the 90° ply direction, and the 3-direction is the stacking direction. The format of the subscripts in Eq. A-3 will be referred to as the full format.

In Eq. A-3, the C-Matrix consists of 36 elements. However, the number of independent elements is reduced to 21 due solely to thermodynamic arguments, as detailed in Nemeth (p. 43–51)¹. The resulting C-Matrix is symmetric. With these reductions, the matrix form in full format for a general anisotropic solid is (Nemeth, p. 64)¹

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1113} & C_{1112} \\ C_{1122} & C_{2222} & C_{2233} & C_{2223} & C_{2213} & C_{2212} \\ C_{1133} & C_{2233} & C_{3333} & C_{3323} & C_{3313} & C_{3312} \\ C_{1123} & C_{2223} & C_{3323} & C_{2323} & C_{2313} & C_{2312} \\ C_{1113} & C_{2213} & C_{3313} & C_{2313} & C_{1313} & C_{1312} \\ C_{1112} & C_{2212} & C_{3312} & C_{2312} & C_{1312} & C_{1212} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix} \quad (\text{A-4})$$

A.2.2 Hooke's Law in Reduced Notation

A more convenient method of representing Eq. A-1 in matrix format uses Voigt notation to reduce the number of subscripts. The subscripts in the full format (Eq. A-4) referred directly to the 1,2,3 coordinate system. Table A-1 shows the relationship between the reduced subscripts and the previously defined 1,2,3 coordinate system.

Table A-1 Reduced notation for subscripts

Full notation ^a	Reduced (Voigt) notation
11	1
22	2
33	3
23	4
13	5
12	6

Notes: Nemeth (p. 96)¹, Schreiber et al. (p. 13)³

^a In full notation, the subscripts directly refer to the previously defined 1,2,3 coordinate system (Section 2.1, Fig. 1).

Using Voigt notation, Eq. A-4 can be rewritten in reduced form by substituting the subscripts as shown in Table A-1, yielding

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ 2\varepsilon_4 \\ 2\varepsilon_5 \\ 2\varepsilon_6 \end{bmatrix} \quad (\text{A-5})$$

A.2.3 Hooke's Law in Mixed Notation

Hooke's law is written above in matrix representation using full notation on the subscripts (Eq. A-4) and also in reduced notation (Eq. A-5). In this report, a so-called mixed notation was preferentially used. In mixed notation, the full indices on the stress and strain matrices were used as in Eq. A-4, while the reduced notation was used for the C-Matrix. Mixed notation was preferred in this report so that the physical meaning of the stress and strain matrices was immediately apparent from their subscripts, while reducing the number of subscripts on the C-Matrix for convenience. The mixed notation format of Hooke's law is (Nemeth, p. 100)¹

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix} \quad (\text{A-6})$$

³ Schreiber E, Anderson OL, Soga N. Elastic constants and their measurement. McGraw-Hill; 1973.

A.3 Matrix Representation of Compliance Tensor

Equation A-2 can be written in matrix format as (Nemeth, p. 39)¹

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix} = \begin{bmatrix} S_{1111} & S_{1122} & S_{1133} & 2S_{1123} & 2S_{1113} & 2S_{1112} \\ S_{2211} & S_{2222} & S_{2233} & 2S_{2223} & 2S_{2213} & 2S_{2212} \\ S_{3311} & S_{3322} & S_{3333} & 2S_{3323} & 2S_{3313} & 2S_{3312} \\ 2S_{2311} & 2S_{2322} & 2S_{2333} & 4S_{2323} & 4S_{2313} & 4S_{2312} \\ 2S_{1311} & 2S_{1322} & 2S_{1333} & 4S_{1323} & 4S_{1313} & 4S_{1312} \\ 2S_{1211} & 2S_{1222} & 2S_{1233} & 4S_{1223} & 4S_{1213} & 4S_{1212} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} \quad (\text{A-7})$$

The compliance tensor is represented by a 6×6 matrix, referred to here as the S-Matrix. As with the C-Matrix, the S-Matrix also reduces to 21 independent elements and is symmetric due to thermodynamic arguments (Nemeth, p. 63).¹ The procedure described above for the C-Matrix, in moving from the full matrix notation (Section A.2.1) to the mixed matrix notation (Section A.2.3), can be replicated for the S-Matrix. The mixed notation for the S-Matrix is (Nemeth, p. 100).¹

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{12} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{13} & S_{23} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{14} & S_{24} & S_{34} & S_{44} & S_{45} & S_{46} \\ S_{15} & S_{25} & S_{35} & S_{45} & S_{55} & S_{56} \\ S_{16} & S_{26} & S_{36} & S_{46} & S_{56} & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix}$$

**Appendix B. The C-Matrix, S-Matrix, and Engineering Constants
(Moduli and Poisson's Ratios) for Various Material
Symmetries**

In this appendix, the matrix representation of the elasticity and compliance tensors (C-Matrix and S-Matrix, respectively) are listed for orthotropic (Section B.1), tetragonal (Section B.2), transversely isotropic (Section B.3), and completely isotropic materials (Section B.4). The subscripts used in the matrix representations follow the mixed format that was described for general anisotropic materials in Appendix A. This appendix summarizes a tutorial provided by Nemeth¹. An additional reference is the textbook from Malvern².

For each of the material symmetries, the S-Matrix is also written in terms of engineering constants (elastic modulus, shear modulus, and Poisson's ratios). This allows the engineering constants to be calculated directly from the S-Matrix. For example, the inverted form of Hooke's law in matrix form for an orthotropic material is (Nemeth, p. 246).¹

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} \quad (\text{B-1})$$

In Eq. B-1, the S-Matrix is a 6×6 matrix, referred to as $[S_{ij}]$. For an orthotropic material, the S-Matrix in Eq. B-1 can be replaced with the following S-Matrix (Nemeth, p. 422)¹:

$$[S_{ij}] = \begin{bmatrix} \frac{1}{E_{11}} & -\frac{\nu_{21}}{E_{22}} & -\frac{\nu_{31}}{E_{33}} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_{11}} & \frac{1}{E_{22}} & -\frac{\nu_{32}}{E_{33}} & 0 & 0 & 0 \\ -\frac{\nu_{13}}{E_{11}} & -\frac{\nu_{23}}{E_{22}} & \frac{1}{E_{33}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{31}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \quad (\text{B-2})$$

Each element in the S-Matrix of Eq. B-1 is related to the corresponding element in the S-Matrix of Eq. B-2. For example, equating the element at row = column = 1 ($i = j = 1$) gives $1/E_{11} = S_{11}$. Therefore, representing the S-Matrix by

¹ Nemeth MP. An in-depth tutorial on constitutive equations for elastic anisotropic materials. National Aeronautics and Space Administration; 2011. Report No: NASA/TM-2011-217314.

² Malvern LE. Introduction to the mechanics of a continuous medium. Prentice-Hall; 1969.

engineering constants, as in Eq. B-2, enables direct computation of the engineering constants when the values of the S-Matrix elements are known.

In addition, representing the S-Matrix by engineering constants also allows some of the engineering constants to be related to each other. For example, Eq. B-1 indicates that the S-Matrix of orthotropic materials is symmetric. Imposing symmetry on Eq. B-2 allows some of Poisson's ratios to be linearly related to each other, thus reducing the number of independent engineering constants. Relationships of this type between the Poisson's ratios are detailed for each material class below.

B.1 Orthotropic Materials

For orthotropic materials, the C-Matrix and S-Matrix each have nine independent constants. The C-Matrix is given in the mixed format matrix representation of Hooke's law for orthotropic materials (Nemeth p. 246):

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix} \quad (\text{B-3})$$

The S-Matrix is given in the corresponding inverted form of Hooke's law, written above in Eq. B-1. The S-Matrix can also be written in terms of engineering constants as was shown in Eq. B-2.

Equation B-1 indicates that the element at row = 1, column = 2 (S_{12}) is equal to the element at row = 2, column = 1. Imposing this on Eq. B-2 gives

$$\frac{\nu_{21}}{E_{22}} = \frac{\nu_{12}}{E_{11}} \quad (\text{B-4})$$

In addition, the element at row = 1, column = 3 (S_{13}) is equal to the element at row = 3, column = 1. Imposing this on Eq. B-2 gives

$$\frac{\nu_{31}}{E_{33}} = \frac{\nu_{13}}{E_{11}} \quad (\text{B-5})$$

Finally, the element at row = 2, column = 3 (S_{23}) is equal to the element at row = 3, column = 2. Imposing this on Eq. B-2 gives

$$\frac{\nu_{32}}{E_{33}} = \frac{\nu_{23}}{E_{22}} \quad (\text{B-6})$$

Thus, there are nine independent engineering constants: E_{11} , E_{22} , E_{33} , G_{23} , G_{31} , G_{12} , ν_{21} (or ν_{12}), ν_{31} (or ν_{13}), and ν_{32} (or ν_{23}). They can be calculated from the S-Matrix by Eq. B-2.

In addition, since the S-Matrix is the inverse of the C-Matrix, the engineering constants can also be directly calculated from the C-Matrix elements. The relationships between the engineering constants and the C-Matrix elements are derived by first inverting the C-Matrix for orthotropic materials (given in Eq. B-3). Then, the inverse is set equal to the right-hand side of Eq. B-2. The relationships have been documented in Mistou and Karama (p. 1704)³ and Ashman (p. 12–14)⁴ and are presented here:

$$\begin{aligned}
 E_{11} &= \frac{D}{(C_{22}C_{33}-C_{23}^2)}, E_{22} = \frac{D}{(C_{11}C_{33}-C_{13}^2)}, E_{33} = \frac{D}{(C_{11}C_{22}-C_{12}^2)} \\
 G_{23} &= C_{44}, G_{31} = C_{55}, G_{12} = C_{66} \\
 \nu_{12} &= -\frac{C_{13}C_{23}-C_{12}C_{33}}{C_{22}C_{33}-C_{23}^2}, \nu_{21} = -\frac{C_{13}C_{23}-C_{12}C_{33}}{C_{11}C_{33}-C_{13}^2} \\
 \nu_{13} &= -\frac{C_{12}C_{23}-C_{13}C_{22}}{C_{22}C_{33}-C_{23}^2}, \nu_{31} = -\frac{C_{12}C_{23}-C_{13}C_{22}}{C_{11}C_{22}-C_{12}^2} \\
 \nu_{23} &= -\frac{C_{12}C_{13}-C_{23}C_{11}}{C_{11}C_{33}-C_{13}^2}, \nu_{32} = -\frac{C_{12}C_{13}-C_{23}C_{11}}{C_{11}C_{22}-C_{12}^2}
 \end{aligned} \tag{B-7}$$

In Eq. B-7, $D = C_{11}C_{22}C_{33} - C_{11}C_{23}^2 - C_{22}C_{13}^2 - C_{33}C_{12}^2 + 2C_{12}C_{23}C_{13}$.

B.2 Tetragonal Materials

Tetragonal materials are orthotropic materials, therefore having three mutual planes of symmetry defined by three orthogonal axes, with the additional constraint that two of the orthogonal axes are identical. Here, *the 1-direction and 2-direction are considered identical while the 3-direction (stacking direction) is the unique direction due to the layup of the UHMWPE panel (Section 2.2), and corresponds to materials in this study*. The corresponding C-Matrix and S-Matrix are (Nemeth, p. 334).¹

³ Mistou S, Karama M. Determination of the elastic properties of composite materials by tensile testing and ultrasound measurement. J Compos Mater. 2000 Oct;34(20):1696–709.

