Equichordal tight fusion frames and biangular orthopartitionable tight frames

DISSERTATION

Benjamin R. Mayo, Maj, USAF

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Benjamin R. Mayo, B.S., M.S.
Maj, USAF

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Benjamin R. Mayo, B.S., M.S.
Maj, USAF

Committee Membership:

Matthew Fickus, Ph.D.
Chairman

Dursun A. Bulutoglu, Ph.D.
Member

Raymond R. Hill, Ph.D.
Member

ADEDEJI B. BADIRU, Ph.D.
Dean, Graduate School of Engineering and Management

Dedicated to my mother, who fostered a love of math throughout my life, my father, who encouraged a dedication to service and hard work, and my wife and daughter, who put up with the many hours of math and hard work, as well as the sacrifices they made while supporting me in this endeavor. You have made this all possible.

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"The basic process involved is one of learning through failures."

-J.T. Duane

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Benjamin R. Mayo

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Abstract

An equichordal tight fusion frame (ECTFF) is a sequence of equidimensional subspaces of a Euclidean space that achieves equality in Conway, Hardin and Sloane's simplex bound, and so is a type of optimal Grassmannian code. In the special case where its subspaces have dimension one, an ECTFF corresponds to an equiangular tight frame (ETF); such frames have minimal coherence and so are useful for compressed sensing. More generally, an ECTFF will yield a frame with minimal block coherence when its subspaces are pairwise isoclinic, namely when it is an equi-isoclinic tight fusion frame (EITFF). In this dissertation, we generalize the notion of an ETF to that of a biangular orthopartitionable tight frame (BOPTF). Every BOPTF generates an ECTFF and has a coherence that rivals that of an ETF. Generalizing a recent observation of King, we construct a new infinite family of BOPTFs whose ECTFFs are actually equi-isoclinic. Such EITFF-generating BOPTFs are remarkable: having both low coherence and minimal block coherence, they guarantee the efficient recovery of signals that are either sparse or block sparse (without foreknowledge of the sparsity type). We moreover show that such EITFF-generating BOPTFs are special, proving that certain infinite families of BOPTFs (including an infinite number of those constructed by King from semiregular divisible difference sets) generate ECTFFs that are not equi-isoclinic. Along the way, we discover several new methods for constructing and comprehending ECTFFs.

EQUICHORDAL TIGHT FUSION FRAMES AND

BIANGULAR ORTHOPARTITIONABLE TIGHT FRAMES

I. Introduction

Let $\mathbb{F}$ be either $\mathbb{R}$ or $\mathbb{C}$. For any integers $N > 1$ and $D \geq R \geq 1$, let $\{\boldsymbol{\varphi}_{n,r}\}_{n=1,\, r=1}^{N \;\;\, R}$ be a sequence of NR vectors in $\mathbb{F}^D$ with the property that for any $n = 1, \dots, N$, the subsequence $\{\boldsymbol{\varphi}_{n,r}\}_{r=1}^{\mathcal{R}}$ is orthonormal. For any $n_1, n_2 = 1, \dots, N$, the corresponding *cross-Gram* matrix is the $R \times R$ matrix $\boldsymbol{\Phi}_{n_1}^* \boldsymbol{\Phi}_{n_2}$ whose (r_1, r_2)th entry is $\left(\boldsymbol{\Phi}_{n_1}^* \boldsymbol{\Phi}_{n_2}\right)(r_1, r_2) = \langle \boldsymbol{\varphi}_{n_1,r_1}, \boldsymbol{\varphi}_{n_2,r_2} \rangle$. For any such $\{\boldsymbol{\varphi}_{n,r}\}_{n=1,\, r=1}^{N \;\;\, R}$, it is known [3, 39] that

$$\upsilon := \max_{n_1 \neq n_2} \|\boldsymbol{\Phi}_{n_1}^* \boldsymbol{\Phi}_{n_2}\|_2 \geq \tfrac{1}{\sqrt{R}} \max_{n_1 \neq n_2} \|\boldsymbol{\Phi}_{n_1}^* \boldsymbol{\Phi}_{n_2}\|_{\mathrm{Fro}} \geq \sqrt{\tfrac{NR-D}{D(N-1)}}. \tag{1}$$

Here υ is known as the *block coherence* of $\{\boldsymbol{\varphi}_{n,r}\}_{n=1,\, r=1}^{N \;\;\, R}$. As detailed in the next chapter, any such $\{\boldsymbol{\varphi}_{n,r}\}_{n=1,\, r=1}^{N \;\;\, R}$ achieves equality in the right-hand bound of (1) if and only if the sequence $\{\mathcal{U}_n\}_{n=1}^N$ of R-dimensional subspaces of $\mathbb{F}^D$ defined by $\mathcal{U}_n := \operatorname{span}\{\boldsymbol{\varphi}_{n,r}\}_{r=1}^R$ is an *equichordal tight fusion frame* (ECTFF) for $\mathbb{F}^D$, denoted as $\operatorname{ECTFF}(D, N, R)$. Specifically, this occurs if and only if it is a *tight fusion frame* (TFF) for $\mathbb{F}^D$, meaning that the corresponding rank-R projections $\{\mathbf{P}_n\}_{n=1}^N$ onto $\{\mathcal{U}_n\}_{n=1}^N$ satisfy $\sum_{n=1}^N \mathbf{P}_n = A\mathbf{I}$ for some $A > 0$ (tightness), and also has the trait that $\|\mathbf{P}_{n_1} - \mathbf{P}_{n_2}\|_{\mathrm{Fro}}$ is constant over all $n_1 \neq n_2$ (equichordality). Additionally, $\{\boldsymbol{\varphi}_{n,r}\}_{n=1,\, r=1}^{N \;\;\, R}$ achieves equality throughout (1) if and only if $\{\mathcal{U}_n\}_{n \in \mathcal{N}}$ is an *equi-isoclinic tight fusion frame* (EITFF) for $\mathbb{F}^D$, namely an ECTFF with the additional property that there exists $\sigma \geq 0$ such that $\mathbf{P}_{n_1} \mathbf{P}_{n_2} \mathbf{P}_{n_1} = \sigma^2 \mathbf{P}_{n_1}$ for all $n_1 \neq n_2$ (equi-isoclinicity). These fusion frames are denoted EITFF(D, N, R). In the special case where $R = 1$, (1) reduces to the *Welch bound* [40] on the *coherence* μ of any sequence $\{\boldsymbol{\varphi}_n\}_{n=1}^N$ of unit-norm vectors in $\mathbb{F}^D$:

$$\mu := \max_{n_1 \neq n_2} \left|\langle \boldsymbol{\varphi}_{n_1}, \boldsymbol{\varphi}_{n_2} \rangle\right| \geq \sqrt{\tfrac{N-D}{D(N-1)}}. \tag{2}$$

For such a sequence $\{\boldsymbol{\varphi}_n\}_{n=1}^N$, it is known [37] that equality is achieved in (2) if and only if it is an *equiangular tight frame* (ETF) for $\mathbb{F}^D$, meaning that $\sum_{n=1}^N \boldsymbol{\varphi}_n \boldsymbol{\varphi}_n^* = A\mathbf{I}$ for some $A > 0$ (tightness) and $|\langle \boldsymbol{\varphi}_{n_1}, \boldsymbol{\varphi}_{n_2} \rangle|$ is constant for all $n_1 \neq n_2$; these are denoted as ETF(D, N). If this condition is relaxed to allow for a second value, i.e. there exists numbers $B, C \geq 0$ such that $|\langle \boldsymbol{\varphi}_{n_1}, \boldsymbol{\varphi}_{n_2} \rangle| \in \{B, C\}$ for all $n_1 \neq n_2$, $\{\boldsymbol{\varphi}_n\}_{n=1}^N$ is referred to as a *biangular tight frame*; in this case, if either A or B is equal to zero, then $\{\boldsymbol{\varphi}_n\}_{n \in \mathcal{N}}$ is called an *orthobiangular tight frame* (OBTF). OBTFs recently arose as a study of frames with putatively minimal coherence [15].

ETFs and EITFFs naturally arise in *compressed sensing* where sufficiently sparse signals are reconstructed from underdetermined linear measurements [7, 8]. To elaborate, for integers $N \geq K \geq 0$, a vector $\mathbf{x}$ in $\mathbb{F}^N$ is K-*sparse* if

$$\|\mathbf{x}\|_0 := \#\{n = 1, \dots, N : \mathbf{x}(n) \neq 0\} \leq K.$$

Using the *orthogonal matching pursuit* (OMP) algorithm [39], a K-sparse signal $\mathbf{x}$ can be efficiently recovered from $\sum_{n=1}^N \mathbf{x}(n)\boldsymbol{\varphi}_n$, provided $K \leq \frac{1}{2}(\mu^{-1} + 1)$. In light of (2), ETFs have minimal coherence μ making them optimally suited for signal recovery. More generally, a vector $\mathbf{x}$ in $\mathbb{F}^{N \times R}$ is K-*block sparse* if

$$\#\{n = 1, \dots, N : \exists\, r = 1, \dots, R \text{ s.t. } \mathbf{x}(n, r) \neq 0\} \leq K.$$

Using the *block orthogonal matching pursuit* (BOMP) algorithm [9], a K-block sparse signal $\mathbf{x}$ can be efficiently recovered provided $K \leq \frac{1}{2}(\upsilon^{-1}+1)$. In light of (1), EITFFs have minimal block coherence, υ, which makes them optimally suited for signal recovery, provided the signal sparsity is "blocky". Because of this, there has been a lot of recent research devoted to the explicit construction of sequences of vectors with small coherence or block coherence, with a particular interest in ETFs and EITFFs.

1.1 Literature review

ETF construction relates to a variety of mathematical disciplines and has been thoroughly explored with new constructions becoming increasingly rare [21, 19]. A real ETF with *redundancy* two (meaning $N = 2D$) exists if and only if a symmetric conference matrix of size N exists [21]. ETFs with $D = \frac{1}{2}(N \pm \sqrt{N})$ exist whenever there exists a self-adjoint complex Hadamard matrix with constant diagonal [38]. As explored in Chapter IV, harmonic ETFs arise from *difference sets* (DS) of finite abelian groups [33, 37, 42, 6]. The Steiner method explored in Chapter V generates ETFs using a balanced incomplete block design (BIBD) to carefully arrange regular simplices with respect to each other [23]. This method generalizes to produce Tremain ETFs from Steiner triple systems [16] as well as ETFs from hyperovals of projective planes [22]. ETFs also arise from distance-regular antipodal covers of the complete graph (DRACKNs) [4, 18] and association schemes [30, 29]. Outside of complex or real spaces, [1] provides a computer proof of the existence of quaternionic ETFs, some of which were explicitly formulated by [12]. Additionally, as an offshoot of the research in this dissertation, we recently discovered yet another construction of ETFs from generalizations of quantum-information-theoretic objects known as *mutually unbiased bases* (MUBs) [19].

There are fewer known methods for constructing EITFFs. One simple method is to tensor an ETF(D, N) with an *orthonormal basis* (ONB) of size R, or more generally, to take a direct sum of EITFFs where $\frac{R}{D}$ and N are constant. Another method [27] involves converting a complex EITFF(D, N, R) into a real EITFF$(2D, N, 2R)$, as well as converting a quaternionic EITFF(D, N, R) into a complex EITFF$(2D, N, 2R)$. When applied to the quaternionic ETFs from [1, 12], each of which can be considered an EITFF$(D, N, 1)$, [27] yields a complex EITFF$(2D, N, 2)$ and a real EITFF$(4D, N, 4)$. Modifying [27], a complex symmetric conference matrix of size N gives rise to a real EITFF$(N, N, 2)$ [10, 11]. Recently, a harmonic EITFF(D, N, R) was constructed from a *semiregular divisible difference set* (SR-DDS) [32], specifically one arising from a Gordon-Mills-Welch [26] sum of a *semiregular relative difference set* (SR-RDS) and a DS put forth in [28]. It was noted in this last case that such an EITFF would have flat cross-Gram matrices, that is $\left|\left(\mathbf{\Phi}_{n_1}^* \mathbf{\Phi}_{n_2}\right)(r_1, r_2)\right|$ is constant for all $n_1 \neq n_2$, a trait unique to this type of construction thus far.

While ETFs and EITFFs are sought after in compressed sensing, ECTFFs generally are not, since unlike EITFFs they may not have low block coherence υ. However, it is clear from (1) that every EITFF is also an ECTFF, meaning that each of the EITFF constructions mentioned above also yields an ECTFF. Additionally, some of the aforementioned EITFF construction methods generalize to the non-equi-isoclinic case. For example, the method of [27] yields a real ECTFF$(4D, N, 4R)$ and complex ECTFF$(2D, N, 2R)$ from a quaternionic ECTFF(D, N, R). Moreover, applying [32] to infinite families of SR-DDSs like those discussed in Theorem 2.3.6, Corollary 2.3.8, and Result 2.3.9 of [36] yields ECTFFs. One of the earliest known proofs of ECTFF existence is found in [5], which demonstrates the construction of a real ECTFF$(P, \frac{P(P+1)}{2}, \frac{P-1}{2})$ for any prime P provided there exists a Hadamard matrix of size $\frac{P+1}{2}$. Zauner [43] demonstrated construction of real ECTFFs from BIBDs, where a BIBD(V, K, Λ) results in an ECTFF(B, V, R) where $R = \frac{\Lambda(V-1)}{K-1}$ and $B = \frac{VR}{K}$.

This construction is notable as it results in ECTFFs that are never equi-isoclinic. An ECTFF(D, N, R) arises from an ETF(D, M) that can be partitioned into regular simplices for their span. In this case, $R = \left[\frac{D(M-1)}{M-D}\right]^{\frac{1}{2}}$ and $N = \frac{M}{R+1}$ [14]. ECTFFs are not often sought in compressed sensing as the coherence μ and block coherence υ can vary. However, the equichordality of ECTFFs is a requirement for equi-isoclinicity, as can be seen in (1); this suggests that insights into new EITFF constructions may come from the study of ECTFF construction, particularly those with unique traits such as the flat cross-Gram matrices arising from King's construction [32].

1.2 Novel contributions and dissertation outline

In this dissertation, inspired by King [32] and others [15], we introduce the following notion: we say that $\{\boldsymbol{\varphi}_{n,r}\}_{n=1,\,r=1}^{N\quad R}$ in $\mathbb{F}^D$ is a *biangular orthopartitionable tight frame* (BOPTF) for $\mathbb{F}^D$ if

$$\left|\langle \boldsymbol{\varphi}_{n_1,r_1}, \boldsymbol{\varphi}_{n_2,r_2}\rangle\right| = \begin{cases} 1, & n_1 = n_2, r_1 = r_2, \\ 0, & n_1 = n_2, r_1 \neq r_2, \\ \sqrt{\frac{NR-D}{DR(N-1)}}, & n_1 \neq n_2, \end{cases} \tag{3}$$

which we denote as a BOPTF(D, N, R). Note that if $\{\boldsymbol{\varphi}_{n,r}\}_{n=1,\,r=1}^{N\quad R}$ is a BOPTF for $\mathbb{F}^D$, then the coherence $\mu = \sqrt{\frac{NR-D}{DR(N-1)}}$, which is on the order of the Welch bound for NR vectors in $\mathbb{F}^D$: $\sqrt{\frac{NR-D}{D(NR-1)}}$. This suggests that BOPTFs, while not as optimally conditioned as ETFs, are well suited to the OMP algorithm. As noted in Chapter V, for any BOPTF(D, N, R) $\{\boldsymbol{\varphi}_{n,r}\}_{n=1,\,r=1}^{N\quad R}$ for $\mathbb{F}^D$, defining $\mathcal{U}_n := \operatorname{span}\{\boldsymbol{\varphi}_{n,r}\}_{r=1}^{R}$ for each n yields an ECTFF(D, N, R) for $\mathbb{F}^D$. The ECTFFs generated in [32] all arise from BOPTFs, but we took specific notice of the EITFF generated from the harmonic BOPTF from Ionin's construction of a SR-DDS [28]. This EITFF automatically has optimally low block coherence υ, but the fact that it arises from a BOPTF means that it has low coherence μ as well. When used in an application, EITFF-generating BOPTFs permit recovery of either sparse or block-sparse signals, allowing for the relaxation of the signal model to allow for the two possibilities. We generalize King's construction, leading to new infinite families of EITFF-generating BOPTFs from tensor products of ETFs and MUBs. This inspired a related method for constructing new ETFs from ETFs tensored with *mutually unbiased equiangular tight frames* (MUETF) [19]. We further show that not all BOPTFs are as well suited for compressed sensing, finding new infinite families of BOPTFs with terrible υ. In fact, by generalizing the techniques from [31], we show that infinite families of SR-DDSs yield BOPTFs with subspaces that are explicitly not equi-isoclinic. Along the way, we discover numerous new facts relating various BOPTFs, EITFFs, and ECTFFs.

Specifically, in Chapter II, we provide the necessary definitions, methods, and relationships currently known to the field. One such relationship is that every nontrivial ECTFF(D, N, R) has both a Naimark complement and spatial complement, both of which are also ECTFFs. We investigate this in Chapter III to show that taking alternating Naimark and spatial complements of an initial ECTFF can lead to infinite families of ECTFFs. In particular, Theorem 3.1.5 characterizes the families of TFFs that arise from such Naimark-spatial sequences and provides a mechanism for determining when these TFFs can be equi-isoclinic. Chapter IV focuses on harmonic frame constructions over finite abelian groups, with Theorem 4.1.1 introducing families of harmonic EC/EITFFs constructed from difference sets and difference families, and Theorem 4.2.1 generalizing King's construction from [32] in terms of Theorem 4.1.1. As previously mentioned, we note that these latter ECTFFs are generated from BOPTFs. In Chapter V, we investigate the construction of BOPTFs, outlining BOPTF characteristics in Theorem 5.1.1. Inspired by the harmonic EITFFs King generates from Ionin's SR-DDS [32, 28], we introduce a new EITFF-generating

BOPTF construction from infinite families of ETFs and MUBs in Theorem 5.2.1. We note that such EITFFs would have low coherence and optimal block coherence in Theorems 5.3.1 and 5.3.2, increasing their utility in applications like compressed sensing when the sparsity pattern is unknown. We further demonstrate that not all BOPTFs are as useful by constructing infinite families of BOPTFs that generate ECTFFs that are not equi-isoclinic in Theorems 5.4.1 – 5.5.1. Finally in Chapter VI, we summarize these results and put forth potential methods for constructing new EITFF-generating BOPTFs based on our research.

The material in this dissertation has been divided into three separate journal articles. A paper under the title "Certifying the novelty of equichordal tight fusion frames" has been submitted to Linear Algebra and its Applications in which we show that the characterization of EI/ECTFFs via Naimark-spatial chains from Chapter III can be used to certify when an EI/ECTFF construction method is truly novel. We demonstrate this by introducing the harmonic frame constructions from Chapter IV and confirming they are truly new to the field. An article introducing the BOPTF construction methods from Chapter V is currently in preparation for submission this fall. As mentioned above, the method for constructing EITFF-generating BOPTFs in Theorem 5.2.1 from ETFs and MUBs inspired a similar method for constructing ETFs from MUETFs. This work was published in IEEE Transactions on Information Theory [19].

II. Preliminaries

2.1 Frame theory

Let $\mathbb{F}$ be either $\mathbb{R}$ or $\mathbb{C}$ and let $\mathbb{H}$ be a D-dimensional Hilbert space over $\mathbb{F}$ equipped with an inner product $\langle\cdot,\cdot\rangle_{\mathbb{H}}$ that is conjugate linear in the first argument. Let $\mathcal{N}$ be an arbitrary indexing set of size N. Equip $\mathbb{F}^{\mathcal{N}} := \{\mathbf{x} : \mathcal{N} \to \mathbb{F}\}$ with the inner product $\langle \mathbf{x}_1, \mathbf{x}_2\rangle_{\mathbb{F}^{\mathcal{N}}} := \sum_{n\in\mathcal{N}} \overline{\mathbf{x}_1(n)}\mathbf{x}_2(n)$. The *synthesis operator* of a sequence of N vectors $\{\boldsymbol{\varphi}_n\}_{n\in\mathcal{N}}$ in $\mathbb{H}$ is $\boldsymbol{\Phi} : \mathbb{F}^{\mathcal{N}} \to \mathbb{H}$, $\boldsymbol{\Phi}\mathbf{x} := \sum_{n\in\mathcal{N}} \mathbf{x}(n)\boldsymbol{\varphi}_n$. The adjoint of $\boldsymbol{\Phi}$ is the *analysis operator*, $\boldsymbol{\Phi}^* : \mathbb{H} \to \mathbb{F}^{\mathcal{N}}$, $(\boldsymbol{\Phi}^*\mathbf{y})(n) = \langle\boldsymbol{\varphi}_n, \mathbf{y}\rangle_{\mathbb{H}}$. As a special case of this, we sometimes regard a single vector $\boldsymbol{\varphi}$ in $\mathbb{H}$ as the synthesis operator $\boldsymbol{\varphi} : \mathbb{F} \to \mathbb{H}$, where $\boldsymbol{\varphi}(x) = x\boldsymbol{\varphi}$ whose adjoint is the linear functional $\boldsymbol{\varphi}^* : \mathbb{H} \to \mathbb{F}$, $\boldsymbol{\varphi}^*\mathbf{y} = \langle\boldsymbol{\varphi}, \mathbf{y}\rangle$. In general, composing $\boldsymbol{\Phi}$ and $\boldsymbol{\Phi}^*$ gives the *frame operator* $\boldsymbol{\Phi}\boldsymbol{\Phi}^* : \mathbb{H} \to \mathbb{H}$, with $\boldsymbol{\Phi}\boldsymbol{\Phi}^* = \sum_{n\in\mathcal{N}} \boldsymbol{\varphi}_n\boldsymbol{\varphi}_n^*$ and the *Gram matrix* $\boldsymbol{\Phi}^*\boldsymbol{\Phi} : \mathbb{F}^{\mathcal{N}} \to \mathbb{F}^{\mathcal{N}}$, namely the operator whose $\mathcal{N}\times\mathcal{N}$ matrix representation with respect to the standard bases $\{\boldsymbol{\delta}_n\}_{n\in\mathcal{N}}$ for $\mathbb{F}^{\mathcal{N}}$ has the (n_1, n_2)th entry $(\boldsymbol{\Phi}^*\boldsymbol{\Phi})(n_1, n_2) := \langle\boldsymbol{\varphi}_{n_1}, \boldsymbol{\varphi}_{n_2}\rangle$. In the special case where $\mathbb{H} = \mathbb{F}^{\mathcal{D}}$ for some D-element index set, $\boldsymbol{\Phi}$ is the $\mathcal{D}\times\mathcal{N}$ matrix whose nth column is $\boldsymbol{\varphi}_n$, $\boldsymbol{\Phi}^*$ is its $\mathcal{N}\times\mathcal{D}$ conjugate (Hermitian) transpose, and $\boldsymbol{\Phi}\boldsymbol{\Phi}^*$ and $\boldsymbol{\Phi}^*\boldsymbol{\Phi}$ are the $\mathcal{D}\times\mathcal{D}$ and $\mathcal{N}\times\mathcal{N}$ matrices whose entries are the complex dot products of the rows and columns of $\boldsymbol{\Phi}$ with each other, respectively.

We say $\{\boldsymbol{\varphi}_n\}_{n\in\mathcal{N}}$ is a *frame* for $\mathbb{H}$ if there exist A, B with $0 < A \leq B < \infty$ such that $A\|\mathbf{y}\|^2 \leq \sum_{n\in\mathcal{N}} |\langle\boldsymbol{\varphi}_n, \mathbf{y}\rangle|^2 \leq B\|\mathbf{y}\|^2$ for all $\mathbf{y}$ in $\mathbb{H}$. In this finite-dimensional setting this occurs if and only if $\{\boldsymbol{\varphi}_n\}_{n\in\mathcal{N}}$ spans $\mathbb{H}$, namely if and only if the frame operator $\boldsymbol{\Phi}\boldsymbol{\Phi}^*$ is invertible. The optimal frame bounds A, B correspond to the least and greatest eigenvalues of $\boldsymbol{\Phi}\boldsymbol{\Phi}^*$, respectively. We say $\{\boldsymbol{\varphi}_n\}_{n\in\mathcal{N}}$ is a *tight frame* when

$\boldsymbol{\Phi}$ is perfectly conditioned, namely when there exists $A > 0$ such that

$$A\|\mathbf{y}\|^2 = \sum_{n\in\mathcal{N}} |\langle\boldsymbol{\varphi}_n, \mathbf{y}\rangle|^2 = \|\boldsymbol{\Phi}^*\mathbf{y}\|^2, \forall\, \mathbf{y} \in \mathbb{H}.$$

By polarization, this equates to having $A\mathbf{I} = \boldsymbol{\Phi}\boldsymbol{\Phi}^* = \sum_{n\in\mathcal{N}} \boldsymbol{\varphi}_n\boldsymbol{\varphi}_n^*$. In the special case where each $\boldsymbol{\varphi}_n$ is unit-norm it is referred to as a *unit-norm tight frame* (UNTF) for $\mathbb{H}$. For any n in $\mathcal{N}$, notice that the operator $\mathbf{P}_n = \boldsymbol{\varphi}_n\boldsymbol{\varphi}_n^*$ is the rank-1 projection onto the one-dimensional subspace $\mathcal{U}_n = \langle\boldsymbol{\varphi}_n\rangle$, or more specifically, the projection onto its span. This means that $A\mathbf{I} = \sum_{n\in\mathcal{N}} \boldsymbol{\varphi}_n\boldsymbol{\varphi}_n^* = \sum_{n\in\mathcal{N}} \mathbf{P}_n$. Taking the trace of this equation gives

$$DA = \mathrm{Tr}(A\mathbf{I}) = \mathrm{Tr}\left(\sum_{n\in\mathcal{N}} \mathbf{P}_n\right) = \sum_{n\in\mathcal{N}} \mathrm{Tr}(\mathbf{P}_n) = N,$$

therefore $A = \frac{N}{D}$. Now consider the following: for any sequence of N unit-norm vectors in a D-dimensional Hilbert space, $\{\boldsymbol{\varphi}_n\}_{n\in\mathcal{N}}$, we have that

$$\begin{aligned}
0 &\leq \mathrm{Tr}\left[\left(\boldsymbol{\Phi}\boldsymbol{\Phi}^* - \tfrac{N}{D}\mathbf{I}\right)^2\right] \\
&= \mathrm{Tr}\left[\left(\sum_{n\in\mathcal{N}} \mathbf{P}_n - \tfrac{N}{D}\mathbf{I}\right)^2\right] \\
&= \sum_{n\in\mathcal{N}} \sum_{\substack{n'\in\mathcal{N} \\ n'\neq n}} \mathrm{Tr}(\mathbf{P}_n\mathbf{P}_{n'}) + \sum_{n\in\mathcal{N}} \mathrm{Tr}(\mathbf{P}_n^2) - \tfrac{N}{D} \sum_{n\in\mathcal{N}} \mathrm{Tr}(\mathbf{P}_n\mathbf{I}) + \left(\tfrac{N}{D}\right)^2 \mathrm{Tr}(\mathbf{I}) \\
&= \sum_{n\in\mathcal{N}} \sum_{\substack{n'\in\mathcal{N} \\ n'\neq n}} \mathrm{Tr}(\mathbf{P}_n\mathbf{P}_{n'}) + N + \tfrac{1}{D} + D\left(\tfrac{N}{D}\right)^2 \\
&= \sum_{n\in\mathcal{N}} \sum_{\substack{n'\in\mathcal{N} \\ n'\neq n}} \mathrm{Tr}(\mathbf{P}_n\mathbf{P}'_n) - \tfrac{N(N-D)}{D},
\end{aligned}$$

where equality is achieved if and only if $\{\boldsymbol{\varphi}_n\}_{n\in\mathcal{N}}$ is a UNTF for $\mathbb{H}$. Recalling that

$\mathbf{P}_n = \boldsymbol{\varphi}_n\boldsymbol{\varphi}_n^*$, we get the lower bound for

$$\begin{aligned}
\tfrac{N(N-D)}{D} &\leq \sum_{n\in\mathcal{N}}\sum_{\substack{n'\in\mathcal{N}\\ n'\neq n}} \operatorname{Tr}(\mathbf{P}_n\mathbf{P}_{n'})\\
&= \sum_{n\in\mathcal{N}}\sum_{\substack{n'\in\mathcal{N}\\ n'\neq n}} \operatorname{Tr}(\boldsymbol{\varphi}_n\boldsymbol{\varphi}_n^*\boldsymbol{\varphi}_{n'}\boldsymbol{\varphi}_{n'}^*)\\
&= \sum_{n\in\mathcal{N}}\sum_{\substack{n'\in\mathcal{N}\\ n'\neq n}} \operatorname{Tr}(\boldsymbol{\varphi}_{n'}^*\boldsymbol{\varphi}_n\boldsymbol{\varphi}_n^*\boldsymbol{\varphi}_{n'})\\
&= \sum_{n\in\mathcal{N}}\sum_{\substack{n'\in\mathcal{N}\\ n'\neq n}} |\langle\boldsymbol{\varphi}_n,\boldsymbol{\varphi}_n'\rangle|^2.
\end{aligned}$$

Notice that this last term has $N(N-1)$ summands, meaning that if we divide this inequality by $N(N-1)$, we find the average value and can recover (2):

$$\tfrac{N-D}{D(N-1)} \leq \tfrac{1}{N(N-1)}\sum_{n\in\mathcal{N}}\sum_{\substack{n'\in\mathcal{N}\\ n'\neq n}} |\langle\boldsymbol{\varphi}_n,\boldsymbol{\varphi}_{n'}\rangle|^2 \leq \max_{\substack{n\in\mathcal{N}\\ n\neq n'}}|\langle\boldsymbol{\varphi}_n,\boldsymbol{\varphi}_{n'}\rangle|^2.$$

Here equality on the left occurs if and only if $\{\boldsymbol{\varphi}_n\}_{n\in\mathcal{N}}$ is a UNTF for $\mathbb{H}$ and equality throughout occurs if and only if $\{\boldsymbol{\varphi}_n\}_{n\in\mathcal{N}}$ is an ETF for $\mathbb{H}$.

2.2 Fusion frames: a projection-based perspective

As discussed above, every UNTF for $\mathbb{H}$ yields a sequence of rank-1 (orthogonal) projections that sum to a multiple of the identity. As a generalization of this, a sequence of R-dimensional subspaces $\{\mathcal{U}_n\}_{n\in\mathcal{N}}$ of $\mathbb{H}$ with corresponding rank-R projections $\{\mathbf{P}_n\}_{n\in\mathcal{N}}$ is known as a *tight fusion frame* (TFF) for $\mathbb{H}$ if there exists a constant A such that

$$\sum_{n\in\mathcal{N}} \mathbf{P}_n = A\mathbf{I}.$$

Because $\mathrm{Tr}(A\mathbf{I}) = \sum_{n\in\mathcal{N}} \mathrm{Tr}(\mathbf{P}_n)$, we can see that $A = \frac{NR}{D}$. For any sequence $\{\mathcal{U}_n\}_{n\in\mathcal{N}}$ of R-dimensional subspaces of $\mathbb{H}$, note that

$$0 \le \mathrm{Tr}\Big[\Big(\sum_{n\in\mathcal{N}} \mathbf{P}_n - \tfrac{NR}{D}\mathbf{I}\Big)^2\Big] = \sum_{n\in\mathcal{N}} \sum_{\substack{n'\in\mathcal{N}\\ n'\neq n}} \mathrm{Tr}(\mathbf{P}_n\mathbf{P}_{n'}) - \tfrac{NR(NR-D)}{D}.$$

This provides us with the lower bound

$$\tfrac{R(NR-D)}{D(N-1)} \le \tfrac{1}{N(N-1)} \sum_{n\in\mathcal{N}} \sum_{\substack{n'\in\mathcal{N}\\ n'\neq n}} \mathrm{Tr}(\mathbf{P}_n\mathbf{P}_{n'}), \tag{4}$$

with equality held in (4) if and only if $\{\mathcal{U}_n\}_{n\in\mathcal{N}}$ is a TFF for $\mathbb{H}$. The *chordal distance* between any two of these subspaces is the scaled Frobenius (Hilbert-Schmidt) norm of the difference of the corresponding orthogonal projection operators, $\mathbf{P}_{n_1}, \mathbf{P}_{n_2}$:

$$\mathrm{dist_c}(\mathcal{U}_{n_1}, \mathcal{U}_{n_2}) := \tfrac{1}{\sqrt{2}} \|\mathbf{P}_{n_1} - \mathbf{P}_{n_2}\|_{\mathrm{Fro}} = \left\{\tfrac{1}{2}\mathrm{Tr}[(\mathbf{P}_{n_1} - \mathbf{P}_{n_2})^2]\right\}^{\frac{1}{2}}. \tag{5}$$

To continue, we express the chordal distance (5) between two subspaces $\mathcal{U}_{n_1}, \mathcal{U}_{n_2}$ as

$$\mathrm{dist}^2_{\mathrm{c}}(\mathcal{U}_{n_1}, \mathcal{U}_{n_2}) = \tfrac{1}{2}\|\mathbf{P}_{n_1} - \mathbf{P}_{n_2}\|^2_{\mathrm{Fro}} = \tfrac{1}{2}\mathrm{Tr}\left[(\mathbf{P}_{n_1} - \mathbf{P}_{n_2})^2\right] = R - \mathrm{Tr}(\mathbf{P}_{n_1}\mathbf{P}_{n_2}). \tag{6}$$

In particular, $\mathrm{Tr}(\mathbf{P}_{n_1}\mathbf{P}_{n_2})$ is a real number, and so we continue (4) as

$$\begin{aligned}\tfrac{R(NR-D)}{D(N-1)} &\leq \tfrac{1}{N(N-1)}\sum_{n_1\in\mathcal{N}}\sum_{\substack{n_2\in\mathcal{N}\\ n_1'\neq n_2}}\mathrm{Tr}(\mathbf{P}_{n_1}\mathbf{P}_{n_2})\\ &\leq \max_{n_1\neq n_2}\mathrm{Tr}(\mathbf{P}_{n_1}\mathbf{P}_{n_2}) \qquad (7)\\ &= R-\min_{n_1\neq n_2}\mathrm{dist}_{\mathrm{c}}^2(\mathcal{U}_{n_1},\mathcal{U}_{n_2}).\end{aligned}$$

Equality holds throughout here if and only if $\{\mathcal{U}_n\}_{n\in\mathcal{N}}$ is both a TFF for $\mathbb{H}$ (equality in (6)) and $\{\mathcal{U}_n\}_{n\in\mathcal{N}}$ is equichordal, meaning $\mathrm{dist}_{\mathrm{c}}(\mathcal{U}_{n_1},\mathcal{U}_{n_2})$ is constant over all $n_1\neq n_2$, namely if and only if $\{\mathcal{U}_n\}_{n\in\mathcal{N}}$ is an ECTFF for $\mathbb{H}$. Solving for the minimum chordal distance reveals the simplex bound from [3],

$$\min_{n_1\neq n_2}\mathrm{dist}_{\mathrm{c}}(\mathcal{U}_{n_1},\mathcal{U}_{n_2})\leq\left[\tfrac{R(D-R)}{D}\tfrac{N}{N-1}\right]^{\frac{1}{2}}.$$

Thus, when such an ECTFF exists, it is an optimal *Grassmannian code*, that is, an optimal packing of N points on the *Grassmannian (space)* that consists of all R-dimensional subspaces of $\mathbb{H}$. Other types of optimal Grassmannian packings arise from alternative notions of distance. For example, the *spectral distance* between two R-dimensional subspaces $\mathcal{U}_1,\mathcal{U}_2$ of $\mathbb{H}$ is

$$\mathrm{dist}_{\mathrm{s}}(\mathcal{U}_1,\mathcal{U}_2):=\left[1-\|\mathbf{P}_1\mathbf{P}_2\|_2^2\right]^{\frac{1}{2}}. \qquad (8)$$

Before we demonstrate optimal packings with respect to spectral distance, we must first consider the following.

