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Galois Operators for Distributed Logic

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14. ABSTRACT

This is a report on some new operators for Distributed Logic for use in expressing properties of FPGA applications. The Galois operators are variations on the usual necessity and possibility operators. In effect, they complete the first-order logic patterns in the relational semantics for modal logic, although here we extend them for use in Distributed Logic. It was important to bring these into Distributed Logic to increase its expressive power. There is a companion Memo Report, Closure Properties for Galois Operators in Distributed Logic, which shows how the semantic frames for these operators are closed under a smash product and null sum. The companion report also includes all the proofs elided in the current report. This report and the companion report will be used for a logic paper showing the Thomason-Goldblatt theorem for the Galois operators. The original Thomason-Goldblatt theorem covered only the necessity and possibility operators for bog standard modal logic.

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EXECUTIVE SUMMARY

This report defines Galois operators for Distributed Logic. Eventually, this report and a companion report will be used for a logic paper showing the Thomason-Goldblatt theorem for the Galois operators. The original Thomason-Goldblatt theorem covered only the necessity and possibility operators for bog standard modal logic. These operators fill out the first-order logic patterns in the relational semantics for modal logic, although here we extend them for use in Distributed Logic. Distributed Logic is used for proving properties about distributed systems in general and FPGA applications in particular. We have designed Distributed Logic to work with the new FPGA applications language ReWire, which we are also developing.

The usual necessity and possibility operators of modal logic are Galois operators. There are no actual names for the six new operators as they do not have a common use in modal logic although they do appear sporadically there. These operators abstract over first-order properties of their frames. The frames are semantic mathematical entities that are parts of logic models. A frame consists of a collection of points or states, a relation on those points, and a Boolean algebra of sets of points closed under Galois operators applied to the sets. A model is a map from formulas of the logic to the Boolean algebra of sets of a frame. Unwinding the set definitions yields a point-centric view of the semantics. This latter is quite useful in describing systems to which the logic is applied.

The point of Distributed Logic is that cramming everything used for logic into a single logic is a mistake if that logic is supposed to be used for describing and proving aspects of a distributed system. Point or states in one part of the system may have no or little relationship to points in another part. As a consequence, logical formulas describing one part of a system may have no or very restricted relationships with another part. It makes sense to distribute the logic in same manner in which the system is distributed.

Distributed Logic requires a *connection graph* showing the distribution structure of the system under examination. The connection graph specializes Distributed Logic to a particular application in much the same way that any logic is specialized to an application via selecting its propositions. In FPGA applications, the connection graph is the connection structure of the collection of components; each node of the graph corresponds to a component, each arc represents a relationship between two components. The nodes are connected together via signals and are usually run in parallel. This latter is a difficult concept to embed in a logic. The considerations of distributing a logic resulted in Distributed Logic, which is the PI's development. Distributed Logic's solution to this representation problem is to recognize that each component supports its own *local logic* and a *local relation*, often interpreted as a *next state* relation. This latter is captured in the local logic's *local frame*. Local logics are connected to other components using restricted logical connectives, the distributed operators.

The distributed operators for this report are the Galois operators. The connection structure is reified as a graph, which is part of Distributed Logic's apparatus. The semantic evaluation at each node can occur in parallel just as each component executes in parallel. The only time interaction is required is when signals

are exchanged. The exchange of signals gives rise in the semantics to non-trivial distributed relations; one common type is a *concurrence relation* describing when points of two components can occur simultaneously in the components' execution. Distributed Logic itself does not care what kinds of relations are used, concurrence or otherwise, just that they be binary relations.

GALOIS OPERATORS FOR DISTRIBUTED LOGIC

1. INTRODUCTION

This research started from requiring more expressive modal connectives in Distributed Logic [1] for use in high assurance of FPGA applications. Eventually, this report and a companion report will be used for a logic paper showing the Thomason-Goldblatt theorem [2, 3] for the Galois operators. The original Thomason-Goldblatt theorem covered only the necessity and possibility operators for bog standard modal logic. Distributed Logic can be seen as a distributed modal logic. Distributed Logic is quite general and can happily accept *non-normal* modal operators; these are operators that do not distribute across conjunctions and disjunctions of a Boolean logic base. The operators we consider in this paper do distribute across conjunctions and disjunctions, hence are *normal* operators. The difference model-theoretically [1, 4] is that non-normal operators are interpreted using neighborhood systems whereas normal operators are interpreted using relations; these are distributed relations in our case.

A Distributed Logic has a graph of localities as part of the syntactic structure of the logic. Each node in the graph is associated with a *local* modal logic. Distributed Logic adds modal connectives, here called *operators*, between the local logics. A distributed operator takes propositions of a local logic at, say node k and returns propositions of another local logic, say node h. Each local logic is evaluated in a typical modal frame (one per node) and there are distributed Kripke relations linking the nodes for interpreting the distributed operators.

The reason we developed Distributed Logic was because FPGA applications have three large scale features: a collection of components, some connected with signals, and all running in parallel. Each component is a world of states unto itself and shares no internal state with any other component. Hence each component could be reasoned about using a single modal logic, as long as one was not concerned with signals to other components and the components running in parallel. The graph allows us to associate a local logic with each component, and distributed Kripke relations to the effects of signals and parallel operations. From this substrate, it is possible to abstract a bit and use distributed relations for parallel behavior and some other high assurance concepts. Requiring capabilities for expressing high assurance concepts drove us to consider the operators from [5].

1.1 The Logic

The basic logic is detailed in [1]. We use the term Distributed Logic in a general sense such as modal logic; each has several logics that can fall under the term. Distributed Logic also includes modal logic as simple case. A particular distributed logic is actually a collection of *local* modal logics that are connected in a formal way via distributed operators. Each local modal logic has a classical base that admits the usual necessity and possibility operators that abstract over next-state relations (these local operators can be augmented with the Galois operators [5]).

Distributed Logic lends itself well to FPGA applications. Each local logic is seen as being the local logic of a single component. The components are connected via their behavior; that behavior is expressed using

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distributed relations. The distributed operators abstract over those relations. The abstraction takes the form of the evaluation conditions on the operator as shown in the sequel. The use of the term *distributed relation* reflects that a relation is relating two distinct collections of states in different components. This is in contrast to the local *endo-relations* (such as next-state relations); these latter are limited to a single locality

We use the term *locality* to denote a local logic and its underlying component structure as expressed in its states and relations. Thus, the distributed relations connect localities. A common distributed relation is a parallelism relation, called *concurrence*, that represents when states in two different localities can simultaneously occur. Another distributed relation is one that relates all components that share the same clock domain.

This paper also briefly describes extra constructions for grouping together localities: smash product and null sum. Smash products are used to glue together two components that are seen to be intimately run together in that we always want to be concerned with pairs of states, one state from each locality. Null sums are used for combining components together in a disjoint fashion where we only are interested in one without regard to the other, somewhat like an exclusive-or.

In the sequel, we will concentrate mostly on the new Galois operators. The name stems from a certain algebraic property they share. Each operator has a kin. If an operator is seen as abstracting over a relation where the relation is viewed a relation from locality h to locality k, the kin of this operator abstracts over the relation turned around so that this *converse relation* runs from k to h.

1.2 The Interpretations

The technical term *frame* in logic can represent many different situations. We use it primarily to represent a component in an FPGA application. Each frame consists of: (1) a collection of points (also called *states*), (2) a Boolean algebra of sets (where the sets are sets of points), (3) the $\in$ relation between points and elements of the Boolean algebra of sets, and (4) at least one *local relation*. In other words, it is a particular type of classification where the $\models$ relation is the set theoretic membership relation $\in$. The local relation can represent the next state relation when viewing the component as a finite automaton. However, local relations (not the $\in$ relation, the $\in$ relation is not a local relation) can also represent other notions as the need arises. An example of a non-next-state relation is where some states are considered security critical and related to states that are not security critical. Incidentally, requiring at least one local relation is not really much of constraint since it could always be made the identity (diagonal) relation. That relation has little modal import because from any state one can only move along the relation to the same state.

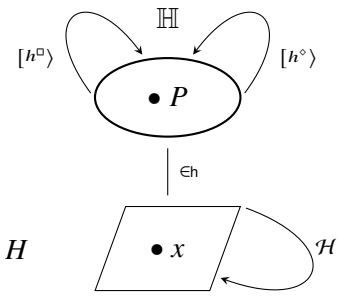
We assume a graph of localities usually denoted with sans serif, i.e., nodes are denoted h, k, etc. At each node is a classification consisting of a local logic, a frame, a satisfaction relations $\models^{\mathsf{h}}$, $\models^{\mathsf{k}}$, and at least one local relation. The local relation can be the identity relation if no feature of a component needs to be represented using a local relation.

Definition 1.2.1 A *local frame* is a structure $H = (H, \mathcal{H}, \mathbb{H})$ such that H is a collection of points called, generically, a *domain*. $\mathcal{H} : \mathsf{h} \rightharpoonup \mathsf{h}$ is a local relation connecting some states of H. We use the same symbol for the frame and its collection of states, and let use disambiguate meaning. $\mathbb{H}$ is a collection of *neighborhoods* or sets of states that are subsets of H and the entire collection is closed under the Boolean operations and under the generic Galois operations $[{}^{h}\rangle : \mathbb{H} \to \mathbb{H}$. Hence $\mathbb{H}$ is a modal set algebra. These operators are used to define special collections of states. They have valuation conditions in the sequel given by distributed

operators, just set the distribution to a single component. The $\in$ relation between states and elements of the Boolean algebra is left implicit.

The set $\top^{\mathbb{H}}$ is the top of the Boolean lattice of sets and $\bot^{\mathbb{H}}$ is the bottom. A caveat, from the requirements of the sequel, $\top^{\mathbb{H}}$ is not H and $\bot^{\mathbb{H}}$ is not $\emptyset$. Special points have been added to the frames.

A good mental picture to remember the definition of a frame is the following diagram where one can think of P as a proposition of a language or as a UCLA proposition. This latter is merely a set of points in a Boolean lattice of sets. In the former case, the $\in$ relation becomes $\models$ and in fact, in interpretations $\in$ is the meaning of $\models$.



Here x is some point in the domain H and P is some UCLA proposition in $\mathbb{H}$, but it is not necessarily the case that $x \in$h P. Two example modal operators $[h^{\Box}\rangle$ and $[h^{\Diamond}\rangle$ are shown. They are the usual modal necessity and possibility operators $\Box$ and $\Diamond$ but put into our notation. The relation $\mathcal{H} : \mathsf{h} \rightharpoonup \mathsf{h}$ is some set $\mathcal{H} \subseteq H \times H$. Note that h refers to the node at which the frame indicated by the diagram lives.

Definition 1.2.2 A *distributed frame* consists of a graph of nodes where each node is a local frame, distributed relations linking the domains, and distributed modal operators linking the set algebras.