⁴ Ashman RB. Ultrasonic determination of the elastic properties of cortical bone: techniques and limitations [thesis]. Tulane University; 1982.

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix} \quad (\text{B-8})$$

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{11} & S_{13} & 0 & 0 & 0 \\ S_{13} & S_{13} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} \quad (\text{B-9})$$

The S-Matrix written in terms of engineering constants was given by Eq. B-2 for orthotropic materials. In order to use it for tetragonal materials, several elements must be equal as indicated in Eq. B-9. Applying these equalities to the elements of Eq. B-2 gives several relationships between the engineering constants as described below. First, Eq. B-9 indicates that the element at row = 1, column = 1 (S_{11}) is equal to the element at row = 2, column = 2. Imposing this on Eq. B-2 gives

$$E_{22} = E_{11} \quad (\text{B-10})$$

Equation B-9 also indicates that the element at row = 4, column = 4 (S_{44}) is equal to the element at row = 5, column = 5. Imposing this on Eq. B-2 gives

$$G_{23} = G_{31} \quad (\text{B-11})$$

The element at row = 1, column = 2 (S_{12}) is equal to the element at row = 2, column = 1. Imposing this on Eq. B-2 yields the relation previously given as Eq. B-4. Substituting Eq. B-10 into Eq. B-4 indicates that

$$\nu_{21} = \nu_{12} \quad (\text{B-12})$$

Finally, Eq. B-9 also indicates that the element at row = 1, column = 3 (S_{13}) is equal to three other elements at the following locations: row = 2 and column = 3, row = 3 and column = 1, row = 3 and column = 2. Imposing these equalities on Eq. B-2 gives

$$\frac{\nu_{31}}{E_{33}} = \frac{\nu_{13}}{E_{11}} = \frac{\nu_{23}}{E_{22}} = \frac{\nu_{32}}{E_{33}} \quad (\text{B-13})$$

Equation B-13 indicates that

$$\nu_{31} = \nu_{32} \quad (\text{B-14})$$

Furthermore, given that $E_{22} = E_{11}$ (Eq. B-10), Eq. B-13 also indicates

$$\nu_{13} = \nu_{23} \quad (\text{B-15})$$

These considerations show that tetragonal materials have six independent engineering constants. Equations B-12 through B-15 indicate that there are two independent Poisson's ratios that consist of ν_{12} (equivalent to ν_{21}) and any one of the following: ν_{31} (equivalent to ν_{32}) or ν_{13} (equivalent to ν_{23}). There are two elastic moduli: E_{11} (equivalent to E_{22}) and E_{33} . Finally, there also two shear moduli: G_{23} (equivalent to G_{31}) and G_{12} .

The engineering constants can be directly related to the C-Matrix using a similar process as with orthotropic materials (Eq. B-7):

$$\begin{aligned} E_{11} = E_{22} &= \frac{D}{(C_{11}C_{33}-C_{13}^2)}, \quad E_{33} = \frac{D}{(C_{11}^2-C_{12}^2)} \\ G_{23} = G_{31} &= C_{44}, \quad G_{12} = C_{66} \\ \nu_{12} = \nu_{21} &= -\frac{C_{13}^2-C_{12}C_{33}}{C_{11}C_{33}-C_{13}^2} \\ \nu_{13} = \nu_{23} &= -\frac{C_{12}C_{13}-C_{13}C_{11}}{C_{11}C_{33}-C_{13}^2} \\ \nu_{31} = \nu_{32} &= -\frac{C_{12}C_{13}-C_{13}C_{11}}{C_{11}^2-C_{12}^2} \end{aligned} \quad (\text{B-16})$$

B.3 Transversely Isotropic Materials

Transversely isotropic materials are similar to tetragonal materials with the additional constraint that there exists a plane of isotropy. The C-Matrix and S-Matrix are given below for transversely isotropic materials where the 1-direction and 2-direction are identical (such that the 3-direction is the unique direction, as for the tetragonal materials in Section B.2) and where there is a plane of isotropy in the 1-2 plane:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{C_{11}-C_{12}}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix} \quad (\text{B-17})$$

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{11} & S_{13} & 0 & 0 & 0 \\ S_{13} & S_{13} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(S_{11} - S_{12}) \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} \quad (\text{B-18})$$

References for Eqs. B-17 and B-18 are Nemeth (p. 426)¹, Freeman (p. 4–5)⁵, and Lubarda and Chen (p. 157–158)⁶.

Comparing the S-Matrix for transversely isotropic materials (Eq. B-18) with the S-Matrix for tetragonal materials (Eq. B-9) indicates that these two types of materials are nearly the same. The sole difference is that for tetragonal materials, the element at row = 6, column = 6 (S_{66}) was an independent element, whereas for transversely isotropic materials it is not independent but is given by

$$S_{66} = 2(S_{11} - S_{12}) \quad (\text{B-19})$$

All of the inter-element relationships in the S-Matrix for tetragonal materials are also present in the S-Matrix of Eq. B-18, indicating that all of the relationships between engineering constants in tetragonal materials (Eqs. B-10 through B-15) are also true for transversely isotropic materials. Applying the additional constraint for transversely isotropic materials (Eq. B-19) to the S-Matrix written in terms of engineering constants (Eq. B-2) gives

$$\frac{1}{G_{12}} = 2 \left(\frac{1}{E_{11}} + \frac{\nu_{21}}{E_{22}} \right) \quad (\text{B-20})$$

Since $E_{22} = E_{11}$ (Eq. B-10), Eq. B-20 gives

$$G_{12} = \frac{E_{11}}{2(1+\nu_{21})} \quad (\text{B-21})$$

Thus, transversely isotropic materials have five independent engineering constants. There are two independent Poisson's ratios that consist of ν_{12} (equivalent to ν_{21}) and any one of the following: ν_{31} (equivalent to ν_{32}) or ν_{13} (equivalent to ν_{23}). There are two independent elastic moduli: E_{11} (equivalent to E_{22}) and E_{33} . Finally, there is a single independent shear modulus: G_{23} (equivalent to G_{31}). These

⁵ Freeman JC. Basic equations for the modeling of gallium nitride (GaN) high electron mobility transistors (HEMTS). National Aeronautics and Space Administration; 2003. Report No: NASA/TM-2003-211983.

⁶ Lubarda V, Chen M. On the elastic moduli and compliances of transversely isotropic and orthotropic materials. J Mech Mater Struct. 2008 Mar 11;3(1):153–171.

engineering constants are the same as for tetragonal materials with the exception that here, G_{12} is not independent but is instead given by Eq. B-21.

B.4 Completely Isotropic Materials

For completely isotropic materials, the C-Matrix and S-Matrix have only two independent constants^{7,8}:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{C_{11}-C_{12}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{C_{11}-C_{12}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{C_{11}-C_{12}}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix} \quad (\text{B-22})$$

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{11} & S_{13} & 0 & 0 & 0 \\ S_{13} & S_{13} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(S_{11} - S_{12}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(S_{11} - S_{12}) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(S_{11} - S_{12}) \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} \quad (\text{B-23})$$

There are only two independent engineering constants: an elastic modulus (E) and Poisson's ratio ν . The S-Matrix can be written in terms of these two constants as^{5,9}:

$$[S_{ij}] = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2(1+\nu)}{E} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2(1+\nu)}{E} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{2(1+\nu)}{E} \end{bmatrix}$$

⁷ Burnley P. Tensors: stress, strain and elasticity. University of Nevada Las Vegas; 2020 June 26 [accessed 2020 June 30]. https://serc.carleton.edu/NAGTWorkshops/mineralogy/mineral_physics/tensors.html.

⁸ Introduction to elasticity/constitutive equations. Wikiversity; 2017 July 26 [accessed 2020 June 30]. https://en.wikiversity.org/wiki/Introduction_to_Elasticity/Constitutive_relations.

⁹ Bower AF. Chapter 3 Constitutive models – relations between stress and strain. Applied Mechanics of Solids. c2008 [accessed 2020 June 30]. http://solidmechanics.org/text/Chapter3_2/Chapter3_2.htm.

Appendix C. Relationships between Wave Velocity and Elastic Constants Based on Kelvin–Christoffel Equation

The ultrasonic wave velocity (V) was related to the elastic constants (C_{ij}) of the panel using the Kelvin–Christoffel equation. The 18 velocity–elasticity relationships used in this report were presented in Table 6 in the form of C_{ij} as a function of V . This appendix presents additional details into how these equations were derived from the Kelvin–Christoffel equation.

First, Section C.1 presents the generalized Kelvin–Christoffel equation for the most generic case (anisotropy). Then, the assumption of orthotropy is applied. The Kelvin–Christoffel equation is simplified by invoking assumptions on the elasticity matrix, C_{ij} for orthotropic materials (Section C.2). Next, the equation is solved in to give relationships in the form of V as a function of C_{ij} for wave propagation in six different propagation directions, which were necessary in order to calculate all of the independent elastic constants (Section C.3). This is done for orthotropic materials in order to provide a more general reference and for direct verification with previous publications, which commonly present or derive the velocity–elasticity relationships for orthotropic materials^{1,2,3,4,5}. The relationships are also here presented for orthotropic materials because tetragonal materials, such as the UHMWPE panels, can be considered as orthotropic with an additional constraint applied. The substitutions necessary to transfer the relationships from orthotropic to tetragonal materials are presented in Section C.4. In addition, all of the relationships for tetragonal materials are explicitly presented in Section C.4.

Finally, velocity–elasticity equations are summarily presented in the form of V as a function of C_{ij} in Section C.5. The reader interested in finding more information regarding a particular velocity–elasticity relationship is referred first to Section C.5. The equation number can be located for the particular relationship in question. Then the equation can be located by its number in Section C.3 (orthotropic) or Section C.4 (tetragonal), and the details regarding the relationship are described immediately above the equation.

¹ Ashman RB. Ultrasonic determination of the elastic properties of cortical bone: techniques and limitations [thesis]. Tulane University; 1982.

² Dandekar DP, Boteler JM, Beaulieu PA. Elastic constants and delamination strength of a glass-fiber-reinforced polymer composite. *Compos Sci Tech*. 1998 Sep 1;58(9):1397–1403.

³ Kohlhauser C. Elasticity tensor determination by means of ultrasonic pulse transmission: application ranges in terms of specimen geometry and microstructure, off-diagonal tensor components, as well as different engineering and biomedical materials [dissertation]. Vienna University of Technology; 2009.

⁴ Mistou S, Karama M. Determination of the elastic properties of composite materials by tensile testing and ultrasound measurement. *J Compos Mater*. 2000 Oct;34(20):1696–1709.

⁵ Van Buskirk WC, Cowin SC, Ward RN. Ultrasonic measurement of orthotropic elastic constants of bovine femoral bone. *J Biomech Eng*. 1981;103(2):67.

C.1 Kelvin–Christoffel Equation for Wave Propagation

A plane wave is considered with a propagation vector $\vec{a}$ given by $[a_1, a_2, a_3]$ and a polarization vector $\vec{p}$ (also referred to as the direction of particle motion) given by $[p_1, p_2, p_3]$. The components of $\vec{a}$ and $\vec{p}$ use the vector notation that was presented in Section 2.1 (Table 1).