Lemma 2.2.1. *For any two rank-R projections $\mathbf{P}_1,\mathbf{P}_2$ on a D-dimensional Hilbert* *space* $\mathbb{H}$,

$$\mathrm{Tr}(\mathbf{P}_1\mathbf{P}_2) \le R\|\mathbf{P}_1\mathbf{P}_2\|_2^2, \tag{9}$$

where equality holds in (9) *if and only if there exists* $\sigma \ge 0$ *such that* $\mathbf{P}_1\mathbf{P}_2\mathbf{P}_1 = \sigma^2\mathbf{P}_1$, *that is, if and only if their images are isoclinic [34].*

Proof. Note that $\|\mathbf{P}_1\mathbf{P}_2\|_2^2 = \sigma_{\max}^2(\mathbf{P}_1\mathbf{P}_2) = \lambda_{\max}(\mathbf{P}_1\mathbf{P}_2(\mathbf{P}_1\mathbf{P}_2)^*) = \lambda_{\max}(\mathbf{P}_1\mathbf{P}_2\mathbf{P}_1)$. Then since $\mathrm{Rank}(\mathbf{P}_1\mathbf{P}_2\mathbf{P}_1) = \mathrm{Rank}(\mathbf{P}_1\mathbf{P}_2) \le R$, and $\mathbf{P}_2\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_1\mathbf{P}_2(\mathbf{P}_1\mathbf{P}_2)^* \succeq 0$,

$$\mathrm{Tr}(\mathbf{P}_1\mathbf{P}_2) = \mathrm{Tr}(\mathbf{P}_1^2\mathbf{P}_2) = \mathrm{Tr}(\mathbf{P}_1\mathbf{P}_2\mathbf{P}_1) \le R\lambda_{\max}(\mathbf{P}_1\mathbf{P}_2\mathbf{P}_1) = R\|\mathbf{P}_1\mathbf{P}_2\|_2^2,$$

as claimed. Next, suppose that $\mathbf{P}_1\mathbf{P}_2\mathbf{P}_1 = \sigma^2\mathbf{P}_1$ for some $\sigma \ge 0$. Then the R largest eigenvalues of $\mathbf{P}_1\mathbf{P}_2\mathbf{P}_1$ are equal, and so (9) holds with equality. Conversely, if equality holds then either $\mathbf{P}_1\mathbf{P}_2\mathbf{P}_1 = \mathbf{0}$, in which case $\mathbf{P}_1\mathbf{P}_2\mathbf{P}_1 = \sigma^2\mathbf{P}_1$ with $\sigma = 0$, or $\mathbf{P}_1\mathbf{P}_2\mathbf{P}_1 \ne \mathbf{0}$. In the latter case, having $\mathrm{Tr}(\mathbf{P}_1\mathbf{P}_2\mathbf{P}_1) = R\lambda_{\max}(\mathbf{P}_1\mathbf{P}_2\mathbf{P}_1)$ implies that $\mathbf{P}_1\mathbf{P}_2\mathbf{P}_1$ is a positive multiple of a rank-R projection. Furthermore, since the image of $\mathbf{P}_1\mathbf{P}_2\mathbf{P}_1$ is contained in that of $\mathbf{P}_1$, it is necessarily a positive multiple of $\mathbf{P}_1$. □

Lemma 2.2.1 gives us a second way to continue (4): for any sequence $\{\mathcal{U}_n\}_{n\in\mathcal{N}}$ of R-dimensional subspaces of $\mathbb{H}$,

$$\begin{aligned}\tfrac{NR(NR-D)}{D} &\le \sum_{n_1\in\mathcal{N}}\sum_{\substack{n_2\in\mathcal{N}\\ n_2\ne n_1}} \mathrm{Tr}(\mathbf{P}_{n_1}\mathbf{P}_{n_2})\\ &\le \sum_{n_1\in\mathcal{N}}\sum_{\substack{n_2\in\mathcal{N}\\ n_2\ne n_1}} R\|\mathbf{P}_{n_1}\mathbf{P}_{n_2}\|_2^2\\ &\le N(N-1)R\max_{n_1\ne n_2}\|\mathbf{P}_{n_1}\mathbf{P}_{n_2}\|_2^2.\end{aligned}$$

This provides a so-called *spectral bound*,

$$\left[\tfrac{NR-D}{D(N-1)}\right]^{\frac{1}{2}} \leq \max_{n_1\neq n_2} \|\mathbf{P}_{n_1}\mathbf{P}_{n_2}\|_2, \tag{10}$$

where $\{\mathcal{U}_n\}_{n\in\mathcal{N}}$ achieves equality in (10) if and only if it is a TFF for $\mathbb{H}$ (4) such that for every n_1, n_2 in $\mathcal{N}$, there exists $\sigma_{n_1,n_2} \geq 0$ such that $\mathbf{P}_{n_1}\mathbf{P}_{n_2}\mathbf{P}_{n_1} = \sigma^2_{n_1,n_2}\mathbf{P}_{n_1}$ (Lemma 2.2.1), where moreover $\|\mathbf{P}_{n_1}\mathbf{P}_{n_2}\|_2^2 = \|\mathbf{P}_{n_1}\mathbf{P}_{n_2}\mathbf{P}_{n_1}\|_2 = \|\sigma^2_{n_1,n_2}\mathbf{P}_{n_1}\|_2 = \sigma^2_{n_1,n_2}$ is constant over all $n_1 \neq n_2$. Specifically, we have equality in (10) if and only if $\{\mathcal{U}_n\}_{n\in\mathcal{N}}$ is an EITFF for $\mathbb{H}$ [17]. In this case, $\mathrm{Tr}(\mathbf{P}_{n_1}\mathbf{P}_{n_2}) = R\|\mathbf{P}_{n_1}\mathbf{P}_{n_2}\|_2^2 = R\sigma^2$ is constant over all $n_1 \neq n_2$, meaning that every EITFF for $\mathbb{H}$ is necessarily an ECTFF. Expressing (10) in terms of the spectral distance (8) yields the spectral packing bound from [5]:

$$\min_{n_1\neq n_2} \mathrm{dist}_{\mathrm{s}}^2(\mathcal{U}_{n_1},\mathcal{U}_{n_2}) = \min_{n_1\neq n_2}\left(1-\|\mathbf{P}_{n_1}\mathbf{P}_{n_2}\|_2^2\right) \leq 1 - \max_{n_1\neq n_2}\|\mathbf{P}_{n_1}\mathbf{P}_{n_2}\|_2^2 \leq 1 - \tfrac{NR-D}{D(N-1)}.$$

2.3 Fusion frames: a basis-based perspective

For an alternative perspective on these same ideas, for any sequence $\{\mathcal{U}_n\}_{n\in\mathcal{N}}$ of R-dimensional subspaces of $\mathbb{H}$, and any R-element index set $\mathcal{R}$, we can let $\{\boldsymbol{\varphi}_{n,r}\}_{r\in\mathcal{R}}$ be any ONB for $\mathcal{U}_n$, and let $\mathbf{\Phi}_n$ be its synthesis operator. Here $\mathbf{P}_n = \mathbf{\Phi}_n\mathbf{\Phi}_n^*$ for each n implying that the frame operator of the concatenation $\{\boldsymbol{\varphi}_{n,r}\}_{n\in\mathcal{N},r\in\mathcal{R}}$ of these N orthonormal sequences is

$$\mathbf{\Phi}\mathbf{\Phi}^* = \sum_{(n,r)\in\mathcal{N}\times\mathcal{R}} \boldsymbol{\varphi}_{n,r}\boldsymbol{\varphi}_{n,r}^* = \sum_{n\in\mathcal{N}}\Bigl(\sum_{r\in\mathcal{R}}\boldsymbol{\varphi}_{n,r}\boldsymbol{\varphi}_{n,r}^*\Bigr) = \sum_{n\in\mathcal{N}}\mathbf{\Phi}_n\mathbf{\Phi}_n^* = \sum_{n\in\mathcal{N}}\mathbf{P}_n,$$

namely the *fusion frame operator* of $\{\mathcal{U}_n\}_{n\in\mathcal{N}}$. In particular, $\{\mathcal{U}_n\}_{n\in\mathcal{N}}$ is a TFF for $\mathbb{H}$ if and only if $\{\boldsymbol{\varphi}_{n,r}\}_{(n,r)\in\mathcal{N}\times\mathcal{R}}$ is a UNTF for $\mathbb{H}$. With this in mind, note that

$$\begin{aligned}\mathrm{Tr}(\mathbf{P}_{n_1}\mathbf{P}_{n_2}) &= \mathrm{Tr}(\boldsymbol{\Phi}_{n_1}\boldsymbol{\Phi}_{n_1}^*\boldsymbol{\Phi}_{n_2}\boldsymbol{\Phi}_{n_2}^*)\\ &= \mathrm{Tr}(\boldsymbol{\Phi}_{n_2}^*\boldsymbol{\Phi}_{n_1}\boldsymbol{\Phi}_{n_1}^*\boldsymbol{\Phi}_{n_2})\\ &= \mathrm{Tr}\big((\boldsymbol{\Phi}_{n_1}^*\boldsymbol{\Phi}_{n_2})^*\boldsymbol{\Phi}_{n_1}^*\boldsymbol{\Phi}_{n_2}\big)\\ &= \|\boldsymbol{\Phi}_{n_1}^*\boldsymbol{\Phi}_{n_2}\|_{\mathrm{Fro}}^2.\end{aligned}$$

With this, we see that the bound from (4) becomes

$$\tfrac{R(NR-D)}{D(N-1)} \le \tfrac{1}{N(N-1)}\sum_{n_1\in\mathcal{N}}\sum_{\substack{n_2\in\mathcal{N}\\ n_1\ne n_2}}\mathrm{Tr}(\mathbf{P}_{n_1}\mathbf{P}_{n_2}) = \tfrac{1}{N(N-1)}\sum_{n_1\in\mathcal{N}}\sum_{\substack{n_2\in\mathcal{N}\\ n_1\ne n_2}}\|\boldsymbol{\Phi}_{n_1}^*\boldsymbol{\Phi}_{n_2}\|_{\mathrm{Fro}}^2, \qquad (11)$$

where equality is achieved if and only if $\{\mathcal{U}_n\}_{n\in\mathcal{N}}$ is a TFF for $\mathbb{H}$, namely if and only if $\{\boldsymbol{\varphi}_{n,r}\}_{n\in\mathcal{N},r\in\mathcal{R}}$ is a UNTF for $\mathbb{H}$. In this setting, the $(\mathcal{N}\times\mathcal{R})\times(\mathcal{N}\times\mathcal{R})$ Gram matrix of $\{\boldsymbol{\varphi}_{n,r}\}_{n\in\mathcal{N},r\in\mathcal{R}}$ has entries

$$\big(\boldsymbol{\Phi}^*\boldsymbol{\Phi}\big)\big((n_1,r_1),(n_2,r_2)\big) = \langle\boldsymbol{\varphi}_{n_1,r_1},\boldsymbol{\varphi}_{n_2,r_2}\rangle = \langle\boldsymbol{\Phi}_{n_1}\boldsymbol{\delta}_{r_1},\boldsymbol{\Phi}_{n_2}\boldsymbol{\delta}_{r_2}\rangle = \big(\boldsymbol{\Phi}_{n_1}^*\boldsymbol{\Phi}_{n_2}\big)(r_1,r_2),$$

and so is naturally regarded as an $\mathcal{N}\times\mathcal{N}$ array (block matrix) of $(\mathcal{R}\times\mathcal{R})$-indexed blocks. Specifically, for any n_1, n_2 in $\mathcal{N}$, the (n_1,n_2)th block is the corresponding cross-Gram matrix $\boldsymbol{\Phi}_{n_1}^*\boldsymbol{\Phi}_{n_2}$ whose entries are inner products of members of our chosen ONBs for $\mathcal{U}_{n_1}$ and $\mathcal{U}_{n_2}$. We can express the chordal and spectral distances between $\mathcal{U}_{n_1}$ and $\mathcal{U}_{n_2}$ in terms of their cross-Gram matrix:

$$\begin{aligned}\mathrm{dist}_{\mathrm{c}}^2(\mathcal{U}_{n_1},\mathcal{U}_{n_2}) &= R-\mathrm{Tr}(\mathbf{P}_{n_1}\mathbf{P}_{n_2}) = R-\|\boldsymbol{\Phi}_{n_1}^*\boldsymbol{\Phi}_{n_2}\|_{\mathrm{Fro}}^2,\\ \mathrm{dist}_{\mathrm{s}}^2(\mathcal{U}_{n_1},\mathcal{U}_{n_2}) &= 1-\|\mathbf{P}_{n_1}\mathbf{P}_{n_2}\|_2^2 = 1-\|\boldsymbol{\Phi}_{n_1}^*\boldsymbol{\Phi}_{n_2}\|_2^2.\end{aligned}$$

To formally prove this latter relationship, note

$$\begin{aligned}
\|\mathbf{P}_{n_1}\mathbf{P}_{n_2}\|_2 &= \|\mathbf{\Phi}_{n_1}\mathbf{\Phi}_{n_1}^*\mathbf{\Phi}_{n_2}\mathbf{\Phi}_{n_2}^*\|_2\\
&\leq \|\mathbf{\Phi}_{n_1}\|_2\|\mathbf{\Phi}_{n_1}^*\mathbf{\Phi}_{n_2}\|_2\|\mathbf{\Phi}_{n_2}^*\|_2\\
&= \|\mathbf{\Phi}_{n_1}^*\mathbf{\Phi}_{n_2}\|_2,\\
\|\mathbf{\Phi}_{n_1}^*\mathbf{\Phi}_{n_2}\|_2 &= \|\mathbf{\Phi}_{n_1}^*\mathbf{\Phi}_{n_1}^*\mathbf{\Phi}_{n_1}^*\mathbf{\Phi}_{n_2}\mathbf{\Phi}_{n_2}^*\mathbf{\Phi}_{n_2}\|_2\\
&\leq \|\mathbf{\Phi}_{n_2}\|_2\|\mathbf{\Phi}_{n_1}\mathbf{\Phi}_{n_1}^*\mathbf{\Phi}_{n_2}\mathbf{\Phi}_{n_2}^*\|_2\|\mathbf{\Phi}_{n_2}\|_2\\
&= \|\mathbf{P}_{n_1}\mathbf{P}_{n_2}\|_2.
\end{aligned}$$

As such the bounds (7) and (10) can alternatively be written as

$$\max_{n_1\neq n_2}\|\mathbf{\Phi}_{n_1}^*\mathbf{\Phi}_{n_2}\|_{\mathrm{Fro}} \geq \sqrt{\tfrac{R(NR-D)}{D(N-1)}}, \quad \max_{n_1\neq n_2}\|\mathbf{\Phi}_{n_1}^*\mathbf{\Phi}_{n_2}\|_2 \geq \sqrt{\tfrac{NR-D}{D(N-1)}}, \tag{12}$$

where as before, equality holds if and only if $\{\mathcal{U}_n\}_{n\in\mathcal{N}}$ is an ECTFF or EITFF for $\mathbb{H}$, respectively. Note that if the left inequality is scaled by $\sqrt{R}$, we can combine both inequalities as in (1). In the special case where $R = 1$ and for each n we have $\mathcal{U}_n = \langle\boldsymbol{\varphi}_n\rangle$ where $\|\boldsymbol{\varphi}_n\| = 1$, both inequalities in (12) reduce to the Welch bound (2), and $\{\mathcal{U}_n\}_{n\in\mathcal{N}}$ is an ECTFF/EITFF for $\mathbb{H}$ if and only if $\{\boldsymbol{\varphi}_n\}_{n\in\mathcal{N}}$ is an ETF for $\mathbb{H}$.

For any $n_1, n_2 \in \mathcal{N}$, let $\{\sigma_{r,n_1,n_2}\}_{r\in\mathcal{R}}$ be the singular values of $\mathbf{\Phi}_{n_1}^*\mathbf{\Phi}_{n_2}$. Since $\|\mathbf{\Phi}_{n_1}^*\mathbf{\Phi}_{n_2}\|_2 \leq \|\mathbf{\Phi}_{n_1}^*\|_2\|\mathbf{\Phi}_{n_2}\|_2 = 1$ these values are bounded between 0 and 1. There thus exists a corresponding sequence of angles $\{\theta_{r,n_1,n_2}\}_{r\in\mathcal{R}}$ with $0 \leq \theta_{r,n_1,n_2} \leq \frac{\pi}{2}$ and $\cos(\theta_{r,n_1,n_2}) = \sigma_{r,n_1,n_2}$ for all $r \in \mathcal{R}$. These are known as the *principal angles* between $\mathcal{U}_{n_1}$ and $\mathcal{U}_{n_2}$. One can show that $\{\mathcal{U}_n\}_{n\in\mathcal{N}}$ is equi-isoclinic if and only if the value of θ_{r,n_1,n_2} is constant over all $n_1 \neq n_2$. In terms of these angles, the chordal distance between $\mathcal{U}_{n_1}$ and $\mathcal{U}_{n_2}$ is

$$\begin{aligned}\operatorname{dist}_{\mathrm{c}}(\mathcal{U}_{n_1},\mathcal{U}_{n_2}) &= \big(R - \operatorname{Tr}(\mathbf{P}_{n_1}\mathbf{P}_{n_2})\big)^{\frac{1}{2}}\\ &= \big(R - \sum_{r\in\mathcal{R}} \cos^2(\theta_{r,n_1,n_2})\big)^{\frac{1}{2}}\\ &= \Big(\sum_{r\in\mathcal{R}} \sin^2(\theta_{r,n_1,n_2})\Big)^{\frac{1}{2}},\end{aligned}$$

while their spectral distance is simply

$$\operatorname{dist}_{\mathrm{s}}(\mathcal{U}_1,\mathcal{U}_2) = \big(1 - \|\mathbf{P}_1\mathbf{P}_2\|_2^2\big)^{\frac{1}{2}} = \big(1-\sigma_{\max}^2\big)^{\frac{1}{2}} = \big(1-\cos^2(\theta_{\min})\big)^{\frac{1}{2}} = \sin(\theta_{\min}).$$

Though EITFFs are challenging to construct in general, some of them are trivial. To elaborate, we can let $\mathbb{H}$ be a D-dimensional Hilbert space, and let $\{\mathcal{U}_n\}_{n\in\mathcal{N}}$ be N copies of the entire space. It is then not hard to see that $\{\mathcal{U}_n\}_{n\in\mathcal{N}}$ is a $\operatorname{TFF}(D,N,D)$, specifically one that is necessarily equi-isoclinic. Another option is to instead let $\{\boldsymbol{\varphi}_{n,r}\}_{n\in\mathcal{N},r\in\mathcal{R}}$ be an ONB for $\mathbb{H}$. In this case $D = NR$, and we can partition $\{\boldsymbol{\varphi}_{n,r}\}_{n\in\mathcal{N},r\in\mathcal{R}}$ into N sets of ONBs for R-dimensional subspaces of $\mathbb{H}$. Let $\{\mathcal{U}_n\}_{n\in\mathcal{N}}$ be these corresponding subspaces, then they constitute a $\operatorname{TFF}(NR,N,R)$, again, one that is necessarily equi-isoclinic. These EITFFs can also be real by letting $\mathbb{H} = \mathbb{R}^D$.

2.4 Naimark and spatial complements

Let $\{\mathcal{U}_n\}_{n\in\mathcal{N}}$ be a $\operatorname{TFF}(D,N,R)$ for $\mathbb{F}^D$ with $D \neq NR$. Additionally for each n in $\mathcal{N}$, let $\{\boldsymbol{\varphi}_{n,r}\}_{r\in\mathcal{R}}$ be an ONB for $\mathcal{U}_n$. Recall that if $\boldsymbol{\Phi}$ is the synthesis operator for $\{\boldsymbol{\varphi}_{n,r}\}_{n\in\mathcal{N},r\in\mathcal{R}}$, then the fusion frame operator is

$$\boldsymbol{\Phi}\boldsymbol{\Phi}^* = \sum_{n\in\mathcal{N}} \boldsymbol{\Phi}_n\boldsymbol{\Phi}_n^* = \sum_{n\in\mathcal{N}} \mathbf{P}_n = \tfrac{NR}{D}\mathbf{I}.$$

So $\frac{D}{NR}\mathbf{\Phi}^*\mathbf{\Phi}$ is an orthogonal projection operator with rank D and $\mathbf{I}-\frac{D}{NR}\mathbf{\Phi}^*\mathbf{\Phi}$ is then an orthogonal projection operator with rank $NR-D$. If we scale this operator by $\frac{NR}{NR-D}$, that is let

$$\mathbf{\Psi}^*\mathbf{\Psi} = \tfrac{NR}{NR-D}\left(\mathbf{I}-\tfrac{D}{NR}\mathbf{\Phi}^*\mathbf{\Phi}\right),$$

then the diagonal blocks of its Gram matrix will be $\mathcal{R}\times\mathcal{R}$ identities. Then

$$\mathbf{\Psi}_{n_1}^*\mathbf{\Psi}_{n_2} = \begin{cases} \mathbf{I}, & n_1 = n_2, \\ -\tfrac{D}{NR-D}\mathbf{\Phi}_{n_1}^*\mathbf{\Phi}_{n_2}, & n_1 \neq n_2. \end{cases}$$

Then $\mathbf{\Psi}$ is the synthesis operator for $\{\boldsymbol{\psi}_{n,r}\}_{(n,r)\in\mathcal{N}\times\mathcal{R}}$, a UNTF$(NR-D, NR)$ for some NR-dimensional Hilbert space $\mathbb{K}$ where $\{\boldsymbol{\psi}_{n,r}\}_{r\in\mathcal{R}}$ is orthonormal for all n. Additionally, defining $\mathcal{V}_n := \operatorname{span}\{\boldsymbol{\psi}_{n,r}\}_{r\in\mathcal{R}}$ for all n in $\mathcal{N}$, then $\{\mathcal{V}_n\}_{n\in\mathcal{N}}$ is a TFF$(NR-D, N, R)$ for $\mathbb{K}$. Since $\mathbf{\Psi}_{n_1}^*\mathbf{\Psi}_{n_2} = -\frac{D}{NR-D}\mathbf{\Phi}_{n_1}^*\mathbf{\Phi}_{n_2}$, then

$$\begin{aligned} \|\mathbf{\Psi}_{n_1}^*\mathbf{\Psi}_{n_2}\|_{\mathrm{Fro}}^2 &= \left(\tfrac{D}{NR-D}\right)^2\|\mathbf{\Phi}_{n_1}^*\mathbf{\Phi}_{n_2}\|_{\mathrm{Fro}}^2, \\ \|\mathbf{\Psi}_{n_1}^*\mathbf{\Psi}_{n_2}\|_2^2 &= \left(\tfrac{D}{NR-D}\right)^2\|\mathbf{\Phi}_{n_1}^*\mathbf{\Phi}_{n_2}\|_2^2. \end{aligned}$$

By (12), if $\{\mathcal{U}_n\}_{n\in\mathcal{N}}$ is an EC/EITFF for $\mathbb{H}$, then $\{\mathcal{V}_n\}_{n\in\mathcal{N}}$ is an EC/EITFF for $\mathbb{K}$.

The *spatial complement* of any sequence $\{\mathcal{U}_n\}_{n\in\mathcal{N}}$ of R-dimensional subspaces of $\mathbb{H}$ is $\{\mathcal{U}_n^\perp\}_{n\in\mathcal{N}}$ (with $D\neq R$). For such a sequence, the orthogonal projection operators on the subspaces are $\{\mathbf{I}-\mathbf{P}_n\}_{n\in\mathcal{N}}$ where $\mathbf{P}_n$ is the orthogonal projection operator onto $\mathcal{U}_n$. Because $\sum_{n\in\mathcal{N}}(\mathbf{I}-\mathbf{P}_n) = N\mathbf{I}-\sum_{n\in\mathcal{N}}\mathbf{P}_n$, it is clear the $\{\mathcal{U}_n\}_{n\in\mathcal{N}}$ is a TFF for $\mathbb{H}$ if and only if its spatial complement is as well. Additionally,

$$\begin{aligned}
\mathrm{dist_c}(\mathcal{U}_{n_1}^{\perp},\mathcal{U}_{n_2}^{\perp}) &= \tfrac{1}{\sqrt{2}}\|(\mathbf{I}-\mathbf{P}_{n_1})-\mathbf{I}-\mathbf{P}_{n_2})\|_{\mathrm{Fro}}\\
&= \tfrac{1}{\sqrt{2}}\|(\mathbf{P}_{n_1}-\mathbf{P}_{n_2})\|_{\mathrm{Fro}}\\
&= \mathrm{dist_c}(\mathcal{U}_{n_1},\mathcal{U}_{n_2}),
\end{aligned}$$

therefore $\{\mathcal{U}_n\}_{n\in\mathcal{N}}$ is an ECTFF if and only if $\{\mathcal{U}_n^{\perp}\}_{n\in\mathcal{N}}$ is as well, specifically, $\{\mathcal{U}_n\}_{n\in\mathcal{N}}$ is an ECTFF(D,N,R) if and only if $\{\mathcal{U}_n^{\perp}\}_{n\in\mathcal{N}}$ is an ECTFF$(D,N,D-R)$. We caution however, that the spatial complement of an EITFF(D,N,R) for $\mathbb{H}$ is not an EITFF $(D,N,D-R)$ for $\mathbb{H}$, except in the special case where $R=\frac{1}{2}D$.

2.5 Compressed sensing

Let $\{\boldsymbol{\varphi}_n\}_{n\in\mathcal{N}}$ be a sequence of N vectors in a D-dimensional Hilbert space $\mathbb{H}$ with $N>D$. The *spark* of $\{\boldsymbol{\varphi}_n\}_{n\in\mathcal{N}}$ is the size of the smallest subset $\mathcal{K}$ of $\mathcal{N}$ such that $\{\boldsymbol{\varphi}_n\}_{n\in\mathcal{K}}$ is linearly dependent:

$$\begin{aligned}
\mathrm{spark}\left(\{\boldsymbol{\varphi}_n\}_{n\in\mathcal{N}}\right) :&= \min\{\#(\mathcal{K}) : \mathcal{K}\subseteq\mathcal{N}, \{\boldsymbol{\varphi}_n\}_{n\in\mathcal{K}} \text{ are linearly dependent}\}\\
&= \min\{\|\mathbf{x}\|_0 : \mathbf{x}\in\mathrm{Null}(\boldsymbol{\Phi}), \mathbf{x}\neq\mathbf{0}\},
\end{aligned}$$

where $\|\mathbf{x}\|_0$ is the cardinality of the *support* of $\mathbf{x}$. The spark is always bounded above by $\dim(\mathcal{H})+1=D+1$, and when $\{\boldsymbol{\varphi}_n\}_{n\in\mathcal{N}}$ achieves this bound, it is known as a *full-spark frame* for $\mathbb{H}$. Let μ be the coherence and $\boldsymbol{\Phi}$ be the synthesis operator for $\{\boldsymbol{\varphi}_n\}_{n\in\mathcal{N}}$. A vector $\mathbf{x}$ is called K-*sparse* if at most K elements of $\mathbf{x}$ are non-zero.

Let $\mathbf{y}=\boldsymbol{\Phi}\mathbf{x}_0$ for some K-sparse $\mathbf{x}_0$. In general, there are an infinite number of solutions $\mathbf{x}$ to $\boldsymbol{\Phi}\mathbf{x}=\mathbf{y}$ since, having $N>D$, $\mathrm{Null}(\boldsymbol{\Phi})$ is non-trivial; however, if $\mathrm{spark}\{\boldsymbol{\varphi}_n\}_{n\in\mathcal{N}}>2K$, then $\mathbf{x}_0$ is the unique K-sparse solution to $\mathbf{y}=\boldsymbol{\Phi}\mathbf{x}$. To see this, suppose that $\text{spark}\{\boldsymbol{\varphi}_n\}_{n\in\mathcal{N}} > 2K$, and $\mathbf{x}$ and $\mathbf{x}_0$ are any two K-sparse solutions to $\mathbf{\Phi}\mathbf{x}' = \mathbf{y}$, then

$$\mathbf{\Phi}(\mathbf{x}_0 - \mathbf{x}) = \mathbf{0},$$

meaning that $\mathbf{x}_0 - \mathbf{x} \in \text{Null}(\mathbf{\Phi})$. If $\mathbf{x}_0 - \mathbf{x} = \mathbf{0}$, this is obviously the case. If we assume otherwise, then

$$\|\mathbf{x}_0 - \mathbf{x}\|_0 \geq \min\{\|\mathbf{x}\|_0 : \mathbf{x} \in \text{Null}(\mathbf{\Phi}), \mathbf{x} \neq \mathbf{0}\} = \text{spark}(\mathbf{\Phi}).$$

However, $\|\mathbf{x}_0 - \mathbf{x}\|_0 \leq \|\mathbf{x}_0\|_0 + \|\mathbf{x}\|_0 \leq 2K$, which is a contradiction.

A sequence $\{\boldsymbol{\varphi}_n\}_{n\in\mathcal{N}}$ is said to have the (K, δ) *restricted isometry property* (RIP) if $0 \leq \delta < 1$ such that $\|\mathbf{\Phi}_{\mathcal{K}}^*\mathbf{\Phi}_{\mathcal{K}} - \mathbf{I}\|_2 < \delta$ for all $\mathcal{K} \subseteq \mathcal{N}$. The Gershgorin circle theorem (GCT) states that if λ is an eigenvalue of an $N \times N$ matrix $\mathbf{A}$, then

$$|\lambda - \mathbf{A}(m,m)| \leq \sum_{\substack{n=1 \\ n\neq m}}^{N} |\mathbf{A}(m,n)|.$$

We can estimate $\|\mathbf{\Phi}_{\mathcal{K}}^*\mathbf{\Phi}_{\mathcal{K}} - \mathbf{I}\|_2$ using the GCT:

$$\|\mathbf{\Phi}_{\mathcal{K}}^*\mathbf{\Phi}_{\mathcal{K}} - \mathbf{I}\|_2 \leq \max_{n_1\in\mathcal{K}} \sum_{\substack{n_2\in\mathcal{K} \\ n_2\neq n_1}} |\langle \boldsymbol{\varphi}_{n_1}, \boldsymbol{\varphi}_{n_2}\rangle| \leq (K-1)\mu. \tag{13}$$

Therefore, if $(K-1)\mu < 1$ (meaning $\text{spark}\{\boldsymbol{\varphi}_n\}_{n\in\mathcal{K}} < \frac{1}{\mu} + 1$), $\mathbf{\Phi}_{\mathcal{K}}^*\mathbf{\Phi}_{\mathcal{K}}$ is invertible, meaning $\{\boldsymbol{\varphi}_n\}_{n\in\mathcal{K}}$ is linearly independent. If this is true for all $\mathcal{K}$, then $\mathbf{\Phi}$ has the (K, δ) RIP.

Compressed sensing techniques focus on the recovery of such K-sparse signals from underdetermined systems, e.g. solve $\mathbf{y} = \mathbf{\Phi}\mathbf{x}_0$ exactly [7, 8]. Two popular methods are *orthogonal matching pursuit* (OMP), and *basis pursuit* (BP) [39]. By Corollary

3.6 of [39], both OMP and BP will recover $\mathbf{x}_0$ exactly as long as $K < \frac{1}{2}\left(\frac{1}{\mu}+1\right)$ (a restriction on (13)).

If the signal $\mathbf{x} \in \mathbb{F}^{\mathcal{N}\times\mathcal{R}}$ for some indexing sets $\mathcal{N}, \mathcal{R}$, and we define $\mathbf{x}_n(r) := \mathbf{x}(n,r)$, we say that $\mathbf{x}$ is K-*block sparse* if $\mathbf{x}_n \neq \mathbf{0}$ for at most K choices of n [9]. The *block coherence* $R\mu_B$ of any sequence $\{\boldsymbol{\varphi}_{n,r}\}_{n\in\mathcal{N},r\in\mathcal{R}}$ of vectors with the property that $\{\boldsymbol{\varphi}_{n,r}\}_{r\in\mathcal{R}}$ is orthonormal for each n in $\mathcal{N}$[9] is

$$R\mu_B := \max_{n_1\neq n_2} \|\mathbf{\Phi}^*_{n_1}\mathbf{\Phi}_{n_2}\|_2. \tag{14}$$

Theorem 3 of [9], gives that BOMP will recover a (K,R) block-sparse signal $\mathbf{x}_0$ if

$$K < \frac{1}{2}\left(\frac{1}{R\mu_B}+1\right).$$

Furthermore, combining (14) and (12) shows that the block coherence is bounded below by $\sqrt{\frac{NR-D}{D(N-1)}}$, with equality if and only if $\mathbf{\Phi}$ is the synthesis operator for an EITFF, making them ideally suited for the BOMP algorithm.

2.6 Harmonic analysis

A *character* $\gamma : \mathcal{G} \to \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ of a finite abelian group $\mathcal{G}$ of order G is a homomorphism from $\mathcal{G}$ into the unit circle. The set of all characters for a finite abelian group $\mathcal{G}$ is called the *Pontryagin dual* of $\mathcal{G}$, denoted as $\widehat{\mathcal{G}}$. It is well-known that $\widehat{\mathcal{G}}$ is itself a finite abelian group under entrywise multiplication, that is, $(\gamma_1\gamma_2)(g) := \gamma_1(g)\gamma_2(g)$. In fact, $\widehat{\mathcal{G}}$ is isomorphic to $\mathcal{G}$, and moreover $g \mapsto (\gamma \to \gamma(g))$ is an isomorphism from $\mathcal{G}$ onto the dual of its dual. The synthesis operator of $\{\gamma\}_{\gamma\in\widehat{\mathcal{G}}}$ is called the *character table* $\mathbf{\Gamma} : \mathbb{C}^{\widehat{\mathcal{G}}} \to \mathbb{C}^{\mathcal{G}}$, which we regard as a $\mathcal{G}\times\widehat{\mathcal{G}}$ matrix Γ with entries $\mathbf{\Gamma}(g,\gamma) := \gamma(g)$. Its adjoint, $\mathbf{\Gamma}^* : \mathbb{C}^{\mathcal{G}} \to \mathbb{C}^{\widehat{\mathcal{G}}}$, $(\mathbf{\Gamma}^*\mathbf{x})(\gamma) = \langle\gamma, \mathbf{x}\rangle$, is the analysis operator of the characters, namely the *discrete Fourier transform* (DFT) on $\mathcal{G}$. It is well known that $\{\gamma\}_{\gamma\in\widehat{\mathcal{G}}}$ is an orthogonal basis for $\mathbb{C}^{\mathcal{G}}$, with $\|\gamma\|^2 = \sum_{g\in\mathcal{G}} |\gamma(g)|^2 = G$ for each γ in $\widehat{\mathcal{G}}$, implying that $\mathbf{\Gamma}\mathbf{\Gamma}^* = G\mathbf{I}_{\mathcal{G}}, \mathbf{\Gamma}^*\mathbf{\Gamma} = G\mathbf{I}_{\widehat{\mathcal{G}}}$. That is, $\frac{1}{\sqrt{G}}\mathbf{\Gamma}$ is unitary.

For any $g \in \mathcal{G}$, the corresponding *translation operator* $\mathbf{T}^g : \mathbb{C}^{\mathcal{G}} \to \mathbb{C}^{\mathcal{G}}$, is defined by $(\mathbf{T}^g\mathbf{x})(g') := \mathbf{x}(g' - g)$. For any $\gamma \in \widehat{\mathcal{G}}$, the corresponding *modulation operator* is $\mathbf{M}^\gamma : \mathbb{C}^{\mathcal{G}} \to \mathbb{C}^{\mathcal{G}}$, $(\mathbf{M}^\gamma\mathbf{x})(g) := \gamma(g)\mathbf{x}(g)$. Because $\widehat{\mathcal{G}}$ is also a finite abelian group, we can also consider instances of translation and modulation operators over $\widehat{\mathcal{G}}$: for any $\gamma \in \widehat{\mathcal{G}}$, we have $\mathbf{T}^\gamma : \mathbb{C}^{\widehat{\mathcal{G}}} \to \mathbb{C}^{\widehat{\mathcal{G}}}$, $(\mathbf{T}^\gamma\mathbf{y})(\gamma') := \mathbf{y}(\gamma'\gamma^{-1})$, while for any $g \in \mathcal{G}$ (under the aforementioned isomorphism between $\mathcal{G}$ and the dual of its dual) we have $\mathbf{M}^g : \mathbb{C}^{\widehat{\mathcal{G}}} \to \mathbb{C}^{\widehat{\mathcal{G}}}$, $(\mathbf{M}^\gamma\mathbf{y})(\gamma) := g(\gamma)\mathbf{y}(\gamma) = \gamma(g)\mathbf{y}(\gamma)$. Translation and modulation are conjugates via the DFT:

$$\mathbf{\Gamma}^*\mathbf{T}^g = \mathbf{M}^{-g}\mathbf{\Gamma}^*, \quad \mathbf{\Gamma}\mathbf{M}^g = \mathbf{T}^{-g}\mathbf{\Gamma}, \quad \mathbf{\Gamma}^*\mathbf{M}^\gamma = \mathbf{T}^\gamma\mathbf{\Gamma}^*, \quad \mathbf{\Gamma}\mathbf{T}^\gamma = \mathbf{M}^\gamma\mathbf{\Gamma}.$$

For any $\mathbf{x} \in \mathbb{C}^{\mathcal{G}}$, the corresponding *filter* is $\mathbf{X} : \mathbb{C}^{\mathcal{G}} \to \mathbb{C}^{\mathcal{G}}$, $\mathbf{X} := \sum_{g\in\mathcal{G}} \mathbf{x}(g)\mathbf{T}^g$. Here $\mathbf{x} = \mathbf{X}\boldsymbol{\delta}_0$ is known as the *impulse response* of $\mathbf{X}$. A $\mathcal{G} \times \mathcal{G}$ matrix $\mathbf{X}$ is a filter if and only if it is $\mathcal{G}$-*circulant*, that is, if and only if there exists some $\mathbf{x} \in \mathbb{C}^{\mathcal{G}}$ with $\mathbf{x}(g, g') = \mathbf{X}(g - g')$ for all $g, g' \in \mathcal{G}$; in this case, $\mathbf{x}$ is necessarily the 0th column of $\mathbf{X}$, namely its impulse response. The composition of two filters is also a filter. In fact, for any $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{C}^{\mathcal{G}}$,

$$\mathbf{X}_1\mathbf{X}_2 = \Big(\sum_{g\in\mathcal{G}} \mathbf{x}_1(g)\mathbf{T}^g\Big)\Big(\sum_{g\in\mathcal{G}} \mathbf{x}_2(g)\mathbf{T}^g\Big) = \sum_{g\in\mathcal{G}} (\mathbf{x}_1 * \mathbf{x}_2)(g)\mathbf{T}^g,$$

where $\mathbf{x}_1 * \mathbf{x}_2 \in \mathbb{C}^{\mathcal{G}}$, $(\mathbf{x}_1 * \mathbf{x}_2)(g) := \sum_{g'\in\mathcal{G}} \mathbf{x}_1(g')\mathbf{x}_2(g - g')$ is the *convolution* of $\mathbf{x}_1$ and $\mathbf{x}_2$. The adjoint of a filter is also a filter: for any $\mathbf{x} \in \mathbb{C}^{\mathcal{G}}$,

$$\mathbf{X}^* = \Big(\sum_{g\in\mathcal{G}} \mathbf{x}(g)\mathbf{T}^g\Big)^* = \sum_{g\in\mathcal{G}} \tilde{\mathbf{x}}(g)\mathbf{T}^g,$$

where $\tilde{\mathbf{x}} \in \mathbb{C}^{\mathcal{G}}$, $\tilde{\mathbf{x}}(g) := \overline{\mathbf{x}(-g)}$ is the *involution* of $\mathbf{x}$. Under DFTs, convolution and involution correspond to entrywise multiplication and conjugation, respectively:

$$\begin{aligned}
\mathbf{\Gamma}^*(\mathbf{x}_1 * \mathbf{x}_2) &= (\mathbf{\Gamma}^*\mathbf{x}_1) \odot (\mathbf{\Gamma}^*\mathbf{x}_2),\\
\mathbf{\Gamma}^*(\mathbf{x}_1 \odot \mathbf{x}_2) &= \tfrac{1}{G}(\mathbf{\Gamma}^*\mathbf{x}_1) * (\mathbf{\Gamma}^*\mathbf{x}_2),\\
\mathbf{\Gamma}^*\tilde{\mathbf{x}} &= \overline{(\mathbf{\Gamma}^*\mathbf{x})},\\
\mathbf{\Gamma}^*\overline{\mathbf{x}} &= (\mathbf{\Gamma}^*\mathbf{x})^{\sim},
\end{aligned}$$

for all $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x} \in \mathbb{C}^{\mathcal{G}}$. From this, it follows that $\mathbf{X}$ is a filter if and only if it is diagonalized by the characters of $\mathcal{G}$ (its Fourier basis), namely if and only if we can write $\mathbf{X} = \frac{1}{G}\mathbf{\Gamma}\mathbf{D}\mathbf{\Gamma}^*$ where $\mathbf{D}$ is the $\widehat{\mathcal{G}} \times \widehat{\mathcal{G}}$ matrix with $\mathbf{D}(\gamma, \gamma) = (\mathbf{\Gamma}^*\mathbf{x})(\gamma)$. The *autocorrelation* of any $\mathbf{x} \in \mathbb{C}^{\mathcal{G}}$ is $\mathbf{x} * \tilde{\mathbf{x}}$, namely the function whose DFT is the entrywise-modulus-squared of that of $\mathbf{x}$:

$$[\mathbf{\Gamma}^*(\mathbf{x} * \tilde{\mathbf{x}})](\gamma) = (\mathbf{\Gamma}^*\mathbf{x})(\gamma)(\mathbf{\Gamma}^*\tilde{\mathbf{x}})(\gamma) = (\mathbf{\Gamma}^*\mathbf{x})(\gamma)\overline{(\mathbf{\Gamma}^*\mathbf{x})(\gamma)} = |(\mathbf{\Gamma}^*\mathbf{x})(\gamma)|^2.$$

For any subset $\mathcal{D}$ of a finite abelian group $\mathcal{G}$, the *characteristic function* of $\mathcal{D}$ is defined as

$$\boldsymbol{\chi}_{\mathcal{D}}(g) = \begin{cases} 1, \ g \in \mathcal{G}, \\ 0, \ g \notin \mathcal{G}, \end{cases}$$

which we can also think of as a vector where the gth entry is 1 whenever g is in $\mathcal{D}$ and 0 otherwise. The autocorrelation of the characteristic function of any such $\mathcal{D} \subseteq \mathcal{G}$, $(\boldsymbol{\chi}_{\mathcal{D}} * \widetilde{\boldsymbol{\chi}}_{\mathcal{D}})(g)$ is equal to the number of (unique) ways that a specific g can be written as the difference of elements of $\mathcal{D}$:

$$
\begin{aligned}
(\boldsymbol{\chi}_{\mathcal{D}} * \tilde{\boldsymbol{\chi}}_{\mathcal{D}})(g) &= \sum_{g' \in \mathcal{G}} \boldsymbol{\chi}_{\mathcal{D}}(g') \tilde{\boldsymbol{\chi}}_{\mathcal{D}}(g - g') \\
&= \sum_{d \in \mathcal{D}} \tilde{\boldsymbol{\chi}}_{\mathcal{D}}(g - d) \\
&= \sum_{d \in \mathcal{D}} \boldsymbol{\chi}_{\mathcal{D}}(d - g) \\
&= \#\{d : d - g \in \mathcal{D}\} \\
&= \#\{(d, d') \in \mathcal{D} \times \mathcal{D} : g = d - d'\}.
\end{aligned} \tag{15}
$$

The *annihilator* of any subgroup $\mathcal{H}$ of $\mathcal{G}$ is $\mathcal{H}^\perp := \{\gamma \in \widehat{\mathcal{G}} : \gamma(h) = 1, \forall\, h \in \mathcal{H}\}$, which is itself a subgroup of the Pontryagin dual $\widehat{\mathcal{G}}$. The mapping $g + \mathcal{H} \mapsto \big(\eta \mapsto \eta(g)\big)$ is a well-known isomorphism onto the Pontryagin dual of $\mathcal{H}^\perp$. In particular, note that $\#(\mathcal{H}^\perp) = \#(\mathcal{G}/\mathcal{H}) = \frac{G}{H}$, where $\#(\mathcal{H}) = H$. The *Poisson summation formula* (PSF) states that for any subgroup $\mathcal{H}$ of $\mathcal{G}$, $\boldsymbol{\Gamma}^* \boldsymbol{\chi}_{\mathcal{H}} = H \boldsymbol{\chi}_{\mathcal{H}^\perp}$ ("the DFT of a comb is a comb"). To see this, note that for any $\eta \in \mathcal{H}^\perp$,

$$
(\boldsymbol{\Gamma} \boldsymbol{\chi}_{\mathcal{H}})(\eta) = \sum_{g \in \mathcal{G}} \overline{\eta(g)} \boldsymbol{\chi}_{\mathcal{H}}(g) = \sum_{h \in \mathcal{H}} 1 = H.
$$

Additionally, we have that

$$
GH = G \|\boldsymbol{\chi}_{\mathcal{H}}\|^2 = \|\boldsymbol{\Gamma}^* \boldsymbol{\chi}_{\mathcal{H}}\|^2 = \sum_{\gamma \in \widehat{\mathcal{G}}} |\boldsymbol{\Gamma}^* \boldsymbol{\chi}_{\mathcal{H}}(\gamma)|^2 = GH + \sum_{\gamma \notin \mathcal{H}^\perp} |\boldsymbol{\Gamma}^* \boldsymbol{\chi}_{\mathcal{H}}(\gamma)|^2,
$$

which implies that $|\boldsymbol{\Gamma}^* \boldsymbol{\chi}_{\mathcal{H}}(\gamma)|^2 = 0$ for all $\gamma \notin \mathcal{H}^\perp$. Applying $\boldsymbol{\Gamma}$ to both sides, we see that the PSF may be equivalently stated as $\boldsymbol{\Gamma} \boldsymbol{\chi}_{\mathcal{H}^\perp} = \frac{G}{H} \boldsymbol{\chi}_{\mathcal{H}}$.