Propositions in the logic are modeled by elements of the set algebras. A distributed frame for modeling two components has two localities. The next diagram depicts this situation (without showing the local modalities or local relations) and two (generic) distributed modalities:

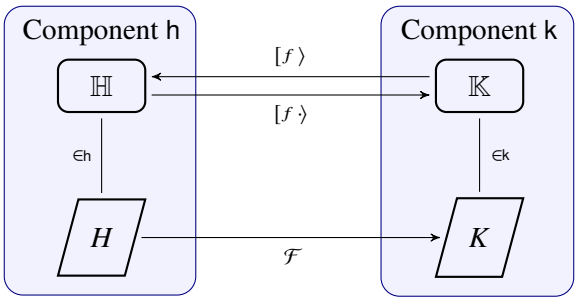


Diagram 1.1: Generic Distribution with Two Components as Localities

where the arrows $[f\rangle$, $[f\cdot\rangle$ can be any of the Galois forward and backward distributed modalities interpreted by the distributed relation $\mathcal{F}$, i.e., the lower case f in the modalities is linked with the script $\mathcal{F}$ relation. Example pairs of modalities are $[f^{\square}\rangle$ and $[f^{\square}\cdot\rangle$ and are necessity operators, and $[f^{\circ}\rangle$ and $[f^{\circ}\cdot\rangle$ are possibility operators.

The diagram is only showing the simplest of distributed frames. In any application, even one with only two localities, there can be any number of distributed relations and any number of distributed modalities.

The relation $\mathcal{F} : \mathsf{h} \rightharpoonup \mathsf{k}$ used in the evaluation of the operators above uses two localities, h and k. These are variables in that any actual FPGA application will fill those localities in by two components; the entire distributed frame will use as many localities as there are components. Also, any one locality can have many endo-relations, and any two localities can be connected by many distributed relations. The modalities involved are restricted by the number of relations we have at our disposal to interpret the modalities. Technically, $\mathcal{F} \subseteq H \times K$, we leave this implicit in the notation $\mathcal{F} : \mathsf{h} \rightharpoonup \mathsf{k}$.

The distributed relation $\mathcal{F}$ (see diagram) is denoted with an arrow but this is mere convention; $\mathcal{F}$ is a two-place relation that, by fiat, is viewed as a morphism from elements of its first position to elements of its second position. The distributed modalities, on the other hand, really are functions although they have special properties required for us to treat them as modalities.

To evaluate logic formulas, we interpret the $\models$ relation as set theoretic membership $\in$ and the proposition P can be either a linguistic proposition, or switching to the set theory interpretation, a UCLA proposition, i.e., a set of points. The following definitions are standard modal logic fare:

- $\models^{\mathsf{h}} P$ if and only if for all $x \in_{\mathsf{h}} \top^{\mathbb{H}}$, it is the case that $x \models^{\mathsf{h}} P$. Equivalently, $\models^{\mathsf{h}} P$ if and only if for all $x \in_{\mathsf{h}} \top^{\mathbb{H}}$ (where $\top^{\mathbb{H}}$ is the top of the Boolean set algebra $\mathbb{H}$), it is the case that $x \models^{\mathsf{h}} P$.
- $\not\models^{\mathsf{h}} P$ if and only if there is some $x \in_{\mathsf{h}} \top^{\mathbb{H}}$ and $x \not\models^{\mathsf{h}} P$.
- A local logic at h is consistent just when for all $x \in_{\mathsf{h}} \top^{\mathbb{H}}$, it is the case that for all propositions P, not both $x \models^{\mathsf{h}} P$ and $x \not\models^{\mathsf{h}} P$.

Modal formulas $[f^{\square}\rangle$ and $[f^{\circ}\rangle$ for necessity and possibility are evaluated in the usual way except we must respect the distributed nature now of these modalities:

$$x \models^{\mathsf{h}} [f^{\square}\rangle Q \text{ iff } \text{ for all } y,\ \mathcal{F}xy \text{ implies } y \models^{\mathsf{k}} Q \qquad x \models^{\mathsf{h}} [f^{\circ}\rangle Q \text{ iff } \text{ there exists } y,\ \mathcal{F}xy \text{ and } y \models^{\mathsf{k}} Q.$$

The other versions $[f^{\square}\cdot\rangle$ and $[f^{\circ}\cdot\rangle$ run in the other direction:

$$y \models^{\mathsf{k}} [f^{\square}\cdot\rangle P \text{ iff } \text{ for all } x,\ \mathcal{F}xy \text{ implies } x \models^{\mathsf{h}} P \qquad y \models^{\mathsf{k}} [f^{\circ}\cdot\rangle Q \text{ iff } \text{ there exists } x,\ \mathcal{F}xy \text{ and } x \models^{\mathsf{h}} Q.$$

Setting $\mathsf{h} = \mathsf{k}$ yields the evaluation conditions for the corresponding local modalities. Hence the distributed modalities are evaluated similar to the local modalities except notice the changes between $\models^{\mathsf{h}}$ and $\models^{\mathsf{k}}$ on the two sides of the evaluations.

2. GALOIS OPERATORS

The operators for the modal logic case are in [5]. Their basic mathematical properties are shown there for a few of the operators. This paper extends the properties from the modal case where there is ever only a single locality to this distributed case for Distributed Logic.

Each operator comes with a *distribution type*, which becomes a bit of a misnomer when put into a Distributed Logic framework because the two uses of "distribution" refer to two kinds of distribution. For Distributed Logic, the term "distribution " refers to the distribution as detailed in a graph of nodes each with its own logic logic and connected by distributed operators and distributed relations. For [5], which was developed quite a bit before Distributed Logic, the term "distribution" refers to how the operator distributes across conjunctions and disjunctions, or in the algebraic case where we replace the logic with a lattice, lattice meets and joins. We will shorten the use of "distribution type" of [5] to merely "type". The type is intimately connected with an operator's residuation properties (see Section 5 Residuation).

The move from logic's conjunction and disjunction to lattice meets and joins is the move from the logic to an algebraic framework. A way to get an algebra out of a logic is to divide out the set of formulas via bi-implication:

$$P \equiv Q \text{ iff } \vdash\ P \supset Q \text{ and } \vdash\ Q \supset P,$$

where one can interpret the (usually) classical implication $\supset$ with a logic's implication and $\vdash$ is provability in the logic. The equivalence classes then are elements of a lattice if the logic supports the usual Boolean identities and an arbitrary modal operator $[-\rangle$. In particular,

$$[P] \wedge [Q] \stackrel{def}{=} [P \wedge Q] \qquad [P] \vee [Q] \stackrel{def}{=} [P \vee Q] \qquad \neg[P] \stackrel{def}{=} [\neg P] \qquad [-\rangle\,[P] \stackrel{def}{=} [[-\rangle\, P]$$

In the sequel, we use P and Q to refer to sets of points of the Boolean algebra (of sets) of a frame, i.e., UCLA propositions. They can also be thought of as propositions. This blurs any distinction between $[P]$ which is an equivalence class of formulas, all of which are equivalent to P, and P as a UCLA proposition consisting of all the points of a model which are members of P.

The rest of this paper assumes an algebraic framework. The propositions are interpreted as elements of the carrier set of an algebra, in our case a Boolean algebra, which is a lattice under intersection $\cap$, union $\cup$, and a form of set complement $\stackrel{\bullet\circ}{\neg}$, augmented with local modal operators and distributed operators. The new form of complement $\stackrel{\bullet\circ}{\neg}$ in place of set complement was due to the extra points added to the domains.

We let $f : \mathsf{h} \rightharpoonup \mathsf{k}$ refer to an arc of a Distributed Logic graph from node h to node k. As such, there are several operators that can be associated with the arc as well as the relation, usually depicted similarly, i.e., $\mathcal{F} : \mathsf{h} \rightharpoonup \mathsf{k}$ where the small case italic f is linked semantically to the interpreting relation $\mathcal{F}$ in a script font.

In the following table, the type of each operator is of the form $\rho \mapsto \sigma$ where $\rho, \sigma \in \{\wedge, \vee\}$. The first line of any operator gives its type and its semantic evaluation condition. The second line gives the canonical definition of the interpreting relation where the x, y are maximal filters of the lattices that support the operator. The elements of the lattice are a, b. The third line shows the closure properties. The wedge sum is only for completeness, we do not use it for the Thomason-Goldblatt theorem (as of this paper's writing). We have annotated the set theoretical membership relation $\in$ with a locality to show in which frame the domain of the membership relation lives, usually one of $\in_{\mathsf{h}}$ or $\in_{\mathsf{k}}$. We also annotate set inclusion in the sequel, when it aids understanding, with the domains that are involved, i.e, $\subseteq$ becomes $\subseteq_{\mathsf{h}}$ or $\subseteq_{\mathsf{k}}$.

Table of all Two-Place Galois Operators

Menu A

$\wedge\mapsto\wedge$ $y \in_{\mathsf{k}} [f^{\flat\cdot}\rangle P$ iff $\forall x(x \in_{\mathsf{h}} P$ or $\mathcal{F}^{\flat\cdot}xy)$
$\mathcal{F}^{\flat\cdot}xy$ iff $\exists a(a \notin_{\mathsf{h}} x$ and $[f^{\flat}\rangle a \in_{\mathsf{k}} y)$
null and wedge sum

$\vee\mapsto\vee$ $y \in_{\mathsf{k}} [f^{\circ\cdot}\rangle P$ iff $\exists x(x \in_{\mathsf{h}} P$ and $\mathcal{F}^{\circ\cdot}xy)$
$\mathcal{F}^{\circ\cdot}xy$ iff $\forall a(a \notin_{\mathsf{h}} x$ or $[f^{\circ}\cdot\rangle a \in_{\mathsf{k}} y)$
smash product

$\vee\mapsto\wedge$ $y \in_{\mathsf{k}} [f^{\perp\cdot}\rangle P$ iff $\forall x(x \notin_{\mathsf{h}} P$ or $\mathcal{F}^{\perp\cdot}xy)$
$\mathcal{F}^{\perp\cdot}xy$ iff $\exists a(a \in_{\mathsf{h}} x$ and $[f^{\perp}\cdot\rangle a \in_{\mathsf{k}} y)$
smash product

$\wedge\mapsto\vee$ $y \in_{\mathsf{k}} [f^{?\cdot}\rangle P$ iff $\exists x(x \notin_{\mathsf{h}} P$ and $\mathcal{F}^{?\cdot}xy)$
$\mathcal{F}^{?\cdot}xy$ iff $\forall a(a \in_{\mathsf{h}} x$ or $[f^{?}\cdot\rangle a \in_{\mathsf{k}} y)$
null sum

