



**VERIFYING AND IMPROVING A FLIGHT
REFERENCE SYSTEM'S PERFORMANCE**

THESIS

Loren E Myers, 1st Lt, USAF
AFIT-ENG-MS-20-M-046

**DEPARTMENT OF THE AIR FORCE
AIR UNIVERSITY**

AIR FORCE INSTITUTE OF TECHNOLOGY

Wright-Patterson Air Force Base, Ohio

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PERFORMANCE

THESIS

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Degree of Master of Science in Electrical Engineering

Loren E Myers, B.S.Comp.E.

1st Lt, USAF

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Loren E Myers, B.S.Comp.E.
1st Lt, USAF

Committee Membership:

Maj Aaron J. Canciani, Ph.D
Chair

John F. Raquet, Ph.D
Member

Sanjeev Gunawardena, Ph.D
Member

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Loren E. Myers

Notation

Scalars, Vectors, and Matrices

Scalars are denoted with upper and lower case characters in italic type. (x, y, z)

Vectors are denoted with lower case letters in boldface type ($\mathbf{x}$) and their elements denoted by the vector with an index subscript in italic type ($x_i \rightarrow$ the i th element of vector $\mathbf{x}$)

Matrices are denoted with upper case letters in boldface type ($\mathbf{F}$) and their elements denoted by the matrix with index subscripts in italic type ($F_{ij} \rightarrow$ the element in row i , column j)

Random Variables and Vectors

Random variables are denoted by san-serif lower case letters ($\mathbf{x}, \mathbf{y}, \theta$)

Random vectors are denoted by boldface san-serif lower case letters ($\mathbf{x}$) and their elements denoted by the vector with an index subscript in san-serif type ($\mathbf{x}_i$)

Expectation the expectation of a random variable is denoted by $E[\cdot]$

Variance the variance of a random variable is denoted by $\text{Var}(\cdot)$

Superscripts and Subscripts

^{T} : Matrix transpose

$^{-1}$: Matrix inverse

$^{\wedge}$: Estimate

$^{-}$: *a priori* estimate (before measurement update)

$^{+}$: *posteriori* estimate (after measurement update)

VERIFYING AND IMPROVING A FLIGHT REFERENCE SYSTEM'S PERFORMANCE

I. Introduction

1.1 Motivation

In the modern era, precise navigation is crucial to warfighting success on the ground, in the air, and at sea. Weapons systems are more dependent on Global Positioning System (GPS) for precise Position, Navigation, and Timing (PNT) than ever before. This dependency makes those same weapons systems vulnerable to GPS jamming and spoofing. In order to combat this dependency, more rugged navigation systems are being designed to operate in GPS degraded environments. However, testing such devices presents its own unique challenge: in a world that uses GPS as the de facto truth when it comes to PNT, how does one verify the accuracy of new systems in a GPS denied environment? The 746th Test Squadron presents an answer to this problem with the Ultra High Accuracy Reference System (UHARS) precision reference system. Consisting of airborne sensors, GPS receivers, a ground based pseudolite positioning system, and an Extended Kalman Filter (EKF) post-processing algorithm, UHARS provides a high accuracy truth reference for navigation test. The most recent addition to UHARS is the Locata Non-GPS Based Positioning System (NGBPS), which increases overall reference system accuracy, and allows for full-spectrum GPS jamming. The position accuracy of UHARS has been determined, but the velocity accuracy has not characterized. A reference system that remains accurate and operational in the absence of GPS creates more realistic test environments for next-generation resilient navigation systems.

1.2 Background

The 746th Test Squadron conducts test and evaluation of Department of Defense (DoD) guidance, navigation, and navigation warfare (NAVWAR) systems. New systems are tested using ground equipment or by flying on a C-12J or T-38A aircraft. The navigation solution of the system under test is stored and compared to a truth solution provided by the 746th Test Squadron. In order to provide an accurate assessment of navigation system accuracy, the truth solution must be of high precision and accuracy.

There is a long pedigree of truth reference systems at 746th Test Squadron dating back to the 1970's [23]. Each of the reference systems has depended on airborne sensors and some positioning service independent from the test aircraft. The current iteration of this system is called UHARS. This system consists of multiple sensors capable of providing a precise navigation solution over approximately 2,500 square miles of the Tularosa Basin and supporting tests over Holloman Air Force Base (AFB) and White Sands Missile Range (WSMR).

UHARS consists of a 746th Test Squadron maintained airborne Inertial Navigation System (INS) and barometer, Differential Global Positioning System (DGPS) measurements, and measurements from a non-GPS based positioning system. The non-GPS based positioning system continues to provide position and velocity updates when test aircraft are being jammed or denied GPS measurements. The technology used is Locata, a Commercial Off the Shelf (COTS) system modified to meet the requirements of the 746th Test Squadron .

Locata operates on the same principle as GPS, but in the license free 2.4 GHz Industry Scientific and Medical (ISM) band [8]. It provides a networked timing infrastructure called TimeLoc [3] which synchronizes network clocks to within nanoseconds. In addition, the use of Time Division Multiple Access (TDMA) defeats the near-far problem present in many ground-based wireless networks. Operating in either master-slave or cascaded configuration, Locata provides the necessary radionavigation alternative to GPS.

Measurements are collected and stored from all the above sources over the course of a

flight test. Afterwards, the measurements are combined in an EKF implemented in the Sensor Processing and Inertial Dynamics Error Reduction (SPIDER) software suite. SPIDER is a modular framework implemented in MATLAB and developed by the Autonomy and Navigation Technology (ANT) center at Air Force Institute of Technology (AFIT). SPIDER allows for rapid implementation of new stochastic filters in a navigation context. By maintaining sensor models and data interface standards, SPIDER makes adding and removing sensors trivial and allows for simulated outages. An INS modeling EKF using a modified Pinson 15 model (incorporating barometer bias parameters) updates using sensor measurements logged in uniquely formatted Spider measurement files and propagates by no more than the user defined maximum propagation time. The EKF solution then provides position, velocity, bias, and error estimates that are applied to the INS position. The incorporation of multiple sensors provides a solution estimate that is more accurate than its constituent measurements.

In order to evaluate Locata performance, the 746th Test Squadron flew several technology demonstration test flights with both the new Locata NGBPS and the legacy CIGTF Reference System (CRS) systems active on the ground and airborne. The 746th Test Squadron then calculated an EKF solution using Range and Range-Rate Subsystem (RRS) measurement updates. Locata Corp calculated their own position solution independent of the 746th Test Squadron process. The two solutions were then compared to each other to establish code and carrier solution Root-Mean-Square (RMS) differences. Final results established that the Locata position solution was within 6 cm RMS of the CRS solution horizontally and within 15 cm RMS of the solution vertically, creating an overall three dimensional root-mean-square (3D-RMS) of 18 cm. This means that the new Locata NGBPS agrees to within 18 cm of the previously established high accuracy truth. Similar studies have not been performed for velocity characterization.

1.3 Problem Definition

While the positional accuracy of the UHARS EKF has been characterized, the velocity accuracy has not. Velocity is a crucial component to navigation test, but unlike position, an independent velocity truth has not been established for UHARS. Systems under test require a known velocity truth and its associated tolerances in order to effectively test and evaluate navigation systems.

1.4 Proposed Solution

This thesis proposes using delta-position velocity measurements derived from Locata and GPS to characterize the velocity accuracy of the UHARS system in different configurations. When broken down, this solution therefore requires four primary efforts:

1. Form an error ceiling for Locata position measurements
2. Demonstrate the utility of a delta-position velocity measurement
3. Assess UHARS velocity estimate accuracy using Locata delta-position velocity measurements
4. Characterize UHARS drift in outage conditions using GPS delta-position velocity measurements

Subtracting Locata position measurements from GPS position measurements demonstrates how closely the systems agree. The errors associated with Locata and GPS will be combined. Assuming that GPS is a perfect truth, the error derived from subtracting these measurements will be a ceiling for maximum Locata error (the error will be larger than the true Locata error). Using the calculated error as a maximum error for Locata will ensure that predictions made will capture worst-case performance.

A delta-position velocity is calculated from a change in position and divided by the time difference between the two position measurements. Studies of GPS based delta-position measurements have been performed in [26], but studied using pseudorange calculations to

determine velocity. Demonstrating that a delta-position velocity derived from position measurements is within a particular tolerance of filter velocity will establish the validity of using a delta-position velocity as an independent truth.

Having established a maximum error for Locata position measurements and demonstrating the validity of a delta-position velocity as an independent truth, the performance of the UHARS filter can be assessed. Assessing the filter with Locata measurements included will be somewhat optimistic, as the Locata measurements will affect both the filter and the truth measurements. Removing Locata from the filter, but keeping GPS will maintain an accurate filter solution which can be compared to a completely independent truth. This will assess filter velocity accuracy.

Finally, operating the filter with Locata and simulated GPS outages will characterize filter INS tuning when the results are compared to an independent truth. In a GPS denied environment, Locata will continue to provide position updates. Therefore filter performance in outage conditions, when only Locata is present, must be characterized. The operational jamming configuration can be simulated by periodically removing GPS measurements from the filter. This is accomplished using Scorpion's built in simulated outage capability. The result is a series of GPS outages with fixed length. For each of these, the filter will continue to produce position and velocity estimates. Using a delta-position velocity truth, the velocity estimate drift may be characterized. The challenge then is which truth to use. In a real-world jamming scenario, aircraft GPS position measurements will be unavailable. Therefore, in a real world test, Locata would be used to form the delta-position velocity truth. However, in a simulated outage those GPS measurements not incorporated into the Scorpion solution are still available. These can be used to calculate an independent delta-position velocity. This truth will provide a more accurate assessment of filter performance in outage conditions, but will not be available in real world jamming scenarios. Therefore, filter performance in GPS outage conditions will be compared to GPS delta-position velocity. This will yield a more rigorous assessment of filter performance, as Locata measurements will still be used to

update the filter solution, and a truth independent from Locata will be used to generate the results.

1.5 Thesis Overview

First, background on the technologies involved and a survey of literature is presented in Chapter II. Algorithms for calculating all necessary measurements and derivation of the error associated with a delta-position measurement is presented in Chapter III. Finally, results and analysis of studies performed is presented in Chapter IV. These studies will first find the error between Locata and GPS position measurements at the same time epochs in order to find the maximum error that could be attributed to these systems. Then, a simple filter solution incorporating either Locata or GPS will be compared to the delta-position velocity calculated from the unincorporated, independent positioning measurements. Next, a flight-configuration filter will be compared to Locata and GPS-derived measurements in order to demonstrate the accuracy of the reference solution when compared to a delta-position measurement. Finally, simulated outages will be injected into the filter as it runs in order to simulate a GPS or Locata denied environment. The filter drift in these simulated conditions will be compared to the predicted error reported by the filter in order to validate filter tuning characteristics. A summation of results and recommendations for improvement are presented in Chapter V.

II. Background and Literature Review

2.1 Reference Frames

Reference frames are necessary for expressing both spatial position and spatial measurements. Different positioning applications require various reference frames, and therefore several common frames have developed over time. The most common are the Geodetic Frame; the Earth-Centered Inertial Frame (ECI); the Earth Centered, Earth Fixed (ECEF); and the Local-Level Frame.

2.1.1 Geodetic Frame

The Geodetic Frame, sometimes referred to as Latitude, Longitude, Altitude (LLA) or Latitude, Longitude, Height (LLH) expresses position relative to an ellipsoid approximating the shape of the geoid. A geoid is a non-ellipsoid description of the Earth that takes into account gravitational potential. The Earth itself is roughly an ellipsoid, but terrain variations and material compositions make the gravitational potential vary at the same altitude above the ellipsoid. Differences between the geoid and an ellipsoid fit only become critical when using geoid-based reference systems such as Mean Sea Level (MSL). Global Positioning System (GPS) uses the World Geodetic System 1984 (WGS-84) datum. The WGS-84 provides 3 axes and four defining parameters to describe the ellipsoid:

- The WGS-84 Z-Axis passes through the International Earth Rotation and Reference Systems Service (IERS) Reference Pole (IRP), which corresponds to the direction of the International Time Bureau Conventional Terrestrial Pole (CTP) with an uncertainty of 0.005 inches.
- The WGS-84 X-Axis is the intersection of the IERS Reference Meridian (IRM) and the plane passing through the origin normal to the Z-Axis. The IRM is coincident with the BIH Zero Meridian with an uncertainty of 0.005 inches.
- The WGS-84 Y-Axis completes a right-handed orthogonal coordinate frame.

(The BIH Zero Meridian and CTP are from epoch 1984.0)

The four following parameters define the shape and rotation of the WGS-84 ellipsoid.

- Semi-Major Axis (a) : 6378137.0 m
- Flattening Factor of the Earth ($1/f$) : 298.257223563
- Nominal Mean Angular Velocity of the Earth (ω) : 7292115×10^{-11} radians/second
- Geocentric Gravitational Constant (including mass of Earth atmosphere) (GM) : $3.986004418 \times 10^{14}$ meters³/seconds²

Geodetic position is expressed as the angle from the IRM (longitude), angle from the plane orthogonal to the Z-axis (latitude), and height above the geoid (altitude). WGS-84 is the convention used for GPS positioning.

2.1.2 Earth-Centered Inertial Frame

The ECI frame is a fixed frame in relation to the larger solar system. The ECI is centered at the center of mass of the earth. Its xy -plane is coincident with the equatorial plane where the $+x$ -axis is fixed with relation to the celestial sphere, the $+z$ -axis is normal to the xy -plane, and the $+y$ -axis completes a right-handed coordinate system. The ECI is typically used to measure the position and orbit of GPS satellites, as in this fixed frame the satellites obey Newtonian mechanics. For GPS applications, the ECI frame of choice is the January 1, 2000 system (J2000), established on 1 January, 2000 at 12:00 Universal Coordinated Time (UTC) (per the US Naval Observatory). The J2000 frame $+x$ -axis points in the direction of the vernal equinox, and the y - and z - axes follow the conventions described above.

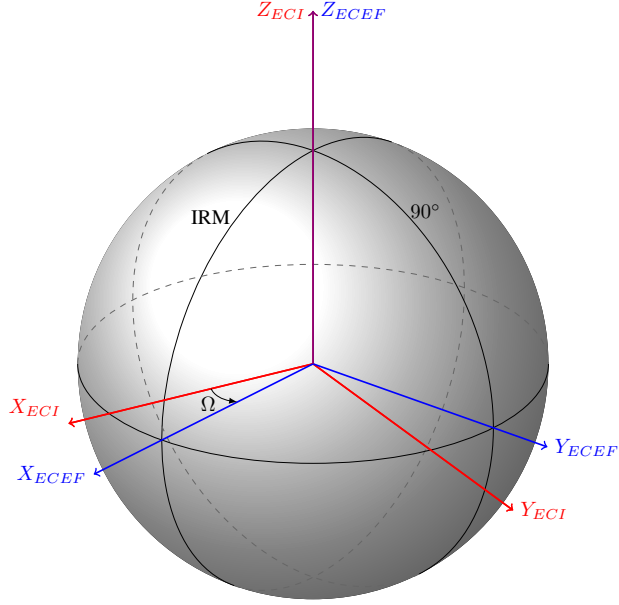


Figure 1: Comparison of ECI and ECEF Frames

2.1.3 Earth-Centered, Earth-Fixed Frame

The ECEF is similar to the ECI frame, but rotates with the Earth, staying fixed relative to the surface of the earth. The ECEF is more typically used to describe the position of a receiver/rover, as frame conversions do not require accounting for the rotation of the surface of the Earth. The ECEF is defined similarly to the ECI with the xy -plane coincident with the equatorial plane, but the $+x$ -axis now points in the direction of 0° latitude (the IRM) and the $+y$ -axis now points in the direction of 90° latitude. The $+z$ -axis is then chosen to point in the direction of the geographic north pole.

2.1.4 Local-Level Frames

While the Geodetic, ECI, and ECEF frames lend themselves well to absolute global position, a more intuitive method of referencing relative position is to use a Local-Level Frame. Local-Level Frames (also known as Local Tangent Coordinates) exist tangent to a

single point relative to the geoid. The most common Local-Level Frames are the North-East-Down (NED) and East-North-Up (ENU) frames. In the case of the NED frame, the x -axis runs tangent to the local line of longitude (North-South), the y -axis runs tangent to the local line of latitude (East-West), and the z -axis runs normal to the geoid. In the ENU frame the $+z$ -axis points away from the geoid normal, and the other axes are switched to complete a right-handed coordinate system.

2.1.5 Body Frame

A Body Frame is a reference frame centered on an object, with the purpose of referencing vectors relative to and in relation to that object. When discussing aircraft, a common convention is the body frame $+x$ -axis travels from the center of mass through the nose of the aircraft, the $+y$ -axis travels out the right wing, and the $+z$ -axis completes the right-handed coordinates, pointing straight down.

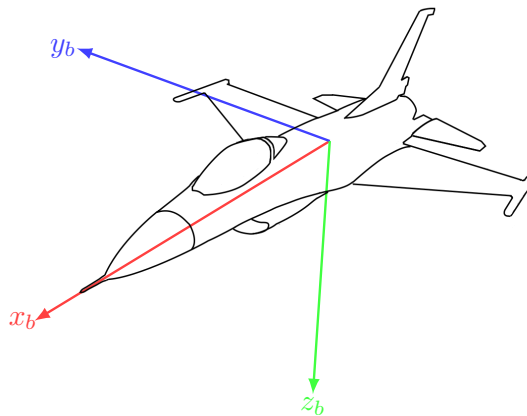


Figure 2: Illustration of typical aircraft body frame

The Body Frame also provides the rotation conventions for aircraft Euler angles, roll being a rotation about the x_b axis, pitch being a rotation about the y_b axis, and yaw being a rotation about the z_b axis.

2.1.6 Frame Conversions

This thesis uses the ECEF, LLA, and NED frames. Conversions between these frames are necessary for either calculations or data representations. First, ECEF to LLA conversions are numerous in type, both closed-form and iterative. This thesis uses Zhu's algorithm, a computationally fast closed-form algorithm [27] to convert from (x, y, z) to (ϕ, λ, h)

$$w = \sqrt{x^2 + y^2} \quad (2.1)$$

$$l = \frac{e^2}{2} \quad (2.2)$$

$$m = \left(\frac{w}{a}\right)^2 \quad (2.3)$$

$$n = \left(\frac{(1 - e^2)z}{b}\right)^2 \quad (2.4)$$

$$i = -\frac{1}{2}2l^2 + m + n \quad (2.5)$$

$$k = l^2 (l^2 - m - n) \quad (2.6)$$

$$q = \frac{(m + n - 4l^2)^3}{216} + mnl^2 \quad (2.7)$$

$$D = \sqrt{(2q - mnl^2) mnl^2} \quad (2.8)$$

$$\beta = \frac{i}{3} - \sqrt[3]{q + D} - \sqrt[3]{q - D} \quad (2.9)$$

$$t = \sqrt{\sqrt{\beta^2 - k} - \frac{(\beta + i)}{2}} - \text{sign}(m - n) \sqrt{\frac{(\beta - i)}{2}} \quad (2.10)$$

$$w_1 = \frac{w}{t + l} \quad (2.11)$$

$$z_1 = \frac{(1 - e^2)z}{t - l} \quad (2.12)$$

$$\phi = \arctan\left(\frac{z_1}{(1 - e^2)w_1}\right) \quad (2.13)$$

$$\lambda = 2 \arctan\left(\frac{w - x}{y}\right) \quad (2.14)$$

$$h = \text{sign}(t - 1 + l) \sqrt{(w - w_1)^2 + (z - z_1)^2} \quad (2.15)$$

In the special case where $w = 0$

$$h = \text{sign}(z)(z - b) \quad (2.16)$$

$$\phi = \text{sign}(z)\frac{\pi}{2} \quad (2.17)$$

Converting from LLA to ECEF is somewhat more straightforward [12]. Converting from (ϕ, λ, h) to (x, y, z) :

$$x = \frac{a \cos \lambda}{\sqrt{1 + (1 - e^2) \tan^2 \phi}} + h \cos \lambda \cos \phi \quad (2.18)$$

$$y = \frac{a \sin \lambda}{\sqrt{1 + (1 - e^2) \tan^2 \phi}} + h \sin \lambda \cos \phi \quad (2.19)$$

$$z = \frac{a (1 - e^2 \sin^2 \phi)}{\sqrt{1 - e^2 \sin^2 \phi}} + h \sin \phi \quad (2.20)$$

Where a and b are respectively the semi-major and semi-minor axis of the geoid.

Converting from LLA to NED consists of a translation to a local frame and a rotation into that frame. Therefore, converting from (x, y, z) to (N, E, D) follows:

1. Establish a local-level origin : $\mathbf{p}_0 = [\phi_0, \lambda_0, h_0]$
2. Determine differences between LLA coordinates and the origin : $\Delta\phi = \phi - \phi_0, \Delta\lambda = \lambda - \lambda_0, \Delta h = h - h_0$
3. Convert the radian difference to a meter difference by multiplying by equivalent radii
 - Radius of Curvature in the Meridian - R_m converts North/South distances
 - Radius of Curvature in the Prime Vertical - R_p , scaled by the cosine of the latitude, converts East/West distances

All told:

$$R_m = \frac{a(1 - e^2)}{(1 - e^2 \sin^2 \phi)^{3/2}} \quad (2.21)$$

$$R_p = \frac{a}{(1 - e^2 \sin^2 \phi)^{1/2}} \quad (2.22)$$

$$p_n = (R_m + h)\Delta\phi \quad (2.23)$$

$$p_e = (R_p + h) \cos \phi \Delta\lambda \quad (2.24)$$

$$p_d = -\Delta h \quad (2.25)$$

Converting from ECEF to NED follows a somewhat similar process:

1. Define origin : $\mathbf{p}_0 = [X_0 \ Y_0 \ Z_0]^T$
2. Create a position vector relative to the new origin, expressed in the ECEF frame:
 $\mathbf{p}^E = [X - X_0, Y - Y_0, Z - Z_0]^T$
3. Rotate the ECEF vector into the NED frame using a Direction Cosine Matrix (DCM)

$$\begin{bmatrix} p_N \\ p_E \\ p_D \end{bmatrix} = \begin{bmatrix} -\sin \phi \cos \lambda & -\sin \phi \sin \lambda & \cos \phi \\ -\sin \lambda & -\cos \lambda & 0 \\ -\cos \phi \cos \lambda & -\cos \phi \sin \lambda & -\sin \phi \end{bmatrix} \begin{bmatrix} p_X^E \\ p_Y^E \\ p_Z^E \end{bmatrix} \quad (2.26)$$

Note, this formula converts a position vector in ECEF to a position vector relative to the local origin in NED. If instead the velocity vector of an object relative to its body frame is expressed in ECEF, that velocity vector is merely rotated into the NED frame based on the latitude and longitude of the aircraft at that time epoch:

$$\begin{bmatrix} v_N \\ v_E \\ v_D \end{bmatrix} = \begin{bmatrix} -\sin \phi \cos \lambda & -\sin \phi \sin \lambda & \cos \phi \\ -\sin \lambda & -\cos \lambda & 0 \\ -\cos \phi \cos \lambda & -\cos \phi \sin \lambda & -\sin \phi \end{bmatrix} \begin{bmatrix} v_X \\ v_Y \\ v_Z \end{bmatrix} \quad (2.27)$$

2.2 Global Navigation Satellite System (GNSS) Positioning

2.2.1 Overview and General Solution

The majority of this section is based on [12]. A Global Navigation Satellite System (GNSS) utilizes Time-of-Arrival (TOA) ranging to perform multilateration positioning. At its core, TOA ranging depends on calculating the distance to a transmitter from a receiver using known speeds of a signal. In the case of GNSS or GPS, the known speed is the speed of light. Knowing both time of transmission and time of reception, the range between the receiver and transmitter can be calculated.

2.2.2 Pseudorange

The pseudorange observable is a combination of the actual range from a satellite to a receiver and all timing errors. This differs from the true geometric range because any errors in time are multiplied by the calculated time of flight, and thus are converted to meters. A complete expression for pseudorange from user u to GNSS satellite j is given by equation (2.28)

$$\rho_j = \sqrt{(x_j - x_u)^2 + (y_j - y_u)^2 + (z_j - z_u)^2} + c\delta t_u \quad (2.28)$$

Where

ρ_j is the pseudorange output of the receiver's code correlation process

x_j, y_j, z_j are the ECEF position of satellite j

x_u, y_u, z_u are the ECEF position of the user receiver u

c is the speed of light

δt_u is the offset of the receiver clock from system time

While only four satellites are required to calculate the user position (solving for the three axes and the receiver clock delay), typically more than four pseudoranges are observed. In

such an overdetermined case, Least-Squares techniques are used to minimize the position error. Additional sources of error beyond receiver clock error include ionospheric delay, tropospheric delay, multipathing, satellite ephemeris error, and Additive White Gaussian Noise (AWGN).

2.2.3 Carrier-Phase Measurements

Carrier-Phase tracking allows for more precise positioning measurements. Carrier-phase measurements use the relative position within the carrier wave to allow for millimeter level accuracy [12]. GPS receivers rely on a Phase Locked Loop (PLL) to monitor GPS signal carrier frequency deviation due to Doppler, meaning that a Doppler frequency shift can be calculated at every GPS epoch. A phase difference of 2π is equivalent to a single cycle, or one wavelength advancement. Integrating carrier Doppler frequency offsets over the duration of an epoch yields the total carrier phase advancement over the same interval. The accumulated Doppler shift is therefore equivalent to the fractional phase advancement or delay. Modern commercial receivers can make carrier-phase measurements with 0.001 cycle precision. Equation (2.29) gives the expression determining carrier-phase measurement (adapted from [12]).

$$\phi_{l1_n} = \phi_{l1_{n-1}} + \int_{t_{n-1}}^{t_n} f_{D_{l1}}(\tau) d\tau + \phi_{r_{l1_n}} \quad (2.29)$$

$$\phi_{l2_n} = \phi_{l2_{n-1}} + \int_{t_{n-1}}^{t_n} f_{D_{l2}}(\tau) d\tau + \phi_{r_{l2_n}} \quad (2.30)$$

Where

ϕ is the accumulated phase at the current epoch

$l1$ and $l2$ are the L1 and L2 frequencies

n and $n - 1$ are the current and last epoch, respectively

f_D is the Doppler frequency as a function of time

ϕ_r is the fractional phase at the current epoch

A is the arbitrary integer plus fractional cycle count at acquisition

While carrier phase measurements can attain much higher accuracy, they are much more sensitive. The primary challenge in carrier-phase positioning is the integer ambiguity N , which is the unknown number of integer cycles that cover the range between the receiver and the satellite at the start of carrier-phase tracking.

2.2.4 Doppler Measurement

Carrier frequency Doppler shift can be used to obtain a the user's three dimensional velocity. A nearly constant user velocity can be modeled by the difference in position over the difference in time $\left(\frac{d\mathbf{p}_u}{dt}\right)$, provided the errors in position at t_1 and t_2 are small relative to the difference of the position change. However, most GNSS receivers velocity is determined using carrier-phase measurements, which allow for precise estimation of the Doppler shift of received signals. Doppler shift is a produced by the relative motion between the satellite and the user. Using satellite ephemeris data to calculate the satellite velocity vector $\mathbf{v}_j$ for satellite j , [12] provides a technique to determine user velocity and receiver clock drift simultaneously.

$$f_{R_j} = f_{T_j} \left\{ 1 - \frac{1}{c} [(\mathbf{v}_j - \dot{\mathbf{p}}_u) \mathbf{a}_j] \right\} \quad (2.31)$$

Where

f_{R_j} is the received frequency from satellite j

f_{T_j} is the transmitted frequency at satellite j

c is the speed of light

$\mathbf{v}_j$ is the calculated velocity vector of satellite j

$\dot{\mathbf{p}}_u$ is the velocity of the user

$\mathbf{a}_j$ is the unit line-of-sight vector from the user to the satellite

Recognizing that received frequency is the transmitted frequency scaled by 1 plus the drift rate of the user clock, $f_{R_j} = f_j(1 + \dot{t}_u)$ can be substituted into (2.31). After rearranging the terms, the Doppler d_j between the user and satellite j can be written as:

$$d_j = \dot{x}_u a_{xj} + \dot{y}_u a_{yj} + \dot{z}_u a_{zj} - c\dot{t}_u \quad (2.32)$$

The above equation can then be solved for velocity and clock drift rate in a manner similar to determining user position [12].

2.2.5 Differential Global Positioning System (DGPS)

DGPS uses multiple receivers to improve location and timing accuracy. DGPS reference stations can provide corrections to ephemeris data, pseudorange values, and broadcast clock information. They can also provide pseudorange and carrier measurements in order to remove errors common to measurements from both receivers. Because GPS error sources are typically highly time and space correlated, measurements from two receivers taken at the same time and in a local area are anticipated to have very similar errors. By forming a single difference or double difference measurement, the following errors can be practically eliminated [19]:

1. Satellite clock errors
2. Satellite ephemeris errors
3. Tropospheric errors
4. Ionospheric errors

Unfortunately, multipath and receiver errors are not time or space correlated and will not be common, even at very small baselines. This means the strength of both these errors will be added to the final signal error. Receiver specific designs, such as choke ring antennas, narrowly spaced correlators, and advanced signal processing must be used to reduce these errors.

2.3 Locata Pseudolite Positioning

Pseudolites (short for “Pseudo-Satellites”) are terrestrial transmitters that mimic GNSS signals to some extent. The first pseudolites were used to validate the GNSS architecture prior to launch of the GPS constellation, transmitting replica GPS messages on the L1 and L2 carrier frequencies [21]. Pseudolites provide the ability to vary transmitter power, frequency, and geometry. However, pseudolite technology faces several significant challenges, including bandwidth availability, timing synchronization, and the near-far problem of positioning efficacy [21, 4, 2].

Bandwidth in pseudolites is limited by their transmission bands. For exact replica GNSS signals, the transmit power must be low enough as to not interfere with extra-terrestrial signals.

Pseudolite designers must also be careful in terms of transmitting power. Traditional pseudolites transmit on L1 and L2 bands, meaning they can easily swamp extra-terrestrial positioning signals. Therefore, the utility of a duplicate GNSS receiver on the ground is limited to a very narrow range so as to not distort or even remove other positioning signals.

