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Closure Properties for Galois Operators in Distributed Logic

DR. GERARD ALLWEIN

Center for High Assurance Computer Systems Branch
Information Technology Division

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14. ABSTRACT

This is a report on two closure properties of some new operators, the Galois operators, for Distributed Logic for use in expressing properties of FPGA applications. The Galois operators are variations on the usual necessity and possibility operators. In effect, they complete the first-order logic patterns in the relational semantics for modal logic, although here we extend them for use in Distributed Logic. It was important to bring these into Distributed Logic to increase its expressive power. There is a companion NRL Memorandum Report, Galois Operators for Distributed Logic, which provides more of the connections with expressing properties of FPGA applications of the Galois operators in Distributed Logic. This report and the companion report will be used for a logic paper showing the Thomason-Goldblatt theorem for the Galois operators. The original Thomason-Goldblatt theorem covers only the necessity and possibility operators for bog standard modal logic.

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EXECUTIVE SUMMARY

This research started from requiring more expressive modal connectives in Distributed Logic [1] for use in high assurance of FPGA applications. Eventually, this report and a companion report will be used for a logic paper showing the Thomason-Goldblatt theorem [2, 3] for the Galois operators. The original Thomason-Goldblatt theorem covered only the necessity and possibility operators for bog standard modal logic. Most of Distributed Logic (at this time) can be seen as a distributed modal logic. However, we have expanded it to include intensional (modal) operators of greater arity than the modal operators.

Distributed Logic is quite general and can happily accept *non-normal* modal operators; these are operators that do not distribute across conjunctions and disjunctions of a Boolean logic base. Even the Boolean base is not strictly necessary although we have not yet extended the logic over other bases. The operators we consider in this paper do distribute across conjunctions and disjunctions, hence are *normal* operators. The difference model-theoretically [1, 4] is that non-normal operators are interpreted using neighborhood systems whereas normal operators are interpreted using relations; these are distributed relations in our case.

The companion report, “Galois Operators for Distributed Logic" contains a fairly condensed version of this report. That report only mentioned the closure properties, this report proves them. This report is rather densely mathematical, not every bit of background could be explained or it would be twice as long. Failures as well as successes are detailed here. The failures are important because they go into some depth as to why a particular construction fails. Closure properties hold for every Galois operator but not every closure property holds for every operator. It was important to figure out which properties failed and why so we had a complete picture of the Galois operators in Distributed Logic.

[2] has the best explanation of the Goldblatt-Thomason theorem: the class of modal frames is closed under disjoint sums, homomorphic images, subframes, and ultraproducts. Here we will only be concerned with sums. Their rendition says each frame can be decomposed into a collection of point-generated frames. By that they mean that for each frame, pick a point and follow the relation from that point including all the visited points. This gives a collection of point-generated frames for each frame. Those get collected into a disjoint sum. For us, we will use a special form of disjoint sum that we call a *null sum*. We treat the binary cases here where only two frames are involved but do include a section showing how to extend the result to an infinite number of frames. Another closure construction is also used, namely *smash product*. Operators whose frames are closed under smash product are related through a new Boolean negation to operators whose frames are closed under null sums. This neatly solves how to get all the frames closed under null sums since half the frames are directly closed under smash products. Goldblatt

The original Goldblatt-Thomason theorem also relied upon frames for possibility being related to frames for necessity through the use of Boolean negation that sets up an isomorphism between the frames.

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CLOSURE PROPERTIES FOR GALOIS OPERATORS IN DISTRIBUTED LOGIC

1. INTRODUCTION

This research started from requiring more expressive modal connectives in Distributed Logic [1] for use in high assurance of FPGA applications. Eventually, this report and a companion report will be used for a logic paper showing the Thomason-Goldblatt theorem [2, 3] for the Galois operators. The original Thomason-Goldblatt theorem covered only the necessity and possibility operators for bog standard modal logic. Distributed Logic can be seen as a distributed modal logic. Distributed Logic is quite general and can happily accept *non-normal* modal operators; these are operators that do not distribute across conjunctions and disjunctions of a Boolean logic base. The operators we consider in this paper do distribute across conjunctions and disjunctions, hence are *normal* operators. The difference model-theoretically [1, 4] is that non-normal operators are interpreted using neighborhood systems whereas normal operators are interpreted using relations; these are distributed relations in our case.

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[2] has the best explanation of the Goldblatt-Thomason theorem: the class of modal frames is closed under disjoint sums, homomorphic images, subframes, and ultraproducts. Here we will only be concerned with two constructions that substitute for disjoint sums. Their rendition says each frame can be decomposed into a collection of point-generated frames. By that they mean that for each frame, pick a point and follow the relation from that point including all the visited points. This gives a collection of point-generated frames for each frame. Those get collected into a disjoint sum. For us, we will use a special form of disjoint sum that we call a *null sum*. We treat the binary case here where only two frames are involved but do include a section showing how to extend the result to an infinite number of frames. Another closure construction is also used, namely *smash product*. Operators whose frames are closed under smash product are related through a new Boolean negation to operators whose frames are closed under null sums. This neatly solves how to get all the frames closed under null sums since half the frame are directly closed under smash products.

The original Goldblatt-Thomason theorem also relied upon frames for possibility being related to frames for necessity through the use of Boolean negation that sets up an isomorphism between the frames. Assume there is a frame for possibility, then via Boolean negation there is an isomorphic frame for necessity. The points of the two frames are the same, but the Boolean algebra of sets in the frame supports possibility in the first instance and necessity in the second. The Boolean negation shows how to map possibility into necessity. Hence our use of the new Boolean negation plus some additional properties of the involved frames is analogous.

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A Distributed Logic has a graph of nodes called *localities* as part of the syntactic structure of the logic. Each node in the graph is associated with a *local* modal logic. Distributed Logic adds modal connectives, here called *operators*, between the local logics. A distributed operator takes propositions of a local logic at, say node k and returns propositions of another local logic, say node h. Each local logic is evaluated in a typical modal frame (one per node) and there are distributed Kripke relations linking the nodes for interpreting the distributed operators. Incidentally, the reason for the switch from *connective* to *operator* is that this paper assumes an algebraic perspective where the term *operator* is apropos. Switching between the two should cause no problems since the interpretation of any connective is an operator and this paper mostly concentrates on interpretations.

The reason we developed Distributed Logic was because FPGA applications have three large scale features: a collection of *components* connected with *signals*, and all running in *parallel*. Each component is a world of states unto itself and shares no internal state with any other component. Hence each component could be reasoned about using a single modal logic, as long as one was not concerned with signals to other components and the components running in parallel. The graph allows us to associate a local logic with each component, and distributed Kripke relations to model the effects of signals and parallel operations. From this substrate, it is possible to abstract a bit and use distributed relations for parallel behavior and some other high assurance concepts. Requiring capabilities for expressing high assurance concepts drove us to consider the operators from [5].

2. THE LOGIC

The basic logic is detailed in [1]. We use the term Distributed Logic in a general sense such as modal logic; each has several logics that can fall under the term. Distributed Logic also includes modal logic as simple case. A particular distributed logic is actually a collection of *local* modal logics that are connected in a formal way via distributed operators. Each local modal logic has a classical base that admits the usual necessity and possibility operators that abstract over next-state relations (these local operators can be augmented with the Galois operators [5]).

Distributed Logic lends itself well to FPGA applications. Each local logic is seen as being the local logic of a single component. The components are connected via their behavior; that behavior is expressed using distributed relations. The distributed operators abstract over those relations. The abstraction takes the form of the evaluation conditions on the operator as shown in the sequel. The use of the term *distributed relation* reflects that a relation is relating two distinct collections of states in different components. This is in contrast to the local *endo-relations* (such as next-state relations); these latter are limited to a single locality

We use the term *locality* to denote a local logic and its underlying component structure as expressed in its states and relations. Thus, the distributed relations connect localities. A common distributed relation is a parallelism relation, called *concurrence*, that represents when states in two different localities can simultaneously occur. Another distributed relation is one that relates all components that share the same clock domain.