Menu B

$\wedge\mapsto\wedge$ $y \in_{\mathsf{k}} [f^{\square\cdot}\rangle P$ iff $\forall x(x \in_{\mathsf{h}} P$ or $\neg\mathcal{F}^{\square\cdot}xy)$
$\mathcal{F}^{\square\cdot}xy$ iff $\forall a(a \in_{\mathsf{h}} x$ or $[f^{\square}\cdot\rangle a \notin_{\mathsf{k}} y)$
smash product and null sum

$\vee\mapsto\vee$ $y \in_{\mathsf{k}} [f^{\sharp\cdot}\rangle P$ iff $\exists x(x \in_{\mathsf{h}} P$ and $\neg\mathcal{F}^{\sharp\cdot}xy)$
$\mathcal{F}^{\sharp\cdot}xy$ iff $\exists a(a \in_{\mathsf{h}} x$ and $[f^{\sharp}\cdot\rangle a \notin_{\mathsf{k}} y)$
smash product and wedge sum

$\vee\mapsto\wedge$ $y \in_{\mathsf{k}} [f^{!\cdot}\rangle P$ iff $\forall x(x \notin_{\mathsf{h}} P$ or $\neg\mathcal{F}^{!\cdot}xy)$
$\mathcal{F}^{!\cdot}xy$ iff $\forall a(a \notin_{\mathsf{h}} x$ or $[f^{!}\cdot\rangle a \notin_{\mathsf{k}} y)$
smash product

$\wedge\mapsto\vee$ $y \in_{\mathsf{k}} [f^{\star\cdot}\rangle P$ iff $\exists x(x \notin_{\mathsf{h}} P$ and $\neg\mathcal{F}^{\star\cdot}xy)$
$\mathcal{F}^{\star\cdot}xy$ iff $\exists a(a \notin_{\mathsf{h}} x$ and $[f^{\star}\cdot\rangle a \notin_{\mathsf{k}} y)$
null sum

Menu C

$\wedge\mapsto\wedge$ $x \in_{\mathsf{h}} [f^{\flat}\rangle Q$ iff $\forall y(y \in_{\mathsf{k}} Q$ or $\mathcal{F}^{\flat}xy)$
$\mathcal{F}^{\flat}xy$ iff $\exists b(b \notin_{\mathsf{k}} y$ and $[f^{\flat}\rangle b \in_{\mathsf{h}} x)$
null and wedge sum

$\vee\mapsto\vee$ $x \in_{\mathsf{h}} [f^{\circ}\rangle Q$ iff $\exists y(y \in_{\mathsf{k}} Q$ and $\mathcal{F}^{\circ}xy)$
$\mathcal{F}^{\circ}xy$ iff $\forall b(b \notin_{\mathsf{k}} y$ or $[f^{\circ}\rangle b \in_{\mathsf{h}} x)$
smash product

$\vee\mapsto\wedge$ $x \in_{\mathsf{h}} [f^{\perp}\rangle Q$ iff $\forall y(y \notin_{\mathsf{k}} Q$ or $\mathcal{F}^{\perp}xy)$
$\mathcal{F}^{\perp}xy$ iff $\exists b(b \in_{\mathsf{k}} y$ and $[f^{\perp}\rangle b \in_{\mathsf{h}} x)$
smash product and wedge sum

$\wedge\mapsto\vee$ $x \in_{\mathsf{h}} [f^{?}\rangle Q$ iff $\exists y(y \notin_{\mathsf{k}} Q$ and $\mathcal{F}^{?}xy)$
$\mathcal{F}^{?}xy$ iff $\forall b(b \in_{\mathsf{k}} y$ or $[f^{?}\rangle b \in_{\mathsf{h}} x)$
null sum

Menu D

$\wedge\mapsto\wedge$ $x \in_{\mathsf{h}} [f^{\square}\rangle Q$ iff $\forall y(y \in_{\mathsf{k}} Q$ or $\neg\mathcal{F}^{\square}xy)$
$\mathcal{F}^{\square}xy$ iff $\forall b(b \in_{\mathsf{k}} y$ or $[f^{\square}\rangle b \notin_{\mathsf{h}} x)$
null sum and wedge sum

$\vee\mapsto\vee$ $x \in_{\mathsf{h}} [f^{\sharp}\rangle Q$ iff $\exists y(y \in_{\mathsf{k}} Q$ and $\neg\mathcal{F}^{\sharp}xy)$
$\mathcal{F}^{\sharp}xy$ iff $\exists b(b \in_{\mathsf{k}} y$ and $[f^{\sharp}\rangle b \notin_{\mathsf{h}} x)$
smash product and wedge sum

$\vee\mapsto\wedge$ $x \in_{\mathsf{h}} [f^{!}\rangle Q$ iff $\forall y(y \notin_{\mathsf{k}} Q$ or $\neg\mathcal{F}^{!}xy)$
$\mathcal{F}^{!}xy$ iff $\forall b(b \notin_{\mathsf{k}} y$ or $[f^{!}\rangle b \notin_{\mathsf{h}} x)$
smash product

$\wedge\mapsto\vee$ $x \in_{\mathsf{h}} [f^{\star}\rangle Q$ iff $\exists y(y \notin_{\mathsf{k}} Q$ and $\neg\mathcal{F}^{\star}xy)$
$\mathcal{F}^{\star}xy$ iff $\exists b(b \notin_{\mathsf{k}} y$ and $[f^{\star}\rangle b \notin_{\mathsf{h}} x)$
null sum

Menus A and B are the "backwards" operators, so termed because the evaluating relation is read from the second position to first position in operator's evaluation.

2.1 Example Interpretations of the Operators

The table below has the operators paired by forwards and backwards on the same line, and two rows per section are wn-pointed-Boolean negations of each other (see section below on additional points and the new Boolean negation). The table that even though the operators can be paired up via negations, their evaluation conditions are certainly not classical negations of each other.

op	Evaluation Condition	op	Evaluation Condition
$[-^{\diamond}\rangle$	$x \in_{\mathsf{h}} [f^{\diamond}\rangle Q$ iff $\exists y(y \in_{\mathsf{k}} Q$ and $\mathcal{F}^{\diamond}xy)$	$[-^{\diamond}\cdot\rangle$	$y \in_{\mathsf{k}} [f^{\diamond}\cdot\rangle P$ iff $\exists x(x \in_{\mathsf{h}} P$ and $\mathcal{F}^{\diamond\cdot}xy)$
$[-^{\square}\rangle$	$x \in_{\mathsf{h}} [f^{\square}\rangle Q$ iff $\forall y(y \in_{\mathsf{k}} Q$ or $\neg\mathcal{F}^{\square}xy)$	$[-^{\square}\cdot\rangle$	$y \in_{\mathsf{k}} [f^{\square}\cdot\rangle P$ iff $\forall x(x \in_{\mathsf{h}} P$ or $\neg\mathcal{F}^{\square\cdot}xy)$
$[-^{\flat}\rangle$	$x \in_{\mathsf{h}} [f^{\flat}\rangle Q$ iff $\forall y(y \in_{\mathsf{k}} Q$ or $\mathcal{F}^{\flat}xy)$	$[-^{\flat}\cdot\rangle$	$y \in_{\mathsf{k}} [f^{\flat}\cdot\rangle P$ iff $\forall x(x \in_{\mathsf{h}} P$ or $\mathcal{F}^{\flat\cdot}xy)$
$[-^{\sharp}\rangle$	$x \in_{\mathsf{h}} [f^{\sharp}\rangle Q$ iff $\exists y(y \in_{\mathsf{k}} Q$ and $\neg\mathcal{F}^{\sharp}xy)$	$[-^{\sharp}\cdot\rangle$	$y \in_{\mathsf{k}} [f^{\sharp}\cdot\rangle P$ iff $\exists x(x \in_{\mathsf{h}} P$ and $\neg\mathcal{F}^{\sharp\cdot}xy)$
$[-^{\perp}\rangle$	$x \in_{\mathsf{h}} [f^{\perp}\rangle Q$ iff $\forall y(y \notin_{\mathsf{k}} Q$ or $\mathcal{F}^{\perp}xy)$	$[-^{\perp}\cdot\rangle$	$y \in_{\mathsf{k}} [f^{\perp}\cdot\rangle P$ iff $\forall x(x \notin_{\mathsf{h}} P$ or $\mathcal{F}^{\perp\cdot}xy)$
$[-^{\star}\rangle$	$x \in_{\mathsf{h}} [f^{\star}\rangle Q$ iff $\exists y(y \notin_{\mathsf{k}} Q$ and $\neg\mathcal{F}^{\star}xy)$	$[-^{\star}\cdot\rangle$	$y \in_{\mathsf{k}} [f^{\star}\cdot\rangle P$ iff $\exists x(x \notin_{\mathsf{h}} P$ and $\neg\mathcal{F}^{\star\cdot}xy)$
$[-^{!}\rangle$	$x \in_{\mathsf{h}} [f^{!}\rangle Q$ iff $\forall y(y \notin_{\mathsf{k}} Q$ or $\neg\mathcal{F}^{!}xy)$	$[-^{!}\cdot\rangle$	$y \in_{\mathsf{k}} [f^{!}\cdot\rangle P$ iff $\forall x(x \notin_{\mathsf{h}} P$ or $\neg\mathcal{F}^{!\cdot}xy)$
$[-^{?}\rangle$	$x \in_{\mathsf{h}} [f^{?}\rangle Q$ iff $\exists y(y \notin_{\mathsf{k}} Q$ and $\mathcal{F}^{?}xy)$	$[-^{?}\cdot\rangle$	$y \in_{\mathsf{k}} [f^{?}\cdot\rangle P$ iff $\exists x(x \notin_{\mathsf{h}} P$ and $\mathcal{F}^{?\cdot}xy)$

The relations, here generically, $\mathcal{F} : \mathsf{h} \rightharpoonup \mathsf{k}$, used in the evaluation of the operators above uses two localities, h and k. These are variables in that any actual FPGA application will fill those localities in by components using as many as are needed. Note that the backward operators have their own relation, i.e., $[f^{\diamond}\rangle$ uses $\mathcal{F}^{\diamond}$ and $[f^{\diamond}\cdot\rangle$ uses $\mathcal{F}^{\diamond\cdot}$. In bog standard modal logic, these two relations would be the same. Due to the pointed domains used in this paper, this is no longer the case.

The relation is typically a *concurrence* relation in that it lists the pairs of states, one from each of h and k that can occur simultaneously. Taking the set complement of this relation then gives a *perpendicular* relation for pairs of states; that is, two states are in the relation if they cannot simultaneously occur. The relation, however, can be interpreted in other ways depending upon the application. Distributed Logic does not care and only manages the mechanics so that an FPGA application determines the ultimate meanings of the terms used in the logic.