The governing equation for the plane wave traveling with a phase velocity V in an elastic solid with mass density ρ is

$$\begin{bmatrix} \Gamma_{11} - \rho V^2 & \Gamma_{12} & \Gamma_{13} \\ \Gamma_{12} & \Gamma_{22} - \rho V^2 & \Gamma_{23} \\ \Gamma_{13} & \Gamma_{23} & \Gamma_{33} - \rho V^2 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (\text{C-1})$$

Equation C-1 will be referred to as the Kelvin–Christoffel equation. The terms Γ_{ij} refer to elements of the Christoffel matrix. For a general anisotropic solid, without any material symmetry, the Γ_{ij} elements are related to the elastic constants (C_{ij}) and the propagation vector $\vec{a}$ by

$$\begin{aligned} \Gamma_{11} &= a_1^2 C_{11} + a_2^2 C_{66} + a_3^2 C_{55} + 2a_2 a_3 C_{56} + 2a_1 a_3 C_{15} + 2a_1 a_2 C_{16} \\ \Gamma_{22} &= a_1^2 C_{66} + a_2^2 C_{22} + a_3^2 C_{44} + 2a_2 a_3 C_{24} + 2a_3 a_1 C_{46} + 2a_1 a_2 C_{26} \\ \Gamma_{33} &= a_1^2 C_{55} + a_2^2 C_{44} + a_3^2 C_{33} + 2a_2 a_3 C_{34} + 2a_3 a_1 C_{35} + 2a_1 a_2 C_{45} \\ \Gamma_{12} &= a_1^2 C_{16} + a_2^2 C_{26} + a_3^2 C_{45} + a_2 a_3 (C_{46} + C_{25}) + a_3 a_1 (C_{14} + C_{56}) + a_1 a_2 (C_{12} + C_{66}) \\ \Gamma_{13} &= a_1^2 C_{15} + a_2^2 C_{46} + a_3^2 C_{35} + a_2 a_3 (C_{45} + C_{36}) + a_3 a_1 (C_{13} + C_{55}) + a_1 a_2 (C_{14} + C_{56}) \\ \Gamma_{23} &= a_1^2 C_{56} + a_2^2 C_{24} + a_3^2 C_{34} + a_2 a_3 (C_{44} + C_{23}) + a_3 a_1 (C_{36} + C_{45}) + a_1 a_2 (C_{25} + C_{46}) \end{aligned} \quad (\text{C-2})$$

Equation C-1 is taken as the starting point for the relationships derived in this appendix. For the derivation of Eq. C-1, the reader is referred to previous publications.^{1,3,6}

C.2 Simplifying the Kelvin–Christoffel Equation for Orthotropic Materials

As presented in Appendix B, an orthotropic material has only nine nonzero independent elastic constants: $C_{11}, C_{22}, C_{33}, C_{44}, C_{55}, C_{66}, C_{12}, C_{13}$, and C_{23} . Therefore, the Γ_{ij} elements written in Eq. C-2 can be simplified by removing all of the zero C_{ij} elements, leaving

⁶ Schreiber E, Anderson OL, Soga N. Elastic constants and their measurement. McGraw-Hill; 1973. p. 25–29.

$$\begin{aligned}
\Gamma_{11} &= a_1^2 C_{11} + a_2^2 C_{66} + a_3^2 C_{55} \\
\Gamma_{22} &= a_1^2 C_{66} + a_2^2 C_{22} + a_3^2 C_{44} \\
\Gamma_{33} &= a_1^2 C_{55} + a_2^2 C_{44} + a_3^2 C_{33} \\
\Gamma_{12} &= a_1 a_2 (C_{12} + C_{66}) \\
\Gamma_{13} &= a_1 a_3 (C_{13} + C_{55}) \\
\Gamma_{23} &= a_2 a_3 (C_{44} + C_{23})
\end{aligned} \tag{C-3}$$

Substituting these elements into Eq. C-1 gives the Kelvin–Christoffel equation for orthotropic materials:

$$\begin{bmatrix} a_1^2 C_{11} + a_2^2 C_{66} + a_3^2 C_{55} - \rho V^2 & a_1 a_2 (C_{12} + C_{66}) & a_1 a_3 (C_{13} + C_{55}) \\ a_1 a_2 (C_{12} + C_{66}) & a_1^2 C_{66} + a_2^2 C_{22} + a_3^2 C_{44} - \rho V^2 & a_2 a_3 (C_{44} + C_{23}) \\ a_1 a_3 (C_{13} + C_{55}) & a_2 a_3 (C_{44} + C_{23}) & a_1^2 C_{55} + a_2^2 C_{44} + a_3^2 C_{33} - \rho V^2 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{C-4}$$

C.3 Eigenanalysis to Find Velocity–Elasticity Equations in Orthotropic Materials

For a given propagation direction, $\vec{a}$, Eq. C-4 indicates that there are three different possible waves, each wave with a different polarization vector. The three roots (eigenvalues) of Eq. C-4 specify the velocities of the three waves. The eigenvector corresponding to each eigenvalue gives the polarization direction. Therefore, each wave will have a unique combination of $\vec{a}$ and $\vec{p}$. For easier identification of this combination, the velocity of the wave in the present derivations is written with two sets of subscripts separated by a comma. The first set of subscripts refers to the propagation direction and the second to the polarization direction.

The wave is classified as purely longitudinal when $\vec{a}$ and $\vec{p}$ are parallel, for which each of the components must be equal⁷:

$$a_1 = p_1, \quad a_2 = p_2, \quad a_3 = p_3 \tag{C-5}$$

The wave is classified as purely transverse when $\vec{a}$ and $\vec{p}$ are perpendicular, for which the dot product of the two vectors must be zero⁷:

$$a_1 p_1 + a_2 p_2 + a_3 p_3 = 0 \tag{C-6}$$

⁷ Nagy PB. Acoustic wave propagation in anisotropic media. Part 5 of lecture notes for a class given in spring 2013 “20-AEEM-7028 Ultrasonic NDE” at University of Cincinnati School of Aerospace Systems. Uploaded 08 Mar 2013 [accessed 24 June 2020]. [http://www.ae.uc.edu/~pnagy/ClassNotes/AEEM7028 Ultrasonic NDE/AEEM-7028 lecture%2C Part 5 Waves in Anisotropic Media.pdf](http://www.ae.uc.edu/~pnagy/ClassNotes/AEEM7028%20Ultrasonic%20NDE/AEEM-7028%20lecture%20Part%205%20Waves%20in%20Anisotropic%20Media.pdf).

The eigenvalues of Eq. C-4 are functions not only of the velocity, but also of some of the elastic constants. Therefore, each eigenvalue is a relationship between velocity and elastic constants for the particular wave under consideration (specified by the combination of $\vec{a}$ and $\vec{p}$). Generally, these velocity–elasticity relationships will involve different elastic constants depending on the propagation direction. In order to calculate all of the different elastic constants, the wave must be propagated in different directions.

Previous studies used six different propagation directions to calculate the nine independent elastic constants of orthotropic materials.^{1,2,3,4,5,8} Table C-1 lists these six directions. In the present analysis, the $[a_1, a_2, a_3]$ components of each propagation direction are substituted into Eq. C-4, and eigen-analysis is performed to find the velocity–elasticity relationships and polarization directions.

Table C-1 Six propagation directions required to calculate elastic constants of orthotropic materials

Propagation direction ($\vec{a}$)	Description	Section
[1,0,0]	+1-direction (0° fiber direction)	C.3.1
[0,1,0]	+2-direction (90° fiber direction)	C.3.2
[0,0,1]	+3-direction (ply stacking direction). Panels had a nominal thickness of 1 inch in the 3-direction.	C.3.3
[1,1,0]	Referred to as the 1–2 direction. Lies in the 1–2 plane and is 45° to both the +1 and +2 axes.	C.3.4
[1,0,1]	Referred to as the 1–3 direction. Lies in the 1–3 plane and is 45° to both the +1 and +3 axes.	C.3.5
[0,1,1]	Referred to as the 2–3 direction. Lies in the 2–3 plane and is 45° to both the +2 and +3 axes.	C.3.6

C.3.1 Propagation in the 0° Direction (+1-direction)

Propagation vector is $\vec{a} = [1,0,0]$. Substituting this into Eq. C-4 gives

$$\begin{bmatrix} C_{11} - \rho V^2 & 0 & 0 \\ 0 & C_{66} - \rho V^2 & 0 \\ 0 & 0 & C_{55} - \rho V^2 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (\text{C-7})$$

The three roots of Eq. C-7 are

$$C_{11} - \rho V^2 = 0 \quad (\text{C-8})$$

⁸ Neighbours JR, Schacher GE. Determination of elastic constants from sound-velocity measurements in crystals of general symmetry. J Appl Phys. 1967 Dec;38(13):5366–5375.

$$C_{66} - \rho V^2 = 0 \quad (\text{C-9})$$

$$C_{55} - \rho V^2 = 0 \quad (\text{C-10})$$

Rearrangement of the **first root** (Eq. C-8) gives $\rho V^2 = C_{11}$. Substituting this in Eq. C-7 gives the following set of equations:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & C_{66} - C_{11} & 0 \\ 0 & 0 & C_{55} - C_{11} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (\text{C-11})$$

This set of equations is satisfied by setting $p_2 = p_3 = 0$, while p_1 can be any number. Therefore, the polarization vector $\vec{p}$ for this root is $\vec{p} = [1,0,0]$. Since both the propagation and polarization vectors are in the 1-direction, the velocity for this root was denoted using subscripts 1,1. Furthermore, the wave is recognized as purely longitudinal by Eq. C-5. Rearrangement of Eq. C-8 gives

$$\rho(V_{1,1})^2 = C_{11} \quad (\text{C-12})$$

Rearrangement of the **second root** (Eq. C-9) gives $\rho V^2 = C_{66}$. Substituting this in Eq. C-7 gives

$$\begin{bmatrix} C_{11} - C_{66} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & C_{55} - C_{66} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (\text{C-13})$$

This set of equations is satisfied by setting $p_1 = p_3 = 0$, while p_2 can be any number. Therefore, the polarization vector $\vec{p}$ for this root is $\vec{p} = [0,1,0]$. Since the propagation direction is in the 1-direction and polarization is in the 2-direction, the velocity for this root was denoted with subscripts 1,2. Furthermore, the wave is recognized as purely transverse since it satisfies Eq. C-6. Rearrangement of Eq. C-9 gives

$$\rho(V_{1,2})^2 = C_{66} \quad (\text{C-14})$$

Rearrangement of the **third root** (Eq. C-10) gives $\rho V^2 = C_{55}$. Substituting this in Eq. C-7 gives

$$\begin{bmatrix} C_{11} - C_{55} & 0 & 0 \\ 0 & C_{66} - C_{55} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (\text{C-15})$$

This set of equations is satisfied by setting $p_1 = p_2 = 0$, while p_3 can be any number. Therefore, the polarization vector $\vec{p}$ for this root is $\vec{p} = [0,0,1]$. Since the

propagation direction is in the 1-direction and polarization is in the 3-direction, the velocity for this root was denoted with subscripts 1,3. Furthermore, the wave is recognized as purely transverse since it satisfies Eq. C-6. Rearrangement of Eq. C-10 gives

$$\rho(V_{1,3})^2 = C_{55} \quad (\text{C-16})$$

C.3.2 Propagation in the 90° Direction (+2-direction)

The propagation vector is $\vec{a} = [0,1,0]$. Substituting this into Eq. C-4 gives

$$\begin{bmatrix} C_{66} - \rho V^2 & 0 & 0 \\ 0 & C_{22} - \rho V^2 & 0 \\ 0 & 0 & C_{44} - \rho V^2 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (\text{C-17})$$