The main thrust of this paper is to describe the Galois operators in the context of Distributed Logic and show their closure properties under *smash product* and *null sum*. Smash products are used to glue together two components that are seen to be intimately run together in that we always want to be concerned with pairs of states, one state from each locality. Null sums are used for combining components together in a disjoint fashion where we only are interested in one without regard to the others, somewhat like an exclusive-or.

The name *Galois operator* stems from a certain algebraic property they share: each operator has a kin. If an operator is seen as abstracting over a binary relation where the relation is viewed a relation from a locality h to a locality k, the kin of this operator abstracts over the relation turned around so that this *converse relation* runs from k to h. This relationship is not a negation relationship.

Traditional possibility and necessity are also Galois operators. Call an operator *forward* if in its evaluation condition, we consider the (binary) relation used in its evaluation to have a direction from its first position to its second position. Call an operator *backward* if we turn the direction of the relation around to use the converse direction. Traditional possibility is a forward possibility that has a kin called a backward necessity, and backward possibility has a kin called forward necessity. The key fact is that the same relation can be used to evaluate both forward possibility and backward necessity. This property holds for all Galois operators and their kins. To force this relationship to obtain, one adds special axioms to the logic, called *residuation axioms*. It is remarkable that this property persists in Distributed Logic where the first position of a binary relation will be in one locality and the second position will be in a different locality.

2.1 The Interpretations

The technical term *frame* in logic can represent many different situations. We use it primarily to represent a component in an FPGA application. Each frame consists of: (1) a collection of *points* (also called *states*), (2) a Boolean algebra of sets (where the sets are sets of points), (3) the $\in$ relation between points and elements of the Boolean algebra of sets, and (4) at least one *local relation* on points. In other words, it is a particular type of *classification* where the $\models$ relation is the set theoretic membership relation $\in$. The local relation can represent the next state relation when viewing the component as a finite automaton. However, local relations (not the $\in$ relation, the $\in$ relation is not a local relation) can also represent other notions as the need arises. An example of a non-next-state relation is where some states are considered security critical and related to states that are not security critical. Incidentally, requiring at least one local relation is not really much of constraint since it could always be made the identity (diagonal) relation. That relation has little modal import because from any state one can only move along the relation to the same state.

We assume a graph of localities usually denoted with sans serif, i.e., nodes are denoted h, k, etc. At each node is a classification consisting of a local logic, a frame, a satisfaction relations $\in_{\mathsf{h}}$, $\in_{\mathsf{k}}$ (or alternately $\overset{\mathsf{h}}{\models}$, $\overset{\mathsf{k}}{\models}$ when we want to emphasize a logic interpretation), and at least one local relation. The local relation can be the identity relation if no feature of a component needs to be represented using a local relation.

Definition 2.1.1 A *local frame* at a node h is a structure $\mathsf{h} = (H, \mathcal{H}, \mathbb{H})$ such that H is a collection of points called, generically, a *domain*. $\mathcal{H} : H \rightharpoonup H$ is a local relation connecting some points of H. We use the same symbol for the frame and its collection of states, and let use disambiguate meaning. $\mathbb{H}$ is a collection of *neighborhoods* or sets of points that are subsets of H and the entire collection is closed under the Boolean operations and under Galois operations, here a single Galois operation is generically indicated with $[h\rangle : \mathbb{H} \rightarrow \mathbb{H}$. Hence $\mathbb{H}$ is a modal set algebra. These operators are used to define special collections of points. They have valuation conditions in the sequel given by distributed operators, just set the distribution to a single component. The $\in_{\mathsf{h}}$ relation between states and elements of the Boolean algebra is left implicit.

The set $\top^{\mathbb{H}}$ is the top of the Boolean lattice of sets and $\perp^{\mathbb{H}}$ is the bottom. A caveat, from the requirements of the sequel, $\top^{\mathbb{H}}$ is not H and $\perp^{\mathbb{H}}$ is not $\emptyset$. Special points will be added to the frames.

The use of the term *neighborhood* comes from topology. A frame is considered a Stone space with a base of clopen sets and hence very disconnected.

A good mental picture to remember the definition of a frame is Diagram 2.1 where one can think of P as a proposition of a language or as a UCLA proposition. This latter is merely a set of points in a Boolean lattice of sets. In the former case, the $\in_{\mathsf{h}}$ relation becomes $\overset{\mathsf{h}}{\models}$ and in fact, in interpretations, $\in_{\mathsf{h}}$ is the meaning of $\overset{\mathsf{h}}{\models}$.

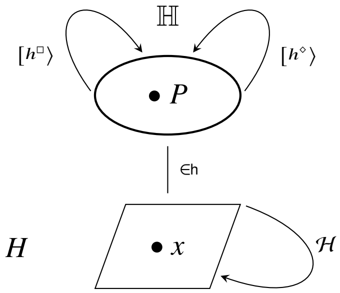

Diagram 2.1: Intutiive View of a Frame at Locality h

Here x is some point in the domain H and P is some UCLA proposition in $\mathbb{H}$, but it is not necessarily the case that $x \in_{\mathsf{h}} P$. Two example modal operators $[h^{\square}\rangle$ and $[h^{\diamond}\rangle$ are shown. They are the usual modal necessity and possibility operators $\square$ and $\diamond$ but put into our notation. The relation $\mathcal{H} : H \rightharpoonup H$ is some set $\mathcal{H} \subseteq H \times H$. Note that h refers to the node at which the frame indicated by the diagram lives.

Definition 2.1.2 A *distributed frame* consists of a graph of nodes where each node is a local frame, distributed relations linking the domains, and distributed modal operators linking the set algebras.

Propositions in the logic are modeled by elements of the set algebras. A distributed frame for modeling two components has two localities. The Diagram 2.2 is distributed frame with localities h and k that depicts this situation (without showing the local modalities or local relations) and two (generic) distributed modalities:

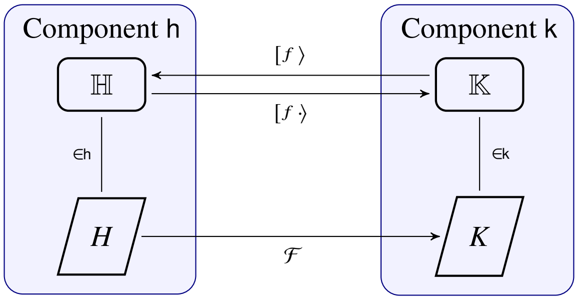

Diagram 2.2: Generic Distribution with Two Components at Localities h and k

where the arrows $[f\rangle$, $[f\cdot\rangle$ can be any of the Galois forward and backward distributed modalities interpreted by the distributed relation $\mathcal{F}$, i.e., the lower case f in the modalities is linked with the script $\mathcal{F}$ relation. Example pairs of modalities are $[f^{\Box}\rangle$ and $[f^{\Box}\cdot\rangle$ and are necessity operators, and $[f^{\diamond}\rangle$ and $[f^{\diamond}\cdot\rangle$ are possibility operators.

The diagram is only showing the simplest of distributed frames. In any application, even one with only two localities, there can be any number of distributed relations and any number of distributed modalities.

The relation $\mathcal{F} : \mathsf{h} \rightharpoonup \mathsf{k}$ used in the evaluation of the operators above uses two localities, h and k. These are variables in that any actual FPGA application will fill those localities in by two components; the entire distributed frame will use as many localities as there are components. Also, any one locality can have many endo-relations, and any two localities can be connected by many distributed relations. The modalities involved are restricted by the number of relations we have at our disposal to interpret the modalities. Technically, $\mathcal{F} \subseteq H \times K$, we leave this implicit in the notation $\mathcal{F} : \mathsf{h} \rightharpoonup \mathsf{k}$.

The distributed relation $\mathcal{F}$ (see diagram) is denoted with an arrow but this is mere convention; $\mathcal{F}$ is a two-place relation that, by fiat, is viewed as a morphism from elements of its first position to elements of its second position. The distributed modalities, on the other hand, really are functions although they have special properties required for us to treat them as modalities.