There can be many flavors of concurrence. One collection of examples is caused by shared resources. Assume there is a resource, say, a port or device or memory. Let k_i be a collection of components that have access to the resource. A controller at h must prevent simultaneous use of the resource by k_i and k_j for $i \neq j$. This spawns a concurrence relation specific to the use of the resource between h and each k_i. There are also concurrence relations strictly among the k_i. The consequence of this scenario is that the logic for the controller will have many modal operators with which to work.

Applications in the broad sense, not necessarily FPGA or computational applications, are not precluded from Distributed Logic. There are other various extensions we have used. The relations can be swapped for neighborhood maps; that is, for $\mathcal{F}xS$ where $\mathcal{F}$ is a function in relation form, x is a state at h, and S is a set of states at k. We have also extended the modal operators to be intensional operators with two input places, the relation $\mathcal{F}$ becomes three placed and distributed among three localities, say, h, k, and l.

In general, we can view the distributed modal operators as short hand for their first-order modeling conditions. As such, they abstract away details that do not effect the reasoning. Since there is a finite collection of operators, there is a finite collection of properties expressible using them. That said, it is always possible to define new relations on an underlying FPGA application and evaluate those operators with respect to the new relations. So the limitation is loosened considerably.

For the rest of this section, we will mostly ignore the backwards looking operators. Any application can simply reorient with respect to the direction of the relations and hence, using the terms proximal and distal for two localities, reorienting simply means swapping proximal and distal.

2.2 $[-^{\diamond}\rangle, [-^{\square}\rangle$: Possibility and Necessity

To say $x \in_{\mathsf{h}} [f^{\diamond}\rangle Q$ is to say that x at h can concur with at least one state at k that makes Q true. This is useful in expressing properties of controllers where the component at k must access some resource but only if k is in a state that makes Q true. The controller state x is then true just in case there is a state in Q under the concurrence relation $\mathcal{F}^{\diamond}$. The set $[f^{\diamond}\rangle Q$ is all such states x with the above property.

To say $x \in_{\mathsf{h}} [f^{\square}\rangle Q$ is to say that if x at h concur with y at k, then that y must make Q true. This too is useful for controllers where one wants to express that states in k that concur under the relation $\mathcal{F}^{\square}$ with x must make Q true. The set $[f^{\square}\rangle Q$ is all such states x with the above property.

2.3 $[-^{\flat}\rangle, [-^{\sharp}\rangle$: Flat and Sharp

Interpret the "flat" relation $\mathcal{F}^{\flat}xy$ as y in component k is inspectable from x in component h. Let Q be a collection of trusted states in k that do not require inspection. Hence $[f^{\flat}\rangle Q$ is that collection of states of h that can inspect the untrusted states from k that are not in Q.

Like the previous case, interpret the relation $\mathcal{F}^{\sharp}xy$ as y in component k is inspectable from x in component h. The formula $[f^{\sharp}\rangle Q$ represents those x such that there is an untrusted state in Q that x can inspect.

2.4 $[f^{!}\rangle, [f^{?}\rangle$: Exclaim and Question

The operator $[f^{!}\rangle$ has the valuation condition

$$x \in_{\mathsf{h}} [f^{!}\rangle Q \text{ iff } \forall y(y \notin_{\mathsf{k}} Q \text{ or } \neg\mathcal{F}^{!}xy).$$

This is best seen if we use the (classical logic) equivalent:

$$x \in_{\mathsf{h}} [f^{!}\rangle Q \text{ iff } \forall y(y \in_{\mathsf{k}} Q \text{ implies } \neg\mathcal{F}^{!}xy).$$

The evaluation condition shows that this operator "localizes" the compliment of the relation to focus on the states in Q. This notion of localization is not the same as the localities we use in Distributed Logic but rather has more of philosophical logic sense. The following operator $[f^{\perp}\rangle$ also has this localization effect.

The operator $[f^{?}\rangle$ has the valuation condition

$$x \in_{\mathsf{h}} [f^{?}\rangle Q \text{ iff } \exists y(y \notin_{\mathsf{k}} Q \text{ and } \mathcal{F}^{?}xy).$$

$[f^{?}\rangle Q$ can be seen to be the recorder of a counter-examples to Q under the relation $\mathcal{F}^{?}$. Concentrating on the collection of states Q, $[f^{?}\rangle Q$ is recording those states that are counter-examples to Q being true and that satisfy the relation $\mathcal{F}^{?}$, where $\mathcal{F}^{?}xy$ indicates a state x that "witnesses" Q being true under the relation.

2.5 $[f^{\perp}\rangle$, $[f^{\star}\rangle$: Perp and Star

The operator $[f^{\perp}\rangle$ has the valuation condition

$$x \in_{\mathsf{h}} [f^{\perp}\rangle Q \text{ iff } \forall y(y \notin_{\mathsf{k}} Q \text{ or } \mathcal{F}^{\perp}xy).$$

This is best seen if we use the (classical logic) equivalent:

$$x \in_{\mathsf{h}} [f^{\perp}\rangle Q \text{ iff } \forall y(y \in_{\mathsf{k}} Q \text{ implies } \mathcal{F}^{\perp}xy).$$

The evaluation condition for this operator also "localizes" the relation.

Think of Q as being a collection of states at locality k, then $[f^{\perp}\rangle Q$ is the view of those states from h such that every one of the states of $[f^{\perp}\rangle Q$ is related to all the states in Q under the distributed relation $\mathcal{F}^{\perp}$. This is useful for cutting down the states required for some aspect of security represented by the relation $\mathcal{F}^{\perp}$ to only those within Q and their inverse image under $\mathcal{F}^{\perp}$, i.e., the states in $[f^{\perp}\rangle Q$.

The operator $[f^{\star}\rangle$ has the valuation condition

$$x \in_{\mathsf{h}} [f^{\star}\rangle Q \text{ iff } \exists y(y \notin_{\mathsf{k}} Q \text{ and } \neg\mathcal{F}^{\star}xy).$$

$[f^{\star}\rangle Q$ can be seen to be the recorder of a counter-example to Q and the relation $\mathcal{F}^{\star}$. Concentrating on he collection of states Q, $[f^{\star}\rangle Q$ is recording those states that are counter-examples to Q being true and that fail the relation $\mathcal{F}^{\star}$, where $\neg\mathcal{F}^{\star}xy$ indicates a state x that "witnesses" Q being false under the relation.

2.6 Security Example for $[f^{\perp}\rangle$

Let L and H be sets of high and low traces. The collection of all traces is $T = L \times H$. There is only a single locality involved here, namely the locality associated with T. The relation at this locality is $\mathcal{F}^{\perp}$, which in this instance is not a distributed relation. Let $X \subseteq T$ represent a system in the sense that it tells us which traces may be paired together. Let x_1 refer to the L side of a pair $x \in T$ and x_2 refer to the H side

$$\mathcal{F}^{\perp}xy \text{ iff } \langle x_1, y_2\rangle \in X$$

Hence, for all $x \in X$,

$$\mathcal{F}^{\perp}xx.$$

Using the definition of $[f^{\perp}\rangle Q$ and $[f^{\perp}\cdot\rangle P$, we get the following

$$[f^{\perp}\rangle Q = \{x \mid \forall y(y \in Q \text{ implies } \mathcal{F}^{\perp}xy)\}, \quad [f^{\perp}\cdot\rangle P = \{y \mid \forall x(x \in P \text{ implies } \mathcal{F}^{\perp}xy)\}.$$

Subsets of T are then propositions for things we can say about the system X. They do not have to be true things.

We have the following Galois connection condition

$$P \subseteq [f^{\perp}\rangle Q \text{ iff } Q \subseteq [f^{\perp}\cdot\rangle P.$$

This is equivalent to

$$P \subseteq [f^{\perp}\rangle\,[f^{\perp}\cdot\rangle P, \qquad Q \subseteq [f^{\perp}\cdot\rangle\,[f^{\perp}\rangle\, Q.$$

Using the Galois connection condition, the two operators together act like closure operators.

Corollary 2.6.1

$$[f^{\perp}\rangle Q = [f^{\perp}\rangle\,([f^{\perp}\cdot\rangle([f^{\perp}\rangle\, Q)), \qquad [f^{\perp}\cdot\rangle P = [f^{\perp}\cdot\rangle([f^{\perp}\rangle\,([f^{\perp}\cdot\rangle P)).$$

A proposition that can be said about the system X is merely some subset $P \subseteq T$. It need not be a proposition that is everywhere true of X. The latter requires that $P \subseteq X$. We can now state what it means for propositions to be non-low-interfering or non-high-interfering.

Definition 2.6.2 For $P, Q \subseteq T$, P is non-low-interfering and Q is non-high-interfering iff (respectively)

$$P \subseteq [f^{\perp}\rangle\, P, \qquad Q \subseteq [f^{\perp}\cdot\rangle Q.$$

Now it is easy to state that X is non-low-interfering (non-high-interfering) since $X \subseteq T$. We simply claim $X \subseteq [f^{\perp}\rangle\, X$ ($X \subseteq [f^{\perp}\cdot\rangle X$). X is separated iff X is non-low-high-interfering.

Note that it is always the case that

$$[f^{\perp}\rangle\, X \subseteq X, \qquad [f^{\perp}\cdot\rangle X \subseteq X.$$

This happens simply because X acts as a ambient universal set from each non-empty P is taken as a subset.

Note also that if $X \neq [f^{\perp}\cdot\rangle X$ or $X \neq [f^{\perp}\rangle\, X$ then

$$X \not\subseteq [f^{\perp}\cdot\rangle X \quad \text{ or } \quad X \not\subseteq [f^{\perp}\rangle\, X.$$

Unwinding the definitions, we get

$$\exists y \in X \text{ and } \exists x \in X \text{ and } \neg \mathcal{F}^{\perp} xy.$$

The two operators will collapse into a single operator if the underlying relation, in our case, $\mathcal{F}^{\perp}$, is symmetric. This argues against attempting to get them to collapse by imposing closure conditions on the algebras.

3. USEFUL NEW CONSTRUCTIONS

The two new constructions for modal logic frames are *smash product* and *null sum*. These represent a jump in the distributed structure for the logic in that the modal operators can now be applied to aggregates of localities by lifting their construction on the individual localities to the aggregates. This is done preserving the distributed structure.

In the sequel, the usage of the term *canonical* refers to some condition on frames when the frames arise from Boolean lattices, usually with Galois operators on the lattices. This usually means that any points of the frame are maximal filters.