For any subgroup $\mathcal{H}$ of $\mathcal{G}$, the corresponding *downsampling* and *upsampling* operators are

$$\downarrow : \mathbb{C}^{\mathcal{G}} \to \mathbb{C}^{\mathcal{H}}, \quad (\downarrow \mathbf{y})(h) := \mathbf{y}(h), \quad \uparrow : \mathbb{C}^{\mathcal{H}} \to \mathbb{C}^{\mathcal{G}}, \quad (\uparrow \mathbf{x})(g) = \begin{cases} \mathbf{x}(g), & g \in \mathcal{H}, \\ 0, & g \notin \mathcal{H}, \end{cases}$$

respectively. Here $\uparrow$ is an isometry with $\uparrow^* = \downarrow$ satisfying $\uparrow^* \uparrow = \downarrow \uparrow = \mathbf{I}$. The DFT (over $\mathcal{G}$) of $\uparrow \mathbf{x} \in \mathbb{C}^{\mathcal{G}}$ is the *periodic extension* of the DFT (over $\mathcal{H}$) of $\mathbf{x} \in \mathbb{C}^{\mathcal{H}}$:

$$(\boldsymbol{\Gamma}_{\mathcal{G}}^* \uparrow \mathbf{x})(\gamma) = \sum_{g \in \mathcal{G}} \overline{\gamma(g)} (\uparrow \mathbf{x})(g) = \sum_{h \in \mathcal{H}} \overline{\gamma(h)} \mathbf{x}(h) = (\boldsymbol{\Gamma}_{\mathcal{H}}^* \mathbf{x})(\gamma).$$

Meanwhile, the DFT (over $\mathcal{H}$) of $\downarrow \mathbf{y} \in \mathbb{C}^{\mathcal{H}}$ is the *periodization* of the DFT (over $\mathcal{G}$) of $\mathbf{y} \in \mathbb{C}^{\mathcal{G}}$:

$$(\boldsymbol{\Gamma}_{\mathcal{H}}^* \downarrow \mathbf{y})(\gamma \mathcal{H}^\perp) = \sum_{h \in \mathcal{H}} \overline{\gamma(h)} (\downarrow \mathbf{y})(h) = \sum_{g \in \mathcal{G}} \overline{\gamma(g)} \boldsymbol{\chi}_{\mathcal{H}}(g) \mathbf{y}(g) = \tfrac{H}{G} \sum_{\eta \in \mathcal{H}^\perp} (\boldsymbol{\Gamma}_{\mathcal{G}}^* \mathbf{y})(\gamma \eta^{-1}).$$

2.7 Difference sets

A subset $\mathcal{D}$ of an abelian group $\mathcal{G}$ is known as a *difference set* (DS) for $\mathcal{G}$ if there exists a constant $\Lambda \geq 0$ such that $\#(\Delta_g) = \Lambda$ for all $g \neq 0 \in \mathcal{G}$, where $\Delta_g := \{(d_1, d_2) \in \mathcal{D} : g = d_1 - d_2\}$. This is often demonstrated via a table indexed by elements of $\mathcal{D}$ whose (d_1, d_2)th entry is $d_1 - d_2$. This is known as a *difference table*.

Example 2.7.1. When $\mathcal{G} = \mathbb{Z}_7$, the set $\mathcal{D} = \{0, 1, 3\}$ has difference table

$-$	0	1	3
0	0	6	4
1	1	0	5
3	3	2	0.

It is clear that each non-zero element of $\mathbb{Z}_7$ appears once in the table, meaning $\Lambda = 1$

and the autocorrelation of $\boldsymbol{\chi}_{\mathcal{D}}$ has entries

$$\left(\boldsymbol{\chi}_{\mathcal{D}} * \tilde{\boldsymbol{\chi}}_{\mathcal{D}}\right)(g) = \begin{cases} 3, \ g = 0, \\ 1, \ g \neq 0. \end{cases}$$

More generally, a set of subsets $\{\mathcal{D}_n\}_{n\in\mathcal{N}}$ of an abelian group $\mathcal{G}$ is a *difference family* (D, N, Λ) for $\mathcal{G}$ if there exists a constant $\Lambda \geq 0$ with $\sum_{n\in\mathcal{N}} \#(\Delta_{n,g}) = \Lambda$ for all $g \neq 0 \in \mathcal{G}$ where $\Delta_{n,g} := \{(d_1, d_2) \in \mathcal{D}_n : g = d_1 - d_2\}$. A difference set can be thought of as a difference family where $N = 1$ (only one member of the family). For a difference family there is a difference table for each member of the family. Unlike a difference set, however, the members of $\mathcal{G}$ do not need to show up equally in each table, but rather equally across all tables.

Example 2.7.2. The family $\{(1, 4); (2, 3)\}$ is a difference family for $\mathcal{G} = \mathbb{Z}_5$. The members are $\mathcal{D}_0 = \{1, 4\}$ and $\mathcal{D}_1 = \{2, 3\}$, with difference tables

$$\begin{array}{c|cc} - & 1 & 4 \\ \hline 1 & 0 & 2 \\ 4 & 3 & 0 \end{array} \quad \begin{array}{c|cc} - & 2 & 3 \\ \hline 2 & 0 & 4 \\ 3 & 1 & 0 \end{array}.$$

Note how not all elements of $\mathbb{Z}_5$ appear in either table, but each element appears once across both (again $\Lambda = 1$).

One way to generate a difference family is to start with a difference set as your first member, then add shifts of that difference set to create more members. Obviously if $\mathcal{D}_0$ is a difference set, then $\mathcal{D}_1 = g + \mathcal{D}_0$ is as well. This is because any $d_1, d_2 \in \mathcal{D}_1$ have corresponding $d_1' d_2' \in \mathcal{D}_0$ with $d_1 - d_2 = (d_1' + g) - (d_2' + g) = d_1' - d_2'$. Lemma 2.7.3 provides a helpful relationship between difference sets, difference families, and the auto-correlation of the characteristic function.

Lemma 2.7.3. *Let $\{\mathcal{D}_n\}_{n\in\mathcal{N}}$ be a family of N subsets of a finite abelian group $\mathcal{G}$.*

(a) A subset $\mathcal{D}_n$ is a difference set if and only if $(\boldsymbol{\chi}_{\mathcal{D}_n} * \boldsymbol{\chi}_{-\mathcal{D}_n})(g)$ is constant for all $g \in \mathcal{G}, g \neq 0$.

(b) $\{\mathcal{D}_n\}_{n\in\mathcal{N}}$ is a difference family if and only if $\sum_{n\mathcal{N}}(\boldsymbol{\chi}_{\mathcal{D}_n} * \boldsymbol{\chi}_{-\mathcal{D}_n})(g)$ is constant for all $g \in \mathcal{G}, g \neq 0$.

Proof. Notice that for any subset $\mathcal{D}_n$, $(\boldsymbol{\chi}_{\mathcal{D}_n} * \boldsymbol{\chi}_{-\mathcal{D}_n})(g)$ is equal to the number of times g shows up in the difference table for $\mathcal{D}_n$. Then for (a), recall that $\mathcal{D}_n$ is a difference set for $\mathcal{G}$ if and only if there exists $\Lambda \geq 0$ such that each non-zero element of $\mathcal{G}$ appears in the difference table Λ times. This happens if and only if $(\boldsymbol{\chi}_{\mathcal{D}_n} * \boldsymbol{\chi}_{-\mathcal{D}_n})(g) = \Lambda$ for all $g \neq 0 \in \mathcal{G}$.

For (b), recall that $\{\mathcal{D}_n\}_{n\in calN}$ is a difference family for $\mathcal{G}$ if and only if there exists a $\Lambda \geq 0$ such that $\sum_{n\in\mathcal{G}} \#(\Delta_{n,g}) = \Lambda$ for all non-zero $g \in \mathcal{G}$. Again, from (15) we see that $\#(\Delta_{n,g}) = (\boldsymbol{\chi}_{\mathcal{D}_n} * \boldsymbol{\chi}_{-\mathcal{D}_n})(g)$. The result follows from this substitution into the definition. □

For a finite abelian group $\mathcal{G}$ of order $G < \infty$, we say that a D-element subset $\mathcal{D} \subset \mathcal{G}$ is a *divisible difference set* $(\frac{G}{H}, H, D, \Lambda_1, \Lambda_2)$ for some subgroup $\mathcal{H}$ ($\mathcal{H}$-DDS) for $\mathcal{G}$ if there exists constants Λ_1, Λ_2 such that $\#(\Delta_g) = \Lambda_1$ for all non-identity members of $\mathcal{H}$ and $\#(\Delta_2) = \Lambda_2$ for all g that are members of $\mathcal{G}$ but not members of $\mathcal{H}$. Difference sets like those in Example 2.7.1 are a special case of DDS where $H = 1$ ($\mathcal{H} = \{0\}$). In the case where $\Lambda_1 = 0$, $\mathcal{D}$ is referred to as an *$\mathcal{H}$-relative difference set* $(\frac{G}{H}, H, D, \Lambda)$ ($\mathcal{H}$-RDS) for $\mathcal{G}$, where $\Lambda = \Lambda_2$.

Example 2.7.4. The set $\mathcal{D} = \{1, 2, 4, 8\}$ is a (6,2,4,2,1) divisible difference set for

$\mathbb{Z}_{12}$, as can be seen in the following difference table:

$-$	1	2	4	8
1	0	11	9	5
2	1	0	10	6
4	3	2	0	8
8	7	6	4	0

Here we see that $\mathcal{H} = \{0, 6\}$, with 6 appearing twice in the difference table (meaning $\Lambda_1 = 2$) and the other elements ($\mathbb{Z}_{12} \backslash \mathcal{H}$) each appear once ($\Lambda_2 = 1$). This equates to $\boldsymbol{\chi}_{\mathcal{D}} * \boldsymbol{\chi}_{-\mathcal{D}} = 4\boldsymbol{\delta}_0 + 2\boldsymbol{\chi}_{\mathcal{H}\backslash\{0\}} + \boldsymbol{\chi}_{\mathcal{G}\backslash\mathcal{H}}$.

An $\mathcal{H}$-DDS is *semiregular* if $\Lambda_1 \neq D$ and $D^2 = \Lambda_2 G$. In this case, there is no choice for the values of Λ_1 or Λ_2, as shown in the following lemma.

Lemma 2.7.5. *Let $\mathcal{G}$ be a finite abelian group with Pontryagin dual $\widehat{\mathcal{G}}$. Let $\mathcal{H}$ be a subgroup of $\mathcal{G}$, and let $\mathcal{D}$ be a non-empty proper subgroup of $\mathcal{G}$. Then $\mathcal{D}$ is a semiregular $\mathcal{H}$-DDS for $\mathcal{G}$ if and only if there exists $C \geq 0$ such that*

$$|(\boldsymbol{\Gamma}^* \boldsymbol{\chi}_{\mathcal{D}})(\gamma)|^2 = \begin{cases} 0, & \gamma \in \mathcal{H}^{\perp}, \gamma \neq 1, \\ C, & \gamma \notin \mathcal{H}^{\perp}, \end{cases}$$

*for all $\gamma \in \widehat{\mathcal{G}}$, $\gamma \neq 1$. In this case, $\mathcal{D}$ is an $\mathcal{H}$-*DDS$(\frac{G}{H}, H, D, \Lambda_1, \Lambda_2)$*, where*

$$\Lambda_1 = \tfrac{D(DH-G)}{G(H-1)}, \quad \Lambda_2 = \tfrac{D^2}{G}, \quad C = \tfrac{DH(G-D)}{G(H-1)}.$$

In the special case where $\mathcal{D}$ is an $\mathcal{H}$-RDS, then

$$\Lambda_1 = 0, \quad \Lambda_2 = \tfrac{D^2}{G} = \tfrac{D}{H}, \quad C = \tfrac{G-D}{H-1}.$$

Proof. Note that $\mathcal{D}$ is an $\mathcal{H}$-DDS if and only if

$$\begin{aligned}\boldsymbol{\chi}_{\mathcal{D}} * \boldsymbol{\chi}_{-\mathcal{D}} &= D\boldsymbol{\delta}_0 + \Lambda_1(\boldsymbol{\chi}_{\mathcal{H}} - \boldsymbol{\delta}_0) + \Lambda_2(\boldsymbol{\chi}_{\mathcal{G}} - \boldsymbol{\chi}_{\mathcal{H}})\\ &= (D-\Lambda_1)\boldsymbol{\delta}_0 + (\Lambda_1 - \Lambda_2)\boldsymbol{\chi}_{\mathcal{H}} + \Lambda_2\mathbf{1}.\end{aligned}$$

Applying the DFT and PSF, this equates to having

$$\begin{aligned}|(\boldsymbol{\Gamma}^*\boldsymbol{\chi}_{\mathcal{D}})(\gamma)|^2 &= (D-\Lambda_1) + (\Lambda_1 - \Lambda_2)H\boldsymbol{\chi}_{\mathcal{H}^\perp}(\gamma) + \Lambda_2 G\boldsymbol{\delta}_0(\gamma)\\ &= \begin{cases} D - \Lambda_1(\Lambda_1-\Lambda_2)H + \Lambda_2 G, & \gamma = 1,\\ D - \Lambda_1(\Lambda_1 - \Lambda_2)H, & \gamma \in \mathcal{H}^\perp\backslash\{\mathbf{1}\},\\ D-\Lambda_1, & \gamma \notin \mathcal{H}^\perp.\end{cases}\end{aligned} \tag{16}$$

When we evaluate this at $\gamma = \mathbf{1}$, we see that $D^2 = D + (H-1)\Lambda_1 + (G-H)\Lambda_2$. Then, when $\mathcal{H} \neq \mathcal{G}$, we have that $\mathcal{D}$ is an $\mathcal{H}$-DDS$(\frac{G}{H}, H, D, \Lambda_1, \Lambda_2)$ for $\mathcal{G}$ if and only if $\Lambda_2 = \frac{1}{G-H}[D(D-1) - (H-1)\Lambda_1]$, and

$$|(\boldsymbol{\Gamma}^*\boldsymbol{\chi}_{\mathcal{D}})(\gamma)|^2 = \begin{cases} D^2 - \Lambda_2 G, & \gamma \in \mathcal{H}^\perp\backslash\{1\},\\ D - \Lambda_1, & \gamma \notin \mathcal{H}^\perp.\end{cases}$$

Now recall that an $\mathcal{H}$-DDS$(\frac{G}{H}, H, D, \Lambda_1, \Lambda_2)$ is semiregular if and only if $D^2 - \Lambda_2 G = 0$ and $D - \Lambda_1 \neq 0$. Note that if both $D^2 - \Lambda_2 G = 0$ and $D - \Lambda_1 = 0$, then (16) becomes

$$D^2 = D - \Lambda_1(\Lambda_1 - \Lambda_2)H + \Lambda_2 G = (D - \Lambda_2)H + D^2,$$

resulting in $\Lambda_2 = D$ and therefore $0 = D^2 - \Lambda_2 G = D(D-G)$, contradicting the assumption that $\mathcal{D}$ is a nonempty proper subset of $\mathcal{G}$. Thus, under our assumptions, semiregularity equates to the sole condition that $D^2 = \Lambda_2 G$. In this case we have that $\frac{D^2}{G} = \Lambda_2 = \frac{1}{G-H}[D(D-1) - (H-1)\Lambda_2]$, meaning that

$$\Lambda_1 = -\tfrac{1}{H-1}\big[\tfrac{D^2(G-H)}{G} - D(D-1)\big] = \tfrac{D(DH-G)}{G(H-1)},$$

and $D-\Lambda_1 = \frac{DH(G-D)}{G(H-1)}$. Together, we see that, under the assumption that $\varnothing \neq \mathcal{D} \neq \mathcal{G}$, a subset $\mathcal{D}$ is a semiregular $\mathcal{H}$-DDS$(\frac{G}{H}, H, D, \Lambda_1, \Lambda_2)$ if and only if

$$\Lambda_1 = \tfrac{D(DH-G)}{G(H-1)}, \quad \Lambda_2 = \tfrac{D^2}{G}, \text{ and } |(\mathbf{\Gamma}^*\boldsymbol{\chi}_{\mathcal{D}})(\gamma)|^2 = \begin{cases} 0, & \gamma \in \mathcal{H}^\perp \backslash \{1\}, \\ \frac{DH(G-D)}{G(H-1)}, & \gamma \notin \mathcal{H}^\perp. \end{cases}$$

In addition, if $\mathcal{D}$ is any subset of $\mathcal{G}$ satisfying $\varnothing \neq \mathcal{D} \subsetneq \mathcal{G}$ and

$$|(\mathbf{\Gamma}^*\boldsymbol{\chi}_{\mathcal{D}})(\gamma)|^2 = \begin{cases} 0,\ \gamma \in \mathcal{H}^\perp \backslash \{1\}, \\ C,\ \gamma \notin \mathcal{H}^\perp, \end{cases}$$

for some constant $C \neq 0$, then

$$\mathbf{\Gamma}^*(\boldsymbol{\chi}_{\mathcal{D}} * \boldsymbol{\chi}_{-\mathcal{D}}) = D^2\boldsymbol{\delta}_1 + C\boldsymbol{\chi}_{\widehat{\mathcal{G}}\backslash\mathcal{H}^\perp} = D^2\boldsymbol{\delta}_1 + C\boldsymbol{\chi}_{\widehat{\mathcal{G}}} - C\boldsymbol{\chi}_{\mathcal{H}^\perp}.$$

Again, applying the character table and PSF,

$$\begin{aligned} \boldsymbol{\chi}_{\mathcal{D}} * \boldsymbol{\chi}_{-\mathcal{D}} &= \tfrac{1}{G}(D^2\mathbf{1} + CG\boldsymbol{\delta}_0 - \tfrac{CG}{H}\boldsymbol{\chi}_{\mathcal{H}}) \\ &= \tfrac{1}{G}(D^2 - \tfrac{CG(H-1)}{H})\boldsymbol{\delta}_0 + \tfrac{1}{G}(D^2 - \tfrac{CG}{H})\boldsymbol{\chi}_{\mathcal{H}\backslash\{0\}} + \tfrac{D^2}{G}\boldsymbol{\chi}_{\mathcal{G}\backslash\mathcal{H}}, \end{aligned}$$

meaning $\mathcal{D}$ is an $\mathcal{H}$-DDS$(\frac{G}{H}, H, D, \Lambda_1, \Lambda_2)$ with $\Lambda_2 = \frac{D^2}{G}$, and is therefore semiregular.

In the special case where $\mathcal{D}$ is an $\mathcal{H}$-RDS, recall that $\Lambda_1 = \frac{D(DH-G)}{G(H-1)} = 0$. Then $DH - G = 0$ and $C = \frac{G-D}{H-1}$. Since all differences between unique elements of $\mathcal{D}$ are elements of $\mathcal{G}\backslash\mathcal{H}$,

$$D(D-1) = D^2 - D = \Lambda_2(G-H) = \Lambda_2 G - \Lambda_2 H.$$

Semiregularity requires that $\Lambda_2 G = D^2$. Substituting into the previous equation shows that

$$D = \Lambda_2 H. \qquad \square$$

2.8 Group divisible designs

A *group divisible design* (GDD) is a set of *vertices* $\mathcal{V}$ along with a set $\mathcal{B}$ of subsets of $\mathcal{V}$ (each with K elements) known as *blocks* as well as a partition $\mathcal{U}$ of $\mathcal{V}$ known as *groupings* such that for any two distinct vertices, either:

(i) they lie in a common grouping and no common block, or

(ii) they do not lie in a common grouping and do lie in exactly Λ common blocks.

We refer to such objects as a GDD(V, K, Λ). Our focus is exclusively on *uniform* GDDs, those with the property that all groupings have the same cardinality M. In this case, we regard $\mathcal{V} = \mathcal{M} \times \mathcal{U}$ where $\mathcal{M}$ is an indexing set of size M. In the case where $M = 1$ (meaning $U = V$), a GDD(V, K, Λ) is commonly known as a BIBD(V, K, Λ). Let $\mathbf{X} \in \{0,1\}^{\mathcal{B}\times\mathcal{V}}$ be the incidence matrix of a GDD, meaning

$$\mathbf{X}(b, v) = \begin{cases} 1, \ v \in b, \\ 0, \ v \notin b. \end{cases}$$

Any uniform GDD(V, K, Λ) has the following features:

(i) $\mathbf{X}\mathbf{1}_{\mathcal{V}} = K\mathbf{1}_{\mathcal{B}}$,

(ii) $\mathbf{X}^T\mathbf{1}_{\mathcal{B}} = R\mathbf{1}_{\mathcal{V}}$ (where $R = \frac{BK}{V} = \frac{\Lambda M(U-1)}{K-1}$),

(iii) $\mathbf{X}^T\mathbf{X} = R\mathbf{I}_{\mathcal{V}} \otimes \mathbf{I}_{\mathcal{M}} + \Lambda(\mathbf{J}_{\mathcal{V}} - \mathbf{I}_{\mathcal{V}}) \otimes \mathbf{J}_{\mathcal{M}} = R\mathbf{I}_{\mathcal{V}} + \Lambda\mathbf{J}_{\mathcal{V}} - \Lambda\mathbf{I}\mathcal{V} \otimes \mathbf{J}_{\mathcal{M}}$.

A *Latin square* is a combinatorial design of N elements in an $N \times N$ array such that each element appears in every row and column exactly once. If two or more Latin squares can be overlaid onto one another in such a way that every possible combination of pairs of elements appears exactly once, then they are all *mutually orthogonal Latin squares* (MOLS). MOLS are in fact a special kind of uniform GDD where $K = U$ [13].

Lemma 2.8.1. *Given a GDD($V, K, 1$) of type M^U (uniform), then $U \geq K$. Moreover, $U = K$ if and only if the U submatrices of the adjacency matrix* $\mathbf{X}$ *represent* $U - 2$ *MOLS.*

Proof. First, assume $K > U$. Then by the pigeonhole principle at least one grouping contains two points v_1, v_2 of the first block. However, this means that v_1, v_2 lie in both a common grouping and a common block, a contradiction.

For the second claim, let us assume that $K = U$. Then the adjacency matrix $\mathbf{X}$ can be divided into U submatrices, each of size $B \times M$ where each column consists of R ones and $B - R$ zeroes. First, notice that

$$R = \tfrac{M(U-1)}{K-1} = M, \qquad\qquad B = \tfrac{VR}{K} = \tfrac{(MU)M}{K} = M^2.$$

Without loss of generality, we can arrange the rows of $\mathbf{X}$ such that the first two blocks are $\mathbf{I}_M \otimes \mathbf{1}_M$ and $\mathbf{1}_M \otimes \mathbf{I}_M$. This means that the remaining $U - 2$ blocks consist of $M \times M$ permutation matrices. Notice that the first two blocks can represent every combination of row and column of an $M \times M$ matrix. For each block, we can assign a symbol to represent a specific row vector, say $\begin{bmatrix} 1 \; 0 \; 0 \ldots \; 0 \end{bmatrix} \to 1$, $\begin{bmatrix} 0 \; 1 \; 0 \ldots \; 0 \end{bmatrix} \to 2$,

$..., \begin{bmatrix} 0 \; 0 \; 0 \ldots \; m \end{bmatrix} \to m$. Then each row of $\mathbf{X}$ becomes a $1 \times U$ row vector of a matrix $\mathbf{\Pi}$. Letting the first two entries of each row represent the row and column entries of an $M \times M$ matrix, the following $U-2$ columns then represent $U-2$ Latin squares, and every combination of (m_1, m_2) where $m_1, m_2 \in \{1, 2, ..., M\}$ is contained in any pair of columns, otherwise known as MOLS. □

2.9 Mutually Unbiased Bases

For an arbitrary indexing set $\mathcal{D}$ of size D, let $\{\mathbf{v}_d\}_{d\in\mathcal{D}}$ and $\{\mathbf{u}_d\}_{d\in\mathcal{D}}$ be distinct ONBs for some D-dimensional Hilbert space $\mathbb{H}$. The bases $\{\mathbf{v}_d\}_{d\in\mathcal{D}}$ and $\{\mathbf{u}_d\}_{d\in\mathcal{D}}$ are said to be *mutually unbiased* if $|\langle \mathbf{v}_{d_1}, \mathbf{u}_{d_2}\rangle|^2 = \frac{1}{D}$ for all d_1, d_2 in $\mathcal{D}$. For the sake of simplicity, we will often refer to a set of MUBs by their synthesis operators, e.g., if $\{\mathbf{V}_n\}_{n\in\mathcal{N}}$ are the synthesis operators for N MUBs for a D-dimensional Hilbert space, we refer to $\{\mathbf{V}_n\}_{n\in\mathcal{N}}$ as an MUB(D, N). For example, $\{\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3\}$ is an MUB$(2,3)$ for $\mathbb{C}^2$:

$$\mathbf{V}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{V}_2 = \tfrac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \mathbf{V}_3 = \tfrac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix},$$

because $\left|\left(\mathbf{V}_{n_1}^* \mathbf{V}_{n_2}\right)(d_1, d_2)\right|^2 = \frac{1}{2}$ for any distinct n_1, n_2 in $\{1, 2, 3\}$ and any d_1, d_2 in $\{1, 2\}$. Note that this also means that $\mathbf{V}_{n_1}^* \mathbf{V}_{n_2} = \frac{1}{\sqrt{2}} \mathbf{J}_2$ for all $n_1 \neq n_2$.

There are three known methods for constructing MUBs. First, Theorem 4.1 from [25] states that for any finite abelian group $\mathcal{G}$ with subgroup $\mathcal{H}$, the existence a semiregular $\mathcal{H}$-RDS(D, H, D, Λ) for $\mathcal{G}$ is equivalent to the existence of $H + 1$ MUBs.

To see this, recall from Lemma 2.7.5 that

$$|(\mathbf{\Gamma}^* \boldsymbol{\chi}_{\mathcal{D}})(\gamma)|^2 = \begin{cases} D^2, & \gamma = 1, \\ 0, & \gamma \in \mathcal{H}^\perp, \gamma \neq 1, \\ \frac{G-D}{H-1}, & \gamma \notin \mathcal{H}^\perp, \end{cases} \tag{17}$$

and that $D = \frac{G}{H}$. Let $\{\boldsymbol{\varphi}_\gamma\}_{\gamma \in \widehat{\mathcal{G}}}$ be the harmonic frame arising from $\mathcal{D}$. Then for any γ_1, γ_2 in $\widehat{\mathcal{G}}$,

$$\begin{aligned} \left|\langle \boldsymbol{\varphi}_{\gamma_1}, \boldsymbol{\varphi}_{\gamma_2} \rangle\right|^2 = \left| \sum_{d \in \mathcal{D}} \overline{\boldsymbol{\varphi}_{\gamma_1}(d)} \boldsymbol{\varphi}_{\gamma_2}(d) \right|^2 &= \left| \sum_{g \in \mathcal{G}} \overline{\gamma_1(g)} \gamma_2(g) \boldsymbol{\chi}_{\mathcal{D}}(g) \right|^2 \\ &= \left| \sum_{g \in \mathcal{G}} \gamma_1^{-1} \gamma_2(g) \boldsymbol{\chi}_{\mathcal{D}}(g) \right|^2 = \left| \tfrac{1}{D} \left(\mathbf{\Gamma}^* \boldsymbol{\chi}_{\mathcal{D}}\right) (\gamma_1^{-1} \gamma_2) \right|^2. \end{aligned}$$

Combining this with (17), we can see that when γ_1 and γ_2 are in the same coset of $\mathcal{H}^\perp$ (meaning that $\gamma_1^{-1}\gamma_2$ is in $\mathcal{H}^\perp$), they form an ONB for their span. Note that $\#(\mathbf{H}^\perp) = \#(\mathcal{G}/\mathcal{H}) = \frac{G}{H} = D$. This means that if we arrange the frame vectors according to the cosets of $\mathcal{H}^\perp$, every coset group forms an ONB for some D-dimensional Hilbert space. Next, note that if γ_1 and γ_2 are not in the same coset of $\mathcal{H}^\perp$ (meaning $\gamma_1^{-1}\gamma_2$ is not in $\mathcal{H}^\perp$), their modulus-squared inner product is

$$\left(\tfrac{1}{D}\right)^2 \tfrac{G-D}{H-1} = \left(\tfrac{H^2}{G^2}\right) \tfrac{G - \frac{G}{H}}{H-1} = \left(\tfrac{H^2}{G^2}\right) \tfrac{GH-G}{H(H-1)} = \left(\tfrac{H^2}{G^2}\right) \tfrac{G(H-1)}{H(H-1)} = \tfrac{H}{G} = \tfrac{1}{D}.$$

Therefore this arrangement constitutes an MUB(D,H). If we include the $\mathcal{D} \times \mathcal{D}$ identity matrix, the number of MUBs becomes $H + 1$. For any prime power Q, the set $\mathcal{D} = \{(x, x^2); x \in \mathbb{F}^Q\}$ is an RDS$(Q, Q, Q, 1)$ for $\mathcal{G} = \mathbb{F}^Q \times \mathbb{F}^Q$ with respect to $\mathcal{H} = \{0\} \times \mathbb{F}^Q$, therefore for any prime power Q, there exists an MUB$(Q, Q+1)$ [41]. Additionally, there is a real MUB$(Q, \frac{Q+1}{2})$ for any prime power Q that is also a power of 4 (the proof of which is outside the scope of this dissertation).

Second, MUBs can be combined to create new MUBs. Let $\{\mathbf{V}_n\}_{n=1}^{N}$ be an MUB(B, N) and $\{\mathbf{V}_m\}_{m=1}^{M}$ be an MUB(C, M) where $M \leq N$. Then $\{\mathbf{U}_m \otimes \mathbf{V}_m\}_{m=1}^{M}$ is an MUB(BC, M). To see this, note that $\big(\mathbf{U}_m \otimes \mathbf{V}_m\big)^*\big(\mathbf{U}_m \otimes \mathbf{V}_m\big) = \mathbf{I}_{BC}$. Since both $\{\mathbf{U}_m\}_{m=1}^{M}$ and $\{\mathbf{V}_m\}_{m=1}^{M}$ are MUBs, for distinct m_1, m_2, $\big|\mathbf{U}_{m_1}^*\mathbf{U}_{m_2}\big|^2 = \frac{1}{B}\mathbf{J}$ and $\big|\mathbf{V}_{m_1}^*\mathbf{V}_{m_2}\big|^2 = \frac{1}{C}\mathbf{J}$. This is similarly true for $\{\mathbf{U}_m \otimes \mathbf{V}_m\}_{m=1}^{M}$ since it follows that $\big|\big(\mathbf{U}_{m_1} \otimes \mathbf{V}_{m_1}\big)^*\big(\mathbf{U}_{m_2} \otimes \mathbf{V}_{m_2}\big)\big|^2 = \big|\big(\mathbf{U}_{m_1}^*\mathbf{U}_{m_2}\big) \otimes \big(\mathbf{V}_{m_1}^*\mathbf{V}_{m_2}\big)\big|^2 = \frac{1}{BC}\mathbf{J}_{D_1D_2}$.

The third method comes from MOLS. If $\{\mathbf{\Psi}_j\}_{j=1}^{J}$ is a set of J MOLS of size N, it is possible to use them to construct $J + 2$ MUBs of size N^2.

Example 2.9.1. Let $\mathbf{\Psi}_1, \mathbf{\Psi}_2$ be 3×3 MOLS, each consisting of the numbers $\{1, 2, 3\}$. Let $\mathbf{\Psi}$ be their overlay.

$\mathbf{\Psi}_1 =$

1	2	3
3	1	2
2	3	1

, $\mathbf{\Psi}_2 =$

1	2	3
2	3	1
3	1	2

, $\mathbf{\Psi} =$

11	22	33
32	13	21
23	31	12

.

For each row r and column c in $\mathbf{\Psi}_1$ and $\mathbf{\Psi}_2$ we create the 1×4 row vector $(r, c, \mathbf{\Psi}_1(r, c), \mathbf{\Psi}_2(r, c))$. We can then concatenate these row vectors into the 9×4

matrix $\mathbf{\Pi}$,

$$\mathbf{\Pi} = \begin{bmatrix} 1\ 1\ 1\ 1 \\ 1\ 2\ 2\ 2 \\ 1\ 3\ 3\ 3 \\ 2\ 1\ 3\ 2 \\ 2\ 2\ 1\ 3 \\ 2\ 3\ 2\ 1 \\ 3\ 1\ 2\ 3 \\ 3\ 2\ 3\ 1 \\ 3\ 3\ 1\ 2 \end{bmatrix}.$$

We now let each each entry $\mathbf{\Pi}(i,j) \in \{1,2,3\}$ correspond with the binary vectors $\{\boldsymbol{\delta}_1^{\mathrm{T}}, \boldsymbol{\delta}_2^{\mathrm{T}}, \boldsymbol{\delta}_3^{\mathrm{T}}\}$. If we replace each entry with the corresponding binary vector, we generate the adjacency matrix $\mathbf{X}$ of a GDD$(12,4,1)$ of type 3^4,

$$\mathbf{X} = \begin{bmatrix} 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0 \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0 \\ 1\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1 \\ 0\ 1\ 0\ 1\ 0\ 0\ 0\ 0\ 1\ 0\ 1\ 0 \\ 0\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 0\ 0\ 0\ 1 \\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 0 \\ 0\ 0\ 1\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 1 \\ 0\ 0\ 1\ 0\ 1\ 0\ 0\ 0\ 1\ 1\ 0\ 0 \\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 0\ 1\ 0 \end{bmatrix}.$$

Note that there are $R = \frac{M(U-1)}{K-1} = 3$ ones per column of $\mathbf{X}$. Let $\mathbf{H}$ be the character table for $\mathcal{G} = \mathbb{Z}_3$. If $\mathbf{x}_v$ is the vth column of $\mathbf{X}$, let $\mathbf{\Psi}_v$ be the 9×3 matrix where the row corresponding to the rth one in $\mathbf{x}_v$ is the rth row of $\mathbf{H}$ while all other rows are filled with zeroes. Let $\boldsymbol{\Psi} = [\boldsymbol{\Psi}_1\, \boldsymbol{\Psi}_2\, \cdots\, \boldsymbol{\Psi}_{12}]$, then $\boldsymbol{\Psi}$ is the 9×36 matrix,

$$\boldsymbol{\Psi} = \left[\begin{array}{ccc|ccc|ccc||ccc|ccc|ccc||c}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \\
1 & \omega & \omega^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & \\
1 & \omega^2 & \omega & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & \omega & \omega^2 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 1 & \omega & \omega^2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \omega & \omega^2 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 1 & \omega^2 & \omega & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \omega & \omega^2 & \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & \omega^2 & \omega & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & \omega & \omega^2 & 0 & 0 & 0 & 1 & \omega^2 & \omega & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & \omega^2 & \omega & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \omega^2 & \omega &
\end{array}\right].$$

This gives an MUB$(9, 4)$.

III. Naimark-spatial orbits and categorization

3.1 Naimark-spatial orbits

Recall from the previous chapter that a Naimark complement of any TFF(D, N, R) with $D \neq NR$, is a TFF$(NR-D, N, R)$, and it is real or equi-isoclinic or equichordal if the original TFF is as well. Additionally, the spatial complement of any TFF(D, N, R) with $D \neq R$ is a TFF$(D, N, D-R)$, and equichordality is preserved (though equi-isoclinicity is not, unless $D = 2R$). The Naimark complement of a Naimark complement (and similarly for a spatial complement) is the original TFF (up to unitary transformation). However, as we explain below, taking alternating Naimark and spatial complements of any TFF(D, N, R) generates an infinite sequence of TFFs with distinct parameters (provided either $N \geq 5$ or $N = 4$ and $D \neq 2R$).

Example 3.1.1. Taking four copies of the scalar 1 in $\mathbb{R}$ yields an ETF(1,4), which can be considered an EITFF(1,4,1). Because $D = 1 = R$, it has no spatial complement. However, the Naimark complement is a real EITFF(3,4,1), with each line containing one of the four vertices of a regular tetrahedron centered at the origin. While its Naimark complement is the original EITFF(1,4,1) its spatial complement is a real ECTFF(3,4,2). Continuing to take alternating Naimark and spatial complements results in an infinite sequence of ECTFFs (all of which are real) with the following (D, N, R) parameters:

$$(\mathbf{1}, \mathbf{4}, \mathbf{1}) \underset{\text{N}}{\leftrightarrow} (3,4,1) \underset{\text{s}}{\leftrightarrow} (3,4,2) \underset{\text{N}}{\leftrightarrow} (5,4,2) \underset{\text{s}}{\leftrightarrow} (5,4,3) \underset{\text{N}}{\leftrightarrow} (7,4,3) \underset{\text{s}}{\leftrightarrow} (7,4,4) \underset{\text{N}}{\leftrightarrow} \cdots .$$

Example 3.1.2. Now consider the real EITFF(4,4,1) consisting of the four canonical axes of $\mathbb{R}^4$. Because $D = 4 = NR$, this EITFF has no Naimark complement, but its spatial complement is a real ECTFF(4,4,3). Continuing to take alternating Naimark and spatial complements yields another infinite sequence of ECTFFs (again, all of which are real) with the following (D, N, R) parameters:

$$(\mathbf{4}, \mathbf{4}, \mathbf{1}) \underset{\text{s}}{\leftrightarrow} (4,4,3) \underset{\text{N}}{\leftrightarrow} (8,4,3) \underset{\text{s}}{\leftrightarrow} (8,4,5) \underset{\text{N}}{\leftrightarrow} (12,4,5) \underset{\text{s}}{\leftrightarrow} (12,4,7) \underset{\text{N}}{\leftrightarrow} (16,4,7) \underset{\text{s}}{\leftrightarrow} \cdots .$$

Example 3.1.3. Sequences like those shown in the previous two examples can also be bi-infinite. It is known that an ETF(3,7) exists [21] and can be considered an EITFF(3,7,1). Note that $NR \neq D \neq R$, and that the sequence of ECTFF parameters from alternating Naimark and spatial complements depends on the order in which they are applied:

$$\cdots \underset{\text{s}}{\leftrightarrow} (11,7,2) \underset{\text{N}}{\leftrightarrow} (3,7,2) \underset{\text{s}}{\leftrightarrow} (\mathbf{3}, \mathbf{7}, \mathbf{1}) \underset{\text{N}}{\leftrightarrow} (4,7,1) \underset{\text{s}}{\leftrightarrow} (4,7,3) \underset{\text{N}}{\leftrightarrow} \cdots . \tag{18}$$

The existence of these infinite families of Naimark and spatial complements makes it difficult to determine if a potentially "new" EI/ECTFF construction is truly novel. We want a way of categorizing a distinct family in such a way as to easily determine if any EI/ECTFF can be generated via such a Naimark-spatial manner from a known construction. To that end, we formalize the following definitions and concepts.

Definition 3.1.4. Let $\nu, \sigma : \mathbb{R}^3 \to \mathbb{R}^3$, where $\nu(D, N, R) := (NR - D, N, R)$, and $\sigma(D, N, R) := (D, N, D - R)$. The *Naimark-spatial sequence* of $(D, N, R) \in \mathbb{Z}^3$ with $N > 1$ is the doubly infinite sequence $\{(D^{(K)}, N, R^{(K)})\}_{K=-\infty}^{\infty}$ where we denote $(D^{(0)}, N, R^{(0)}) = (D, N, R)$ and

$$\nu(D^{(2J+1)}, N, R^{(2J+1)}) = (D^{(2J)}, N, R^{(2J)}) = \sigma(D^{(2J-1)}, N, R^{(2J-1)}), \quad \forall J \in \mathbb{Z}.$$

The *(Naimark-spatial) orbit* of (D, N, R) is its orbit of (D, N, R) under the action of the group generated by ν and σ, namely the set

$$\mathrm{Orb}(D,N,R) := \{(D^{(K)}, N, R^{(K)}) : K \in \mathbb{Z}\}.$$

Notice here that ν and σ are operations on the TFF parameter triples that represent the changes that take place under a Naimark or spatial complement, respectively. Because of this, ν and σ are involutions (their own inverses), and the notation provided in the definition of the Naimark-spatial sequence allows us to easily associate the parameter values associated with taking alternating Naimark and spatial complements, regardless of which complement is taken first:

$$\begin{aligned}
\cdots\ (D^{(-3)}, N, R^{(-3)}) &= \sigma(\nu(\sigma(D,N,R))),\\
(D^{(-2)}, N, R^{(-2)}) &= \nu(\sigma(D,N,R)),\\
(D^{(-1)}, N, R^{(-1)}) &= \sigma(D,N,R),\\
(D^{(0)}, N, R^{(0)}) &= (D,N,R),\\
(D^{(1)}, N, R^{(1)}) &= \nu(D,N,R),\\
(D^{(2)}, N, R^{(2)}) &= \sigma(\nu(D,N,R)),\\
(D^{(3)}, N, R^{(3)}) &= \nu(\sigma(\nu(D,N,R))),\ \cdots.
\end{aligned} \tag{19}$$

Even though the middle parameter N is unchanged by the actions of either ν or σ, it is used to evaluate ν. Our definition of $\mathrm{Orb}(D,N,R)$ is then the orbit of (D,N,R) under the action of the group generated by ν and σ.

For any existing $\mathrm{TFF}(D,N,R)$, the parameters of its Naimark or spatial complements will equal $\nu(D,N,R)$ or $\sigma(D,N,R)$, respectively. Therefore $\mathrm{Orb}(D,N,R)$ contains the parameter triples for any TFF that can be obtained via alternating Naimark and spatial complements. As we have seen, however, not every $\mathrm{TFF}(D,N,R)$ has a

Naimark or spatial complement, therefore $\mathrm{Orb}(D,N,R)$ may contain triples for which no TFF can exist. For example, (1,4,0) is an element of $\mathrm{Orb}(1,4,1)$, but no TFF(1,4,0) exists. However, we can say that a $\mathrm{TFF}(D',N,R')$ exists for every (D',N,R') in $\mathrm{Orb}(D,N,R)$ if and only if it is contained in $\{(D',N,R')\in\mathbb{Z}^3 : 0<R'<D'<NR'\}$. Because ν and σ change the values for D', R', and NR', but $\mathrm{Orb}(D,N,R)$ is invariant under both operations, we can say that this occurs if and only if $\mathrm{Orb}(D,N,R)$ is contained in $\{(D',N,R')\in\mathbb{Z}^3 : D'>0, R'>0\}$.

Here we introduce the function $f_N(D,R)$: $f_N:\mathbb{R}^2\to\mathbb{R}$, where

$$f_N(D,R) := f(D,N,R) = DNR - D^2 - NR^2. \tag{20}$$

(See (6.3) of [1] for a function of (D,N,R) that previously arose in the study of the ECTFFs and only differs from f by a function of N.) Notice that the value of (20) is invariant under ν and σ:

$$\begin{aligned}
f\left(\sigma(D,N,R)\right) &= f(D,N,D-R),\\
&= DN(D-R) - D^2 - N(D-R)^2,\\
&= D^2N - DNR - D^2 - D^2N + 2DNR - NR^2,\\
&= DNR - D^2 - NR^2,\\
&= f(D,N,R).\\
f\left(\nu(D,N,R)\right) &= f(NR-D,N,R),\\
&= (NR-D)NR - (NR-D)^2 - NR^2,\\
&= (NR)^2 - DNR - (NR)^2 + 2DNR - D^2 - NR^2,\\
&= DNR - D^2 - NR^2,\\
&= f(D,N,R).
\end{aligned}$$

This means that for all (D', N, R') in $\mathrm{Orb}(D, N, R)$, $f_n(D', R') = f_N(D, R)$, and this is also true for all $\mathrm{TFF}(D', N, R')$ with parameters in $\mathrm{Orb}(D, N, R)$.