The near-far problem arises from relative power of transmission signals. In a pseudolite network, if all transmitters are operating at the same power level there is a near and far boundary of utility for any single transmitter. If the receiver is too close to a transmitter, that transmitter will overpower other transmitters in the network. If the receiver is too far away, the signal strength of that transmitter is too weak to be useful.

2.3.1 Locata

The Locata system overcomes several of these challenges using specific design choices. Because the L1 frequency is protected, the ability to transmit on it worldwide is very limited. Any transmission that would be authorized would also have to be at a low enough power to not interfere with extra-terrestrial GNSS signals. Locata circumvents this issue by transmitting on the Industry Scientific and Medical (ISM) band (2.4-2.4835 GHz). As an

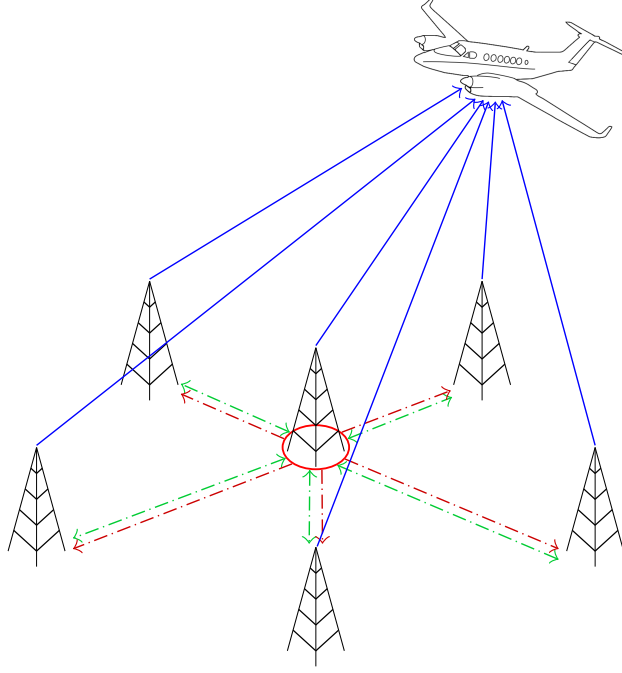


Figure 3: Conceptual diagram of LocataNet Operation with an airborne receiver. The master Locata transmitter is circled in red. Blue lines indicate ISM position messages, green arrows represent timing synchronization, and red arrows represent command and control, and meteorological signals within the LocataNet.

added effect, the LocataLite design allows for transmission of two frequencies inside the ISM band, each with two spatially-diverse Pseudo Random Noise (PRN) codes [21].

In order to counter the near-far problem, Locata uses Time Division Multiple Access (TDMA) methods to reduce signal interference. The specific configuration for Locata involves 100 microsecond timeslots, aggregated into 1 millisecond frames, and 200 millisecond superframes. Each transmitter in a Locata network or subnetwork only transmits during its assigned timeslots. Transmitter timeslot allocations are also randomized frame to frame in order to smooth and distribute residual interference not removed by the base TDMA architecture [8].

One defining characteristic of Locata that separates it from other pseudolite technology is the time synchronization algorithm referred to as “TimeLoc”. TimeLoc is described in [20] and listed by [3, 10]. The algorithm is summarized below:

1. LocataLite A transmits a C/A Code using a particular PRN.

2. LocataLite B acquires, tracks, and measures LocataLite A's C/A code and carrier-phase measurement.
3. LocataLite B generates its own C/A code on a different carrier and with a different PRN from LocataLite A.
4. LocataLite B calculates the difference between the tracked signal and its own signal.
5. LocataLite B adjusts its local oscillator using Direct Digital Synthesis (DDS) to bring the carrier and code differences to 0.
6. LocataLite B continuously monitors clock drift and re-adjusts its oscillator to remain locked with Locatalite A's oscillator.
7. Using precisely known positions of both receivers, LocataLite B can correct for the geometric offset between itself and LocataLite A.

Note that in Fig. 3 and the description of TimeLoc, the LocataNet is configured in a basic Master-Slave convention. However, it is possible to operate Locata in a cascaded configuration when not all transmitters have line-of-sight to the master transmitter. In this case, Locata will cascade time synchronization signals from the master site to slave sites with other slave sites forwarding information. Locata tracks the number of RF hops from the master station to each slave and automatically calculate the timing quality of each sites time-synchronized clock [8]. Because the LocataNet automatically adapts its topology to maintain TimeLoc, there are many configurations in which Locata would operate in a cascaded mode; a notional case is presented in Fig. 4

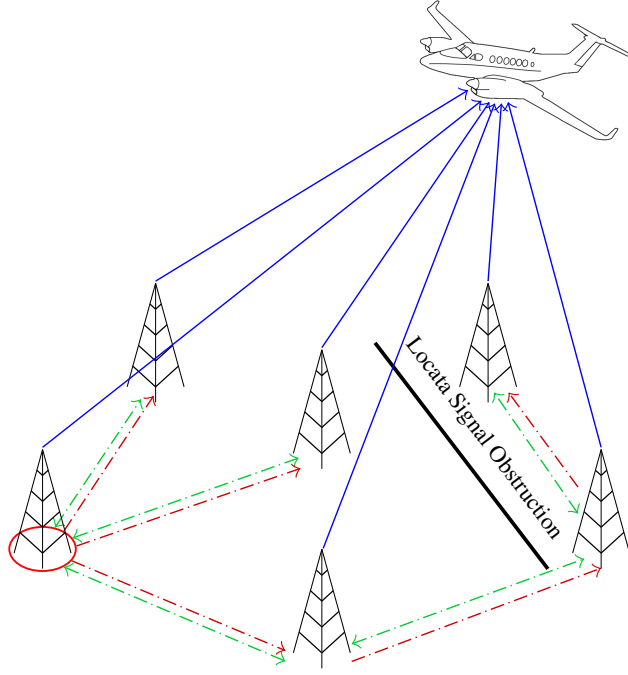


Figure 4: Conceptual diagram of LocataNet Operation with an airborne receiver and cascaded operation due to signal obstruction. The master Locata transmitter is circled in red. Blue lines indicate ISM position messages, green arrows represent timing synchronization, and red arrows represent command and control, and meteorological signals within the LocataNet.

2.3.2 Pseudolite Measurements

Shockley investigated unidentified errors and laid out a framework for error estimation and mitigation from Locata in [22]. The largest source of unidentified error was found to be tropospheric modeling. These errors can be mitigated by implementing a tropospheric scale-factor estimator, reducing Locata horizontal position error by 12cm 2D-RMS. The drawback is that the tropospheric model suffers a bootstrapping issue, where the position of a Locata rover must be very well known in order to estimate the tropospheric error.

Following Shockley’s work, Ciampa investigated pseudolite phase measurement failure detection methodologies in [7]. Two methods were presented: residual monitoring and a moving window method in the update step of the Extended Kalman Filter (EKF). Both methods relied on integrating inertial measurements into an existing pseudolite network. Ultimately, the moving window method detected more cycle slips than the residual mon-

itoring method, but the residual monitoring method was able to detect cycle slips faster. The ultimate recommendation was to implement both methods using a high-quality inertial measurement unit (IMU) (specifically the Honeywell HG-764). Residual monitoring would quickly detect cycle slips albeit with a higher failure rate, and cycle slips could be validated using the moving window method.

2.4 Stochastic Estimation

Mathematical models are commonly presented as deterministic. However, real-world systems neither follow deterministic dynamics, nor receive deterministic measurements. Rather, true systems propagate both deterministic and stochastic components. This limits the accuracy of any control or measurement system to the probability distribution of its stochastic components. Therefore, real-world systems require stochastic estimators and controllers to be properly characterized.

Because stochastic components are by nature random variables of unknown distributions, a general closed form solution of stochastic estimation problems is intractable. Solutions that are implemented make assumptions about the distributions of the random variables or use approximation methods to replicate said distributions.

2.4.1 Stochastic Models

Linear deterministic systems are typically modeled by the equation:

$$\dot{\mathbf{x}}(t) = \mathbf{F}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \quad (2.33)$$

This equation is simply a system of linear equations represented in matrix form. Therefore, the dynamics governing that system will be present in $\mathbf{F}$ and the equations required to translate control input into state change will be present in $\mathbf{B}$. Because the system is deterministic, given the starting state and a series of inputs, the system will evolve identically and

predictably every time the states are calculated. However, by adding a degree of randomness the system becomes stochastic. Physical and electrical systems are inherently random due to atmospheric disturbances, manufacturing variability, and electrical or thermal noise. This means that the model above will not appropriately capture or describe the behavior of a real-world system under real-world conditions. For that, stochastic models are required.

Models only require a few modifications in order to change from deterministic to stochastic. While they may look superficial, adding the stochastic terms has some deeper implications. The addition of two noise processes, one ($\mathbf{w}$) representing process noise and one ($\mathbf{v}$) representing measurement noise, allows for random noise of a known or estimated strength to be added to every state transition and measurement update. The measurement mapping matrix $\mathbf{G}(t)$ is an n -by- s matrix to shape s random processes into n different state components (e.g. a random process modeling barometer bias would affect both an altitude state and a vertical velocity state).

$$\dot{\mathbf{x}}(t) = \mathbf{F}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) + \mathbf{G}(t)\mathbf{w}(t) \quad (2.34)$$

$$\mathbf{z}(t_i) = \mathbf{H}(t_i)\mathbf{x}(t_i) + \mathbf{v}(t_i) \quad (2.35)$$

The larger implication here is that now each state is not a perfectly known value, but a sufficiently described Probability Density Function (PDF). Therefore, at each time step each state in the state vector $\mathbf{x}$ is described by a Gaussian distribution (for more information on why Gaussian distributions and AWGN are the only suitable uses for linear systems, see [17]). Utilizing the definition of Gaussian distributions and Bayes' rule, we may then develop a method to propagate these stochastic processes forward in time as well as incorporate stochastic measurements.

2.4.2 Kalman filtering

The Kalman filter is a linear unbiased stochastic estimator. For linear systems, the Kalman filter is the *optimal* unbiased stochastic estimator. The Kalman filter has many advantages over other stochastic estimators, particularly that it can be implemented recursively, only requiring data from the previous time step in order to estimate the current state. The Kalman filter depends on several assumptions: all equations are linear, all states are Gaussian, and all noise sources are strictly AWGN. The Kalman filter is typically conceptualized as two distinct steps: propagate and update. Propagation is merely transitioning from one epoch to another using the known system dynamics. The associated errors will grow over the propagation time period. A Kalman filter algorithm could only propagate over the entire course of operation; the result would be a very uncertain estimate based on the initial state estimates. However the strength of the Kalman filter lies in the update process, which uses the state estimate and noisy sensor measurements to decrease the uncertainty of the state estimates. The Kalman filter can be continuous or discrete time, but typically measurements are discrete inputs, restricting implementation to the discrete version.

2.4.2.1 Propagate Step

Kalman filtering depends on an initial state vector $\mathbf{x}_0$. However, because the initial state cannot be perfectly known, it is modeled as a normally distributed random vector $\mathbf{x}_0$ with mean $\hat{\mathbf{x}}_0$ and covariance $\mathbf{P}_0$. This initial estimate is propagated forward using a state process $\mathbf{x}$ which satisfies:

$$\dot{\mathbf{x}}(t) = \mathbf{F}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) + \mathbf{G}(t)\mathbf{w}(t) \quad (2.36)$$

Where

- $\mathbf{x}$ is the stochastic process describing the system states
- $\mathbf{F}$ is the system dynamics matrix
- $\mathbf{B}$ is the input or control matrix

- $\mathbf{u}$ is the input vector
- $\mathbf{G}$ maps the process noise into the system states
- $\mathbf{w}$ is the process noise that is added to the system states during propagation with strength $\mathbf{Q}$

Note that [17] defines the strength of the process noise $\mathbf{Q}$ as a matrix of piecewise continuous functions defined as:

$$\mathbf{Q}(t)\delta(t - t') = \mathbb{E}[\mathbf{w}(t)\mathbf{w}(t')^T] \quad (2.37)$$

The state process definition in (2.36) is in continuous time. To operate in discrete time, the system dynamics matrix is converted to a discrete state transition matrix $\Phi(t_i, t_{i-1})$, calculated by:

$$\Phi(t_i, t_{i-1}) = e^{\mathbf{F}(\Delta t)} \quad (2.38)$$

where $\Delta t = t_i - t_{i-1}$. The control matrix $\mathbf{B}(\tau)$ and input vector $\mathbf{u}(\tau)$ are also calculated for t_i using the integral

$$\mathbf{B}_d \mathbf{u}_d = \int_{t_{i-1}}^{t_i} \Phi(t_i, \tau) \mathbf{B}(\tau) \mathbf{u}(\tau) d\tau \quad (2.39)$$

and finally the state at t_i can be written as:

$$\mathbf{x}(t_i) = \Phi(t_i, t_{i-1})\mathbf{x}(t_{i-1}) + \mathbf{B}_d \mathbf{u}_d + \mathbf{w}_d(t_{i-1}) \quad (2.40)$$

Because this is the generalized case, we define the previous state estimate and covariance at t_{i-1} as *a posteriori* estimates conditioned on all measurements $(0, 1, \dots, i-1)$. [17] states this more concisely as:

$$\hat{\mathbf{x}}(t_{i-1}^+) = \mathbb{E}[\mathbf{x}(t_{i-1}) | \mathbf{Z}(t_{i-1}) = \mathbf{Z}_{i-1}] \quad (2.41)$$

$$\mathbf{P}(t_{i-1}^+) = \mathbb{E} \left[[\mathbf{x}(t_{i-1}) - \hat{\mathbf{x}}(t_{i-1}^+)] [\mathbf{x}(t_{i-1}) - \hat{\mathbf{x}}(t_{i-1}^+)]^T | \mathbf{Z}(t_{i-1}) = \mathbf{Z}_{i-1} \right] \quad (2.42)$$

Propagating forward in time then finds the distribution of $\mathbf{x}$ at t_i conditioned on all measurements $\mathbf{Z}_0, \mathbf{Z}_1, \dots, \mathbf{Z}_{i-1}$. Because the system is linear and its states are Gaussian, the complete distribution can be described by a mean and covariance. Therefore, using the linearity of the expectation operator, the expectation of (2.40) simplifies to:

$$\hat{\mathbf{x}}(t_i^-) = \Phi(t_i, t_{i-1})\hat{\mathbf{x}}(t_{i-1}^+) + \mathbf{B}_d \mathbf{u}_d \quad (2.43)$$

Note that the estimate is only affected by the system dynamics (expressed as the state transition matrix) and inputs to the system (expressed as the discretized input matrix and vector). Similarly, from [17] the covariance from t_{i-1} to t_i is the covariance conditioned on all previous measurements and given by:

$$\mathbf{P}(t_i^-) = \Phi(t_i, t_{i-1})\mathbf{P}(t_{i-1}^+)\Phi^T(t_i, t_{i-1}) + \int_{t_{i-1}}^{t_i} \Phi(t_i, \tau) + \mathbf{G}(\tau)\mathbf{Q}(\tau)\mathbf{G}^T(\tau)\Phi(t_i, \tau)d\tau \quad (2.44)$$

Using the Van Loan method of discretization for $\mathbf{Q}$ to yield $\mathbf{Q}_d$ [5] and other discretization methods for $\mathbf{G}$ (typically $\mathbf{G}_d$ is assumed to be the identity matrix) [17], the continuous equation (2.44) becomes the discrete equation:

$$\mathbf{P}(t_i^-) = \Phi(t_i, t_{i-1})\mathbf{P}(t_{i-1}^+)\Phi^T(t_i, t_{i-1}) + \mathbf{G}_d(t_{i-1})\mathbf{Q}_d(t_{i-1})\mathbf{G}_d^T(t_{i-1}) \quad (2.45)$$

as a side note, $\mathbf{P}(t_i^-)$ is not only the conditional covariance of the state, but also incorporates the error committed by using the conditional mean as an estimator of the state (because the conditional covariance contains $[\mathbf{x}(t_i) - \hat{\mathbf{x}}(t_i^-)]$ and $\hat{\mathbf{x}}(t_i^-)$ is the estimated state)

2.4.2.2 Update

Using the propagate function above, the Kalman filter can give state estimates and covariances at any time step. However, as stated earlier, the strength of the Kalman filter

lies in the incorporation of measurements, normally called the update step. The system model for measurements is given in [17] as:

$$\mathbf{z}(t_i) = \mathbf{H}(t_i)\mathbf{x}(t_i) + \mathbf{v}(t_i) \quad (2.46)$$

Where

- $\mathbf{z}$ is the observation vector used to update states
- $\mathbf{H}$ is the observation matrix, used to map the state vector to a measurement
- $\mathbf{v}$ is the sensor noise, added to each measurement and defined by strength $\mathbf{R}$

Similarly to the white noise in the propagate equation, the white noise in the update equation is given by a strength $\mathbf{R}$ given by:

$$\mathbf{R}(t)\delta(t - t') = \text{E}[\mathbf{v}(t)\mathbf{v}(t')^\top] \quad (2.47)$$

Through an extended derivation using Bayes Rule to condition $\mathbf{z}_i$ on all measurements from 0 to i and the *a priori* state estimate $\hat{\mathbf{x}}(t_i^-)$. See [17] for this extended derivation. Summarized, we can find an expected measurement by passing the state estimate through the observation matrix $\mathbf{H}$. A residual is then formed where the expected measurement, conditioned on the *a priori* state estimate, is subtracted from the realized measurement at t_i .

$$\mathbf{r}(t_i) = \mathbf{z}(t_i) - \mathbf{H}\hat{\mathbf{x}}(t_i^-) \quad (2.48)$$

The covariance of the residual is given by:

$$\mathbf{S} = \mathbf{R}(t_i) + \mathbf{H}\mathbf{P}(t_i^-)\mathbf{H}^\top \quad (2.49)$$

This error is then corrected using a weighting factor called the Kalman Gain, given as:

$$\mathbf{K} = \mathbf{P}(t_i^-) \mathbf{H}^T [\mathbf{R}(t_i) + \mathbf{H} \mathbf{P}(t_i^-) \mathbf{H}^T]^{-1} \quad (2.50)$$

Using the above definitions, the complete update equations are:

$$\hat{\mathbf{x}}(t_i^+) = \hat{\mathbf{x}}(t_i^-) + \mathbf{K}(t_i) [\mathbf{z}(t_i) - \mathbf{H} \hat{\mathbf{x}}(t_i^-)] \quad (2.51)$$

$$\mathbf{P}(t_i^+) = \mathbf{P}(t_i^-) - \mathbf{K}(t_i) \mathbf{H} \mathbf{P}(t_i^-) \quad (2.52)$$

This section was adapted from [17], [7], and [14].

2.4.3 The Extended Kalman filter

Systems cannot always be adequately estimated by a Kalman filter. If a system contains significant non-linearities, such characteristics must be captured by a non-linear measurement function, a non-linear propagate function, or both. The EKF presents an approach to solve problems that do not meet the strict linearity requirements of the Kalman filter. The broad picture approach to the EKF is to linearize all differentiable functions at every time step. The Jacobian of the non-linear functions, a matrix of first-order partial derivatives, serves as a “linearization function” that can be evaluated parametrically as the filter executes. The linearization point for every execution is the estimated mean and covariance. The resulting propagate and update equations resemble the Kalman filter equations with some slight modifications [17], [7], and [14]:

$\hat{\mathbf{x}}_{i+1}^- = \mathbf{f}(\hat{\mathbf{x}}(t_{i-1}), \mathbf{u}(t_i))$	State Propagation Equation	
$\mathbf{P}_{i+1}^- = \Phi_i \mathbf{P}_i^+ \Phi_i^T + \mathbf{Q}_{d_i}$	Covariance Propagation Equation	
$\hat{\mathbf{x}}_{i+1}^+ = \hat{\mathbf{x}}_{i+1}^- + \mathbf{K}_{i+1} [\mathbf{z}(t_{i+1}) - \mathbf{h}(\hat{\mathbf{x}}_{i+1}^-, t_{i+1})]$	State Update Equation	(2.53)
$\mathbf{P}_{i+1}^+ = (\mathbf{I} - \mathbf{K}_{i+1} \mathbf{H}_{i+1}) \mathbf{P}_{i+1}^-$	Covariance Update Equation	
$\mathbf{K}_{i+1} = \mathbf{P}_{i+1}^- \mathbf{H}_{i+1}^T (\mathbf{R}_{i+1} + \mathbf{H}_{i+1} \mathbf{P}_{i+1}^- \mathbf{H}_{i+1}^T)^{-1}$	Kalman Gain Equation	

This section was adapted from [18], [7], and [14].

2.5 Delta-position Measurements

2.5.1 Terminology

For clarity's sake, prior to describing position-based velocity research in depth we shall define some terms that will be used in distinct ways throughout the following sections. First, *displacement* describes the physical distance between two points in space. In the context of this thesis, displacement is only used to describe the true distance between the position solution at time t_1 and time t_2 . Displacement is a theoretical value. A *change in position* is used to describe any calculation that approximates displacement. The two change in position measurements used in this thesis are integrated velocity and delta-position. A *delta-position* measurement is the calculated distance between two points in space. Delta-position is calculated by subtracting a position solution $\mathbf{p}_1$ at time t_1 and a position solution $\mathbf{p}_2$ at time t_2 . Delta-position therefore, is a measurement with the associated errors from the positioning service (Locata or GPS). *Delta-position velocity* is the delta-position measurement divided by the time between the two epochs, resulting in the average velocity over that period.

2.5.2 Previous Research

There have been several efforts to create a precise velocity measurements from radionavigation sources (specifically GPS) most noticeably in [24], [6].

The equation of motion for velocity is the change in position divided by the change in time:

$$\mathbf{v} = \frac{\Delta \mathbf{p}}{\Delta t} \quad (2.54)$$

A delta-position velocity is the difference in position at two time epochs, divided by the difference in the time epochs:

$$\mathbf{v}_{\Delta t} = \frac{\mathbf{p}_2 - \mathbf{p}_1}{t_2 - t_1} \quad (2.55)$$

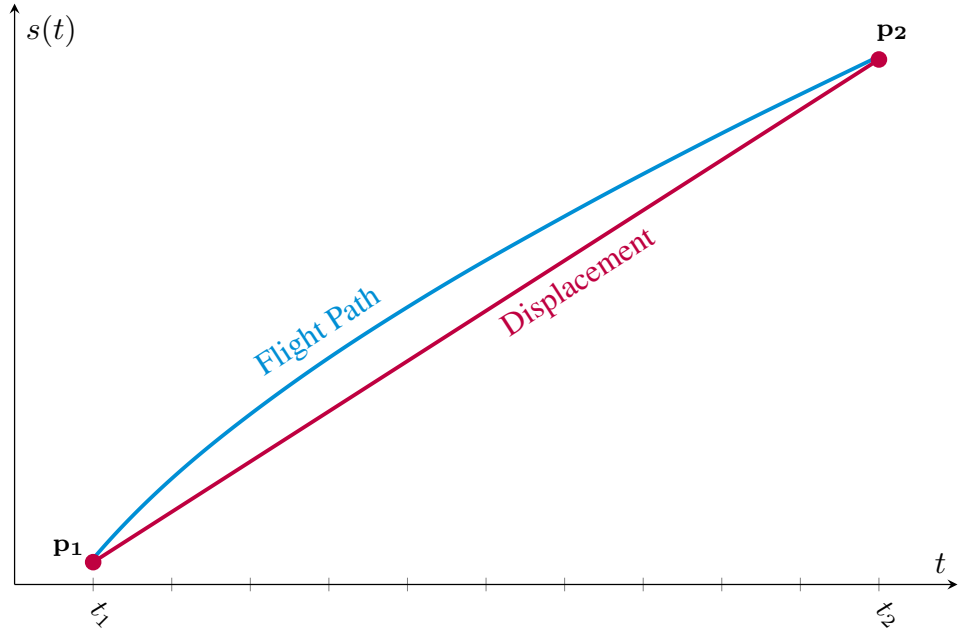


Figure 5: Demonstration of displacement (red) vs. flight path (blue) from position $\mathbf{p}_1$ to $\mathbf{p}_2$ over the time interval t_1 to t_2

This measurement functions as the average velocity over the time period in question.

$\mathbf{p}_2 - \mathbf{p}_1$ is the change in position from t_1 to t_2 , but velocity can not be assumed as constant over the time period t_1 to t_2 . As figure 5 shows, if the true flight path is not straight line, then the velocity vectors in each axis will not be constant. However, from t_1 to t_2 the average velocity in each axis will be equal to the change in position in each axis divided by time. Therefore, a delta-position measurement is more accurately the change in position between the two time steps weighted by the time difference between the two time steps.

2.6 Inertial Navigation Systems

Inertial Navigation System (INS) provide another form of navigation that uses dead reckoning based on gyroscopic and accelerometer sensor data. In order to provide three dimensional navigation data, an INS requires at minimum three orthogonal accelerometers to measure specific force and three orthogonal gyroscopes to measure angular rate. In order to determine acceleration, the force of gravity must be removed. Typically this requires an altimeter input, as attempting to use INS altitude leads to a bootstrapping problem where altitude errors feed gravity errors, which then feed altitude errors. Integrating specific force after the removal of gravity produces acceleration; integrating acceleration yields position. As acceleration is the first derivative of velocity and the second derivative of position, accelerometer data can be integrated over time to produce both velocity and position [25].

Gyroscopes in an INS measure angular rates in the inertial frame. Integrating gyro measurements yields Euler angles describing the attitude of the object. The rotation of the earth must be removed here similarly to the specific force of gravity in accelerometer calculations. The angular rate of the body with respect to the navigation frame (ω_{nb}^b) is therefore the difference between the body rate in the inertial frame (ω_{ib}^b) and the estimated earth rate expressed in the navigation frame with respect to the inertial frame (ω_{in}^n), transformed using the navigation to body DCM (C_n^b).

Inertial navigation systems, being dead-reckoning systems, depend on knowing a precise starting position and attitude and tracking relative motion from that starting position and

attitude. A distinct advantage of an INS is that it cannot be jammed and contains everything to calculate relative position from the starting point. However, because the measurements of an INS are integrated to produce position and attitude, measurement errors, even AWGN build up over time. This means that the error in an INS will be constantly increasing. The quality of an INS is typically determined by the drift rate of its sensors. An outline of INS grades is presented in Table 1 [11].

Table 1: Comparison of different grades of INS with the associated sensor drift rates. (Adapted from [11])

Sensor	Performance Parameters	Tactical	Navigation	Maritime (Submarine)
Angular Rate Sensors	Bias ($^{\circ}/\text{hr}$)	1-10	0.1-1.0	0.002-0.01
	Scale Factor (ppm)	200-500	100-200	5-50
	Noise Floor ($\frac{^{\circ}/\text{hr}}{\sqrt{\text{Hz}}}$)	0.2-0.5	0.05-0.2	0.002-0.005
Accelerometers	Bias (μg)	200-500	50-100	5-10
	Scale Factor (ppm)	400-1000	100-200	10-20
	Noise Floor ($\frac{\mu g/\text{hr}}{\sqrt{\text{Hz}}}$)	200-400	50	5-10

2.7 746th Test Squadron Reference Systems

The 746th Test Squadron is the Lead Test Organization for GPS and INS testing. The 746th Test Squadron tests systems for the Department of Defense (DoD), Federal Aviation Administration (FAA), and National Aeronautics and Space Administration (NASA) [23]. The 746th Test Squadron maintains numerous lab, ground, flight, and sled test capabilities. Of particular relevance is the flight test reference system. The 746th Test Squadron utilizes all of the above technologies to form an extremely precise, integrated reference system for

Guidance, Navigation, and Control (GNC) flight test at Holloman AFB, NM. This reference system is used to test inertial and GPS navigation systems in operational environments, including GPS-denied scenarios.

2.7.1 Legacy Systems

Numerous systems have served as the reference baseline for the 746th Test Squadron and the Central Inertial and GPS Test Facility (CIGTF) since its inception. The majority of these systems are designed to test individual elements of inertial, GPS, and blended navigation systems. The Advanced Inertial Test Laboratory (AITL), specifically designed to be seismically quiescent and equipped with platforms, three-axis tables, and centrifuges designed to test and validate sensitive inertial components, is a perfect example of this paradigm. Other examples include the Climatic Laboratory, Shock/Vibration Laboratory, and Navigation Test and Evaluation Laboratory. However, the need for robust evaluation of systems during developmental and operational flight test drives the necessity of an integrated range reference system that can appropriately evaluate a complete GNC system during flight [23].

The first deployed range reference system was the Completely Integrated Reference Instrumentation System (CIRIS), introduced in the early 1970s. CIRIS was a van-based test system with ground reference capable of 10 meter accuracy. In the early 1990s, the CIGTF High Accuracy Post-processing Reference System (CHAPS) replaced CIRIS [16]. CHAPS was capable of sub-meter accuracy and was able to operate in a GPS jammed environment. Independently, the Air Force Test Center (AFTC) at Edwards AFB developed the GPS Aided Inertial Navigation Reference (GAINR) system, which was capable of utilizing an Embedded GPS/INS (EGI). Combining the existing CHAPS and GAINR systems resulted in the CIGTF Reference System (CRS) [16].

The immediate predecessor of Ultra High Accuracy Reference System (UHARS), CRS consisted of a reference pallet, Satellite Reference Station (SRS), and post-processing hard-

ware. The EGI portion of the reference pallet contained a Honeywell H-764GU EGI modified to produce 256 Hz gyroscope and accelerometer measurements, a Rockwell Collins GEM IV all-in-view GPS receiver, as well as an inertial sensor assembly incorporating Honeywell Digital Laser Gyros [15]. The GPS receiver subsystem of the reference pallet contained a commercial 12-channel dual frequency carrier Ashtech Z-12 receiver. The INS portion contained a Honeywell H423 strapdown INS with ring laser gyroscopes and was capable of a 0.1 nautical mile drift per hour. Altitude aiding for the INS was a combination of GPS altitude and baro-altimeter measurements. The CR-100 Range and Range-Rate Subsystem (RRS) consisted of multiple transponders and one or more airborne interrogators. This system was capable of deploying up to 255 unique transponders, all of which were portable and configurable. The GPS Environment Measurement Subsystem (GEMS) measured real-time GPS jamming signal levels, specifically the GPS Jam-to-Signal (J/S) ratio. Finally, the Data Acquisition System (DAS) initialized the entire pallet, monitored subsystem operation, performed sensor fusion, and recorded all flight test data. [16]

2.7.2 Ultra High Accuracy Reference System (UHARS)

The current reference system in place at the 746th Test Squadron is referred to as the UHARS. UHARS is an EKF solution, implemented in Scorpion and utilizing INS, GPS, and Locata measurements (referred to in [14], [1], [13], [9] as the Non-GPS Based Positioning System (NGBPS)). An overview of UHARS can be found in Figure 6.