The three roots of Eq. C-17 are

$$C_{66} - \rho V^2 = 0 \quad (\text{C-18})$$

$$C_{22} - \rho V^2 = 0 \quad (\text{C-19})$$

$$C_{44} - \rho V^2 = 0 \quad (\text{C-20})$$

Rearranging the **first root** (Eq. C-18) gives $\rho V^2 = C_{66}$. Substituting this in Eq. C-17 gives

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & C_{22} - C_{66} & 0 \\ 0 & 0 & C_{44} - C_{66} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (\text{C-21})$$

This set of equations is satisfied by setting $p_2 = p_3 = 0$, while p_1 can be any number. Therefore, the polarization vector $\vec{p}$ for this root is $\vec{p} = [1,0,0]$. Since the propagation direction is the 2-direction and the polarization direction is the 1-direction, the velocity for this wave was denoted using subscripts 2,1. The wave is recognized as purely transverse by Eq. C-6. Rearrangement of Eq. C-18 gives

$$\rho(V_{2,1})^2 = C_{66} \quad (\text{C-22})$$

Rearranging the **second root** (Eq. C-19) gives $\rho V^2 = C_{22}$. Substituting this in Eq. C-17 gives

$$\begin{bmatrix} C_{66} - C_{22} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & C_{44} - C_{22} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (\text{C-23})$$

This set of equations is satisfied by setting $p_1 = p_3 = 0$, while p_2 can be any number. Therefore, the polarization vector $\vec{p}$ for this root is $\vec{p} = [0,1,0]$. Since both the propagation and polarization are in the 2-direction, the velocity for this wave was denoted using subscripts 2,2. The wave is recognized as purely longitudinal by Eq. C-5. Rearrangement of Eq. C-19 gives

$$\rho(V_{2,2})^2 = C_{22} \quad (C-24)$$

Rearranging the **third root** (Eq. C-20) gives $\rho V^2 = C_{44}$. Substituting this in Eq. C-17 gives

$$\begin{bmatrix} C_{66} - C_{44} & 0 & 0 \\ 0 & C_{22} - C_{44} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (C-25)$$

This set of equations is satisfied by setting $p_1 = p_2 = 0$, while p_3 can be any number. Therefore, the polarization vector $\vec{p}$ for this root is $\vec{p} = [0,0,1]$. Since the propagation direction is the 2-direction and the polarization direction is the 3-direction, the velocity for this wave was denoted using subscripts 2,3. The wave is recognized as purely transverse by Eq. C-6. Rearrangement of Eq. C-20 gives

$$\rho(V_{2,3})^2 = C_{44} \quad (C-26)$$

C.3.3 Propagation in the Stacking Direction (+3-direction)

The propagation vector is $\vec{a} = [0,0,1]$. Substituting this into Eq. C-4 gives

$$\begin{bmatrix} C_{55} - \rho V^2 & 0 & 0 \\ 0 & C_{44} - \rho V^2 & 0 \\ 0 & 0 & C_{33} - \rho V^2 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (C-27)$$

The three roots of Eq. C-27 are

$$C_{55} - \rho V^2 = 0 \quad (C-28)$$

$$C_{44} - \rho V^2 = 0 \quad (C-29)$$

$$C_{33} - \rho V^2 = 0 \quad (C-30)$$

Rearranging the **first root** (Eq. C-28) gives $\rho V^2 = C_{55}$. Substituting this in Eq. C-27 gives

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & C_{44} - C_{55} & 0 \\ 0 & 0 & C_{33} - C_{55} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (C-31)$$

This set of equations is satisfied by setting $p_2 = p_3 = 0$, while p_1 can be any number. Therefore, the polarization vector $\vec{p}$ for this root is $\vec{p} = [1,0,0]$. Since the propagation direction is the 3-direction and the polarization direction is the 1-direction, the velocity for this wave was denoted using subscripts 3,1. The wave is recognized as purely transverse by Eq. C-6. Rearrangement of Eq. C-18 gives

$$\rho(V_{3,1})^2 = C_{55} \quad (\text{C-32})$$

Rearranging the **second root** (Eq. C-29) gives $\rho V^2 = C_{44}$. Substituting this in Eq. C-27 gives

$$\begin{bmatrix} C_{55} - C_{44} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & C_{33} - C_{44} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (\text{C-33})$$

This set of equations is satisfied by setting $p_1 = p_3 = 0$, while p_2 can be any number. Therefore, the polarization vector $\vec{p}$ for this root is $\vec{p} = [0,1,0]$. Since propagation is in the 3-direction and polarization in the 2-direction, the velocity for this wave was denoted using subscripts 3,2. The wave is recognized as purely transverse by Eq. C-6. Rearrangement of Eq. C-19 gives

$$\rho(V_{3,2})^2 = C_{44} \quad (\text{C-34})$$

Rearranging the **third root** (Eq. C-30) gives $\rho V^2 = C_{33}$. Substituting this in Eq. C-27 gives

$$\begin{bmatrix} C_{55} - C_{33} & 0 & 0 \\ 0 & C_{44} - C_{33} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (\text{C-35})$$

This set of equations is satisfied by setting $p_1 = p_2 = 0$, while p_3 can be any number. Therefore, the polarization vector $\vec{p}$ for this root is $\vec{p} = [0,0,1]$. Since propagation and polarization are both in the 3-direction, the velocity for this wave was denoted using subscripts 3,3. The wave is recognized as purely longitudinal by Eq. C-5. Rearrangement of Eq. C-30 gives

$$\rho(V_{3,3})^2 = C_{33} \quad (\text{C-36})$$

C.3.4 Propagation in the 1–2 Direction

The propagation vector is $\vec{a} = [1,1,0]$. This direction is in the 1–2 plane at 45° to both the +1-direction and the +2-direction. Equivalently, using directional cosines, propagation is $\vec{a} = [\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0]$. Substituting this into Eq. C-4 gives

$$\begin{bmatrix} \frac{C_{11}+C_{66}}{2} - \rho V^2 & \frac{C_{12}+C_{66}}{2} & 0 \\ \frac{C_{12}+C_{66}}{2} & \frac{C_{22}+C_{66}}{2} - \rho V^2 & 0 \\ 0 & 0 & \frac{C_{44}+C_{55}}{2} - \rho V^2 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (\text{C-37})$$

The three roots of Eq. C-37 are

$$\frac{C_{11}+C_{22}+2C_{66}+\sqrt{(C_{11}-C_{22})^2+4(C_{12}+C_{66})^2}}{4} - \rho V^2 = 0 \quad (\text{C-38})$$

$$\frac{C_{11}+C_{22}+2C_{66}-\sqrt{(C_{11}-C_{22})^2+4(C_{12}+C_{66})^2}}{4} - \rho V^2 = 0 \quad (\text{C-39})$$

$$\frac{C_{44}+C_{55}}{2} - \rho V^2 = 0 \quad (\text{C-40})$$

Rearranging the **first root** (Eq. C-38) gives

$$\rho V^2 = \frac{C_{11}+C_{22}+2C_{66}+\sqrt{(C_{11}-C_{22})^2+4(C_{12}+C_{66})^2}}{4} \quad (\text{C-41})$$

Substituting this for ρV^2 in Eq. C-37 yields the following set of equations:

$$\begin{bmatrix} \frac{C_{11}-C_{22}-\Delta}{4} & \frac{C_{12}+C_{66}}{2} & 0 \\ \frac{C_{12}+C_{66}}{2} & \frac{C_{22}-C_{11}-\Delta}{4} & 0 \\ 0 & 0 & \frac{2(C_{44}+C_{55}-C_{66})-C_{22}-C_{11}-\Delta}{4} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad (\text{C-42})$$

where $\Delta = \sqrt{(C_{11} - C_{22})^2 + 4(C_{12} + C_{66})^2}$. The set of equations given by Eq. C-42 is satisfied by the following polarization vector:

$$\vec{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} \frac{2(C_{12}+C_{66})}{C_{22}-C_{11}+\Delta} \\ 1 \\ 0 \end{bmatrix} \quad (\text{C-43})$$

Equation C-43 indicates that the polarization direction is dependent on the material properties, since it is a function of elements of the C-Matrix. In the case that $C_{11} = C_{22}$, then p_1 in Eq. C-43 becomes 1 and the polarization vector becomes $\vec{p} = [1,1,0]$. In this case, the polarization vector would be parallel to the propagation vector $\vec{a}$ and the wave would be purely longitudinal by Eq. C-5. Therefore, for the general case that $C_{11} \neq C_{22}$, the wave is classified as quasi-longitudinal.

The subscripts used to refer to the velocity for this wave are described here. The first set of subscripts corresponds to the propagation direction, which is $\vec{a} = [1,1,0]$. This direction corresponds to a vector at 45° between the +1-direction and the +2-direction. Therefore, this direction is also referred to as the 1–2 direction,

and the first set of subscripts is written as “12”. The second set of subscripts corresponds to the polarization direction. The case when $C_{22} = C_{11}$ is used for the sake of simplified notation. For this case, the polarization direction is $\vec{p} = [1,1,0]$. This is the same direction as the propagation. Therefore, the second set of subscripts is also “12”. Using these two subscripts, the velocity is written with the notation $V_{12,12}^Q$. The superscript Q denotes the quasi, rather than pure, nature of the wave. Using this notation in Eq. C-41 gives

$$\rho(V_{12,12}^Q)^2 = \frac{C_{11}+C_{22}+2C_{66}+\sqrt{(C_{11}-C_{22})^2+4(C_{12}+C_{66})^2}}{4} \quad (C-44)$$

Rearranging the **second root** (Eq. C-39) gives

$$\rho V^2 = \frac{C_{11}+C_{22}+2C_{66}-\sqrt{(C_{11}-C_{22})^2+4(C_{12}+C_{66})^2}}{4} \quad (C-45)$$

Substituting this for ρV^2 in Eq. C-37 yields the following set of equations:

$$\begin{bmatrix} \frac{C_{11}-C_{22}+\Delta}{4} & \frac{C_{12}+C_{66}}{2} & 0 \\ \frac{C_{12}+C_{66}}{2} & \frac{C_{22}-C_{11}+\Delta}{4} & 0 \\ 0 & 0 & \frac{2(C_{44}+C_{55}-C_{66})-C_{22}-C_{11}+\Delta}{4} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad (C-46)$$

where Δ is the same as in Eq. C-42. The set of equations given by Eq. C-46 is satisfied by the following polarization vector:

$$\vec{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} -1 \cdot \frac{2(C_{12}+C_{66})}{C_{11}-C_{22}+\Delta} \\ 1 \\ 0 \end{bmatrix} \quad (C-47)$$

Equation C-47 indicates that the polarization direction is dependent on the material properties, since it is a function of elements of the C-Matrix. In the case that $C_{11} = C_{22}$, then p_1 in Eq. C-47 becomes -1 and the polarization vector becomes $\vec{p} = [-1,1,0]$. In this case, the polarization vector would be orthogonal to the propagation vector $\vec{a}$ and the wave would be purely transverse by Eq. C-6: $a_1 p_1 + a_2 p_2 + a_3 p_3 = 1 \cdot (-1) + 1 \cdot 1 + 0 \cdot 0 = 0$. Therefore, for the general case that $C_{11} \neq C_{22}$, the wave is classified as quasi-transverse.