3.1 Additional Points for Domains

A domain at single locality is called a *base domain*. The new constructions requires some additional points to be added to the base domains and then the sets are constructed inductively for aggregates. Specifically, the new points are contained in sets for the domain at every locality h and in aggregates of localities described by smash products and null sums. The new points at a locality h are the *well-points*, denoted $H^\bullet$, and the *null-points*, denoted H°. Each is a set of points. For base domains, these sets are singletons. For aggregated domains, these sets are built up inductively during the smash product and null sum constructions. It is always the case that

$$H^\bullet \cap H^\circ = \emptyset.$$

Both $H^\bullet$ and H° can be thought of as containing error states. It is easiest to discuss using base domains; these domains use singleton sets: $H^\bullet = \{\bullet_H\}$ and $H^\circ = \{\circ_H\}$. $\bullet_H$ is the state that makes every proposition true and $\circ_H$ is the state that makes every proposition false. Considering the *always true proposition* and *always false proposition*, $\bullet_H$ represents a meta-stable state, and $\circ_H$ represents a state which essentially has no value, something like a tri-state where the third state is high impedance.

Any component will probably have states that are considered error states. The important feature is that these error states are relative to the component, hence $\bullet_H$ and $\circ_H$ exist for each component h. The way these relate to connections within a component is that connections can only be referenced by propositions. Hence a connection labeled x comes with at least two propositions, $x = 0$ and $x = 1$. The state $\bullet_H$ turns both propositions true and $\circ_H$ turns both propositions false.

The consequence of treating states in this way is that states are not simply a readout of all the values on all the connections within a component. They value propositions, not connections. The propositions are the entities that can declare a value on a connection. This makes possible the representation of error states.

Lattice theoretically, $\bullet_H$ is the non-proper filter containing all elements of the lattice. $\circ_H$ is the non-proper filter containing no elements of the lattice. Let $\bot_\mathsf{h}$ and $\top_\mathsf{h}$ represent the bottom and top elements of a Boolean algebra (as a lattice) at locality h. $\bullet_H$ is the upper set determined by the bottom element and $\circ_H$ is the empty set. In symbols, $\bullet_H = \uparrow\bot_\mathsf{h}$ and $\circ_H = A_\mathsf{h} - \downarrow\top_\mathsf{h} = \emptyset$ where A_h is the carrier set of the lattice. Hence $\uparrow\bot_\mathsf{h}$ is the collection $\{a \mid \bot_\mathsf{h} \leq a\}$ and $\downarrow\top_\mathsf{h} = \{a \mid a \leq \top_\mathsf{h}\}$.

All propositions are considered well-pointed and null-pointed. By that we mean that for any proposition P at h,

$$P = (H^\bullet \cup P) - H^\circ.$$

That is adding the points in $H^\bullet$ to P adds no new points, and subtracting the points in H° takes away no points. This is under the assumption above that $H^\bullet \cap H^\circ = \emptyset$.

The constructions $\otimes$ and $\parr$ below require that we generalize pointed domains or *universes*. The latter term is handy for set theoretic considerations.

Definition 3.1.1 Every ground domain H comes with a well-point and null-point sets. For this paper, these are

$$H^\bullet = \{\bullet_H\} \qquad (H_1 \otimes H_2)^\bullet = \{\langle \bullet_{H_1}, \bullet_{H_2}\rangle\} \qquad (H_1 \parr H_2)^\bullet = \{\langle \bullet_{H_1}, \circ_{H_2}\rangle, \langle \circ_{H_1}, \bullet_{H_2}\rangle\}.$$

Every domain H comes with a null-point set. For this paper, these are

$$H^\circ = \{\circ_H\} \qquad (H_1 \otimes H_2)^\circ = (\{\circ_{H_1}\} \times H_2) \cup (H_1 \times \{\circ_{H_2}\}) \qquad (H_1 \parr H_2)^\circ = \{\langle \circ_{H_1}, \circ_{H_2}\rangle\}.$$

The bounds are, for any H whether of the form $H_1 \otimes H_2$ or $H_1 \parr H_2$ or neither (base domains),

$$\bot^{\mathbb{H}} = H^\bullet \qquad \top^{\mathbb{H}} = H - H^\circ.$$

These constructions must be iterated:

Definition 3.1.2 Iterated point sets are inductively defined via the following two clauses.

Base Case:

$$H^\bullet = \{\bullet_H\} \quad H^\circ = \{\circ_H\}.$$

Inductive step:

$$(H_1 \otimes H_2)^\bullet = {H_1}^\bullet \times {H_2}^\bullet \quad (H_1 \otimes H_2)^\circ = ({H_1}^\circ \times H_2) \cup (H_1 \times {H_2}^\circ).$$

and

$$(H_1 \parr H_2)^\bullet = ({H_1}^\bullet \times {H_2}^\circ) \cup ({H_1}^\circ \times {H_2}^\bullet) \quad (H_1 \parr H_2)^\circ = {H_1}^\circ \times {H_2}^\circ.$$

3.2 Smash Product

Smash product represents a form of tensor product for pointed sets. The smash product $P_1 \otimes P_2$ of two propositions P_1 and P_2 of domains h_1 and h_2 respectively allows us to treat $P_1 \otimes P_2$ as one unit while the internal structure of the domains h_1 and h_2 is respected. The propositions P_1 and P_2 are considered conjoined propositions in that a state of the smash product produces a value for each P_1 and P_2.

Definition 3.2.1 (xdefinition:smash_product)

$$\langle x, y\rangle \in H_1 \otimes H_2 \text{ iff } \langle x, y\rangle \in (H_1 - H_1^\bullet) \times (H_2 - H_2^\bullet) \cup H_\otimes^\bullet.$$

and for Q_i well- and null-pointed,

$$\langle x, y\rangle \in P_1 \otimes P_2 \text{ iff } \langle x, y\rangle \in (P_1 - H_1^\bullet) \times (P_2 - H_2^\bullet) \cup H_\otimes^\bullet.$$

As can be seen from $P_1 \otimes P_2$, the points $H_i^\bullet$ are removed before the cross-product and new points are added afterwards, but only the $-^\bullet$ points are used in this fashion. The $-^\circ$ points in each domain are left in the cross-products.

3.3 Null Sum

The null sum represents a form of disjoint sum for pointed sets. The null sum $P_1 \oslash P_2$ of P_1 and P_2 of domains h_1 and h_2 respectively treats P_1 and P_2 as either one or the other at a state but not both. A state here is taken from one of h_1 or h_2. A state from h_1 that makes P true in $P_1 \oslash P_2$ does not make P_2 false since to make P_2 false requires a state from h_2. This represents that a state in one component, h_1, cannot make a proposition about another component h_2 true or false simply because states in component h_1 are independent of states in h_2.

This representation resulted from the problems of getting $[f^\star\rangle$ to work well with lattice bounds and have closure under some kind of sum representation. It is tempting to let the universe for UCLA propositions be $\top^{\mathbb{H}}$ and evaluating

$$x \notin_{\mathsf{h}} Q \text{ iff } x \in_{\mathsf{h}} \top^{\mathbb{H}} - Q.$$

The assumption is that x is in the universe, so $x \notin_{\mathsf{h}_1} H^\circ$ since $H^\circ \cap \top^{\mathbb{H}} = \emptyset$. The condition for the operator $[f^\star\rangle$ is

$$x \in_{\mathsf{h}} [f^\star\rangle Q \text{ iff } \exists y(y \notin_{\mathsf{k}} Q \text{ and } \neg \mathcal{F}^\star xy),$$

where $f : \mathsf{h} \rightharpoonup \mathsf{k}$. We could not now rely on a point not being in $\top^{\mathbb{K}}$. This becomes a problem when $Q = \top^{\mathbb{K}}$ because then $y \in_{\mathsf{k}} K - K^\circ$ and that $y \notin_{\mathsf{k}} \top^{\mathbb{K}}$, yet $K - K^\circ = \top^{\mathbb{K}}$. The clear implication is that the universe must contain K°.

Definition 3.3.1

$$\langle x, y\rangle \in H_1 \oslash H_2 \text{ iff } \langle x, y\rangle \in (H_1 \times H_2^\circ) \cup (H_1^\circ \times H_2).$$

We also have for P_i well- and null-pointed,

$$\langle x, y\rangle \in P_1 \oslash P_2 \text{ iff } \langle x, y\rangle \in (P_1 \times H_2^\circ) \cup (H_1^\circ \times P_2).$$

This representation differs from the usual definition of a disjoint sum where the markers from H_i° are not usually from the domains they are marking and the markers for H_1 are from H_2° not H_1°. This allowed us to give the markers frame theoretic properties. The markers are better termed *tickets*. A consequence of this representation that $\langle x, y\rangle \in P_1 \oslash P_2$ results in either $x \in_{\mathsf{h}_1} P_1$ and $y \notin_{\mathsf{h}_2} P_2$ or $x \notin_{\mathsf{h}_1} P_1$ and $y \in_{\mathsf{h}_2} P_2$ simply because both P_1 and P_2 are null-pointed.

3.4 Galois Operators, Smash Products, and Null Sums

One of the main reasons for distributed logic is that states in each component are independent in terms of their construction but not independent in terms of their behavior if components are interacting. The constructions smash product and null sum allow one to transcend the distributed structure of a device in a controlled manner.

All domains h have both $H^\bullet$ and H° points. The constructions and some special frame conditions for each modal operator keep the these points from interfering with each other. Some of the operators satisfy commutation across smash products and others satisfy commutation across null sums in the sense

$$[f\rangle\,(Q_1 \otimes Q_2) = [f_1\rangle\,Q_1 \otimes [f_2\rangle\,Q_2 \qquad\qquad [f\rangle\,(Q_1 \oslash Q_2) = [f_1\rangle\,Q_1 \oslash [f_2\rangle\,Q_2$$

where $[f\rangle$ is a generic distributed modal operator, $\mathcal{F}$ is some Boolean combination of $\mathcal{F}_1$ and $\mathcal{F}_2$, and the f in the operator $[f\rangle$ of the logic is linked via valuations to $\mathcal{F}$ in the frame. Each frame for a particular operator will fix the Boolean combination to be either conjunction or disjunction of the (possibly complemented) relations. These commutation properties are commonly termed *preservation properties*.

The sets $H^\bullet$ and H° allow for the constructions to be iterated. With the original Thomason-Goldblatt theorem, there was only the disjoint sum construction. Since this was allowed to be of any arity and because there was only a single construction, iteration was never an issue. However here, where there are two constructions, iteration does become an issue because of the these special sets. They must be accounted for because $(H_1 \otimes H_2)^\bullet$ and $(H_1 \oslash H_2)^\bullet$ must involve $H_1^\bullet$ and $H_2^\bullet$. Similarly, H_1° and H_2°.

Any UCLA proposition P, i.e., a members of a set lattice in a frame, is called *wn-pointed* (well- and null-pointed) at h if

$$H^\bullet \subseteq_{\mathsf{h}} P \qquad H^\circ \cap P = \emptyset.$$

The smash product and null sum preserve wn-pointedness.

4. NEW BOOLEAN COMPLEMENT

The above characterization clearly will not allow the Boolean set-complement $\neg Q$ to be wn-pointed if Q is wn-pointed. The reason is clear, if $H^\bullet \subseteq_{\mathsf{h}} P$, then $H^\bullet \not\subseteq_{\mathsf{h}} H - P$, yet this latter is the typical Boolean negation of set lattices. We need a new negation.