The main changes from CRS to UHARS was in the replacement of the CR-100 RRS. This system was replaced with a variant of the commercial Locata system. The requirements of the reference system upgrade called for a more accurate sub-meter reference system for GPS denied or degraded truth reference. This would allow the 746th Test Squadron to jam GPS-dependent systems and still receive truth data.

Indeed, the LocataNet deployment at White Sands Missile Range (WSMR) is significantly modified above and beyond its Commercial Off the Shelf (COTS) counterparts; Table 2

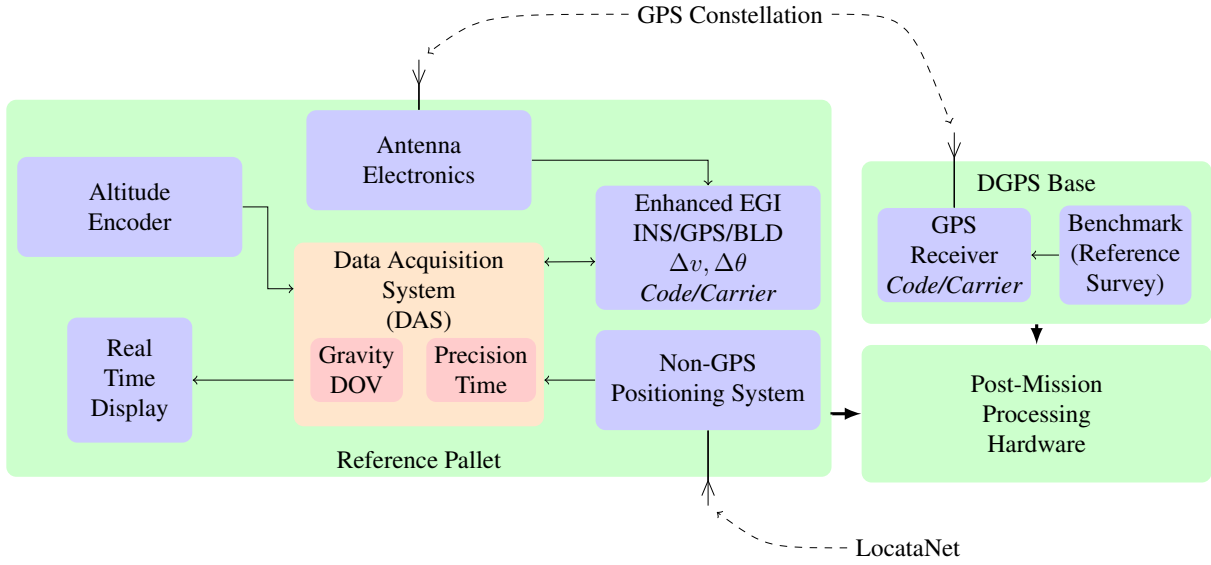


Figure 6: Architecture of the UHARS (Adapted from [13])

elucidates these differences. Summarily, the deployed LocataNet at WSMR is tailor-made for flight test operations. More details on the site-specific modifications can be found in [1],[9],[13], and [7]

Table 2: Comparison of system requirements (Commercial Off the Shelf Locata vs. 746th TS Non-GPS Based Positioning System)

	Locata	NGBPS
Tx Power	~ 100 mW	~ 10 W
Transceiver Spacing	≤ 10 km	Up to 35 km
Dynamics	Low-Rate (Ground Based Vehicles)	High-Rate (Aircraft)
Antenna	Radiating Monopole	Quadrifilar helix
Tropospheric Delay	Negligible	≈ 280 ppm (13.5 m at max range)
Timing Configuration	Master-Slave TimeLoc	Cascaded Network TimeLoc

Initial implementation of UHARS is heavily studied by Kawecki in [14]. Specifically, the filter design and tuning in Sensor Processing and Inertial Dynamics Error Reduction (SPIDER) is laid out.

Prior to the efforts of this project, the accuracy of the UHARS velocity measurement had not been validated. However, utilizing delta-position Measurements, the velocity of the filter can now be compared to an independent source of velocity information.

2.8 Summary

The challenge specific to this thesis is the problem of determining a velocity measurement inside of a Scorpion implemented filter while still using the SPIDER suite front end. The technology and research outlined above will be leveraged in order to produce as accurate a delta-position velocity measurement as possible. Using DGPS as a truth, the error equations outlined in section 3.2.2 can be verified as credible. The Delta-position measurements can then be used as a truth to characterize the EKF implemented in Scorpion that reads SPIDER data from the UHARS reference system in order to produce a highly accurate velocity measurement.

III. Methodology

3.1 Overview

To reiterate, the goal of this thesis is to characterize the accuracy of UHARS velocity estimates. This goal can be broken down into several tasks, outlined below.

Task 1: Form an error ceiling for Locata position measurements

Task 2: Demonstrate the utility of a delta-position velocity measurement as a truth

Task 3: Assess accuracy of nominal UHARS velocity estimates

Task 4: Characterize UHARS drift and INS tuning in outage conditions

If each task is completed successfully, then the goal of characterizing UHARS velocity accuracy has been met.

Each of these tasks in turn require their own separate developmental efforts and test methodologies. Starting with the current assessment of UHARS accuracy, an overall strategy was formulated to get from the current state of UHARS characterization (position only using CRS as a reference) to the desired state of UHARS characterization (position and velocity using CRS and delta-position velocities as reference). A “roadmap” demonstrating how each individual effort leads to and culminates in project completion is outlined in Fig. 7.

First, Locata position measurements must be characterized in terms of a maximum possible error. A direct comparison of radionavigation position measurements is preferable to the already performed comparisons between Locata and UHARS as outlined in section 2.7.2. While Locata has been characterized using UHARS calculated position estimates, stochastic filters contain a number of variables that must be controlled. In addition, characterizing UHARS performance is at the core of this thesis research meaning it is more prudent to use a measurement other than UHARS estimates to determine the maximum error in Locata position. Therefore, DGPS position measurements will be used to establish a maximum

error for Locata. Subtracting Locata measurements from DGPS measurements will result in a series of positions with variance determined by the errors in both Locata and GPS. Therefore, the variance of the DGPS minus Locata study will fully encapsulate the errors in Locata and form a ceiling for the worst-case error present in Locata. Using this error ceiling, the total error in Locata delta-position velocity measurements can be estimated, and the error model established in section 3.2.2 can be validated.

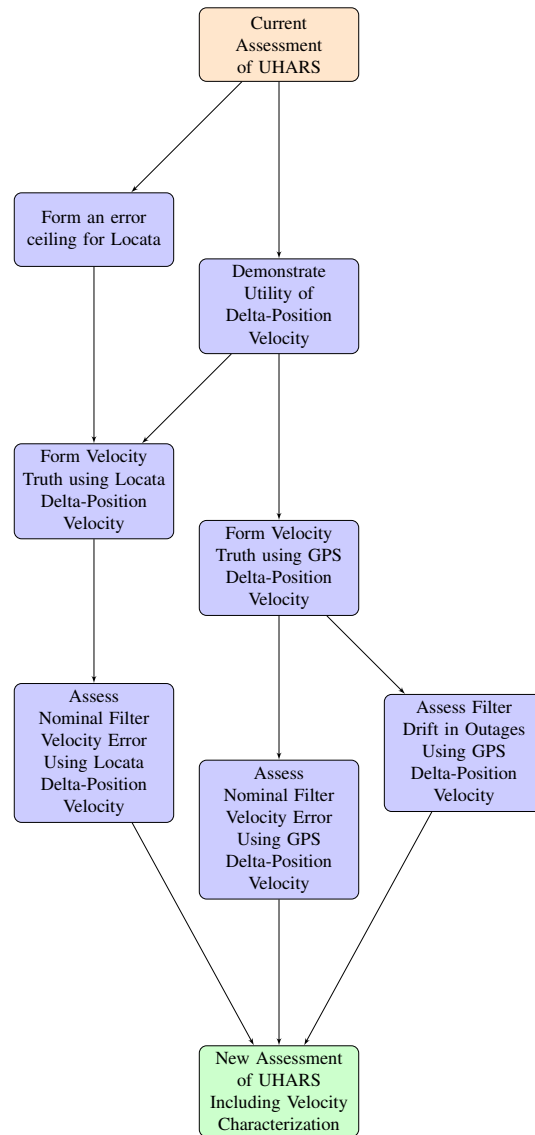


Figure 7: Flowchart of technical efforts completed in order to characterize UHARS velocity accuracy in different configurations and environments.

Next, the utility of the delta-position velocity measurement must be demonstrated. The first step is to derive an error equation that can predict the error present in delta-position velocity measurements. This error equation will provide the theoretical basis for a delta-position velocity measurement. After establishing the theoretical foundation, GPS and Locata delta-position velocity measurements are compared to a flight configuration filter. This requires the ability to average the filter velocity. Because a delta-position velocity measurement is a change in position divided by a change in time, the resulting velocity measurement is an average velocity over the time interval. This means the filter's velocity over this time period must also be averaged. Under constant acceleration, a linear interpolation can be used to determine the average velocity over a time interval, as the corresponding velocity is a straight line. However, under non-constant acceleration, the velocity results of the filter must be integrated in order to determine the change in position and that must be divided by the time difference. If the averaged filter velocity is reasonable given the established error ceiling for Locata and the error equation derived in section 3.2.2, then delta-position velocity measurements can be considered valid.

After determining the error associated with Locata position measurements and demonstrating the utility of the delta-position velocity measurement, velocity truths may be established by calculating both GPS and Locata delta-position velocities. These truths will act as the independent reference for filter results. Several different averaging periods are used when creating delta-position velocity measurements; averaging periods of 1 second, 10 seconds, and 100 seconds were selected. These different time intervals show how delta-position error phenomenology changes over time, and may be used operationally in different ways. Delta-position velocities with a 1 second averaging period may be used as filter measurement updates in future iterations of UHARS, whereas velocities with a 100 second averaging period may be used in a post-processed, smoothed solution to create a more accurate reference.

Nominal filter accuracy will be assessed by comparing filters with one measurement source to the independent delta-position velocity. This means a GPS-aided filter will be compared

to a Locata delta-position velocity and a Locata-aided filter will be compared to a GPS delta-position velocity. Using independent velocity sources removes any agreement between position updates in the filter and corresponding delta-position velocity measurements. For example, a filter incorporating GPS position measurements may be correlated with delta-position velocities generated by that same GPS data set. Comparing the GPS filter to a Locata delta-position velocity instead keeps the position measurements and delta-position velocities completely independent.

Finally, pseudo-Monte Carlo simulations characterize filter performance in GPS outage conditions. This not only determines how well the UHARS filter models INS drift, but also assesses operational filter performance in a GPS denied environment. In a real-world jammed environment, GPS will not be available as a truth. However, in a simulated jammed environment, those GPS measurements not incorporated into the filter may be used to generate a delta-position velocity truth, and that truth may be used to assess filter performance in the absence of GPS updates.

After completing all the above tasks, the filter velocity estimates will be well characterized, both in nominal variance and in outage drift. Determining filter effectiveness will require both qualitative and quantitative analysis of filter performance.

3.2 Delta-Position Velocity Error Equation

3.2.1 Motivation

Delta-position measurements are made up of two position measurements and two time measurements. Position measurements have inherent errors associated with them. Therefore, a delta-position measurement will have associated errors as well. It is trivial to realize that the errors present in a delta-position velocity will be related to the errors present in the constituent position measurements. Formulating an expression for delta-position velocity error forms a theoretical basis for the accuracy of the delta-position velocity. If position measurements of known typical variance are used to generate delta-position measurements,

the variance of those measurements may be predicted or validated using a delta-position velocity error equation. As no error equation for position-derived delta-position velocity is present in literature, such an equation must be derived before studying delta-position velocity accuracy.

3.2.2 Derivation

The error associated with a delta-position measurement is derived from several sources. Because the delta-position measurement is derived from two independent position measurements, the errors associated with each individual measurement contribute to the overall error of the delta-position measurement. These individual measurement errors can be classified as time-correlated and AWGN. Succinctly, equation (3.1) expresses the delta-position measurement between epoch t_1 and epoch t_2 :

$$\Delta \mathbf{p}_m = (\mathbf{p}_{2t} + \delta \mathbf{p}_2 + \mathbf{w}_2) - (\mathbf{p}_{1t} + \delta \mathbf{p}_1 + \mathbf{w}_1) \quad (3.1)$$

Where

- $\mathbf{p}_{x_t}$ is the true position at that particular point in time
- $\delta \mathbf{p}_x$ are the time-correlated errors associated with GPS
- $\mathbf{w}_x$ is AWGN

In deriving the error equation, several assumptions were made.

1. Position measurements take the form $\mathbf{p}_m = \mathbf{p}_t + \delta \mathbf{p}_c + w$
2. $\Delta \mathbf{p}_{1t}$ and $\Delta \mathbf{p}_{2t}$ are scalars with an associated error of zero
3. $\delta \mathbf{p}_1$ & $\delta \mathbf{p}_2$ are Gaussian
4. All noise ($\mathbf{w}_1, \mathbf{w}_2$) is Additive, White, and Gaussian, and i.i.d.

Because of above assumptions, the Delta Position measurement is a sum of Gaussian Random Variables and scalars which, is itself a Gaussian distributed random variable. Therefore, the final Delta Position measurement is expected to be Gaussian and characterized by its

mean and variance (μ, σ^2) . Therefore, a delta-position measurement can be appropriately characterized by finding expressions for both its mean (or expected value) and variance independently.

Expected Value of Delta Position Measurement

(1)	$\Delta \mathbf{p}_m = (\mathbf{p}_{2t} + \delta \mathbf{p}_{2c} + \mathbf{w}_2) - (\mathbf{p}_{1t} + \delta \mathbf{p}_{1c} + \mathbf{w}_1)$	Given
(2)	$E[\Delta \mathbf{p}_m] = E[(\mathbf{p}_{2t} + \delta \mathbf{p}_{2c} + \mathbf{w}_2) - (\mathbf{p}_{1t} + \delta \mathbf{p}_{1c} + \mathbf{w}_1)]$	Expected Value
(3)	$E[\Delta \mathbf{p}_m] = E[(\mathbf{p}_{2t} - \mathbf{p}_{1t}) + (\delta \mathbf{p}_{2c} - \delta \mathbf{p}_{1c}) + (\mathbf{w}_2 - \mathbf{w}_1)]$	Rearrange
(4)	$E[\Delta \mathbf{p}_m] = E[\mathbf{p}_{2t} - \mathbf{p}_{1t}] + E[\delta \mathbf{p}_{2c} - \delta \mathbf{p}_{1c}] + E[\mathbf{w}_2 - \mathbf{w}_1]$	Linearity of Expected Value
(5)	$E[\Delta \mathbf{p}_m] = E[\mathbf{p}_{2t}] - E[\mathbf{p}_{1t}] + E[\delta \mathbf{p}_{2c} - \delta \mathbf{p}_{1c}] + E[\mathbf{w}_2 - \mathbf{w}_1]$	Linearity of Expected Value
(6)	$E[\Delta \mathbf{p}_m] = \underbrace{\mathbf{p}_{2t} - \mathbf{p}_{1t}}_{\Delta \mathbf{p}_t} + E[\delta \mathbf{p}_{2c} - \delta \mathbf{p}_{1c}] + E[\mathbf{w}_2 - \mathbf{w}_1]$	Expected value of a constant
(7)	$E[\Delta \mathbf{p}_m] = \Delta \mathbf{p}_t + E[\delta \mathbf{p}_{2c}] - E[\delta \mathbf{p}_{1c}] + E[\mathbf{w}_2 - \mathbf{w}_1]$	Linearity of Expected Value
(8)	$E[\Delta \mathbf{p}_m] = \Delta \mathbf{p}_t + E[\delta \mathbf{p}_{2c}] - E[\delta \mathbf{p}_{1c}] + E[\mathbf{w}_2] - E[\mathbf{w}_1]$	Linearity of Expected Value
(9)	$E[\Delta \mathbf{p}_m] = \Delta \mathbf{p}_t + E[\delta \mathbf{p}_{2c}] - E[\delta \mathbf{p}_{1c}] + 0 - 0$	AWGN is zero mean
(10)	$E[\Delta \mathbf{p}_m] = \Delta \mathbf{p}_t + (E[\delta \mathbf{p}_{2c}] - E[\delta \mathbf{p}_{1c}])$	

In words, the final equation above tells us: “The expected value of a delta-position measurement is the true delta-position plus the difference in the expected values of the time correlated errors”

Variance of Delta Position Measurement

$$(1) \quad \text{Var}(\Delta \mathbf{p}_m) = \text{E} \left[(\Delta \mathbf{p}_m - \text{E}[\Delta \mathbf{p}_m])^2 \right]$$

$$(2) \quad \text{Var}(\Delta \mathbf{p}_m) = \text{E} \left[((\mathbf{p}_{2_t} + \delta \mathbf{p}_{2_c} + \mathbf{w}_2) - (\mathbf{p}_{1_t} + \delta \mathbf{p}_{1_c} + \mathbf{w}_1) - \text{E}[\Delta \mathbf{p}_m])^2 \right]$$

$$(3) \quad \text{Var}(\Delta \mathbf{p}_m) = \text{E} \left[(\mathbf{p}_{2_t} + \delta \mathbf{p}_{2_c} + \mathbf{w}_2) - (\mathbf{p}_{1_t} + \delta \mathbf{p}_{1_c} + \mathbf{w}_1) - (\Delta \mathbf{p}_t + (\text{E}[\delta \mathbf{p}_{2_c}] - \text{E}[\delta \mathbf{p}_{1_c}]))^2 \right]$$

$$(4) \quad \text{Var}(\Delta \mathbf{p}_m) = \text{E} \left[(\mathbf{p}_{2_t} + \delta \mathbf{p}_{2_c} + \mathbf{w}_2) - (\mathbf{p}_{1_t} + \delta \mathbf{p}_{1_c} + \mathbf{w}_1) - (\mathbf{p}_{2_t} - \mathbf{p}_{1_t} + (\text{E}[\delta \mathbf{p}_{2_c}] - \text{E}[\delta \mathbf{p}_{1_c}]))^2 \right]$$

$$(5) \quad \text{Var}(\Delta \mathbf{p}_m) = \text{E} \left[(\mathbf{p}_{2_t} + \delta \mathbf{p}_{2_c} + \mathbf{w}_2 - \mathbf{p}_{1_t} - \delta \mathbf{p}_{1_c} - \mathbf{w}_1 - \mathbf{p}_{2_t} + \mathbf{p}_{1_t} - (\text{E}[\delta \mathbf{p}_{2_c}] - \text{E}[\delta \mathbf{p}_{1_c}]))^2 \right]$$

$$(6) \quad \text{Var}(\Delta \mathbf{p}_m) = \text{E} \left[((\delta \mathbf{p}_{2_c} + \mathbf{w}_2 - \delta \mathbf{p}_{1_c} - \mathbf{w}_1 - (\text{E}[\delta \mathbf{p}_{2_c}] - \text{E}[\delta \mathbf{p}_{1_c}]))^2 \right]$$

$$(7) \quad \text{Var}(\Delta \mathbf{p}_m) = \text{E} \left[((\delta \mathbf{p}_{2_c} + \mathbf{w}_2 - \delta \mathbf{p}_{1_c} - \mathbf{w}_1 - \text{E}[\delta \mathbf{p}_{2_c}] + \text{E}[\delta \mathbf{p}_{1_c}])^2 \right]$$

$$(8) \quad \text{Var}(\Delta \mathbf{p}_m) = \text{E} \left[((\delta \mathbf{p}_{2_c} - \delta \mathbf{p}_{1_c} - \text{E}[\delta \mathbf{p}_{2_c}] + \text{E}[\delta \mathbf{p}_{1_c}] + \mathbf{w}_2 - \mathbf{w}_1)^2 \right]$$

$$(9) \quad \begin{aligned} \text{Var}(\Delta \mathbf{p}_m) = & \text{E}[\delta \mathbf{p}_{2_c}^2 - \delta \mathbf{p}_{1_c} \delta \mathbf{p}_{2_c} - \delta \mathbf{p}_{2_c} \text{E}[\delta \mathbf{p}_{2_c}] + \delta \mathbf{p}_{2_c} \text{E}[\delta \mathbf{p}_{1_c}] + \delta \mathbf{p}_{2_c} \mathbf{w}_2 - \delta \mathbf{p}_{2_c} \mathbf{w}_1 \\ & - \delta \mathbf{p}_{1_c} \delta \mathbf{p}_{2_c} + \delta \mathbf{p}_{1_c}^2 + \delta \mathbf{p}_{1_c} \text{E}[\delta \mathbf{p}_{2_c}] - \delta \mathbf{p}_{1_c} \text{E}[\delta \mathbf{p}_{1_c}] - \delta \mathbf{p}_{1_c} \mathbf{w}_2 + \delta \mathbf{p}_{1_c} \mathbf{w}_1 \\ & - \delta \mathbf{p}_{2_c} \text{E}[\delta \mathbf{p}_{2_c}] + \delta \mathbf{p}_{1_c} \text{E}[\delta \mathbf{p}_{2_c}] + \text{E}[\delta \mathbf{p}_{2_c}]^2 - \text{E}[\delta \mathbf{p}_{1_c}] \text{E}[\delta \mathbf{p}_{2_c}] - \mathbf{w}_2 \text{E}[\delta \mathbf{p}_{2_c}] + \mathbf{w}_1 \text{E}[\delta \mathbf{p}_{2_c}] \\ & + \delta \mathbf{p}_{2_c} \text{E}[\delta \mathbf{p}_{1_c}] - \delta \mathbf{p}_{1_c} \text{E}[\delta \mathbf{p}_{1_c}] - \text{E}[\delta \mathbf{p}_{1_c}] \text{E}[\delta \mathbf{p}_{2_c}] + \text{E}[\delta \mathbf{p}_{1_c}]^2 + \mathbf{w}_2 \text{E}[\delta \mathbf{p}_{1_c}] - \mathbf{w}_1 \text{E}[\delta \mathbf{p}_{1_c}] \\ & + \delta \mathbf{p}_{2_c} \mathbf{w}_2 - \delta \mathbf{p}_{1_c} \mathbf{w}_2 - \text{E}[\delta \mathbf{p}_{2_c}] \mathbf{w}_2 + \text{E}[\delta \mathbf{p}_{1_c}] \mathbf{w}_2 + \mathbf{w}_2^2 - \mathbf{w}_1 \mathbf{w}_2 \\ & - \delta \mathbf{p}_{2_c} \mathbf{w}_1 + \delta \mathbf{p}_{1_c} \mathbf{w}_1 + \text{E}[\delta \mathbf{p}_{2_c}] \mathbf{w}_1 - \text{E}[\delta \mathbf{p}_{1_c}] \mathbf{w}_1 - \mathbf{w}_1 \mathbf{w}_2 + \mathbf{w}_1^2] \end{aligned}$$

$$(10) \quad \begin{aligned} \text{Var}(\Delta \mathbf{p}_m) = & \text{E}[\delta \mathbf{p}_{2_c}^2 + \delta \mathbf{p}_{1_c}^2 + \text{E}[\delta \mathbf{p}_{2_c}]^2 + \text{E}[\delta \mathbf{p}_{1_c}]^2 + \mathbf{w}_2^2 + \mathbf{w}_1^2 \\ & - 2\delta \mathbf{p}_{1_c} \delta \mathbf{p}_{2_c} - 2\delta \mathbf{p}_{2_c} \text{E}[\delta \mathbf{p}_{2_c}] + 2\delta \mathbf{p}_{2_c} \text{E}[\delta \mathbf{p}_{1_c}] - 2\delta \mathbf{p}_{1_c} \text{E}[\delta \mathbf{p}_{1_c}] + 2\delta \mathbf{p}_{1_c} \text{E}[\delta \mathbf{p}_{2_c}] \\ & + 2\delta \mathbf{p}_{2_c} \mathbf{w}_2 - 2\delta \mathbf{p}_{2_c} \mathbf{w}_1 - 2\delta \mathbf{p}_{1_c} \mathbf{w}_2 + 2\delta \mathbf{p}_{1_c} \mathbf{w}_1 \\ & - 2\text{E}[\delta \mathbf{p}_{1_c}] \text{E}[\delta \mathbf{p}_{2_c}] - 2\mathbf{w}_2 \text{E}[\delta \mathbf{p}_{2_c}] + 2\mathbf{w}_1 \text{E}[\delta \mathbf{p}_{2_c}] \\ & + 2\mathbf{w}_2 \text{E}[\delta \mathbf{p}_{1_c}] - 2\mathbf{w}_1 \text{E}[\delta \mathbf{p}_{1_c}] - 2\mathbf{w}_1 \mathbf{w}_2] \end{aligned}$$

$$(11) \quad \begin{aligned} \text{Var}(\Delta \mathbf{p}_m) = & \text{E}[\delta \mathbf{p}_{2_c}^2] + \text{E}[\delta \mathbf{p}_{1_c}^2] + \text{E}[\text{E}[\delta \mathbf{p}_{2_c}]^2] + \text{E}[\text{E}[\delta \mathbf{p}_{1_c}]^2] + \text{E}[\mathbf{w}_2^2] + \text{E}[\mathbf{w}_1^2] \\ & - \text{E}[2\delta \mathbf{p}_{1_c} \delta \mathbf{p}_{2_c}] - \text{E}[2\delta \mathbf{p}_{2_c} \text{E}[\delta \mathbf{p}_{2_c}]] + \text{E}[2\delta \mathbf{p}_{2_c} \text{E}[\delta \mathbf{p}_{1_c}]] - \text{E}[2\delta \mathbf{p}_{1_c} \text{E}[\delta \mathbf{p}_{1_c}]] + \text{E}[2\delta \mathbf{p}_{1_c} \text{E}[\delta \mathbf{p}_{2_c}]] \\ & + \text{E}[2\delta \mathbf{p}_{2_c} \mathbf{w}_2] - \text{E}[2\delta \mathbf{p}_{2_c} \mathbf{w}_1] - \text{E}[2\delta \mathbf{p}_{1_c} \mathbf{w}_2] + \text{E}[2\delta \mathbf{p}_{1_c} \mathbf{w}_1] \\ & - \text{E}[2\text{E}[\delta \mathbf{p}_{1_c}] \text{E}[\delta \mathbf{p}_{2_c}]] - \text{E}[2\mathbf{w}_2 \text{E}[\delta \mathbf{p}_{2_c}]] + \text{E}[2\mathbf{w}_1 \text{E}[\delta \mathbf{p}_{2_c}]] \\ & + \text{E}[2\mathbf{w}_2 \text{E}[\delta \mathbf{p}_{1_c}]] - \text{E}[2\mathbf{w}_1 \text{E}[\delta \mathbf{p}_{1_c}]] - \text{E}[2\mathbf{w}_1 \mathbf{w}_2] \end{aligned}$$

Definition of Variance

Substituting for $\Delta \mathbf{p}_m$ (given)

Substituting for $\text{E}[\Delta \mathbf{p}_m]$ ((10) above)

Definition of true delta position ($\Delta \mathbf{p}_t$)