For any integer $N > 0$, this function is a quadratic form of (D, R):

$$f_N(D, R) := f(D, N, R) = DNR - D^2 - NR^2 = \begin{bmatrix} D & R \end{bmatrix} \begin{bmatrix} -1 & \frac{N}{2} \\ \frac{N}{2} & -N \end{bmatrix} \begin{bmatrix} D \\ R \end{bmatrix}. \quad (21)$$

This 2×2 matrix has characteristic polynomial

$$\begin{vmatrix} \lambda + 1 & -\frac{N}{2} \\ -\frac{N}{2} & \lambda + N \end{vmatrix} = (\lambda + 1)(\lambda + N) - \frac{N^2}{4} = \lambda^2 + (N + 1)\lambda - \frac{N(N-4)}{4},$$

which has eigenvalues $\lambda = \frac{1}{2}\{-(N+1) \pm [(N+1)^2 + N(N-4)]^{\frac{1}{2}}\}$. Notice that the lesser eigenvalue $\frac{1}{2}\{-(N_1) - [(N+1)^2 + N(N-4)]^{\frac{1}{2}}\}$ is negative, while its greater eigenvalue $\frac{1}{2}\{-(N_1) + [(N+1)^2 + N(N-4)]^{\frac{1}{2}}\}$ will be zero when $N = 4$, negative when $N < 4$, and positive for $N > 0$. In these three cases, the level set $\{(D, R) \in \mathbb{R}^2 : f_N(D, R) = C\}$ for some $C \in \mathbb{R}$ generates an ellipse, one or two parallel lines, or a hyperbola, respectively. From any point on this level set, taking the Naimark complement moves to another point on the level set horizontally, while taking a spatial complement moves to another point vertically. Note that this will be the same point if $D = NR - D$ or $R = D - R$, respectively. Applying alternating Naimark and spatial involutions moves from point to point along the level set via an alternating sequence of corresponding horizontal and vertical steps. These alternating paths only generate a finite number of distinct points when $1 < N < 4$; meanwhile, when $N \geq 4$, the nature of the path is dependent on the value of C. When $C < 0$, the path infinitely bounces between two parallel lines when $N = 4$, but when $N > 4$, the path alternates along the two connected components of a hyperbola. However, when $C > 0$ (which is only possible when $N > 4$), the path interlaces along a single connected component of a hyperbola that lies in the first quadrant. In this last case, we will show that the orbit contains a unique point (D_0, N, R_0) that is considered *minimal*, meaning that

$$0 < D_0 \leq NR_0 - D_0 \text{ and } 0 < R_0 \leq D_0 - R_0, \text{ i.e., } 0 < \tfrac{2D_0}{N} \leq R_0 \leq \tfrac{D_0}{2}. \tag{22}$$

We summarize these results in Theorem 3.1.5, and prove each component in the following sections.

Theorem 3.1.5. *Take (D, N, R) such that $1 \leq R \leq D \leq NR$ and $N > 1$. Let $f(D, N, R) = DNR - D^2 - NR^2$. Then*

(a) If $f(D, N, R) \leq 0$, then the existence of real and/or equichordal and/or equi-isoclinic TFF(D, N, R) *is fully characterized. In particular, any such TFF is obtained via alternating Naimark-spatial complements of a trivial* TFF(D_0, N, R_0)*, namely one with either $D_0 = R_0$ or $D_0 = NR_0$.*

(b) If $f(D, N, R) > 0$, then every TFF(D, N, R) *can be obtained via iterated alternating Naimark-spatial complements of one whose parameters (D_0, N, R_0) are minimal in the sense of* (22).

In this latter case, any such TFF(D, N, R) *can only be equi-isoclinic if $D = D_0$ and $D \in \{D_0, NR_0 - D_0\}$, and will only be real and/or equichordal if the* TFF(D_0, N, R_0) *is as well.*

3.2 Naimark-spatial orbits when $N \in \{2, 3\}$

As mentioned before, when $N \in \{2, 3\}$, the orbit of any triple (D, N, R) under ν and σ is necessarily finite. For example, when $N = 2$, said orbit contains at most eight points:

$$\begin{aligned}\cdots \underset{\sigma}{\leftrightarrow}&(D,2,R) \underset{\nu}{\leftrightarrow} (2R-D,2,R) \underset{\sigma}{\leftrightarrow} (2R-D,2,R-D) \underset{\nu}{\leftrightarrow} (-D,2,R-D)\\ &\underset{\sigma}{\leftrightarrow}(-D,2,-R) \underset{\nu}{\leftrightarrow} (D-2R,2,-R) \underset{\sigma}{\leftrightarrow} (D-2R,2,D-R) \underset{\nu}{\leftrightarrow} (D,2,D-R)\\ &\underset{\sigma}{\leftrightarrow}(D,2,R) \underset{\nu}{\leftrightarrow} \cdots . \end{aligned} \tag{23}$$

Notice that if a TFF$(D,2,R)$ exists, a TFF$(-D,2,-R)$ cannot. Similarly, when $N=3$, Orb$(D,3,R)$ consists of at most twelve distinct points:

$$\begin{aligned}\cdots \underset{\sigma}{\leftrightarrow}&(D,3,R) \underset{\nu}{\leftrightarrow} (3R-D,3,R) \underset{\sigma}{\leftrightarrow} (3R-D,3,2R-D) \underset{\nu}{\leftrightarrow} (3R-2D,3,2R-D)\\ &\underset{\sigma}{\leftrightarrow}(3R-2D,3,R-D) \underset{\nu}{\leftrightarrow} (-D,3,R-D) \underset{\sigma}{\leftrightarrow} (-D,3,-R) \underset{\nu}{\leftrightarrow} (D-3R,3,-R)\\ &\underset{\sigma}{\leftrightarrow}(D-3R,3,D-2R) \underset{\nu}{\leftrightarrow} (2D-3R,3,D-2R) \underset{\sigma}{\leftrightarrow} (2D-3R,3,D-R)\\ &\underset{\nu}{\leftrightarrow}(D,3,D-R) \underset{\sigma}{\leftrightarrow} (D,3,R) \underset{\nu}{\leftrightarrow} \cdots . \end{aligned} \tag{24}$$

Again, we see that Orb$(D,3,R)$ contains both points $(D,3,R)$ and $(-D,3,-R)$. Due to this, the set of triples generated via this Naimark-spatial chain from a TFF$(D,2,R)$ or TFF$(D,3,R)$ are properly contained within Orb$(D,2,R)$ and Orb$(D,3,R)$, respectively. Given that the parameters of these TFFs must be integers, it logically follows that there are few choices for D and R when $1<N<4$.

Theorem 3.2.1. *Let D and R be positive integers.*

(a) A TFF$(D,2,R)$ *exists if and only if $R \in \{\frac{D}{2}, D\}$. Such TFFs are necessarily equi-isoclinic, and can be chosen to be real.*

(b) A TFF$(D,3,R)$ *exists if and only if $R \in \{\frac{D}{3}, \frac{D}{2}, \frac{2D}{3}, D\}$. Such TFFs are necessarily equi-isoclinic when $R \in \{\frac{D}{3}, \frac{D}{2}, D\}$, equichordal when $R=\frac{2D}{3}$, and can be chosen to be real.*

Proof. Let $\mathbb{H}$ be a D-dimensional Hilbert space. Recall from Chapter II that any $\text{TFF}(D, N, R)$ with $R \in \{\frac{D}{N}, D\}$ is automatically equi-isoclinic, and such a trivial $\text{EITFF}(D, N, R)$ exists whenever $(D, N, R) \in \mathbb{Z}^3$ satisfies $D > 0, N > 1$ as well as $R \in \{\frac{D}{N}, D\}$. Then for (a), it is sufficient to show that $R \in \{\frac{D}{2}, D\}$ for any possible $\text{TFF}(D, 2, R)$.

Let $\{\mathcal{U}_1, \mathcal{U}_2\}$ be a TFF for $\mathbb{H}$, with corresponding rank-R projections $\mathbf{P}_1$ and $\mathbf{P}_2$. We know that these projections satisfy $\mathbf{P}_1 + \mathbf{P}_2 = \frac{2R}{D}\mathbf{I}$ (tightness). This suggests that 1 is an eigenvalue for $\frac{2R}{D}\mathbf{I} - \mathbf{P}_2$, because $\mathbf{P}_1 = \frac{2R}{D}\mathbf{I} - \mathbf{P}_2$. Therefore $1 \in \{\frac{2R}{D} - 1, \frac{2R}{D}\}$. Solving for R shows that $R = D$ or $R = \frac{D}{2}$, respectively.

For (b), we first show that if a $\text{TFF}(D, 3, R)$ exists, then it is either the case that $R = \frac{D}{3}$ or $R > \frac{D}{2}$. To see this, let $\{\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3\}$ be such a TFF for $\mathbb{H}$ with corresponding rank-R projections $\{\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3\}$. Again, we know that $\mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3 = \frac{3R}{D}\mathbf{I}$ and therefore $\mathbf{P}_1 + \mathbf{P}_2 = \frac{3R}{D}\mathbf{I} - \mathbf{P}_3$. If we consider $\mathbf{P}_1 + \mathbf{P}_2$ as an operator, it has eigenvalues $\{\frac{3R}{D} - 1, \frac{3R}{D}\}$ with multiplicities R and $D - R$, respectively. Let λ be the largest eigenvalue of $\mathbf{P}_1 + \mathbf{P}_2$, meaning $\lambda = \frac{3R}{D}$. If $\mathcal{U}_1 \subseteq \mathcal{U}_2^{\perp}$, then $\mathbf{P}_1 + \mathbf{P}_2$ is the projection onto $\mathcal{U}_1 + \mathcal{U}_2$, meaning $\lambda = \frac{3R}{D} = 1$ and $R = \frac{D}{3}$. Otherwise, there must exist $\mathbf{y}_1, \mathbf{y}_2$ with $\mathbf{y}_1 \in \mathcal{U}_1$ and $\mathbf{y}_2 \in \mathcal{U}_2$ and $\langle \mathbf{y}_1, \mathbf{y}_2 \rangle \neq 0$. In this case, $\lambda = \frac{3R}{D} > 1$.

Here we make use of a well-known fact from matrix analysis. Let $\mathbf{B}$ be any self-adjoint operator on $\mathbb{H}$. Without loss of generality, its eigenvalues $\{b_d\}_{d=1}^{D}$ can be ordered as $b_1 \geq b_2 \geq \ldots \geq b_{d-1} \geq b_d$. For some $\boldsymbol{\varphi} \in \mathbb{H}$ with $\|\boldsymbol{\varphi}\| = 1$, let $\mathbf{A} = \mathbf{B} + \boldsymbol{\varphi}\boldsymbol{\varphi}^*$ be the rank-1 "update" of $\mathbf{B}$. The eigenvalues for $\mathbf{A}$, $\{a_d\}_{d=1}^{D}$ will *interlace* with $\{b_d\}_{d=1}^{D}$, that is

$$a_1 \geq b_1 \geq a_2 \geq b_2 \geq \ldots \geq a_{d-1} \geq b_{d-1} \geq a_d \geq b_d.$$

Here since $\lambda > 1$, meaning that it is strictly greater than the largest eigenvalue of $\mathbf{P}_1$, and $\mathbf{P}_1 + \mathbf{P}_2$ is obtained from $\mathbf{P}_1$ via R iterative rank-1 "updates" we know that the multiplicity of λ, $D-R$, is at most R, therefore $R \geq \frac{D}{2}$.

With this, we note that if a TFF$(D, 3, R)$ exists, then either $R \in \{\frac{D}{3}, \frac{D}{2}, D\}$ or $\frac{D}{2} < R < D$. For the latter case, recall that the Naimark complement for such a TFF would be a TFF$(D, 3, D-R)$ with $D - R < D - \frac{D}{2} = \frac{D}{2}$. From before, we know that $D - R = \frac{D}{3}$, meaning $R = \frac{2D}{3}$. As first noted, any TFF$(D, 3, R)$ with $R \in \{\frac{D}{3}, D\}$ must be equi-isoclinic. Additionally, if $R = \frac{D}{2}$, a TFF$(D, 3, \frac{D}{2})$ must also be equi-isoclinic, as it is the Naimark complement of a TFF$(\frac{D}{2}, 3, \frac{D}{2})$, which is necessarily equi-isoclinic as it has equal "D" and "R" parameters. Additionally, a TFF$(D, N, \frac{2D}{R})$ cannot be equi-isoclinic because $\frac{D}{2} < R < D$, but it is the spatial complement of a TFF$(D, 3, \frac{D}{3})$, which as noted above is necessarily equi-isoclinic (and therefore equichordal). Conversely, the construction methods at the beginning of this proof show that a trivial real TFF(D, N, R) exists whenever $R \in \{\frac{D}{3}, D\}$. One can also take the Naimark complement of a trivial real TFF$(\frac{D}{2}, 3, \frac{D}{3})$ to create a real TFF$(D, 3, \frac{D}{2})$, or take the spatial complement of a trivial real TFF$(D, 3, \frac{D}{3})$ to yield a TFF$(D, 3, \frac{2D}{3})$. $\square$

From this it is clear that at most only a small portion of an orbit such as (23) or (24) will correspond to parameters where a TFF can exist. Specifically, because a TFF$(R, 2, R)$ has no spatial complement and a TFF$(2R, 2, R)$ has no Naimark complement, the parameters of any TFF$(D, 2, R)$ exist on one of two types of "Naimark-spatial paths," each only of length two:

$$(R, 2, R) \underset{\text{N}}{\leftrightarrow} (R, 2, R), \qquad (2R, 2, R) \underset{\text{s}}{\leftrightarrow} (2R, 2, R). \tag{25}$$

Similarly, because there is no Naimark complement for a TFF$(3R, 3, R)$ and no spatial complement for a TFF$(R, 3, R)$, the parameters of any TFF$(D, 3, R)$ exist in a path of length four. These are of the form

$$(3R, 3, R) \underset{\text{s}}{\leftrightarrow} (3R, 3, 2R) \underset{\text{N}}{\leftrightarrow} (3R, 3, 2R) \underset{\text{s}}{\leftrightarrow} (3R, 3, R) \tag{26}$$

when $R \in \{\frac{D}{3}, \frac{2D}{3}\}$ or the form

$$(R, 3, R) \underset{\text{N}}{\leftrightarrow} (2R, 3, R) \underset{\text{s}}{\leftrightarrow} (2R, 3, R) \underset{\text{N}}{\leftrightarrow} (R, 3, R) \tag{27}$$

when $R \in \{\frac{D}{2}, D\}$. We see so far that the Naimark-spatial orbit structure from Definition 3.1.4 does not provide much insight for a TFF(D, N, R) with $N < 4$, and this will also be the case when $f(D, N, R) = 0$. However, the utility of these Naimark-spatial orbits will be clear with respect to the study of TFFs with either $f(D, N, R) > 0$ or when $N \geq 4$ and $f(D, N, R) < 0$.

3.3 Naimark-spatial orbits when $f(D, N, R) = 0$

As the next theorem shows, TFF(D, N, R) with $f(D, N, R) = 0$ are rare.

Theorem 3.3.1. *If $f(D, N, R) := DNR - D^2 - NR^2 = 0$ for some $(D, N, R) \in \mathbb{Z}^3$ with $D > 0$, $N > 1$ and $R > 0$ then $N = 4$ and $D = 2R$. In this case,* Orb(D, N, R) *is a singleton set.*

Proof. Let N be any positive integer. The quadratic form f_N of (21) arises from a negative definite matrix whenever $N < 4$. Because of this, if $(D, N, R) \in \mathbb{Z}^3$ satisfies $f_N(D, R) = f(D, N, R) = 0$ the fact that $D > 0, N > 1$, and $R > 1$ implies that $N \geq 4$. Using the quadratic formula, we can determine that f_N is the union of the two lines in the (D, R)-plane with slopes $\frac{1}{2}\left[1 \pm \left(\frac{N-4}{N}\right)^{\frac{1}{2}}\right]$. Notice that if $N > 4$, then $\left(\frac{N-4}{N}\right)^{\frac{1}{2}} = \frac{1}{N^2}\left[N(N-4)\right]^{\frac{1}{2}}$ must be irrational. This can be seen by completing the square, $N(N-4) = (N-2)^2 - 4$; the only perfect squares whose difference is 4 are 4 and 0. This contradicts the fact that D and R are positive integers, requiring $N = 4$, in which case the lines are both equal with slope $\frac{1}{2}$. We see that (D, N, R) is necessarily $(2R, 4, R)$, a point that is fixed under the operations of both ν and σ:

$$\nu(2R, 4, R) = (4R - 2R, 4, R) = (2R, 4, R),$$
$$\sigma(2R, 4, R) = (2R, 4, 2R - R) = (2R, 4, R).$$

Therefore, $\mathrm{Orb}(D, N, R) = \mathrm{Orb}(2R, 4, R) = \{(2R, 4, R)\}$ is a singleton set. □

Because $D = 2R$, we see from this result that if such a $\mathrm{TFF}(D, N, R)$ with $f(D, N, R) = 0$ exists, both its Naimark and spatial complements exist. Moreover, if said $\mathrm{TFF}(D, N, R)$ is real and/or equichordal and/or equi-isoclinic, so are the Naimark and spatial complements.

3.4 Naimark-spatial orbits when $N \geq 4$ and $f(D, N, R) < 0$

When $N \geq 4$ and $C < 0$, the level set of the form $\{(D, R) \in \mathbb{R}^2 : f_N(D, R) = C\}$ consists of either the two connected components of a single hyperbola ($N > 4$) or a union of two parallel lines ($N = 4$). For any fixed N, applying ν or σ to any (D, N, R) in the level set corresponds to moving horizontally or vertically, respectively, from one of these connected components to the other. The Naimark-spatial path generated by iteratively applying ν and σ in an alternating fashion yields an infinite "staircase" of (D, R) pairs of arbitrarily large positive and negative values. It is clear that any (D, R) pair with a non-positive entry in either coordinate cannot correspond to the parameters of a $\mathrm{TFF}(D, N, R)$; however, we now show that any remaining "positive pairs" in this staircase correspond to the parameters of a TFF if and only if the minimal pair (D_0, R_0) of this set equates to a trivial $\mathrm{TFF}(D_0, N, R_0)$. Specifically, this occurs if and only if either $D_0 = R_0$ or $D_0 = NR_0$.

Theorem 3.4.1. *If* $f(D,N,R) := DNR - D^2 - NR^2 < 0$ *for some* $(D,N,R) \in \mathbb{Z}^3$ *with* $D > 0$*,* $N \geq 4$ *and* $R > 0$ *then*

$$\mathrm{Orb}^+(D,N,R) := \{(D',N,R') \in \mathrm{Orb}(D,N,R) : D' > 0, R' > 0\}$$

is an infinite proper subset of $\mathrm{Orb}(D,N,R)$*, and contains no* (D_0,N,R_0) *that satisfies* (22)*, but does contain a unique point* (D_0,N,R_0) *such that* $D_0 \leq D'$ *and* $R_0 \leq R'$ *for all* $(D',N,R') \in \mathrm{Orb}^+(D,N,R)$*.*

If $R_0 \notin \{\frac{D_0}{N}, D_0\}$ *then for any* $(D',N,R') \in \mathrm{Orb}(D,N,R)$*, no* $\mathrm{TFF}(D',N,R')$ *exists. If instead* $R_0 \in \{\frac{D_0}{N}, D_0\}$ *then a* $\mathrm{TFF}(D_0,N,R_0)$ *exists, and* $\mathrm{Orb}^+(D,N,R)$ *is the set of* (D',N,R') *for which there exists a* $\mathrm{TFF}(D',N,R')$ *that can be obtained from it via iterated alternating Naimark and spatial complements. In this case, any such* $\mathrm{TFF}(D',N,R')$ *is necessarily equichordal, but can only be equi-isoclinic if* $R' = R_0$ *and* $D' \in \{D_0, NR_0 - D_0\}$*. Moreover, in this case* (D_0,N,R_0) *is the only point* $(D',N,R') \in \mathrm{Orb}^+(D,N,R)$ *with* $R' \in \{\frac{D'}{N}, D'\}$*.*

Proof. Let $(D,N,R) \in \mathbb{R}^3$ with $N \geq 4$ and $f(D,N,R) < 0$. First note that either

$$(D < NR - D \text{ and } R > D - R) \text{ or } (D > NR - D \text{ and } R < D - R). \tag{28}$$

To see this, consider

$$\begin{aligned}(NR - 2D)(D - 2R) &= DNR - 2D^2 - 2NR^2 + 4DR \\ &= (DNR - D^2 - NR^2) - (D^2 - 4DR + NR^2) \\ &= f(D,N,R) - (D - 2R)^2 - (N-4)R^2 \\ &< 0,\end{aligned}$$

therefore the quantities $NR - 2D$ and $D - 2R$ cannot have the same sign. For any fixed $(D, N, R) \in \mathbb{Z}^3$ with $N \geq 4$ and $f(D, N, R) < 0$, we can assume without loss of generality that $D < NR - D$ (otherwise, we could apply ν to achieve this). By (28) we see that $D < NR - D$, meaning that we can assume that the Naimark-spatial sequence $\{(D^{(K)}, N, R^{(K)}\}_{K=-\infty}^{\infty}$ of (D, N, R) (from Definition 3.1.4) has $D^{(0)} < D^{(1)}$. Note that if we were to replace (D, N, R) with $\nu(D, N, R)$, this would shift and reverse $\{D^{(K)}, N, R^{(K)}\}_{K=-\infty}^{\infty}$ but $\operatorname{Orb}(D, N, R) = \operatorname{Orb}(D^{(K)}, N, R^{(K)} : K \in \mathbb{Z})$ would remain unchanged. Since $f(D, N, R)$ is constant throughout this sequence, that is, $f(D^{(K)}, N, R^{(K)}) = f(D, N, R)$ for all K, combining this assumption with (28) shows that

$$D^{(2J-1)} = D^{(2J)} < D^{(2J+1)}, \quad R^{(2J-1)} < R^{(2J)} = R^{(2J+1)}, \quad \forall J \in \mathbb{Z}.$$

Notably, both $\{D^{(K)}\}_{K=-\infty}^{\infty}$ and $\{R^{(K)}\}_{K=-\infty}^{\infty}$ are nondecreasing and unbounded both above and below. We then define

$$K_0 := \min\{K : D^{(K)} > 0 \text{ and } R^{(K)} > 0\}, \quad D_0 := D^{(K_0)}, \quad R_0 := R^{(K_0)}.$$

With this in mind, the positive orbit

$$\operatorname{Orb}^+(D, N, R) := \{(D', N, R') \in \operatorname{Orb}(D, N, R) : D' > 0, R' > 0\},$$

can be re-expressed as

$$\operatorname{Orb}^+(D, N, R) = \{(D^{(K)}, N, R^{(K)}) : K \geq K_0\},$$

and so is an infinite proper subset of $\operatorname{Orb}(D, N, R)$. From this it is also clear that $(D_0, N, R_0) \in \operatorname{Orb}^+(D, N, R)$ satisfies $D_0 \leq D'$ and $R_0 \leq R'$ for all possible $(D', N, R') \in \operatorname{Orb}^+(D, N, R)$, and that such a point is necessarily unique. We caution however that (D_0, N, R_0) is not "minimal" in the sense of (22). Indeed, for any

$(D_0, N, R_0) \in \text{Orb}(D, N, R)$ we have $N \geq 4$ and $f(D_0, N, R_0) = f(D, N, R) < 0$, implying via (28) a contradiction of (22).

Continuing, note that since (D_0, N, R_0) lies in $\text{Orb}^+(D, N, R)$ while the point $(D^{(K_0-1)}, N, R^{(K_0-1)})$ does not, we have both $D_0 > 0$ and $R_0 > 0$ while either $D^{(K_0-1)} \leq 0$ or $R^{(K_0-1)} \leq 0$. The ramifications of this fact depend on whether K_0 is even or odd. If K_0 is even,

$$(D^{(K_0-1)}, N, R^{(K_0-1)}) = \sigma(D^{(K_0)}, N, R^{(K_0)}) = \sigma(D_0, N, R_0) = (D_0, N, D_0 - R_0),$$

implying $D^{(K_0-1)} = D_0 > 0$ and so $D_0 - R_0 = R^{(K_0-1)} \leq 0$. If instead K_0 is odd,

$$(D^{(K_0-1)}, N, R^{(K_0-1)}) = \nu(D^{(K_0)}, N, R^{(K_0)}) = \nu(D_0, N, R_0) = (NR_0 - D_0, N, R_0),$$

implying $R^{(K_0-1)} = R_0 > 0$ and so $NR_0 - D_0 = D^{(K_0-1)} \leq 0$. Thus, in general, either $D_0 \leq R_0$ or $NR_0 \leq D_0$. Note that both of these inequalities cannot hold simultaneously: since $R_0 > 0$, having $NR_0 \leq D_0 \leq R_0$ implies $N \leq 1$, a contradiction. In particular, any $\text{TFF}(D_0, N, R_0)$ has either a Naimark or spatial complement but not the other.

Note that if a $\text{TFF}(D_0, N, R_0)$ exists then the parameters of any $\text{TFF}(D', N, R')$ constructed from it via iterated alternating Naimark and spatial complements satisfy $D' > 0$, $R' > 0$ and $(D', N, R') \in \text{Orb}(D, N, R)$, that is, $(D', N, R') \in \text{Orb}^+(D, N, R)$. Conversely, if a $\text{TFF}(D_0, N, R_0)$ exists then for any $(D', N, R') \in \text{Orb}^+(D, N, R)$, a $\text{TFF}(D', N, R')$ can be constructed from it in this way. To be precise, writing $(D', N, R') = (D^{(K_1)}, N, R^{(K_1)})$ where $K_1 \geq K_0$, a $\text{TFF}(D', N, R')$ arises from a $\text{TFF}(D_0, N, R_0)$ via $K_1 - K_0$ such complements in total, beginning with a Naimark complement when K_0 is even and with a spatial complement when K_0 is odd. Here we see that by the same logic, if a $\text{TFF}(D', N, R')$ exists for some arbitrary

$(D', N, R') \in \mathrm{Orb}^+(D, N, R)$ then a $\mathrm{TFF}(D_0, N, R_0)$ can be constructed from it by applying these same complements in the reverse order.

Now note that if $R_0 \notin \{\frac{D_0}{N}, D_0\}$ then since $D_0 \leq R_0$ or $NR_0 \leq D_0$ in general, either $D_0 < R_0$ or $NR_0 < D_0$. In either case, recall from Theorem 3.2.1 that such a $\mathrm{TFF}(D_0, N, R_0)$ does not exist. This in turn implies that no $\mathrm{TFF}(D', N, R')$ exists for any $(D', N, R') \in \mathrm{Orb}(D, N, R)$: such a TFF is nonsensical if either $D' \leq 0$ or $R' \leq 0$, while if $D' > 0$ and $R' > 0$ then $(D', N, R') \in \mathrm{Orb}^+(D, N, R)$, and so the existence of a $\mathrm{TFF}(D', N, R')$ would imply that a $\mathrm{TFF}(D_0, R, N_0)$ exists, a contradiction.

If instead $R_0 \in \{\frac{D_0}{N}, D_0\}$, as we assume to be the case for the remainder of this proof, recall from Theorem 3.2.1 that a $\mathrm{TFF}(D_0, N, R_0)$ exists, and that any such TFF is necessarily equi-isoclinic. In this case, as already noted above, $\mathrm{Orb}^+(D, N, R)$ is the set of (D', N, R') for which there exists a $\mathrm{TFF}(D', N, R')$ that can be obtained from it via iterated alternating Naimark and spatial complements, and so any such $\mathrm{TFF}(D', N, R')$ is necessarily equichordal.

Next, we show that if $(D', N, R') \in \mathrm{Orb}^+(D, N, R)$ satisfies $R' \in \{\frac{D'}{N}, D'\}$ then it is necessarily the case that $(D', N, R') = (D_0, N, R_0)$. If $D' = NR'$ then both $\sigma(D', N, R') = (NR', N, (N-1)R')$ and (D', N, R') have all positive entries while $\nu(D', N, R') = (0, N, R')$ does not. In this case, (D', N, R') is necessarily the minimally-indexed member of $\{(D^{(K)}, N, R^{(K)})\}_{K=-\infty}^{\infty}$ that has all-positive entries, that is, $(D', N, R') = (D^{(K_0)}, N, R^{(K_0)}) = (D_0, N, R_0)$. Similarly, if $D' = R'$ then both $\nu(D', N, R') = ((N-1)R', N, R')$ and (D', N, R') have all positive entries while $\sigma(D', N, R') = (R', N, 0)$ does not, and so

$$(D', N, R') = (D^{(K_0)}, N, R^{(K_0)}) = (D_0, N, R_0).$$

To conclude, fix any $(D', N, R') \in \mathrm{Orb}^+(D, N, R)$ for which an $\mathrm{EITFF}(D', N, R')$ exists. We will show that $R' = R_0$ and $D' \in \{D_0, NR_0 - D_0\}$. If $D' = NR'$, then as noted above, $(D', N, R') = (D_0, N, R_0)$ and therefore we have $R' = R_0$ and $D' = D_0 \in \{D_0, NR_0 - D_0\}$. If instead $D' < NR'$ then this EITFF(D', N, R') has a Naimark complement, and at least one of these two Naimark complementary TFFs is an EITFF(D'', N, R') where $D'' \in \{D', NR' - D'\}$ and $D'' \leq NR' - D''$. In fact, since $(D'', N, R') \in \mathrm{Orb}(D, N, R)$ we have $f(D'', N, R') = f(D, N, R) < 0$ and so (28) gives $D'' < NR' - D''$ and $R' > D'' - R'$. Now recall from Theorem 3.2.1 that any such EITFF(D'', N, R') with $D'' < 2R'$ necessarily has $D'' = R'$. Since $(D'', N, R') \in \mathrm{Orb}^+(D, N, R)$ satisfies $D'' = R'$ recall from above that we have that $(D'', N, R') = (D_0, N, R_0)$. Thus, $R' = R_0$ and $D_0 = D''$ where $D'' \in \{D', NR' - D'\} = \{D', NR_0 - D'\}$, that is, $D' \in \{D_0, NR_0 - D_0\}$. □

Combining Theorems 3.2.1 and 3.4.1 fully characterizes the existence of all (real and/or equichordal and/or equi-isoclinic) TFF(D, N, R) with $f(D, N, R) < 0$. It is important to note here that the nonexistence implications of Theorem 3.4.1 show that satisfying the basic requirement of $R \leq D \leq NR$ is not sufficient to prove a TFF(D, N, R) exists. For example, even though $2 \leq 7 \leq (4)(2)$, no TFF$(7, 4, 2)$ exists as its Naimark complement would be a TFF$(1, 4, 2)$ and $2 \not\leq 1$. Similarly, no TFF$(5, 4, 4)$ exists even though $4 \leq 5 \leq (4)(4)$, otherwise its spatial complement TFF$(5, 4, 1)$ would have to exist, and $5 \not\leq (4)(1)$. Rather than imply anything about the existence of a TFF(D, N, R) directly, having $f(D, N, R) < 0$ simply implies that the existence problem is already settled.

3.5 Naimark-spatial orbits when $f(D, N, R,) > 0$

Due to the importance of this case in the study of still-open problems regarding the existence of (real and/or equichordal and/or equi-isoclinic) TFFs, we begin with an example. For the aforementioned EITFF$(3,7,1)$ considered in (18),

$$f_7(3,1) = f(3,7,1) = (3)(7)(1) - (3)^2 - (7)(1)^2 = 5 > 0.$$

In this case $\{(D,R) \in \mathbb{R}^2 : f_7(D,R) = 7DR - D^2 - 7R^2 = 5\}$, the corresponding level set of the quadratic form f_7, is a hyperbola that contains $(3,1)$. As can be seen in Figure 1, one of the two connected components of this hyperbola is contained in the subset $\{(D,R) \in \mathbb{R}^2 : 0 < R < D < 7R\}$ of the first quadrant of the (D,R)-plane. (Its other connected component is the negation of this one.) As such, any ECTFF$(D,7,R)$ whose (D,R) parameters lie on this hyperbola has both a Naimark and spatial complement. Their (D',R') parameters are obtained by moving horizontally or vertically, respectively, from (D,R) to another point on this hyperbola. Interestingly, the path from $(3,1)$ to $(4,1)$ to $(4,3)$ to $(17,3)$, etc., (alternating complements, beginning with Naimark) "interweaves" with that from $(3,1)$ to $(3,2)$ to $(11,2)$ to $(11,9)$, etc., (alternating complements, beginning with spatial). From this graph, it is clear that Orb$(3,7,1)$ is infinite and moreover contains a unique point $(D_0,7,R_0)$ such that $D_0 \leq D$ and $R_0 \leq R$ for all $(D,7,R) \in$ Orb$(3,7,1)$, namely $(D_0,7,R_0) = (3,7,1)$. It is also clear that $(3,7,1)$ is also the only point of Orb$(3,7,1)$ that lies both below and to the left of its spatial and Naimark-complementary cousins, namely that satisfies (22). In the next result, we formally state and prove these and other claims in general.

Theorem 3.5.1. *If $f(D,N,R) := DNR - D^2 - NR^2 > 0$ for some $(D,N,R) \in \mathbb{Z}^3$ with $D > 0$, $N > 1$ and $R > 0$ then $N > 4$ and* Orb(D,N,R) *is an infinite subset of $\{(D',N,R') \in \mathbb{Z}^3 : D' > 0, R' > 0\}$ that contains a unique point (D_0,N,R_0) that satisfies* (22). *Moreover, $D_0 \leq D'$ and $R_0 \leq R'$ for all $(D',N,R') \in$* Orb(D,N,R).

Here, if a TFF(D,N,R) *exists then* Orb(D,N,R) *is the set of (D',N,R') for*

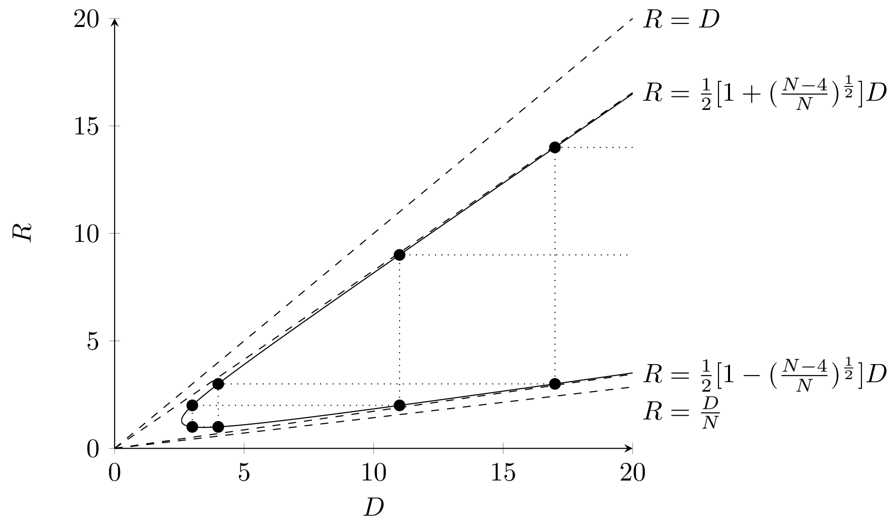

Figure 1. The Naimark-spatial orbit of an $\mathrm{ETF}(3,7)$, regarded as an $\mathrm{EITFF}(3,7,1)$. Each node indicates the (D,R) parameters of an $\mathrm{ECTFF}(D,7,R)$ that can be obtained from the $\mathrm{EITFF}(3,7,1)$ via iterated alternating Naimark and spatial complements; see (18). The hyperbola on which they lie is a level set of the quadratic form $f_7(D,R) = f(D,7,R) = 7DR - D^2 - 7R^2$ of (21) at "elevation" $f_7(3,1) = 5 > 0$. Here, since the function f of (20) is invariant under the Naimark and spatial involutions of Definition 3.1.4, taking a Naimark or spatial complement of one of these ECTFFs corresponds to traversing horizontally or vertically, respectively, from one node on this hyperbola to another. Since $f_7(3,1) = 5 > 0$, this component of this hyperbola lies between the lines with slopes $\frac{1}{2}[1 \pm (\frac{N-4}{N})^{\frac{1}{2}}]$ that form the zero set of this quadratic form. These two slopes are themselves between $\frac{1}{N}$ and 1, meaning any $\mathrm{ECTFF}(D,7,R)$ whose (D,R) parameters lie on this hyperbola has both a Naimark and spatial complement. From this graph, it is intuitively obvious that this orbit is infinite and contains exactly one point $(D_0,R_0) = (3,1)$ that lies both below and to the left of its nearest neighbors, that is, is minimal in the sense of (22). This point is also minimal in the sense that both $D_0 \leq D$ and $R_0 \leq R$ for all other nodes (D,R). To be clear, however, there are points on this component of this hyperbola with lesser values of D and R. In Theorem 3.5.1, we verify that these phenomena occur in general whenever $f(D,N,R) > 0$.

which there exists a $\mathrm{TFF}(D',N,R')$ *that can be obtained from it via iterated alternating Naimark and spatial complements. Any such* $\mathrm{TFF}(D',N,R')$ *can only be equi-isoclinic if* $R' = R_0$ *and* $D' \in \{D_0, NR_0 - D_0\}$.

Proof. Recall that the matrix that gives rise to the quadratic form f_N of (21) is negative semidefinite when $N \in \{2,3,4\}$. As such, if $f(D,N,R) > 0$ for some $(D,N,R) \in \mathbb{Z}^3$ with $N > 1$ then $N > 4$. Accordingly, for the remainder of the proof let $N > 4$ be a fixed integer. As noted in the proof of Theorem 3.3.1, for such N, the zero set of f_N is the union of the two (distinct) lines in the (D, R)-plane with slopes $\frac{1}{2}[1 \pm (\frac{N-4}{N})^{\frac{1}{2}}]$. These slopes are bounded above by 1 and below by $\frac{1}{N}$: since $N(N-4) < (N-2)^2$ we have $(\frac{N-4}{N})^{\frac{1}{2}} < \frac{N-2}{N} = 1 - \frac{2}{N}$, and so $\frac{1}{N} < \frac{1}{2}[1 - (\frac{N-4}{N})^{\frac{1}{2}}]$. Note f_N is negative at nonzero points of these bounding lines:

$$f_N(D, \tfrac{D}{N}) = DN\tfrac{D}{N} - D^2 - N(\tfrac{D}{N})^2 = -\tfrac{D^2}{N} < 0,$$
$$f_N(D, D) = DND - D^2 - ND^2 = -D^2 < 0, \tag{29}$$

for all $D \neq 0$. Meanwhile it is positive at the nonzero points on the line of slope $\frac{1}{2}$:

$$f_N(D, \tfrac{D}{2}) = DN\tfrac{D}{2} - D^2 - N(\tfrac{D}{2})^2 = (\tfrac{N}{4} - 1)D^2 > 0,$$

for all $D \neq 0$. In particular, $\{(D, R) \in \mathbb{R}^2 : f_N(D, R) > 0\}$ is a subset of the union of the first and third quadrants, with its first-quadrant component lying within the cone $0 < \frac{D}{N} < R < D$, that is,

$$\{(D, R) \in \mathbb{R}^2 : D > 0,\ R > 0,\ f_N(D, R) > 0\} \subseteq \{(D, R) \in \mathbb{R}^2 : 0 < R < D < NR\}. \tag{30}$$

At this stage, fix positive integers D and R such that $f(D, N, R) > 0$, let $C = f(D, N, R)$, and define $\mathrm{Orb}(D, N, R)$ as in Definition 3.1.4. Note that (D, N, R) is a point in the set

$$\{(D', N, R') \in \mathbb{Z}^3 : D' > 0,\ R' > 0,\ f(D', N, R') = C\}. \tag{31}$$

To show that $\mathrm{Orb}(D, N, R)$ is contained in $\{(D', N, R') \in \mathbb{Z}^3 : D > 0,\ R > 0\}$ as claimed it thus suffices to show that (31) is invariant under both ν and σ. To see this, note that if $(D', N, R') \in \mathbb{Z}^3$ then $\nu(D', N, R') = (NR' - D', N, R')$ and

$\sigma(D', N, R') = (D', N, D' - R')$ lie in $\mathbb{Z}^3$ and satisfy

$$f(\nu(D', N, R')) = f(\sigma(D', N, R')) = f(D', N, R') = C.$$

Moreover, by (30), $0 < R' < D' < NR'$ and so all entries of both $\nu(D', N, R')$ and $\sigma(D', N, R')$ are positive.

We next show that $\mathrm{Orb}(D, N, R)$ contains a point (D_0, N, R_0) that is minimal in the sense of (22). Let $R_0 := \min\{R' : (D', N, R') \in \mathrm{Orb}(D, N, R)\}$. (This is well-defined since $\mathrm{Orb}(D, N, R)$ is contained in $\{(D', N, R') \in \mathbb{Z}^3 : D > 0,\ R > 0\}$.) Take D_0 such that $(D_0, N, R_0) \in \mathrm{Orb}(D, N, R)$. By applying ν if necessary, we may assume without loss of generality that $0 < D_0 \leq NR_0 - D_0$, namely that (D_0, N, R_0) satisfies the first condition of (22). At the same time, note that since $(D_0, N, D_0 - R_0) = \sigma(D_0, N, R_0) \in \mathrm{Orb}(D, N, R)$, the definition of R_0 gives that $0 < R_0 \leq D_0 - R_0$, namely that (D_0, N, R_0) also satisfies the second condition of (22).