4.1 The New Negation

Definition 4.1.1 $\overset{\bullet\circ}{\neg} Q$ is the *wn-complement* or *wn-negation* defined by

$$\overset{\bullet\circ}{\neg} Q \overset{def}{=} \top^{\mathbb{H}} \overset{\bullet\circ}{-} Q \overset{def}{=} (\top^{\mathbb{H}} - Q) \cup H^{\bullet}.$$

An alternate definition is

$$x \in_{\mathsf{h}} \overset{\bullet\circ}{\neg} Q \text{ iff } (x \notin_{\mathsf{h}} Q \text{ and } x \notin_{\mathsf{h}} H^{\circ}) \text{ or } x \in_{\mathsf{h}} H^{\bullet}.$$

The alternate definition points out that the top of the set lattice $\top^{\mathbb{H}}$ is not the universe H since $x \notin_{\mathsf{h}} Q$ allows for $x \in_{\mathsf{h}} H^{\circ}$. This matters in places where non-membership in a UCLA proposition is used.

Lemma 4.1.2 *UCLA sets produced from UCLA sets using* $\overset{\bullet\circ}{\neg}$ *are wn-pointed.*

Theorem 4.1.3 *Assume* Q*,* Q_1*, and* Q_2 *are well- and null-pointed at locality* h*, then* $\overset{\bullet\circ}{\neg}$ *is a Boolean negation:*

$$\overset{\bullet\circ}{\neg}(Q_1 \wedge Q_2) = \overset{\bullet\circ}{\neg} Q_1 \vee \overset{\bullet\circ}{\neg} Q_2 \qquad \overset{\bullet\circ}{\neg}(Q_1 \vee Q_2) = \overset{\bullet\circ}{\neg} Q_1 \wedge \overset{\bullet\circ}{\neg} Q_2$$
$$Q \wedge \overset{\bullet\circ}{\neg} Q = \bot_{\mathsf{h}} \qquad Q \vee \overset{\bullet\circ}{\neg} Q = \top_{\mathsf{h}}$$
$$\overset{\bullet\circ}{\neg}\overset{\bullet\circ}{\neg} Q = Q$$

4.2 Complements for the Galois Operators

[2] has the best explanation of the Goldblatt-Thomason theorem, there they are concentrating on disjoint sums. Their rendition says each frame can be decomposed into a collection of point-generated frames. By that they mean that for each frame, pick a point and follow the relation from that point including all the visited points. This gives a collection of point-generated frames for each frame. Those get collected into a disjoint sum. For us, we will use a null sum. The representation we will ultimately use is

$$\vec{x} \in_{\mathsf{h}} \Pi_i H_i, \text{ such that } \exists! j \forall i (i \neq j \text{ implies } \vec{x}_i \in_{\mathsf{h}_i} H_i^{\circ}).$$

This leaves the possibility that for $i = j$, $\vec{x}_j \notin_{\mathsf{h}_i} H_j^{\circ}$.

Consider the proofs for $\overset{\bullet\circ}{\neg}[f^{\perp}\rangle Q = [f^{\star}\rangle \overset{\bullet\circ}{\neg} Q$ (see sequel). The main supporting Lemma says that $\mathcal{F}^{\perp}$ and $\mathcal{F}^{\star}$ are the same relation as long as none of the special points are involved. If we do include those points, then the two relations are not the same.

It might be best to consider $\overset{\bullet\circ}{\neg}$ to be a morphism between frames, one with $\mathcal{F}^{\perp}$ and the other with $\mathcal{F}^{\star}$. This is also somewhat dodgy since $\overset{\bullet\circ}{\neg}$ does not preserve those relations on the nose due to the special points. Also, $\overset{\bullet\circ}{\neg}$ is not mapping points, it is mapping algebras. So it might be better to see it as

$$\overset{\bullet\circ}{\neg} : ((H, \mathcal{H}, \mathbb{H}), \mathcal{F}^{\perp}, (K, \mathcal{K}, \mathbb{K})) \rightarrow ((H, \mathcal{H}, \mathbb{H}), \mathcal{G}^{\star}, (K, \mathcal{K}, \mathbb{K}))$$

where for all the standard points, $\mathcal{F}^{\perp} = \mathcal{G}^{\star}$, but for the non-standard points, the relations are determined by some extra frame conditions involving the special points. The frame conditions are spelled out in the companion report.

The situation is still not resolved, however. It appears we can morph $\mathcal{F}^{\perp}$ into $\mathcal{G}^{\star}$ by ignoring the frame conditions for $\mathcal{F}^{\perp}$, which requires throwing all those tuples involving special points out of the relation and substituting the tuples demanded of the frame conditions for $\mathcal{G}^{\star}$.

The result is that we can pair up the frames according to which are related by $\overset{\bullet\circ}{\neg}$. This is no different than the original Goldblatt-Thomason theorem that only had one kind of frame to consider if you ignore the set algebras. Technically, the set algebras only needed to support a single modal operator. The $[-^{\Box}\rangle$ operator is related by Boolean negation to $[-^{\circ}\rangle$ in that the negation distributes across $\vee$ and $[-^{\Box}\rangle$. This closed the $[-^{\Box}\rangle$ frames with respect to disjoint sums.

For our situation, consider the following example. First, the relevant frame conditions. From the companion report,

Frame Condiiton 4.2.1 The null frame conditions for $\mathcal{F}^{\perp}$ are

$$(1) \quad \forall y(\neg \mathcal{F}^{\perp} H^{\circ} y) \qquad (2) \quad \forall x(\neg \mathcal{F}^{\perp} x K^{\circ})$$

where $\mathcal{F}^{\perp} H^{\circ} y$ stands for $\forall x \in_{\mathsf{h}} H^{\circ}(\mathcal{F}^{\perp} x y)$ and a similar statement holds for $\mathcal{F}^{\perp} x K^{\circ}$.

Frame Condiiton 4.2.2 The well frame conditions for $\mathcal{F}^{\perp}$ are

$$(1) \quad \forall x(x \notin_{\mathsf{h}} H^{\circ} \text{ implies } \mathcal{F}^{\perp} x K^{\bullet}) \qquad (2) \quad \forall y(y \notin_{\mathsf{k}} K^{\circ} \text{ implies } \mathcal{F}^{\perp} H^{\bullet} y).$$

where $\mathcal{F}^{\perp} H^{\bullet} y$ stands for $\forall x \in_{\mathsf{h}} H^{\bullet}(\mathcal{F}^{\perp} x y)$ and a similar statement holds for $\mathcal{F}^{\perp} x K^{\bullet}$.

We can define the following sets:

$$\mathcal{F}_{\circ}^{\perp} \overset{def}{=} \{\langle x, y\rangle \mid x \in_{\mathsf{h}} H^{\circ}\} \cup \{\langle x, y\rangle \mid y \in_{\mathsf{k}} K^{\circ}\}.$$

and

$$\mathcal{F}_{\bullet}^{\perp} \overset{def}{=} \{\langle x, y\rangle \mid x \notin_{\mathsf{h}} H^{\circ} \text{ and } y \in_{\mathsf{k}} K^{\bullet}\} \cup \{\langle x, y\rangle \mid y \notin_{\mathsf{k}} K^{\circ} \text{ and } x \in_{\mathsf{h}} H^{\bullet}\}.$$

Regarding the first set of the second condition, this picks out points that satisfy Frame Condition 4.2.2. That is, assume x is such that $x \notin_{\mathsf{h}} H^{\circ}$ implies $\mathcal{F}^{\perp} x K^{\bullet}$, and that $y \in_{\mathsf{k}} K^{\circ}$. Hence $\mathcal{F}^{\perp} x y$ from that condition, but also $\langle x, y\rangle \in \mathcal{F}_{\bullet}^{\perp}$ from the set definition. Next, assume $\langle x, y\rangle \in \mathcal{F}_{\bullet}^{\perp}$ from the set condition, then $x \notin_{\mathsf{h}} H^{\circ}$ and so $\mathcal{F}^{\perp} x K^{\bullet}$. Since $y \in_{\mathsf{k}} K^{\bullet}$, then $\mathcal{F}^{\perp} x y$ from this Frame Condition.

Let

$$\mathrm{FC}^{\perp} \overset{def}{=} \mathcal{F}_{\circ}^{\perp} \cup \mathcal{F}_{\bullet}^{\perp} \qquad \mathrm{FC}^{\star} \overset{def}{=} \mathcal{F}_{\circ}^{\star} \cup \mathcal{F}_{\bullet}^{\star}$$

where $\mathcal{F}_\circ^\star$ and $\mathcal{F}_\bullet^\star$ are defined similarly to $\mathcal{F}_\circ^\perp$ and $\mathcal{F}_\bullet^\perp$. The relationship between the associated relations is now clear:

$$(\mathcal{F}^\perp - \mathrm{FC}^\perp) \cup \mathrm{FC}^\star = \mathcal{F}^\star \qquad (\mathcal{F}^\star - \mathrm{FC}^\star) \cup \mathrm{FC}^\perp = \mathcal{F}^\perp.$$

In effect, we can transform a frame using $\mathcal{F}^\perp$ into a frame using $\mathcal{F}^\star$ by throwing out the tuples associated with the special points in the $\mathcal{F}^\perp$ frame and replacing them with the tuples associated with the special points in the $\mathcal{F}^\star$ frame.

The conclusion is that we can transform $\mathcal{F}^\perp$ frames into $\mathcal{F}^\star$ frames and thus bypass the issue that $\mathcal{F}^\perp$ frames are not closed under null sums.