To evaluate logic formulas, we interpret the $\models$ relation as set theoretic membership $\in$ and the proposition P can be either a linguistic proposition, or switching to the set theory interpretation, a UCLA proposition, i.e., a set of points. The following definitions are standard modal logic fare:

- $\overset{\mathsf{h}}{\models} P$ if and only if for all $x \in_{\mathsf{h}} \top^{\mathbb{H}}$, it is the case that $x \overset{\mathsf{h}}{\models} P$. Equivalently, $\overset{\mathsf{h}}{\models} P$ if and only if for all $x \in_{\mathsf{h}} \top^{\mathbb{H}}$ (where $\top^{\mathbb{H}}$ is the top of the Boolean set algebra $\mathbb{H}$), it is the case that $x \overset{\mathsf{h}}{\models} P$.
- $\overset{\mathsf{h}}{\not\models} P$ if and only if there is some $x \in_{\mathsf{h}} \top^{\mathbb{H}}$ and $x \overset{\mathsf{h}}{\not\models} P$.
- A local logic at h is consistent just when for all $x \in_{\mathsf{h}} \top^{\mathbb{H}}$, it is the case that for all propositions P, not both $x \overset{\mathsf{h}}{\models} P$ and $x \overset{\mathsf{h}}{\not\models} P$.

Modal formulas $[f^{\Box}\rangle$ and $[f^{\diamond}\rangle$ for necessity and possibility are evaluated in the usual way except we must respect the distributed nature now of these modalities:

$$x \overset{\mathsf{h}}{\models} [f^{\Box}\rangle Q \text{ iff for all } y,\ \mathcal{F}xy \text{ implies } y \overset{\mathsf{k}}{\models} Q \qquad x \overset{\mathsf{h}}{\models} [f^{\diamond}\rangle Q \text{ iff there exists } y,\ \mathcal{F}xy \text{ and } y \overset{\mathsf{k}}{\models} Q.$$

The other versions $[f^{\Box}\cdot\rangle$ and $[f^{\diamond}\cdot\rangle$ run in the other direction:

$$y \overset{\mathsf{k}}{\models} [f^{\Box}\cdot\rangle P \text{ iff for all } x,\ \mathcal{F}xy \text{ implies } x \overset{\mathsf{h}}{\models} P \qquad y \overset{\mathsf{k}}{\models} [f^{\diamond}\cdot\rangle Q \text{ iff there exists } x,\ \mathcal{F}xy \text{ and } x \overset{\mathsf{h}}{\models} Q.$$

Setting $\mathsf{h} = \mathsf{k}$ yields the evaluation conditions for the corresponding local modalities. Hence the distributed modalities are evaluated similar to the local modalities except notice the changes between $\overset{\mathsf{h}}{\models}$ and $\overset{\mathsf{k}}{\models}$ on the two sides of the evaluations.

2.2 Galois Operators

The operators for the modal logic case are in [5]. Their basic mathematical properties are shown there for a few of the operators. This paper extends the properties from the modal case where there is ever only a single locality to this distributed case for Distributed Logic.

Each operator comes with a *distribution type*, which becomes a bit of a misnomer when put into a Distributed Logic framework because the two uses of "distribution" refer to two kinds of distribution. For Distributed Logic, the term "distribution " refers to the distribution as detailed in a graph of nodes each with its own logic logic and connected by distributed operators and distributed relations. For [5], which was developed quite a bit before Distributed Logic, the term "distribution" refers to how the operator distributes across conjunctions and disjunctions, or in the algebraic case where we replace the logic with a lattice, lattice meets and joins. We will shorten the use of "distribution type" of [5] to merely "type". The type is intimately connected with an operator's residuation properties (see Section 6 Residuation).

The move from logic's conjunction and disjunction to lattice meets and joins is the move from the logic to an algebraic framework. A way to get an algebra out of a logic is to divide out the set of formulas via bi-implication:

$$P \equiv Q \text{ iff} \vdash P \supset Q \text{ and} \vdash Q \supset P,$$

where one can interpret the (usually) classical implication $\supset$ with a logic's implication and $\vdash$ is provability in the logic. The equivalence classes then are elements of a lattice if the logic supports the usual Boolean identities. In particular,

$$[P] \wedge [Q] \overset{def}{=} [P \wedge Q] \qquad [P] \vee [Q] \overset{def}{=} [P \vee Q] \qquad \neg[P] \overset{def}{=} [\neg P].$$

The modal operators are defined similarly.

In the sequel, we use P and Q to refer to sets of points of the Boolean algebra (of sets) of a frame, i.e., UCLA propositions. They can also be thought of as propositions. This blurs any distinction between $[P]$ which is an equivalence class of formulas, all of which are equivalent to P, and P as a UCLA proposition consisting of all the points of a model which are members of P.

The rest of this paper assumes an algebraic framework. The propositions are interpreted as elements of the carrier set of an algebra, in our case a Boolean algebra, which is a lattice under intersection $\cap$, union $\cup$, and a form of set complement $\overset{\bullet\circ}{\neg}$, augmented with local modal operators and distributed operators. The new form of set complement $\overset{\bullet\circ}{\neg}$ in place of the usual set complement was necessary due to the extra points added to the domains.

We let $f : \mathsf{h} \rightharpoonup \mathsf{k}$ refer to an arc of a Distributed Logic graph from node h to node k. As such, there are several operators that can be associated with the arc as well as the relation, usually depicted similarly, i.e., $\mathcal{F} : \mathsf{h} \rightharpoonup \mathsf{k}$ where the small case italic f is linked semantically to the interpreting relation $\mathcal{F}$ in a script font.

In Table 2.1, the type of each operator is of the form $\rho \mapsto \sigma$ where $\rho, \sigma \in \{\wedge, \vee\}$. The first line of any operator gives its type and its semantic evaluation condition. The second line gives the canonical definition of the interpreting relation where the x, y are maximal filters of the lattices that support the operator. The elements of the lattice are a, b. The third line shows the closure properties. We have annotated the set theoretical membership relation $\in$ with a locality to show in which frame the domain of the membership relation lives, usually one of $\in_{\mathsf{h}}$ or $\in_{\mathsf{k}}$. We also annotate set inclusion in the sequel, when it aids understanding, with the domains that are involved, i.e, $\subseteq$ becomes $\subseteq_{\mathsf{h}}$ or $\subseteq_{\mathsf{k}}$.

Table 2.1: **Table of all Two-Place Galois Operators**

Menu A

$\wedge\mapsto\wedge$	$y \in_{\mathsf{k}} [f^{\flat}\cdot\rangle P$ iff $\forall x(x \in_{\mathsf{h}} P$ or $\mathcal{F}^{\flat\cdot}xy)$ $\mathcal{F}^{\flat\cdot}xy$ iff $\exists a(a \notin_{\mathsf{h}} x$ and $[f^{\flat}\cdot\rangle a \in_{\mathsf{k}} y)$ null and wedge sum	$\vee\mapsto\vee$	$y \in_{\mathsf{k}} [f^{\diamond}\cdot\rangle P$ iff $\exists x(x \in_{\mathsf{h}} P$ and $\mathcal{F}^{\diamond\cdot}xy)$ $\mathcal{F}^{\diamond\cdot}xy$ iff $\forall a(a \notin_{\mathsf{h}} x$ or $[f^{\diamond}\cdot\rangle a \in_{\mathsf{k}} y)$ smash product
$\vee\mapsto\wedge$	$y \in_{\mathsf{k}} [f^{\perp}\cdot\rangle P$ iff $\forall x(x \notin_{\mathsf{h}} P$ or $\mathcal{F}^{\perp\cdot}xy)$ $\mathcal{F}^{\perp\cdot}xy$ iff $\exists a(a \in_{\mathsf{h}} x$ and $[f^{\perp}\cdot\rangle a \in_{\mathsf{k}} y)$ smash product	$\wedge\mapsto\vee$	$y \in_{\mathsf{k}} [f^{?}\cdot\rangle P$ iff $\exists x(x \notin_{\mathsf{h}} P$ and $\mathcal{F}^{?\cdot}xy)$ $\mathcal{F}^{?\cdot}xy$ iff $\forall a(a \in_{\mathsf{h}} x$ or $[f^{?}\cdot\rangle a \in_{\mathsf{k}} y)$ null sum