As we now explain, (D_0, N, R_0) is also a minimal point of $\mathrm{Orb}(D, N, R)$ in the traditional sense, that is, it satisfies $D_0 \leq D'$ and $R_0 \leq R'$ for all possible $(D', N, R') \in \mathrm{Orb}(D, N, R)$. To see this, let $\{(D^{(K)}, N, R^{(K)})\}_{K=-\infty}^{\infty}$ be the Naimark spatial sequence of (D_0, N, R_0) as defined by (19) when $(D^{(0)}, N, R^{(0)}) := (D_0, N, R_0)$. We caution that since (D_0, N, R_0) is not necessarily equal to (D, N, R), their respective Naimark-spatial sequences might not be equal. That said, because we know that $(D_0, N, R_0) \in \mathrm{Orb}(D, N, R)$, each of these two sequences can be obtained by shifting and/or reversing the other, and $\mathrm{Orb}(D_0, N, R_0) = \mathrm{Orb}(D, N, R)$ regardless.

To proceed, we formally prove something evidenced in graphs of such orbits, such as Figure 1: when $N > 4$, any rightwards move in $\mathrm{Orb}(D, N, R)$ is followed by one upwards, while any upwards move is followed by one rightwards. To be precise, if $(D', N, R') \in \mathrm{Orb}(D, N, R)$ and $D' \leq NR' - D'$ (that is, if the first entry of (D', N, R')

is no more than that of $\nu(D', N, R')$), then since $N > 4$ (and so $\frac{N}{2} < N-2$),

$$D' \leq \tfrac{N}{2}R' < (N-2)R' = (N-1)R' - R', \quad \text{i.e.,} \quad R' < (N-1)R' - D',$$

namely that the third entry of $\nu(D', N, R')$ is less than that of

$$\sigma(\nu(D', N, R')) = \sigma(NR' - D', N, R') = (NR' - D', (N-1)R' - D'). \tag{32}$$

Similarly, if $(D', N, R') \in \mathrm{Orb}(D, N, R)$ and $R' \leq D' - R'$ (that is, if the third entry of (D', N, R') is no more than that of $\sigma(D', N, R')$) then

$$NR' \leq \tfrac{N}{2}D' < (N-2)D' = (N-1)D' - D', \quad \text{i.e.,} \quad D' < (N-1)D' - NR',$$

namely that the first entry of $\sigma(D', N, R')$ is less than that of

$$\nu(\sigma(D', N, R')) = \nu(D', N, D' - R') = ((N-1)D' - NR', D' - R'). \tag{33}$$

In particular, since $D_0 \leq NR_0 - D_0$, iteratively applying these facts to the positively-indexed portion of the Naimark-spatial sequence (19) of (D_0, N, R_0) gives

$$\begin{aligned} D_0 = D^{(0)} \leq D^{(1)} = D^{(2)} < D^{(3)} = D^{(4)} < D^{(5)} = \cdots, \\ R_0 = R^{(0)} = R^{(1)} < R^{(2)} = R^{(3)} < R^{(4)} = R^{(5)} < \cdots, \end{aligned} \tag{34}$$

where $D^{(2J-1)} = D^{(2J)} < D^{(2J+1)}$ and $R^{(2J-1)} < D^{(2J)} = D^{(2J+1)}$ for all $J \geq 1$. Since $R_0 \leq D - R_0$, iteratively applying these same facts to the negatively-indexed portion of this sequence also gives

$$\begin{aligned} &D_0 = D^{(0)} = D^{(-1)} < D^{(-2)} = D^{(-3)} < D^{(-4)} = D^{(-5)} < \cdots , \\ &R_0 = R^{(0)} \leq R^{(-1)} = R^{(-2)} < R^{(-3)} = R^{(-4)} < R^{(-5)} = \cdots , \end{aligned} \tag{35}$$

where $D^{(-2J+1)} < D^{(-2J)} = D^{(-2J-1)}$ and $R^{(-2J+1)} = D^{(-2J)} < D^{(-2J-1)}$ for all $J \geq 1$. (We leave formal inductive proofs of these facts to the interested reader.) In particular, $\operatorname{Orb}(D, N, R)$ has infinite cardinality, and both $D_0 \leq D'$ and $R_0 \leq R'$ for all $(D', N, R') \in \operatorname{Orb}(D, N, R)$. In fact, our argument implies that any such $(D_0, N, R_0) \in \operatorname{Orb}(D, N, R)$ that satisfies (22) necessarily has both

$$D_0 = \min\{D' : (D', N, R') \in \operatorname{Orb}(D, N, R)\},$$

$$R_0 = \min\{R' : (D', N, R') \in \operatorname{Orb}(D, N, R)\},$$

and so such a point (D_0, N, R_0) is necessarily unique.

To prove the final parts of this result, recall that since $f_N(D, R) > 0$, $\operatorname{Orb}(D, N, R)$ is contained in $\{(D', N, R') \in \mathbb{Z}^3 : D' > 0, R' > 0\}$, and so any $\operatorname{TFF}(D', N, R')$ with $(D', N, R') \in \operatorname{Orb}(D, N, R)$ has both a Naimark and spatial complement. Moreover, since any member of $\operatorname{Orb}(D, N, R)$ can be obtained from any other via iterated alternating Naimark and spatial involutions, if a (real and/or equichordal) $\operatorname{TFF}(D', N, R')$ exists for any $(D', N, R') \in \operatorname{Orb}(D, N, R)$ then one exists for all $(D', N, R') \in \operatorname{Orb}(D, N, R)$. In this case, $\operatorname{Orb}(D, N, R)$ is the set of (D', N, R') for which there exists a $\operatorname{TFF}(D', N, R')$ that can be obtained from a $\operatorname{TFF}(D, N, R)$ in this manner. Only a scant few of these might be equi-isoclinic. To see this, assume an $\operatorname{EITFF}(D', N, R')$ exists for some $(D', N, R') \in \operatorname{Orb}(D, N, R)$. By taking its Naimark complement if necessary, we then have that an $\operatorname{EITFF}(D'', N, R')$ exists for some $(D'', N, R') \in \operatorname{Orb}(D, N, R)$ that satisfies $D'' \leq NR' - D''$ where

$D'' \in \{D', NR' - D'\}$. From Section 2, recall that an EITFF(D'', N, R') can only exist if either $2R' \leq D''$ or $D'' = R'$. The latter case cannot happen here since (29) gives $f(D'', N, D'') < 0$ whereas the invariance of f over $\mathrm{Orb}(D, N, R)$ guarantees $f(D'', N, R') = f(D, N, R) > 0$. Thus, $2R' \leq D''$. Since $(D'', N, R') \in \mathrm{Orb}(D, N, R)$ satisfies both $D'' \leq NR' - D''$ and $2R' \leq D''$, it is minimal (22), implying $R' = R_0$ and $D'' = D_0$ (and so $D' \in \{D_0, NR_0 - D_0\}$), as claimed. □

As we have just seen, when $f(D, N, R) > 0$, there are only ever at most two choices of triples in the infinite orbit of (D, N, R) for which a corresponding EITFF might exist (and at most one such choice when $D_0 = NR_0 - D_0$), namely those corresponding to the lowest point(s) $\{(D_0, R_0), (NR_0 - D_0, R_0)\}$ on the "Naimark-spatial path" on its associated hyperbola. We remark that more generally, by combining (34) and (35) with how the singular values of cross-Gram matrices evolve with respect to Naimark and spatial complements as shown in Section 2 of [20], one finds that for any $J > 1$, there are at least J distinct principal angles between any two subspaces of any TFF$(D^{(K)}, N, R^{(K)})$ when either $K \geq 2J$ or $K \leq -2J - 1$ (under the assumptions and notation of Theorem 3.5.1 and its above proof).

We also remark that when $f(D, N, R) > 0$, (34) and (35) imply that (D_0, N, R_0) is the only point (D', N, R') of $\mathrm{Orb}(D, N, R)$ that might be a fixed point of either ν or σ, that is, have either $D' = NR' - D'$ or $R' = D - R'$. Here, unlike in Theorem 3.3.1, it cannot be a fixed point of both: if $D_0 = NR_0 - D_0$ and $R_0 = D_0 - R_0$ then we have that $4R_0 = 2D_0 = NR_0$ implying $N_0 = 4$ and $D_0 = 2R_0$, and subsequently $f(D, N, R) = f(D_0, N, R_0) = f(2R_0, 4, R_0) = 0$, a contradiction. In such cases, the graph of the orbit is not "braided" like that of Figure 1, but rather appears as two copies of the same path emanating from (D_0, R_0). For context, note that in the proof of Theorem 3.4.1 we showed that no point (D, N, R) with $N \geq 4$ and $f(D, N, R) < 0$ can be a fixed point of either ν or σ.

We further note that the mapping $(D,R) \mapsto (NR-D,(N-1)R-D)$ essentially seen in (32) is a linear transformation with inverse $(D,R) \mapsto ((N-1)D-NR, D-R)$, cf. (33). It turns out that its 2×2 matrix representation is diagonalizable provided $N > 4$. This permits one to derive explicit closed-form expressions for all members of a Naimark-spatial sequence in terms of (D,N,R) and K. We do not do so here, since in our opinion, their technical nature only serves to obscure the delicate yet elementary arguments we have used up to this point.

3.6 Existence of tight fusion frames

As we have exhausted the cases above, we conclude this chapter with a proof for Theorem 3.1.5:

Proof of Theorem 3.1.5. Fix any $\mathrm{TFF}(D,N,R)$ where $(D,N,R) \in \mathbb{Z}^3$ satisfies $D > 0$, $N > 1$ and $R > 0$. We usually assume $N > 1$ by convention to avoid dividing by 0 in (1). That said, if one wishes to more generally consider TFFs for $\mathcal{H}$ that consist of a single subspace $\mathcal{U}$, note that necessarily $\mathcal{U} = \mathcal{H}$. This $\mathrm{TFF}(D,1,D)$ has neither a Naimark or spatial complement, but is vacuously equi-isoclinic, and $(D_0,N,R_0) = (D,1,D)$ satisfies $R_0 = D \in \{D\} = \{\frac{D_0}{N}, D_0\}$. That is, (i) applies even in this degenerate case.

Now for the moment assume that this $\mathrm{TFF}(D,N,R)$ arises from a $\mathrm{TFF}(D_0,N,R_0)$ via iterated alternating Naimark and spatial complements. Clearly, if one of these two TFFs is real and/or equichordal then the other is as well. Further recall that since f is invariant with respect to ν and σ, $f(D,N,R) = f(D_0,N,R_0)$. In this case, if $R_0 \in \{\frac{D_0}{N}, D_0\}$ then we have that $f(D,N,R) < 0$ since it is either the case that $f(D,N,R) = f(NR_0,N,R_0) = -NR_0^2$ or $f(D,N,R) = f(R_0,N,R_0) = -R_0^2$. If instead (D_0,N,R_0) satisfies (22) then $0 < 4R_0 \le 2D_0 \le NR_0$ and so $N \ge 4$, at which point Theorem 3.4.1 implies that $f(D_0,N,R_0) \ge 0$. We now use Theorems 3.2.1–3.5.1

to show that this $\mathrm{TFF}(D,N,R)$ indeed arises in exactly one of these two ways.

If $N=2$ then $f(D,N,R)<0$ and Theorem 3.2.1 gives that $R\in\{\frac{D}{2},D\}$, meaning that we can choose our desired $\mathrm{TFF}(D_0,N,R_0)$ to be this $\mathrm{TFF}(D,N,R)$. To be clear, in light of (25), we are also free to choose our $\mathrm{TFF}(D_0,N,R_0)$ to be the spatial or Naimark complement of this $\mathrm{TFF}(D,N,R)$ when $R=\frac{D}{2}$ and $R=D$, respectively. Regardless, $(D_0,N,R_0)=(D,N,R)$. Here, the necessary conditions that $R=R_0$ and $D\in\{D_0,NR_0-D_0\}$ are automatically satisfied.

If $N=3$ then $f(D,N,R)<0$ and Theorem 3.2.1 gives that R is $\frac{D}{3}$, $\frac{D}{2}$, $\frac{2D}{3}$, or D. When either $R=\frac{D}{3}$ or $R=D$ we can just let our desired $\mathrm{TFF}(D_0,N,R_0)$ be this $\mathrm{TFF}(D,N,R)$. If instead $R=\frac{D}{2}$ then (27) suggests we let our $\mathrm{TFF}(D_0,N,R_0)$ be the Naimark complement of this $\mathrm{TFF}(D,N,R)$, implying that we have $R_0=D_0$, since $(D_0,N,R_0)=\nu(D,3,\frac{D}{2})=(\frac{D}{2},3,\frac{D}{2})$. Similarly, if $R=\frac{2D}{3}$ then (26) suggests we let our $\mathrm{TFF}(D_0,N,R_0)$ be the spatial complement of this $\mathrm{TFF}(D,N,R)$, implying $(D_0,N,R_0)=\sigma(D,3,\frac{2D}{3})=(D,3,\frac{D}{3})$ and so $R_0=\frac{D_0}{N}$. In fact, by (26) and (27), when R is $\frac{D}{3}$, $\frac{D}{2}$, $\frac{2D}{3}$ or D we must choose (D_0,N,R_0) to be

$$(D,3,\tfrac{D}{3}),\qquad(\tfrac{D}{2},3,\tfrac{D}{2}),\qquad(D,3,\tfrac{D}{3}),\qquad(D,3,D),$$

respectively. To be clear, though there is a unique choice of (D_0,N,R_0) in each case, the corresponding $\mathrm{TFF}(D_0,N,R_0)$ is not necessarily unique. When $R=\frac{D}{2}$ for example, both the Naimark complement of our given $\mathrm{TFF}(D,N,R)$ and the Naimark complement of its spatial complement are suitable $\mathrm{TFF}(D_0,N,R_0)$. In all but the third of these four cases, (D,N,R) satisfies $R=R_0$ and $D\in\{D_0,NR_0-D_0\}$. Meanwhile, in the third case, $R=\frac{2D}{3}$, implying $R<D<2R$ and so no $\mathrm{EITFF}(D,N,R)$ exists. Thus, when $N=3$, if an $\mathrm{EITFF}(D,N,R)$ exists then indeed $R=R_0$ and $D\in\{D_0,NR_0-D_0\}$, as needed.

Next consider the case where $N\geq 4$ and $f(D,N,R)<0$. Here, Theorem 3.4.1

gives that this $\mathrm{TFF}(D, N, R)$ arises via iterated alternating Naimark complements from a $\mathrm{TFF}(D_0, N, R_0)$ with $R_0 \in \{\frac{D_0}{N}, D_0\}$ that is necessarily equi-isoclinic. Theorem 3.4.1 gives that this (D_0, N, R_0) is unique: if $(D', N, R') \in \mathrm{Orb}^+(D, N, R)$ satisfies $R' \in \{\frac{D'}{N}, D'\}$ then $(D', N, R') = (D_0, N, R_0)$ where

$$D_0 = \min\{D' : (D', N, R') \in \mathrm{Orb}^+(D, N, R)\},$$
$$R_0 = \min\{R' : (D', N, R') \in \mathrm{Orb}^+(D, N, R)\}.$$

In fact, recalling from the proof of Theorem 3.4.1 that no member of the Naimark-spatial sequence of (D, N, R) is a fixed point of either ν or σ, our $\mathrm{TFF}(D_0, N, R_0)$ is itself unique. Theorem 3.4.1 further gives that this $\mathrm{TFF}(D, N, R)$ is equi-isoclinic only if $R = R_0$ and $D \in \{D_0, NR_0 - D_0\}$.

Next, in the case where $f(D, N, R) = 0$, Theorem 3.3.1 gives that $N = 4$ and $D = 2R$. In this case, letting $\mathrm{TFF}(D_0, N, R_0)$ be this $\mathrm{TFF}(D, N, R)$ we have $(D_0, N, R_0) = (D, N, R) = (2R, 4, R)$ which satisfies (22). More generally, we could take $\mathrm{TFF}(D_0, N, R_0)$ to be any TFF obtained from this $\mathrm{TFF}(D, N, R)$ via iterated alternating Naimark and spatial complements. Regardless, $(D_0, N, R_0) = (2R, 4, R)$. Here, $R = R_0$ and the condition $D \in \{D_0, NR_0 - D_0\}$ is automatically satisfied since $D = 2R$ and $\{D_0, NR_0 - D_0\} = \{2R\}$.

Finally, in the case where $f(D, N, R) > 0$, we can see from Theorem 3.5.1 that this $\mathrm{TFF}(D, N, R)$ arises via iterated alternating Naimark and spatial complements from a $\mathrm{TFF}(D_0, N, R_0)$ where (D_0, N, R_0) is the unique member of $\mathrm{Orb}(D, N, R)$ that satisfies (22). (In fact, a careful analysis of the proof of Theorem 3.5.1 reveals that this $\mathrm{TFF}(D_0, N, R_0)$ is itself unique when $4R_0 < 2D_0 < NR_0$, but is only unique up to spatial or Naimark complements when either $2R_0 = D_0$ or $2D_0 = NR_0$, respectively.) Theorem 3.5.1 moreover gives that this $\mathrm{TFF}(D, N, R)$ can only be equi-isoclinic if $R = R_0$ and $D \in \{D_0, NR_0 - D_0\}$. $\square$

IV. Harmonic fusion frame construction

4.1 Difference sets and families

Given a finite abelian group $\mathcal{G}$ with nonempty subset $\mathcal{D}$ of size D, the *harmonic frame* that corresponds to $\mathcal{G}$ is the restriction of the character table $\mathbf{\Gamma}$ of $\mathcal{G}$ to the subset $\mathcal{D}$, namely $\{\boldsymbol{\varphi}_\gamma\}_{\gamma\in\widehat{\mathcal{G}}} \subseteq \mathbb{C}^{\mathcal{D}}$ with $\boldsymbol{\varphi}_\gamma(d) := \frac{1}{\sqrt{D}}\gamma(d)$. Effectively, a harmonic frame is built by extracting the rows of the character table that correspond to the subset D and reforming them into the synthesis operator of the frame. Harmonic frames are studied to exploit both the underlying character structure and the fact that any harmonic frame is automatically tight:

$$\begin{aligned}\mathbf{\Phi}\mathbf{\Phi}^*(d,d') &= \sum_{\gamma\in\widehat{\mathcal{G}}}\mathbf{\Phi}(d,\gamma)\mathbf{\Phi}^*(\gamma,d')\\ &= \tfrac{1}{D}\sum_{\gamma\in\widehat{\mathcal{G}}}\mathbf{\Gamma}(d,\gamma)\mathbf{\Gamma}^*(\gamma,d')\\ &= \tfrac{1}{D}(\mathbf{\Gamma}\mathbf{\Gamma}^*)(d,d')\\ &= \tfrac{G}{D}\mathbf{I}(d,d').\end{aligned}$$

Our focus here is to determine how the harmonic frame generated from the character table $\mathbf{\Gamma}$ is affected by our choice of the set $\mathcal{D}$ in relation to a subgroup $\mathcal{H}$. To that end, we will investigate how the $\mathcal{D}$ is related to cosets of $\mathcal{H}$.

Theorem 4.1.1. *Let $\widehat{\mathcal{G}}$ be the Pontryagin dual of an abelian group $\mathcal{G}$ of $G<\infty$. Let $\mathcal{H}$ be a subgroup of $\mathcal{G}$ of order H, and $\mathcal{D}$ be a subset of $\mathcal{G}$ with cardinality $D\neq 0$. Let $\{\boldsymbol{\varphi}_{\gamma\eta}\}_{\eta\in\mathcal{H}^\perp}$ be the harmonic frame arising from $\mathcal{D}$, with $\boldsymbol{\varphi}_\gamma\in\mathbb{C}^{\mathcal{D}}, \boldsymbol{\varphi}_\gamma := \frac{1}{\sqrt{D}}\gamma(d)$. For any $g\in\mathcal{G}$ let $\mathcal{D}_g := \mathcal{H}\cap(\mathcal{D}-g)$. Let $\mathcal{U}_\gamma = \operatorname{span}\{\boldsymbol{\varphi}_{\gamma\eta}\}_{\eta\in\mathcal{H}^\perp}$, and for any $\gamma\in\widehat{\mathcal{G}}$, let $\mathbf{\Phi}_\gamma$ be the synthesis operator for $\{\boldsymbol{\varphi}_{\gamma\eta}\}_{\eta\in\mathcal{H}^\perp}$, namely the $\mathcal{D}\times\mathcal{H}^\perp$-indexed matrix with entries $\mathbf{\Phi}_\gamma(d,\eta) = \boldsymbol{\varphi}_{\gamma\eta}(d) = \frac{1}{\sqrt{D}}\gamma(d)\eta(d)$.*

Then $\{\mathcal{U}_\gamma\}_{\gamma\mathcal{H}^\perp\in\widehat{\mathcal{G}}/\mathcal{H}^\perp}$ *is a* $\mathrm{TFF}(D,H,\frac{G}{H})$ *for* $\mathbb{C}^{\mathcal{D}}$ *if and only if* $\#(\mathcal{D}_g)=\frac{DH}{G}$ *for all* $g\in\mathcal{G}$*. Furthermore, under this assumption,* $\{\mathcal{U}_\gamma\}_{\gamma\mathcal{H}^\perp\in\widehat{\mathcal{G}}/\mathcal{H}^\perp}$ *is an:*

(a) $\mathrm{ECTFF}(D,H,\frac{G}{H})$ *for* $\mathbb{C}^{\mathcal{D}}$ *if and only if* $\{\mathcal{D}_g\}_{g+\mathcal{H}\in\mathcal{G}/\mathcal{H}}$ *is a difference family for* $\mathcal{H}$*;*

(b) $\mathrm{EITFF}(D,H,\frac{G}{H})$ *for* $\mathbb{C}^{\mathcal{D}}$ *if and only if each* $\mathcal{D}_g$ *is a difference set for* $\mathcal{H}$*.*

Proof. For the first claim, we have $\mathbf{\Phi}_\gamma(d,\eta)=\boldsymbol{\varphi}_{\gamma\eta}(d)=\gamma\eta(d)=\gamma(d)\eta(d)$ for all γ in $\widehat{\mathcal{G}}$, d in $\mathcal{D}$ and η in $\mathcal{H}^\perp$. Now consider the Gram matrix of $\{\boldsymbol{\varphi}_{\gamma\eta}\}_{\eta\in\mathcal{H}^\perp}$:

$$\left(\mathbf{\Phi}_\gamma^*\mathbf{\Phi}_\gamma\right)(\eta,\eta')=\sum_{d\in\mathcal{D}}\overline{\boldsymbol{\varphi}_{\gamma\eta}(d)}\boldsymbol{\varphi}_{\gamma\eta'}(d)=\tfrac{1}{D}\sum_{g\in\mathcal{G}}\overline{\eta(\eta')^{-1}(d)}\boldsymbol{\chi}_{\mathcal{D}}(g)=\tfrac{1}{D}\left(\mathbf{\Gamma}^*\boldsymbol{\chi}_{\mathcal{D}}\right)(\eta(\eta')^{-1}).$$

Notice that this only depends on elements of $\mathcal{H}^\perp$ and is therefore invariant with respect to γ. This means that $\{\boldsymbol{\varphi}_{\gamma\eta}\}_{\eta\in\mathcal{H}^\perp}$ is orthonormal for all γ in $\widehat{\mathcal{G}}$ if and only if

$$\left(\mathbf{\Gamma}^*\boldsymbol{\chi}_{\mathcal{D}}\right)(\eta(\eta')^{-1})=\begin{cases}D, & \eta=\eta',\\ 0, & \eta\neq\eta',\end{cases}\qquad\forall\,\eta,\eta'\in\mathcal{H}^\perp,$$

namely if and only if

$$\left(\mathbf{\Gamma}^*\boldsymbol{\chi}_{\mathcal{D}}\right)(\gamma)=\begin{cases}D, & \gamma=\mathbf{1},\\ 0, & \gamma\in\mathcal{H}^\perp,\gamma\neq\mathbf{1},\end{cases}$$

or equivalently, that $\boldsymbol{\chi}_{\mathcal{H}^\perp}\odot\left(\mathbf{\Gamma}^*\boldsymbol{\chi}_{\mathcal{D}}\right)=D\boldsymbol{\delta}_1$. This tells us something about the nature of $\mathcal{D}$ and $\mathcal{H}^\perp$, and we can use the PSF to determine the equivalent relationship between $\mathcal{D}$ and $\mathcal{H}$:

$$\begin{aligned}D\mathbf{1}=\mathbf{\Gamma}\left(D\boldsymbol{\delta}_1\right)=\mathbf{\Gamma}\left(\boldsymbol{\chi}_{\mathcal{H}^\perp}\odot\left(\mathbf{\Gamma}^*\boldsymbol{\chi}_{\mathcal{D}}\right)\right)&=\tfrac{1}{G}(\mathbf{\Gamma}\boldsymbol{\chi}_{\mathcal{H}^\perp})*(\mathbf{\Gamma}\mathbf{\Gamma}^*\boldsymbol{\chi}_{\mathcal{D}})\\&=\tfrac{1}{G}\left(\tfrac{G}{H}\boldsymbol{\chi}_{\mathcal{H}}\right)*(G\boldsymbol{\chi}_{\mathcal{D}})=\tfrac{G}{H}\left(\boldsymbol{\chi}_{\mathcal{H}}*\boldsymbol{\chi}_{\mathcal{D}}\right).\end{aligned}$$

Here we have that $\left(\boldsymbol{\chi}_{\mathcal{H}} * \boldsymbol{\chi}_{\mathcal{D}}\right)(g) = \frac{DH}{G}$ for all $g \in \mathcal{G}$. This shows that $\{\mathcal{U}_n\}_{n\in\mathcal{N}}$ is orthonormal if and only if $\left(\boldsymbol{\chi}_{\mathcal{H}} * \boldsymbol{\chi}_{\mathcal{D}}\right)(g)$ is constant for all $g \in \mathcal{G}$. Specifically, this happens if and only if for all $g \in \mathcal{G}$,

$$\begin{aligned}\tfrac{DH}{G} = \left(\boldsymbol{\chi}_{\mathcal{H}} * \boldsymbol{\chi}_{\mathcal{D}}\right)(g) = \sum_{g'\in\mathcal{G}} \boldsymbol{\chi}_{\mathcal{H}}(g')\boldsymbol{\chi}_{\mathcal{D}}(g-g') = \sum_{g''\in\mathcal{G}} \boldsymbol{\chi}_{\mathcal{H}}(-g'')\boldsymbol{\chi}_{\mathcal{D}}(g+g'')\\ = \#(\mathcal{H} \cap (\mathcal{D}-g)) = \#\left(\mathcal{D} \cap (\mathcal{H}+g)\right) = \#\left(\mathcal{D}_g\right).\end{aligned}$$

Therefore $\{\boldsymbol{\varphi}_{\gamma\eta}\}_{\eta\in\mathcal{H}^\perp}$ is orthonormal if and only if $\#(\mathcal{D}_g)$ is constant for all $g \in \mathcal{G}$. More specifically, $\#(\mathcal{D}_g) = \frac{DH}{G}$ and $\{\mathcal{U}_\gamma\}_{\gamma\mathcal{H}^\perp\in\widehat{\mathcal{G}}/\mathcal{H}^\perp}$ is a $\mathrm{TFF}(D, H, \frac{G}{H})$ for $\mathbb{C}^{\mathcal{D}}$.

For the remainder of this proof, we assume that for all $g \in \mathcal{G}$, $\#(\mathcal{D}_g) = \frac{DH}{G}$ (a constant). We already have that $\boldsymbol{\Phi}_\gamma^*\boldsymbol{\Phi}_\gamma = \mathbf{I}_{\mathcal{H}^\perp}$ for all $\gamma \in \widehat{\mathcal{G}}$. For any $\gamma_1, \gamma_2 \in \widehat{\mathcal{G}}$, consider the corresponding cross-Gram matrix:

$$\begin{aligned}(\boldsymbol{\Phi}_{\gamma_1}^*\boldsymbol{\Phi}_{\gamma_2})(\eta_1,\eta_2) &= \langle \boldsymbol{\varphi}_{\gamma_1\eta_1}, \boldsymbol{\varphi}_{\gamma_2\eta_2}\rangle\\ &= \tfrac{1}{D}\sum_{g\in\mathcal{G}} \overline{(\gamma_1\gamma_2^{-1}\eta_1\eta_2^{-1})(g)}\boldsymbol{\chi}_{\mathcal{D}}(g)\\ &= \tfrac{1}{D}(\boldsymbol{\Gamma}^*\boldsymbol{\chi}_{\mathcal{D}})(\gamma_1\gamma_2^{-1}\eta_1\eta_2^{-1}).\end{aligned}$$

Here we notice that for any fixed γ_1, γ_2, the value of $(\boldsymbol{\Phi}_{\gamma_1}^*\boldsymbol{\Phi}_{\gamma_2})(\eta_1,\eta_2)$ depends only on the quotient of the characters. That is, $\boldsymbol{\Phi}_{\gamma_1}^*\boldsymbol{\Phi}_{\gamma_2}$ is an $\mathcal{H}^\perp$-circulant matrix (filter) with impulse response $\mathbf{f}$ where $\mathbf{f}(\eta) = \frac{1}{D}(\boldsymbol{\Gamma}^*\boldsymbol{\chi}_{\mathcal{D}})(\gamma_1\gamma_2^{-1}\eta)$. We can thus diagonalize $\boldsymbol{\Phi}_{\gamma_1}^*\boldsymbol{\Phi}_{\gamma_2}$ with the Fourier basis. To elaborate, it is well known that $g + \mathcal{H} \mapsto (\eta \mapsto \eta(g))$ is an isomorphism from $\mathcal{G}/\mathcal{H}$ onto the Pontryagin dual of $\mathcal{H}^\perp$. Under this identification, the DFT of $\mathbf{f}$ over $\mathcal{H}^\perp$ becomes

$$(\boldsymbol{\Gamma}_{\mathcal{H}^\perp}^*\mathbf{f})(g+\mathcal{H}^\perp) = \sum_{\eta\in\mathcal{H}^\perp} \overline{\eta(g)}\mathbf{f}(\eta) = \tfrac{1}{D}\sum_{\gamma\in\widehat{\mathcal{G}}} \overline{\gamma(g)}(\boldsymbol{\Gamma}^*\chi_{\mathcal{D}})(\gamma_1\gamma_2^{-1}\gamma)\boldsymbol{\chi}_{\mathcal{H}^\perp}(\gamma).$$

We now express this in terms of the character table of $\mathcal{G}$, denoted $\mathbf{\Gamma}$:

$$
\begin{aligned}
(\mathbf{\Gamma}^*_{\mathcal{H}^\perp}\mathbf{f})(g+\mathcal{H}) &= \tfrac{1}{D}\sum_{\gamma\in\widehat{\mathcal{G}}}\Big[\boldsymbol{\chi}_{\mathcal{H}^\perp}\odot(\mathbf{T}^{\gamma_1^{-1}\gamma_2}\mathbf{\Gamma}^*\boldsymbol{\chi}_{\mathcal{D}})\Big](\gamma)\gamma(-g)\\
&= \tfrac{1}{D}\Big\{\mathbf{\Gamma}\Big[\boldsymbol{\chi}_{\mathcal{H}^\perp}\odot(\mathbf{T}^{\gamma_1^{-1}\gamma_2}\mathbf{\Gamma}^*\boldsymbol{\chi}_{\mathcal{D}})\Big]\Big\}(-g).
\end{aligned}
$$

Here distributing $\mathbf{\Gamma}$ gives

$$
\mathbf{\Gamma}\Big[\boldsymbol{\chi}_{\mathcal{H}^\perp}\odot(\mathbf{T}^{\gamma_1^{-1}\gamma_2}\mathbf{\Gamma}^*\boldsymbol{\chi}_{\mathcal{D}})\Big] = \tfrac{1}{G}(\mathbf{\Gamma}\boldsymbol{\chi}_{\mathcal{H}^\perp})*(\mathbf{\Gamma}\mathbf{T}^{\gamma_1^{-1}\gamma_2}\mathbf{\Gamma}^*\boldsymbol{\chi}_{\mathcal{D}}) = \tfrac{G}{H}\boldsymbol{\chi}_{\mathcal{H}}*(\mathbf{M}^{\gamma_1^{-1}\gamma_2}\boldsymbol{\chi}_{\mathcal{D}}).
$$

Altogether, we see that $\big\{\tfrac{G}{DH}[\boldsymbol{\chi}_{\mathcal{H}}*(\mathbf{M}^{\gamma_1^{-1}\gamma_2}\boldsymbol{\chi}_{\mathcal{D}})](g)\big\}_{g+\mathcal{H}\in\mathcal{G}/\mathcal{H}}$ is the spectrum of the cross-Gram matrix $\mathbf{\Phi}^*_{\gamma_1}\mathbf{\Phi}_{\gamma_2}$. As expected, when $\gamma_1=\gamma_2$, every eigenvalue becomes

$$
\tfrac{G}{DH}(\boldsymbol{\chi}_{\mathcal{H}}*\boldsymbol{\chi}_{\mathcal{D}})(-g) = \tfrac{G}{DH}\#(\mathcal{D}_{-g}) = \tfrac{G}{DH}\left(\tfrac{DH}{G}\right) = 1,
$$

since $\mathbf{\Phi}^*_\gamma\mathbf{\Phi}_\gamma = \mathbf{I}$. To further simplify this expression for the spectrum of $\mathbf{\Phi}^*_{\gamma_1}\mathbf{\Phi}_{\gamma_2}$, note that for any $\gamma\in\widehat{\mathcal{G}}$,

$$
\begin{aligned}
(\boldsymbol{\chi}_{\mathcal{H}}*(\mathbf{M}^{\gamma}\boldsymbol{\chi}_{\mathcal{D}}))(g) &= \sum_{g'\in\mathcal{G}}\boldsymbol{\chi}_{\mathcal{H}}(g')\gamma(g-g')\boldsymbol{\chi}_{\mathcal{D}}(g-g')\\
&= \sum_{g''\in\mathcal{G}}\boldsymbol{\chi}_{\mathcal{H}}(-g'')\gamma(g+g'')\boldsymbol{\chi}_{\mathcal{D}}(g+g'').
\end{aligned}
$$

Notice that $\boldsymbol{\chi}_{\mathcal{H}}(-g)=\boldsymbol{\chi}_{-\mathcal{H}}(g)=\boldsymbol{\chi}_{\mathcal{H}}(g)$ since $\mathcal{H}$ is a subgroup. Then we have that

$$
\begin{aligned}
(\boldsymbol{\chi}_{\mathcal{H}}*(\mathbf{M}^{\gamma}\boldsymbol{\chi}_{\mathcal{D}}))(g) &= \gamma(g)\sum_{g''\in\mathcal{G}}\boldsymbol{\chi}_{\mathcal{H}}(g'')\overline{\gamma(g'')}\boldsymbol{\chi}_{\mathcal{D}-g}(g'')\\
&= \gamma(g)(\mathbf{\Gamma}^*\boldsymbol{\chi}_{\mathcal{H}\cap(\mathcal{D}-g)})(\gamma)\\
&= \gamma(g)(\mathbf{\Gamma}^*\boldsymbol{\chi}_{\mathcal{D}_g})(\gamma).
\end{aligned}
$$

As such, the spectrum of $\mathbf{\Phi}_{\gamma_1}^*\mathbf{\Phi}_{\gamma_2}$ is $\left\{\frac{G}{DH}(\gamma_1^{-1}\gamma_2)(g)\left(\mathbf{\Gamma}^*\boldsymbol{\chi}_{\mathcal{D}_g}\right)(\gamma_1^{-1}\gamma_2)\right\}_{g+\mathcal{H}^\perp\in\mathcal{G}/\mathcal{H}}$ for any $\gamma_1,\gamma_2\in\widehat{\mathcal{G}}$.

Recall that $\{\mathcal{U}_n\}_{n\in\mathcal{N}}$ is an ECTFF if and only if the Frobenius norm of the cross-Gram matrices are all constant. When $\gamma_1\neq\gamma_2$, let $\gamma=\gamma_1^{-1}\gamma_2$:

$$\begin{aligned}\tfrac{D^2H^2}{G^2}\|\mathbf{\Phi}_{\gamma_1}^*\mathbf{\Phi}_{\gamma_2}\|_{\text{Fro}}^2 &= \sum_{g+\mathcal{H}\in\mathcal{G}/\mathcal{H}}|(\boldsymbol{\chi}_{\mathcal{H}}*(\mathbf{M}^\gamma\boldsymbol{\chi}_{\mathcal{D}})(-g))|^2\\ &= \sum_{g+\mathcal{H}\in\mathcal{G}/\mathcal{H}}|\gamma(g)(\mathbf{\Gamma}^*\boldsymbol{\chi}_{\mathcal{D}_g})(\gamma)|^2,\\ &= \sum_{g+\mathcal{H}\in\mathcal{G}/\mathcal{H}}(\mathbf{\Gamma}^*(\boldsymbol{\chi}_{\mathcal{D}_g}*\tilde{\boldsymbol{\chi}}_{\mathcal{D}_g}))(\gamma).\end{aligned}$$

Then $\{\mathcal{U}_\gamma\}_{\gamma\mathcal{H}^\perp\in\widehat{\mathcal{G}}/\mathcal{H}^\perp}$ is an ECTFF if and only if $\left(\mathbf{\Gamma}^*\left(\sum_{g+\mathcal{H}\in\mathcal{G}/\mathcal{H}}\boldsymbol{\chi}_{\mathcal{D}_g}*\tilde{\boldsymbol{\chi}}_{\mathcal{D}_g}\right)\right)(\gamma)$ is constant for all $\gamma\neq 1$ in $\widehat{\mathcal{G}}$. Moreover, this occurs if and only if for all $g'\neq 0$, it is the case that $\sum_{g+\mathcal{H}\in\mathcal{G}/\mathcal{H}}(\boldsymbol{\chi}_{\mathcal{D}_g}*\tilde{\boldsymbol{\chi}}_{\mathcal{D}_g})(g')$ is constant. From Lemma 2.7.3, this happens if and only if $\{\mathcal{D}_g\}_{g\in\mathcal{G}}$ is a difference family for $\mathcal{H}$.

For (b), recall that $\{\mathcal{U}_\gamma\}_{\gamma\mathcal{H}^\perp\in\widehat{\mathcal{G}}/\mathcal{H}^\perp}$ is an EITFF if and only if $\#(\mathcal{D}_g)$ is constant for all $g\in\mathcal{G}$ (tightness) and there exists a $\lambda\geq 0$ with $\frac{G^2}{D^2H^2}|(\mathbf{\Gamma}^*\boldsymbol{\chi}_{\mathcal{D}_g})(\gamma_1\gamma_2^{-1})|^2=\lambda$ for all $\gamma_1\mathcal{H}^\perp\neq\gamma_2\mathcal{H}^\perp$ and all $g+\mathcal{H}\in\mathcal{G}/\mathcal{H}$. This occurs if and only if $|(\mathbf{\Gamma}^*\boldsymbol{\chi}_{\mathcal{D}_g}(\gamma)|$ is constant over all $\gamma\notin\mathcal{H}^\perp$ and all $g+\mathcal{H}\in\mathcal{G}/\mathcal{H}$.

Recall that $\mathcal{D}_g\subseteq\mathcal{H}$, meaning that we can write $\boldsymbol{\chi}_{\mathcal{D}_g}=\uparrow\boldsymbol{\chi}'_{\mathcal{D}_g}$ where $\boldsymbol{\chi}'_{\mathcal{D}_g}\in\mathbb{C}^{\mathcal{H}}$. Then $\mathbf{\Gamma}_{\mathcal{G}}^*\uparrow\boldsymbol{\chi}'_{\mathcal{D}_g}=\mathbf{\Gamma}_{\mathcal{H}}^*\boldsymbol{\chi}'_{\mathcal{D}_g}$ is the periodic extension of $\mathbf{\Gamma}_{\mathcal{H}}^*\boldsymbol{\chi}'_{\mathcal{D}_g}\in\mathbb{C}^{\widehat{\mathcal{H}}}$. Note that $\mathbb{C}^{\widehat{\mathcal{H}}}\cong\mathbb{C}^{\widehat{\mathcal{G}}/\mathcal{H}^\perp}$, then

$$|(\mathbf{\Gamma}_{\mathcal{G}}^*\boldsymbol{\chi}_{\mathcal{D}_g})(\gamma)|=|(\mathbf{\Gamma}_{\mathcal{G}}^*\uparrow\boldsymbol{\chi}'_{\mathcal{D}_g})(\gamma)|=|(\mathbf{\Gamma}_{\mathcal{H}}^*\boldsymbol{\chi}'_{\mathcal{D}_g})(\gamma)|,\ \forall\ \gamma\mathcal{H}^\perp\in\widehat{\mathcal{G}}/\mathcal{H}^\perp.$$

Therefore $|(\mathbf{\Gamma}_{\mathcal{G}}^*\boldsymbol{\chi}_{\mathcal{D}_g}(\gamma)|$ is constant over all $\gamma\notin\mathcal{H}^\perp$ and all $g+\mathcal{H}\in\mathcal{G}/\mathcal{H}$ if and only if $|(\mathbf{\Gamma}_{\mathcal{H}}^*\boldsymbol{\chi}'_{\mathcal{D}_g}(\gamma)|$ is constant over all $\gamma\mathcal{H}^\perp\neq\mathcal{H}^\perp$ and all $g\in\mathcal{G}$, which happens if and only if $\mathcal{D}_g$ is a difference set for $\mathcal{H}$ for all $g\in\mathcal{G}$. □

Example 4.1.2. Consider the difference family from Example 2.7.2: $\mathcal{D}_0 = \{1, 4\}$ and $\mathcal{D}_1 = \{2, 3\}$ is a difference family for $\mathcal{H} = \mathbb{Z}_5$. In order to construct the harmonic frame, we identify $\mathcal{G}$ with $\mathbb{Z}_{10}$ and $\mathcal{H}$ as $\langle 2\rangle = \{0, 2, 4, 6, 8\}$ which is isomorphic to $\mathbb{Z}_5$. Then $\mathcal{D} = 2\mathcal{D}_0 \cup (2\mathcal{D}_1 + 1) = \{2, 8, 5, 7\}$. Next, consider $\mathcal{D}_g = \mathcal{H} \cap (\mathcal{D} - g)$:

$$\mathcal{D}_0 = \{2, 8\}, \quad \mathcal{D}_1 = \{4, 6\}, \quad \mathcal{D}_2 = \{0, 6\} = \mathcal{D}_0 - 2, \quad \mathcal{D}_3 = \{2, 4\} = \mathcal{D}_1 - 1 \ldots$$

This pattern repeats, with $\#(\mathcal{D}_g)$ constant for all g (tightness). Notice that $\mathcal{D}_0$ and $\mathcal{D}_1$ can be used to generate the other $\mathcal{D}_g$ sets by appropriate shifts of elements of $\mathcal{H}$. In this case, $\mathcal{G}/\mathcal{H} = \{g+\mathcal{H} : g \in \mathcal{G}\} = \{\{0, 2, 4, 6, 8\}, \{1, 3, 5, 7, 9\}\} = \{(0+\mathcal{H}), (1+\mathcal{H})\}$. Because $\{\mathcal{D}_0, \mathcal{D}_1\}$ is a difference family for $\mathcal{H}$, Theorem 4.1.1 states that the harmonic frame generated by $\mathcal{D}$ is an ECTFF$(4, 2, 5)$ for $\mathbb{F}^4$.