The subscripts used in denoting the velocity for this wave are explained here. The propagation vector is in the direction $[1,1,0]$. Therefore, the first set of subscripts is “12” (see discussion above Eq. C-44). The second set of subscripts refers to the polarization direction. The case when $C_{22} = C_{11}$ is used for the sake of simplified notation. For this case, the polarization direction is $\vec{p} = [-1,1,0]$. This direction corresponds to a vector at 45° between the -1 -direction and the $+2$ -direction. In this

report, the -1 -direction is also referred to as the $\bar{1}$ -direction, where the overbar indicates negative direction. Therefore, the second set of subscripts is given by “ $\bar{1}2$ ”. Using these two subscripts, the velocity is written with the notation $V_{12,\bar{1}2}^Q$. The superscript Q denotes the quasi, rather than pure, nature of the wave. Using this notation in Eq. C-45 gives

$$\rho(V_{12,\bar{1}2}^Q)^2 = \frac{C_{11}+C_{22}+2C_{66}-\sqrt{(C_{11}-C_{22})^2+4(C_{12}+C_{66})^2}}{4} \quad (C-48)$$

Rearranging the **third root** (Eq. C-40) gives

$$\rho V^2 = \frac{C_{44}+C_{55}}{2} \quad (C-49)$$

Substituting this for ρV^2 in Eq. C-37 gives

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot \begin{bmatrix} C_{11} - C_{44} - C_{55} + C_{66} & C_{12} + C_{66} & 0 \\ C_{12} + C_{66} & C_{22} - C_{44} - C_{55} + C_{66} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (C-50)$$

This set of equations is satisfied by setting $p_1 = p_2 = 0$, while p_3 can be any number. Therefore, the polarization vector $\vec{p}$ for this wave is $\vec{p} = [0,0,1]$. The polarization is orthogonal to the propagation, and therefore this wave corresponds to a pure transverse wave (Eq. C-6).

The velocity is denoted with the first set of subscripts corresponding to the propagation direction. The propagation direction for this wave is referred to with the subscripts “12” (see discussion above Eq. C-44.) The second set of subscripts corresponds to the polarization direction, which is in the 3-direction. Therefore, the velocity is denoted as $V_{12,3}$. Using this notation, Eq. C-49 becomes

$$\rho(V_{12,3})^2 = \frac{C_{44}+C_{55}}{2} \quad (C-51)$$

C.3.5 Propagation in the 1–3 Direction

The propagation vector is $\vec{a} = [1,0,1]$. This direction is in the 1–3 plane at 45° to both the $+1$ -direction and the $+3$ -direction. Using directional cosines, the propagation direction is equivalently written as $\vec{a} = [\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}]$. Substituting this into Eq. C-4 gives

$$\begin{bmatrix} \frac{C_{11}+C_{55}}{2} - \rho V^2 & 0 & \frac{C_{13}+C_{55}}{2} \\ 0 & \frac{C_{44}+C_{66}}{2} - \rho V^2 & 0 \\ \frac{C_{13}+C_{55}}{2} & 0 & \frac{C_{33}+C_{55}}{2} - \rho V^2 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (C-52)$$

The three roots of Eq. C-52 are

$$\frac{C_{11}+C_{33}+2C_{55}+\sqrt{(C_{11}-C_{33})^2+4(C_{13}+C_{55})^2}}{4} - \rho V^2 = 0 \quad (\text{C-53})$$

$$\frac{C_{11}+C_{33}+2C_{55}-\sqrt{(C_{11}-C_{33})^2+4(C_{13}+C_{55})^2}}{4} - \rho V^2 = 0 \quad (\text{C-54})$$

$$\frac{C_{44}+C_{66}}{2} - \rho V^2 = 0 \quad (\text{C-55})$$

Rearranging the **first root** (Eq. C-53) gives

$$\rho V^2 = \frac{C_{11}+C_{33}+2C_{55}+\sqrt{(C_{11}-C_{33})^2+4(C_{13}+C_{55})^2}}{4} \quad (\text{C-56})$$

Substituting this for ρV^2 in Eq. C-52 yields the following set of equations:

$$\begin{bmatrix} \frac{C_{11}-C_{33}-\Psi}{4} & 0 & \frac{C_{13}+C_{55}}{2} \\ 0 & \frac{2(C_{44}-C_{55}+C_{66})-C_{11}-C_{33}-\Psi}{4} & 0 \\ \frac{C_{13}+C_{55}}{2} & 0 & \frac{C_{33}-C_{11}-\Psi}{4} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad (\text{C-57})$$

where $\Psi = \sqrt{(C_{11} - C_{33})^2 + 4(C_{13} + C_{55})^2}$. The set of equations given by Eq. C-57 is satisfied by the following polarization vector:

$$\vec{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} \frac{2(C_{13}+C_{55})}{C_{33}-C_{11}+\Psi} \\ 0 \\ 1 \end{bmatrix} \quad (\text{C-58})$$

Equation C-58 indicates that the polarization direction is dependent on the material properties, since it is a function of elements of the C-Matrix. In the case that $C_{33} = C_{11}$, then p_1 in Eq. C-58 becomes 1 and the polarization vector becomes $\vec{p} = [1,0,1]$. In this case, the polarization vector would be parallel to the propagation vector $\vec{a}$ and the wave would be purely longitudinal (Eq. C-5). Therefore, for the general case that $C_{33} \neq C_{11}$, the wave is classified as quasi-longitudinal.

The subscripts used to refer to the velocity for this wave are described here. The first set of subscripts corresponds to the propagation direction, which is $\vec{a} = [1,0,1]$. This direction corresponds to a vector at 45° between the +1-direction and the +3-direction. Therefore, this direction is also referred to as the 1–3 direction, and the first set of subscripts is written as “13”. The second set of subscripts corresponds to the polarization direction. The case when $C_{33} = C_{11}$ is used for the sake of simplified notation. For this case, the polarization direction is $\vec{p} = [1,0,1]$. This is the same direction as the propagation. Therefore, the second set of subscripts is also “13”. Using these two subscripts, the velocity is written with the notation

$V_{13,13}^Q$. The superscript Q denotes the quasi, rather than pure, nature of the wave. Using this notation in Eq. C-56 gives

$$\rho(V_{13,13}^Q)^2 = \frac{c_{11}+c_{33}+2c_{55}+\sqrt{(c_{11}-c_{33})^2+4(c_{13}+c_{55})^2}}{4} \quad (C-59)$$

Rearranging the **second root** (Eq. C-39) gives

$$\rho V^2 = \frac{c_{11}+c_{33}+2c_{55}-\sqrt{(c_{11}-c_{33})^2+4(c_{13}+c_{55})^2}}{4} \quad (C-60)$$

Substituting this for ρV^2 in Eq. C-52 yields the following set of equations:

$$\begin{bmatrix} \frac{c_{11}-c_{33}+\Psi}{4} & 0 & \frac{c_{13}+c_{55}}{2} \\ 0 & \frac{2(c_{44}-c_{55}+c_{66})-c_{11}-c_{33}+\Psi}{4} & 0 \\ \frac{c_{13}+c_{55}}{2} & 0 & \frac{c_{33}-c_{11}+\Psi}{4} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad (C-61)$$

where Ψ is the same as in Eq. C-57. The set of equations given by Eq. C-61 is satisfied by the following polarization vector:

$$\vec{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} -1 \cdot \frac{2(c_{13}+c_{55})}{c_{11}-c_{33}+\Psi} \\ 0 \\ 1 \end{bmatrix} \quad (C-62)$$

Equation C-62 indicates that the polarization direction is dependent on the material properties, since it is a function of elements of the C-Matrix. In the case that $c_{11} = c_{33}$, then p_1 in Eq. C-62 becomes -1 and the polarization vector becomes $\vec{p} = [-1, 0, 1]$. In this case, the polarization vector would be orthogonal to the propagation vector $\vec{a}$ and the wave would be purely transverse by Eq. C-6: $a_1 p_1 + a_2 p_2 + a_3 p_3 = 1 \cdot (-1) + 0 \cdot 0 + 1 \cdot 1 = 0$. Therefore, for the general case that $c_{11} \neq c_{33}$, the wave is classified as quasi-transverse.

The subscripts used in denoting the velocity for this wave are explained here. The propagation vector is in the direction $[1, 0, 1]$. Therefore, the first set of subscripts is “13” (see discussion above Eq. C-59). The second set of subscripts refers to the polarization direction. The case when $c_{33} = c_{11}$ is used for the sake of simplified notation. For this case, the polarization direction is $\vec{p} = [-1, 0, 1]$. This direction corresponds to a vector at 45° between the -1 -direction and the $+3$ -direction. In this report, the -1 -direction is also referred to as the $\bar{1}$ -direction, where the overbar indicates negative direction. Therefore, the second set of subscripts is given by “ $\bar{1}3$ ”. Using these two subscripts, the velocity is written with the notation $V_{13,\bar{1}3}^Q$. The superscript Q denotes the quasi, rather than pure, nature of the wave. Using this notation in Eq. C-60 gives

$$\rho(V_{13,13}^Q)^2 = \frac{c_{11}+c_{33}+2c_{55}-\sqrt{(c_{11}-c_{33})^2+4(c_{13}+c_{55})^2}}{4} \quad (C-63)$$

Rearranging the **third root** (Eq. C-55) gives

$$\rho V^2 = \frac{c_{44}+c_{55}}{2} \quad (C-64)$$

Substituting this for ρV^2 in Eq. C-52 gives

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot \begin{bmatrix} c_{11} - c_{44} + c_{55} - c_{66} & 0 & c_{13} + c_{55} \\ 0 & 0 & 0 \\ c_{13} + c_{55} & 0 & c_{33} - c_{44} + c_{55} - c_{66} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (C-65)$$

This set of equations is satisfied by setting $p_1 = p_3 = 0$, while p_2 can be any number. Therefore, the polarization vector $\vec{p}$ for this wave is $\vec{p} = [0,1,0]$. The polarization is orthogonal to the propagation, and therefore this wave corresponds to a pure transverse wave (Eq. C-6).

The velocity is denoted with the first set of subscripts corresponding to the propagation direction. Propagation direction is referred to with the subscripts “13” (see discussion above Eq. C-59). The second set of subscripts corresponds to the polarization direction, which is in the 2-direction. Therefore, the velocity is denoted as $V_{13,2}$. Using this notation, Eq. C-64 becomes

$$\rho(V_{13,2})^2 = \frac{c_{44}+c_{66}}{2} \quad (C-66)$$

C.3.6 Propagation in the 2–3 Direction

The propagation vector is $\vec{a} = [0,1,1]$. This direction is in the 2–3 plane at 45° to both the +2-direction and the +3-direction. Using directional cosines, the propagation vector is equivalently written as $\vec{a} = [0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]$. Substituting this into Eq. C-4 gives

$$\begin{bmatrix} \frac{c_{55}+c_{66}}{2} - \rho V^2 & 0 & 0 \\ 0 & \frac{c_{22}+c_{44}}{2} - \rho V^2 & \frac{c_{23}+c_{44}}{2} \\ 0 & \frac{c_{23}+c_{44}}{2} & \frac{c_{33}+c_{44}}{2} - \rho V^2 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (C-67)$$

The three roots of Eq. C-67 are

$$\frac{c_{55}+c_{66}}{2} - \rho V^2 = 0 \quad (C-68)$$

$$\frac{c_{22}+c_{33}+2c_{44}+\sqrt{(c_{22}-c_{33})^2+4(c_{23}+c_{44})^2}}{4} - \rho V^2 = 0 \quad (C-69)$$

$$\frac{C_{22}+C_{33}+2C_{44}-\sqrt{(C_{22}-C_{33})^2+4(C_{23}+C_{44})^2}}{4}-\rho V^2=0 \quad (\text{C-70})$$

Rearranging the **first root** (Eq. C-68) gives

$$\rho V^2 = \frac{C_{55}+C_{66}}{2} \quad (\text{C-71})$$

Substituting this for ρV^2 in Eq. C-67 gives

$$\left(\frac{1}{2}\right) \cdot \begin{bmatrix} 0 & 0 & 0 \\ 0 & C_{22}+C_{44}-C_{55}-C_{66} & C_{23}+C_{44} \\ 0 & C_{23}+C_{44} & C_{33}+C_{44}-C_{55}-C_{66} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (\text{C-72})$$

This set of equations is satisfied by setting $p_2 = p_3 = 0$, while p_1 can be any number. Therefore, the polarization vector $\vec{p}$ for this wave is $\vec{p} = [1,0,0]$. The polarization is orthogonal to the propagation, and therefore this wave corresponds to a pure transverse wave (Eq. C-6).