Distributing signs

Cancelling terms

Distributing signs

Rearrange Terms

Polynomial Expansion

Combining Terms

Linearity of Expectation

$$+ 2\mathbb{E}[\delta\mathbf{p}_{1_c}]\mathbb{E}[\mathbf{w}_2] - 2\mathbb{E}[\delta\mathbf{p}_{1_c}]\mathbb{E}[\mathbf{w}_1] - 2\mathbb{E}[\mathbf{w}_1\mathbf{w}_2]$$

$$(18) \quad \text{Var}(\Delta\mathbf{p}_m) = \mathbb{E}[\delta\mathbf{p}_{2_c}^2] - \mathbb{E}[\delta\mathbf{p}_{2_c}]^2 + \mathbb{E}[\delta\mathbf{p}_{1_c}^2] - \mathbb{E}[\delta\mathbf{p}_{1_c}]^2$$

$$- 2\mathbb{E}[\delta\mathbf{p}_{1_c}\delta\mathbf{p}_{2_c}] + 2\mathbb{E}[\delta\mathbf{p}_{1_c}]\mathbb{E}[\delta\mathbf{p}_{2_c}] + 2\mathbb{E}[\delta\mathbf{p}_{2_c}]\mathbb{E}[\delta\mathbf{p}_{1_c}]$$

$$+ 2\mathbb{E}[\delta\mathbf{p}_{2_c}\mathbf{w}_2] - 2\mathbb{E}[\delta\mathbf{p}_{2_c}\mathbf{w}_1] - 2\mathbb{E}[\delta\mathbf{p}_{1_c}\mathbf{w}_2] + 2\mathbb{E}[\delta\mathbf{p}_{1_c}\mathbf{w}_1]$$

$$- 2\mathbb{E}[\delta\mathbf{p}_{1_c}]\mathbb{E}[\delta\mathbf{p}_{2_c}] - 2\mathbb{E}[\delta\mathbf{p}_{2_c}]\mathbb{E}[\mathbf{w}_2] + 2\mathbb{E}[\delta\mathbf{p}_{2_c}]\mathbb{E}[\mathbf{w}_1]$$

$$+ 2\mathbb{E}[\delta\mathbf{p}_{1_c}]\mathbb{E}[\mathbf{w}_2] - 2\mathbb{E}[\delta\mathbf{p}_{1_c}]\mathbb{E}[\mathbf{w}_1] + \mathbb{E}[\mathbf{w}_2^2] + \mathbb{E}[\mathbf{w}_1^2] - 2\mathbb{E}[\mathbf{w}_1\mathbf{w}_2]$$

$$(19) \quad \text{Var}(\Delta\mathbf{p}_m) = \text{Var}(\delta\mathbf{p}_{2_c}) + \text{Var}(\delta\mathbf{p}_{1_c}) - 2\mathbb{E}[\delta\mathbf{p}_{1_c}\delta\mathbf{p}_{2_c}] + 2\mathbb{E}[\delta\mathbf{p}_{1_c}]\mathbb{E}[\delta\mathbf{p}_{2_c}]$$

$$+ 2\mathbb{E}[\delta\mathbf{p}_{2_c}\mathbf{w}_2] - 2\mathbb{E}[\delta\mathbf{p}_{2_c}\mathbf{w}_1] - 2\mathbb{E}[\delta\mathbf{p}_{1_c}\mathbf{w}_2] + 2\mathbb{E}[\delta\mathbf{p}_{1_c}\mathbf{w}_1]$$

$$- 2\mathbb{E}[\delta\mathbf{p}_{2_c}]\mathbb{E}[\mathbf{w}_2] + 2\mathbb{E}[\delta\mathbf{p}_{2_c}]\mathbb{E}[\mathbf{w}_1]$$

$$+ 2\mathbb{E}[\delta\mathbf{p}_{1_c}]\mathbb{E}[\mathbf{w}_2] - 2\mathbb{E}[\delta\mathbf{p}_{1_c}]\mathbb{E}[\mathbf{w}_1] + \mathbb{E}[\mathbf{w}_2^2] + \mathbb{E}[\mathbf{w}_1^2] - 2\mathbb{E}[\mathbf{w}_1\mathbf{w}_2]$$

$$(20) \quad \text{Var}(\Delta\mathbf{p}_m) = \text{Var}(\delta\mathbf{p}_{2_c}) + \text{Var}(\delta\mathbf{p}_{1_c}) - 2 \text{Cov}(\delta\mathbf{p}_{1_c}, \delta\mathbf{p}_{2_c})$$

$$+ 2\mathbb{E}[\delta\mathbf{p}_{2_c}\mathbf{w}_2] - 2\mathbb{E}[\delta\mathbf{p}_{2_c}\mathbf{w}_1] - 2\mathbb{E}[\delta\mathbf{p}_{1_c}\mathbf{w}_2] + 2\mathbb{E}[\delta\mathbf{p}_{1_c}\mathbf{w}_1]$$

$$- 2\mathbb{E}[\delta\mathbf{p}_{2_c}]\mathbb{E}[\mathbf{w}_2] + 2\mathbb{E}[\delta\mathbf{p}_{2_c}]\mathbb{E}[\mathbf{w}_1]$$

$$(21) \quad \text{Var}(\Delta\mathbf{p}_m) = \text{Var}(\delta\mathbf{p}_{2_c}) + \text{Var}(\delta\mathbf{p}_{1_c}) - 2 \text{Cov}(\delta\mathbf{p}_{1_c}, \delta\mathbf{p}_{2_c})$$

$$+ 2\mathbb{E}[\delta\mathbf{p}_{2_c}\mathbf{w}_2] - 2\mathbb{E}[\delta\mathbf{p}_{2_c}]\mathbb{E}[\mathbf{w}_2] + 2\mathbb{E}[\delta\mathbf{p}_{1_c}\mathbf{w}_1] - 2\mathbb{E}[\delta\mathbf{p}_{1_c}]\mathbb{E}[\mathbf{w}_1]$$

$$- 2\mathbb{E}[\delta\mathbf{p}_{1_c}\mathbf{w}_2] + 2\mathbb{E}[\delta\mathbf{p}_{1_c}]\mathbb{E}[\mathbf{w}_2] - 2\mathbb{E}[\delta\mathbf{p}_{2_c}\mathbf{w}_1] + 2\mathbb{E}[\delta\mathbf{p}_{2_c}]\mathbb{E}[\mathbf{w}_1]$$

$$+ \mathbb{E}[\mathbf{w}_2^2] + \mathbb{E}[\mathbf{w}_1^2] - 2\mathbb{E}[\mathbf{w}_1\mathbf{w}_2]$$

$$(22) \quad \text{Var}(\Delta\mathbf{p}_m) = \text{Var}(\delta\mathbf{p}_{2_c}) + \text{Var}(\delta\mathbf{p}_{1_c}) - 2 \text{Cov}(\delta\mathbf{p}_{1_c}, \delta\mathbf{p}_{2_c})$$

$$+ 2 \text{Cov}(\delta\mathbf{p}_{2_c}, \mathbf{w}_2) + \text{Var}(\delta\mathbf{p}_{1_c}) + 2 \text{Cov}(\delta\mathbf{p}_{1_c}, \mathbf{w}_1) - 2 \text{Cov}(\delta\mathbf{p}_{1_c}, \mathbf{w}_2) - 2 \text{Cov}(\delta\mathbf{p}_{2_c}, \mathbf{w}_1)$$

$$+ \mathbb{E}[\mathbf{w}_2^2] + \mathbb{E}[\mathbf{w}_1^2] - 2\mathbb{E}[\mathbf{w}_1\mathbf{w}_2]$$

$$(23) \quad \text{Var}(\Delta\mathbf{p}_m) = \text{Var}(\delta\mathbf{p}_{2_c}) + \text{Var}(\delta\mathbf{p}_{1_c}) - 2 \text{Cov}(\delta\mathbf{p}_{1_c}, \delta\mathbf{p}_{2_c})$$

$$+ \mathbb{E}[\mathbf{w}_2^2] + \mathbb{E}[\mathbf{w}_1^2] - 2\mathbb{E}[\mathbf{w}_1\mathbf{w}_2]$$

$$(24) \quad \text{Var}(\Delta\mathbf{p}_m) = \text{Var}(\delta\mathbf{p}_{2_c}) + \text{Var}(\delta\mathbf{p}_{1_c}) - 2 \text{Cov}(\delta\mathbf{p}_{1_c}, \delta\mathbf{p}_{2_c}) + \mathbb{E}[\mathbf{w}_2^2] + \mathbb{E}[\mathbf{w}_1^2] - 2\mathbb{E}[\mathbf{w}_1]\mathbb{E}[\mathbf{w}_2]$$

$$(25) \quad \text{Var}(\Delta\mathbf{p}_m) = \text{Var}(\delta\mathbf{p}_{2_c}) + \text{Var}(\delta\mathbf{p}_{1_c}) - 2 \text{Cov}(\delta\mathbf{p}_{1_c}, \delta\mathbf{p}_{2_c}) + \mathbb{E}[\mathbf{w}_2^2] + \mathbb{E}[\mathbf{w}_1^2]$$

Rearrange terms

$$\mathbb{E}[X^2] - \mathbb{E}[X]^2 = \text{Var}(X)$$

-See Proof (A)-

$$\text{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$

-See Proof (B)-

Rearrange terms

$$\text{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$

-See Proof (B)-

White Noise is not correlated
with time-correlated errors

For independent vars X and Y ,
 $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$

Expectation of White Noise is 0

Alternate forms of the above equation are:

$$\text{Var}(\Delta \mathbf{p}_m) = \text{Var}(\delta \mathbf{p}_{2_c} - \delta \mathbf{p}_{1_c}) + E[\mathbf{w}_2^2] + E[\mathbf{w}_1^2] \quad \text{See Proof (E)}$$

$$\text{Var}(\Delta \mathbf{p}_m) = E[(\delta \mathbf{p}_{2_c} - E[\delta \mathbf{p}_{2_c}])^2] + E[(\delta \mathbf{p}_{1_c} - E[\delta \mathbf{p}_{1_c}])^2] - E[\delta \mathbf{p}_{2_c} \delta \mathbf{p}_{1_c}] - E[\delta \mathbf{p}_{2_c}]E[\delta \mathbf{p}_{1_c}] + E[\mathbf{w}_2^2] + E[\mathbf{w}_1^2]$$

By definition of Variance and see Proof (B)

Proof (A) - Alternate Formula for Variance (a.k.a. the König-Huygens Formula)

Prove $\text{Var}(X) = E[X^2] - E[X]^2$

- | | | |
|------|---|--|
| (1) | $\text{Var}(X) = E[(X - E[X])^2]$ | Definition of Variance |
| (2) | $\text{Var}(X) = E[(X - \mu_X)^2]$ | Expected Value of X yields the mean of X |
| (3) | $\text{Var}(X) = E[X^2 - 2X\mu_X + \mu_X^2]$ | Polynomial Expansion |
| (4) | $\text{Var}(X) = E[X^2] - E[2X\mu_X] + E[\mu_X^2]$ | Linearity of Expectation |
| (5) | $\text{Var}(X) = E[X^2] - E[2X\mu_X] + \mu_X^2$ | Expectation of a constant is that constant |
| (6) | $\text{Var}(X) = E[X^2] - 2\mu_X E[X] + \mu_X^2$ | Factoring out constants (linearity) |
| (7) | $\text{Var}(X) = E[X^2] - 2\mu_X \cdot \mu_X + \mu_X^2$ | Expected Value of an R.V. X yields the mean of X |
| (8) | $\text{Var}(X) = E[X^2] - 2\mu_X^2 + \mu_X^2$ | |
| (9) | $\text{Var}(X) = E[X^2] - \mu_X^2$ | |
| (10) | $\text{Var}(X) = E[X^2] - E[X]^2$ | Q.E.D. |

Proof (B) - Alternate Formula for Covariance

Prove $\text{Cov}(X, Y) = E[XY] - E[X]E[Y]$

- | | | |
|-----|--|---|
| (1) | $\text{Cov}(X, Y) = E[(X - E[X])(Y - E[Y])]$ | Definition of Covariance |
| (2) | $\text{Cov}(X, Y) = E[XY - XE[Y] - YE[X] + E[X]E[Y]]$ | Polynomial Expansion |
| (3) | $\text{Cov}(X, Y) = E[XY - X\mu_Y - Y\mu_X + \mu_X\mu_Y]$ | Expected Value of an R.V. X yields the mean of X |
| (4) | $\text{Cov}(X, Y) = E[XY] - E[X\mu_Y] - E[Y\mu_X] + E[\mu_X\mu_Y]$ | Linearity of Expectation |
| (5) | $\text{Cov}(X, Y) = E[XY] - E[X\mu_Y] - E[Y\mu_X] + \mu_X\mu_Y$ | Expectation of a constant is that constant |
| (6) | $\text{Cov}(X, Y) = E[XY] - \mu_Y E[X] - \mu_X E[Y] + \mu_X\mu_Y$ | Factoring out constants (linearity) |
| (7) | $\text{Cov}(X, Y) = E[XY] - E[X]E[Y] - E[X]E[Y] + E[X]E[Y]$ | The mean of X is given by the Expected Value of X |
| (8) | $\text{Cov}(X, Y) = E[XY] - E[X]E[Y]$ | Q.E.D. |

Proof (C) - Variance of Sum of Random Variables

Prove $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) - 2 \text{Cov}(X, Y)$

$$(1) \quad \text{Var}(X + Y) = \text{E} \left[((X + Y) - \text{E}[X + Y])^2 \right]$$

$$(2) \quad \text{Var}(X + Y) = \text{E} \left[(X + Y - \text{E}[X] - \text{E}[Y])^2 \right]$$

$$(3) \quad \begin{aligned} \text{Var}(X + Y) = \text{E}[X^2 + XY - X\text{E}[X] - X\text{E}[Y] + XY + Y^2 - Y\text{E}[X] - Y\text{E}[Y] - X\text{E}[X] \\ - Y\text{E}[Y] + \text{E}[X]^2 + \text{E}[X]\text{E}[Y] - X\text{E}[Y] - X\text{E}[Y] - Y\text{E}[Y] + \text{E}[X]\text{E}[Y] + \text{E}[Y]^2] \end{aligned}$$

$$(4) \quad \begin{aligned} \text{Var}(X + Y) = \text{E}[X^2 + Y^2 + 2XY \\ - 2X\text{E}[X] - 2Y\text{E}[Y] - 2X\text{E}[Y] - 2Y\text{E}[X] \\ + 2\text{E}[X]\text{E}[Y] + \text{E}[X]^2 + \text{E}[Y]^2] \end{aligned}$$

$$(5) \quad \text{Var}(X + Y) = \text{E}[X^2 + Y^2 + 2XY$$

$$\quad - 2X\mu_X - 2Y\mu_Y - 2X\mu_Y - 2Y\mu_X \\ + 2\mu_X\mu_Y + \mu_X^2 + \mu_Y^2]$$

$$(6) \quad \begin{aligned} \text{Var}(X + Y) = \text{E}[X^2] + \text{E}[Y^2] + \text{E}[2XY] \\ - \text{E}[2X\mu_X] - \text{E}[2Y\mu_Y] - \text{E}[2X\mu_Y] - \text{E}[2Y\mu_X] \\ + \text{E}[2\mu_X\mu_Y] + \text{E}[\mu_X^2] + \text{E}[\mu_Y^2] \end{aligned}$$

$$(6) \quad \begin{aligned} \text{Var}(X + Y) = \text{E}[X^2] + \text{E}[Y^2] + \text{E}[2XY] \\ - \text{E}[2X\mu_X] - \text{E}[2Y\mu_Y] - \text{E}[2X\mu_Y] - \text{E}[2Y\mu_X] \\ + 2\mu_X\mu_Y + \mu_X^2 + \mu_Y^2 \end{aligned}$$

$$(7) \quad \begin{aligned} \text{Var}(X + Y) = \text{E}[X^2] + \text{E}[Y^2] + 2\text{E}[XY] \\ - 2\mu_X\text{E}[X] - 2\mu_Y\text{E}[Y] - 2\mu_Y\text{E}[X] - 2\mu_X\text{E}[Y] \\ + 2\mu_X\mu_Y + \mu_X^2 + \mu_Y^2 \end{aligned}$$

$$(8) \quad \begin{aligned} \text{Var}(X + Y) = \text{E}[X^2] + \text{E}[Y^2] + 2\text{E}[XY] \\ - 2\text{E}[X]^2 - 2\text{E}[Y]^2 - 2\text{E}[X]\text{E}[Y] - 2\text{E}[X]\text{E}[Y] \\ + 2\text{E}[X]\text{E}[Y] + \text{E}[X]^2 + \text{E}[Y]^2 \end{aligned}$$

$$(9) \quad \text{Var}(X + Y) = \text{E}[X^2] + \text{E}[Y^2] + 2\text{E}[XY] - \text{E}[X]^2 - \text{E}[Y]^2 - 2\text{E}[X]\text{E}[Y]$$

$$(10) \quad \text{Var}(X + Y) = \underbrace{\text{E}[X^2] - \text{E}[X]^2}_{\text{Var}(X)} + \underbrace{\text{E}[Y^2] - \text{E}[Y]^2}_{\text{Var}(Y)} + \underbrace{2\text{E}[XY] - 2\text{E}[X]\text{E}[Y]}_{\text{Cov}(X, Y)}$$

$$(11) \quad \text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) - 2 \text{Cov}(X, Y)$$

Definition of Variance

Linearity of Expectation

Polynomial Expansion

Combine Like Terms

Expected Value of

an R.V. X yields the mean of X

Linearity of Expectation

Expectation of a constant
is that constant

Factoring out constants (linearity)

The mean of X is given
by the Expected Value of X
Cancel terms
Rearrange terms

Q.E.D.

3.3 Filter Design and Data Sets

3.3.1 Available Measurements

The UHARS reference pallet contains several navigation systems that provide measurements for post-processing and filtering. First, a Honeywell EGI provides angular rotation rates and force measurements. A Trimble Force 524D GPS receiver provides position and velocity measurements, and the LocataNet NGBPS provides position and velocity updates. An unspecified baro-altimeter provides altitude updates. There are also the options to incorporate zero velocity and clock correction measurements, but neither data set was provided.

3.3.2 Filter Design

The implemented EKF utilized a Scorpion core and SPIDER measurements. Scorpion is the follow-on software to SPIDER, implemented in Kotlin and Java rather than MATLAB. SPIDER measurements are files formatted to the SPIDER Interface Control Document (ICD). This means that the stock Scorpion implementation must be modified to accept measurements from a different file format. The EKF loads all measurements in from a file using the built-in measurement sequencer class. First, the EKF initializes the solution to the first INS position and velocity. Then, the filter checks to see if the next measurement timestamp is within the designated maximum propagation time. If the next measurement is within the maximum propagation time, then the filter will propagate to that time. If not, the next check is whether or not a maximum time propagation will cross over a whole second epoch. If so, the filter will propagate to that whole second. This modification enables the MATLAB analysis scripts to align times appropriately; without it present, it is not practical to compare time intervals of equal length due to a variable number of epochs between the beginning and the end. Propagating to the whole second allows for, at the very least, second-level alignment of the time vectors when comparing outage statistics. Finally, if neither the next measurement nor a whole second is within the maximum propagation time, then the filter

propagates the maximum propagation time (see Fig. 8).

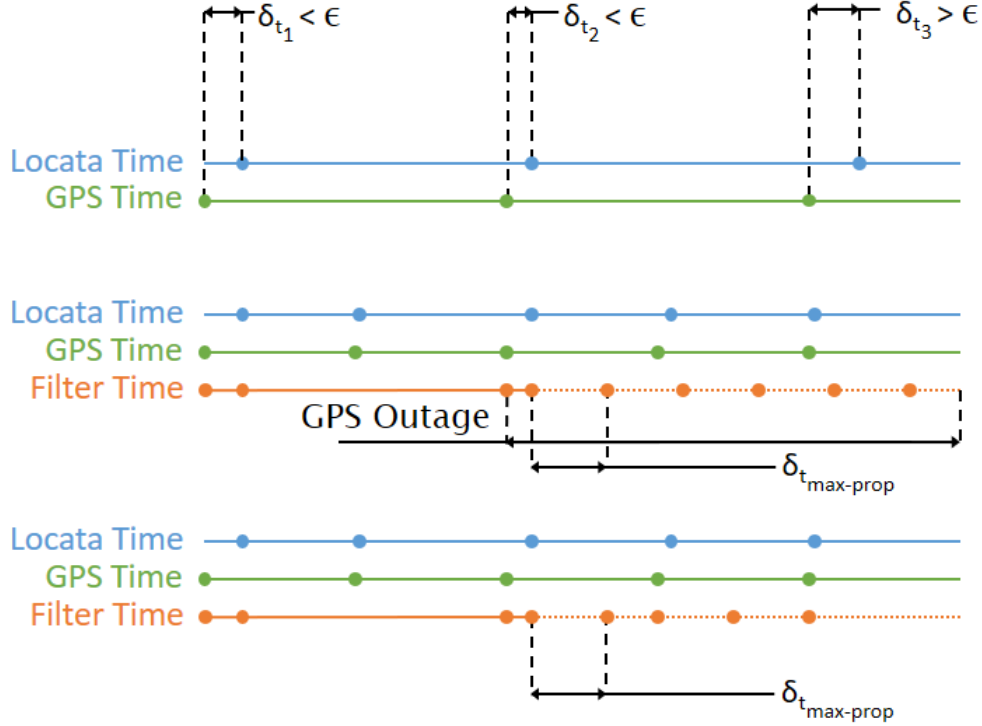


Figure 8: Relationship of different filter and measurement timelines, illustrating the need to propagate to whole-second epochs.

Scorpion routines automatically linearize the motion and measurement functions and applies them. Filter states are then saved for post-solution smoothing (if active).

3.3.3 Filter Data Sets Generated

Because of the complex nature of this system, and the numerous measurements available, several different test cases were studied. The first comparison performed was that of the raw signals, specifically the raw positioning signals. Comparing GPS and Locata position, delta-position, and delta-position Velocity will validate the above equation for delta-position error and will provide a maximum anticipated error for each positioning source.

An EKF was implemented with different combinations of measurements and then compared to measurements not incorporated into the solution. The comparison configurations

are summarized in Table 3. By completing a filtered and smoothed solution then comparing to an independent source, the accuracy of each source can be characterized.

Test Case 1, the most basic, uses the operational configuration, all sensor measurements incorporated with areas of non-coverage removed. Because of high dynamics, the Locata system periodically loses carrier-phase lock with the Locata transmitters. At high pitch rates, even the Novatel DGPS loses a degree of accuracy and can drop out. In these cases, the gaps in Locata and GPS availability are identified and the intermittent times are NaN masked (similar to XOR masking, that portion is multiplied by NaN so it does not contribute to the error statistics and is not plotted on the error plots). This configuration presents the ideal, best-case accuracy both of position and velocity. The relevant statistics studied here are the mean and standard deviation of the position and velocity errors. This reveals the accuracy of the filter using GPS as a truth, assuming position errors are Gaussian.

The next configurations, Test Case 2 and 3, studied a filter aided only by the barometer and one of the two location measurements, GPS position or Locata position. The resulting filter outputs are compared to the position and delta-position velocity of the independent positioning signal. That is, when the filter is aided by GPS, the results are compared to Locata position and delta-position velocity. When aided by Locata, the filter's results are compared to the GPS position and delta-position velocity. These configurations reveal the performance of the filter at its most basic level and allow characterization of the performance of each positioning system. If the maximum error of the positioning services is found to be within the expected range, then these results can be expected to be almost identical in terms of numerical or statistical accuracy. Once again, the statistics studied here are mean and standard deviation of errors and the Root-Mean-Square (RMS) of errors in each axis. This also again assumes that errors are Gaussian and white, but will reveal

After comparing filter aided by each measurement independently in a nominal configuration, outage cases (Test Case 4 and 5) are considered. The case we are most interested in is a filter configuration with all measurements incorporated, but with periodic simulated

Table 3: Comparison configurations using UHARS data sets

Test Case	Filter Measurements	Comparison Source
1	Novatel DGPS 524D GPS Locata Barometer	Novatel DGPS
2	Novatel DGPS 524D GPS Barometer	Locata
3	Locata Barometer	Novatel DGPS
4	Novatel DGPS (with Outages) 524D GPS (with Outages) Barometer	Locata
5	Locata (with Outages) Barometer	Novatel DGPS

GPS outages. This test case utilizes the Scorpion built-in signal outage capability. Outages can be generated with a fixed outage length and reset length, with cycles repeating over the length of measurement availability. In this case, the number of complete cycles is limited by the real-world availability of the Locata measurements, as position measurements from Locata are only available when the aircraft is over the Locata service area. Simulating these outages multiple times over a single flight creates a pseudo-Monte Carlo study, where an ensemble of solution drift can be overlaid all from one flight test. This configuration will reveal filter inaccuracies related to error in the INS error estimation and modeling. Because these conditions occur under dynamics, it is necessary to bin each implemented outage based on the condition of flight over the course of the outage. The procedure for doing so can be found in section 3.8.

3.4 Setup and Pre-processing

Prior to any calculations, the filter and GPS positions are rotated from the geodetic frame to the ECEF frame. The filter and GPS velocity vectors are rotated from the NED frame to the ECEF frame. This means all calculations can be carried out in the ECEF frame in order

to remove any errors due to the fact that the NED frame rotates as a function of position. In addition, converting geodetic position differences from radians to meters requires calculation of the geodetic radii of curvature at each point, which only approximates the errors in meters and introduces unacceptable errors.

Delta-position calculation first requires a common timeline, achieved by selecting filter time stamps that are aligned with the GPS measurement epochs. In this case, GPS measurements are incorporated at a 1 Hz nominal rate. The filter, propagating at 10 Hz, allows for alignment to this timeline. After a common timeline is established, delta-position is calculated by simply differencing two GPS locations. This difference is not an instantaneous velocity at either time step, but rather an average velocity over the length of time between the two times. Therefore, if the calculated delta-position velocity were to be compared to the instantaneous velocity at either time, the error would be much higher. In general, this calculated delta-position velocity would be most accurate at the center of the time interval. Note that the delta-position velocities are not moving averages; the entire timeline studied is subdivided into discrete windows of constant width and the average velocity is calculated for that entire window independent of the window before or after it. Therefore, as the averaging period increases, the delta-position velocity measurement rate decreases. A 1 second averaging period produces a 1 Hz measurement, a 10 second averaging period produces a 0.1 Hz measurement, and so forth.

This leads to the issue of comparing the delta-position velocity to the filter velocity; comparing average velocity over one second to instantaneous velocity in a 10 Hz solution would not yield useful results. However, if a separate averaging algorithm is implemented with the filter velocity measurement, then the average velocities will align both in magnitude and time. This averaging is accomplished using trapezoidal integration to determine change in position from velocity. Dividing by the interval length then yields an averaged velocity. The averaging period plays a large role in the final error. If a one second averaging period is used, the division is by 1 second, whereas a 100 second averaging period, in which two

position measurements 100 seconds apart are differenced, the division is by 100 seconds. This means that a truly Gaussian position measurement would result in a smaller error over 100 seconds vs 1 second averaging period, as the variance of the time-correlated errors in the delta-position measurement will decrease over time, and the strength of the white noise will stay the same.

3.5 Algorithm for Direct Comparison Between Locata and GPS

The first step was characterizing the maximum error of a delta-position measurement when compared between Locata and GPS measurements. In order to determine the maximum possible error associated with a delta-position measurement, the delta-position velocities were calculated for both DGPS and Locata measurements. The algorithm for such a calculation follows:

1. Load Waypoint DGPS measurements
2. Apply lever arm corrections to Waypoint measurements
3. Convert Novatel DGPS measurements from LLA to ECEF using Waypoint LLA
4. Load Locata measurements
5. Remove degraded Locata measurements (horizontal $\sigma^2 > 1.6$ cm, vertical $\sigma^2 > 11$ cm)
6. Apply lever arm corrections to Locata
7. Convert Locata measurements from LLA to ECEF using Locata measured LLA
8. Find common times between the Locata and Novatel DGPS measurements within 10^{-8} sec
9. For the length of the aligned time vector:
 - (a) Calculate $\Delta x = \frac{x_{i+1} - x_i}{t_{i+1} - t_i}$, $\Delta y = \frac{y_{i+1} - y_i}{t_{i+1} - t_i}$, and $\Delta z = \frac{z_{i+1} - z_i}{t_{i+1} - t_i}$ for Locata Position and GPS Position
 - (b) Calculate $\Delta t = t_{i+1} - t_i$
 - (c) Calculate $v_{\Delta \text{pos}} = \frac{\Delta \mathbf{p}}{\Delta t}$
 - (d) Calculate the valid time as $t_i + \frac{t_{i+1} - t_i}{2}$

10. Subtract the Locata positions measurements from GPS position measurements
11. Subtract the Locata velocity measurements from GPS position measurements
12. Subtract the Locata delta-position Velocity from the GPS delta-position velocity
13. Rotate the errors from the ECEF frame to the NED frame using Waypoint LLA to determine origin

In this case, lever arms are expressed in the body frame (nose, right wing, down). Lever arms are present because positioning system solutions are calculated at their antenna center. This position solution must then be translated to be co-located with either the origin of the body frame or with the IMU. However, if the distance from the antenna to the IMU is miscalculated, then navigation solutions will be a small distance away from the true center, leading to complex errors. Such errors are a constant offset in straight and level flight, but change when under dynamics. A conceptual lever arm error condition is depicted in Figure 9. Note that real-world errors are more complex because the roll, pitch, and yaw of the aircraft rotate antennas relative to each other, meaning that real-world lever arms are more difficult to determine.

Reported lever arms were corrected by performing a lever arm sensitivity analysis. The reported lever arm was established as a best-guess estimate. Filter solutions were calculated using a 1 cm offset in each body frame axis. A new best-guess estimate was then established based on the lowest three dimensional root-mean-square (3D-RMS) of each solution. Using this new estimate, a sensitivity analysis was then performed in each body axis with a 0.5 cm offset. This process continued, halving the distance offset until reaching 0.003125 cm disturbances.

The delta-position velocity was calculated using three averaging periods. Because the expression for a delta-position averaged velocity is: $v_{\Delta\text{pos}} = \frac{\Delta\mathbf{p}_m}{\Delta t}$, and the associated variance is: $\text{Var}(\Delta\mathbf{p}_m) = \frac{1}{t_2 - t_1} (\text{Var}(\delta\mathbf{p}_{2c}) + \text{Var}(\delta\mathbf{p}_{1c}) - 2\text{Cov}(\delta\mathbf{p}_{1c}, \delta\mathbf{p}_{2c}) + \text{E}[\mathbf{w}_2^2] + \text{E}[\mathbf{w}_1^2])$

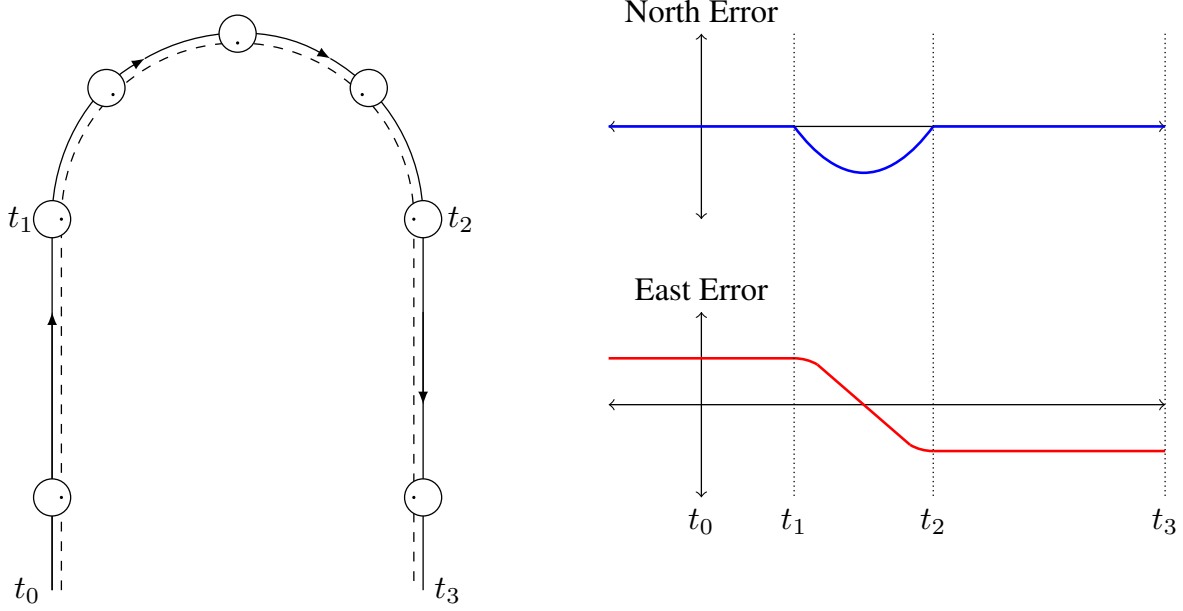


Figure 9: Conceptual, 2D representation of lever-arm position errors in a turning aircraft

3.6 Algorithm for Generating Delta-Position and Delta-Position Velocity

The delta-position and delta-position velocity are two novel measurements generated in this thesis. As discussed in section 2.5.2, a delta-position measurement is the a calculated change in position divided by a calculated change in time. One item of note, the Locata or GPS measurements are taken only when within 10^{-8} s of a filter estimate epoch. Interpolation was not used to due to interpolation error form real-world outages. Fig. 10 shows the effect linear interpolation has on Locata measurements. Note that here the Locata 10 Hz measurements and GPS 20 Hz measurements have been downsampled to 1 Hz for clarity.