Let $\omega = \exp\left(\frac{2\pi i}{10}\right)$. Then the character table for $\mathcal{G}$ is

$$\boldsymbol{\Gamma} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 & -1 & \omega^6 & \omega^7 & \omega^8 & \omega^9 \\ 1 & \omega^2 & \omega^4 & \omega^6 & \omega^8 & 1 & \omega^2 & \omega^4 & \omega^6 & \omega^8 \\ 1 & \omega^3 & \omega^6 & \omega^9 & \omega^2 & -1 & \omega^8 & \omega^1 & \omega^4 & \omega^7 \\ 1 & \omega^4 & \omega^8 & \omega^2 & \omega^6 & 1 & \omega^4 & \omega^8 & \omega^2 & \omega^6 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & \omega^6 & \omega^2 & \omega^8 & \omega^4 & 1 & \omega^6 & \omega^2 & \omega^8 & \omega^4 \\ 1 & \omega^7 & \omega^4 & \omega & \omega^8 & -1 & \omega^2 & \omega^9 & \omega^6 & \omega^3 \\ 1 & \omega^8 & \omega^6 & \omega^4 & \omega^2 & 1 & \omega^8 & \omega^6 & \omega^4 & \omega^2 \\ 1 & \omega^9 & \omega^8 & \omega^7 & \omega^6 & -1 & \omega^4 & \omega^3 & \omega^2 & \omega \end{bmatrix}.$$

Finally, to construct the frame, reduce the character table to the rows indexed by $\mathcal{D}$.

$$\begin{matrix} \mathbf{0} \\ \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{4} \\ \mathbf{5} \\ \mathbf{6} \\ \mathbf{7} \\ \mathbf{8} \\ \mathbf{9} \end{matrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 & -1 & \omega^6 & \omega^7 & \omega^8 & \omega^9 \\ 1 & \omega^2 & \omega^4 & \omega^6 & \omega^8 & 1 & \omega^2 & \omega^4 & \omega^6 & \omega^8 \\ 1 & \omega^3 & \omega^6 & \omega^9 & \omega^2 & -1 & \omega^8 & \omega^1 & \omega^4 & \omega^7 \\ 1 & \omega^4 & \omega^8 & \omega^2 & \omega^6 & 1 & \omega^4 & \omega^8 & \omega^2 & \omega^6 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & \omega^6 & \omega^2 & \omega^8 & \omega^4 & 1 & \omega^6 & \omega^2 & \omega^8 & \omega^4 \\ 1 & \omega^7 & \omega^4 & \omega & \omega^8 & -1 & \omega^2 & \omega^9 & \omega^6 & \omega^3 \\ 1 & \omega^8 & \omega^6 & \omega^4 & \omega^2 & 1 & \omega^8 & \omega^6 & \omega^4 & \omega^2 \\ 1 & \omega^9 & \omega^8 & \omega^7 & \omega^6 & -1 & \omega^4 & \omega^3 & \omega^2 & \omega \end{bmatrix} \rightarrow \begin{matrix} \mathbf{2} \\ \mathbf{5} \\ \mathbf{7} \\ \mathbf{8} \end{matrix} \begin{bmatrix} 1 & \omega^2 & \omega^4 & \omega^6 & \omega^8 & 1 & \omega^2 & \omega^4 & \omega^6 & \omega^8 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & \omega^7 & \omega^4 & \omega & \omega^8 & -1 & \omega^2 & \omega^9 & \omega^6 & \omega^3 \\ 1 & \omega^8 & \omega^6 & \omega^4 & \omega^2 & 1 & \omega^8 & \omega^6 & \omega^4 & \omega^2 \end{bmatrix}.$$

Recall that $\mathcal{D}_g$ from Theorem 4.1.1 are grouped according to cosets of $\mathcal{H}^\perp = \{0, 5\}$ in this case. Therefore, if we group the columns of our matrix by these cosets, we will form the ONBs for the subspaces:

$$\begin{array}{c} \begin{matrix} \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} & \mathbf{6} & \mathbf{7} & \mathbf{8} & \mathbf{9} \end{matrix} \\ \begin{bmatrix} 1 & \omega^2 & \omega^4 & \omega^6 & \omega^8 & 1 & \omega^2 & \omega^4 & \omega^6 & \omega^8 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & \omega^7 & \omega^4 & \omega & \omega^8 & -1 & \omega^2 & \omega^9 & \omega^6 & \omega^3 \\ 1 & \omega^8 & \omega^6 & \omega^4 & \omega^2 & 1 & \omega^8 & \omega^6 & \omega^4 & \omega^2 \end{bmatrix} \end{array} \rightarrow \begin{array}{c} \begin{matrix} \mathbf{0} & \mathbf{5} & \mathbf{1} & \mathbf{6} & \mathbf{2} & \mathbf{7} & \mathbf{3} & \mathbf{8} & \mathbf{4} & \mathbf{9} \end{matrix} \\ \left[\begin{array}{cc|cc|cc|cc|cc} 1 & 1 & \omega^2 & \omega^2 & \omega^4 & \omega^4 & \omega^6 & \omega^6 & \omega^8 & \omega^8 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\ 1 & -1 & \omega^7 & \omega^2 & \omega^4 & \omega^9 & \omega & \omega^6 & \omega^8 & \omega^3 \\ 1 & 1 & \omega^8 & \omega^8 & \omega^6 & \omega^6 & \omega^4 & \omega^4 & \omega^2 & \omega^2 \end{array}\right] \end{array}.$$

We can confirm that this frame is equichordal from the values of the Gram matrix; however, as complex numbers can be difficult to compare, it is often easier to plot the values of the Gram matrix on a color scale as seen in Figure 2. Each block represents the color scale of the corresponding cross-Gram matrix Frobenius norm, therefore constant colors in the off-diagonal indicate an ECTFF.

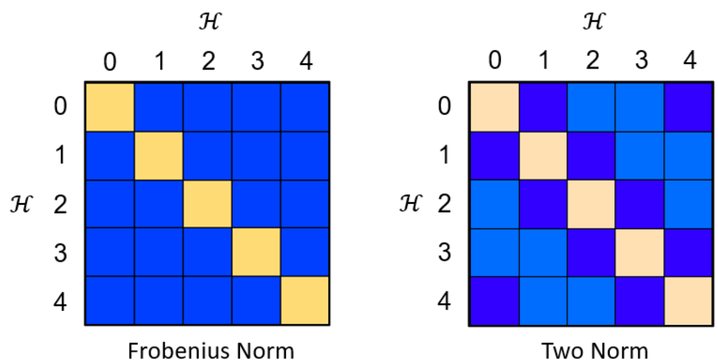

Figure 2. Color-scaled Frobenius and Two-norms for the cross-Gram matrices in Example 4.1.2

4.2 Harmonic frames from divisible difference sets

Theorem 24 of [32] gives that the harmonic frame constructed from $\mathcal{D}$, a semiregular $\mathcal{H}$-DDS results in an ECTFF or EITFF with *flat* cross-Gram matrices, meaning that the Gram matrix $\mathbf{\Phi}^*\mathbf{\Phi}$ has block identities down the diagonal and all other entries have equal modulus. As King demonstrates, such frames would clearly yield ECTFFs as these constant off-diagonal entries would automatically give the cross-Gram matrices the same Frobenius norm. Theorem 4.1.1, however, shows that such a harmonic frame would be an ECTFF if and only if it comes from a difference family. In light of this, we frame King's method in the context of Theorem 4.1.1.

Theorem 4.2.1. *For any non-empty proper subset $\mathcal{D}$ and subgroup $\mathcal{H}$ of a finite abelian group $\mathcal{G}$, define $\{\boldsymbol{\varphi}_{\gamma\eta}\}_{\eta\in\mathcal{H}^\perp}$, $\mathcal{D}_g$, $\mathbf{\Phi}_\gamma$, and $\mathcal{U}_\gamma$ as in Theorem 4.1.1. Then $\mathcal{D}$ is a semiregular $\mathcal{H}$-DDS if and only if $\mathbf{\Phi}_\gamma^*\mathbf{\Phi}_\gamma = \mathbf{I}$ for all $\gamma \in \widehat{\mathcal{G}}$ and $|\left(\mathbf{\Phi}_{\gamma_1}^*\mathbf{\Phi}_{\gamma_2}\right)(\eta_1,\eta_2)|$ is constant for all $\eta_1,\eta_2 \in \mathcal{H}^\perp$ whenever $\gamma_1\mathcal{H}^\perp \neq \gamma_2\mathcal{H}^\perp$.*

In this case $\{\mathcal{U}_\gamma\}_{\gamma\mathcal{H}^\perp\in\widehat{\mathcal{G}}/\mathcal{H}^\perp}$ is an ECTFF$(D, H, \frac{G}{H})$ *for $\mathbb{C}^\mathcal{D}$, and is moreover an* EITFF$(D, H, \frac{G}{H})$ *for $\mathbb{C}^\mathcal{D}$ if and only if we further have that each $\mathcal{D}_g$ is a difference set for $\mathcal{H}$.*

Proof. For any $\gamma_1, \gamma_2 \in \widehat{\mathcal{G}}$, $\eta_1, \eta_2 \in \mathcal{H}^\perp$, we have that

$$(\mathbf{\Phi}_{\gamma_1}^* \mathbf{\Phi}_{\gamma_2})(\eta_1, \eta_2) = \langle \boldsymbol{\varphi}_{\gamma_1\eta_1}, \boldsymbol{\varphi}_{\gamma_2\eta_2} \rangle = \tfrac{1}{D}(\mathbf{\Gamma}^* \boldsymbol{\chi}_{\mathcal{D}})(\gamma_1\gamma_2^{-1}\eta_1\eta_2^{-1}).$$

Then by our assumption, we have that $\mathbf{\Phi}_\gamma^* \mathbf{\Phi}_\gamma = \mathbf{I}$ for all $\gamma \in \widehat{\mathcal{G}}$ and also that $\mathbf{\Phi}_{\gamma_1}^* \mathbf{\Phi}_{\gamma_2}$ is flat whenever $\gamma_1\mathcal{H}^\perp \neq \gamma_2\mathcal{H}^\perp$ if and only if

$$\tfrac{1}{D}(\mathbf{\Gamma}^* \boldsymbol{\chi}_{\mathcal{D}})(\eta_1\eta_2^{-1}) = (\mathbf{\Phi}_\gamma^* \mathbf{\Phi}_\gamma)(\eta_1, \eta_2) = \begin{cases} 1, \ \eta_1 = \eta_2, \\ 0, \ \eta_1 \neq \eta_2, \end{cases}$$

and also there exists $B \geq 0$ such that

$$\left|\tfrac{1}{D}(\mathbf{\Gamma}^* \boldsymbol{\chi}_{\mathcal{D}})(\gamma_1\gamma_2^{-1}\eta_2\eta_2^{-1})\right|^2 = |(\mathbf{\Phi}_{\gamma_1}^* \mathbf{\Phi}_{\gamma_2})(\eta_1\eta_2)|^2 = B,$$

for all $\gamma_1, \gamma_2 \in \widehat{\mathcal{G}}$, $\eta_1, \eta_2 \in \mathcal{H}^\perp$ such that $\gamma_1\mathcal{H}^\perp \neq \gamma_2\mathcal{H}^\perp$ (meaning $\gamma_1\gamma_2^{-1}$ is not an element of $\mathcal{H}^\perp$). This equates to having

$$\left|(\mathbf{\Gamma}^* \boldsymbol{\chi}_{\mathcal{D}})(\gamma)\right|^2 = \begin{cases} D^2, & \gamma = 1, \\ 0, & \gamma \in \mathcal{H}^\perp, \gamma \neq 1, \\ D^2 B, & \gamma \notin \mathcal{H}^\perp. \end{cases}$$

Since $(\mathbf{\Gamma}^* \boldsymbol{\chi}_{\mathcal{D}})(1) = D$ for any subset, Lemma 2.7.5 shows that this is equivalent to having $\mathcal{D}$ be a semiregular $\mathcal{H}$-DDS for $\mathcal{G}$.

Now assume that this is the case, that is, that $\mathcal{D}$ is a semiregular $\mathcal{H}$-DDS. Applying Lemma 2.7.5 further provides that $D^2 B = C - \frac{DH(G-D)}{G(H-1)}$, and therefore whenever $\gamma_1\mathcal{H}^\perp \neq \gamma_2\mathcal{H}^\perp$,

$$\|\mathbf{\Phi}_{\gamma_1}^* \mathbf{\Phi}_{\gamma_2}\|_{\mathrm{Fro}}^2 = \sum_{\eta_1 \in \mathcal{H}^\perp} \sum_{\eta_2 \in \mathcal{H}^\perp} |(\mathbf{\Phi}_{\gamma_1}^* \mathbf{\Phi}_{\gamma_2})(\eta_1, \eta_2)|^2 = \left(\tfrac{G}{H}\right)^2 B = \tfrac{G^2}{H^2} \tfrac{H(G-D)}{DG(H-1)} = \tfrac{G(G-D)}{DH(H-1)},$$

which is the generalized Welch bound (10) for $(D, N, R) = \big(D, H, \frac{G}{H}\big)$:

$$\tfrac{R(NR-D)}{D(N-1)} = \tfrac{G(G-D)}{DH(H-1)}.$$

This means that $\{\mathcal{U}_n\}_{n\in\mathcal{N}}$ achieves equality in (7), and is therefore an ECTFF, as claimed. Applying Theorem 4.1.1 then establishes that $\{\mathcal{D}_g\}_{g+\mathcal{H}\in\mathcal{G}/\mathcal{H}}$ is a difference family for $\mathcal{H}$, additionally that $\{\mathcal{U}_n\}_{n\in\mathcal{N}}$ is an EITFF if and only if each $\mathcal{D}_g$ is a difference set for $\mathcal{H}$. $\square$

Example 4.2.2. Let $\mathcal{G} = \mathbb{Z}_3 \times \mathbb{Z}_3$. The subset $\mathcal{D} = \{(x, x^2) : x \in \mathbb{Z}_3\}$ is then an $\mathcal{H}$-RDS(3,3,3,1) for $\mathcal{H} = \{0\} \times \mathbb{Z}_3$:

−	00	11	21
00	00	22	12
11	11	00	20
21	21	10	00

The subset $\mathcal{E} = \{01, 02\}$ is a difference set for $\mathcal{H}$. Then consider the difference table for $\mathcal{D} + \mathcal{E} = \{01, 12, 22, 02, 10, 20\}$:

−	01	12	22	02	10	20
01	00	22	12	02	21	11
12	11	00	20	10	02	22
22	21	10	00	20	12	02
02	01	20	10	00	22	12
10	12	01	21	12	00	20
20	22	11	01	21	10	00

Let $\omega = \exp\left(\frac{2\pi i}{3}\right)$, then the character table for $\mathcal{G}$ is

$$\mathbf{\Gamma}_{\mathcal{G}} = \mathbf{\Gamma}_{\mathbb{Z}_3} \times \mathbf{\Gamma}_{\mathbb{Z}_3} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix} = \left[\begin{array}{ccc|ccc|ccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & 1 & \omega & \omega^2 & 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega & 1 & \omega^2 & \omega & 1 & \omega^2 & \omega \\ \hline 1 & 1 & 1 & \omega & \omega & \omega & \omega^2 & \omega^2 & \omega^2 \\ 1 & \omega & \omega^2 & \omega & \omega^2 & 1 & \omega^2 & 1 & \omega \\ 1 & \omega^2 & \omega & \omega & 1 & \omega^2 & \omega^2 & \omega & 1 \\ \hline 1 & 1 & 1 & \omega^2 & \omega^2 & \omega^2 & \omega & \omega & \omega \\ 1 & \omega & \omega^2 & \omega^2 & 1 & \omega & \omega & \omega^2 & 1 \\ 1 & \omega^2 & \omega & \omega^2 & \omega & 1 & \omega & 1 & \omega^2 \end{array}\right].$$

The harmonic frame is the restriction of the character table's rows to those corresponding to $\mathcal{D}$:

$$\mathbf{\Phi} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & 1 & \omega & \omega^2 & 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega & 1 & \omega^2 & \omega & 1 & \omega^2 & \omega \\ 1 & \omega^2 & \omega & \omega & 1 & \omega^2 & \omega^2 & \omega & 1 \\ 1 & 1 & 1 & \omega^2 & \omega^2 & \omega^2 & \omega & \omega & \omega \\ 1 & \omega & \omega^2 & \omega^2 & 1 & \omega & \omega & \omega^2 & 1 \end{bmatrix}.$$

The modulus squared Gram matrix shows that $\mathbf{\Phi}$ is the synthesis operator for an

EITFF with flat cross-Gram matrices:

$$|\mathbf{\Phi}^*\mathbf{\Phi}|^2 = \frac{1}{12}\left[\begin{array}{ccc|ccc|ccc} 12 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 12 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 12 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 12 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 12 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 12 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 12 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 12 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 12 \end{array}\right].$$

The construction above is not unique. In fact, such a construction guarantees an $\mathcal{H}$-DDS [28]. King uses this $\mathcal{H}$-DDS construction to generate EITFFs with flat cross-Gram matrices [32], and we again put this in the context of Theorem 4.1.1.

Theorem 4.2.3. *Let $\mathcal{D}$ be an SR-DDS for $\mathcal{H} \leq \mathcal{G}$. Suppose that for each $g \in \mathcal{G}$, the set $\mathcal{D}_g = \mathcal{H} \cap (\mathcal{D} - g)$ is a DS for $\mathcal{H}$ and each $\mathcal{D}_g$ is a shift of another, i.e. for any $g_1, g_2 \in \mathcal{G}$, there is some $h(g_1, g_2) \in \mathcal{H}$ such that $\mathcal{D}_{g_1} = \mathcal{D}_{g_2} + h(g_1, g_2)$.*

This occurs if and only if $\mathcal{D} = \mathcal{A} + \mathcal{B}$ where $\mathcal{A}$ is a DS for $\mathcal{H}$ and $\mathcal{B}$ is an $\mathcal{H}$-RDS for $\mathcal{G}$.

Proof. First, we partition $\mathcal{G}$ according to cosets of $\mathcal{H}$ ($\bar{g} = g + \mathcal{H}$),

$$\mathcal{G} = \bigsqcup_{\bar{g} \in \mathcal{G}/\mathcal{H}} (\bar{g}).$$

Because $\mathcal{D} \subsetneq \mathcal{G}$, we have that

$$\mathcal{D} = \mathcal{D} \cap \mathcal{G} = \mathcal{D} \cap \Big[\bigsqcup_{\bar{g} \in \mathcal{G}/\mathcal{H}} (\bar{g}) \Big] = \bigsqcup_{\bar{g} \in \mathcal{G}/\mathcal{H}} [\mathcal{D} \cap (\bar{g})] = \bigsqcup_{\bar{g} \in \mathcal{G}/\mathcal{H}} (g + \mathcal{D}_g).$$

Let $\mathcal{B}$ be one choice of coset representatives of $\mathcal{G}/\mathcal{H}$. Then since each $\mathcal{D}_g$ is a shift of the others, let $\mathcal{A} = \mathcal{D}_b$ for some $b \in \mathcal{B}$, meaning

$$\mathcal{D} = \bigsqcup_{\bar{g} \in \mathcal{G}/\mathcal{H}} (g + \mathcal{D}_g) = \bigsqcup_{b \in \mathcal{B}} b + \mathcal{A}.$$

This means that

$$\boldsymbol{\chi}_{\mathcal{D}} = \sum_{b \in \mathcal{B}} \boldsymbol{\chi}_{b+\mathcal{A}} = \sum_{b \in \mathcal{B}} \mathbf{T}^b \boldsymbol{\chi}_{\mathcal{A}} = \boldsymbol{\chi}_{\mathcal{A}} * \boldsymbol{\chi}_{\mathcal{B}}.$$

Therefore $g \in \mathcal{G}$ can be written as $a + b = g$ where $a \in \mathcal{A}$ and $b \in \mathcal{B}$ if and only if $g \in \mathcal{D}$. In this case, a, b are unique to g. Furthermore,

$$|(\boldsymbol{\Gamma}_{\mathcal{G}}^* \boldsymbol{\chi}_{\mathcal{D}})(\gamma)|^2 = \big|\big[\boldsymbol{\Gamma}_{\mathcal{G}}^* \big(\boldsymbol{\chi}_{\mathcal{A}} * \boldsymbol{\chi}_{\mathcal{B}}\big)\big](\gamma)\big|^2 = \big|\big(\boldsymbol{\Gamma}_{\mathcal{G}}^* \boldsymbol{\chi}_{\mathcal{A}}\big)(\gamma)\big|^2 |(\boldsymbol{\Gamma}_{\mathcal{G}}^* \boldsymbol{\chi}_{\mathcal{B}})(\gamma)|^2. \tag{36}$$

From Lemma 2.7.5, we know that $|(\boldsymbol{\Gamma}_{\mathcal{G}}^* \boldsymbol{\chi}_{\mathcal{D}})(\gamma)|^2 = \frac{DH(G-D)}{G(H-1)}$ if $\gamma \notin \mathcal{H}^{\perp}$. To find $|(\boldsymbol{\Gamma}_{\mathcal{H}}^* \boldsymbol{\chi}_{\mathcal{A}})(\gamma)|^2$, first note that $\lambda_{\mathcal{A}} = \frac{H}{G}\Lambda_1 = \frac{DH(DH-G)}{G^2(H-1)}$ since $\mathcal{D}$ is the disjoint union of shifts of $\mathcal{A}$. Then, considering $\boldsymbol{\chi}_{\mathcal{A}} \in \mathbb{C}^{\mathcal{G}}$,

$$\boldsymbol{\chi}_{\mathcal{A}} * \boldsymbol{\chi}_{-\mathcal{A}} = \tfrac{DH}{G}\boldsymbol{\delta}_0 + \tfrac{DH(DH-G)}{G^2(H-1)}\boldsymbol{\chi}_{\mathcal{H}\setminus\{0\}} + 0\boldsymbol{\chi}_{\mathcal{G}\setminus\mathcal{H}} = \tfrac{DH^2(G-D)}{G^2(H-1)}\boldsymbol{\delta}_0 + \tfrac{DH(DH-G)}{G^2(H-1)}\boldsymbol{\chi}_{\mathcal{H}}.$$

Taking the DFT of both sides and applying the PSF gives

$$\big|\big(\boldsymbol{\Gamma}^* \boldsymbol{\chi}_{\mathcal{A}}\big)\big|^2 = \tfrac{DH^2(G-D)}{G^2(H-1)}\boldsymbol{\chi}_{\widehat{\mathcal{G}}} + \tfrac{DH^2(DH-G)}{G^2(H-1)}\boldsymbol{\chi}_{\mathcal{H}^{\perp}}.$$

Then we can see that

$$\big|\big(\boldsymbol{\Gamma}^* \boldsymbol{\chi}_{\mathcal{A}}\big)(\gamma)\big|^2 = \begin{cases} \frac{D^2H^2}{G^2}, & \gamma \in \mathcal{H}^{\perp}, \\ \frac{DH^2(G-D)}{G^2(H-1)}, & \gamma \notin \mathcal{H}^{\perp}. \end{cases}$$

Substituting into (36) when $\gamma \notin \mathcal{H}^\perp$ gives

$$\left|(\boldsymbol{\Gamma}^* \boldsymbol{\chi}_{\mathcal{B}})(\gamma)\right|^2 = \left|\frac{(\boldsymbol{\Gamma}^* \boldsymbol{\chi}_{\mathcal{D}})(\gamma)}{(\boldsymbol{\Gamma}^* \boldsymbol{\chi}_{\mathcal{A}})(\gamma)}\right|^2 = \left(\tfrac{DH(G-D)}{G(H-1)}\right)\left(\tfrac{G^2(H-1)}{DH^2(G-D)}\right) = \tfrac{G}{H}.$$

Recall from Lemma 2.7.5 that $\mathcal{B}$ is an $\mathcal{H}$-RDS if and only if

$$|(\boldsymbol{\Gamma}^* \boldsymbol{\chi}_{\mathcal{B}})(\gamma)|^2 = \begin{cases} 0, & \gamma \in \mathcal{H}^\perp, \gamma \neq 1, \\ \tfrac{G-B}{H-1}, & \gamma \notin \mathcal{H}^\perp, \end{cases}$$

for all $\gamma \neq 1 \in \widehat{\mathcal{G}}$. When $\gamma \neq 1 \in \mathcal{H}^\perp$, (36) gives the first condition. For the second, note that $B = \#(\mathcal{B}) = \frac{G}{H}$, and

$$\tfrac{G-B}{H-1} = \tfrac{G}{H}. \qquad \square$$

Corollary 4.2.4. *For any prime power Q there is an* EITFF(QK, Q, Q) *whenever a DS(Q, K) exists for $\mathbb{F}^Q$.*

Proof. Let $\mathcal{A}$ be a DS(Q, K) for $\mathbb{F}^Q$. Recall that for any prime power Q, the set $\mathcal{B} = \{(x, x^2); x \in \mathbb{F}^Q\}$ is an RDS$(Q, Q, Q, 1)$ for $\mathcal{G} = \mathbb{F}^Q \times \mathbb{F}^Q$ with respect to $\mathcal{H} = \{0\} \times \mathbb{F}^Q$. Then the set $\mathcal{D} = (\{0\} \times \mathcal{A}) + \mathcal{B})$ is an SR-DDS of the type shown in [28], and the harmonic frame generated will be of the form shown in Theorem 4.2.3. $\square$

Corollary 4.2.4 leads to the following two infinite families of EITFFs. When $Q \equiv 3 \mod 4$, $\mathcal{A} = \{x^2 : x \in \mathbb{F}_Q\}$ is known as a Paley DS [2]. Here we have that $K = \frac{Q-1}{2}$ meaning an EITFF$(\frac{Q(Q-1)}{2}, Q, Q)$ exists. When $Q = 2^{2J}$ (a power of 4), there exists a McFarland DS$\big(2^{2J}, 2^{J-1}(2^J - 2)\big)$ [35, 36, 2], which then results in an EITFF$\big(2^{3J-1}(2^J - 1), 2^{2J}, 2^{2J}\big)$. We explore the McFarland DS construction in the next section.

4.3 McFarland divisible difference sets

In this section, we focus on the harmonic frame generated from a specific type of DDS. As previously shown, a McFarland DS can be used to construct an EITFF, but in Section 2.3 of [36], Pott refers to any DDS constructed by the arrangement of hyperplanes over finite vector spaces as a "McFarland" DDS. We want to know if such a DDS can ever generate an EC/EITFF, but first we must demonstrate how this method relates to the group structures we previously used.

Lemma 4.3.1. *Let $\mathcal{V}$ be a finite vector space over $\mathbb{F}_Q$ where Q is a prime power. Let $\mathrm{B} : \mathcal{V} \times \mathcal{V} \to \mathbb{F}_Q$ be any nondegenerate bilinear form. Letting P denote the characteristic of $\mathbb{F}_Q$, the function $\mathrm{tr}_{Q\backslash P}(\mathrm{B}) : \mathcal{V} \times \mathcal{V} \to \mathbb{F}_P$ is also a nondegenerate bilinear form, meaning we can identify $\mathcal{V}$ with its Pontryagin dual via the isomorphism $v_2 \mapsto \Big(v_1 \mapsto \exp\big(\frac{2\pi i}{P}\,\mathrm{tr}_{Q\backslash P}(\mathrm{B}(v_1, v_2))\big)\Big)$.*

Moreover, under this identification, the orthogonal complement of any subspace $\mathcal{U}$ of $\mathcal{V}$ (regarded as a vector space over $\mathbb{F}_Q$) with respect to B, that is

$$\mathcal{U}^\perp := \{v_2 \in \mathcal{V} : \mathrm{B}(v_1, v_2) = 0,\ \forall\, v_1 \in \mathcal{U}\},$$

is equal to the annihilator of $\mathcal{U}$ regarded as a subgroup of (the additive group) $\mathcal{V}$, i.e.

$$\mathcal{U}^\perp := \{v_2 \in \mathcal{V} : \exp\big(\tfrac{2\pi i}{P}\,\mathrm{tr}_{Q\backslash P}(\mathrm{B}(v_1, v_2))\big) = 0,\ \forall\, v_1 \in \mathcal{U}\}.$$

In particular, letting $\mathbf{\Gamma}_{\mathcal{V}}$ be the $\mathcal{V} \times \mathcal{V}$ representation of the character table of $\mathcal{V}$ under this representation, defined by $\mathbf{\Gamma}_{\mathcal{V}}(v_1, v_2) := \exp\big(\frac{2\pi i}{p}\,\mathrm{tr}_{Q\backslash P}(\mathrm{B}(v_1, v_2))\big)$, the Poisson summation formula states $\mathbf{\Gamma}_{\mathcal{U}}^* \boldsymbol{\chi}_{\mathcal{U}} = \#(\mathcal{U}) \boldsymbol{\chi}_{\mathcal{U}^\perp}$.

Proof. Take any c_1, c_2 in $\mathbb{F}_P$ and v_1, v_2, v_3 in $\mathcal{V}$ and note that, through the bilinearity of B,

$$[\mathrm{tr}_{Q\setminus P}(\mathrm{B})](c_1v_1 + c_2v_2, v_3) = c_1[\mathrm{tr}_{Q\setminus P}(\mathrm{B})](v_1, v_3) + c_2[\mathrm{tr}_{Q\setminus P}(\mathrm{B})](v_2, v_3),$$

which is similarly true in the other coordinate, establishing bilinearity. For nondegeneracy, it is clear that $\mathrm{Null}(\mathrm{tr}_{Q\setminus P}(\mathrm{B})) \subseteq \mathrm{Null}(\mathrm{B})$ because $\mathrm{tr}_{Q\setminus P}(0) = 0$. Next, take any $v \in \mathcal{V}$ such that $\mathrm{tr}_{Q\setminus P}(\mathrm{B}(u, v)) = 0$ for all u in $\mathcal{U}$. Assume that there exists some $u_1 \in \mathcal{U}$ such that $\mathrm{B}(u_1, v) \neq 0$. Take $\beta \in \mathbb{F}_Q$ such that $\mathrm{tr}_{Q\setminus P}(\beta) = 1$. Let $u_2 = \beta[\mathrm{B}(u_1, v)]^{-1}u_1$. This lies in $\mathcal{U}$ since $\beta[\mathrm{B}(u_1, v)]^{-1} \in \mathbb{F}_Q$ and $u_1 \in \mathcal{U}$. Then

$$\begin{aligned}
0 &= \mathrm{tr}_{Q\setminus P}(\mathrm{B}(u_0, v)) \\
&= \mathrm{tr}_{Q\setminus P}(\mathrm{B}(\beta[\mathrm{B}(u_1, v)]^{-1}u_1, v)) \\
&= \mathrm{tr}_{Q\setminus P}(\beta[\mathrm{B}(u_1, v)]^{-1}\,\mathrm{B}(u_1, v)) \\
&= \mathrm{tr}_{Q\setminus P}(\beta) \\
&= 1,
\end{aligned}$$

which is a contradiction, therefore $\mathrm{tr}_{Q\setminus P}(\mathrm{B})$ inherits nondegeneracy from B. This means that $\hat{\mathrm{B}} = \mathrm{tr}_{Q\setminus P}(\mathrm{B})$ is an irreducible $\mathbb{F}_P$-valued nondegenerate bilinear form on $\mathcal{V}$. For all v_1, v_2, v_3 in $\mathcal{V}$, consider the mapping $\boldsymbol{\varphi}_{v_2}(v_1) := \exp(\frac{2\pi i}{P}\hat{\mathrm{B}}(v_1, v_2))$. This is a homomorphism from $\mathcal{V}$ onto $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$:

$$\begin{aligned}
\boldsymbol{\varphi}_{v_3}(v_1 + v_2) &= \exp(\tfrac{2\pi i}{P}\hat{\mathrm{B}}(v_1 + v_2, v_3)) \\
&= \exp(\tfrac{2\pi i}{P}\hat{\mathrm{B}}(v_1, v_3))\exp(\tfrac{2\pi i}{P}\hat{\mathrm{B}}(v_2, v_3)) \\
&= \boldsymbol{\varphi}_{v_3}(v_1)\boldsymbol{\varphi}_{v_3}(v_2).
\end{aligned}$$

The mapping $\boldsymbol{\psi}(v_3) = \boldsymbol{\varphi}_{v_3}$ is a homomorphism from $\mathcal{V}$ onto its Pontryagin dual, $\hat{\mathcal{V}}$:

$$\begin{aligned}[\boldsymbol{\psi}(v_2+v_3)](v_1) &= \boldsymbol{\varphi}_{v_2+v_3}(v_1) \\ &= \exp(\tfrac{2\pi i}{P}\hat{\mathrm{B}}(v_1, v_2+v_3)) \\ &= \boldsymbol{\varphi}_{v_2}(v_1)\boldsymbol{\varphi}_{v_3}(v_1) \\ &= [\boldsymbol{\psi}(v_2)\boldsymbol{\psi}(v_3)](v_1).\end{aligned}$$

Note that if v_2 is in $\ker(\boldsymbol{\psi})$, then $\boldsymbol{\psi}(v_2) = \boldsymbol{\varphi}_{v_2} = \mathbf{1}$, meaning that this homomorphism is injective. Therefore for any v_1 in $\mathcal{V}$, $1 = \boldsymbol{\varphi}_{v_2}(v_1) = \exp(\frac{2\pi i}{P}\hat{\mathrm{B}}(v_1, v_2))$, which implies that $v_2 = 0$ since $\hat{\mathrm{B}}$ is nondegenerate. Since $\#(\mathcal{V}) = \#(\hat{\mathcal{V}})$, $\boldsymbol{\psi}$ is an isomorphism, meaning we can regard the $\mathcal{V} \times \hat{\mathcal{V}}$ character table of $\mathcal{V}$ as the $\mathcal{V} \times \mathcal{V}$ matrix whose (v_1, v_2) entry is

$$\boldsymbol{\Gamma}_{\mathcal{V}}(v_1, v_2) = \exp(\tfrac{2\pi i}{P}\hat{\mathrm{B}}(v_1, v_2)) = \exp(\tfrac{2\pi i}{P}\,\mathrm{tr}_{Q\backslash P}(\mathrm{B}(v_1, v_2))).$$

In this context, any subspace $\mathcal{U}$ of $\mathcal{V}$ as a vector space over $\mathbb{F}_Q$ is also a subgroup of $\mathcal{V}$. The annihilator of $\mathcal{U}$, $\mathcal{U}_A^\perp$ is then

$$\begin{aligned}\mathcal{U}_A^\perp &= \{v \in \mathcal{V} : \exp(\hat{\mathrm{B}}(u, v)) = 1, \forall\, u \in \mathcal{U}\} \\ &= \{v \in \mathcal{V} : \mathrm{tr}_{Q\backslash P}(\mathrm{B}(u, v)) = 0, \forall\, u \in \mathcal{U}\} \\ &= \{v \in \mathcal{V} : \mathrm{B}(u, v) = 0, \forall\, u \in \mathcal{U}\} \\ &= \mathcal{U}_C^\perp,\end{aligned}$$

where $\mathcal{U}_C^\perp$ is the orthogonal complement of $\mathcal{U}$ as a subspace of $\mathcal{V}$, meaning we can denote both as $\mathcal{U}^\perp$. Under this identification, the Poisson summation formula gives $\boldsymbol{\Gamma}_{\mathcal{V}}^* \boldsymbol{\chi}_{\mathcal{U}} = \#(\mathcal{U})\boldsymbol{\chi}_{\mathcal{U}^\perp}$, as stated. $\square$

Theorem 4.3.2. *Let Q be any prime power and J be any integer with $J \geq 2$. Let $R = \frac{Q^J-1}{Q-1}$, $\mathcal{G} = \mathbb{Z}_R \times \mathbb{F}_{Q^J}$, and $\mathcal{D} = \{(r,z) \in \mathbb{Z}_R \times \mathbb{F}_{Q^J} : z \in \langle \alpha^r \rangle^\perp\}$, where the orthogonal complement of $\langle \alpha^r \rangle := \{c\alpha^r : c \in \mathbb{F}_Q\}$ is taken with respect to the nondegenerate bilinear form $\mathrm{B}(z_1, z_2) := \mathrm{tr}_{Q^J \setminus Q}(z_1 z_2)$. Then letting $\mathcal{H} = \{0\} \times \mathbb{F}_{Q^J}$, $\mathcal{D}$ is a semiregular $\mathcal{H}$-DDS(R, Q^J, RQ^{J-1}) and applying Theorem 4.2.1 yields a harmonic ECTFF(RQ^{J-1}, Q^J, R).*

Proof. Let P be the characteristic of $\mathbb{F}_Q$ (and $\mathbb{F}_{Q^J}$ as well), then we can consider $\mathbb{F}_{Q^J}$ to be a vector space over both $\mathbb{F}_Q$ and $\mathbb{F}_P$. For all z_1, z_2 in $\mathbb{F}_{Q^J}$, we define $\mathrm{B} : \mathbb{F}_{Q^J} \times \mathbb{F}_{Q^J} \to \mathbb{F}_Q$, where $\mathrm{B}(z_1, z_2) := \mathrm{tr}_{Q^J \setminus Q}(z_1 z_2)$, is an $\mathbb{F}_Q$-valued nondegenerate bilinear form on $\mathbb{F}_{Q^J}$. Furthermore, let

$$\hat{\mathrm{B}}(z_1, z_2) := \mathrm{tr}_{Q \setminus P}(\mathrm{B}(z_1, z_2) = \mathrm{tr}_{Q^J \setminus P}(z_1 z_2).$$

By Lemma 4.3.1, $\hat{\mathrm{B}}$ is an irreducible $\mathbb{F}_P$-valued nondegenerate bilinear form on $\mathbb{F}_{Q^J}$, and for any z_1, z_2 in $\mathbb{F}_{Q^J}$, the mapping $z_2 \mapsto (z_1 \mapsto \exp(\frac{2\pi i}{P} \mathrm{tr}_{Q^J \setminus P}(z_1 z_2)))$ is an isomorphism from $\mathbb{F}_{Q^J}$ onto its dual. As shown in Example 4.1.2, for r_1, r_2 in $\mathbb{Z}_R$, the mapping $r_2 \mapsto (r_1 \mapsto \exp(\frac{2\pi i}{R} r_1 r_2))$ is an isomorphism from $\mathbb{Z}_R$ onto its dual as well.

Now let $\mathcal{G} = \mathbb{Z}_R \times \mathbb{F}_{Q^J}$, then

$$\widehat{\mathcal{G}} = (\mathbb{Z}_R \times \mathbb{F}_{Q^J})\widehat{\ } \simeq \widehat{\mathbb{Z}}_R \times \widehat{\mathbb{F}}_{Q^J} \simeq \mathbb{Z}_R \times \mathbb{F}_{Q^J}.$$

For all $(r_1, z_1), (r_2, z_2)$ in $\mathcal{G}$, the mapping

$$(r_2, z_2) \mapsto \big((r_1, z_1) \mapsto \exp(\tfrac{2\pi i}{R}) \exp(\tfrac{2\pi i}{P} \mathrm{tr}_{Q^J \setminus P}(z_1 z_2))\big)$$

is an isomorphism from $\mathcal{G}$ onto its dual. Let $\mathcal{H} = \{0\} \times \mathbb{F}_{Q^J}$, then under our identification of $\mathcal{G}$ with $\widehat{\mathcal{G}}$,

$$\begin{aligned}\mathcal{H}^{\perp} &= \{(r_2, z_2) \in \mathcal{G} : \exp(\tfrac{2\pi i}{R} r_1 r_2) \exp(\tfrac{2\pi i}{P} \operatorname{tr}_{Q^J \backslash P}(z_1 z_2)) = 1, \forall\, (r_1, z_1) \in \{0\} \times \mathbb{F}_{Q^J}\} \\ &= \{(r_2, z_2) \in \mathcal{G} : \exp(\tfrac{2\pi i}{P} \operatorname{tr}_{Q^J \backslash P}(z_1 z_2)) = 1, \forall\, (r_1, z_1) \in \mathbb{F}_{Q^J}\} \\ &= \mathbb{Z}_R \times \mathbb{F}_{Q^J}^{\perp},\end{aligned}$$

where $\mathbb{F}_{Q^J}^{\perp}$ is the orthogonal complement of $\mathbb{F}_{Q^J}$ as a subspace of itself. Therefore $\mathcal{H}^{\perp} = \mathbb{Z}_R \times \{0\}$, and all cosets of $\mathcal{H}^{\perp}$ are of the form $\mathbb{Z}_R \times \{z_2\}$ for z_2 in $\mathbb{F}_{Q^J}$.