The velocity is denoted with the first set of subscripts corresponding to the propagation direction. The propagation direction is referred to with the subscripts “23” since $\vec{a}$ is at 45° to both the +2-direction and the +3-direction. The second set of subscripts corresponds to the polarization direction, which is in the 1-direction. Therefore, the velocity is denoted as $V_{23,1}$. Using this notation, Eq. C-71 becomes

$$\rho(V_{23,1})^2 = \frac{C_{55}+C_{66}}{2} \quad (\text{C-73})$$

Rearranging the **second root** (Eq. C-69) gives

$$\rho V^2 = \frac{C_{22}+C_{33}+2C_{44}+\sqrt{(C_{22}-C_{33})^2+4(C_{23}+C_{44})^2}}{4} \quad (\text{C-74})$$

Substituting this for ρV^2 in Eq. C-67 yields the following set of equations:

$$\begin{bmatrix} \frac{2(C_{55}+C_{66}-C_{44})-C_{22}-C_{33}-\Omega}{4} & 0 & 0 \\ 0 & \frac{C_{22}-C_{33}-\Omega}{4} & \frac{C_{23}+C_{44}}{2} \\ 0 & \frac{C_{23}+C_{44}}{2} & \frac{C_{33}-C_{22}-\Omega}{4} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad (\text{C-75})$$

where $\Omega = \sqrt{(C_{22}-C_{33})^2+4(C_{23}+C_{44})^2}$. The set of equations given by Eq. C-75 is satisfied by the following polarization vector:

$$\vec{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{2(C_{23}+C_{44})}{C_{33}-C_{22}+\Omega} \\ 1 \end{bmatrix} \quad (\text{C-76})$$

Equation C-76 indicates that the polarization direction is dependent on the material properties, since it is a function of elements of the C-Matrix. In the case that $C_{33} = C_{22}$, then p_2 in Eq. C-76 becomes 1 and the polarization vector becomes $\vec{p} = [0,1,1]$. In this case, the polarization vector would be parallel to the propagation vector $\vec{a}$ and the wave would be purely longitudinal (Eq. C-5). Therefore, for the general case that $C_{33} \neq C_{22}$, the wave is classified as quasi-longitudinal.

The subscripts used to refer to the velocity for this wave are described here. The first set of subscripts corresponds to the propagation direction, which is $\vec{a} = [0,1,1]$. This direction corresponds to a vector at 45° between the +2-direction and the +3-direction. Therefore, this direction is also referred to as the 2–3 direction, and the first set of subscripts is written as “23”. The second set of subscripts corresponds to the polarization direction. The case when $C_{33} = C_{22}$ is used for the sake of simplified notation. For this case, the polarization direction is $\vec{p} = [0,1,1]$. This is the same direction as the propagation. Therefore, the second set of subscripts is also “23”. Using these two subscripts, the velocity is written with the notation $V_{23,23}^Q$. The superscript Q denotes the quasi, rather than pure, nature of the wave. Using this notation in Eq. C-74 gives

$$\rho(V_{23,23}^Q)^2 = \frac{C_{22}+C_{33}+2C_{44}+\sqrt{(C_{22}-C_{33})^2+4(C_{23}+C_{44})^2}}{4} \quad (C-77)$$

Rearranging the **third root** (Eq. C-70) gives

$$\rho V^2 = \frac{C_{22}+C_{33}+2C_{44}-\sqrt{(C_{22}-C_{33})^2+4(C_{23}+C_{44})^2}}{4} \quad (C-78)$$

Substituting this for ρV^2 in Eq. C-67 yields the following set of equations:

$$\begin{bmatrix} \frac{2(C_{55}+C_{66}-C_{44})-C_{22}-C_{33}+\Omega}{4} & 0 & 0 \\ 0 & \frac{C_{22}-C_{33}+\Omega}{4} & \frac{C_{23}+C_{44}}{2} \\ 0 & \frac{C_{23}+C_{44}}{2} & \frac{C_{33}-C_{22}+\Omega}{4} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad (C-79)$$

where Ω is the same as in Eq. C-75. The set of equations given by Eq. C-79 is satisfied by the following polarization vector:

$$\vec{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \cdot \frac{2(C_{23}+C_{44})}{C_{22}-C_{33}+\Omega} \\ 1 \end{bmatrix} \quad (C-80)$$

Equation C-80 indicates that the polarization direction is dependent on the material properties, since it is a function of elements of the C-Matrix. In the case that $C_{22} = C_{33}$, then p_2 in Eq. C-80 becomes -1 and the polarization vector becomes $\vec{p} =$

$[0, -1, 1]$. In this case, the polarization vector would be orthogonal to the propagation vector $\vec{a}$ and the wave would be purely transverse by Eq. C-6: $a_1 p_1 + a_2 p_2 + a_3 p_3 = 0 \cdot 0 + 1 \cdot (-1) + 1 \cdot 1 = 0$. Therefore, for the general case that $C_{22} \neq C_{33}$, the wave is classified as quasi-transverse.

The subscripts used in denoting the velocity for this wave are explained here. The propagation vector is in the direction $[0, 1, 1]$. Therefore, the first set of subscripts is “23” (see discussion above Eq. C-77). The second set of subscripts refers to the polarization direction. The case when $C_{33} = C_{22}$ is used for the sake of simplified notation. For this case, the polarization direction is $\vec{p} = [0, -1, 1]$. This direction corresponds to a vector at 45° between the -2 -direction and the $+3$ -direction. In this report, the -2 -direction is also referred to as the $\bar{2}$ -direction, where the overbar indicates negative direction. Therefore, the second set of subscripts is given by “ $\bar{2}3$ ”. Using these two subscripts, the velocity is written with the notation $V_{23, \bar{2}3}^Q$. The superscript Q denotes the quasi, rather than pure, nature of the wave. Using this notation in Eq. C-78 gives

$$\rho(V_{23, \bar{2}3}^Q)^2 = \frac{C_{22} + C_{33} + 2C_{44} - \sqrt{(C_{22} - C_{33})^2 + 4(C_{23} + C_{44})^2}}{4} \quad (\text{C-81})$$

C.4 Velocity–Elasticity Relationships for Tetragonal Materials

The velocity–elasticity relationships for tetragonal materials are derived in an analogous manner as was done for orthotropic materials. First, the Γ_{ij} elements of the Kelvin–Christoffel equation are simplified from the general forms given by Eq. C-2 for anisotropic materials. Those general expressions of the Γ_{ij} elements included all of the 21 elements of the C-Matrix for anisotropic materials. However, the C-Matrix of tetragonal materials is greatly simplified compared to that of anisotropic materials. As a visual comparison, the C-Matrix for anisotropic materials is reproduced below, which was presented as Eq. A-6 in Appendix A:

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix}$$

Below is the C-Matrix for tetragonal materials, which was presented as Eq. 4:

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix}$$

Comparing these two C-Matrices indicates that tetragonal materials have only six independent elastic constants: C_{11} , C_{33} , C_{44} , C_{66} , C_{12} , and C_{23} , and all other C_{ij} elements are zero. In addition, comparison of the C-Matrices above indicates that some of the elastic constants that are independent for anisotropic materials are not independent for tetragonal materials. The following elastic constants of anisotropic materials are equivalent in tetragonal materials:

$$C_{22} = C_{11}, C_{55} = C_{44}, C_{23} = C_{13} \quad (\text{C-82})$$

Therefore, the expressions of the Γ_{ij} elements given by Eq. C-2 can be reduced for tetragonal materials by removing the zero C_{ij} elements and applying the equalities of Eq. C-82, giving

$$\begin{aligned} \Gamma_{11} &= a_1^2 C_{11} + a_2^2 C_{66} + a_3^2 C_{55} \\ \Gamma_{22} &= a_1^2 C_{66} + a_2^2 C_{11} + a_3^2 C_{44} \\ \Gamma_{33} &= (a_1^2 + a_2^2) C_{44} + a_3^2 C_{33} \\ \Gamma_{12} &= a_1 a_2 (C_{12} + C_{66}) \\ \Gamma_{13} &= a_1 a_3 (C_{13} + C_{44}) \\ \Gamma_{23} &= a_2 a_3 (C_{44} + C_{13}) \end{aligned} \quad (\text{C-83})$$

Using these simplified Γ_{ij} elements in Eq. C-1 reduces the Kelvin–Christoffel equation for tetragonal materials to the following:

$$\begin{bmatrix} a_1^2 C_{11} + a_2^2 C_{66} + a_3^2 C_{44} - \rho V^2 & a_1 a_2 (C_{12} + C_{66}) & a_1 a_3 (C_{13} + C_{44}) \\ a_1 a_2 (C_{12} + C_{66}) & a_1^2 C_{66} + a_2^2 C_{11} + a_3^2 C_{44} - \rho V^2 & a_2 a_3 (C_{44} + C_{13}) \\ a_1 a_3 (C_{13} + C_{44}) & a_2 a_3 (C_{44} + C_{13}) & (a_1^2 + a_2^2) C_{44} + a_3^2 C_{33} - \rho V^2 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (\text{C-84})$$

Sections C.4.1 through C.4.6 present the velocity–elasticity relationships for the same six propagation directions as were used for orthotropic materials. The six propagation directions were summarized in Table C-1. For orthotropic materials, these six directions were necessary in order to relate ultrasonic velocity to each of the nine independent C-Matrix elements of orthotropic materials. Tetragonal materials have only six independent C-Matrix elements, and so fewer propagation directions are necessary to relate the six elements to an ultrasonic velocity.

However, the analysis is still presented for all six directions for comparison with orthotropic materials.

The velocity–elasticity relationships for tetragonal materials were derived following an analogous eigenanalysis to that which was presented in Section C.3 for orthotropic materials. This was done computationally. However, a simplified method is to start with the relationships and polarization vectors derived for orthotropic materials and apply the equalities given by Eq. C-82. This method will be referred to here as the substitution method. By comparison with the computational analysis, it was found that the substitution method can be used with only a few exceptions. Sections C.4.1 through C.4.6 generally present the substitution method, with the additional caveats noted.