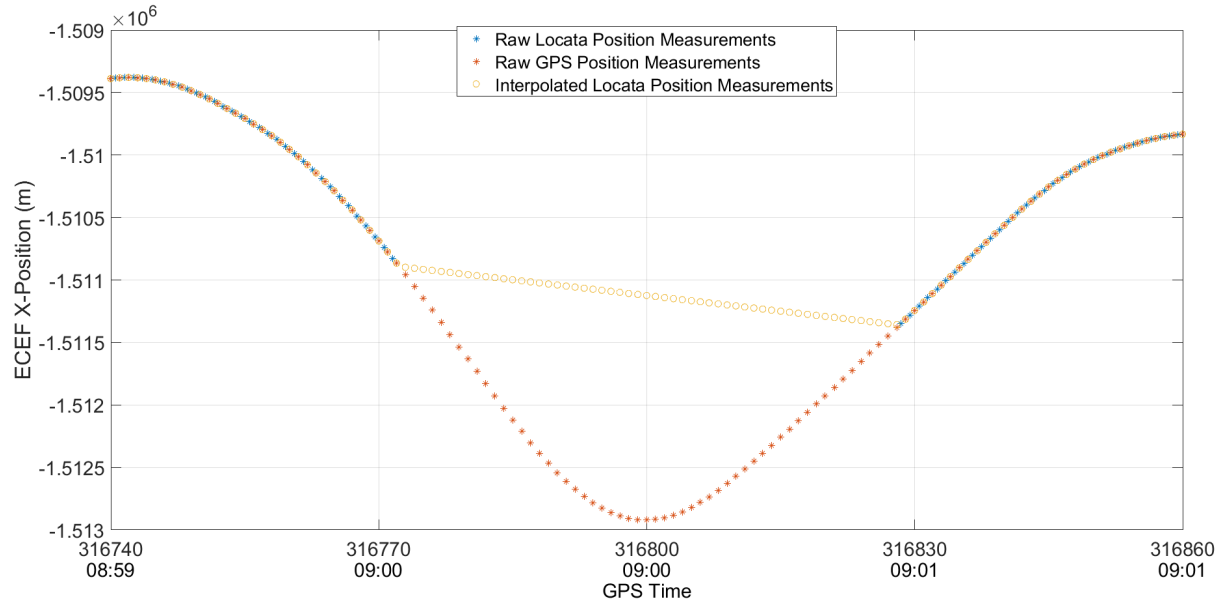


Figure 10: Raw Locata, raw GPS, and interpolated Locata measurements demonstrating the divergence between interpolated Locata values and GPS measurements during real-world Locata outages due to aircraft antenna occlusion.

When the test aircraft turns at the extremes of the test area, the Locata antenna (situated on the bottom of the aircraft) faces completely away from the LocataNet transceivers. All signals to the Locata antenna are blocked by the aircraft body. Locata measurements are lost until the LocataNet transceivers are no longer occluded by the body of the aircraft. When Locata experiences a real-world outage due to antenna occlusion, linearly interpolated values draw a straight line from the last good reading before the outage to the first good reading after it. While scale is difficult to tell in this figure, the first interpolated value differs from GPS by 90 meters. At its peak, the Locata interpolated value differs from the GPS measurement by 1.7 km. Therefore, an intersect of Locata and GPS times is used to align the position measurements. Not only does this remove values that are too far apart in time, but it removes real-world outages caused by loss of Locata signals.

The procedure followed to create timestamped delta-position measurements is outlined below.

1. Load Waypoint DGPS measurements
2. Apply lever arm corrections to Waypoint measurements
3. Convert Novatel DGPS measurements from LLA to ECEF using Waypoint LLA
4. Load Locata measurements
5. Remove degraded Locata measurements (horizontal $\sigma^2 > 1.6$ cm or vertical $\sigma^2 > 11$ cm)
6. Apply lever arm corrections to Locata
7. Convert Locata measurements from LLA to ECEF using Locata LLA
8. Find common times between the Locata and Novatel DGPS measurements within 10^{-8} sec¹
9. For the length of the aligned time vector:
 - (a) Calculate $\Delta p_x = x_{i+1} - x_i$, $\Delta p_y = y_{i+1} - y_i$, and $\Delta p_z = z_{i+1} - z_i$ for Locata Position and GPS Position
 - (b) Calculate $\Delta t = t_{i+1} - t_i$
 - (c) Calculate $v_{\Delta \text{pos}} = \frac{\Delta p}{\Delta t}$
 - (d) Calculate the valid time as $t_i + \frac{t_{i+1} - t_i}{2}$

¹Locata measurement timestamps are frequently not perfectly aligned with GPS seconds, this tolerance includes measurements that are within 10 nanoseconds

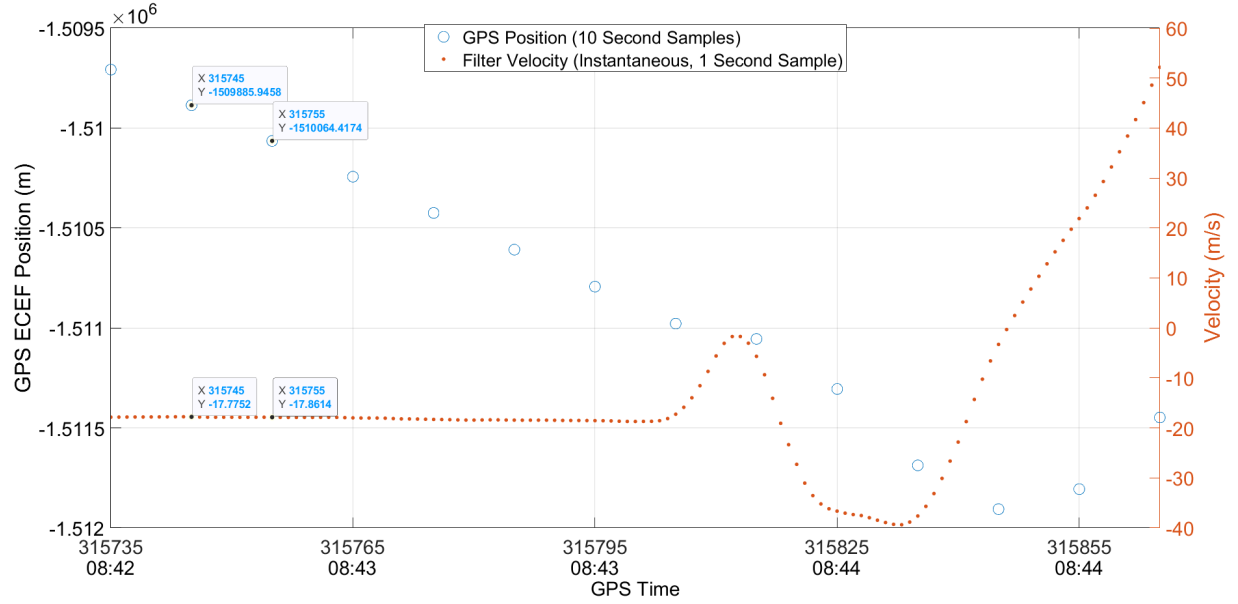


Figure 11: Demonstration of the need for velocity averaging. This plot shows a 0.1 Hz sampling of the GPS ECEF x -position in blue and the Scorpion filter velocity in orange. The aircraft is under little to no acceleration in the x -axis from 08:42 to $\sim$ 08:43.5, varied dynamics from $\sim$ 08:43.5 to 08:44, and somewhat constant acceleration from 08:44 to the end of the interval. The annotated values are in a near-zero acceleration condition.

3.7 Algorithm for Averaging Filter Velocity

Filter velocity estimates must be averaged in order to be compared to delta-position velocity. Under constant acceleration, a linearly interpolated average velocity $\left(\frac{v_2 - v_1}{2}\right)$ is valid in determining the average velocity between v_2 and v_1 (because velocity is a straight line). However, under non-constant acceleration average velocity must be calculated as displacement divided by time. This necessitates integrating all the velocity measurements between v_1 and v_2 in order to calculate change in position. As an example, consider Figs. 11, 12, and 13.

Figs. 11 - 13 show the GPS measured ECEF position and the Scorpion estimated velocity over a portion of flight. The dynamics shown will demonstrate the effectiveness of a linearly-interpolated average velocity under different conditions. Under near zero acceleration, such as that annotated in Fig. 11, results in a constant velocity and a linearly increasing or

decreasing x position. Taking the values shown in Fig. 11, the delta-position velocity and filter average velocity (interpolated) are calculated as:

$$\begin{aligned}
v_{\Delta_{\text{pos}}} &= \frac{p_2 - p_1}{t_2 - t_1} & v_{\text{avg}} &= \frac{v_1 + v_2}{2} \\
v_{\Delta_{\text{pos}}} &= \frac{-1510243.3571 \text{ m} - (-1510064.4174 \text{ m})}{315755 \text{ s} - 315745 \text{ s}} & v_{\text{avg}} &= \frac{-17.8614 \text{ m/s} + (-17.9829 \text{ m/s})}{2} \\
v_{\Delta_{\text{pos}}} &= -17.894 \text{ m/s} & v_{\text{avg}} &= -17.922 \text{ m/s}
\end{aligned}$$

Under near-zero acceleration, the results differ by 2.8 cm/s. If however, the velocity values from 315745 s to 315755 s are integrated and divided by the time difference (10 s), the result is instead within 0.003 cm/s.

Forgoing the long-form calculation for every case, the results of linearly interpolated average velocity, integrated average velocity, and delta-position velocity are summarized in Table 4. The results clearly demonstrate that while the linearly interpolated average velocities have errors of multiple meters per second, the errors of the integrated velocities have errors of less than a centimeter per second in every dynamics case.

Table 4: Comparison of GPS delta-position velocity to averaged filter velocity using different algorithms and under different dynamics as shown in Figs. 11-13.

Case	Delta-Position	Linear-Interp	Integrated
Near-Constant, Near-Zero Acceleration	-17.894 m/s	-17.922 m/s	-17.894 m/s
Near-Constant, Non-Zero Acceleration	35.795 m/s	37.020 m/s	35.796 m/s
Non-Constant Acceleration	-7.636 m/s	-11.46 m/s	-7.633 m/s

In addition to the velocity estimates in each axis, the EKF produces a variance output for each state at the same rate it outputs position and velocity. This variance must be averaged over each delta-position time interval as well in order to appropriately capture predicted filter performance during comparisons. The methodology for averaging both filter velocity and filter variance is described below:

1. Determine the aligned time vector via step 8 in section 3.6
2. For the length of the aligned time vector minus 1 ($i = 1$ to $n - 1$)
 - (a) Find the index j of the filter time that equals the aligned time at i
 - (b) Find the index k of the filter time that equals the aligned time at $i + 1$
 - (c) Trapezoidally integrate the filter velocity v_x , v_y , and v_z from j to k using filter time as the interval weight
 - (d) Take the average of the filter variances $\sigma_{v_x}^2$, $\sigma_{v_y}^2$, and $\sigma_{v_z}^2$ dividing by $k - j$
 - (e) Calculate the valid time as $t_i + \frac{t_{i+1} - t_i}{2}$

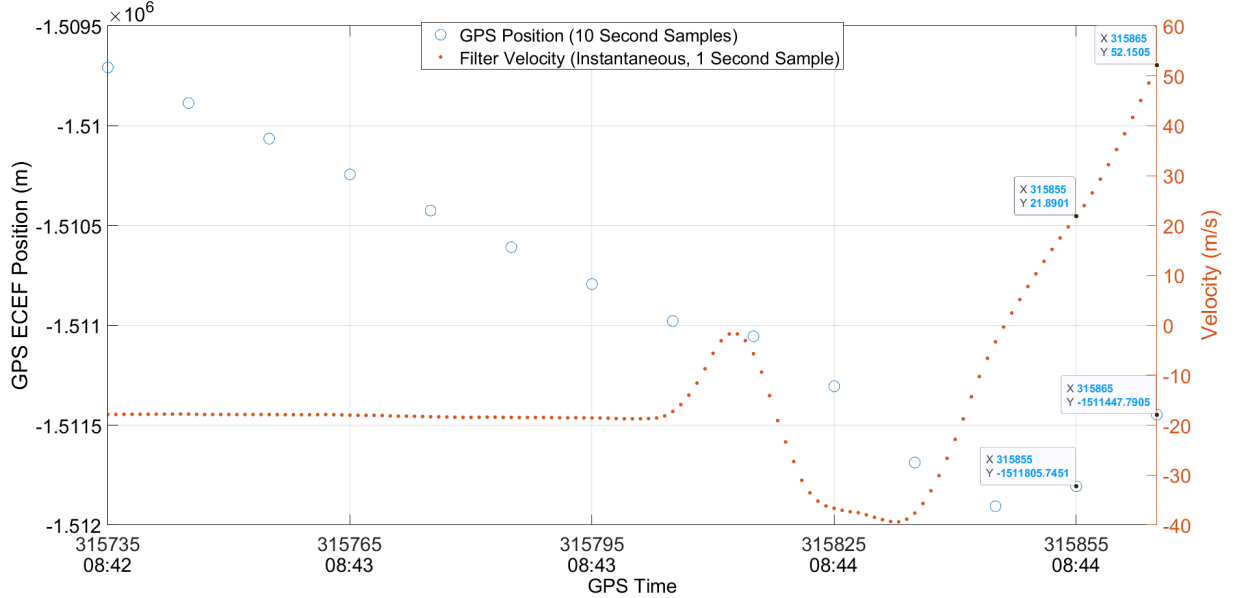


Figure 12: Demonstration of the need for velocity averaging. This plot shows a 0.1 Hz sampling of the GPS ECEF x -position in blue and the Scorpion filter velocity in orange. The annotated values are in a near-constant acceleration condition.

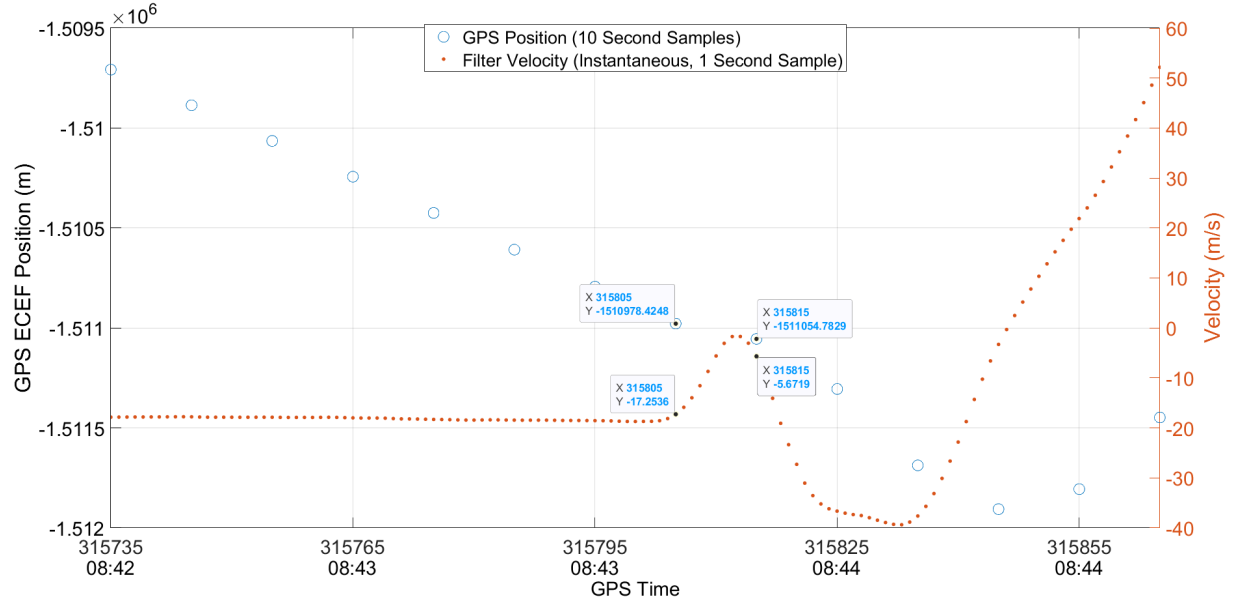


Figure 13: Demonstration of the need for velocity averaging. This plot shows a 0.1 Hz sampling of the GPS ECEF x -position in blue and the Scorpion filter velocity in orange. The annotated values are in a non-constant acceleration condition

3.8 Algorithm for Separating Outages Based on Flight Dynamics

The Pseudo-Monte Carlo procedure is implemented in Scorpion, but the results must be binned in post-filtering. The four conditions considered for Test Cases 4 and 5 are: straight and level, level turning, climbing and diving, and other dynamics. The definitions for each case are outlined in Table 5. Calculation of these values is performed by using the body frame to navigation frame DCM ($\mathbf{C}_b^n$) calculated by Scorpion. The Scorpion solution propagates at 10 Hz, meaning that this DCM is calculated every 0.1 s. Taking a DCM at time t_0 , ($\mathbf{C}_{b_0}^n$) and multiplying by the inverse of the DCM at the next time step, t_1 ($(\mathbf{C}_{b_1}^n)^T$) creates a DCM that rotates from the body frame at time t_0 to the body frame at time t_1 ($\mathbf{C}_{b_0}^{b_1}$). Using the expression from [25] to convert from DCM to Roll-Pitch-Yaw values, the amount of rotation in the body frame can then be calculated over the period of time t_0 to t_1 . Knowing the amount of rotation and the time interval then allows for tuning the script to appropriately capture environments that cover every phase of flight, Straight and Level,

Level Turns, Constant Heading Climbs and Dives, and Other High-Dynamics Environments.

Table 5: Binning limits for flight dynamics conditions in Pseudo-Monte Carlo simulations. Note that the Climbs and Dives environment is defined as constant-heading climbs and dives, where a low horizontal angular rate ensures the heading is constant, and a climb or dive is ensured by either a high angular rate (> 0.05 rad/s) in the vertical direction or a change in altitude above the threshold (15 m)

State of Flight	Dynamics Condition
Straight and Level	Horizontal Angular Rate (RMS) < 0.02 rad/s AND Pitch Rate (RMS) < 0.05 rad/s AND Altitude Change < 15 m
Level Turning	Horizontal Angular Rate (RMS) > 0.02 rad/s AND Pitch Rate (RMS) < 0.05 rad/s AND Altitude Change < 15 m
Climbs and Dives	Horizontal Angular Rate (RMS) < 0.02 rad/s AND Pitch Rate (RMS) > 0.05 rad/s OR Altitude Change > 15 m
Other Dynamics	Horizontal Angular Rate (RMS) > 0.02 rad/s OR Pitch Rate (RMS) > 0.05 rad/s OR Altitude Change > 15 m

3.9 Summary

Having identified the tasks necessary to characterize the velocity accuracy of UHARS and having deconstructed those tasks into a set of interrelated technological efforts, the methods, data, and algorithms required have been established. The theoretical error of a delta-position velocity will form the basis for studies involving delta-position velocities. Direct comparison algorithm will allow for the characterization of Locata position accuracy. Delta-position velocity and average velocity algorithms will allow for both demonstrating the utility of the

delta-position velocity measurement and for characterizing the nominal accuracy of UHARS velocity estimates. Finally, conducting pseudo-Monte Carlo simulations with the ability to bin each outage based on its flight dynamics will allow for the assessment of filter performance and INS tuning across all scenarios.

IV. Results and Analysis

4.1 Overview

Recall that the ultimate goal of this thesis is to characterize the velocity of UHARS in order to increase reference system performance. This is to increase test precision and accuracy when evaluating new navigation technologies. At the beginning of Chapter 3, a flow chart of technical efforts was outlined. That flowchart is re-presented here in Fig. 14. Note that the results sections for each step have been annotated next to their box. This chapter will walk through this chart from top to bottom, presenting results and analysis systematically. The culminating result will be a characterization of UHARS velocity estimates, which will be summarized in section 4.8.

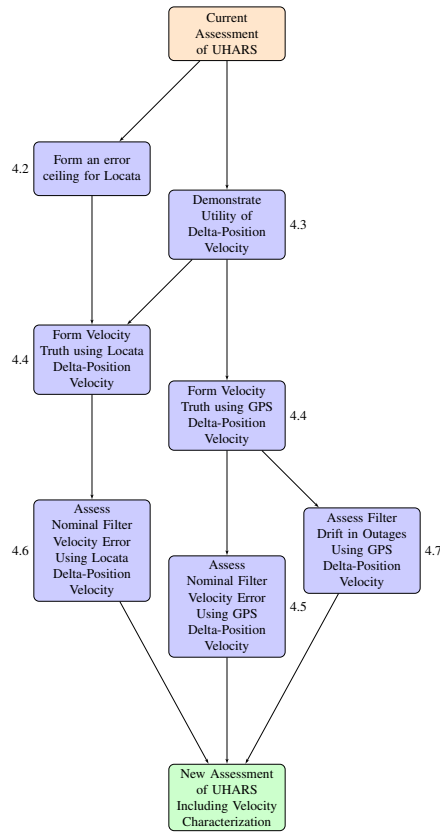


Figure 14: Flowchart of technical efforts completed in order to characterize UHARS velocity accuracy. Note that the relevant results section has been annotated next to each task.

4.2 GPS vs Locata Direct Comparison Analysis

GPS and Locata position differed by close to the anticipated amount. The most error can be observed in the local frame down direction, which has been observed in the deployed reference system, and the root cause has not been determined. Charts of position error can be found in Fig. 16 and statistics are shown in Table 6. GPS and Locata differ by much less than the amount found in the initial tech demo of the NGBPS. With variances on the order of one or two centimeters in the horizontal direction, Locata can be used as a good position source for determining delta-position velocity.

Table 6: Statistics of Locata and Waypoint GPS comparisons. By assuming the complete measurement error is sourced from either one or the other, the maximum error for each measurement source is established.

Comparison	3D-RMS	Dir.	RMS	Mean (μ)	S.D. (σ)
Position	12.5 cm	North	2.80cm	2.27 cm	1.63 cm
		East	1.51 cm	2.10 cm	1.49 cm
		Down	12.1 cm	-11.6 cm	3.51 cm
Velocity	14.9 cm/s	North	4.18 cm/s	-0.17 cm/s	4.18 cm/s
		East	5.86 cm/s	-1.81 cm/s	5.58 cm/s
		Down	13.1 cm/s	-0.99 cm/s	13.1 cm/s

One finding when comparing Locata and Waypoint is that the lever arms applied to one of the two sources was incorrect. Note that lever arm errors in velocity measurements differ from lever arm errors in position measurements. Position error lever arm discrepancies manifest as offsets when in straight and level flight. These offsets change based on the relation of the body frame to the NED or ECEF frame. In the case of velocity measurements, however, straight and level flight express the same velocities in the NED frame regardless of sensor position in the body frame. Therefore, the only way to detect lever arm discrepancies in the velocity measurement is to observe high-value errors under dynamics. Such errors were observed after high variance measurements were masked, meaning lever arm errors were present.

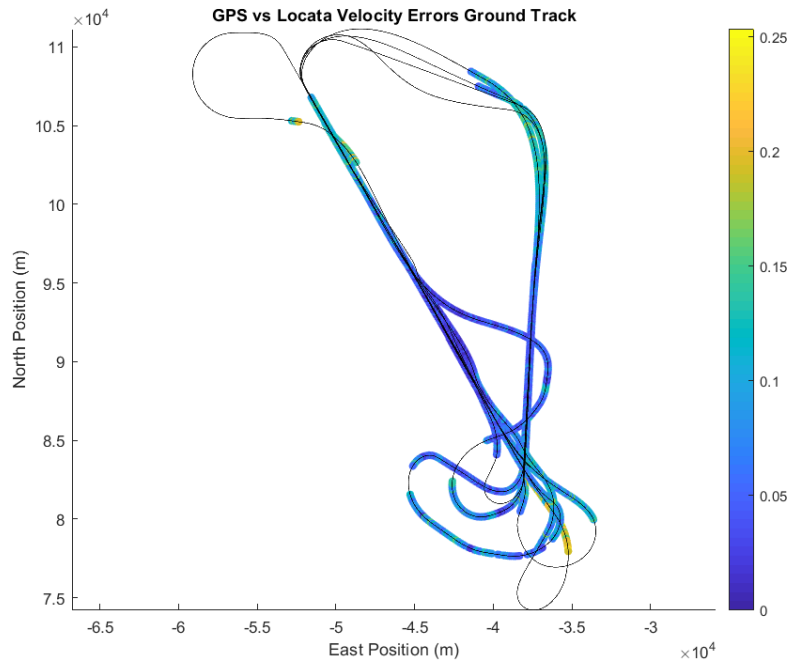
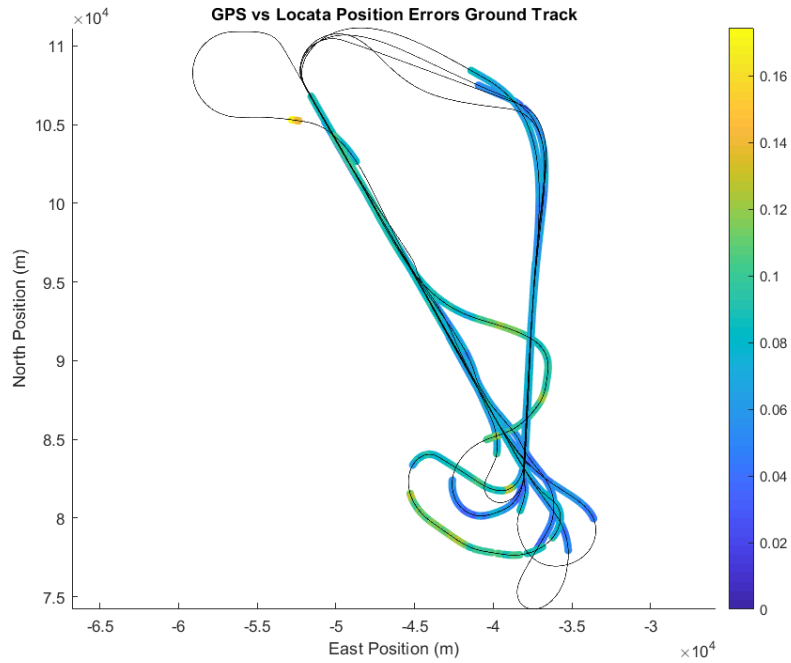


Figure 15: GPS and Locata differences overlaid on test flight ground trace. Differences are 3D-RMS errors. Areas of just a black line are areas where GPS or Locata are not available. Position errors are on the top, velocity errors on the bottom

Two sets of lever arms were given with the source data, switching from one to the other resolved the observed lever arm discrepancies. A lever arm sensitivity analysis was also performed, where the lever arm was varied by 1 cm in each axis and a Scorpion filter was run. The 3D-RMS of the filter position error (compared to GPS) was calculated for all seven runs, and the minimum was selected as the new lever arm to study around. The perturbation value was halved, and the analysis re-completed. This process was repeated until the perturbation value reached 0.000625 cm. While this analysis did minimize the 3D-RMS of the position error, this increase was artificial due to the unknown Locata phenomenon that causes high errors in the down direction.

In addition, Locata measurements are not available in turns at the extreme of the Locata net. Because the Locata antenna is located on the bottom of the aircraft, it points away from the range during a turn. This results in outages where no position measurements are available from Locata. This phenomenon is visible in Fig. 15, where outages are observed at the north and south extremes in turns, but not the middle of the range in turns. In the middle of the range Locata error still increases, but measurements remain available.