Menu B

$\wedge\mapsto\wedge$	$y \in_{\mathsf{k}} [f^{\square}\cdot\rangle P$ iff $\forall x(x \in_{\mathsf{h}} P$ or $\neg\mathcal{F}^{\square\cdot}xy)$ $\mathcal{F}^{\square\cdot}xy$ iff $\forall a(a \in_{\mathsf{h}} x$ or $[f^{\square}\cdot\rangle a \notin_{\mathsf{k}} y)$ smash product and null sum	$\vee\mapsto\vee$	$y \in_{\mathsf{k}} [f^{\sharp}\cdot\rangle P$ iff $\exists x(x \in_{\mathsf{h}} P$ and $\neg\mathcal{F}^{\sharp\cdot}xy)$ $\mathcal{F}^{\sharp\cdot}xy$ iff $\exists a(a \in_{\mathsf{h}} x$ and $[f^{\sharp}\cdot\rangle a \notin_{\mathsf{k}} y)$ smash product and wedge sum
$\vee\mapsto\wedge$	$y \in_{\mathsf{k}} [f^{!}\cdot\rangle P$ iff $\forall x(x \notin_{\mathsf{h}} P$ or $\neg\mathcal{F}^{!\cdot}xy)$ $\mathcal{F}^{!\cdot}xy$ iff $\forall a(a \notin_{\mathsf{h}} x$ or $[f^{!}\cdot\rangle a \notin_{\mathsf{k}} y)$ smash product	$\wedge\mapsto\vee$	$y \in_{\mathsf{k}} [f^{\star}\cdot\rangle P$ iff $\exists x(x \notin_{\mathsf{h}} P$ and $\neg\mathcal{F}^{\star\cdot}xy)$ $\mathcal{F}^{\star\cdot}xy$ iff $\exists a(a \notin_{\mathsf{h}} x$ and $[f^{\star}\cdot\rangle a \notin_{\mathsf{k}} y)$ null sum

Menu C

$\wedge\mapsto\wedge$	$x \in_{\mathsf{h}} [f^{\flat}\rangle Q$ iff $\forall y(y \in_{\mathsf{k}} Q$ or $\mathcal{F}^{\flat}xy)$ $\mathcal{F}^{\flat}xy$ iff $\exists b(b \notin_{\mathsf{k}} y$ and $[f^{\flat}\rangle\, b \in_{\mathsf{h}} x)$ null and wedge sum	$\vee\mapsto\vee$	$x \in_{\mathsf{h}} [f^{\diamond}\rangle Q$ iff $\exists y(y \in_{\mathsf{k}} Q$ and $\mathcal{F}^{\diamond}xy)$ $\mathcal{F}^{\diamond}xy$ iff $\forall b(b \notin_{\mathsf{k}} y$ or $[f^{\diamond}\rangle\, b \in_{\mathsf{h}} x)$ smash product
$\vee\mapsto\wedge$	$x \in_{\mathsf{h}} [f^{\perp}\rangle Q$ iff $\forall y(y \notin_{\mathsf{k}} Q$ or $\mathcal{F}^{\perp}xy)$ $\mathcal{F}^{\perp}xy$ iff $\exists b(b \in_{\mathsf{k}} y$ and $[f^{\perp}\rangle\, b \in_{\mathsf{h}} x)$ smash product and wedge sum	$\wedge\mapsto\vee$	$x \in_{\mathsf{h}} [f^{?}\rangle Q$ iff $\exists y(y \notin_{\mathsf{k}} Q$ and $\mathcal{F}^{?}xy)$ $\mathcal{F}^{?}xy$ iff $\forall b(b \in_{\mathsf{k}} y$ or $[f^{?}\rangle\, b \in_{\mathsf{h}} x)$ null sum

Menu D

$\wedge\mapsto\wedge$	$x \in_{\mathsf{h}} [f^{\square}\rangle Q$ iff $\forall y(y \in_{\mathsf{k}} Q$ or $\neg\mathcal{F}^{\square}xy)$ $\mathcal{F}^{\square}xy$ iff $\forall b(b \in_{\mathsf{k}} y$ or $[f^{\square}\rangle\, b \notin_{\mathsf{h}} x)$ null sum and wedge sum	$\vee\mapsto\vee$	$x \in_{\mathsf{h}} [f^{\sharp}\rangle Q$ iff $\exists y(y \in_{\mathsf{k}} Q$ and $\neg\mathcal{F}^{\sharp}xy)$ $\mathcal{F}^{\sharp}xy$ iff $\exists b(b \in_{\mathsf{k}} y$ and $[f^{\sharp}\rangle\, b \notin_{\mathsf{h}} x)$ smash product and wedge sum
$\vee\mapsto\wedge$	$x \in_{\mathsf{h}} [f^{!}\rangle Q$ iff $\forall y(y \notin_{\mathsf{k}} Q$ or $\neg\mathcal{F}^{!}xy)$ $\mathcal{F}^{!}xy$ iff $\forall b(b \notin_{\mathsf{k}} y$ or $[f^{!}\rangle\, b \notin_{\mathsf{h}} x)$ smash product	$\wedge\mapsto\vee$	$x \in_{\mathsf{h}} [f^{\star}\rangle Q$ iff $\exists y(y \notin_{\mathsf{k}} Q$ and $\neg\mathcal{F}^{\star}xy)$ $\mathcal{F}^{\star}xy$ iff $\exists b(b \notin_{\mathsf{k}} y$ and $[f^{\star}\rangle\, b \notin_{\mathsf{h}} x)$ null sum

Menus A and B are the "backward" operators, so termed because the evaluating relation is read from the second position to first position in operator's evaluation.

3. POINTED DOMAIN OPERATORS

The addition of extra points over the usual flat basic domains as frames for logic has the following intuitive Diagram 3.1 where there are two domains H_i and K_i at localities h_i and k_i respectively for $i \in I$ (I is some arbitrary index set)

Diagram 3.1: Hasse Diagrams at Localities h_i and k_i

The extra points are $\bullet_{H_i}$, $\circ_{H_i}$, $\bullet_{K_i}$, and $\circ_{K_i}$. The lines of the diagram are part of a Hasse diagram of the domains as partial orders. The order reads up the page so that for $x \leq y$ in the order, y is higher in the Hasse diagram. Hence a traditional frame for modal logic has no extra points and hence no lines are necessary in its Hasse diagram. This sort of domain is called a *flat* domain. Ordered domains are used in non-standard logic and these frames with their additional points are forms of ordered domains.

We picture two domains because the stock situation explicating the semantics of Distributed Logic uses an arc $f : \mathsf{h} \rightharpoonup \mathsf{k}$ in the graph where h and k range over $\{\mathsf{h}_i, \mathsf{k}_i \mid i \in I\}$ where I indexes the nodes in the graph of the logic. Each FPGA application supplies the component list which we abstract into the set I. The arcs are used when relationships between components must be expressed.

In the sequel, we will have need for special sets of points in the domains. These sets arise because of the new constructions used: smash product and null sum. The former is a variant of a cross-product where certain points are collapsed to a single point and the latter is a variant of disjoint sum where the tags of the sum are actually points of the frames involved in the sum and hence may support mathematic properties rather than being merely tags.