4.3 Complements

The complements we use are the new Boolean complements using $^{\bullet\circ}\neg$. The follow Lemma holds:

Lemma 4.3.1 *Canonically, assume $x \notin_{\mathsf{h}} \{\bullet_H, \circ_H\}$ and $y \notin_{\mathsf{k}} \{\bullet_K, \circ_K\}$. The algebraic identities on the left imply the relational properties on the right*

Relational Complement Properties

Algebraic Identity	*Property*	*Algebraic Identity*	*Property*
$\neg[f^\flat\rangle\, b = [f^\sharp\rangle \neg b$	$\mathcal{F}^\flat = \mathcal{F}^\sharp$	$\neg[f^\flat\cdot\rangle\, b = [f^\sharp\cdot\rangle \neg b$	$\mathcal{F}^{\flat\cdot} = \mathcal{F}^{\sharp\cdot}$
$\neg[f^\perp\rangle\, b = [f^\star\rangle \neg b$	$\mathcal{F}^\perp = \mathcal{F}^\star$	$\neg[f^\perp\cdot\rangle\, b = [f^\star\cdot\rangle \neg b$	$\mathcal{F}^{\perp\cdot} = \mathcal{F}^{\star\cdot}$
$\neg[f^\square\rangle\, b = [f^\circ\rangle \neg b$	$\mathcal{F}^\square = \mathcal{F}^\circ$	$\neg[f^\square\cdot\rangle\, b = [f^\circ\cdot\rangle \neg b$	$\mathcal{F}^{\square\cdot} = \mathcal{F}^{\circ\cdot}$
$\neg[f^!\rangle\, b = [f^?\rangle \neg b$	$\mathcal{F}^! = \mathcal{F}^?$	$\neg[f^!\cdot\rangle\, b = [f^?\cdot\rangle \neg b$	$\mathcal{F}^{!\cdot} = \mathcal{F}^{?\cdot}$

and assuming the frame conditions in the companion report for each operator, the relational properties (as new frame conditions) on the left imply the set theoretic identities on the right, and we assume $f : \mathsf{h} \rightharpoonup \mathsf{k}$:

Set Complement Properties

Property	*Set Identity*	*Algebraic Identity*	*Property*
$\mathcal{F}^\flat = \mathcal{F}^\sharp$	$^{\bullet\circ}\neg\,[f^\flat\rangle\, Q = [f^\sharp\rangle\, {}^{\bullet\circ}\neg\, Q$	$\mathcal{F}^{\flat\cdot} = \mathcal{F}^{\sharp\cdot}$	$^{\bullet\circ}\neg\,[f^\flat\cdot\rangle\, P = [f^\sharp\cdot\rangle\, {}^{\bullet\circ}\neg\, P$
$\mathcal{F}^\perp = \mathcal{F}^\star$	$^{\bullet\circ}\neg\,[f^\perp\rangle\, Q = [f^\star\rangle\, {}^{\bullet\circ}\neg\, Q$	$\mathcal{F}^{\perp\cdot} = \mathcal{F}^{\star\cdot}$	$^{\bullet\circ}\neg\,[f^\perp\cdot\rangle\, P = [f^\star\cdot\rangle\, {}^{\bullet\circ}\neg\, P$
$\mathcal{F}^\square = \mathcal{F}^\circ$	$^{\bullet\circ}\neg\,[f^\square\rangle\, Q = [f^\circ\rangle\, {}^{\bullet\circ}\neg\, Q$	$\mathcal{F}^{\square\cdot} = \mathcal{F}^{\circ\cdot}$	$^{\bullet\circ}\neg\,[f^\square\cdot\rangle\, P = [f^\circ\cdot\rangle\, {}^{\bullet\circ}\neg\, P$
$\mathcal{F}^! = \mathcal{F}^?$	$^{\bullet\circ}\neg\,[f^!\rangle\, Q = [f^?\rangle\, {}^{\bullet\circ}\neg\, Q$	$\mathcal{F}^{!\cdot} = \mathcal{F}^{?\cdot}$	$^{\bullet\circ}\neg\,[f^!\cdot\rangle\, P = [f^?\cdot\rangle\, {}^{\bullet\circ}\neg\, P$

Corollary 4.3.2 *The following equations are valid and equivalent corollaries of the Set Complement Properties:*

$$^{\bullet\circ}\neg\,[f^\perp\rangle\, Q = [f^\star\rangle\, {}^{\bullet\circ}\neg\, Q \qquad [f^\perp\rangle\, Q = {}^{\bullet\circ}\neg\,[f^\star\rangle\, {}^{\bullet\circ}\neg\, Q \qquad {}^{\bullet\circ}\neg\,[f^\star\rangle\, Q = [f^\perp\rangle\, {}^{\bullet\circ}\neg\, Q \qquad [f^\star\rangle\, Q = {}^{\bullet\circ}\neg\,[f^\perp\rangle\, {}^{\bullet\circ}\neg\, Q.$$

$$\overset{\bullet\circ}{\neg}[f^{!}\rangle Q = [f^{?}\rangle \overset{\bullet\circ}{\neg} Q \qquad [f^{!}\rangle Q = \overset{\bullet\circ}{\neg}[f^{?}\rangle \overset{\bullet\circ}{\neg} Q \qquad \overset{\bullet\circ}{\neg}[f^{?}\rangle Q = [f^{!}\rangle \overset{\bullet\circ}{\neg} Q \qquad [f^{?}\rangle Q = \overset{\bullet\circ}{\neg}[f^{!}\rangle \overset{\bullet\circ}{\neg} Q.$$

$$\overset{\bullet\circ}{\neg}[f^{\flat}\rangle Q = [f^{\sharp}\rangle \overset{\bullet\circ}{\neg} Q \qquad [f^{\flat}\rangle Q = \overset{\bullet\circ}{\neg}[f^{\sharp}\rangle \overset{\bullet\circ}{\neg} Q \qquad \overset{\bullet\circ}{\neg}[f^{\sharp}\rangle Q = [f^{\flat}\rangle \overset{\bullet\circ}{\neg} Q \qquad [f^{\sharp}\rangle Q = \overset{\bullet\circ}{\neg}[f^{\flat}\rangle \overset{\bullet\circ}{\neg} Q.$$

$$\overset{\bullet\circ}{\neg}[f^{\square}\rangle Q = [f^{\diamond}\rangle \overset{\bullet\circ}{\neg} Q \qquad [f^{\square}\rangle Q = \overset{\bullet\circ}{\neg}[f^{\diamond}\rangle \overset{\bullet\circ}{\neg} Q \qquad \overset{\bullet\circ}{\neg}[f^{\diamond}\rangle Q = [f^{\square}\rangle \overset{\bullet\circ}{\neg} Q \qquad [f^{\diamond}\rangle Q = \overset{\bullet\circ}{\neg}[f^{\square}\rangle \overset{\bullet\circ}{\neg} Q.$$

Similar properties obtain with respect to the backwards operators.

4.4 Using Negation to Show Closure under Null Sums

This is a brief example of to use the new Boolean negation to show that $[-^{\perp}\rangle$ is closed under null sums. The frame is not directly closed under null sums. However, if we are allowed to make a detour to the frames that support $[-^{\star}\rangle$, then we can show closure. First, we need a Lemma:

Lemma 4.4.1

$$\big((H_1 \otimes H_2) \overset{\bullet\circ}{-} (Q_1 \otimes Q_2)\big) = (H_1 \overset{\bullet\circ}{-} Q_1) \otimes (H_2 \overset{\bullet\circ}{-} Q_2).$$

Given the definition of $\overset{\bullet\circ}{-}$, this Lemma is the equivalent of

$$\overset{\bullet\circ}{\neg}(Q_1 \otimes Q_2) = \overset{\bullet\circ}{\neg} Q_1 \otimes \overset{\bullet\circ}{\neg} Q_2.$$

We use this Lemma in the following argument:

$$\begin{aligned}
[f^{\perp}\rangle (Q_1 \otimes Q_2) &= \overset{\bullet\circ}{\neg}\overset{\bullet\circ}{\neg}[f^{\perp}\rangle (Q_1 \otimes Q_2) \\
&= \overset{\bullet\circ}{\neg}[f^{\star}\rangle \overset{\bullet\circ}{\neg}(Q_1 \otimes Q_2) \\
&= \overset{\bullet\circ}{\neg}[f^{\star}\rangle (\overset{\bullet\circ}{\neg} Q_1 \otimes \overset{\bullet\circ}{\neg} Q_2) \\
&= \overset{\bullet\circ}{\neg}([f_1{}^{\star}\rangle \overset{\bullet\circ}{\neg} Q_1 \otimes [f_2{}^{\star}\rangle \overset{\bullet\circ}{\neg} Q_2) \\
&= \overset{\bullet\circ}{\neg}(\overset{\bullet\circ}{\neg}[f_1{}^{\perp}\rangle Q_1 \otimes \overset{\bullet\circ}{\neg}[f_2{}^{\perp}\rangle Q_2) \\
&= \overset{\bullet\circ}{\neg}\overset{\bullet\circ}{\neg}([f_1{}^{\perp}\rangle Q_1 \otimes [f_2{}^{\perp}\rangle Q_2) \\
&= [f_1{}^{\perp}\rangle Q_1 \otimes [f_2{}^{\perp}\rangle Q_2
\end{aligned}$$

5. RESIDUATION

The Galois operators may also be paired up according to their residuation properties as opposed to their Boolean negation properties.

5.1 Galois Operator Residuation Properties

The Galois operators are either monotonic or antitonic:

Definition 5.1.1 An operator $[f\rangle$ for $f : \mathsf{h} \rightharpoonup \mathsf{k}$ is *monotonic* if

$$a \leq_{\mathsf{k}} b \text{ implies } [f\rangle a \leq_{\mathsf{h}} [f\rangle b,$$

and *antitonic* if

$$a \leq_{\mathsf{k}} b \text{ implies } [f\rangle b \leq_{\mathsf{h}} [f\rangle a,$$

The operators and their tonicity properties are as follows where the first four lines show monotone operators and the second four lines show antitone operators.

Tonicity Property	
Forward Operator	**Backward Operator**
$a \leq_{\mathsf{k}} b$ implies $[f^{\square}\rangle a \leq_{\mathsf{h}} [f^{\square}\rangle b$	$a \leq_{\mathsf{h}} b$ implies $[f^{\square}\cdot\rangle a \leq_{\mathsf{k}} [f^{\square}\cdot\rangle b$
$a \leq_{\mathsf{k}} b$ implies $[f^{\circ}\rangle a \leq_{\mathsf{h}} [f^{\circ}\rangle b$	$a \leq_{\mathsf{h}} b$ implies $[f^{\circ}\cdot\rangle a \leq_{\mathsf{k}} [f^{\circ}\cdot\rangle b$
$a \leq_{\mathsf{k}} b$ implies $[f^{\sharp}\rangle a \leq_{\mathsf{h}} [f^{\sharp}\rangle b$	$a \leq_{\mathsf{h}} b$ implies $[f^{\sharp}\cdot\rangle a \leq_{\mathsf{k}} [f^{\sharp}\cdot\rangle b$
$a \leq_{\mathsf{k}} b$ implies $[f^{\flat}\rangle a \leq_{\mathsf{h}} [f^{\flat}\rangle b$	$a \leq_{\mathsf{h}} b$ implies $[f^{\flat}\cdot\rangle a \leq_{\mathsf{k}} [f^{\flat}\cdot\rangle b$
$a \leq_{\mathsf{k}} b$ implies $[f^{!}\rangle b \leq_{\mathsf{h}} [f^{!}\rangle a$	$a \leq_{\mathsf{h}} b$ implies $[f^{!}\cdot\rangle b \leq_{\mathsf{k}} [f^{!}\cdot\rangle a$
$a \leq_{\mathsf{k}} b$ implies $[f^{?}\rangle b \leq_{\mathsf{h}} [f^{?}\rangle a$	$a \leq_{\mathsf{h}} b$ implies $[f^{?}\cdot\rangle b \leq_{\mathsf{k}} [f^{?}\cdot\rangle a$
$a \leq_{\mathsf{k}} b$ implies $[f^{\perp}\rangle b \leq_{\mathsf{h}} [f^{\perp}\rangle a$	$a \leq_{\mathsf{h}} b$ implies $[f^{\perp}\cdot\rangle b \leq_{\mathsf{k}} [f^{\perp}\cdot\rangle a$
$a \leq_{\mathsf{k}} b$ implies $[f^{\star}\rangle b \leq_{\mathsf{h}} [f^{\star}\rangle a$	$a \leq_{\mathsf{h}} b$ implies $[f^{\star}\cdot\rangle b \leq_{\mathsf{k}} [f^{\star}\cdot\rangle a$