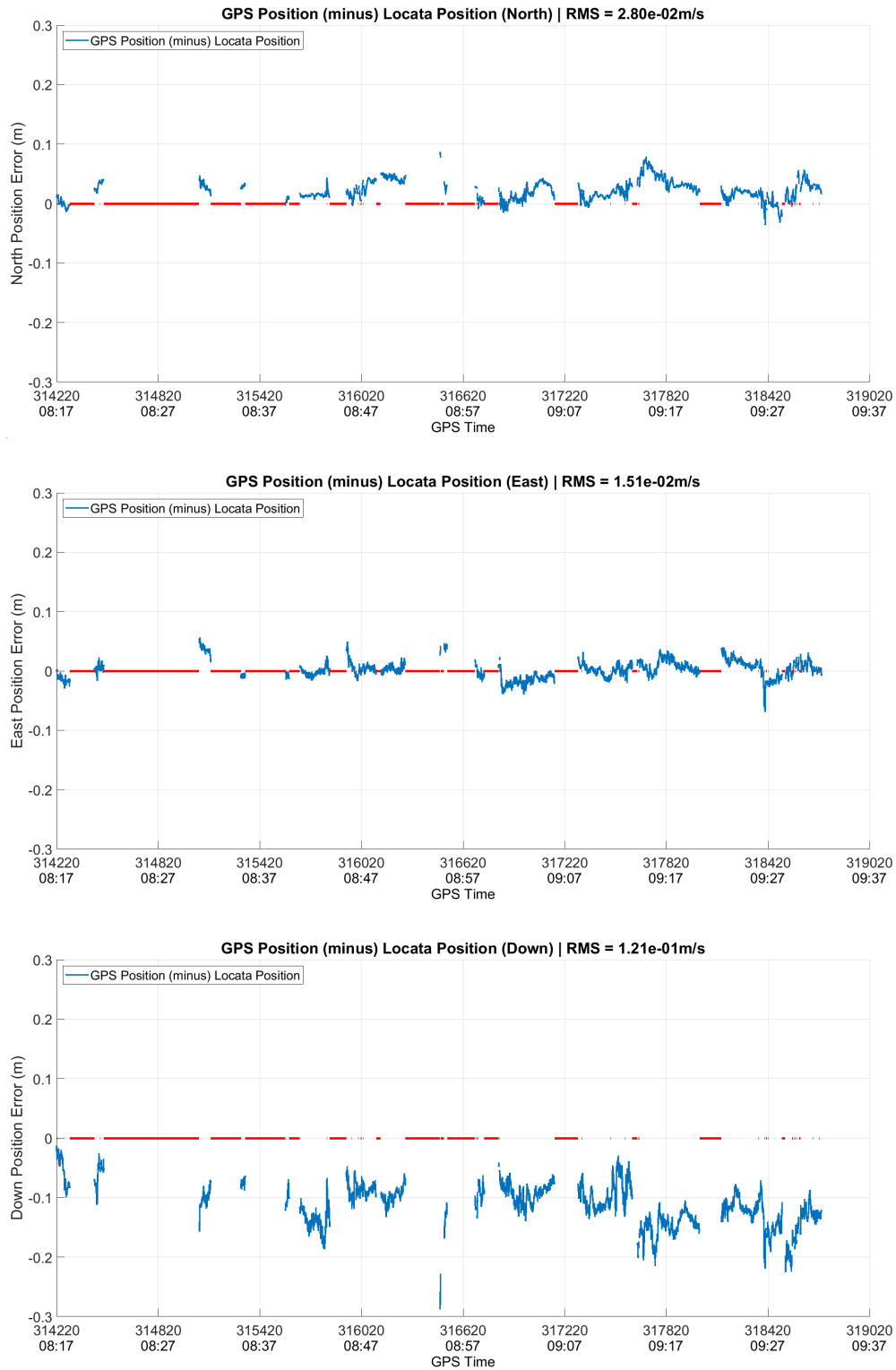


Figure 16: Comparison of GPS position measurements and Locata position measurements. Note that the Locata measurements are subtracted from the GPS measurements and areas covered by red lines are periods of unavailability from either GPS or Locata.

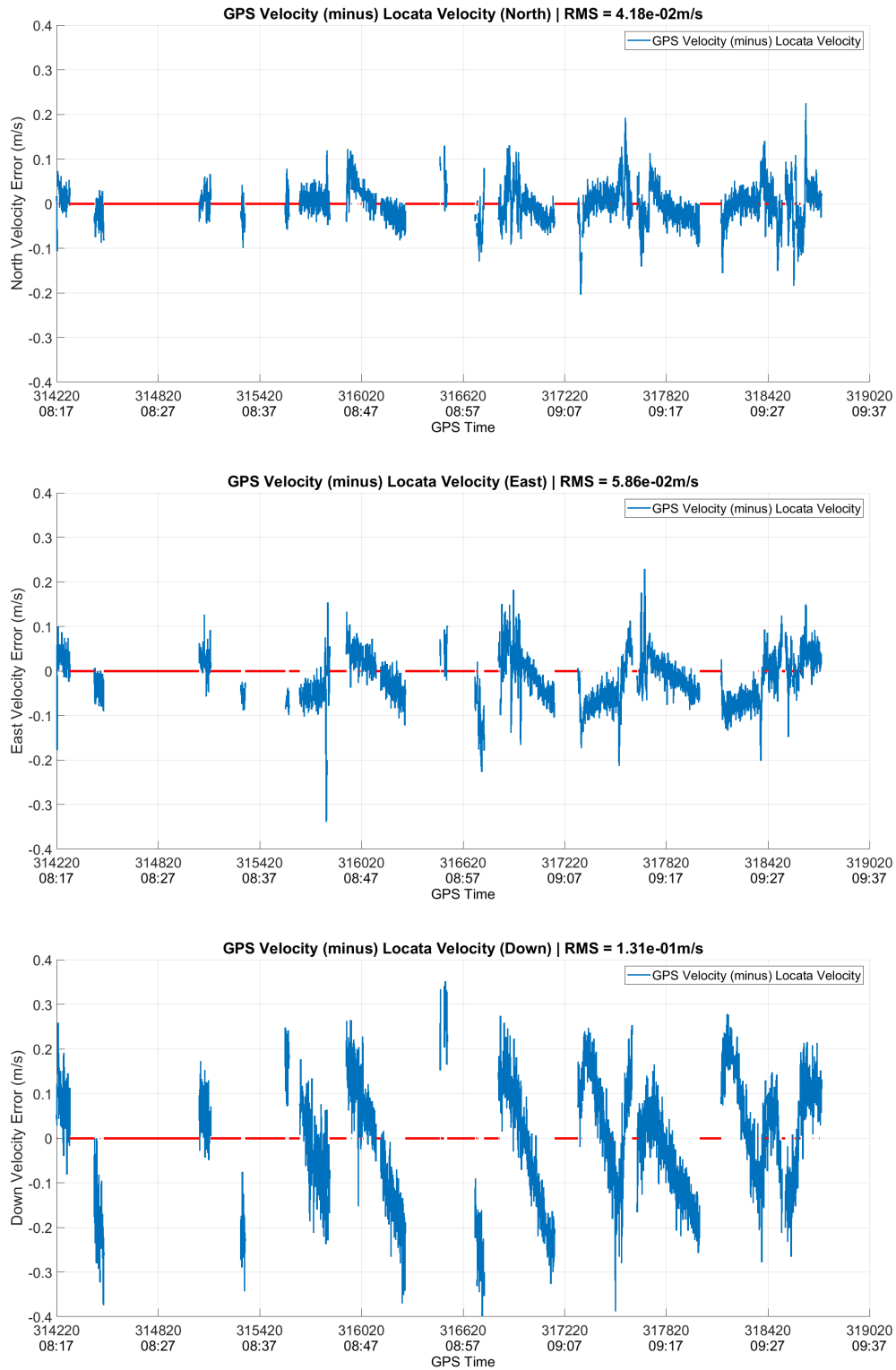


Figure 17: Comparison of GPS velocity measurements and Locata velocity measurements. Note that the Locata measurements are subtracted from the GPS measurements, and areas covered by red lines are periods of unavailability from either GPS or Locata.

Comparing the velocity measurements that both Waypoint and Locata produce reveals a time correlated error in the Locata down velocity measurement. Throughout flight, Locata measurements become inaccurate and then unavailable in high dynamics. Turns typically result in a degradation then loss of Locata measurements. However, when the Locata measurements become available, they start lower than the Waypoint reported velocity and then decrease throughout the available period until the signal is lost in the next turn. In order to verify this error, each period of Locata availability was studied separately.

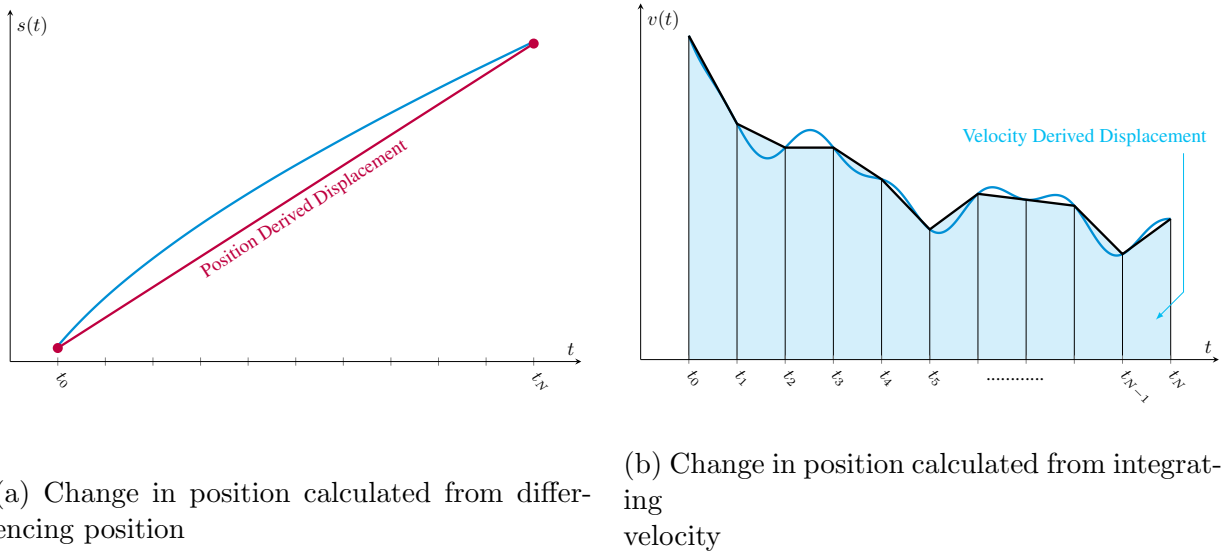


Figure 18: Comparison of the two techniques used to derive change in position over each one second interval in order to determine error in Locata velocity measurements

For each period of Locata availability, the velocity measurements from Locata were compared to the position measurements. First, the Locata position was converted from the LLA frame to the ECEF frame and the Locata velocity measurements were converted from the NED frame to the ECEF frame. Next, the whole-second Locata position measurements were differenced in order to calculate the change in position over a one second interval. Next, the Locata reported velocity was evaluated using trapezoidal integration over the same one second intervals. The position-derived change in position and the velocity-derived change in position were then plotted and compared in each axis. Therefore, over the course of signal

availability, the reported position changes were directly compared to the calculated change in position from the velocity measurements. The velocity accuracy results are shown in Fig. 19), and comparison statistics are found in Table 6.

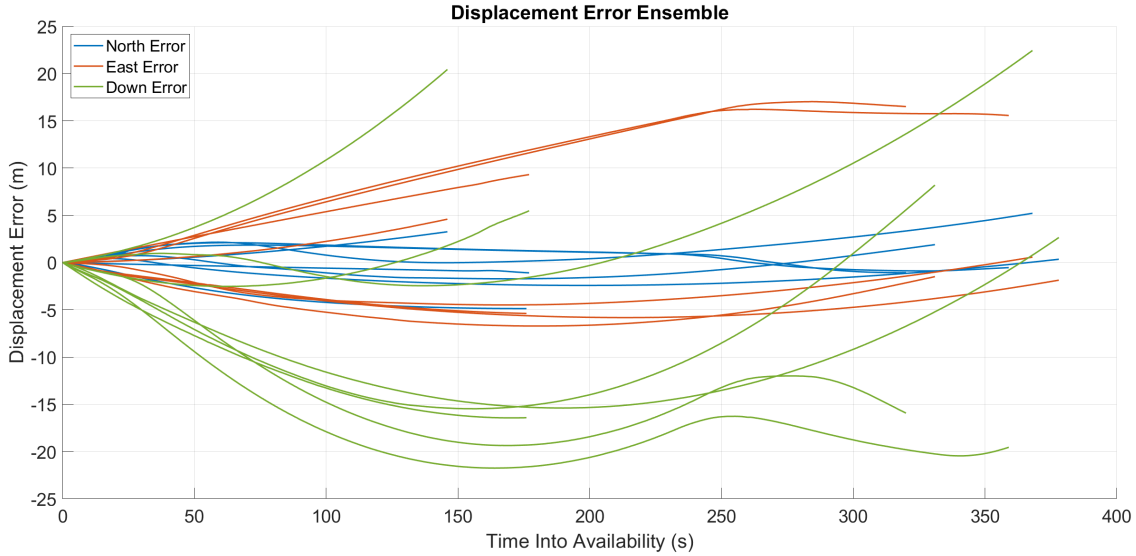


Figure 19: Ensemble of Locata displacement error. Position based delta-position is subtracted from velocity based change in position ($\varepsilon = d_{\text{vel}} - d_{\text{pos}}$)

4.3 Flight-Configuration Filter Compared to GPS Delta-Position Velocity

Having set the error ceiling for Locata, the next step is to demonstrate the utility of the delta-position measurement. This is done by comparing the nominal flight-configuration filter to the GPS delta-position and delta-position velocity. This characterizes the filter position and velocity output by using the independent GPS delta-position Velocity as a reference. While performing this comparison, it was noted that previous versions of this filter purposefully downsampled sensor measurements to 1 Hz, most likely in an attempt to reduce white noise from GPS and Locata. Comparisons used both this downsampled flight-configuration filter (statistics presented in Table 7) and a flight-configuration filter incorporating sensor measurements at their native rate (statistics presented in Table 8)

This study revealed, most prominently, the increase in accuracy when sensors are incorporated at their native sample rate. However, incorporating both GPS and Locata at their native rates (20 Hz and 10 Hz, respectively) increased velocity accuracy by an order of magnitude or better. The 3D-RMS of NED position decreased from 11.6 cm to 1.59 cm and the 3D-RMS of the worst case velocity (1s filter average compared to 1s delta-position Velocity) decreased from 12.1 cm m/s to 0.876 cm m/s. The significant decrease in 3D-RMS for each measurement, coupled with a corresponding decrease in standard deviation and RMS in each axis (RMS and standard deviations continue to agree within approx 5%) demonstrates the value of running the filter with native-rate sensor updates in the future.

Examining the native-rate flight configuration filter further, the results seem promising for future delta-position and delta-position Velocity utilization. 3D-RMS position error is at the centimeter level, an improvement over previous accuracy. In addition, the horizontal

Table 7: Statistics comparing flight-configuration filter and GPS derived delta-position Velocity measurements. These filter results were obtained using GPS and Locata measurements at a downsampled rate of 1 Hz. The result next to each velocity is the 3D-RMS for the associated averaging period.

	RMS	Mean (μ)	Std. Dev. (σ)	N
NED Position (cm)	11.6 cm			
North	1.50 cm	-0.15 cm	1.49 cm	5842
East	1.61 cm	-0.04 cm	1.61 cm	5842
Down	11.4 cm	0.26 cm	11.4 cm	5842
NED Velocity - 1s (cm/s)	12.1 cm/s			
North	2.17 cm/s	-0.01 cm/s	2.17 cm/s	4795
East	1.43 cm/s	-0.01 cm/s	1.43 cm/s	4795
Down	11.8 cm/s	-0.16 cm/s	11.8 cm/s	4795
NED Velocity - 10s (cm/s)	4.32 cm/s			
North	0.30 cm/s	$ \mu < 0.01$ cm	0.30 cm/s	438
East	0.28 cm/s	-0.02 cm/s	0.28 cm/s	438
Down	4.30 cm/s	-0.17 cm/s	4.30 cm/s	438
NED Velocity - 100s (cm/s)	1.75 cm/s			
North	0.07 cm/s	$ \mu < 0.01$ cm/s	0.07 cm/s	18
East	0.10 cm/s	-0.04 cm/s	0.09 cm/s	18
Down	1.75 cm/s	-0.43 cm/s	1.74 cm/s	18

position accuracy is at the millimeter level. Vertical accuracy has improved as well; while not on-par with lateral accuracy, it was decreased from tens of centimeters to almost single centimeter accuracy.

The delta-position measurement accuracy is more promising still. With 1 second averaging periods, the delta-position Velocity and filter velocity 3D-RMS is under 1 cm. Once again, lateral accuracy is better than vertical accuracy, about half the error value. As averaging period lengthens, errors continue to go down, with the best case being the 100 second averaging period, boasting millimeter-level 3D-RMS error between filter velocity and delta-position Velocity. However, the number of uninterrupted GPS intervals overlapping with full signal availability is low. The filter only runs for a 97.35 min duration (5841 seconds) but only 18 periods exist where GPS is available within 10 centimeter reported variance. This limits the operational value of a 100 second delta-position Velocity measurement update in

Table 8: Table of statistics comparing flight-configuration filter and GPS derived delta-position and delta-position Velocity measurements. These filter results were obtained using GPS and Locata measurements at their native rates (20 Hz and 10 Hz respectively). The result next to each velocity is the 3D-RMS for the associated averaging period.

	RMS	Mean (μ)	Std. Dev. (σ)	N
NED Position (cm)	1.59 cm			
North	0.49 cm	-0.07 cm	0.49 cm	5842
East	0.55 cm	$ \mu < 0.01$ cm	0.55 cm	5842
Down	1.41 cm	0.17 cm	1.40 cm	5842
NED Velocity - 1s (cm/s)	0.88 cm/s			
North	0.29 cm/s	$ \mu < 0.01$ cm/s	0.29 cm/s	4795
East	0.45 cm/s	-0.02 cm/s	0.45 cm/s	4795
Down	0.69 cm/s	0.02 cm/s	0.69 cm/s	4795
NED Velocity - 10s (cm/s)	0.32 cm/s			
North	0.06 cm/s	$ \mu < 0.01$ cm/s	0.06 cm/s	438
East	0.08 cm/s	-0.02 cm/s	0.08 cm/s	438
Down	0.31 cm/s	0.02 cm/s	0.31 cm/s	438
NED Velocity - 100s (cm/s)	0.11 cm/s			
North	0.02 cm/s	0.02 cm/s	0.02 cm/s	18
East	0.02 cm/s	-0.02 cm/s	0.01 cm/s	18
Down	0.11 cm/s	0.03 cm/s	0.11 cm/s	18

the EKF. The 10 second averaging period appears to be a good compromise between the other two averaging possibilities. Accurate to the millimeter level or better, and available every 10 seconds. Graphical representations of the filter velocity vs delta-position Velocities of various averaging periods are shown in Figs. 20, 21, and 22 .

Examining the 1 second averaging period (Fig. 20), one item of note is the fact that accuracy is low when Locata position solutions become available and increases, until Locata becomes unavailable once again. This likely stems from the accuracy of Locata position measurements, which will degrade in high dynamic situations before Locata is lost entirely.

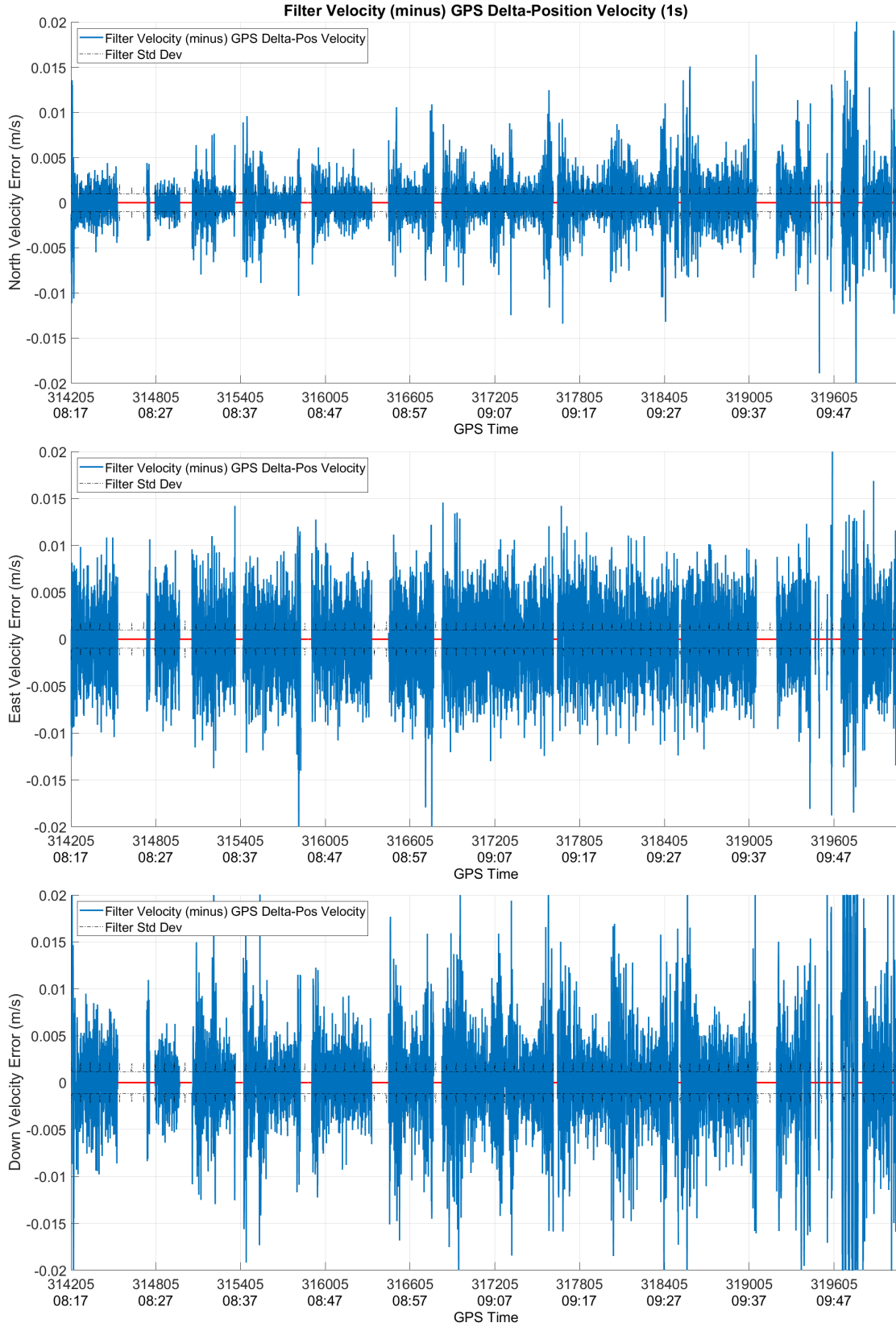


Figure 20: Comparison of a flight configuration filter velocity (incorporating GPS, Locata, barometer, and INS) and delta-position Velocity. Sensor measurements are incorporated at their native rates.

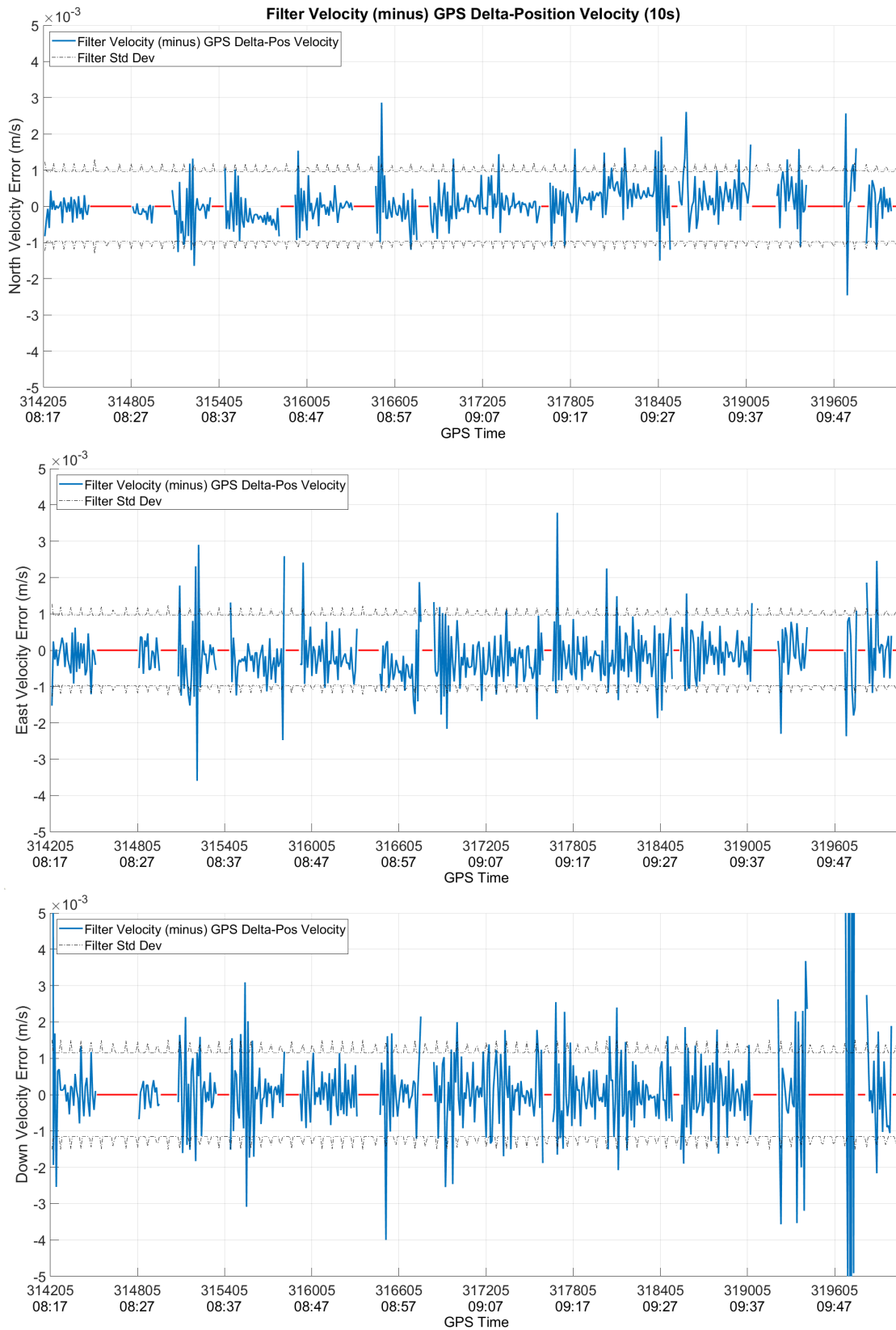


Figure 21: Comparison of a flight configuration filter velocity (incorporating GPS, Locata, barometer, and INS) and delta-position Velocity. Sensor measurements are incorporated at their native rates.

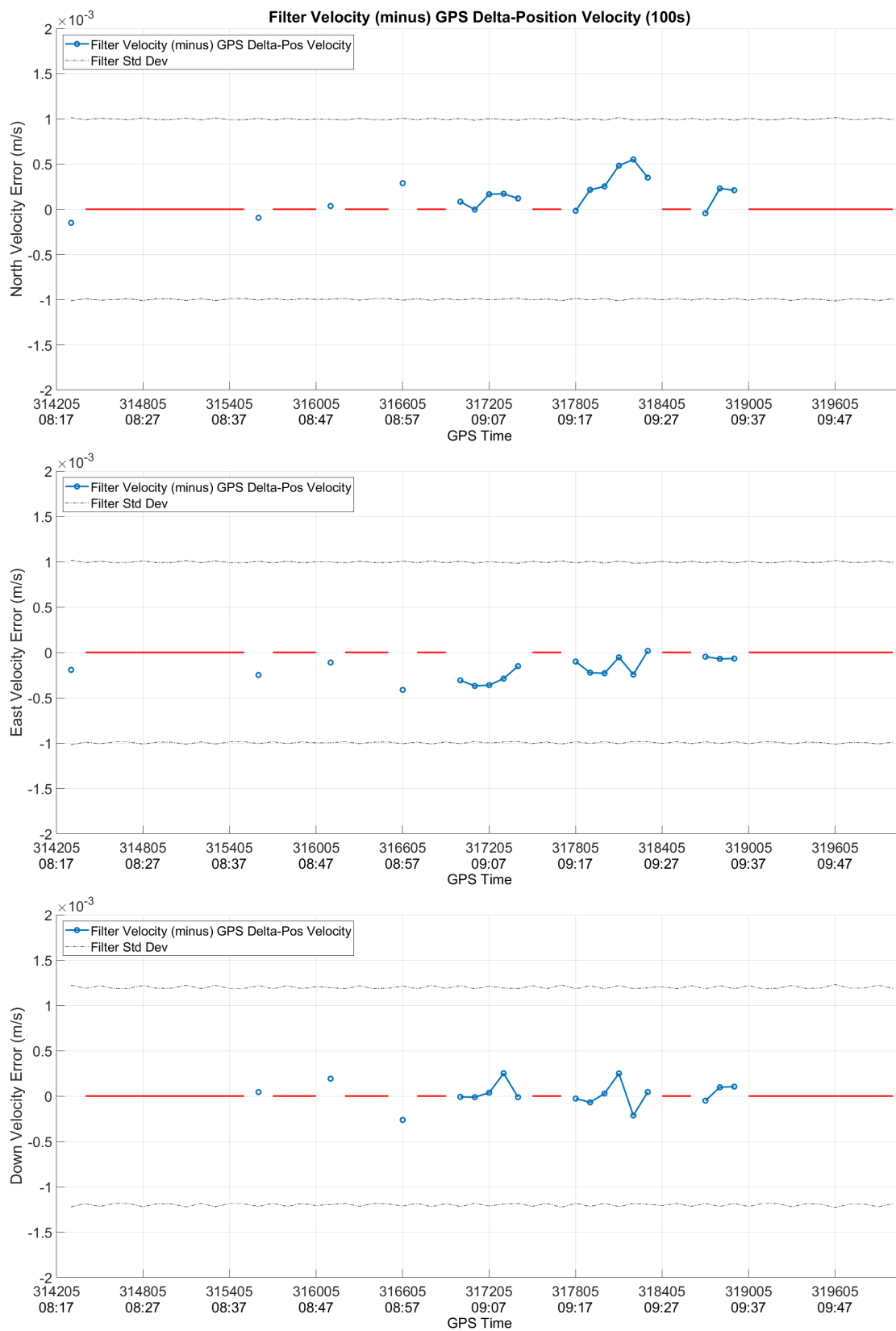


Figure 22: Comparison of a flight configuration filter velocity (incorporating GPS, Locata, barometer, and INS) and delta-position Velocity. Sensor measurements are incorporated at their native rates.

4.4 Comparison of GPS and Locata Delta-Position and Delta-Position Velocity

Having established an error ceiling for Locata and demonstrated the utility of the delta-position measurement, delta-position velocity truths may be established from Locata and GPS. In order to understand delta-position phenomenology, both delta-positions and delta-position velocities were compared. Delta-position velocities were only calculated when there was sufficient availability of the underlying measurements to cover the entire averaging period. These results are shown in Figs. 23, 24, 25. Final accuracy results used to characterize maximum error in DGPS or Locata measurements are summarized in Table 9. This study proved the most promising. The errors between position and velocity measurements from GPS and Locata are not zero mean and have standard deviations centimeters wide. However, the delta-position measurements are consistently close to zero mean, at worst being within 1 or 2 millimeters.