Choose α such that $\mathbb{F}_{Q^J}^{\times} = \langle \alpha \rangle$, and α^R is a member of $\mathbb{F}_Q^{\times}$. We can then enumerate the Q^{J-1}-dimensional hyperplanes according to the orthogonal complements of powers of $\alpha : \{\langle \alpha^r \rangle^{\perp}\}_{r \in \mathbb{Z}_R}$. Construct a subset $\mathcal{D}$ of $\mathcal{G}$ as follows:

$$\mathcal{D} := \sqcup_{r \in \mathbb{Z}_R} \{r\} \times \langle \alpha^r \rangle^{\perp} = \sqcup_{r \in \mathbb{Z}_R} \{r\} \times \{z \in \mathbb{F}_{Q^J} : \operatorname{tr}_{Q^J \backslash Q}(\alpha^r z) = 0\}.$$

We now claim that $\mathcal{D}$ is then a semiregular DDS. To see this, let $\mathbf{\Gamma}_{\mathcal{G}}$ be the DFT for $\mathcal{G}$, then

$$\mathbf{\Gamma}_{\mathcal{G}} \boldsymbol{\chi}_{\mathcal{D}} = \mathbf{\Gamma}_{\mathcal{G}} \sum_{r \in \mathbb{Z}_R} \boldsymbol{\chi}_{\{r\} \times \langle \alpha^r \rangle^{\perp}} = \sum_{r \in \mathbb{Z}_R} \mathbf{\Gamma}_{\mathcal{G}} (\boldsymbol{\delta}_r \otimes \boldsymbol{\chi}_{\langle \alpha^r \rangle^{\perp}}) = \sum_{r \in \mathbb{Z}_R} (\mathbf{\Gamma}_{\mathbb{Z}_R}^* \boldsymbol{\delta}_r) \otimes (\mathbf{\Gamma}_{\mathbb{F}_{Q^J}}^* \boldsymbol{\chi}_{\langle \alpha^r \rangle^{\perp}}).$$

Because $\langle \alpha^r \rangle^{\perp}$ is a subspace of $\mathbb{F}_{Q^J}$, applying the Poisson summation formula (as expressed in Lemma 4.3.1) to $\mathbf{\Gamma}_{\mathbb{F}_{Q^J}}^* \boldsymbol{\chi}_{\langle \alpha^r \rangle^{\perp}} = (Q^{J-1}) \boldsymbol{\chi}_{\langle \alpha^r \rangle}$. We see then that

$$\mathbf{\Gamma}_{\mathcal{G}} \boldsymbol{\chi}_{\mathcal{D}} = \sum_{r \in \mathbb{Z}_R} (\mathbf{\Gamma} *_{\mathbb{Z}_R} \boldsymbol{\delta}_r) \otimes (\mathbf{\Gamma}_{\mathbb{F}_{Q^J}}^* \boldsymbol{\chi}_{\langle \alpha^r \rangle^{\perp}}) = Q^{J-1} \sum_{r \in \mathbb{Z}_R} (\mathbf{\Gamma}_{\mathbb{Z}_R}^* \delta_r) \otimes \boldsymbol{\chi}_{\langle \alpha^r \rangle}.$$

When considered pointwise, we find that

$$(\mathbf{\Gamma}_{\mathcal{G}}^* \boldsymbol{\chi}_{\mathcal{D}})(r_2, z_2) = Q^{J-1} \sum_{r \in \mathbb{Z}_R} \big(\mathbf{\Gamma}_{zr}^* \boldsymbol{\delta}_r\big)(r_2) \boldsymbol{\chi}_{\langle \alpha^r \rangle}(z_2) = Q^{J-1} \sum_{r \in \mathbb{Z}_R} \exp(\tfrac{2\pi i}{R} r_2 r) \boldsymbol{\chi}_{\langle \alpha^r \rangle}(z_2).$$

From this, we can see that when $(r_2, z_2) = (0, 0)$,

$$\left|\big(\mathbf{\Gamma}_{\mathcal{G}}\boldsymbol{\chi}_{\mathcal{D}}\big)(0,0)\right|^2 = \left|Q^{J-1}\sum_{r\in\mathbb{Z}_R}\exp(\tfrac{2\pi i}{R}0r)\boldsymbol{\chi}_{\langle\alpha^r\rangle}(0)\right|^2 = Q^{2(J-1)}\left|\sum_{r\in\mathbb{Z}_R}1\right|^2 = RQ^{2(J-1)}.$$

In the case where $z_2 = 0$ but $r_2 \neq 0$, we see that

$$\left|\big(\mathbf{\Gamma}_{\mathcal{G}}\boldsymbol{\chi}_{\mathcal{D}}\big)(r_2,0)\right|^2 = \left|Q^{J-1}\sum_{r\in\mathbb{Z}_R}\exp(\tfrac{2\pi i}{R}r_2 r)\boldsymbol{\chi}_{\langle\alpha^r\rangle}(0)\right|^2 = Q^{2(J-1)}\left|\sum_{r'\in\mathbb{Z}_R}\exp(\tfrac{2\pi i}{R}r')\right|^2 = 0.$$

Finally, in the case where $z_2 \neq 0$, let r_0 be the unique r in $\mathbb{Z}_R$ such that z_2 is in $\langle\alpha^{r_0}\rangle$. Therefore $\boldsymbol{\chi}_{\langle\alpha^r\rangle}(z_2) = 0$ for all r in $\mathbb{Z}_R$ where $r \neq r_0$. In this case,

$$\left|\big(\mathbf{\Gamma}_{\mathcal{G}}\boldsymbol{\chi}_{\mathcal{D}}\big)(r_2,z_2)\right|^2 = Q^{2(J-1)}\left|\exp(\tfrac{2\pi i}{R}r_1 r_0)\right|^2 = Q^{2(J-1)}.$$

Since $\mathcal{H}^\perp = \mathbb{Z}_R \times \{0\}$, we see that the last case is equivalent to z_2 not in $\mathcal{H}^\perp$, satisfying Lemma 2.7.5. The set $\mathcal{D}$ is therefore an $\mathcal{H}$-DDS$(R, Q^J, RQ^J - 1)$, and applying Theorem 4.2.1 will result in an ECTFF(RQ^{J-1}, Q^J, R). □

Note, the SR-DDS construction in Theorem 4.3.2 is similar to that in Theorem 2.3.6 of [36]; however, it is included here to verify the construction of [32] in the context of this research. This McFarland SR-DDS is explored further in the next chapter and the frame that arises is further explored in Theorem 5.5.1.

V. Biangular orthopartitionable tight frames

5.1 Fundamental facts

As previously mentioned, the EC/EITFFs in Sections 4.2 and 4.3 all result from harmonic frames with the trait that the inner product of any two frame vectors from distinct subspaces have constant modulus. We refer to such frames as BOPTFs in Chapter I. As we explain below, BOPTFs have low coherence in general, and some of them – those that generate EITFFs – also have minimal block coherence. In this chapter, we introduce two BOPTF constructions, one that yields EITFFs and another that generates non-equi-isoclinic ECTFFs. Additionally, we show just how special these former BOPTFs are, comparing their performance in compressed sensing applications to the latter. First, we formalize the relationships and traits of BOPTFs that we provided in Chapter I.

Theorem 5.1.1. *Let $\mathcal{N},\mathcal{R}$ be arbitrary indexing sets of size N and R, respectively. Let $\{\boldsymbol{\varphi}_{n,r}\}_{n\in\mathcal{N},r\in\mathcal{R}}$ be a* BOPTF(D,N,R) *for a D-dimensional Hilbert space $\mathbb{H}$, namely that it satisfies* (3)*. Then*

(a) $\{\boldsymbol{\varphi}_{n,r}\}_{n\in\mathcal{N},r\in\mathcal{R}}$ *is a* UNTF(D,NR) *for $\mathbb{H}$ with coherence* $\sqrt{\frac{NR-D}{DR(N-1)}}$.

(b) $\{\mathcal{U}_n\}_{n\in\mathcal{N}}$, $\mathcal{U}_n := \operatorname{span}\{\boldsymbol{\varphi}_{n,r}\}_{r\in\mathcal{R}}$ *is an* ECTFF(D,N,R) *for $\mathbb{H}$.*

(c) If $D < NR$ and $\{\boldsymbol{\psi}_{n,r}\}_{n\in\mathcal{N},r\in\mathcal{R}}$ is any Naimark complement of $\{\boldsymbol{\varphi}_{n,r}\}_{(n,r)\in\mathcal{N}\times\mathcal{R}}$, then $\{\boldsymbol{\psi}_{n,r}\}_{n\in\mathcal{N},r\in\mathcal{R}}$ is a BOPTF$(NR-D,N,R)$ *for its span.*

Proof. Let $\boldsymbol{\Phi}$ be the synthesis operator for $\{\boldsymbol{\varphi}_{n,r}\}_{n\in\mathcal{N},r\in\mathcal{R}}$. In this case we have that

$\mathbf{\Phi}\left((n,r),d\right) = \mathbf{\Phi}_n(r,d)$. For (a), note that

$$\begin{aligned}\tfrac{1}{N(N-1)}\sum_{n_1\in\mathcal{N}}\sum_{\substack{n_2\in\mathcal{N}\\ n_1'\neq n_2}}\|\mathbf{\Phi}_n^*\mathbf{\Phi}_{n'}\|_{\mathrm{Fro}}^2 &= \tfrac{1}{N(N-1)}\sum_{n_1\in\mathcal{N}}\sum_{\substack{n_2\in\mathcal{N}\\ n_1'\neq n_2}}\sum_{r_1\in\mathcal{R}}\sum_{r_2\in\mathcal{R}}|\langle\boldsymbol{\varphi}_{n_1,r_1},\boldsymbol{\varphi}_{n_2,r_2}\rangle|^2\\ &= \tfrac{1}{N(N-1)}\sum_{n_1\in\mathcal{N}}\sum_{\substack{n_2\in\mathcal{N}\\ n_1'\neq n_2}}R^2\left(\tfrac{NR-D}{DR(N-1)}\right)\\ &= \tfrac{R(NR-D)}{D(N-1)},\end{aligned}$$

which satisfies (11), therefore $\{\boldsymbol{\varphi}_{n,r}\}_{n\in\mathcal{N},r\in\mathcal{R}}$ is a UNTF(D,NR). From (3), for each n in $\mathcal{N}$, $\{\boldsymbol{\varphi}_{n,r}\}_{r\in\mathcal{R}}$ is an ONB for $\mathcal{U}_n := \operatorname{span}\{\boldsymbol{\varphi}_{n,r}\}_{r\in\mathcal{R}}$. Then $\{\mathcal{U}_n\}_{n\in\mathcal{N}}$ is a TFF(D,N,R) for $\mathbb{H}$. To see the coherence, note that when $(n_1,r_1)\neq(n_2,r_2)$, $\left|\langle\boldsymbol{\varphi}_{n_1,r_1},\boldsymbol{\varphi}_{n_2,r_2}\rangle\right|^2 \in \left\{0,\tfrac{NR-D}{DR(N-1)}\right\}$. Without loss of generality, we can assume that $NR-D\geq 0$. It follows that $\max\limits_{(n_1,r_1)\neq(n_2,r_2)}\left|\langle\boldsymbol{\varphi}_{n_1,r_1},\boldsymbol{\varphi}_{n_2,r_2}\rangle\right| = \sqrt{\tfrac{NR-D}{DR(N-1)}}$.

For (b), note that for all n_1,n_2 in $\mathcal{N}$ with $n_1\neq n_2$,

$$\|\mathbf{\Phi}_{n_1}^*\mathbf{\Phi}_{n_2}\|_{\mathrm{Fro}}^2 = \sum_{r_1\in\mathcal{R}}\sum_{r_2\in\mathcal{R}}|\langle\boldsymbol{\varphi}_{n_1,r_1},\boldsymbol{\varphi}_{n_2,r_2}\rangle|^2 = \tfrac{R(NR-D)}{D(N-1)},$$

therefore $\max\limits_{n_1\neq n_2}\|\mathbf{\Phi}_{n_1}^*\mathbf{\Phi}_{n_2}\|_{\mathrm{Fro}}^2 = \tfrac{R(NR-D)}{D(N-1)}$. Therefore $\{\mathcal{U}_n\}_{n\in\mathcal{N}}$ is an ECTFF(D,N,R) by the first bound in (12).

For (c), let $\mathbf{\Psi}$ be the synthesis operator for $\{\boldsymbol{\psi}_{n,r}\}_{n\in\mathcal{N},r\in\mathcal{R}}$, and recall from Chapter II that $\mathbf{\Psi}^*\mathbf{\Psi} = \mathbf{I}-\tfrac{D}{NR}\mathbf{\Phi}^*\mathbf{\Phi}$. Since $\mathbf{\Phi}_n^*\mathbf{\Phi}_n = \mathbf{I}$ for all n in $\mathcal{N}$, $\mathbf{\Psi}_n^*\mathbf{\Psi}_n$ simplifies to $\mathbf{I}-\tfrac{D}{NR}\mathbf{I} = \tfrac{NR-D}{NR}\mathbf{I}$. We want the bases $\{\boldsymbol{\psi}_{n,r}\}_{r\in\mathcal{R}}$ to be orthonormal for each n, so we will scale $\boldsymbol{\Psi}$ by $\sqrt{\frac{NR}{NR-D}}$. Then

$$\begin{aligned}
\left|\left(\boldsymbol{\Psi}_{n_1}^*\boldsymbol{\Psi}_{n_2}\right)(r_1,r_2)\right|^2 &= \left|\tfrac{NR}{NR-D}\left(\mathbf{I}-\tfrac{D}{NR}\boldsymbol{\Phi}_{n_1}^*\boldsymbol{\Phi}_{n_2}\right)(r_1,r_2)\right|^2 \\
&= \begin{cases} \left(\frac{NR}{NR-D}\right)^2\left(1-\frac{D}{NR}\right)^2, & n_1=n_2, r_1=r_2, \\ 0, & n_1=n_2, r_1\neq r_2, \\ \left(\frac{NR}{NR-D}\right)^2\left(\frac{D}{NR}\right)^2\left(\frac{NR-D}{DR(N-1)}\right), & n_1\neq n_2, \end{cases} \\
&= \begin{cases} 1, & n_1=n_2, r_1=r_2, \\ 0, & n_1=n_2, r_1\neq r_2, \\ \frac{D}{R(NR-D)(N-1)}, & n_1\neq n_2, \end{cases}
\end{aligned}$$

which satisfies (3) for a BOPTF$(NR-D, N, R)$. □

Notice that if $R=1$, the coherence of a BOPTF$(D,N,1) = \sqrt{\frac{N-D}{D(N-1)}}$, i.e. any ETF$(D,N)$ is necessarily a BOPTF$(D,N,1)$. This result is not unexpected, however it helps us to demonstrate how ETFs can give rise to other BOPTFs in the next section.

5.2 Equiangular tight frames and mutually unbiased bases

Recall from Theorem 4.2.3 that the SR-DDS was constructed via the Gordon-Mills-Welch [26] sum of a DS, $\mathcal{A}$, and an SR-RDS, $\mathcal{B}$ [28]. From Theorem 4.1.1, we know that the harmonic frame arising from $\mathcal{A}$ would be an ETF. Recall that the columns of a harmonic frame arising from $\mathcal{B}$ can be arranged into a sequence of MUBs [25]. Let $\{\boldsymbol{\varphi}_n\}_{n\in\mathcal{N}}$ be the harmonic ETF arising from $\mathcal{A}$, and $\boldsymbol{\Psi}$ be the synthesis operator for one of the bases arising from $\mathcal{B}$. Then $\left[\boldsymbol{\varphi}_1\otimes\boldsymbol{\Psi}\ \boldsymbol{\varphi}_2\otimes\boldsymbol{\Psi}\ \cdots\ \boldsymbol{\varphi}_n\otimes\boldsymbol{\Psi}\right]$ is the synthesis operator for a UNTF, the vectors of which can be arranged into a sequence of orthonormal bases for an EITFF [24]. We now generalize these two constructions.

Theorem 5.2.1. *If $\{\boldsymbol{\varphi}_n\}_{n\in\mathcal{N}}$ is an* ETF(D_1,N) *for some D_1-dimensional Hilbert space $\mathbb{H}$ and $\{\boldsymbol{\Psi}_n\}_{n\in\mathcal{N}}$ is an* MUB(D_2,N) *for some D_2-dimensional Hilbert space $\mathbb{J}$, then for each n in $\mathcal{N}$, $\boldsymbol{\Phi}_n := \{\boldsymbol{\varphi}_n \otimes \boldsymbol{\Psi}_n\}$ is an ONB for a D_2-dimensional subspace of some D_1D_2-dimensional Hilbert Space $\mathbb{K}$, and $\{\boldsymbol{\Phi}_n\}_{n\in\mathcal{N}}$ is a* BOPTF(D_1D_2,N,D_2) *for $\mathbb{K}$ that generates an* EITFF(D_1D_2,N,D_2) *for $\mathbb{K}$. If $\{\boldsymbol{\varphi}_n\}_{n\in\mathcal{N}}$ and $\{\boldsymbol{\Psi}_n\}_{n\in\mathcal{N}}$ are real, then $\{\boldsymbol{\Phi}_n\}_{n\in\mathcal{N}}$ will be real as well.*

Proof. For ETF(D_1,N) $\{\boldsymbol{\varphi}_n\}_{n\in\mathcal{N}}$, the coherence $\mu = \sqrt{\frac{N-D_1}{D_1(N-1)}}$. For $\{\boldsymbol{\Psi}_n\}_{n\in\mathcal{N}}$, recall that $|\boldsymbol{\Psi}_{n_1}^*\boldsymbol{\Psi}_{n_2}| = \sqrt{\frac{1}{D_2}}\mathbf{J}_{D_2}$ for any n_1, n_2 in $\mathcal{N}$ with $n_1 \neq n_2$. Let $\boldsymbol{\Phi}_n = \boldsymbol{\varphi}_n \otimes \boldsymbol{\Psi}_n$ for all n in $\mathcal{N}$. Let $\boldsymbol{\Phi} = \begin{bmatrix}\boldsymbol{\Phi}_1 & \boldsymbol{\Phi}_2 & \dots & \boldsymbol{\Phi}_n\end{bmatrix}$, then

$$\begin{aligned}
\boldsymbol{\Phi}\boldsymbol{\Phi}^* &= \sum_{n\in\mathcal{N}} \boldsymbol{\Phi}_n\boldsymbol{\Phi}_n^* \\
&= \sum_{n\in\mathcal{N}} \left(\boldsymbol{\varphi}_n \otimes \boldsymbol{\Psi}_n\right)\left(\boldsymbol{\varphi}_n \otimes \boldsymbol{\Psi}_n\right)^* \\
&= \sum_{n\in\mathcal{N}} (\boldsymbol{\varphi}_n\boldsymbol{\varphi}_n^*) \otimes \left(\boldsymbol{\Psi}_n\boldsymbol{\Psi}_n^*\right) \\
&= A\mathbf{I} \otimes \mathbf{I},
\end{aligned}$$

showing that $\{\boldsymbol{\Phi}_n\}_{n\in\mathcal{N}}$ is a TFF. Now consider,

$$\boldsymbol{\Phi}_n^*\boldsymbol{\Phi}_n = \left(\boldsymbol{\varphi}_n \otimes \boldsymbol{\Psi}_n\right)^*\left(\boldsymbol{\varphi}_n \otimes \boldsymbol{\Psi}_n\right) = (\boldsymbol{\varphi}_n^*\boldsymbol{\varphi}_n) \otimes \left(\boldsymbol{\Psi}_n^*\boldsymbol{\Psi}_n\right) = \|\boldsymbol{\varphi}_n\|^2\mathbf{I} = \mathbf{I}.$$

This means that every diagonal block of $\boldsymbol{\Phi}^*\boldsymbol{\Phi}$ is an identity matrix of size D_1D_2. Similarly, when $n_1 \neq n_2$, the entrywise modulus $\left|\boldsymbol{\Phi}_{n_1}^*\boldsymbol{\Phi}_{n_2}\right|$ of the cross-Gram matrix

$\mathbf{\Phi}_{n_1}^*\mathbf{\Phi}_{n_2}$ is

$$\begin{aligned}\left|\mathbf{\Phi}_{n_1}^*\mathbf{\Phi}_{n_2}\right| &= \left|\left(\boldsymbol{\varphi}_{n_1}\otimes\mathbf{\Psi}_{n_1}\right)^*\left(\boldsymbol{\varphi}_{n_2}\otimes\mathbf{\Psi}_{n_2}\right)\right| \\ &= \left|(\boldsymbol{\varphi}_{n_1}^*\boldsymbol{\varphi}_{n_2})\right|\otimes\left|\left(\mathbf{\Psi}_{n_1}^*\mathbf{\Psi}_{n_2}\right)\right| \\ &= \sqrt{\tfrac{1}{D_2}}|\langle\boldsymbol{\varphi}_{n_1},\boldsymbol{\varphi}_{n_2}\rangle|\mathbf{J} \\ &= \sqrt{\tfrac{N-D_1}{D_1D_2(N-D_1)}}\mathbf{J}.\end{aligned}$$

Therefore all entries of the cross-Gram matrices $\mathbf{\Phi}_{n_1}^*\mathbf{\Phi}_{n_2}$ satisfy (3), and $\{\mathbf{\Phi}_n\}_{n\in\mathcal{N}}$ is a BOPTF(D_1D_2, N, D_2). For equi-isoclinicity, note that

$$\begin{aligned}\|\mathbf{\Phi}_{n_1}^*\mathbf{\Phi}_{n_2}\|_2 &= \|\left(\boldsymbol{\varphi}_{n_1}\otimes\mathbf{\Psi}_{n_1}\right)^*\left(\boldsymbol{\varphi}_{n_2}\otimes\mathbf{\Psi}_{n_2}\right)\|_2 \\ &= \|(\boldsymbol{\varphi}_{n_1}^*\boldsymbol{\varphi}_{n_2})\otimes\left(\mathbf{\Psi}_{n_1}^*\mathbf{\Psi}_{n_2}\right)\|_2 \\ &= \sqrt{\tfrac{N-D_1}{D_1(N-1)}}\|\tfrac{1}{D_2}\mathbf{J}_{D_2}\|_2 \\ &= \sqrt{\tfrac{N-D_1}{D_1(N-1)}}.\end{aligned}$$

So $\{\mathbf{\Phi}_n\}_{n\in\mathcal{N}}$ generates an EITFF(D_1D_2, N, D_2). Because $\mathbf{\Phi}_n = \boldsymbol{\varphi}_n\otimes\mathbf{\Psi}_n$, if $\{\boldsymbol{\varphi}_n\}_{n\in\mathcal{N}}$ and $\{\mathbf{\Psi}_n\}_{n\in\mathcal{N}}$ are all real, $\{\mathbf{\Phi}_n\}_{n\in\mathcal{N}}$ will be as well. □

While the EITFF-generating BOPTFs from Theorem 4.2.3 were known [32], the BOPTFs from Theorem 5.2.1 do not necessarily arise from divisible difference sets. In fact, for any ETF(D_1, N), we can use the construction in Theorem 5.2.1 with any MUB(D_2, M) as long as $M \geq N$ as any N MUBs can be used in the construction. In the next section, we will show just how special these BOPTFs are when compared to those that yield to non-equi-isoclinic ECTFFs.

5.3 Compressed sensing performance

As we have previously discussed, the BOPTF vectors from Theorem 5.2.1 can be arranged into ONBs for subspaces that are EITFFs for their span. As such, these frames have optimal block coherence with respect to compressed sensing applications such as BOMP. Recall that the coherence of any BOPTF is on the order of the Welch bound, making them well suited (if not optimal) for compressed sensing applications like OMP. Theorem 5.3.1 shows just how useful these EITFF-generating BOPTFs are with respect to signal recovery.

Theorem 5.3.1. *Let* $\{\boldsymbol{\varphi}_{n,r}\}_{n\in\mathcal{N},r\in\mathcal{R}}$ *be a* BOPTF(D,N,R) *for* $\mathbb{F}^D$ *whose subspaces* $\{\mathcal{U}_n\}_{n\in\mathcal{N}}$ *are equi-isoclinic. For any* $\mathbf{x}\in\mathbb{F}^{\mathcal{N}\times\mathcal{R}}$*, then the following are true:*

(a) Using $\{\boldsymbol{\varphi}_{n,r}\}_{n\in\mathcal{N},r\in\mathcal{R}}$ *in the BOMP algorithm guarantees the recovery of the signal* $\mathbf{x}$ *that is more complex than using OMP by a factor of* $\sqrt{R}$*, if the sparsity patterned is blocky.*

(b) If the sparsity pattern of $\mathbf{x}$ *is not blocky, using* $\{\boldsymbol{\varphi}_{n,r}\}_{n\in\mathcal{N},r\in\mathcal{R}}$ *guarantees the recovery of any signal* $\mathbf{x}$ *that is more complex than one guaranteed by an arbitrary EITFF with the same parameters by a factor of* $\sqrt{R}$*.*

Proof. Recall that the coherence μ of any BOPTF(D,N,R) is

$$\mu=\sqrt{\tfrac{NR-D}{DR(N-1)}}=\tfrac{1}{\sqrt{R}}\sqrt{\tfrac{NR-D}{D(N-1)}},$$

where $\sqrt{\tfrac{NR-D}{D(N-1)}}=\upsilon$, the block coherence of an EITFF(D,N,R). For (a), we note that any (K,R)-block sparse $\mathbf{x}\in\mathbb{F}^{\mathcal{N}\times\mathcal{R}}$ is (at worst) KR-sparse. Using $\{\boldsymbol{\varphi}_{n,r}\}_{n\in\mathcal{N},r\in\mathcal{R}}$, we can recover [39] a KR-sparse $\mathbf{x}$ as long as

$$KR<\tfrac{1}{2}\left(\sqrt{\tfrac{DR(N-1)}{NR-D}}+1\right),\quad\text{i.e.,}\quad K<\tfrac{1}{2}\left(\tfrac{1}{\sqrt{R}}\sqrt{\tfrac{D(N-1)}{NR-D}}+\tfrac{1}{R}\right).$$

However, because $\{\boldsymbol{\varphi}_{n,r}\}_{n\in\mathcal{N},r\in\mathcal{R}}$ generates an EITFF(D,N,R) with block coherence $\upsilon = \sqrt{\frac{NR-D}{D(N-1)}}$, if we know that $\mathbf{x}$ is (K,R)-block sparse, [9] guarantees recovery as long as

$$K < \tfrac{1}{2}\left(\tfrac{1}{\upsilon}+1\right) = \tfrac{1}{2}\left(\sqrt{\tfrac{D(N-1)}{NR-D}}+1\right),$$

allowing K to be approximately $\sqrt{R}$ larger under this assumption.

For (b), note that if we cannot assume a block-sparse pattern, any EITFF(D,N,R) can recover [9] a K-sparse $\mathbf{x}\in\mathbb{F}^{\mathcal{N}\times\mathcal{R}}$ provided

$$K < \tfrac{1}{2}\left(\sqrt{\tfrac{D(N-1)}{NR-D}}+1\right). \tag{37}$$

In this case, we consider a K-sparse $\mathbf{x}$ as (K',R)-block sparse where $K'\le K$. For the EITFF(D,N,R) arising from $\{\boldsymbol{\varphi}_{n,r}\}_{n\in\mathcal{N},r\in\mathcal{R}}$, however, we can recover [39] a K-sparse $\mathbf{x}$ whenever

$$K < \tfrac{1}{2}\left(\sqrt{\tfrac{DR(N-1)}{NR-D}}+1\right) = \tfrac{1}{2}\left(\sqrt{R}\sqrt{\tfrac{D(N-1)}{NR-D}}+1\right),$$

which is again, approximately $\sqrt{R}$ larger than (37). □

Next, we show that not only are BOPTFs low coherence, they achieve optimal coherence among orthobiangular tight frames.

Theorem 5.3.2. *Let $\mathcal{N},\mathcal{R}$ be arbitrary indexing sets of size N and R, respectively. Let $\{\boldsymbol{\varphi}_{n,r}\}_{n\in\mathcal{N},r\in\mathcal{R}}$ be a unit-norm frame over a D-dimensional Hilbert space, $\mathbb{H}$. If every $\boldsymbol{\varphi}_{n,r}$ is orthonormal to exactly $R-1$ over vectors in $\{\boldsymbol{\varphi}_{n,r}\}_{n\in\mathcal{N},r\in\mathcal{R}}$, then*

$$\sqrt{\tfrac{NR-D}{DR(N-1)}} \le \max_{\substack{(n_1,r_1),\in(\mathcal{N}\times\mathcal{R})\\(n_1,r_1)\neq(n_2,r_2)}} |\langle\boldsymbol{\varphi}_{n_1,r_1},\boldsymbol{\varphi}_{n_2,r_2}\rangle|. \tag{38}$$

Proof. Let $\mathbf{\Phi}$ be the $D \times (\mathcal{N} \times \mathcal{R})$ synthesis operator for $\{\boldsymbol{\varphi}_{n,r}\}_{n\in\mathcal{N}, r\in\mathcal{R}}$. Note that

$$\begin{aligned}
0 &\leq \operatorname{Tr}\left[\left(\mathbf{\Phi}\mathbf{\Phi}^* - \tfrac{NR}{D}\mathbf{I}\right)^2\right] \\
&= \operatorname{Tr}\left[(\mathbf{\Phi}\mathbf{\Phi}^*)^2\right] - \tfrac{2NR}{D}\operatorname{Tr}(\mathbf{\Phi}\mathbf{\Phi}^*) + \left(\tfrac{NR}{D}\right)^2 \operatorname{Tr}(\mathbf{I}) \\
&= \operatorname{Tr}\left[(\mathbf{\Phi}^*\mathbf{\Phi})^2\right] - \tfrac{2NR}{D}\operatorname{Tr}(\mathbf{\Phi}^*\mathbf{\Phi}) + \tfrac{(NR)^2}{D} \\
&= \|\mathbf{\Phi}^*\mathbf{\Phi}\|_{\text{Fro}}^2 - \tfrac{(NR)^2}{D} \\
&= \sum_{n_1\in\mathcal{N}} \sum_{n_2\in\mathcal{N}} \|\mathbf{\Phi}_{n_1}^*\mathbf{\Phi}_{n_2}\|_{\text{Fro}}^2 - \tfrac{(NR)^2}{D} \\
&= \sum_{n_1\in\mathcal{N}} \sum_{\substack{n_2\in\mathcal{N} \\ n_1\neq n_2}} \sum_{r_1\in\mathcal{R}} \sum_{r_2\in\mathcal{R}} |\langle \boldsymbol{\varphi}_{n_1,r_1}, \boldsymbol{\varphi}_{n_2,r_2}\rangle|^2 + NR - \tfrac{(NR)^2}{D} \\
&= \sum_{n_1\in\mathcal{N}} \sum_{\substack{n_2\in\mathcal{N} \\ n_1\neq n_2}} \sum_{r_1\in\mathcal{R}} \sum_{r_2\in\mathcal{R}} |\langle \boldsymbol{\varphi}_{n_1,r_1}, \boldsymbol{\varphi}_{n_2,r_2}\rangle|^2 - NR\left(\tfrac{NR}{D} - 1\right).
\end{aligned}$$

Subtracting the constant from both sides gives

$$\begin{aligned}
\tfrac{NR(NR-D)}{D} &\leq \sum_{n_1\in\mathcal{N}} \sum_{\substack{n_2\in\mathcal{N} \\ n_1\neq n_2}} \sum_{r_1\in\mathcal{R}} \sum_{r_2\in\mathcal{R}} |\langle \boldsymbol{\varphi}_{n_1,r_1}, \boldsymbol{\varphi}_{n_2,r_2}\rangle|^2 \\
&\leq N(N-1)R^2 \max_{\substack{n_1\neq n_2\in\mathcal{N} \\ r_1,r_2\in\mathcal{R}}} |\langle \boldsymbol{\varphi}_{n_1,r_1}, \boldsymbol{\varphi}_{n_2,r_2}\rangle|^2.
\end{aligned}$$

Finally, we solve for the coherence, μ:

$$\sqrt{\tfrac{NR-D}{DR(N-1)}} \leq \max_{\substack{n_1\neq n_2\in\mathcal{N} \\ r_1,r_2\in\mathcal{R}}} |\langle \boldsymbol{\varphi}_{n_1,r_1}, \boldsymbol{\varphi}_{n_2,r_2}\rangle| = \max_{(n_1,r_1)\neq(n_2,r_2)\in(\mathcal{N}\times\mathcal{R})} |\langle \boldsymbol{\varphi}_{n_1,r_1}, \boldsymbol{\varphi}_{n_2,r_2}\rangle| = \mu,$$

revealing the bound as stated. □

By combining Theorems 5.1.1 and 5.3.2, we see then that any BOPTF(D, N, R) achieves the minimum possible coherence (38) for such a unit-norm frame. In light of this, one might think that any BOPTF would also be well suited for block coherence-based algorithms, but as we will show in the following sections, this is not the case.

5.4 Steiner biangular orthopartitionable tight frames

In this section, we generalize the construction of Steiner ETFs from [23] to generate Steiner BOPTFs from GDDs. We will show that the ECTFFs that arise from Steiner BOPTFs can never be equi-isoclinic, and have "optimally" terrible block coherence. We begin by establishing the notation for the Steiner method for ETF construction.

Let $\mathbf{X} \in \{0,1\}^{\mathcal{B}\times\mathcal{V}}$ be the incidence matrix for a BIBD$(V,K,1)$ with vector set $\mathcal{V}$ and block set $\mathcal{B}$. Let $\#(\mathcal{B}) = B = \frac{VR}{K}$ where $R = \frac{V-1}{K-1}$. Let $\mathcal{R}$ be any indexing set of cardinality R, then for any v in $\mathcal{V}$, the corresponding *embedding operator* $\mathbf{E}_v$ is the $\mathcal{B}\times\mathcal{R}$ matrix whose columns are standard basis vectors that sum to the vth column of $\mathbf{X}$. Note that $\mathbf{E}_v^*\mathbf{E}_v = \mathbf{I}_{\mathcal{R}}$ and

$$(\mathbf{E}_v\mathbf{E}_v^*)\,(b_1,b_2) = \begin{cases} 1, \ b_1 = b_2 \text{ and } v \in b_1, \\ 0, \text{ else.} \end{cases}$$

Then $\sum_{v\in\mathcal{V}} \mathbf{E}_v\mathbf{E}_v^* = K\mathbf{I}_{\mathcal{B}}$ where K is the number of vertices in the bth block.

Let $\{\boldsymbol{\psi}_t\}_{t\in\mathcal{T}}$ be any flat ETF$(R,R{+}1)$ for $\mathbb{F}^{\mathcal{R}}$, then the corresponding Steiner ETF is $\{\mathbf{E}_v\boldsymbol{\psi}_t\}_{v\in\mathcal{V},t\in\mathcal{T}}$. Without loss of generality, we assume $\|\boldsymbol{\psi}_t\| = 1$ and $|\boldsymbol{\psi}_t(r)| = \frac{1}{\sqrt{R}}$ for any $t \in \mathcal{T}$ and $r \in \mathcal{R}$. To see tightness, consider

$$\begin{aligned}\sum_{v\in\mathcal{V}}\sum_{t\in\mathcal{T}}(\mathbf{E}_v\boldsymbol{\psi}_t)(\mathbf{E}_v\boldsymbol{\psi}_t)^* &= \sum_{v\in\mathcal{V}}\mathbf{E}_v\big(\sum_{t\in\mathcal{T}}\boldsymbol{\psi}_t\boldsymbol{\psi}_t^*\big)\mathbf{E}_v^* \\ &= \big(\tfrac{R+1}{R}\mathbf{I}_{\mathcal{R}}\big)\sum_{v\in\mathcal{V}}\mathbf{E}_v\mathbf{E}_v^* \\ &= \big(\tfrac{R+1}{R}\big)K\mathbf{I}_{\mathcal{B}} \\ &= \tfrac{V(R+1)}{B}\mathbf{I}_{\mathcal{B}}.\end{aligned}$$

To see equiangularity, consider $v_1, v_2 \in \mathcal{V}$ and $t_1, t_2 \in \mathcal{T}$. Notice that $\left|\langle \mathbf{E}_{v_1}\boldsymbol{\psi}t_1, \mathbf{E}_{v_2}\boldsymbol{\psi}_{t_2}\rangle\right| = \left|\langle \boldsymbol{\psi}t_1, \mathbf{E}_{v_1}^*\mathbf{E}_{v_2}\boldsymbol{\psi}_{t_2}\rangle\right|$. Then for $v_1 = v_2$,

$$\left|\langle \boldsymbol{\psi}_{t_1}, \mathbf{E}_{v_1}^*\mathbf{E}_{v_2}\boldsymbol{\psi}_{t_2}\rangle\right| = \left|\langle \boldsymbol{\psi}_{t_1}, \boldsymbol{\psi}_{t_2}\rangle\right| = \begin{cases} 1,\ t_1 = t_2, \\ \frac{1}{R}\ t_1 \neq t_2. \end{cases}$$

When $v_1 \neq v_2$,

$$\left|\langle \mathbf{E}_{v_1}\boldsymbol{\psi}_{t_1}, \mathbf{E}_{v_2}\boldsymbol{\psi}_{t_2}\rangle\right| = \left|\langle \boldsymbol{\delta}_{r_1}(v_1, v_2)^*\boldsymbol{\psi}_{t_1}, \boldsymbol{\delta}_{r_2}(v_1, v_2)^*\boldsymbol{\psi}_{t_2}\rangle\right| = \left|\overline{\boldsymbol{\psi}_{t_1}(r_1)}\boldsymbol{\psi}_{t_2}(r_2)\right| = \tfrac{1}{R}.$$

In the following theorems, we generalize this Steiner embedding technique to construct Steiner BOPTFs.

Theorem 5.4.1. *Let* $\{\mathbf{E}_v\}_{v\in\mathcal{V}}$ *be the embedding operators for a* BIBD$(V, K, 1)$*. If* $\{\boldsymbol{\psi}_r\}_{r\in\mathcal{R}}$ *is a flat ONB (a "normalized" Hadamard matrix), then* $\{\mathbf{E}_v\boldsymbol{\psi}_r\}_{v\in\mathcal{V}, r\in\mathcal{R}}$ *is a BOPTF for its span.*

Proof. First, note that it is tight:

$$\sum_{v\in\mathcal{V}}\sum_{r\in\mathcal{R}}\big(\mathbf{E}_v\boldsymbol{\psi}_r\big)\big(\mathbf{E}_v\boldsymbol{\psi}_r\big)^* = \sum_{v\in\mathcal{V}}\mathbf{E}_v\Big(\sum_{r\in\mathcal{R}}\boldsymbol{\psi}_r\boldsymbol{\psi}_r^*\Big)\mathbf{E}_v^* = \sum_{v\in\mathcal{V}}\mathbf{E}_v\mathbf{E}_v^* = K\mathbf{I}.$$

Second, note that for $v_1, v_2 \in \mathcal{V}$ and $r_1, r_2 \in \mathcal{R}$, when $v_1 \neq v_2$

$$\left|\langle \mathbf{E}_{v_1}\boldsymbol{\psi}_{r_1}, \mathbf{E}_{v_2}\boldsymbol{\psi}_{r_2}\rangle\right| = \left|\langle \boldsymbol{\delta}_{v_1}\boldsymbol{\psi}_{r_1}, \boldsymbol{\delta}_{v_2}\boldsymbol{\psi}_{r_2}\rangle\right| = \left|\overline{\boldsymbol{\psi}_{v_1}(r_1)}\boldsymbol{\psi}_{v_2}(r_2)\right| = \tfrac{1}{R}. \qquad (39)$$

When $v_1 = v_2$,

$$\left|\langle \mathbf{E}_{v_1}\boldsymbol{\psi}_{r_1}, \mathbf{E}_{v_2}\boldsymbol{\psi}_{r_2}\rangle\right| = \left|\langle \boldsymbol{\psi}_{r_1}, \mathbf{E}_{v_1}^*\mathbf{E}_{v_2}\boldsymbol{\psi}_{r_2}\rangle\right| = \left|\langle \boldsymbol{\psi}_{r_1}, \boldsymbol{\psi}_{r_2}\rangle\right| = \begin{cases} 1,\ r_1 = r_2, \\ 0,\ r_1 \neq r_2. \end{cases}$$

This yields an ECTFF(B,V,R) that arises from a BOPTF(B,V,R). □

These BOPTFs are not new. In fact, the construction from Theorem 5.4.1 is a generalization of those generated in [43], which have been shown to be made up of smaller simplices [14]. While Zauner's construction utilizes this Steiner method with a BIBD(V,K,Λ), we found that we can generalize this method by applying the Steiner method with a GDD$(V,K,1)$ and a Hadamard, thus generating a new infinite family of BOPTFs.