3.1 Ambient Spaces

In the original Goldblatt-Thomason theorem, they use a disjoint sum and in [3], he uses distribution across those sums. Hence he must prove

$$[^{h^\circ}\rangle (Q_1 + Q_2) = [^{h^\circ}\rangle Q_1 + [^{h^\circ}\rangle Q_2.$$

The only way for this to make sense is for both sides to be of the correct type. Hence, if $a \in_{\mathsf{h}} [^{h^\circ}\rangle (Q_1 + Q_2)$, then a must be of the form $\langle c_1, c_2 \rangle$ where either c_1 or c_2 (but not both) is a tag. The reason is that it must be shown that $a \in_{\mathsf{h}} [^{h^\circ}\rangle Q_1 + [^{h^\circ}\rangle Q_2$. and hence a must be a tagged pair. In the other direction,

$a \in_{\mathsf{h}} [h^{\diamond}\rangle Q_1 + [h^{\diamond}\rangle Q_2$. implies a is also of the form $\langle c_1, c_2\rangle$ with either c_1 or c_2 (but not both) being a tag. For it to be so that $a \in_{\mathsf{h}} [h^{\diamond}\rangle (Q_1 + Q_2)$, a must retain its tag. The point is that there is an ambient space $H_1 + H_2$ of tagged pairs and $[h^{\diamond}\rangle (Q_1 + Q_2) \subseteq_{\mathsf{h}} H_1 + H_2$ and $[h^{\diamond}\rangle Q_1 + [h^{\diamond}\rangle Q_2 \subseteq_{\mathsf{h}} H_1 + H_2$.

We shall require special ambient spaces for constructions of smash products and null sums below.

3.2 Points

We use the symbols $\otimes$ and $\ovee$ for smash products and null sums respectively. To adequately characterize them, we first define some distinguished sets.

Definition 3.2.1 Canonically, if A_{h} is the carrier set of a Boolean algebra at h,

$$\bullet_H = A_{\mathsf{h}} \qquad \circ_H = \emptyset_{\mathsf{h}}.$$

Technically, all empty sets are the same so that $\emptyset_{\mathsf{h}} = \emptyset = \emptyset_{\mathsf{k}}$ for $\mathsf{h} \neq \mathsf{k}$. The appellations h and k help to indicate which domain is under consideration in any mathematical statement involving them. In the sequel, we will elide the appellations since the context makes clear which domains are involved.

The constructions $\otimes$ and $\ovee$ below require that we generalized pointed domains or universes. Canonically, we term sets of the form $\{\bullet_H\}$ *well-point* sets and sets of the form $\{\circ_H\}$ *null-point* sets. Abstractly, we use $H^\bullet$ to be the well-point set at h and H° to be the null-point set at h.

Later on we will define smash product $\otimes$ and null sum $\ovee$. For now, we will only define these constructions on the well and null-point sets. To shorten the prose, we use the term *point set* to refer to both well and null-point sets.

Definition 3.2.2 A *ground domain* is one that is not constructed using $\otimes$ or $\ovee$. Iterated point sets are inductively defined via the following two clauses.

Base Case: this case is for ground domains.

$$H^\bullet = \{\bullet_H\} \quad H^\circ = \{\circ_H\}.$$

Inductive step:

$$(H_1 \otimes H_2)^\bullet = {H_1}^\bullet \times {H_2}^\bullet \quad (H_1 \otimes H_2)^\circ = ({H_1}^\circ \times H_2) \cup (H_1 \times {H_2}^\circ).$$

and

$$(H_1 \ovee H_2)^\bullet = ({H_1}^\bullet \times {H_2}^\circ) \cup ({H_1}^\circ \times {H_2}^\bullet) \quad (H_1 \ovee H_2)^\circ = {H_1}^\circ \times {H_2}^\circ.$$

The first two layers of point sets covering the ground domains and one layer above them are

$$H^{\bullet} = \{\bullet_H\} \qquad (H_1 \otimes H_2)^{\bullet} = \{\langle \bullet_{H_1}, \bullet_{H_2} \rangle\} \qquad (H_1 \circledvee H_2)^{\bullet} = \{\langle \bullet_{H_1}, \circ_{H_2} \rangle, \langle \circ_{H_1}, \bullet_{H_2} \rangle\}.$$

and

$$H^{\circ} = \{\circ_H\} \qquad (H_1 \otimes H_2)^{\circ} = (\{\circ_{H_1}\} \times H_2) \cup (H_1 \times \{\circ_{H_2}\}) \qquad (H_1 \circledvee H_2)^{\circ} = \{\langle \circ_{H_1}, \circ_{H_2} \rangle\}.$$

We need the capability of infinite smash products and null sums in the sequel, so we must make a further extension:

Definition 3.2.3 Infinite iterated point sets are inductively defined via the following two clauses.

Base Case:

$$H^{\bullet} = \{\bullet_H\} \quad H^{\circ} = \{\circ_H\}.$$

Inductive step:

$$(\otimes_i H_i)^{\bullet} = \prod_i H_i^{\bullet} \qquad (\circledvee_i H_i)^{\bullet} = \bigcup_j \prod_i \begin{cases} H_i^{\bullet} & i = j \\ H_i^{\circ} & \text{otherwise} \end{cases}$$

$$(\otimes_i H_i)^{\circ} = \bigcup_j \prod_i \begin{cases} H_i^{\circ} & i = j \\ H_i & \text{otherwise} \end{cases} \qquad (\circledvee_i H_i)^{\circ} = \prod_i H_i^{\circ}.$$

Let

$$\underline{u}_i H_i^{\bullet} = \textstyle\prod_j \begin{cases} H_j^{\bullet} & \text{if } i = j \\ H_i^{\circ} & \text{otherwise} \end{cases}$$

then we can rephrase $\circledvee_i {H_i}^{\bullet}$ with

$$\circledvee_i {H_i}^{\bullet} = \bigcup_i \underline{u}_i H_i^{\bullet}.$$

Sometimes it is useful to unpack this using elements:

$$\langle x \rangle \in\!\mathsf{h}\; H_{\circledvee}^{\bullet} \text{ iff } \exists j! \in I\ (\langle x \rangle \in\!\mathsf{h}\; \underline{u}_j H_j^{\bullet}).$$

where $\langle x \rangle$ stands for a sequence of x_i for some ambient indexing set I. The quantifier $\exists! j$ stands for there exists exactly one j.

We make some abbreviations to make the text a bit easier to read.

Definition 3.2.4 When $\divideontimes \in \{\otimes, \varovee\}$, then $H^{\bullet}_{\divideontimes}$ and $H^{\circ}_{\divideontimes}$ refer to $(\divideontimes_i H_i)^{\bullet}$ and $(\divideontimes_i H_i)^{\circ}$ respectively. Also, $H^{\bullet}_i$ and H°_i refer to the ith component when the format of H is already assumed.

Lemma 3.2.5 *For all H,*

$$H^{\bullet} \cap H^{\circ} = \emptyset$$

Proof: The proof is an easy induction on the structure of $H^{\bullet}$ and H°. The property clearly holds for the base case. $(\otimes_i H_i)^{\circ}$ always contains an instance of H°_j for some j whereas $(\otimes_i H_i)^{\bullet}$ never does. Similarly, $(\varovee_i H_i)^{\bullet}$ always contains and instance $H^{\bullet}_j$ for some j whereas $(\varovee_i H_i)^{\circ}$ never does. ■

Definition 3.2.6 All UCLA propositions are *well- and null-pointed*, abbreviated *wn-pointed*, if they satisfy

$$Q = (Q - H^{\circ}) \cup H^{\bullet}.$$

From now on, all UCLA propositions not the result of modal operators are considered to be wn-pointed. The modal operators will require each require a Lemma to show this property holds.