This accuracy validates the theory that time-correlated errors in the delta-position measurement are essentially nullified by the subtraction of the two position measurements when they are close together in time. It lends credence to the claim that the error of a delta-position measurement is dominated by the noise associated with the raw position measurements, and that the variance of the time-correlated errors is stationary enough to effectively disappear from a short enough measurement interval. In addition, the time correlated errors in the position measurements are appropriately modeled and can not be reduced any more. In this case, over a longer averaging period, the time correlated errors are also divided by the change in time, that is a hypothetical un-modeled time correlated error that contributes decimeter errors to the delta-position measurement between two epochs 100s apart will be divided by 100s and therefore “diluted” to a millimeter-per-second level error. The result is that a delta-position Velocity with averaging periods greater than one second will experience error attenuation due to the change in time between the two position measurements. With the delta-position Velocity error level established at several millimeters-per-second in the worst case (1 second averaging), we can confidently compare filter velocity results to delta-position

velocity results to within several millimeters.

Table 9: Statistics of Locata and Waypoint GPS delta-position and delta-position velocity measurements.

Comparison	3D-RMS	Dir.	RMS	Mean (μ)	S.D. (σ)
Delta-Position : 1s	1.12 cm	North	0.36 cm	$ \mu < 0.01$ cm	0.36 cm
		East	0.55 cm	$ \mu < 0.01$ cm	0.55 cm
		Down	0.90 cm	$ \mu < 0.01$ cm	0.90 cm
Delta-Position : 10s	2.26 cm	North	8.55 cm	-0.0255 cm	8.57 cm
		East	9.33 cm	-0.0802 cm	9.31 cm
		Down	1.87 cm	$ \mu < 0.01$ cm	1.88 cm
Delta-Position : 100s	3.11 cm	North	1.22 cm	0.290 cm	1.25 cm
		East	1.34 cm	-0.185 cm	1.41 cm
		Down	2.53 cm	-0.162 cm	2.68 cm
Delta-Position Velocity : 1s	1.12 cm/s	North	0.36 cm/s	$ \mu < 0.01$ cm/s	3.59 cm/s
		East	0.55 cm/s	$ \mu < 0.01$ cm/s	5.52 cm/s
		Down	0.90 cm/s	$ \mu < 0.01$ cm/s	9.04 cm/s
Delta-Position Velocity : 10s	0.23 cm/s	North	0.09 cm/s	$ \mu < 0.01$ cm/s	0.09 cm/s
		East	0.09 cm/s	$ \mu < 0.01$ cm/s	0.09 cm/s
		Down	0.02 cm/s	$ \mu < 0.01$ cm/s	0.19 cm/s
Delta-Position Velocity : 100s	0.03 cm/s	North	0.09 cm/s	$ \mu < 0.01$ cm/s	0.09 cm/s
		East	0.01 cm/s	$ \mu < 0.01$ cm/s	0.01 cm/s
		Down	0.03 cm/s	$ \mu < 0.01$ cm/s	0.03 cm/s



Figure 23: Comparison of 1 second averaged GPS delta-position velocity measurements and 1 second averaged Locata delta-position velocity measurements. Note that the GPS measurements are subtracted from the Locata measurements, and areas covered by red lines are periods of unavailability from either GPS or Locata.



Figure 24: Comparison of 10 second averaged GPS delta-position velocity measurements and 10 second averaged Locata delta-position velocity measurements. Note that the vertical scale of these plots are $\frac{1}{5}$ the scale of those in Fig. 23

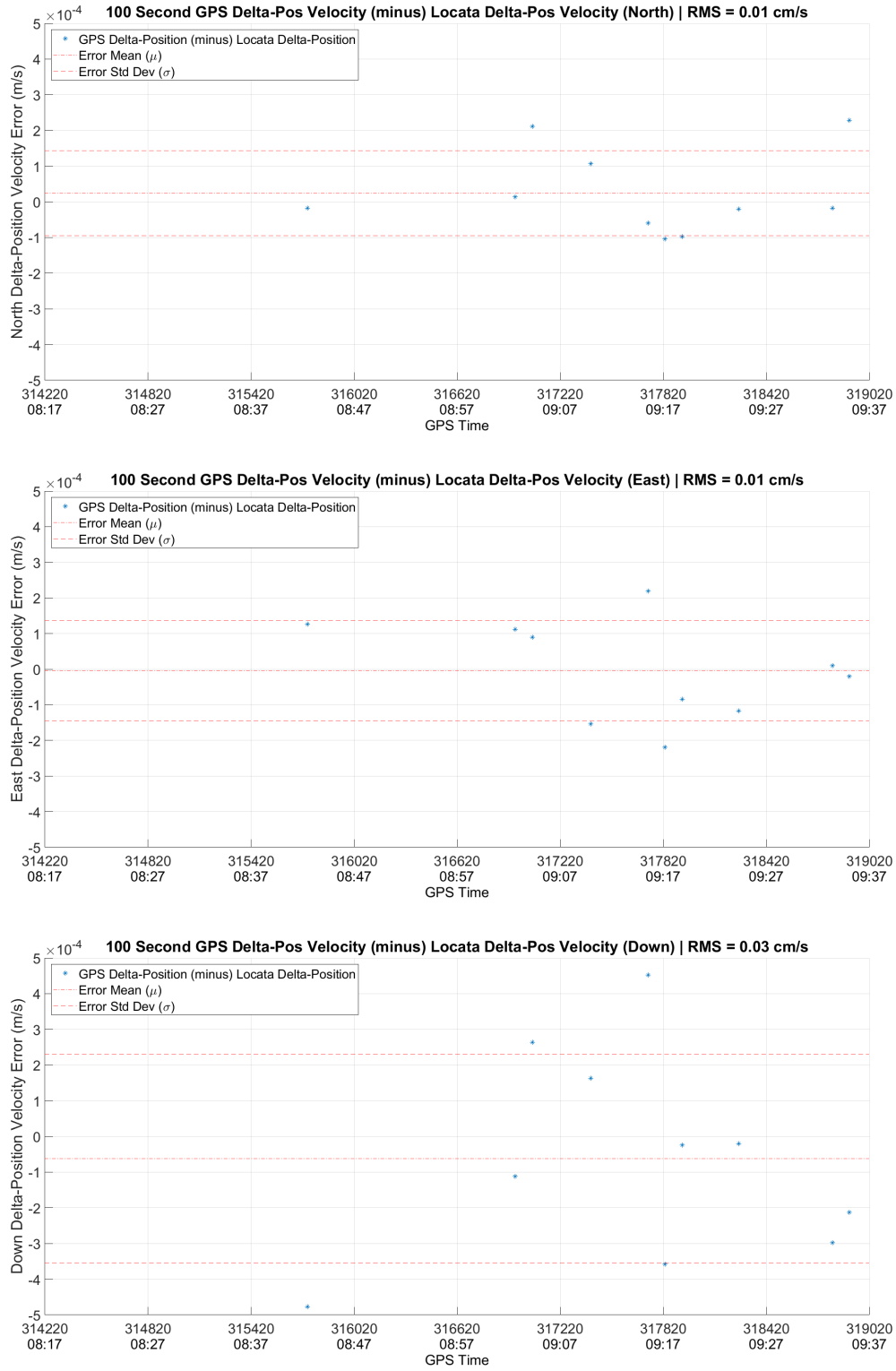


Figure 25: Comparison of 100 second averaged GPS delta-position velocity measurements and 100 second averaged Locata delta-position velocity measurements. Note that the vertical scale of these plots are $\frac{1}{100}$ the scale of those found in Fig. 23

4.5 Locata-Aided Filter Compared to GPS Delta-Position Velocity

Having formulated independent velocity truths, the error of UHARS velocity estimates in a nominal condition may be assessed. First, a Locata aided filter compared to a GPS delta-position velocity. These filter results were generated using Locata measurements, the UHARS INS and barometer measurements. Because GPS measurements are disregarded, the resulting filter output is completely independent from GPS. Statistics are presented in Table 10

Table 10: Table of statistics comparing Locata aided filter results and GPS position measurements as well as GPS delta-position velocity measurements. The result next to each velocity is the 3D-RMS for the associated averaging period.

	RMS	Mean (μ)	Std. Dev. (σ)	N
NED Position (cm)	153 cm			
North	88.4 cm	-7.34 cm	88.1 cm	5842
East	89.0 cm	8.40 cm	88.6 cm	5842
Down	86.9 cm	24.6 cm	83.4 cm	5842
NED Velocity - 1s (cm/s)	1.19 cm/s			
North	0.46 cm/s	$ \mu < 0.01$ cm/s	0.47 cm/s	4795
East	0.63 cm/s	$ \mu < 0.01$ cm/s	0.63 cm/s	4795
Down	0.89 cm/s	0.04 cm/s	0.89 cm/s	4795
NED Velocity - 10s (cm/s)	0.55 cm/s			
North	0.21 cm/s	0.01 cm/s	0.21 cm/s	438
East	0.21 cm/s	$ \mu < 0.01$ cm/s	0.21 cm/s	438
Down	0.45 cm/s	0.04 cm/s	0.46 cm/s	438
NED Velocity - 100s (cm/s)	0.06 cm/s			
North	0.04 cm/s	0.01 cm/s	0.04 cm/s	18
East	0.03 cm/s	$ \mu < 0.01$ cm/s	0.03 cm/s	18
Down	0.03 cm/s	$ \mu < 0.01$ cm/s	0.03 cm/s	18

The difference in position comparing a Locata aided filter to GPS is higher than the difference in position when comparing the underlying Locata and GPS signals. What is odd about these differences is that each axis' standard deviation is on the order of tens of centimeters, whereas in the raw signals comparison the error in the North and East axes were centimeter level with the Down error being tens of centimeters. This is likely due to

the known Locata offset propagating through the filter. However, delta-position velocity performance is promising. The error in each axis has a variance below 1 cm/s, with the lowest variance being only a few mm/s at the 100s averaging period. This low error at a longer averaging period demonstrates a very accurate estimate of velocity. Note however, that a 100 second velocity error of 5 mm/s is truly a displacement error of 5 cm over 100 seconds when using delta-position velocity. The other item of note is that the velocity measurements incorporated from Locata have the time-correlated error pointed out in section 4.2, which could add to filter error. All told however, the velocity accuracy of the filter can be confidently placed below 1 cm/s at 10 and 100 second averaging periods.

4.6 GPS-Aided Filter Compared to Locata Delta-Position Velocity

In the same manner as above, nominal filter error can be assessed by comparing a GPS a Locata generated delta-position measurement. The results should be close to the results in the previous section. Comparing a GPS aided filter to the Locata position and Locata delta-position velocity results in some higher errors when Locata positional accuracy is not taken into account. The Locata reference signals, without a 10 centimeter precision threshold, give the errors listed in table 11. Compared to the results after implementing the Locata threshold (table 12), there are some apparent tradeoffs.

The most noticeable improvement is a decrease in 3D-RMS for all measurements that were studied, sometimes up to half. This means that removing the Locata signals did provide a better reference truth for the GPS aided filter. However, removing Locata measurements sometimes increased errors in the horizontal axes, such as with the 10 second averaged delta-position velocity. This means the decrease in vertical error was more significant than the increase in horizontal error (resulting in an overall lower 3D-RMS). The other drawback to removing low-precision Locata measurements is that not enough Locata measurements are available to calculate any 100 second average delta-position velocity measurements. This means that Locata can not provide a velocity truth at longer averaging times.

Table 11: Table of statistics comparing GPS aided filter results and Locata position measurements as well as Locata delta-position velocity measurements. These results compare filter results every time there is a Locata measurement available, regardless of Locata reported accuracy. The result next to each velocity is the 3D-RMS for the associated averaging period.

	RMS	Mean (μ)	Std. Dev. (σ)	N
NED Position (cm)	13.0 cm			
North	2.96 cm	2.26 cm	1.92 cm	1428
East	2.43 cm	0.29 cm	2.42 cm	1428
Down	12.4 cm	-11.4 cm	4.90 cm	1428
NED Velocity - 1s (cm/s)	0.88 cm/s			
North	0.34 cm/s	$ \mu < 0.01$ cm/s	0.34 cm/s	1347
East	0.37 cm/s	-0.02 cm/s	0.37 cm/s	1347
Down	0.72 cm/s	$ \mu < 0.01$ cm/s	0.72 cm/s	1347
NED Velocity - 10s (cm/s)	0.20 cm/s			
North	0.06 cm/s	$ \mu < 0.01$ cm/s	0.06 cm/s	77
East	0.07 cm/s	-0.03 cm/s	0.06 cm/s	77
Down	0.18 cm/s	$ \mu < 0.01$ cm/s	0.18 cm/s	77
NED Velocity - 100s (cm/s)	0.02 cm/s			
North	0.02 cm/s	0.02 cm/s	0.00 cm/s	1
East	RMS < 0.01 cm/s	$ \mu < 0.01$ cm/s	0.00 cm/s	1
Down	0.01 cm/s	0.01 cm/s	0.00 cm/s	1

Once again, the accuracy of the filter velocity is dependent on the averaging period of the filter results and the corresponding time difference between the position measurements used to calculate the delta-position velocity. While the 100 second averaging period is not usable in after removing degraded Locata measurements, the 10 second averaging period yields an overall error of 1 mm 3D-RMS, with sub-millimeter errors in each axis. To reiterate, this sub-millimeter per second error is the result of a centimeter-level delta-position error divided by a 10 second time difference. This means that over 10 seconds, a centimeter level error accumulated due to the integration of the filter velocity or from the errors associated with the Locata (truly a combination of both), and this error was decreased by a a factor of 10 after division by the time difference.

In all cases the results were better than the corresponding studies performed using a

Locata aided filter and a GPS delta-position velocity. While this may stem from a more accurate velocity measurement from GPS vs the velocity measurement from Locata, the frequency of updates likely makes a difference as well. The GPS measurements occur at 20 Hz, vs Locata's 10 Hz, meaning that the filter is corrected more frequently. Using this study, the filter's velocity variance can confidently be placed at less than 1 cm/s for all averaging periods.

Table 12: Table of statistics comparing GPS aided filter results and Locata position measurements as well as Locata delta-position velocity measurements. These results compare filter results every time there is a Locata measurement available with a reported accuracy of $< 0.1\text{m}$. The result next to each velocity is the 3D-RMS for the associated averaging period.

	RMS	Mean (μ)	Std. Dev. (σ)	N
NED Position (cm)	12.3 cm			
North	2.86 cm	2.33 cm	1.66 cm	1055
East	1.84 cm	0.36 cm	1.81 cm	1055
Down	11.8 cm	-11.2 cm	3.53 cm	1055
NED Velocity - 1s (cm/s)	0.86 cm/s			
North	0.39 cm/s	$ \mu < 0.01 \text{ cm/s}$	0.39 cm/s	649
East	0.39 cm/s	-0.02 cm/s	0.39 cm/s	649
Down	0.65 cm/s	0.02 cm/s	0.65 cm/s	649
NED Velocity - 10s (cm/s)	0.10 cm/s			
North	0.04 cm/s	$ \mu < 0.01 \text{ cm/s}$	0.04 cm/s	26
East	0.05 cm/s	-0.03 cm/s	0.04 cm/s	26
Down	0.08 cm/s	$ \mu < 0.01 \text{ cm/s}$	0.08 cm/s	26
NED Velocity - 100s (cm/s)	N/A			
North	N/A	N/A	N/A	0
East	N/A	N/A	N/A	0
Down	N/A	N/A	N/A	0

4.7 GPS-Aided Filter in Outage Conditions (Pseudo-Monte Carlo Simulation)

Having established a baseline accuracy for filter performance in nominal conditions, the next study examined nominal configuration with GPS outages. This mirrors the operational configuration in which a nominal UHARS configuration is established and the test aircraft is jammed. In an operational scenario, GPS measurements would not be available. However, in this case, the GPS measurements removed from the filter are available to form a delta-position velocity. This means the velocity truth used in this study is completely independent from the filter during an outage, and will better characterize filter performance. This study used the pseudo-Monte Carlo technique outlined in Chapter 3. Results are presented in Figures 26 - 33.

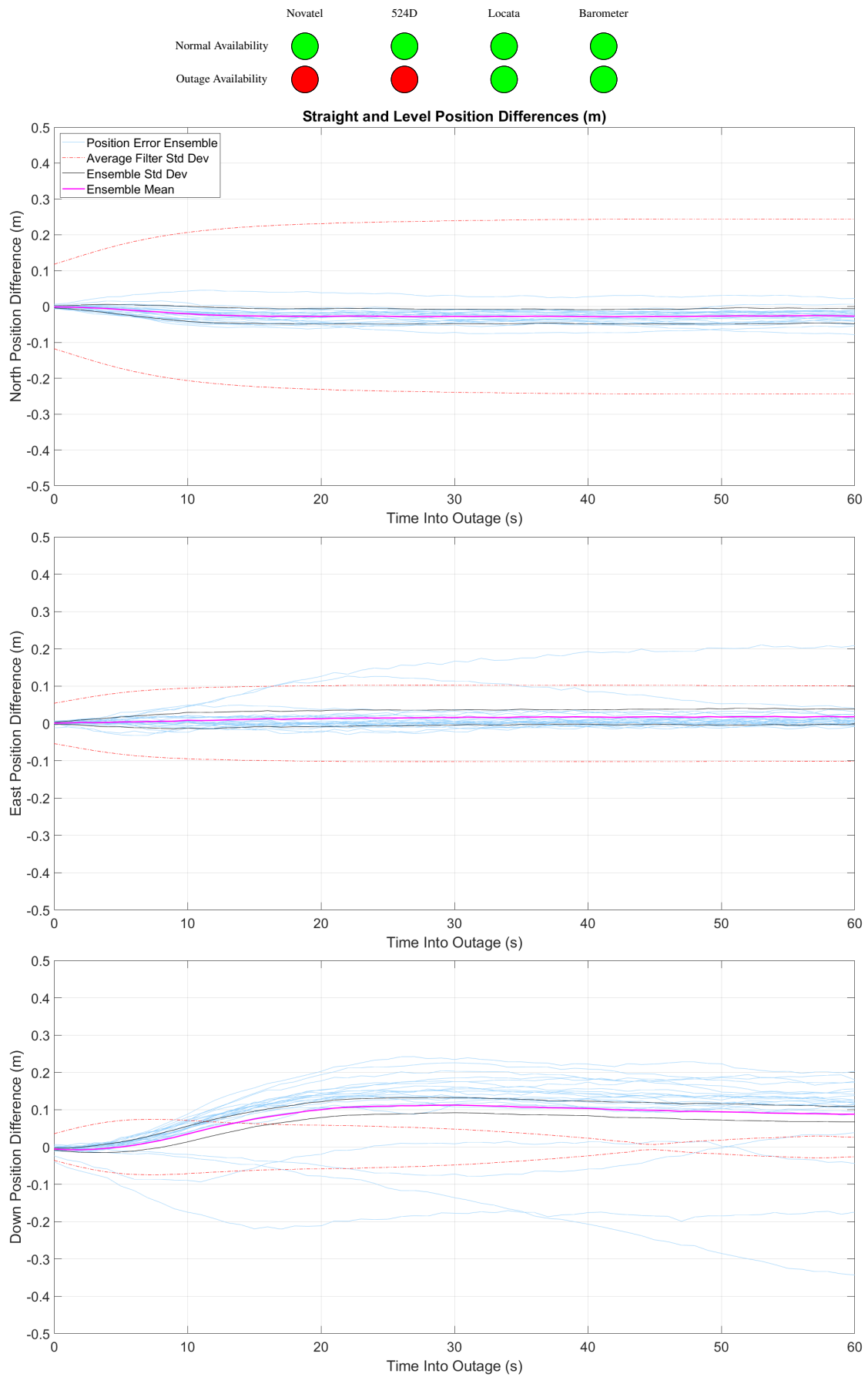


Figure 26: Position Pseudo-Monte Carlo simulation of a flight configuration filter operating in GPS outage conditions during straight and level flight.

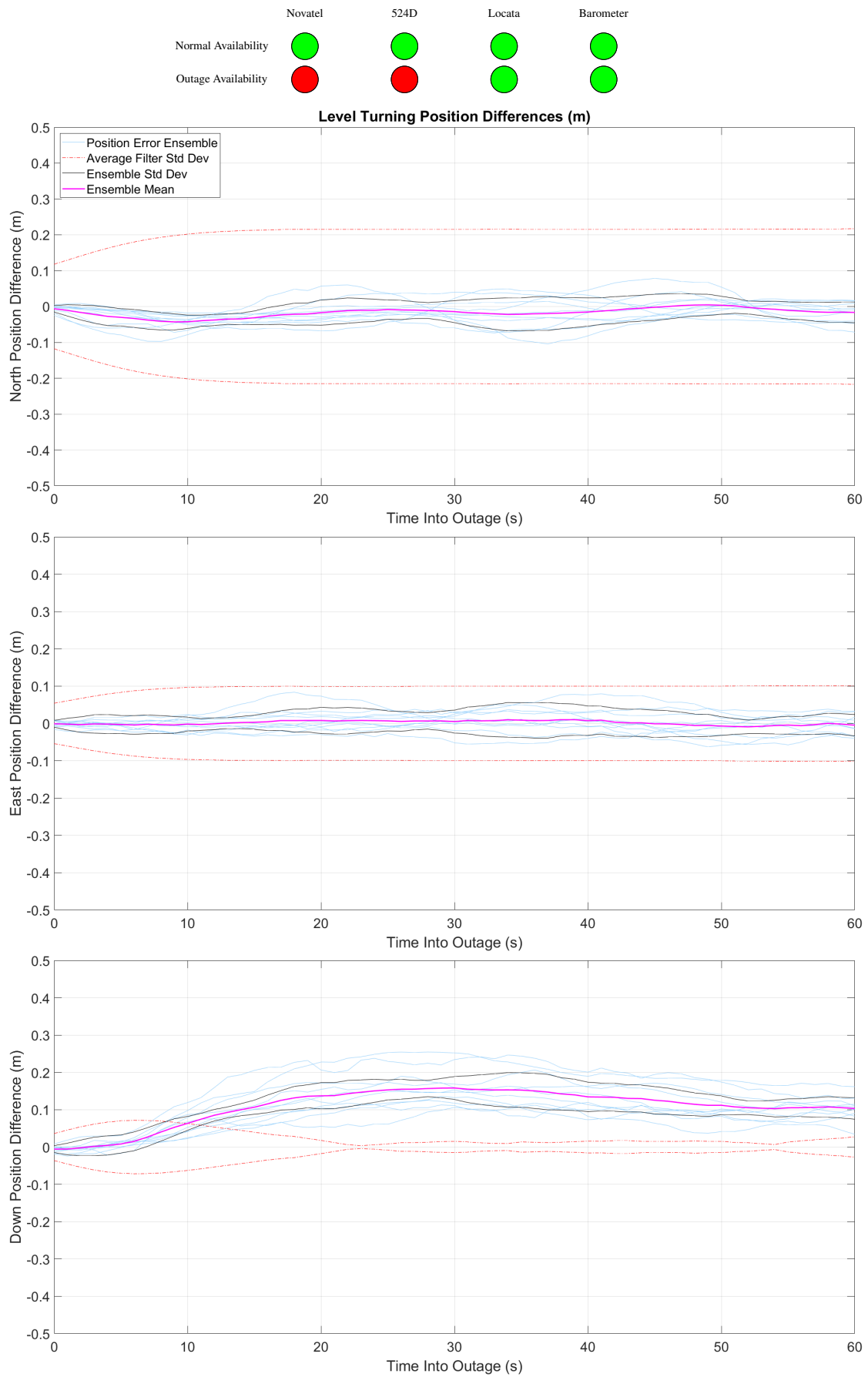


Figure 27: Position Pseudo-Monte Carlo simulation of a flight configuration filter operating in GPS outage conditions during level turning flight.

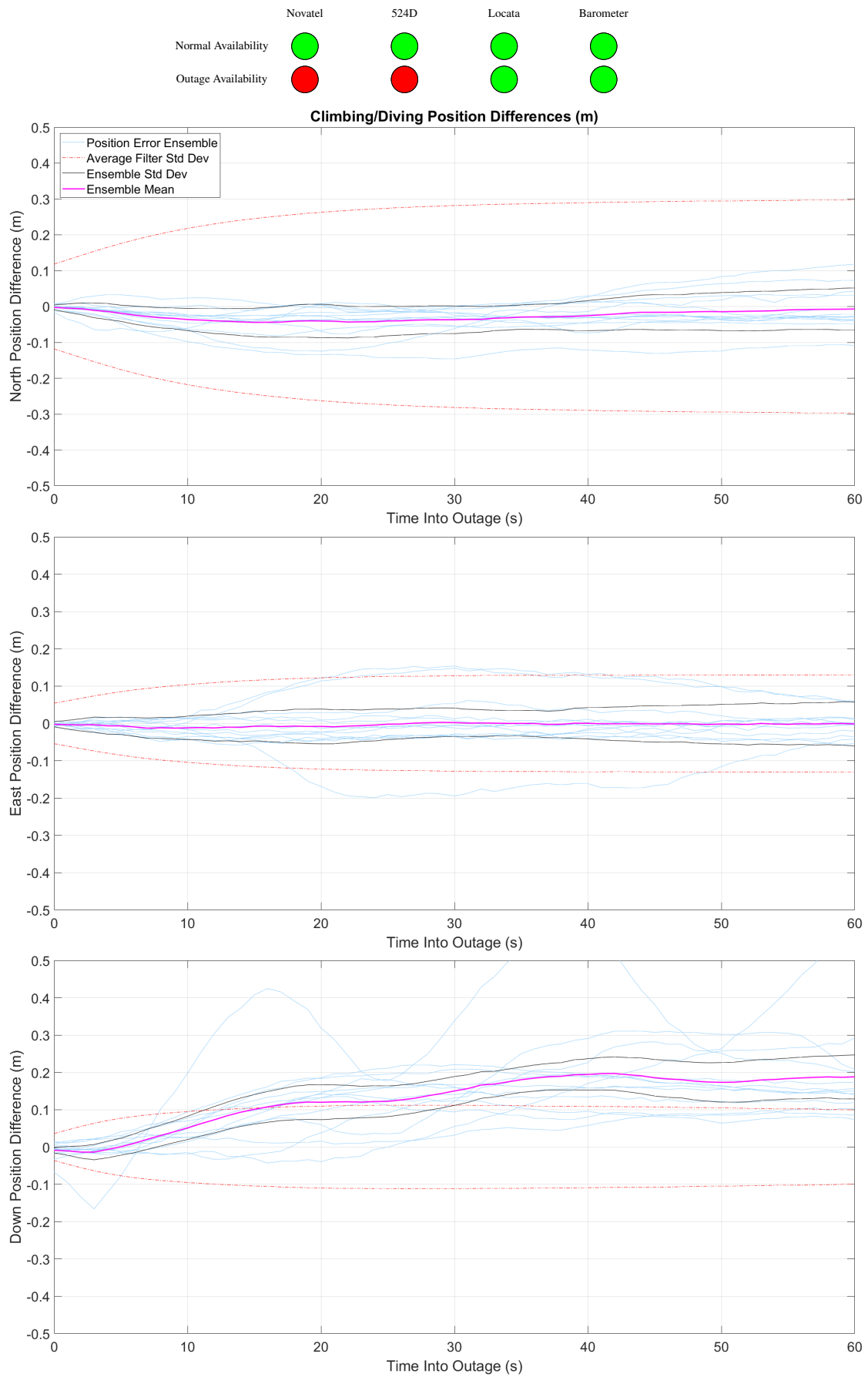


Figure 28: Position Pseudo-Monte Carlo simulation of a flight configuration filter operating in GPS outage conditions during climbing and diving flight.

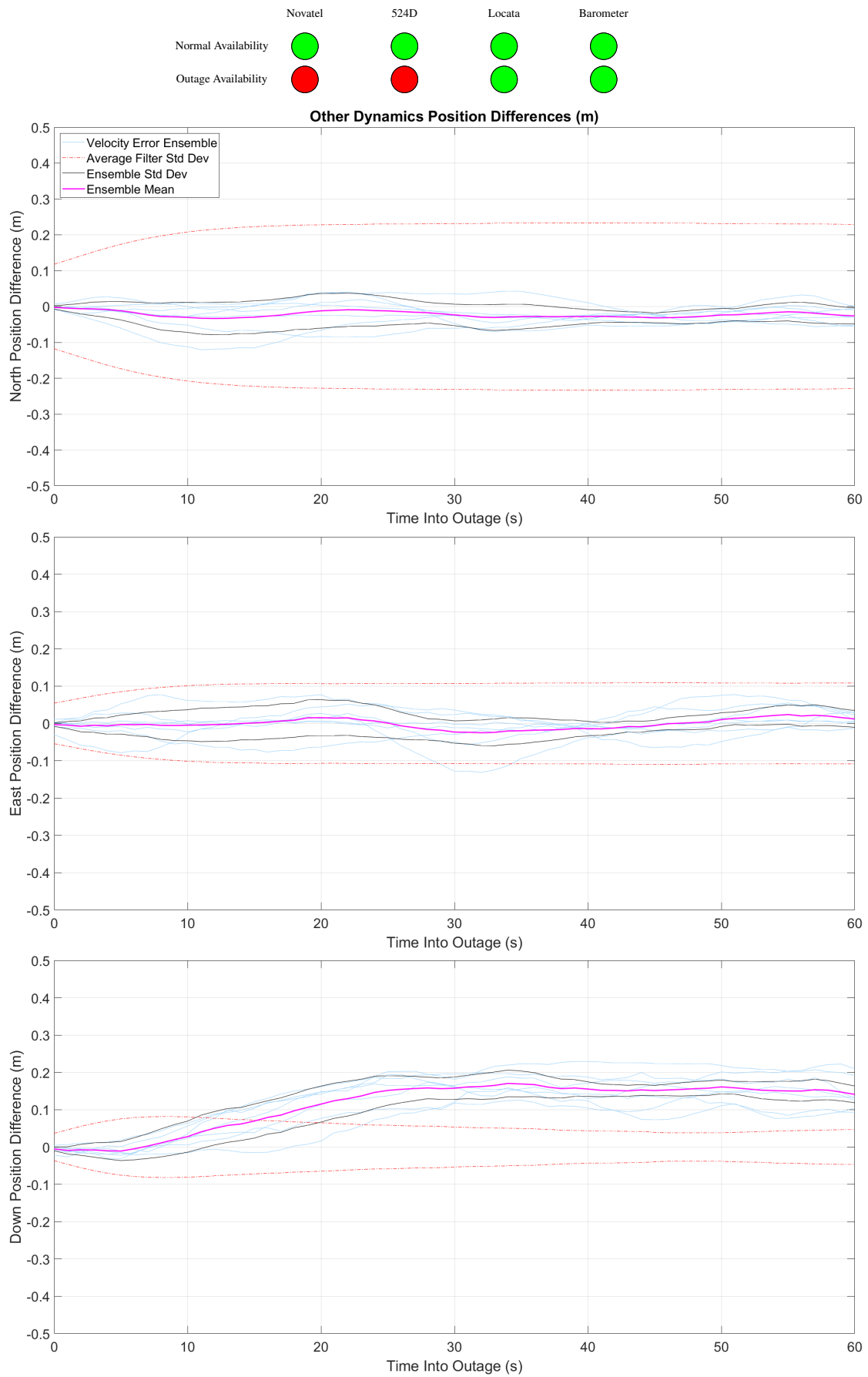


Figure 29: Position Pseudo-Monte Carlo simulation of a flight configuration filter operating in GPS outage conditions during high dynamics flight.