C.4.1 Propagation in the 0° Direction (+1-direction)

The velocities and polarization vectors for the three waves are taken directly from Section C.3.1 with the substitutions of Eq. C-82. This gives

$$\rho(V_{1,1})^2 = C_{11} \quad (\text{C-85})$$

$$\rho(V_{1,2})^2 = C_{66} \quad (\text{C-86})$$

$$\rho(V_{1,3})^2 = C_{44} \quad (\text{C-87})$$

C.4.2 Propagation in the 90° Direction (+2-direction)

The velocities and polarization vectors for the three waves are taken directly from Section C.3.1 with the substitutions of Eq. C-82. This gives

$$\rho(V_{2,1})^2 = C_{66} \quad (\text{C-88})$$

$$\rho(V_{2,2})^2 = C_{11} \quad (\text{C-89})$$

$$\rho(V_{2,3})^2 = C_{44} \quad (\text{C-90})$$

C.4.3 Propagation in the Stacking Direction (+3-direction)

For this propagation direction, there is a caveat to using the relationships derived for orthogonal materials (Section C.3.3). The propagation direction is $\vec{a} = [0,0,1]$. Substituting this into Eq. C-84 gives

$$\begin{bmatrix} C_{44} - \rho V^2 & 0 & 0 \\ 0 & C_{44} - \rho V^2 & 0 \\ 0 & 0 & C_{33} - \rho V^2 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (\text{C-91})$$

There are only two unique eigenvalues for Eq. C-91:

$$C_{44} - \rho V^2 = 0 \quad (\text{C-92})$$

$$C_{33} - \rho V^2 = 0 \quad (\text{C-93})$$

Equation C-93 corresponds to the third root in Section C.3.3 for orthogonal materials. The same velocity–elasticity relationship is derived:

$$\rho(V_{3,3})^2 = C_{33} \quad (\text{C-94})$$

Equation C-92 is a double root. Rearrangement gives $\rho V^2 = C_{44}$. Substituting this in Eq. C-91 gives

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & C_{33} - C_{44} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (\text{C-95})$$

These equations are satisfied by setting $p_3 = 0$ while both p_1 and p_2 can be any number. For example, any of the following polarization vectors satisfy Eq. C-95:

$$\vec{p} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{p} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \vec{p} = \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \vec{p} = \begin{bmatrix} 1 \\ 1/4 \\ 0 \end{bmatrix} \quad (\text{C-96})$$

This indicates that polarization is in any direction along the plane spanning the 1-direction and the 2-direction, which is the plane of the ply (Fig. 1). The wave is recognized as purely transverse by Eq. C-6; taking $\vec{p} = [1,1,0]$ as an example, Eq. C-6 becomes $a_1 p_1 + a_2 p_2 + a_3 p_3 = 0 \cdot 1 + 0 \cdot 1 + 1 \cdot 0 = 0$.

The velocity is denoted with two subscripts. The first subscript is 3, corresponding to the propagation direction. The second subscript corresponds to the polarization direction. For this wave, the second subscript is the perpendicular symbol, $\perp$, to indicate that the polarization direction is in any direction in the plane perpendicular to the propagation direction. Using this notation, the velocity–elasticity equation for this wave is

$$\rho(V_{3,\perp})^2 = C_{44} \quad (\text{C-97})$$

C.4.4 Propagation in the 1–2 Direction

For this propagation direction, the substitution method cannot be used to find the velocity–elasticity relationships for tetragonal materials by starting from the corresponding velocity–elasticity equations derived for orthotropic materials. The

reasons are due to simplification of the roots, as explained below. Therefore, the eigenanalysis is presented here.

The propagation vector is $\vec{a} = [1,0,1]$. This direction is in the plane spanned by the +1-direction and +2-direction and is 45° to both of these directions. Using directional cosines, the propagation vector is equivalently written as $\vec{a} = [\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0]$. Substituting this into Eq. C-84 gives

$$\begin{bmatrix} \frac{C_{11}+C_{66}}{2} - \rho V^2 & \frac{C_{12}+C_{66}}{2} & 0 \\ \frac{C_{12}+C_{66}}{2} & \frac{C_{11}+C_{66}}{2} - \rho V^2 & 0 \\ 0 & 0 & C_{44} - \rho V^2 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (\text{C-98})$$

The three roots of Eq. C-98 are

$$\frac{C_{11}+C_{12}}{2} + C_{66} - \rho V^2 = 0 \quad (\text{C-99})$$

$$\frac{C_{11}-C_{12}}{2} - \rho V^2 = 0 \quad (\text{C-100})$$

$$C_{44} - \rho V^2 = 0 \quad (\text{C-101})$$

Rearranging the **first root** (Eq. C-99) gives $\rho V^2 = (C_{11} + C_{12})/2 + C_{66}$. Substituting this for ρV^2 in Eq. C-98 gives

$$\left(\frac{1}{2}\right) \cdot \begin{bmatrix} -C_{12} - C_{66} & C_{12} + C_{66} & 0 \\ C_{12} + C_{66} & -C_{12} - C_{66} & 0 \\ 0 & 0 & -C_{11} - C_{12} + 2(C_{44} - C_{66}) \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (\text{C-102})$$

This set of equations is satisfied by setting $p_3 = 0$, while p_1 must equal p_2 . Therefore, the polarization vector can be written as $\vec{p} = [1,1,0]$. This polarization vector is parallel to the propagation direction, and therefore the wave is classified as purely longitudinal by Eq. C-5 ($a_1 = p_1 = 1$, $a_2 = p_2 = 1$, $a_3 = p_3 = 0$).

The subscripts used to refer to the velocity for this wave are described here. The first set of subscripts corresponds to the propagation direction, which is $\vec{a} = [1,1,0]$. This direction corresponds to a vector at 45° between the +1-direction and the +2-direction. Therefore, this direction is also referred to as the 1–2 direction, and the first set of subscripts is written as “12”. The second set of subscripts corresponds to the polarization direction. For this wave, the polarization direction is the same as the propagation direction, and the second set of subscripts is also “12”. Using this notation in Eq. C-99 gives

$$\rho(V_{12,12})^2 = \frac{C_{11}+C_{12}}{2} + C_{66} \quad (\text{C-103})$$

Rearranging the **second root** (Eq. C-100) gives $\rho V^2 = (C_{11} - C_{12})/2$. Substituting this for ρV^2 in Eq. C-98 gives

$$\left(\frac{1}{2}\right) \cdot \begin{bmatrix} C_{12} + C_{66} & C_{12} + C_{66} & 0 \\ C_{12} + C_{66} & C_{12} + C_{66} & 0 \\ 0 & 0 & C_{12} - C_{11} + 2C_{44} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (\text{C-104})$$

This set of equations is satisfied by setting $p_3 = 0$ and $p_1 = -p_2$. Therefore, the polarization vector can be written as $\vec{p} = [1, -1, 0]$. The wave is purely transverse, since $p_1 a_1 + p_2 a_2 + p_3 a_3 = 1 \cdot 1 + (-1) \cdot 1 + 0 \cdot 1 = 0$ (Eq. C-6).

The subscripts used to refer to the velocity for this wave are described here. The first set of subscripts corresponds to the propagation direction and was written as “12” (see discussion above C-103). The second set of subscripts corresponds to the polarization direction. For this wave, the polarization direction is the same as the propagation direction, and the second set of subscripts is also “12”. Using this notation in Eq. C-99 gives

$$\rho(V_{12,12})^2 = \frac{C_{11} - C_{12}}{2} \quad (\text{C-105})$$

Rearranging the **third root** (Eq. C-101) gives $\rho V^2 = C_{44}$. Substituting this for ρV^2 in Eq. C-98 gives

$$\left(\frac{1}{2}\right) \cdot \begin{bmatrix} C_{11} - 2C_{44} + C_{66} & C_{12} + C_{66} & 0 \\ C_{12} + C_{66} & C_{11} - 2C_{44} + C_{66} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (\text{C-106})$$

This set of equations is satisfied by setting $p_1 = p_2 = 0$, while p_3 can be any number. Therefore, the polarization vector for this wave is $\vec{p} = [0, 0, 1]$. The wave corresponds to a purely transverse wave by Eq. C-6.

The velocity is denoted with the first set of subscripts corresponding to the propagation direction. The propagation direction for this wave is referred to with the subscripts “12” (see discussion above Eq. C-103.) Second set of subscripts corresponds to the polarization direction, which is in the 3-direction. Therefore, the velocity is denoted as $V_{12,3}$. Using this notation, the velocity–elasticity relationship is

$$\rho(V_{12,3})^2 = C_{44} \quad (\text{C-107})$$

The reason that the substitution method could not be used with the solutions for orthotropic materials is discussed here. The substitution method would attempt to find the elasticity-velocity relationships for tetragonal materials by starting from the corresponding relationships for orthotropic materials. For the propagation

direction under consideration (in the 1–2 direction), the three roots for orthotropic materials were given in Section C.3.4 as Eqs. C-38, C-39, and C-40. The substitution method would then modify these roots using the relationships of Eq. C-82. Specifically, the substitutions necessary would be to substitute C_{11} for C_{22} in Eqs. C-38 and C-39 and to substitute C_{44} for C_{55} in Eq. C-40. However, Eqs. C-38 and C-39 contain square roots that prevent further simplification. For example, after substituting C_{11} for C_{22} , Eq. C-38 becomes

$$\frac{2C_{11}+2C_{66}+\sqrt{4(C_{12}+C_{66})^2}}{4} - \rho V^2 = 0 \quad (\text{C-108})$$

Since C_{12} can be either positive or negative, the square root term can only be simplified to $\sqrt{4(C_{12} + C_{66})^2} = 2\|C_{12} + C_{66}\|$, where the $\|x\|$ operator indicates the magnitude of x . Therefore, the root could not be further simplified into the form found above for tetragonal materials (Eq. C-99).

C.4.5 Propagation in the 1–3 Direction

The velocities and polarization vectors for the three waves are taken directly from Section C.3.5 with the substitutions of Eq. C-82. As for orthotropic materials, the first two of these waves are not purely longitudinal or transverse. The velocities include the Q superscript to denote quasi. Only the third wave is pure.

For the **first root**, the following elasticity-velocity relationship is obtained from Eq. C-59 after substitution:

$$\rho(V_{13,13}^Q)^2 = \frac{C_{11}+C_{33}+2C_{44}+\sqrt{(C_{11}-C_{33})^2+4(C_{13}+C_{44})^2}}{4} \quad (\text{C-109})$$

The accompanying polarization vector was given by Eq. C-58 for orthotropic materials. Substitution yields the corresponding polarization for tetragonal materials as

$$\vec{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} \frac{2(C_{13}+C_{44})}{C_{33}-C_{11}+\sqrt{(C_{11}-C_{33})^2+4(C_{13}+C_{44})^2}} \\ 0 \\ 1 \end{bmatrix} \quad (\text{C-110})$$

For the **second root**, the following elasticity-velocity relationship is obtained from Eq. C-63 after substitution:

$$\rho(V_{13,\bar{1}3}^Q)^2 = \frac{C_{11}+C_{33}+2C_{44}-\sqrt{(C_{11}-C_{33})^2+4(C_{13}+C_{44})^2}}{4} \quad (\text{C-111})$$

The accompanying polarization vector was given by Eq. C-62 for orthotropic materials. Substitution yields the corresponding polarization for tetragonal materials as

$$\vec{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} -1 \cdot \frac{2(C_{13}+C_{44})}{C_{11}-C_{33}+\sqrt{(C_{11}-C_{33})^2+4(C_{13}+C_{44})^2}} \\ 0 \\ 1 \end{bmatrix} \quad (C-112)$$

The **third root** is found from Eq. C-66 by simple substitution. For orthotropic materials, polarization was in the +2-direction, which remains unchanged for tetragonal materials:

$$\rho(V_{13,2})^2 = \frac{C_{44}+C_{66}}{2} \quad (C-113)$$

C.4.6 Propagation in the 2–3 Direction

The velocities and polarization vectors for the three waves are taken directly from Section C.3.6 with the substitutions of Eq. C-82. As for orthotropic materials, the second and third of these waves are not purely longitudinal or transverse. The velocities include the Q superscript to denote quasi. Only the first wave is pure.