The following properties are direct consequences of the definition:

Lemma 3.2.7 *Note that since* $H^{\circ} \cap \top^{\mathbb{H}} = \emptyset$ *and* $H^{\bullet} \subseteq_{\mathsf{h}} \top^{\mathbb{H}}$, *then for all Q,*

$$Q \subseteq_{\mathsf{h}} \top^{\mathbb{H}} \qquad H^{\bullet} \subseteq_{\mathsf{h}} Q \qquad H^{\circ} \cap Q = \emptyset \qquad Q_1 \cap Q_2 \subseteq_{\mathsf{h}} \top^{\mathbb{H}} \qquad Q_1 \cup Q_2 \subseteq_{\mathsf{h}} \top^{\mathbb{H}}.$$

The bounds for $\mathbb{H}$ *are*

$$\bot^{\mathbb{H}} = H^{\bullet} \qquad \top^{\mathbb{H}} = H - H^{\circ}.$$

Lemma 3.2.8

$$H - (Q \cup H^{\circ}) = \top^{\mathbb{H}} - Q.$$

Proof:

$$\begin{aligned} H - (Q \cup H^{\circ}) &= (H - Q) \cap (H - H^{\circ}) \\ &= (H - Q) \cap \top^{\mathbb{H}} \\ &= (H \cap \top^{\mathbb{H}}) - (Q \cap \top^{\mathbb{H}}) \\ &= \top^{\mathbb{H}} - Q \end{aligned}$$

where the second to the third line is justified by the following argument. Let $x \in_{\mathsf{h}} (H - Q) \cap \top^{\mathbb{H}}$, then $x \notin_{\mathsf{h}} Q$ and $x \in_{\mathsf{h}} \top^{\mathbb{H}}$. Since all points are in H, then $x \in_{\mathsf{h}} H \cap \top^{\mathbb{H}}$. Since $x \notin_{\mathsf{h}} Q$, then $x \notin_{\mathsf{h}} Q \cap \top^{\mathbb{H}}$. Therefore,

$x \in\mathsf{h} (H \cap \top^{\mathbb{H}}) - (Q \cap \top^{\mathbb{H}})$. Next, let $x \in\mathsf{h} (H \cap \top^{\mathbb{H}}) - (Q \cap \top^{\mathbb{H}})$, so $x \in\mathsf{h} H \cap \top^{\mathbb{H}}$ and $x \notin\mathsf{h} Q \cap \top^{\mathbb{H}}$. Therefore $x \in\mathsf{h} \top^{\mathbb{H}}$, and $x \notin\mathsf{h} Q$. So $x \in\mathsf{h} (H - Q) \cap \top^{\mathbb{H}}$.

■

Lemma 3.2.9

$$(\top^{\mathbb{H}} - Q) \cup H^{\bullet} = \top^{\mathbb{H}} - (Q - H^{\bullet}).$$

Proof: Let $x \in\mathsf{h} (\top^{\mathbb{H}} - Q) \cup H^{\bullet}$, then either $x \in\mathsf{h} \top^{\mathbb{H}} - Q$ or $x \in\mathsf{h} H^{\bullet}$. Assume the former, then $x \in\mathsf{h} \top^{\mathbb{H}}$, and $x \notin\mathsf{h} Q$. Thus $x \notin\mathsf{h} Q - H^{\bullet}$ and $x \in\mathsf{h} \top^{\mathbb{H}} - (Q - H^{\bullet})$. If $x \in\mathsf{h} H^{\bullet}$, then $x \in\mathsf{h} \top^{\mathbb{H}}$ and $x \notin\mathsf{h} Q - H^{\bullet}$,. Hence $x \in\mathsf{h} \top^{\mathbb{H}} - (Q - H^{\bullet})$.

Next, let $x \in\mathsf{h} \top^{\mathbb{H}} - (Q - H^{\bullet})$, then $x \in\mathsf{h} \top^{\mathbb{H}}$ and $x \notin\mathsf{h} Q - H^{\bullet}$. If $x \in\mathsf{h} H^{\bullet}$, then $x \in\mathsf{h} (\top^{\mathbb{H}} - Q) \cup H^{\bullet}$. If $x \notin\mathsf{h} H^{\bullet}$ then $x \notin\mathsf{h} Q$. So $x \in\mathsf{h} \top^{\mathbb{H}} - Q$ and $(\top^{\mathbb{H}} - Q) \cup H^{\bullet}$.

■

3.3 New Boolean Complement

The above characterization clearly will not allow the Boolean set-complement $\neg Q$ to be wn-pointed if Q is wn-pointed. We need a new negation.

Definition 3.3.1 $\overset{\bullet\circ}{\neg} Q$ is the *wn-complement* or *wn-negation* defined by

$$\overset{\bullet\circ}{\neg} Q \overset{def}{=} H \overset{\bullet\circ}{-} Q \overset{def}{=} (\top^{\mathbb{H}} - Q) \cup H^{\bullet}.$$

An alternate definition is

$$x \in\mathsf{h} \overset{\bullet\circ}{\neg} Q \text{ iff } (x \notin\mathsf{h} Q \text{ and } x \notin\mathsf{h} H^{\circ}) \text{ or } x \in\mathsf{h} H^{\bullet}.$$

The alternate definition points out that the top of the set lattice $\top^{\mathbb{H}}$ is not the universe H since $x \notin\mathsf{h} Q$ allows for $x \in\mathsf{h} H^{\circ}$, yet $x \in\mathsf{h} H$ and $H^{\circ} \cap \top^{\mathbb{H}} = \emptyset$. This matters in places where non-membership in a UCLA proposition is used.

Lemma 3.3.2 *UCLA sets produced from UCLA sets using* $\overset{\bullet\circ}{\neg}$ *are wn-pointed.*

Proof: This an easy consequence of Definition 3.3.1. ■

Lemma 3.3.3 *Assume* Q, Q_1, *and* Q_2 *are wn-pointed at locality* h, *then* $\overset{\bullet\circ}{\neg}$ *is a Boolean negation:*

$$\begin{aligned} \overset{\bullet\circ}{\neg}(Q_1 \wedge Q_2) &= \overset{\bullet\circ}{\neg} Q_1 \vee \overset{\bullet\circ}{\neg} Q_2 & \qquad \overset{\bullet\circ}{\neg}(Q_1 \vee Q_2) &= \overset{\bullet\circ}{\neg} Q_1 \wedge \overset{\bullet\circ}{\neg} Q_2 \\ Q \wedge \overset{\bullet\circ}{\neg} Q &= \bot^{\mathbb{H}} & \qquad Q \vee \overset{\bullet\circ}{\neg} Q &= \top^{\mathbb{H}} \\ \overset{\bullet\circ}{\neg}\overset{\bullet\circ}{\neg} Q &= Q \end{aligned}$$

Proof: First $\overset{\bullet\circ}{\neg}(Q_1 \cap Q_2) = \overset{\bullet\circ}{\neg} Q_1 \cup \overset{\bullet\circ}{\neg} Q_2$:

$$\begin{aligned} \overset{\bullet\circ}{\neg}(Q_1 \cap Q_2) &= (\top^{\mathbb{H}} - (Q_1 \cap Q_2)) \cup H^{\bullet} \\ &= \big((\top^{\mathbb{H}} - Q_1) \cup H^{\bullet}\big) \cup \big((\top^{\mathbb{H}} - Q_2) \cup H^{\bullet}\big) \\ &= \overset{\bullet\circ}{\neg} Q_1 \cup \overset{\bullet\circ}{\neg} Q_2 \end{aligned}$$

Next, $\overset{\bullet\circ}{\neg}(Q_1 \cup Q_2) = \overset{\bullet\circ}{\neg} Q_1 \cap \overset{\bullet\circ}{\neg} Q_2$:

$$\begin{aligned} \overset{\bullet\circ}{\neg}(Q_1 \cup Q_2) &= \top^{\mathbb{H}} - (Q_1 \cup Q_2) \\ &= (\top^{\mathbb{H}} - Q_1) \cap (\top^{\mathbb{H}} - Q_2) \\ &= \overset{\bullet\circ}{\neg} Q_1 \cap \overset{\bullet\circ}{\neg} Q_2 \end{aligned}$$