Position performance in straight and level flight (Figure 26) appears to be the best. In this case, the filter drift was well contained within the predicted variance bounds. In a sample size of 23 outages, none exceeded the filter predicted variance bounds in the north direction (100% in band), two cases exceeded the filter predicted variance in the East direction (91% in band), but all exceeded the filter bounds in the Down direction (0% in band). This high exceedance in the Down direction occurs across all dynamics environments, and has approximately the pattern of exceedance in each case. While the filter error almost always starts at zero, it increases to a particular offset and stays fairly constant for the remainder of the outage. This could be due to an error between GPS and Locata or GPS and the barometer measurements. At the initial loss of GPS measurements, the filter and GPS agree very closely. However, continued updates with Locata and barometer measurements continue to push the filter away from the GPS measured altitude. This is substantiated by the results of comparing raw GPS and Locata measurements; Figure 16 shows a high offset between GPS and Locata altitude measurements. This offset is reversed in figure 26 because GPS is being subtracted from a Locata aided filter, Figure 16 shows Locata being subtracted from GPS. This would account for the persistent error in the down direction throughout all outages.

Turning position error was marginally better than straight and level position error. Out of the 10 outage ensemble, there were no exceedances in the north direction (100% in band), no exceedances in the east direction (100% in band) and every outage exceeding in the down direction (0% in band) following a similar pattern as that described above. Climbing and diving outages saw worse performance. 12 outages fell into the constant heading climb or dive category, with none exceeding in the north direction (100% in band), three exceeding in the east direction (66% in band) and all exceeding in the down direction. The phenomenon of note here however, is that the ensemble standard deviation grows and shrinks periodically, a phenomenon that is not shown in the filter predicted standard deviation. Locata precision does oscillate quite a bit during climbing and diving, but this should not.

Finally high dynamics situations, typically a combination of turns with a climb or dive, performed somewhat better than level turns. This is also a function of the sample size however. With only 7 samples, there were 0 exceedances in the North direction (100% in band), 1 exceedance in the East direction (85% in band), and 7 in the Down direction (0% in band)

Velocity performance is even more promising. As shown above, filter velocity stays in band in all cases except the vertical direction when climbing and diving. Not only this, but the filter is very pessimistic about velocity performance, sometimes being more than 5 cm/s away from the true filter performance. This speaks very strongly to the filter velocity estimate in an outage. Ensemble standard deviation rarely exceeds 2 cm/s, even in the most demanding dynamics.

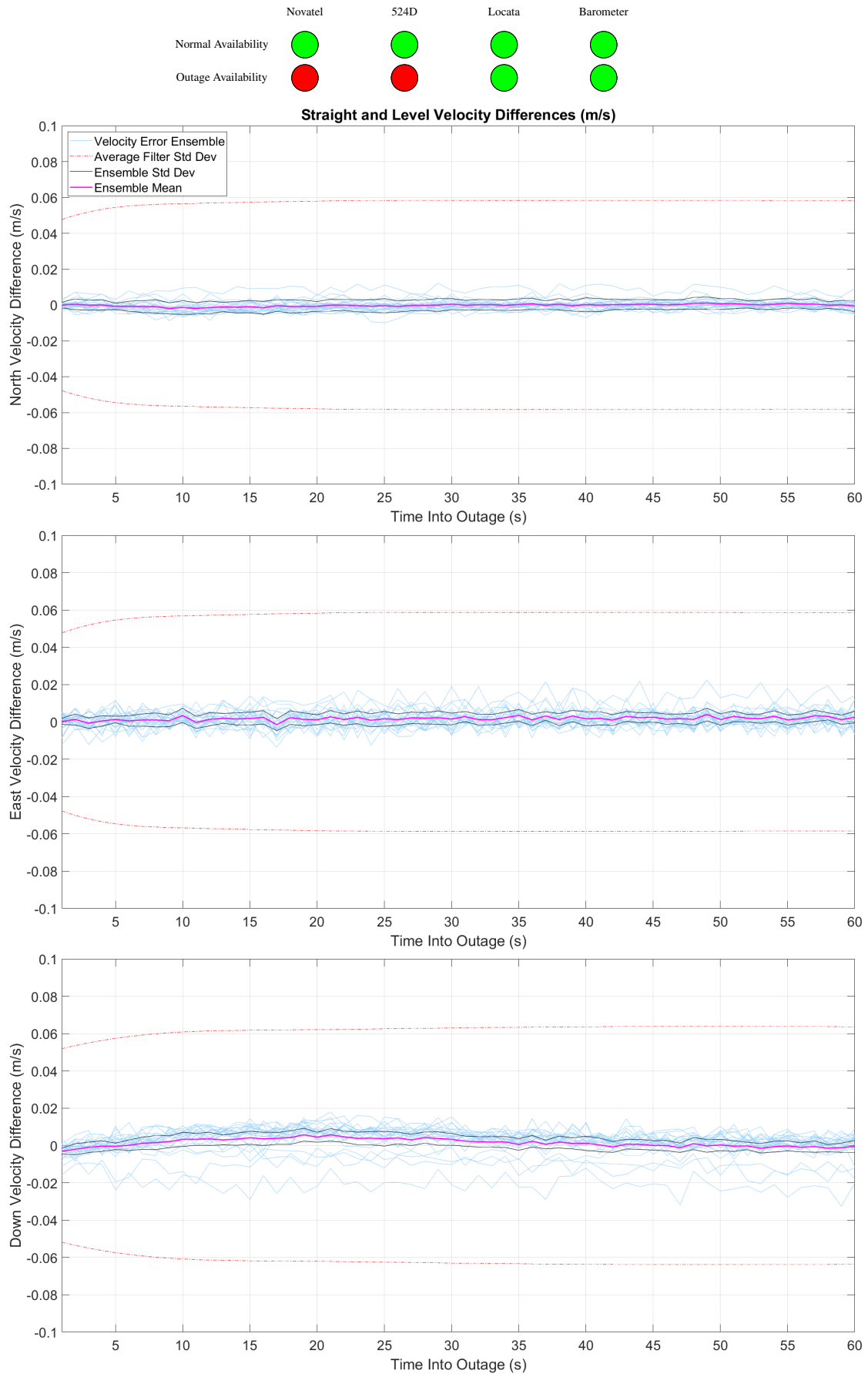


Figure 30: Velocity Pseudo-Monte Carlo simulation of a flight configuration filter operating in GPS outage conditions during straight and level flight.

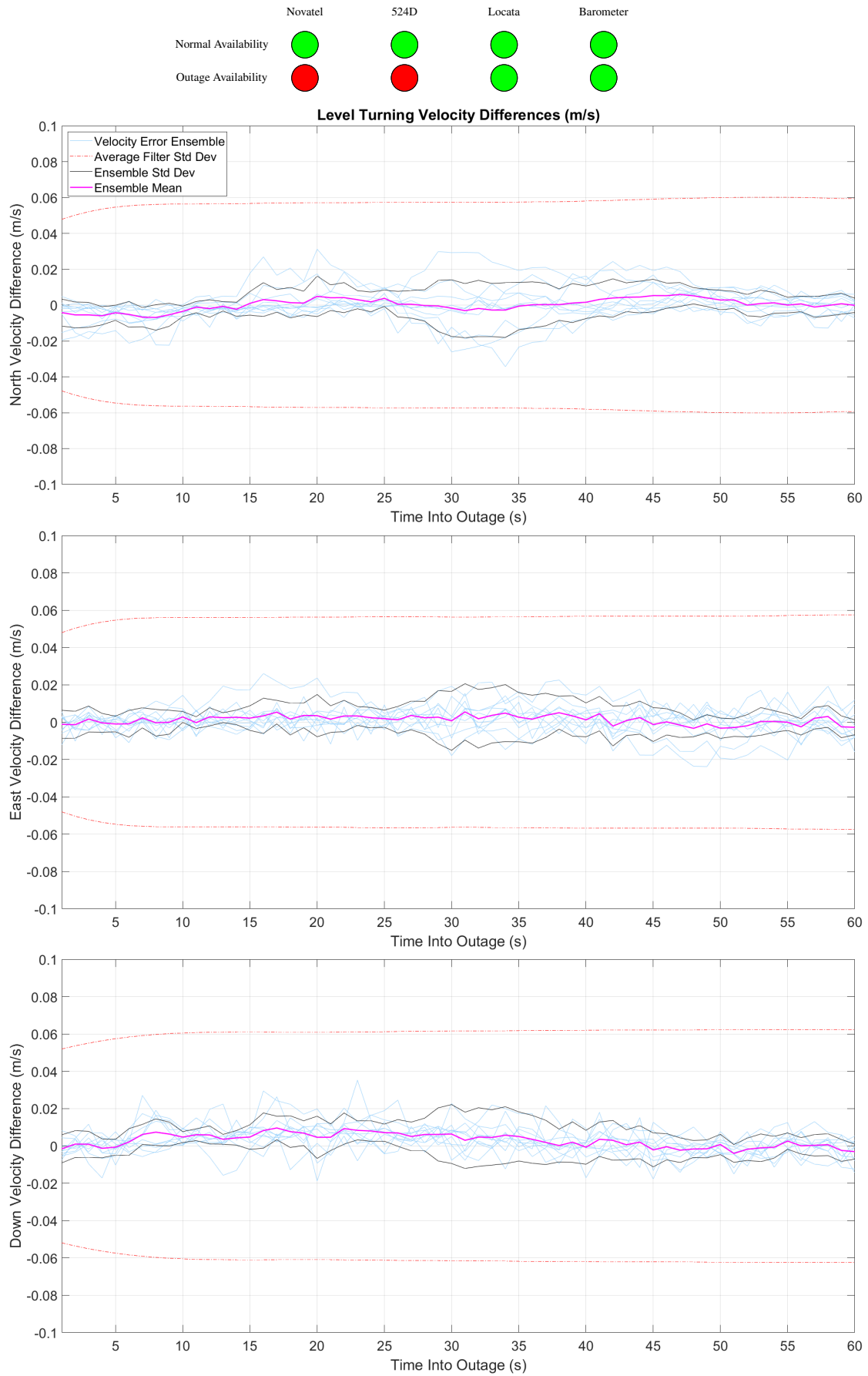


Figure 31: Velocity Pseudo-Monte Carlo simulation of a flight configuration filter operating in GPS outage conditions during level turning flight.

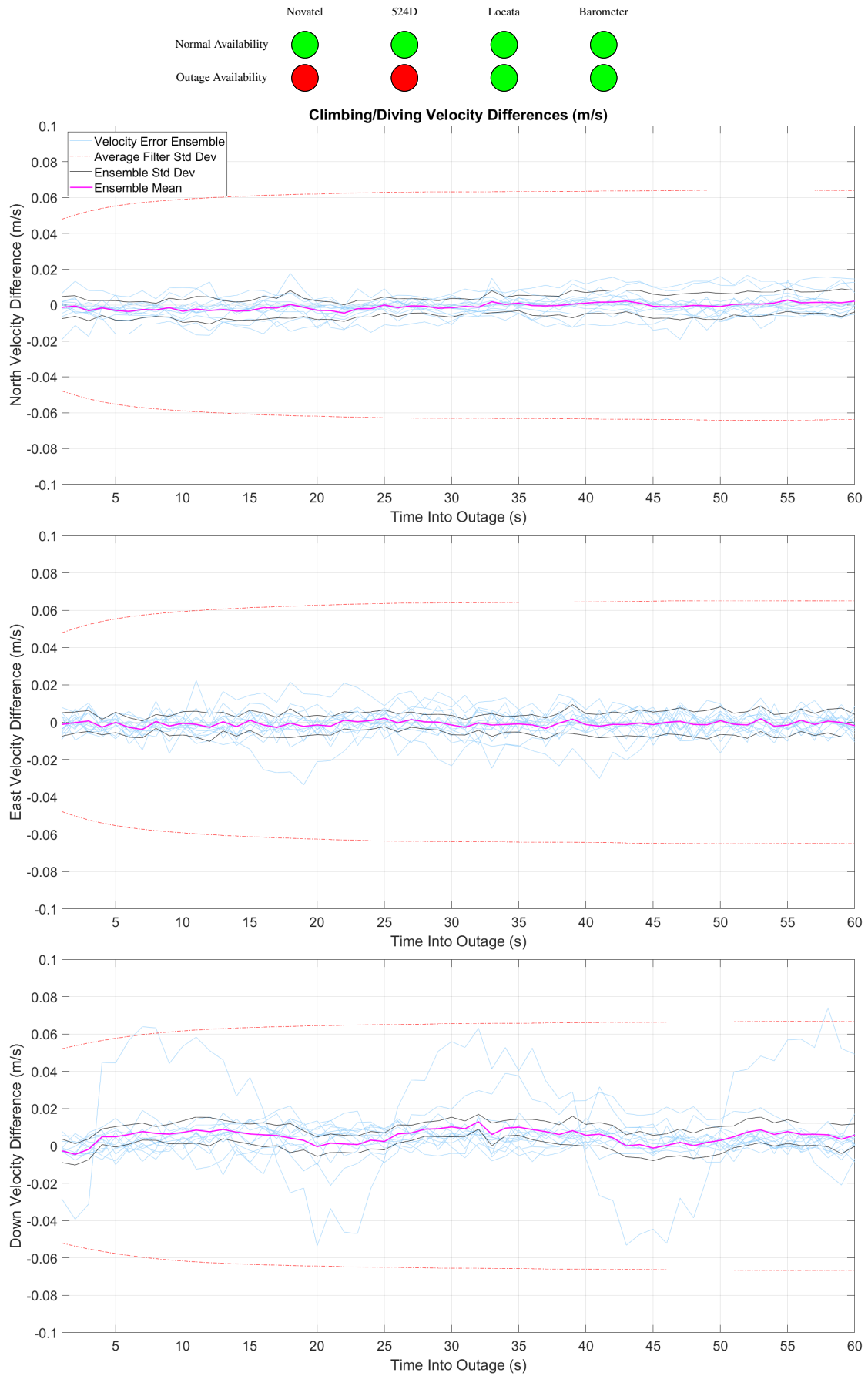


Figure 32: Position Pseudo-Monte Carlo simulation of a flight configuration filter operating in GPS outage conditions during climbing and diving flight.

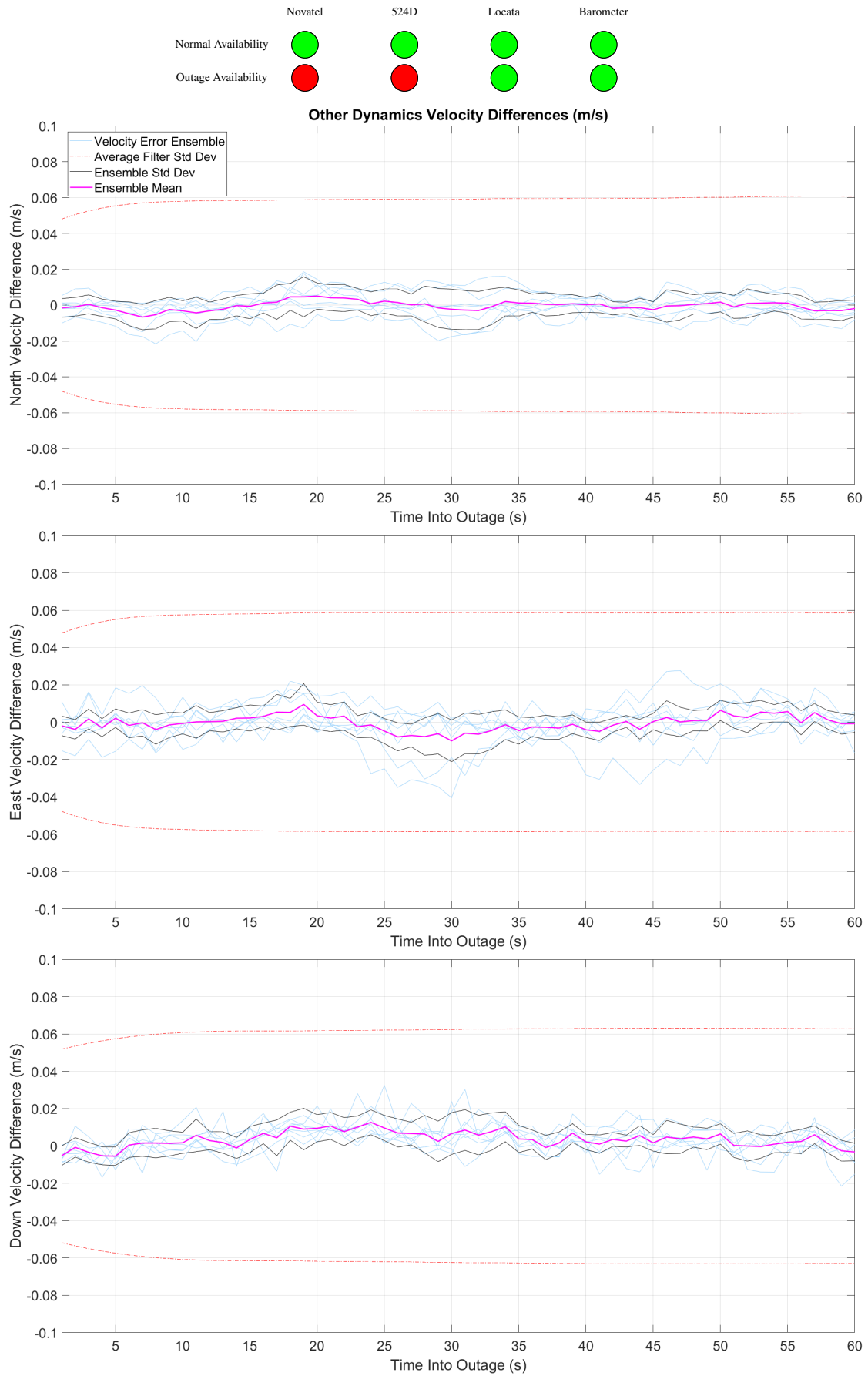


Figure 33: Position Pseudo-Monte Carlo simulation of a flight configuration filter operating in GPS outage conditions during high dynamics flight.

4.8 Summary

Having completed all the steps presented in the methodology flow chart (Fig. 7), all the components required to fully characterize UHARS velocity estimates are in place. First, Locata position measurements have an error of approximately 2 cm in the horizontal direction, and approximately 12 cm in the vertical direction. Locata availability is challenged at the extremes of the Locata network when aircraft turns can result in the antenna pointing away from the range. Delta-position measurements can assess filter accuracy to within 2 mm/s at 100 second averaging periods in the horizontal direction, and 11 mm/s in the vertical direction using a GPS delta-position velocity. Characterizing filter performance in a nominal state with Locata measurements and GPS delta-position velocity truth demonstrates a centimeter per second or sub-centimeter per second level accuracy depending on averaging interval. Most errors likely stem from the highly variable Locata velocity measurement being incorporated into the filter. A GPS aided filter with Locata delta-position velocity truth demonstrated a sub-centimeter per second accuracy at all averaging periods. Most improvements were likely due to the higher rate of GPS measurements and the more accurate velocity measurements. In addition, Locata can be difficult to use to form a delta-position truth at long averaging periods due to real-world availability. Finally, an assessment of filter performance in outage conditions demonstrated good tuning suitable for low dynamics, but not capable of accurately capturing filter performance in high dynamics. The velocity accuracy proved to be very pessimistic however. Velocity errors were fully encapsulated by the filter variance estimates, sometimes by as much as 5 cm/s pessimistically. Overall, filter velocity accuracy is currently underestimated, and a tighter tolerance can be placed on velocity estimates.

V. Conclusions and Recommendations

5.1 Conclusions

This document presented an method for determining average velocity using delta-position measurements from GPS and Locata. This velocity is intended to be used as a truth for navigation testing at the 746th Test Squadron (CIGTF). Comparisons of raw position measurements from GPS and Locata revealed centimeter level differences horizontally and tens of centimeter differences vertically. Comparing delta-position velocities however, showed consistent 3D-RMS values of centimeter per second or better. The lowest error is achieved when the averaging period is long. At 10 seconds and 100 seconds, GPS delta-position velocity and Locata delta-position velocity have less than 1 cm/s RMS error.

In a flight configuration, with all sensor measurements utilized, compared to a GPS delta-position velocity the UHARS solution achieved similar results, with centimeter per second level 3D-RMS error at all three averaging periods. However, incorporating measurements at their native rates increased this accuracy by an order of magnitude. Full rate measurements yielded sub-centimeter per second 3D-RMS error at every averaging period., with a low of 0.2 mm/s horizontally at 100 seconds. This provides confidence that a delta-position velocity is a viable method for truthing the UHARS EKF solution.

A Locata aided filter, incorporating only Locata, INS, and barometer measurements, performed similarly when compared to GPS delta-position velocity. The 3D-RMS error is centimeter per second level at 1 second averaging periods, and is sub-centimeter per second level at 10 and 100 second averaging periods. Individual axes see a millimeter per second level RMS and standard deviation at all averaging periods. This once again provides confidence that the delta-position velocity effectively truths the UHARS solution velocity.

Comparing a GPS aided filter to a Locata delta-position velocity demonstrated sub-centimeter per second 3D-RMS errors at all averaging periods with millimeter per second horizontal errors at the 10 second and 100 second averaging periods. These are the first

results that truly demonstrate that delta-position velocity measurements calculated from a non-GPS based positioning system can effectively truth a GPS dependent system. However, challenges remain both in the vertical accuracy and in the availability of Locata measurements. Because of the availability of Locata positioning signals, only one complete 100 second averaged delta-position velocity could be calculated. Removing degraded measurements then removed this single measurement as well, meaning that a Locata delta-position velocity will face real-world availability and quality problems when truthing a UHARS solution.

Finally, when GPS outages were injected into a flight-configuration filter, results were promising in the characterization of the filter's error. Position ensembles remained within the standard deviation bounds across all environments the appropriate amount with the exception of vertical position. This error can be attributed to the offset between GPS and Locata in terms of their height measurements. The most challenging environment appeared to be climbs and dives, which could be attributed to the quality of the delta-position velocity as well. Velocity ensembles performed even better, remaining well within the standard deviation bounds and remaining practically zero-mean throughout outage conditions. The most challenging environment for velocity appears to be climbing and diving, especially high frequency climbs and dives, which can lead to exceedances of the filters calculated error. This study is the most useful to the end user; it characterizes filter performance in the complete absence of GPS measurements. In GPS denied conditions, the filter is consistently pessimistic about system performance, meaning the filter tuning could be adjusted to reduce the filter predicted velocity variance.

Overall, a millimeter per second level accurate truth can be achieved with the appropriate inputs and averaging periods. 1 second averaging periods are well suited for a real-time measurement that could be incorporated into a navigation filter, but they suffer from high noise. 100 second values could be used to help smooth a post-processed solution, but while accurate, they are few and far between. 10 second averaged results are well suited for both. Highly accurate, yet still frequently available, they could be used to increase filter estimate

accuracy in either application. The utility of this truth is somewhat limited by real-world availability of Locata measurements, but can still be used in sufficiently long flights to characterize the accuracy of navigation systems in GPS degraded environments.

5.2 Recommendations

There exist several areas of improvement that must be studied before a delta-position velocity measurement can be operationally fielded. Each of these areas will serve to advance research and improve UHARS performance.

- GPS outages should be studied using Locata delta-position velocity as the truth. This mirrors what signals are available during a real-world jamming scenario and can offer insight into operations moving forward
- The UHARS EKF should be retuned to better characterize velocity performance. The results of this thesis demonstrate that even in GPS denied environments, the UHARS accurately predicts velocity performance. This means that the variance or standard deviation of the filter velocity output can be significantly decreased through filter tuning
- The height error in Locata should be studied more closely. While it is possible to have different tolerances for horizontal and vertical positions, it is not ideal.
- The delta-position velocity should be expanded to other implementations in UHARS and truth references. For example, integrating Locata measurements could provide a change in position measurement for real-world GPS outages due to cycle slips
- A delta-position velocity measurement may be incorporated into the UHARS EKF.
- Most importantly, the generation of delta-position velocities must be automated to a higher degree and integrated with the existing Spider framework in order to be suitable for operational implementation

Acronyms

3D-RMS three dimensional root-mean-square. 5, 56, 67, 68, 69, 74, 75, 81, 85, 86, 87, 88,

101

AFIT Air Force Institute of Technology. 5

AFTC Air Force Test Center. 35

AITL Advanced Inertial Test Laboratory. 35

ANT Autonomy and Navigation Technology. 5

AWGN Additive White Gaussian Noise. 17, 25, 26, 34, 43

CHAPS CIGTF High Accuracy Post-processing Reference System. 35

CIGTF Central Inertial and GPS Test Facility. 35, 101

CIRIS Completely Integrated Reference Instrumentation System. 35

COTS Commercial Off the Shelf. 4, 36

CRS CIGTF Reference System. 5, 35, 36, 39

DAS Data Acquisition System. 36

DCM Direction Cosine Matrix. 15, 33, 63

DDS Direct Digital Synthesis. 22

DGPS Differential Global Positioning System. iv, 4, 19, 38, 39, 40, 52, 55, 59, 80

DoD Department of Defense. 4, 34

ECEF Earth Centered, Earth Fixed. 9, 11, 13, 14, 15, 16, 53, 55, 56, 59, 60, 62, 63, 67, 72

EGI Embedded GPS/INS. 35, 36, 50

EKF Extended Kalman Filter. 3, 5, 6, 23, 30, 36, 38, 50, 51, 62, 76, 101, 103

ENU East-North-Up. 12

FAA Federal Aviation Administration. 34

GAINR GPS Aided Inertial Navigation Reference. 35

GEMS GPS Environment Measurement Subsystem. 36

GNC Guidance, Navigation, and Control. 35

GNSS Global Navigation Satellite System. 16, 18, 20

GPS Global Positioning System. 3, 4, 6, 7, 8, 9, 10, 16, 17, 19, 20, 31, 32, 34, 35, 36, 40, 41, 42, 43, 50, 51, 52, 53, 54, 55, 57, 58, 59, 60, 61, 62, 63, 67, 69, 70, 71, 73, 74, 75, 80, 81, 82, 83, 84, 85, 86, 88, 89, 90, 91, 92, 93, 94, 96, 97, 98, 99, 100, 101, 102, 103

ICD Interface Control Document. 50

IERS International Earth Rotation and Reference Systems Service. 9

IMU inertial measurement unit. 24, 56

INS Inertial Navigation System. 4, 5, 7, 33, 34, 36, 39, 42, 50, 53, 65, 85, 101

ISM Industry Scientific and Medical. 4, 20, 21

LLA Latitude, Longitude, Altitude. 9, 13, 14, 55, 56, 59, 72

MSL Mean Sea Level. 9

NASA National Aeronautics and Space Administration. 34

NAVWAR navigation warfare. 4

NED North-East-Down. 12, 13, 14, 15, 53, 54, 56, 67, 72, 74

NGBPS Non-GPS Based Positioning System. 3, 5, 36, 50, 67

PDF Probability Density Function. 25

PLL Phase Locked Loop. 17

PNT Position, Navigation, and Timing. 3

PRN Pseudo Random Noise. 21

RMS Root-Mean-Square. 5, 52, 74, 101

RRS Range and Range-Rate Subsystem. 5, 36

SPIDER Sensor Processing and Inertial Dynamics Error Reduction. 5, 37, 38, 50

SRS Satellite Reference Station. 35

TDMA Time Division Multiple Access. 4, 21

TOA Time-of-Arrival. 16

UHARS Ultra High Accuracy Reference System. 3, 4, 6, 7, 35, 36, 37, 38, 39, 40, 41, 42,
50, 64, 65, 66, 85, 89, 100, 101, 102, 103

UTC Universal Coordinated Time. 10

WGS-84 World Geodetic System 1984. 9, 10

WSMR White Sands Missile Range. 4, 36, 37

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14. ABSTRACT						
The 746th Test Squadron (746 TS) at Holloman AFB, NM operates the Ultra High Accuracy Reference System (UHARS) as part of its mission positioning and navigation test. This research presents a method for verifying the performance of a flight reference system using a Delta-Position velocity derived from radionavigation positioning measurements. The algorithm presented may utilize Global Positioning System (GPS) or the Locata ground based positioning system. In the latter case, Locata provides a velocity truth independent from GPS. The accuracy of Locata and GPS are assessed and UHARS velocity measurements are characterized both in nominal and GPS denied applications.						
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U	U	U	U	119	19b. TELEPHONE NUMBER (include area code) (937) 255-3636, x4618; aaron.canciani@afit.edu	