Theorem 5.4.2. *Let* $\{\mathbf{E}_{u,m}\}_{u\in\mathcal{U},m\in\mathcal{M}}$ *be the embedding operators for a* GDD$(V,K,1)$ *of type* M^U*, and* $\{\boldsymbol{\psi}_r\}_{r\in\mathcal{R}}$ *is a flat ONB (a "normalized" Hadamard matrix), then* $\{\mathbf{E}_{u,m}\boldsymbol{\psi}_r\}_{u\in\mathcal{U},m\in\mathcal{M},r\in\mathcal{R}}$ *is a BOPTF.*

Proof. Tightness is achieved as with Theorem 5.4.1:

$$\sum_{\substack{u\in\mathcal{U},\\ m\in\mathcal{M}}}\sum_{r\in\mathcal{R}}(\mathbf{E}_{u,m}\boldsymbol{\psi}_r)(\mathbf{E}_{u,m}\boldsymbol{\psi}_r)^* = \sum_{\substack{u\in\mathcal{U}\\ m\in\mathcal{M}}}\mathbf{E}_{u,m}\Big(\sum_{r\in\mathcal{R}}\boldsymbol{\psi}_r\boldsymbol{\psi}_r^*\Big)\mathbf{E}_{u,m}^* = \sum_{\substack{u\in\mathcal{U}\\ m\in\mathcal{M}}}\mathbf{E}_{u,m}\mathbf{E}_{u,m}^* = K\mathbf{I}.$$

Recall that $\mathbf{E}_{u,m}^*\mathbf{E}_{u,m} = \mathbf{I}_{\mathcal{R}}$, and $\mathbf{E}_{u_1,m_1}^*\mathbf{E}_{u_2,m_2} = \mathbf{0}$ if $u_1 = u_2$, but $m_1 \neq m_2$. Combining this with (39) we see that

$$\begin{aligned}\big|\langle\mathbf{E}_{u_1,m_1}\boldsymbol{\psi}_{r_1},\mathbf{E}_{u_2,m_2}\boldsymbol{\psi}_{r_2}\rangle\big| &= \big|\langle\boldsymbol{\psi}_{r_1},\mathbf{E}_{u_1,m_1}^*\mathbf{E}_{u_2,m_2}\boldsymbol{\psi}_{r_2}\rangle\big| \\ &= \begin{cases}1,\ u_1=u_2, m_1=m_2, r_1=r_2,\\ 0,\ u_1=u_2, m_1\neq m_2,\ \text{or}\ u_1=u_2, r_1\neq r_2,\\ \frac{1}{R}\ u_1\neq u_2.\end{cases}\end{aligned}$$

This gives a BOPTF (B,U,MR) that is also an ECTFF(B,U,MR), specifically one that does not come from the construction of [43]. □

For each of the previous cases, we claim that the subspaces that arise from these Steiner BOPTFs are equichordal (via the nature of BOPTFs) but cannot be equi-isoclinic. To see this, let $\{\mathbf{E}_v\}_{v\in\mathcal{V}}$ be the embedding operators for a BIBD(V, K, Λ). Furthermore, for each v, let $\mathbf{e}_{v,r}(b) := \mathbf{E}_v(b, r)$ and $\mathcal{U}_v := \operatorname{span}\{\mathbf{e}_{v,r}\}_{r\in\mathcal{R}}$. Then because $\sum_{v\in\mathcal{V}} \mathbf{E}_v\mathbf{E}_v^* = K\mathbf{I}$, $\{\mathcal{U}_v\}_v \in \mathcal{V}$ is a TFF for $\mathbb{R}^B$. Next consider the vertices $v_1, v_2 \in \mathcal{V}$, $v_1 \neq v_2$. In this case, $\mathbf{E}_{v_1}^*\mathbf{E}_{v_2}$ is a $\{0, 1\}$-valued $\mathcal{R} \times \mathcal{R}$ matrix with exactly Λ entries with value 1. This means that $\|\mathbf{E}_{v_1}^*\mathbf{E}_{v_2}\|_{\text{Fro}} = \sqrt{\Lambda}$ with each column and row of $\mathbf{E}_{v_1}^*\mathbf{E}_{v_2}$ containing at most a single 1. Then $\left(\mathbf{E}_{v_1}^*\mathbf{E}_{v_2}\right)^*\left(\mathbf{E}_{v_1}^*\mathbf{E}_{v_2}\right)$ is a diagonal matrix with Λ ones on its diagonal, meaning that it has Λ principal angles of 0 and $R-\Lambda$ principal angles of $\frac{\pi}{2}$. Therefore, for any BIBD(V, K, Λ) we obtain an ECTFF(B, V, R) (this is the same method used in [43]). This means that any Steiner BOPTF will have the worst possible block coherence due to the overlap between the subspaces that arise from it. To see how this impacts the performance of such a BOPTF in compressed sensing, consider a BOPTF(B, U, MR_G) from Theorem 5.4.2. Note that this BOPTF has coherence

$$\mu = \sqrt{\tfrac{UMR_G - B}{BMR_G(U-1)}} = \sqrt{\tfrac{BK-B}{BMR_G(U-1)}} = \sqrt{\tfrac{K-1}{MR_G(U-1)}} = \sqrt{\tfrac{1}{R_G^2}} = \tfrac{1}{R_G}.$$

This means that a BOPTF(B, U, MR_G) can recover a K-sparse $\mathbf{x} \in \mathbb{F}^{U\times B}$ provided that $K < \frac{1}{2}\left(\frac{1}{\mu} + 1\right) = \frac{R_G+1}{2}$ [39]. However, under the assumption that the signal has a block structure, a (K, R)-block sparse signal would be (at worst) KR-sparse. This BOPTF could recover a KR-sparse $\mathbf{x} \in \mathbb{F}^{U\times B}$ if $KMR_G = KR < \frac{R_G+1}{2}$, meaning that $K < \frac{1}{2}\left(\frac{R_G+1}{R_G}\right)\frac{1}{M} < 1$, therefore K cannot be an integer, and this BOPTF cannot guarantee recovery of a (K, R)-block sparse signal. We emphasize the poor performance of Steiner BOPTFs because, as we demonstrate in the next section, some methods for constructing BOPTFs result in frames that are unitarily equivalent to a Steiner frame.

5.5 Steiner-McFarland equivalence

Recall that the harmonic BOPTF arising from the McFarland DDS from Theorem 4.3.2 generated a ECTFF for its span. Though it is not clear from its construction, this BOPTF also arises from the Steiner method, meaning that it will have similarly poor performance with respect to block coherence. Before we prove this equivalence, we introduce one final classification of GDDs. A uniform GDD$(V, K, 1)$ of type $\mathcal{M}^{\mathcal{U}}$ is *resolvable* if the set of blocks $\mathcal{B}$ partitions into further partitions of $\mathcal{V}$, that is if $\mathcal{B}$ is a disjoint union of *parallel classes* of $\mathcal{V}$. Since each block consists of K vertices, each parallel class consists of $\frac{V}{K} = \frac{MU}{K}$ blocks and there are $\frac{BK}{V} = \frac{VR}{V} = R$ parallel classes. We write $\mathcal{V} = \mathcal{U} \times \mathcal{M}$ and $\mathcal{B} = \mathcal{R} \times \mathcal{S}$ where $\mathcal{R}$ and $\mathcal{S}$ are indexing sets of size R and $S = \frac{V}{K}$, respectively. Let $\{\mathbf{E}_{u,m}\}_{u\in\mathcal{U},m\in\mathcal{M}}$ be the embedding operators of the GDD as before. Let $\{\boldsymbol{\varphi}_r\}_{r\in\mathcal{R}}$ be a flat ONB of dimension R (such as a scaled Hadamard matrix).

Let $\mathbf{\Phi}$ and $\mathbf{\Psi}$ be the synthesis operators for $\{\boldsymbol{\varphi}_r\}_{r\in\mathcal{R}}$ and $\{\mathbf{E}_{u,m}\boldsymbol{\varphi}_r\}_{u\in\mathcal{U},m\in\mathcal{M},r\in\mathcal{R}}$, respectively. Recall that

$$\mathbf{E}_v(b_1, r_2) = \mathbf{E}_v\big((r_1, s_1), r_2\big) = \begin{cases} 1, \ r_1 = r_2 \text{ and } v \in (r_1, s_1), \\ 0, \text{ else.} \end{cases}$$

Note that $\mathbf{\Psi}\big(b_1, (u_2, m_2, r_2)\big) = \mathbf{\Psi}\big((r_1, s_1), (u_2, m_2, r_2)\big) = \big(\mathbf{E}_{u_2,m_2}\boldsymbol{\varphi}_{r_2}\big)(r_1, s_1)$, meaning that each row is indexed according to the specific element (s_1) of the parallel class (r_1) and each column is indexed according to the associated member (m_2) of which grouping (u_2) and the appropriate member of the ONB (r_2). Then

$$\big(\mathbf{E}_{u_2,m_2}\boldsymbol{\varphi}_{r_2}\big)(r_1, s_1) = \sum_{r\in\mathcal{R}} \mathbf{E}_{u_2,m_2}\big((r_1, s_1), r\big)\boldsymbol{\varphi}_{r_2}(r) = \mathbf{E}_{u_2,m_2}\big((r_1, s_1), r_1\big)\boldsymbol{\varphi}_{r_2}(r_1).$$

Since this depends on the vector (u_2, m_2) being in the block indexed by (r_1, s_1) (which we will call b_{r_1,s_1}), we can write this as

$$\mathbf{E}_{u_2,m_2}\big((r_1,s_1),r_1\big)\boldsymbol{\varphi}_{r_2}(r_1) = \begin{cases} \boldsymbol{\Phi}(r_1,r_2), \ (u_2,m_2) \in b_{r_1,s_1}, \\ \quad 0, \qquad \text{else}, \end{cases}$$
$$= \mathbf{X}\big((r_1,s_1),(u_2,m_2)\big)\boldsymbol{\Phi}(r_1,r_2).$$

We can then consider this to be the $\big(s_1,(u_2,m_2,r_2)\big)$ entry of the tensor product of the incidence matrix of the r_1th parallel class of the GDD and the r_1th row of $\boldsymbol{\Phi}$. This categorization of the frame entries is necessary in the following theorem to show how the Steiner construction relates to the McFarland construction.

Theorem 5.5.1. *Let $\mathcal{D} = \{(r,z) \in \mathbb{Z}_R \times \mathbb{F}_{Q^J} : z \in \langle\alpha^r\rangle^\perp\}$ be the semiregular $\mathcal{H}$-DDS(R,Q^J,RQ^{J-1}) for $\mathcal{G} = \mathbb{Z}_R \times \mathbb{F}_{Q^J}$ from Theorem 4.3.2. Applying Theorem 4.2.3 yields a BOPTF(RQ^{J-1},Q^J,R) that is unitarily equivalent to a Steiner BOPTF that arises by applying Theorem 5.4.1 to a classical affine geometry over $\mathbb{F}_{Q^J}$.*

Proof. Let Q, J, R, $\mathcal{G}$, $\mathcal{H}$, and $\mathcal{D}$ be the same as those in Theorem 4.3.2. Next we construct the affine geometry. Let $\mathcal{V} = \mathbb{F}_{Q^J}$, and take β in $\mathbb{F}_{Q^J}$ such that $\mathrm{tr}_{Q^J\backslash Q}(\beta) = 1$. We define the block set $\mathcal{B} = \{(r,y) \in \mathbb{Z}_R \times \mathbb{F}_{Q^J} : y \in \langle\beta\alpha^{-r}\rangle^\perp\}$. Then for any (r,y) in $\mathcal{B}$ and z in $\mathbb{F}_{Q^J}$, we can construct a $\mathcal{B} \times \mathcal{V}$ incidence matrix $\mathbf{X}$ as

$$\mathbf{X}\big((r,y),z\big) := \begin{cases} 1, \ z \in y + \langle\alpha^r\rangle, \\ 0, \ \text{else}. \end{cases}$$

For any fixed r, the $(r,0)$ block is $\langle\alpha^r\rangle$, the rth the one-dimensional subspace of $\mathbb{F}_{Q^J}$. Varying y over $\langle\beta\alpha^{-r}\rangle^\perp$ partitions $\mathbb{F}_{Q^J}$ into a parallel class. To see this, note that for any z in $\mathbb{F}_Q$, we can write

$$z = \mathrm{tr}_{Q^J\backslash Q}(\beta\alpha^{-r}z)\alpha^r + [z - \mathrm{tr}_{Q^J\backslash Q}(\beta\alpha^{-r}z)\alpha^r],$$

where $\mathrm{tr}_{Q^J\backslash Q}(\beta\alpha^{-r}z)\alpha^r$ is clearly in $\langle\alpha^r\rangle$ and $[z-\mathrm{tr}_{Q^J\backslash Q}(\beta\alpha^{-r}z)\alpha^r]$ is in $\langle\beta\alpha^{-r}\rangle^\perp$:

$$\begin{aligned}\mathrm{tr}_{Q^J\backslash Q}\big(\beta\alpha^{-r}(z-\mathrm{tr}_{Q^J\backslash Q}(\beta\alpha^{-r}z)\alpha^r)\big) &= \mathrm{tr}_{Q^J\backslash Q}(\beta\alpha^{-r}z)-\mathrm{tr}_{Q^J\backslash Q}(\beta\alpha^{-r}z)\,\mathrm{tr}_{Q^J\backslash Q}(\beta)\\ &= \mathrm{tr}_{Q^J\backslash Q}(\beta\alpha^{-r}z)-\mathrm{tr}_{Q^J\backslash Q}(\beta\alpha^{-r}z)\\ &= 0.\end{aligned}$$

This means that z is in the coset $[z-\mathrm{tr}_{Q^J\backslash Q}(\beta\alpha^{-r}z)\alpha^r]+\langle\alpha^r\rangle$, and if z were also in the coset $y+\langle\alpha^r\rangle$ then

$$z-\mathrm{tr}_{Q^J\backslash Q}(\beta\alpha^{-r}z)\alpha^r+\langle\alpha^r\rangle = y+\langle\alpha^r\rangle \iff y-[z-\mathrm{tr}_{Q^J\backslash Q}(\beta\alpha^{-r}z)\alpha^r]\in\langle\alpha^r\rangle.$$

Additionally, both $[z-\mathrm{tr}_{Q^J\backslash Q}(\beta\alpha^{-r}z)\alpha^r]$ and y are in $\langle\beta\alpha^{-r}\rangle^\perp$, which would mean that they are both in $\langle\alpha^r\rangle\cap\langle\beta\alpha^{-r}\rangle^\perp$. However, if z' is in $\langle\alpha^r\rangle\cap\langle\beta\alpha^{-r}\rangle^\perp$ then $z'=c\alpha^r$ for some c in $\mathbb{F}_Q$ and

$$0=\mathrm{tr}_{Q^J\backslash Q}(\beta\alpha^{-r}z')=\mathrm{tr}_{Q^J\backslash Q}(\beta\alpha^{-r}c\alpha^r)=c\,\mathrm{tr}_{Q^J\backslash Q}(\beta)=c.$$

This occurs if and only if $y=z-\mathrm{tr}_{Q^J\backslash Q}(\beta\alpha^{-r}z)\alpha^r$, meaning that $\langle\beta\alpha^{-r}\rangle^\perp$ is a natural set of coset representatives (i.e. a transversal) of $\mathbb{F}_{Q^J}$ with respect to the subgroup $\langle\alpha^r\rangle$. If we let $\mathbf{X}_r$ be the $\langle\beta\alpha^{-r}\rangle^\perp\times\mathbb{F}_{Q^J}$ submatrix of $\mathbf{X}$ (representing the rth parallel class), then

$$\mathbf{X}_r(y,z)=\begin{cases}1,\ y=z-\mathrm{tr}_{Q^J\backslash Q}(\beta\alpha^{-r}z)\alpha^r,\\ 0,\ \text{else}.\end{cases}$$

The mapping $z\mapsto z-\mathrm{tr}_{Q^J\backslash Q}(\beta\alpha^{-r}z)\alpha^r$ is a linear mapping from $\mathbb{F}_{Q^J}$ onto $\langle\beta\alpha^{-r}\rangle^\perp$ with rank $J-1=\dim\big(\langle\beta\alpha^{-r}\rangle^\perp\big)$. This implies that the nullity of the mapping is 1, and for any y in $\langle\beta\alpha^{-r}\rangle^\perp$ there are exactly Q choices of z in $\mathbb{F}_{Q^J}$ such that

$y = z - \mathrm{tr}_{Q^J \backslash Q}(\beta\alpha^{-r}z)\alpha^r$, but for each z in $\mathbb{F}_{Q^J}$ this y is unique. This means that each row of $\mathbf{X}_r$ has exactly Q ones and that each column of $\mathbf{X}$ has a single one in each parallel class.

For any $z_1 \neq z_2$ in $\mathbb{F}_{Q^J}$, we know that $z_1 - z_2 \neq 0$, so there exists a unique r in $\mathbb{Z}_R$ such that $z_1 - z_2$ is in $\langle\alpha^r\rangle$. Then for this r, there exists a c in $\mathbb{F}_Q$ such that $z_1 - z_2 = c\alpha^r$, and

$$\mathrm{tr}_{Q^J \backslash Q}(\beta\alpha^{-r}(z_1 - z_2))\alpha^r = \mathrm{tr}_{Q^J \backslash Q}(\beta\alpha^{-r}(c\alpha^r))\alpha^r = c\,\mathrm{tr}_{Q^J \backslash Q}(\beta)\alpha^r = c\alpha^r = z_1 - z_2.$$

We know that there is a unique y in $\langle\beta\alpha^{-r}\rangle^\perp$ such that $y = z_1 - \mathrm{tr}_{Q^J \backslash Q}(\beta\alpha^{-r}z_1)\alpha^r$, and combining this with the previous result shows that

$$\begin{aligned} y &= [z_1 - \mathrm{tr}_{Q^J \backslash Q}(\beta\alpha^{-r}z_1)\alpha^r] + [(z_1 - z_2) - \mathrm{tr}_{Q^J \backslash Q}(\beta\alpha^{-r}(z_1 - z_2))\alpha^r] \\ &= z_2 - \mathrm{tr}_{Q^J \backslash Q}(\beta\alpha^{-r}z_2)\alpha^r. \end{aligned}$$

This means that there is both a unique r in $\mathbb{Z}_R$ and unique y in $\langle\beta\alpha^{-r}\rangle^\perp$ such that $y = z_1 - \mathrm{tr}_{Q^J \backslash Q}(\beta\alpha^{-r}z_1)\alpha^r$ and $y = z_2 - \mathrm{tr}_{Q^J \backslash Q}(\beta\alpha^{-r}z_2)\alpha^r$. Then the affine geometry is also a *resolvable* BIBD (RBIBD), denoted RBIBD$(Q^J, Q, 1)$.

To construct the Steiner BOPTF, note that a RBIBD(V, K, Λ) has the following relations: $R = \frac{\Lambda(V-1)}{K-1} = \frac{Q^J-1}{Q-1}$ and $B = \#(\mathcal{B}) = \frac{VR}{K} = \frac{Q^J R}{Q} = Q^{J-1}R$, which fit our assumed values. For any z in $\mathbb{F}_{Q^J}$, an embedding operator is a $\mathcal{B} \times \mathbb{Z}_R$ $\{0,1\}$-valued matrix $\mathbf{E}_z$ with entries

$$\begin{aligned} \mathbf{E}_z\big((r_2, y_2), r_3\big) &= \begin{cases} 1, \ r_2 = r_3, z \in y_2 + \langle\alpha^{r_2}\rangle, \\ 0, \ \text{else}, \end{cases} \\ &= \begin{cases} 1, \ r_2 = r_3, y_2 = z - \mathrm{tr}_{Q^J \backslash Q}(\beta\alpha^{-r_2})\alpha^{r_2}, \\ 0, \ \text{else}. \end{cases} \end{aligned}$$

Additionally, the $\mathbb{Z}_R \times \mathbb{Z}_R$ character table of $\mathbf{\Gamma}_{\mathbb{Z}_R}$ of $\mathbb{Z}_R$ has entries

$$\mathbf{\Gamma}_{\mathbb{Z}_R}(r_2, r_3) = \exp(\tfrac{2\pi i}{R} r_2 r_3).$$

We can then construct the synthesis operator $\mathbf{\Psi}$ as a $\mathcal{B} \times (\mathbb{Z}_R \times \mathbb{F}_{Q^J})$ matrix with entries $\mathbf{\Psi}\big((r_2, y_2), (r_3, z_3)\big) = (\mathbf{E}_{z_3}\mathbf{\Gamma}_{\mathbb{Z}_R})\big((r_2, y_2), r_3\big)$. This simplifies to

$$\sum_{r \in \mathbb{Z}_R} \mathbf{E}_{z_3}\big((r_2, y_2), r\big)\mathbf{\Gamma}_{\mathbb{Z}_R}(r, r_3) = \sum_{r \in \mathbb{Z}_R} \exp(\tfrac{2\pi i}{R} r r_3) \begin{cases} 1, \ r_2 = r, \\ \quad y_2 = z_3 - \operatorname{tr}_{Q^J \backslash Q}(\beta\alpha^{-r_2} z_3)\alpha^{r_2}, \\ 0, \text{ else.} \end{cases}$$

This simplifies even further as the summand values are 0 when $r \neq r_2$:

$$\mathbf{\Psi}\big((r_2, y_2), (r_3, z_3)\big) = \begin{cases} \exp(\tfrac{2\pi i}{R} r_2 r_3), & y_2 = z_3 - \operatorname{tr}_{Q^J \backslash Q}(\beta\alpha^{-r_2} z_3)\alpha^{r_2}, \\ 0, & \text{else.} \end{cases}$$

Next, define $\mathbf{H}$ as the $\mathcal{D} \times \mathcal{B}$ matrix (where $\mathcal{D}$ is the subset of $\mathcal{G}$ used to generate the harmonic BOPTF $\mathbf{\Phi}$ from Theorem 4.3.2) with entries

$$\mathbf{H}\big((r_1, x_1), (r_2, y_2)\big) = \begin{cases} \exp(\tfrac{2\pi i}{P} \operatorname{tr}_{Q^J \backslash P}(x_1 y_2)), & r_1 = r_2, \\ 0, & \text{else.} \end{cases}$$

Notice that $\mathbf{H}$ is a block diagonal matrix. Let $\mathbf{H}_r$ be the rth diagonal block submatrix, indexed by $\langle \alpha^r \rangle^\perp$ (from $\mathcal{D}$) on the rows and $\langle \beta\alpha^{-r} \rangle^\perp$ (from $\mathcal{B}$) on the columns. $\mathbf{H}_r$ is in fact a complex Hadamard matrix, specifically the character table for $\langle \alpha^r \rangle^\perp$ (regarded as a subgroup of $\mathbb{F}_{Q^J}$) with entries $\mathbf{H}_r(x_1, y_2) = \exp(\frac{2\pi i}{P} \operatorname{tr}_{Q^J \backslash P}(x_1 y_2))$. To see this, note that for any $y_2 \in \langle \beta\alpha^{-r} \rangle^\perp$, the mapping $\gamma_{y_2} : \langle \alpha^r \rangle^\perp \to \mathbb{T}$ where $\gamma_{y_2}(x_1) = \exp(\frac{2\pi i}{P} \operatorname{tr}_{Q^J \backslash P}(x_1 y_2))$ is well-defined and a homomorphism: note that for any x_0, x_1 in $\langle \alpha^r \rangle^\perp$,

$$\gamma_{y_2}(x_0 + x_1) = \exp(\tfrac{2\pi i}{P} \operatorname{tr}_{Q^J \setminus P}((x_0 + x_1)y_2)) = \gamma_{y_2}(x_0)\gamma_{y_2}(x_1).$$

The mapping $\boldsymbol{\varphi} : \langle \beta\alpha^{-r} \rangle^\perp \to (\langle \alpha^r \rangle^\perp)\widehat{}$ is also a homomorphism: for any $y_2, y_3 \in \langle \beta\alpha^{-r} \rangle^\perp$ and any x_1 in $\langle \alpha^r \rangle^\perp$,

$$\begin{aligned}
[\boldsymbol{\varphi}(y_2 + y_3)](x_1) &= \gamma_{y_2+y_3}(x_3) \\
&= \exp(\tfrac{2\pi i}{P} \operatorname{tr}_{Q^J \setminus P}(x_1(y_2 + y_3))) \\
&= \gamma_{y_2}(x_1)\gamma_{y_3}(x_1) \\
&= [\boldsymbol{\varphi}(y_2)\boldsymbol{\varphi}(y_3)](x_1).
\end{aligned}$$

Because $\langle \beta\alpha^{-r} \rangle^\perp$ and $\langle \alpha^r \rangle^\perp$ have the same cardinality, it is sufficient to show that $\ker(\boldsymbol{\varphi}) = \{0\}$. If y_2 is in $\ker(\boldsymbol{\varphi})$, then $\gamma_{y_2} = \boldsymbol{\varphi}(y_2) = \mathbf{1}$, therefore

$$1 = \gamma_{y_2}(x_1) = \exp(\tfrac{2\pi i}{P} \operatorname{tr}_{Q^J \setminus P}(x_1 y_2)),$$

for all x_1 in $\langle \alpha^r \rangle^\perp$. This implies that $\operatorname{tr}_{Q^J \setminus P}(x_1 y_2) = 0$, which implies that $y_2 = 0$ since $\operatorname{tr}_{Q^J \setminus P}$ is nondegenerate.

Therefore, since $\mathbf{H}_r$ is a scalar of a unitary matrix, and $\mathbf{H}$ is block diagonal, we see that $\mathbf{H}^{-1} = \frac{1}{Q^{J-1}}\mathbf{H}^*$. Now consider the $\mathcal{D} \times (\mathbb{Z}_R \times \mathbb{F}_{Q^J})$ matrix $\mathbf{H\Psi}$, with entries

$$\begin{aligned}
&(\mathbf{H\Psi})\big((r_1,x_1),(r_3,z_3)\big)\\
&\quad= \sum_{(r_2,y_2)\in\mathcal{B}} \mathbf{H}\big((r_1,x_1),(r_2,y_2)\big)\mathbf{\Psi}\big((r_2,y_2),(r_3,z_3)\big)\\
&\quad= \sum_{(r_2,y_2)\in\mathcal{B}} \begin{Bmatrix} \exp(\frac{2\pi i}{P}\operatorname{tr}_{Q^J\backslash P}(x_1y_2)), & r_1 = r_2,\\ 0, & \text{else,}\end{Bmatrix} \mathbf{\Psi}\big((r_2,y_2),(r_3,z_3)\big)\\
&\quad= \sum_{(r_2,y_2)\in\mathcal{B}} \exp(\tfrac{2\pi i}{P}\operatorname{tr}_{Q^J\backslash P}(x_1y_2)) \begin{cases} \exp(\frac{2\pi i}{R}r_2r_3), & \begin{matrix} r_1 = r_2,\\ y_2 = z_3 - \operatorname{tr}_{Q^J\backslash Q}(\beta\alpha^{-r_2}z_3)\alpha^{r_2},\end{matrix}\\ 0, & \text{else,}\end{cases}\\
&\quad= \sum_{y_2\in\langle\beta\alpha^{-r_1}\rangle^\perp} \begin{cases} \exp(\frac{2\pi i}{P}\operatorname{tr}_{Q^J\backslash P}(x_1x_2))\exp(\frac{2\pi i}{R}r_1r_3), & y_2 = z_3 - \operatorname{tr}_{Q^J\backslash Q}(\beta\alpha^{-r_1}z_3)\alpha^{r_1},\\ 0, & \text{else,}\end{cases}\\
&\quad= \exp(\tfrac{2\pi i}{P}\operatorname{tr}_{Q^J\backslash P}(x_1(z_3 - \operatorname{tr}_{Q^J\backslash Q}(\beta\alpha^{-r_1}z_3)\alpha^{r_1}))\exp(\tfrac{2\pi i}{R}r_1r_3),\\
&\quad= \exp(\tfrac{2\pi i}{P}\operatorname{tr}_{Q^J\backslash P}(x_1z_3))\exp(\tfrac{2\pi i}{R}r_1r_3),\\
&\quad= \mathbf{\Phi}\big((r_1,x_1),(r_3,z_3)\big). \qquad \square
\end{aligned}$$

VI. Conclusions and future work

Though ETFs and EITFFs are optimally suited for techniques such as OMP and BOMP, respectively, we have shown that objects like EITFF-generating BOPTFs may be better suited for real-world applications of compressed sensing. This is because they provide better flexibility with respect to the assumptions on the signal being recovered. In addition to constructing new infinite families of EITFF-generating BOPTFs, we have also shown that not all BOPTFs are as useful as these with respect to block coherence.

Along with our focus on BOPTFs, we have also introduced new methods for constructing infinite families of ECTFF from difference families and GDDs. Figure 3 summarizes some relevant known methods for constructing ECTFFs and EITFFs prior to our work, while Figure 4 expands this outline, highlighting our contributions in red and providing the associated theorems. Beyond the construction of new frames and fusion frames, we provide a method for validating the novelty of these new frames in general.

As we have shown, BOPTFs, particularly those with equi-isoclinic subspaces, have great potential for compressed sensing, and new methods for their construction are certainly worth pursuing. Inspired by Theorem 5.5.1, we suggest that it is possible

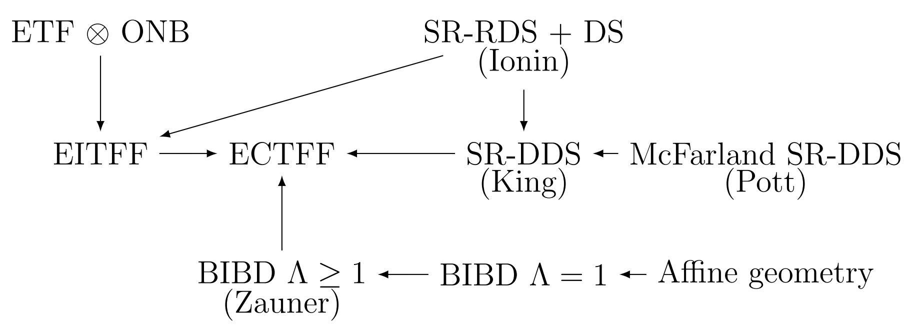

Figure 3. A summary of relevant EI/ECTFF construction methods, prior to our work.

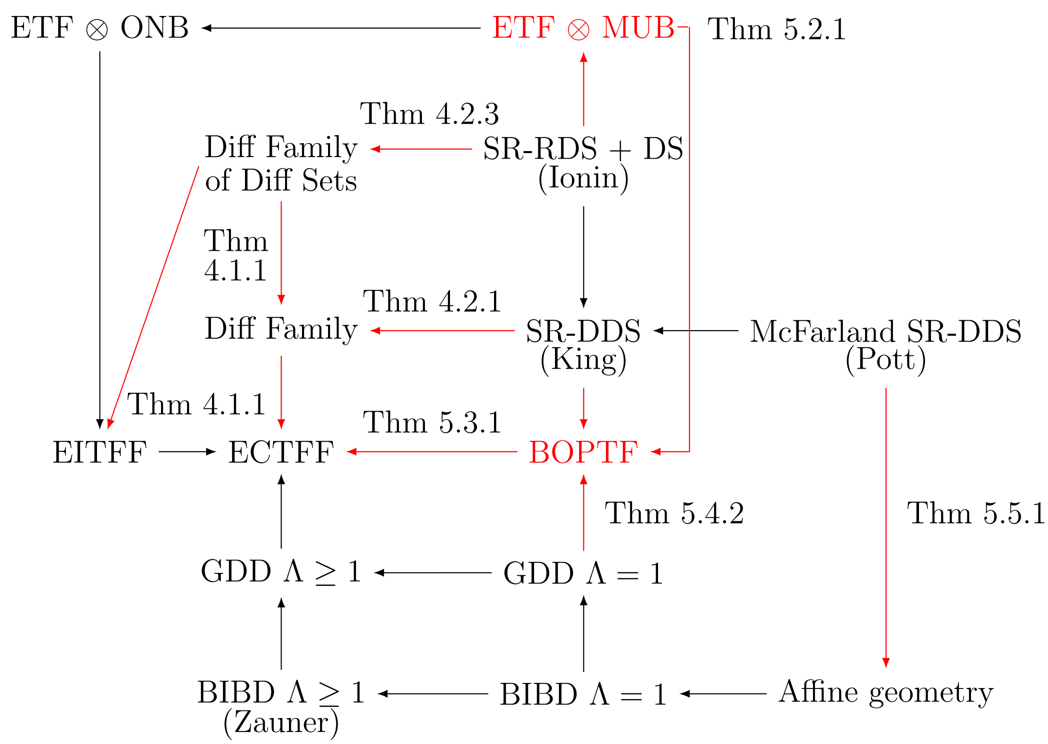

Figure 4. EI/ECTFF construction methods with our contributions highlighted in red.

to find new EITFF-generating BOPTF constructions by investigating which EITFF constructions can arise from a BOPTF. Here, the challenge is to find bases for a given sequence of subspaces that have small coherence. Perhaps a good first step is to develop a numerical algorithm for computing such bases and apply it to small known examples of EITFFs, and then analyze the results to see if any new patterns emerge. It would also be wise to carefully search for the conditions necessary for an EI/ECTFF to have arisen from a BOPTF.

It would also be prudent to pursue yet more methods of constructing BOPTFs, and then determine if any of them are capable of producing equi-isoclinic subspaces. Here, one good first step would be to determine what other methods for constructing ETFs can be generalized to yield BOPTFs: we have already generalized Steiner ETFs to non-EITFF-generating BOPTFs (Theorem 5.4.1) and generalized the "ETF-tensor-

MUETF" construction of [19] to ones that generate EITFFs. In fact, investigating more connections between ETFs and BOPTFs could also lead to new ETFs.

To help inform the search for new BOPTFs, it would help to find more necessary conditions on their existence. The number of vectors in a BOPTF should be restricted by an analogue of Gerzon's bound. The cross-Gram matrices of a real BOPTF are necessarily scalar multiples of Hadamard matrices of size R, implying that R is either 2 or a multiple of 4. Also, every real BOPTF is a real OBTF, meaning its parameters must satisfy certain integrality conditions [15]. Restrictions on the existence of complex BOPTFs beyond Gerzon's will likely be too difficult to find in the near future due to the fact that the set of all complex numbers of a given modulus is a continuum.

Bibliography

1. H. Cohn, A. Kumar, G. Minton, Optimal simplices and codes in projective spaces, Geom. Topol. 20 (2016) 1289–1357.

2. C. J. Colbourn, CRC Handbook of Combinatorial Designs. CRC press, (2007).

3. J. H. Conway, R. H. Hardin, N. J. A. Sloane, Packing lines, planes, etc.: packings in Grassmannian spaces, Exp. Math. 5 (1996) 139–159.

4. G. Coutinho, C. Godsil, H. Shirazi, H. Zhan, Equiangular lines and covers of the complete graph, Linear Algebra Appl. 488 (2016) 264–283.

5. I. S. Dhillon, J. R. Heath, T. Strohmer, J. A. Tropp, Constructing packings in Grassmannian manifolds via alternating projection, Exp. Math. 17 (2008) 9–35.

6. C. Ding, T. Feng, A generic construction of complex codebooks meeting the Welch bound, IEEE Trans. Inform. Theory 53 (2007) 4245–4250.

7. D. L. Donoho, Compressed sensing, IEEE Trans. Inform. Theory 52.4 (2006) 1289-1306.

8. D. L. Donoho, M. Elad, Optimally sparse representation in general (nonorthogonal) dictionaries via l1 minimization, Proc. Natl. Acad. Sci. USA 100 (2003) 2197–2202.

9. Y. C. Eldar, P. Kuppinger, H. Bölcskei, Block-sparse signals: uncertainty relations and efficient recovery, IEEE Trans. Sig. Proc. 58 (2010) 3042–3054.

10. B. Et-Taoui, Complex conference matrices, complex Hadamard matrices and complex equiangular tight frames, Springer Proc. Math. Stat. 148 (2016) 181–191.

11. B. Et-Taoui, Infinite family of equi-isoclinic planes in Euclidean odd dimensional spaces and of complex symmetric conference matrices of odd orders, Linear Algebra Appl. 556 (2018) 373-380.

12. B. Et-Taoui, Quaternionic equiangular lines, Adv. Geom. 20 (2020) 273–284.

13. M. Fickus, J. Jasper, Equiangular tight frames from group divisible designs, Des. Codes Cryptogr. 87 (2019) 1673–1697.

14. M. Fickus, J. Jasper, E. J. King, D. G. Mixon, Equiangular tight frames that contain regular simplices, Linear Algebra Appl. 555 (2018) 98–138.

15. M. Fickus, J. Jasper, D. G. Mixon, Packings in real projective spaces, SIAM J. Appl. Algebra Geom. 2.3 (2018) 377–409.

16. M. Fickus, J. Jasper, D. G. Mixon, J. Peterson, Tremain equiangular tight frames, J. Comb. Theory. 153 (2018) 54–66.

17. M. Fickus, J. Jasper, D. G. Mixon, C. E. Watson, A brief introduction to equi-chordal and equi-isoclinic tight fusion frames, Proc. SPIE 10394 (2017) 103940T/1–9.

18. M. Fickus, J. Jasper, D. G. Mixon, J. Peterson, C. E. Watson, Polyphase equiangular tight frames and abelian generalized quadrangles, Appl. Comput. Harmon. Anal. 47 (2019) 628–661.

19. M. Fickus, B. R. Mayo, Mutually unbiased equiangular tight frames, IEEE Trans. Inform. Theory. 67 (2020) 1656–1667.

20. M. Fickus, B. R. Mayo, C. E. Watson, Certifying the novelty of equichordal tight fusion frames, arXiv:2103.03192 (2021).

21. M. Fickus, D. G. Mixon, Tables of the existence of equiangular tight frames, arXiv:1504.00253 (2016).

22. M. Fickus, D. G. Mixon, J. Jasper, Equiangular tight frames from hyperovals, IEEE Trans. Inform. Theory. 62 (2016) 5225–5236.

23. M. Fickus, D. G. Mixon, J. C. Tremain, Steiner equiangular tight frames, Linear Algebra Appl. 436 (2012) 1014–1027.

24. M. Fickus, C. A. Schmitt, Harmonic equiangular tight frames comprised of regular simplices, Linear Algebra Appl. 586 (2020) 130–169.

25. C. Godsil, A. Roy, Equiangular lines, mutually unbiased bases, and spin models, European J. Combin. 30 (2009) 246–262.

26. B. Gordon, W. H. Mills, L. R. Welch, Some new difference sets, Canad. J. Math. 14 (1962) 614–625.

27. S. G. Hoggar, New sets of equi-isoclinic n-planes from old, Proc. Edinb. Math. Soc. 20 (1977) 287–291.

28. Y. J. Ionin, A technique for constructing divisible difference sets, J. Geom. 67 (2000) 164–172.

29. J. W. Iverson, J. Jasper, D. G. Mixon, Optimal line packings from nonabelian groups, Disc. Comp. Geom. 63 (2020) 731–763.

30. J. W. Iverson, J. Jasper, D. G. Mixon, Optimal projective packings from association schemes, Proc. Internatl. Soc. Opt. Photon. 10394 (2017).

31. J. Jasper, D. G. Mixon, M. Fickus, Kirkman equiangular tight frames and codes, IEEE Trans. Inform. Theory. 60 (2014) 170–181.

32. E. J. King, New constructions and characterizations of flat and almost flat Grassmannian fusion frames, arXiv:1612.05784 (2016).

33. H. König, Cubature formulas on spheres, Math. Res. 107 (1999) 201–212.

34. P. W. H. Lemmens, J. J. Seidel, Equiangular lines, J. Algebra 24 (1973) 494–512.

35. R. L. McFarland, A family of difference sets in non-cyclic groups, J. Combin. Theory Ser. A 15.1 (1973) 1–10.

36. A. Pott, Finite geometry and character theory, Lecture Notes in Math. (1995).

37. T. Strohmer, R. W. Heath, Grassmannian frames with applications to coding and communication, Appl. Comput. Harmon. Anal. 14 (2003) 257–275.

38. F. Szöllősi, Complex Hadamard matrices and equiangular tight frames, Linear Algebra Appl. 438 (2013) 1962–1967.

39. J. A. Tropp, Greed is good: algorithmic results for sparse approximation, IEEE Trans. Inform. Theory 50 (2004) 2231–2242.

40. L. R. Welch, Lower bounds on the maximum cross correlation of signals, IEEE Trans. Inform. Theory 20 (1974) 397–399.

41. P. Wocjan, T. Beth, New construction of mutually unbiased bases in square dimensions, Quantum Inf. Comp. 5 (2005) 93–101.

42. P. Xia, S. Zhou, G. B. Giannakis, Achieving the Welch bound with difference sets, IEEE Trans. Inform. Theory 51 (2005) 1900–1907.

43. G. Zauner, Quantum designs: Foundations of a noncommutative design theory, Ph.D. Thesis, University of Vienna, 1999.

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Air Force Institute of Technology
Graduate School of Engineering and Management (AFIT/EN)
2950 Hobson Way
WPAFB OH 45433-7765

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Department of Mathematics and Statistics
2950 Hobson Way
WPAFB OH 45433-7765
DSN 271-0690, COMM 937-255-3636
Email: benjamin.mayo@afit.edu

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14. ABSTRACT

An equichordal tight fusion frame (ECTFF) is a sequence of equidimensional subspaces of a Euclidean space that achieves equality in Conway, Hardin and Sloane's simplex bound, and so is a type of optimal Grassmannian code. In the special case where its subspaces have dimension one, an ECTFF corresponds to an equiangular tight frame (ETF); such frames have minimal coherence and so are useful for compressed sensing. More generally, an ECTFF will yield a frame with minimal block coherence when its subspaces are pairwise isoclinic, namely when it is an equi-isoclinic tight fusion frame (EITFF). In this dissertation, we generalize the notion of an ETF to that of a biangular orthopartitionable tight frame (BOPTF). Every BOPTF generates an ECTFF and has a coherence that rivals that of an ETF. Generalizing a recent observation of King, we construct a new infinite family of BOPTFs whose ECTFFs are actually equi-isoclinic. Such EITFF-generating BOPTFs are remarkable: having both low coherence and minimal block coherence, they guarantee the efficient recovery of signals that are either sparse or block sparse (without foreknowledge of the sparsity type). We moreover show that such EITFF-generating BOPTFs are special, proving that certain infinite families of BOPTFs (including an infinite number of those constructed by King from semiregular divisible difference sets) generate ECTFFs that are not equi-isoclinic. Along the way, we discover several new methods for constructing and comprehending ECTFFs.

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19b. TELEPHONE NUMBER *(include area code)*

(937)255-3636 x4513; matthew.fickus@afit.edu