Next, $Q \cap \overset{\bullet\circ}{\neg} Q = \bot^{\mathbb{H}}$:

$$\begin{aligned} Q \cap \overset{\bullet\circ}{\neg} Q &= Q \cap \big((\top^{\mathbb{H}} - Q) \cup H^{\bullet}\big) \\ &= \big(Q \cap (\top^{\mathbb{H}} - Q)\big) \cup (Q \cap H^{\bullet}) \\ &= \emptyset \cup H^{\bullet} \\ &= H^{\bullet} = \bot^{\mathbb{H}} \end{aligned}$$

Next, $Q \cup \overset{\bullet\circ}{\neg} Q = \top^{\mathbb{H}}$:

$$\begin{aligned} Q \cup \overset{\bullet\circ}{\neg} Q &= Q \cup \big((\top^{\mathbb{H}} - Q) \cup H^{\bullet}\big) \\ &= Q \cup (\top^{\mathbb{H}} - Q) \\ &= \top^{\mathbb{H}} \end{aligned}$$

Lastly, $\overset{\bullet\circ}{\neg}\overset{\bullet\circ}{\neg} Q = Q$:

$$\begin{aligned} \overset{\bullet\circ}{\neg}\overset{\bullet\circ}{\neg} Q &= (\top^{\mathbb{H}} - \overset{\bullet\circ}{\neg} Q) \cup H^{\bullet} \\ &= \Big(\top^{\mathbb{H}} - \big((\top^{\mathbb{H}} - Q) \cup H^{\bullet}\big)\Big) \cup H^{\bullet} \\ &= \Big(\big(\top^{\mathbb{H}} - (\top^{\mathbb{H}} - Q)\big) \cap (\top^{\mathbb{H}} - H^{\bullet})\Big) \cup H^{\bullet} \\ &= \big(Q \cap (\top^{\mathbb{H}} - H^{\bullet})\big) \cup H^{\bullet} \\ &= (Q \cup H^{\bullet}) \cap ((\top^{\mathbb{H}} - H^{\bullet}) \cup H^{\bullet}) \\ &= Q \cap \big(\top^{\mathbb{H}} - (H^{\bullet} - H^{\bullet})\big) \\ &= Q \cap \top^{\mathbb{H}} \\ &= Q \end{aligned}$$

■

Lemma 3.3.4 *Sets in the Boolean UCLA lattice, excluding those produced by modal operators, are wn-pointed.*

Proof: This a consequence of Lemmas 3.2.7, 3.3.2, and 3.3.3. ■

Lemma 3.3.5

$$x \notin\mathsf{h} \overset{\bullet\circ}{\neg} Q \text{ iff } (x \in\mathsf{h} H^{\circ} \text{ or } x \in\mathsf{h} Q) \text{ and } x \notin\mathsf{h} H^{\bullet}.$$

Proof:

$$\begin{aligned} x \in\mathsf{h} \overset{\bullet\circ}{\neg} Q &\text{ iff } x \in\mathsf{h} (\top^{\mathbb{H}} - Q) \cup H^{\bullet} \\ &\text{ iff } x \in\mathsf{h} ((H - H^{\circ}) - Q) \cup H^{\bullet} \\ &\text{ iff } x \in\mathsf{h} (H - H^{\circ}) - Q \text{ or } x \in\mathsf{h} H^{\bullet} \\ &\text{ iff } (x \in\mathsf{h} H - H^{\circ} \text{ and } x \notin\mathsf{h} Q) \text{ or } x \in\mathsf{h} H^{\bullet} \\ &\text{ iff } (x \notin\mathsf{h} H^{\circ} \text{ and } x \notin\mathsf{h} Q) \text{ or } x \in\mathsf{h} H^{\bullet} \\ x \notin\mathsf{h} \overset{\bullet\circ}{\neg} Q &\text{ iff } (x \in\mathsf{h} H^{\circ} \text{ or } x \in\mathsf{h} Q) \text{ and } x \notin\mathsf{h} H^{\bullet} \end{aligned}$$

■

3.4 Smash Product

Definition 3.4.1

$$\langle x, y\rangle \in\mathsf{h} H_1 \otimes H_2 \text{ iff } \langle x, y\rangle \in\mathsf{h} (H_1 - H_1^{\bullet}) \times (H_2 - H_2^{\bullet}) \cup H_{\otimes}^{\bullet}.$$

and for Q_i well- and null-pointed,

$$\langle x, y\rangle \in\mathsf{h} Q_1 \otimes Q_2 \text{ iff } \langle x, y\rangle \in\mathsf{h} (Q_1 - H_1^{\bullet}) \times (Q_2 - H_2^{\bullet}) \cup H_{\otimes}^{\bullet}.$$

$H_{\otimes}^{\circ} \cap (Q_1 \otimes Q_2) = \emptyset$, this latter because if Q_1 and Q_2 are wn-pointed, then $Q_1 \cap H_1^{\circ} = \emptyset$ and $Q_2 \cap H_2^{\circ} = \emptyset$. Hence $Q_1 \otimes Q_2$ is wn-pointed.

The ambient space for $Q_1 \otimes Q_2$ is $H_1 \otimes H_2$ and not $H_1 \times H_2$. The former is generated from the latter by dividing out by the relation $(H_1^{\bullet} \times H_2) \cup (H_1 \times H_2^{\bullet})$, i.e.,

$$H_1 \otimes H_2 = (H_1 \times H_2)/(H_1^{\bullet} \times H_2) \cup (H_1 \times H_2^{\bullet}).$$

At the lowest level, the relation amounts to

$$\langle x, \bullet_{H_2}\rangle \equiv \langle \bullet_{H_1}, \bullet_{H_2}\rangle \equiv \langle \bullet_{H_1}, y\rangle.$$

The following Lemmas are a necessary prelude to the closure properties for smash product.

Lemma 3.4.2

$$Q_1 \otimes Q_2 \subseteq Q_1 \times Q_2.$$

Proof: If $\langle x, y\rangle \in_{h} Q_1 \otimes Q_2$ then either $\langle x, y\rangle \in_{h} (Q_1 - H_1^{\bullet}) \times (Q_2 - H_2^{\bullet})$ or $\langle x, y\rangle \in_{h} H_1^{\bullet} \times H_2^{\bullet}$. If the latter, then since Q_1 and Q_2 are well-pointed, $x \in_{h_1} H_1^{\bullet}$ and $y \in_{h_2} H_2^{\bullet}$. In this case, $\langle x, y\rangle \in_{h} Q_1 \times Q_2$. If the former, then $x \in_{h_1} Q_1$ and $y \in_{h_2} Q_2$ and hence $\langle x, y\rangle \in_{h} Q_1 \times Q_2$ ■

The following Lemma points out that using $\top^{\mathbb{H}}$ rather that H cleans up the nasty business of dealing with H°.

Lemma 3.4.3 *Assume $x \in_{h_1} Q_1$ and $y \in_{h_2} Q_2$, then*

- *$x \in_{h_1} H_1^{\circ}$ implies $\langle x, y\rangle \notin_{h} Q_1 \otimes Q_2$,*
- *$y \in_{h_2} H_2^{\circ}$ implies $\langle x, y\rangle \notin_{h} Q_1 \otimes Q_2$.*

Proof: Let $x \in_{h_1} H_1^{\circ}$. Towards a reductio, assume $\langle x, y\rangle \in_{h} Q_1 \otimes Q_2$. From Lemma 3.4.2, $\langle x, y\rangle \in_{h} Q_1 \times Q_2$. Since Q_1 is null pointed, $\langle x, y\rangle \notin_{h} Q_1 \times Q_2$. This is a contradiction and $\langle x, y\rangle \notin_{h} Q_1 \otimes Q_2$. The other statement is similar.