The **first root** is found from Eq. C-73 by simple substitution. For orthotropic materials, polarization was in the +1-direction, which remains unchanged for tetragonal materials:

$$\rho(V_{23,1})^2 = \frac{C_{44}+C_{66}}{2} \quad (C-114)$$

For the **second root**, the following elasticity-velocity relationship is obtained from Eq. C-77 after substitution:

$$\rho(V_{23,23}^Q)^2 = \frac{C_{11}+C_{33}+2C_{44}+\sqrt{(C_{11}-C_{33})^2+4(C_{13}+C_{44})^2}}{4} \quad (C-115)$$

The accompanying polarization vector was given by Eq. C-76 for orthotropic materials. Substitution yields the corresponding polarization for tetragonal materials as:

$$\vec{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{2(C_{13}+C_{44})}{C_{33}-C_{11}+\sqrt{(C_{11}-C_{33})^2+4(C_{13}+C_{44})^2}} \\ 1 \end{bmatrix} \quad (C-116)$$

For the **third root**, the following elasticity-velocity relationship is obtained from Eq. C-81 after substitution:

$$\rho(V_{23,\bar{2}3}^Q)^2 = \frac{c_{11}+c_{33}+2c_{44}-\sqrt{(c_{11}-c_{33})^2+4(c_{13}+c_{44})^2}}{4} \quad (\text{C-117})$$

The accompanying polarization vector was given by Eq. C-80 for orthotropic materials. Substitution yields the corresponding polarization for tetragonal materials as

$$\vec{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \cdot \frac{2(c_{13}+c_{44})}{c_{11}-c_{33}+\sqrt{(c_{11}-c_{33})^2+4(c_{13}+c_{44})^2}} \\ 1 \end{bmatrix} \quad (\text{C-118})$$

C.5 Summary of Velocity–Elasticity Relationships

C.5.1 Orthotropic Materials

Table C-2 summarizes the relationships between the velocity of the ultrasonic wave and the C-Matrix elements for orthotropic materials as derived in Section C.3.

Table C-2 Relationships specifying ultrasonic velocity as a function of material properties for orthotropic materials

Equation	Eq. Num. ^a	Wave Type ^b	$\vec{a}$	$\vec{p}$
$\rho(V_{1,1})^2 = C_{11}$	C-12	L	[1,0,0]	[1,0,0]
$\rho(V_{1,2})^2 = C_{66}$	C-14	T	[1,0,0]	[0,1,0]
$\rho(V_{1,3})^2 = C_{44}$	C-16	T	[1,0,0]	[0,0,1]
$\rho(V_{2,1})^2 = C_{66}$	C-22	T	[0,1,0]	[1,0,0]
$\rho(V_{2,2})^2 = C_{11}$	C-24	L	[0,1,0]	[0,1,0]
$\rho(V_{2,3})^2 = C_{44}$	C-26	T	[0,1,0]	[0,0,1]
$\rho(V_{3,1})^2 = C_{55}$	C-32	T	[0,0,1]	[1,0,0]
$\rho(V_{3,2})^2 = C_{44}$	C-34	T	[0,0,1]	[0,1,0]
$\rho(V_{3,3})^2 = C_{33}$	C-36	L	[0,0,1]	[0,0,1]
$4\rho(V_{12,12}^Q)^2 = C_{11} + C_{22} + 2C_{66} + \sqrt{(C_{11} - C_{22})^2 + 4(C_{12} + C_{66})^2}$	C-44	QL	[1,1,0]	[A,1,0]
$4\rho(V_{12,12}^Q)^2 = C_{11} + C_{22} + 2C_{66} - \sqrt{(C_{11} - C_{22})^2 + 4(C_{12} + C_{66})^2}$	C-48	QT	[1,1,0]	[-A',1,0]
$\rho(V_{12,3})^2 = (C_{44} + C_{55})/2$	C-51	T	[1,1,0]	[0,0,1]
$4\rho(V_{13,13}^Q)^2 = C_{11} + C_{33} + 2C_{55} + \sqrt{(C_{11} - C_{33})^2 + 4(C_{13} + C_{55})^2}$	C-59	QL	[1,0,1]	[B,0,1]
$4\rho(V_{13,13}^Q)^2 = C_{11} + C_{33} + 2C_{55} - \sqrt{(C_{11} - C_{33})^2 + 4(C_{13} + C_{55})^2}$	C-63	QT	[1,0,1]	[-B',0,1]
$\rho(V_{13,2})^2 = (C_{66} + C_{44})/2$	C-66	T	[1,0,1]	[0,1,0]
$4\rho(V_{23,23}^S)^2 = C_{22} + C_{33} + 2C_{44} + \sqrt{(C_{22} - C_{33})^2 + 4(C_{23} + C_{44})^2}$	C-77	QL	[0,1,1]	[0,D,1]
$4\rho(V_{23,23}^S)^2 = C_{22} + C_{33} + 2C_{44} - \sqrt{(C_{22} - C_{33})^2 + 4(C_{23} + C_{44})^2}$	C-81	QT	[0,1,1]	[0,-D',1]
$\rho(V_{23,1})^2 = (C_{66} + C_{55})/2$	C-73	T	[0,1,1]	[1,0,0]

$$\begin{aligned} \text{Notes: } A &= \frac{2(C_{12}+C_{66})}{C_{22}-C_{11}+\sqrt{(C_{11}-C_{22})^2+4(C_{12}+C_{66})^2}}, A' = \frac{2(C_{12}+C_{66})}{C_{11}-C_{22}+\sqrt{(C_{11}-C_{22})^2+4(C_{12}+C_{66})^2}} \\ B &= \frac{2(C_{13}+C_{55})}{C_{33}-C_{11}+\sqrt{(C_{11}-C_{33})^2+4(C_{13}+C_{55})^2}}, B' = \frac{2(C_{13}+C_{55})}{C_{11}-C_{33}+\sqrt{(C_{11}-C_{33})^2+4(C_{13}+C_{55})^2}} \\ D &= \frac{2(C_{23}+C_{44})}{C_{33}-C_{22}+\sqrt{(C_{22}-C_{33})^2+4(C_{23}+C_{44})^2}}, D' = \frac{2(C_{23}+C_{44})}{C_{22}-C_{33}+\sqrt{(C_{22}-C_{33})^2+4(C_{23}+C_{44})^2}} \end{aligned}$$

^a Refers to the equation number in Appendix C where the velocity–elasticity relationship is written in the form presented here. For information regarding a particular relationship, or the accompanying polarization vector, the relevant section can be found by locating the particular equation number in Section C.3. Then, the reader is referred to the discussion immediately above each equation.

^b L: longitudinal (Eq. C-5), T: transverse (Eq. C-6), Q: quasi (not pure).

C.5.2 Tetragonal Materials

Table C-3 summarizes the relationships between the velocity of the ultrasonic wave and the C-Matrix elements for tetragonal materials as derived in Section C.4.

Table C-3 Relationships specifying ultrasonic velocity as a function of material properties for tetragonal materials

Equation	Eq. Num. ^a	Wave Type ^b	$\vec{a}$	$\vec{p}$
$\rho(V_{1,1})^2 = C_{11}$	C-85	L	[1,0,0]	[1,0,0]
$\rho(V_{1,2})^2 = C_{66}$	C-86	T	[1,0,0]	[0,1,0]
$\rho(V_{1,3})^2 = C_{44}$	C-87	T	[1,0,0]	[0,0,1]
$\rho(V_{2,1})^2 = C_{66}$	C-88	T	[0,1,0]	[1,0,0]
$\rho(V_{2,2})^2 = C_{11}$	C-89	L	[0,1,0]	[0,1,0]
$\rho(V_{2,3})^2 = C_{44}$	C-90	T	[0,1,0]	[0,0,1]
$\rho(V_{3,1})^2 = C_{44}$	C-97	T	[0,0,1]	[0,1,0]
$\rho(V_{3,3})^2 = C_{33}$	C-94	L	[0,0,1]	[0,0,1]
$\rho(V_{12,12})^2 = (C_{11} + C_{12})/2 + C_{66}$	C-103	L	[1,1,0]	[1,1,0]
$\rho(V_{12,\bar{1}2})^2 = (C_{11} - C_{12})/2$	C-105	T	[1,1,0]	[-1,1,0]
$\rho(V_{12,3})^2 = C_{44}$	C-107	T	[1,1,0]	[0,0,1]
$4\rho(V_{13,13}^Q)^2 = C_{11} + C_{33} + 2C_{44} + \sqrt{(C_{11} - C_{33})^2 + 4(C_{13} + C_{44})^2}$	C-109	QL	[1,0,1]	[Φ ,0,1]
$4\rho(V_{13,\bar{1}3}^Q)^2 = C_{11} + C_{33} + 2C_{44} - \sqrt{(C_{11} - C_{33})^2 + 4(C_{13} + C_{44})^2}$	C-111	QT	[1,0,1]	[- Φ ',0,1]
$\rho(V_{13,2})^2 = (C_{66} + C_{44})/2$	C-113	T	[1,0,1]	[0,1,0]
$4\rho(V_{23,23}^S)^2 = C_{11} + C_{33} + 2C_{44} + \sqrt{(C_{11} - C_{33})^2 + 4(C_{13} + C_{44})^2}$	C-114	QL	[0,1,1]	[0, Φ ,1]
$4\rho(V_{23,\bar{2}3}^S)^2 = C_{11} + C_{33} + 2C_{44} - \sqrt{(C_{11} - C_{33})^2 + 4(C_{13} + C_{44})^2}$	C-115	QT	[0,1,1]	[0,- Φ ',1]
$\rho(V_{23,1})^2 = (C_{66} + C_{44})/2$	C-117	T	[0,1,1]	[1,0,0]
Notes: $\Phi = \frac{2(C_{13}+C_{44})}{C_{33}-C_{11}+\sqrt{(C_{11}-C_{33})^2+4(C_{13}+C_{44})^2}}$ and $\Phi' = \frac{2(C_{13}+C_{44})}{C_{11}-C_{33}+\sqrt{(C_{11}-C_{33})^2+4(C_{13}+C_{44})^2}}$				

^a Refers to the equation number in Appendix C where the velocity–elasticity relationship is written in the form presented here. For information regarding a particular relationship, or the accompanying polarization vector, the relevant section can be found by locating the particular equation number in Section C.4. Then, the reader is referred to the discussion immediately above each equation.

^b L: longitudinal (Eq. C-5), T: transverse (Eq. C-6), Q: quasi (not pure).

List of Symbols, Abbreviations, and Acronyms

ARL	Army Research Laboratory
BFD	backface deflection
BNC	Bayonet Neill–Concelman
CFRP	carbon fiber–reinforced plastics
DEVCOM	US Army Combat Capabilities Development Command
EE	Echo-to-Echo method of measuring TOF
FP	First Pulse method of measuring TOF
LHS	left-hand side
NASA	National Aeronautics and Space Administration
NISW	normal incidence shear wave
PADP	polyether diol-aliphatic diisocyanate polyurethane matrix type
SSE	solid state extruded
TOF	time of flight
UD	unidirectional
UHMWPE	ultra-high molecular weight polyethylene
V50	velocity of penetration
WMRD	Weapons and Materials Research Directorate

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