Theorem 5.1.2 *In the context of a monotonic and antitonic operators, the residuation properties on the left are equivalent to the inequalities on the right:*

Res. Ops	***Residuation Property***	***Inequalities***
$[-^{\square}\rangle, [-^{\circ}\cdot\rangle$	$[f^{\circ}\cdot\rangle a \leq_{\mathsf{k}} b$ iff $a \leq_{\mathsf{h}} [f^{\square}\rangle b$	$a \leq_{\mathsf{h}} [f^{\square}\rangle [f^{\circ}\cdot\rangle a$ and $[f^{\circ}\cdot\rangle [f^{\square}\rangle b \leq_{\mathsf{k}} b$
$[-^{\square}\cdot\rangle, [-^{\circ}\rangle$	$[f^{\circ}\rangle b \leq_{\mathsf{h}} a$ iff $b \leq_{\mathsf{k}} [f^{\square}\cdot\rangle a$	$b \leq_{\mathsf{k}} [f^{\square}\cdot\rangle [f^{\circ}\rangle b$ and $[f^{\circ}\rangle [f^{\square}\cdot\rangle a \leq_{\mathsf{h}} a$
$[-^{\flat}\cdot\rangle, [-^{\sharp}\rangle$	$[f^{\sharp}\rangle b \leq_{\mathsf{h}} a$ iff $b \leq_{\mathsf{k}} [f^{\flat}\cdot\rangle a$	$b \leq_{\mathsf{k}} [f^{\flat}\cdot\rangle [f^{\sharp}\rangle b$ and $[f^{\sharp}\rangle [f^{\flat}\cdot\rangle a \leq_{\mathsf{h}} a$
$[-^{\flat}\rangle, [-^{\sharp}\cdot\rangle$	$[f^{\sharp}\cdot\rangle a \leq_{\mathsf{k}} b$ iff $a \leq_{\mathsf{h}} [f^{\flat}\rangle b$	$a \leq_{\mathsf{h}} [f^{\flat}\rangle [f^{\sharp}\cdot\rangle a$ and $[f^{\sharp}\rangle [f^{\flat}\rangle b \leq_{\mathsf{k}} b$
$[-^{?}\rangle, [-^{?}\cdot\rangle$	$[f^{?}\cdot\rangle a \leq_{\mathsf{k}} b$ iff $[f^{?}\rangle b \leq_{\mathsf{h}} a$	$[f^{?}\rangle [f^{?}\cdot\rangle a \leq_{\mathsf{h}} a$ and $[f^{?}\cdot\rangle [f^{?}\rangle b \leq_{\mathsf{k}} b$
$[-^{\star}\rangle, [-^{\star}\cdot\rangle$	$[f^{\star}\cdot\rangle a \leq_{\mathsf{k}} b$ iff $[f^{\star}\rangle b \leq_{\mathsf{h}} a$	$[f^{\star}\rangle [f^{\star}\cdot\rangle a \leq_{\mathsf{h}} a$ and $[f^{\star}\cdot\rangle [f^{\star}\rangle b \leq_{\mathsf{k}} b$
$[-^{\perp}\rangle, [-^{\perp}\cdot\rangle$	$b \leq_{\mathsf{k}} [f^{\perp}\cdot\rangle a$ iff $a \leq_{\mathsf{h}} [f^{\perp}\rangle b$	$a \leq_{\mathsf{h}} [f^{\perp}\rangle [f^{\perp}\cdot\rangle a$ and $b \leq_{\mathsf{k}} [f^{\perp}\cdot\rangle [f^{\perp}\rangle b$
$[-^{!}\rangle, [-^{!}\cdot\rangle$	$b \leq_{\mathsf{k}} [f^{!}\cdot\rangle a$ iff $a \leq_{\mathsf{h}} [f^{!}\rangle b$	$a \leq_{\mathsf{h}} [f^{!}\rangle [f^{!}\cdot\rangle a$ and $b \leq_{\mathsf{k}} [f^{!}\cdot\rangle [f^{!}\rangle b$

Residuation is intimately connected to the type:

Theorem 5.1.3 *For lattices, the tonicity properties and residuation implies the operator type:*

Type	***Property***	***Type***	***Property***
$\vee\mapsto\vee$	$[f^{\diamond}\cdot\rangle(a\vee b)=[f^{\diamond}\cdot\rangle a\vee[f^{\diamond}\cdot\rangle b$	$\vee\mapsto\vee$	$[f^{\diamond}\rangle(a\vee b)=[f^{\diamond}\rangle a\vee[f^{\diamond}\rangle b$
$\wedge\mapsto\wedge$	$[f^{\Box}\rangle(a\wedge b)=[f^{\Box}\rangle a\wedge[f^{\Box}\rangle b$	$\wedge\mapsto\wedge$	$[f^{\Box}\cdot\rangle(a\wedge b)=[f^{\Box}\cdot\rangle a\wedge[f^{\Box}\cdot\rangle b$
$\wedge\mapsto\wedge$	$[f^{\flat}\cdot\rangle(a\wedge b)=[f^{\flat}\cdot\rangle a\wedge[f^{\flat}\cdot\rangle b$	$\wedge\mapsto\wedge$	$[f^{\flat}\rangle(a\wedge b)=[f^{\flat}\rangle a\wedge[f^{\flat}\rangle b$
$\vee\mapsto\vee$	$[f^{\sharp}\rangle(a\vee b)=[f^{\sharp}\rangle a\vee[f^{\sharp}\rangle b$	$\vee\mapsto\vee$	$[f^{\sharp}\cdot\rangle(a\vee b)=[f^{\sharp}\cdot\rangle a\vee[f^{\sharp}\cdot\rangle b$
$\vee\mapsto\wedge$	$[f^{\perp}\cdot\rangle(a\vee b)=[f^{\perp}\cdot\rangle a\wedge[f^{\perp}\cdot\rangle b$	$\wedge\mapsto\vee$	$[f^{\star}\rangle(a\wedge b)=[f^{\star}\rangle a\vee[f^{\star}\rangle b$
$\vee\mapsto\wedge$	$[f^{\perp}\rangle(a\vee b)=[f^{\perp}\rangle a\wedge[f^{\perp}\rangle b$	$\wedge\mapsto\vee$	$[f^{\star}\rangle(a\wedge b)=[f^{\star}\rangle a\vee[f^{\star}\rangle b$
$\vee\mapsto\wedge$	$[f^{!}\cdot\rangle(a\vee b)=[f^{!}\cdot\rangle a\wedge[f^{!}\cdot\rangle b$	$\wedge\mapsto\vee$	$[f^{?}\rangle(a\wedge b)=[f^{?}\rangle a\vee[f^{?}\rangle b$
$\vee\mapsto\wedge$	$[f^{!}\rangle(a\vee b)=[f^{!}\rangle a\wedge[f^{!}\rangle b$	$\wedge\mapsto\vee$	$[f^{?}\cdot\rangle(a\wedge b)=[f^{?}\cdot\rangle a\vee[f^{?}\cdot\rangle b$

5.2 Residuation and the Set Theoretic Properties

Lemma 5.2.1 *Assuming the tonicity properties of the Galois operators, the following relational identities hold*

Relational Identities	
$\mathcal{F}^{\Box}=\mathcal{F}^{\diamond\cdot}$	$\mathcal{F}^{?}=\mathcal{F}^{?\cdot}$
$\mathcal{F}^{\Box\cdot}=\mathcal{F}^{\diamond}$	$\mathcal{F}^{\star}=\mathcal{F}^{\star\cdot}$
$\mathcal{F}^{\sharp}=\mathcal{F}^{\flat\cdot}$	$\mathcal{F}^{\perp}=\mathcal{F}^{\perp\cdot}$
$\mathcal{F}^{\sharp\cdot}=\mathcal{F}^{\flat}$	$\mathcal{F}^{!}=\mathcal{F}^{!\cdot}$

The following theorem shows that the set theoretic operators satisfy their required residuation properties:

Theorem 5.2.2

Res. Ops	***Property***	***Res. Ops***	***Property***
$[-^{\Box}\rangle,[-^{\diamond}\cdot\rangle$	$[f^{\diamond}\cdot\rangle P\subseteq_{\mathsf{k}}Q$ iff $P\subseteq_{\mathsf{h}}[f^{\Box}\rangle Q$	$[-^{\flat}\cdot\rangle,[-^{\sharp}\rangle$	$[f^{\sharp}\rangle Q\subseteq_{\mathsf{h}}P$ iff $Q\subseteq_{\mathsf{k}}[f^{\flat}\cdot\rangle P$
$[-^{\Box}\cdot\rangle,[-^{\diamond}\rangle$	$[f^{\diamond}\rangle Q\subseteq_{\mathsf{h}}P$ iff $Q\subseteq_{\mathsf{k}}[f^{\Box}\cdot\rangle P$	$[-^{\flat}\rangle,[-^{\sharp}\cdot\rangle$	$[f^{\sharp}\cdot\rangle P\subseteq_{\mathsf{k}}Q$ iff $P\subseteq_{\mathsf{h}}[f^{\flat}\rangle Q$
$[-^{?}\rangle,[-^{?}\cdot\rangle$	$[f^{?}\cdot\rangle P\subseteq_{\mathsf{k}}Q$ iff $[f^{?}\rangle Q\subseteq_{\mathsf{h}}P$	$[-^{\star}\rangle,[-^{\star}\cdot\rangle$	$[f^{\star}\cdot\rangle P\subseteq_{\mathsf{k}}Q$ iff $[f^{\star}\rangle Q\subseteq_{\mathsf{h}}P$
$[-^{\perp}\rangle,[-^{\perp}\cdot\rangle$	$Q\subseteq_{\mathsf{k}}[f^{\perp}\cdot\rangle P$ iff $P\subseteq_{\mathsf{h}}[f^{\perp}\rangle Q$	$[-^{!}\rangle,[-^{!}\cdot\rangle$	$Q\subseteq_{\mathsf{k}}[f^{!}\cdot\rangle P$ iff $P\subseteq_{\mathsf{h}}[f^{!}\rangle Q$

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