■

Corollary 3.4.4

$$\langle x, y\rangle \in_{h} Q_1 \otimes Q_2 \text{ implies } x \notin_{h_1} H_1^{\circ} \text{ and } y \notin_{h_2} H_2^{\circ}.$$

Lemma 3.4.5

$$\langle x, y\rangle \in_{h} Q_1 \otimes Q_2 \text{ implies } (x_1 \notin_{h_1} H_1^{\bullet} \text{ and } x_2 \notin_{h_2} H_2^{\bullet}) \text{ or } (x_1 \in_{h_1} H_1^{\bullet} \text{ and } x_2 \in_{h_2} H_2^{\bullet})$$

Proof: Assume $\langle x, y\rangle \in_{h} Q_1 \otimes Q_2$, and let $(x_1 \in_{h_1} H_1^{\bullet}$ or $x_2 \in_{h_2} H_2^{\bullet})$. If $x_1 \in_{h_1} H_1^{\bullet}$ or $x_2 \in_{h_1} H_2^{\bullet}$, then by the definition of $Q_1 \otimes Q_2$, it must be that both $x_1 \in_{h_1} H_1^{\bullet}$ and $x_2 \in_{h_2} H_2^{\bullet}$. From classical logic, the result follows.

■

Lemma 3.4.6 *Let* $x \in_{\mathsf{h}_1} H_1$ *and* $y \in_{\mathsf{h}_2} H_2$*, then*

$$x \notin_{\mathsf{h}_1} Q_1 \text{ implies } \langle x, y\rangle \notin_{\mathsf{h}} Q_1 \otimes Q_2 \qquad y \notin_{\mathsf{h}_2} Q_2 \text{ implies } \langle x, y\rangle \notin_{\mathsf{h}} Q_1 \otimes Q_2$$

Proof: Let $x \in_{\mathsf{h}_1} H_1$ and $y \in_{\mathsf{h}_2} H_2$, and assume $x \notin_{\mathsf{h}_1} Q_1$. Hence $\langle x, y\rangle \notin_{\mathsf{h}} Q_1 \times Q_2$. From Lemma 3.4.2, $\langle x, y\rangle \notin_{\mathsf{h}} Q_1 \otimes Q_2$. The other statement is similar.

■

Lemma 3.4.7 *If* Q_1 *and* Q_2 *are well-pointed, then*

$$\langle x, y\rangle \in_{\mathsf{h}} (H_1 \otimes H_2) - (Q_1 \otimes Q_2) \text{ implies } x \notin_{\mathsf{h}_1} H_1^\bullet \text{ and } y \notin_{\mathsf{h}_2} H_2^\bullet.$$

Note, we have $(H_1 \otimes H_2) - (Q_1 \otimes Q_2)$*, not* $(H_1 \otimes H_2) \dot{-} (Q_1 \otimes Q_2)$*.*

Proof: Assume $\langle x, y\rangle \in_{\mathsf{h}} (H_1 \otimes H_2) - (Q_1 \otimes Q_2)$, therefore $\langle x, y\rangle \in_{\mathsf{h}} H_1 \otimes H_2$ and $\langle x, y\rangle \notin_{\mathsf{h}} Q_1 \otimes Q_2$. Note that $H_\otimes^\bullet \subseteq_{\mathsf{h}} Q_1 \otimes Q_2$ by definition, hence $H_\otimes^\bullet \cap \big((H_1 \otimes H_2) - (Q_1 \otimes Q_2)\big) = \emptyset$. Therefore towards a reductio, if $x \in_{\mathsf{h}_1} H_1^\bullet$, then from Definition 3.4.1 (smash product), we have $\langle x, y\rangle \in_{\mathsf{h}} H_\otimes^\bullet$, whence $\langle x, y\rangle \in_{\mathsf{h}} Q_1 \otimes Q_2$ since $Q_1 \otimes Q_2$ is well-pointed. This is a contradiction. The case for $y \in_{\mathsf{h}_2} H_2^\bullet$ is similar.

■

The following Lemma is a bit technical and used for shortening proofs involving a Galois operator distributing over smash product.

Lemma 3.4.8 *Let* Q_1, Q_2 *be well-pointed and* $Q_1 \times Q_2 \subseteq_{\mathsf{h}} H_1 \otimes H_2$*, then*

$$(x \in_{\mathsf{h}_1} Q_1 - H_1^\bullet \text{ or } x \in_{\mathsf{h}_1} H_1^\bullet) \text{ and } (y \in_{\mathsf{h}_2} Q_2 - H_2^\bullet \text{ or } y \in_{\mathsf{h}_2} H_2^\bullet)$$

implies

$$\langle x, y\rangle \in_{\mathsf{h}} Q_1 \otimes Q_2$$

Proof:

1 $x \in\mathsf{h}_1\, U_1 - H_1^\bullet$ or $a_1 \in\mathsf{h}_1\, H_1^\bullet$ . . . . assume

2 $y \in\mathsf{h}_2\, U_2 - H_2^\bullet$ or $a_2 \in\mathsf{h}_2\, H_2^\bullet$ . . . . assume

3 $(x \in\mathsf{h}_1\, U - H_1^\bullet$ and $y \in\mathsf{h}_2\, U_2 - H_2^\bullet)$ or . . . . CL, lines 2, 3
$(x \in\mathsf{h}_1\, U_1 - H_1^\bullet$ and $y \in\mathsf{h}_2\, H_2^\bullet)$ or
$(y \in\mathsf{h}_2\, U_2 - H_2^\bullet$ and $x \in\mathsf{h}_1\, H_1^\bullet)$ or
$(x \in\mathsf{h}_1\, H_1^\bullet$ and $y \in\mathsf{h}_2\, H_2^\bullet)$

> 4 $x \in\mathsf{h}_1\, U_1 - H_1^\bullet$ and $y \in\mathsf{h}_2\, H_2^\bullet$ . . . . assume
>
> 5 $x \notin\mathsf{h}_1\, H_1^\bullet$ and $y \in\mathsf{h}_2\, H_2^\bullet$ . . . . set th., line 4
>
> 6 $\rightarrow\leftarrow$ . . . . Contradiction, $\langle x, y\rangle \in\mathsf{h}\, H_1 \otimes H_2$, line 5

> 7 $y \in\mathsf{h}_2\, U_2 - H_2^\bullet$ and $x \in\mathsf{h}_1\, H_1^\bullet$ . . . . assume
>
> 8 $y \notin\mathsf{h}_2\, H_2^\bullet$ and $x \in\mathsf{h}_1\, H_1^\bullet$ . . . . set th., line 7
>
> 9 $\rightarrow\leftarrow$ . . . . Contradiction, $\langle x, y\rangle \in\mathsf{h}\, H_1 \otimes H_2$, line 8

10 $\big(x \in\mathsf{h}_1\, U_1 - H_1^\bullet$ and $y \in\mathsf{h}_2\, U_2 - H_2^\bullet\big)$ or . . . . CL, lines 3, 4, 7
$\langle x, y\rangle \in\mathsf{h}\, H_1^\bullet \times H_2^\bullet$

11 $\big(x \in\mathsf{h}_1\, U_1 - H_1^\bullet$ and $y \in\mathsf{h}_2\, U_2 - H_2^\bullet\big)$ or $\langle x, y\rangle \in\mathsf{h}\, H_\otimes^\bullet$ . . . . def. $H_\otimes^\bullet$, line 10

12 $\langle x, y\rangle \in\mathsf{h}\, \big((U_1 - H_1^\bullet) \times (U_2 - H_2^\bullet)\big) \cup H_\otimes^\bullet$ . . . . set th., line 11

13 $\langle x, y\rangle \in\mathsf{h}\, U_1 \otimes U_2$ . . . . def. $\otimes$, line 12

■

Lemma 3.4.9

$$(\langle x, y\rangle \notin\mathsf{h}\, H_\otimes^\bullet \text{ and } \langle x, y\rangle \in\mathsf{h}\, H_1 \otimes H_2) \text{ implies } (x \notin\mathsf{h}_1\, H_1^\bullet \text{ and } y \notin\mathsf{h}_2\, H_2^\bullet).$$

Proof: Assume $\langle x, y\rangle \notin\mathsf{h}\, H_\otimes^\bullet$ and $\langle x, y\rangle \in\mathsf{h}\, H_1 \otimes H_2$. Let $x \in\mathsf{h}_1\, H_1^\bullet$. From the definition of $H_1 \otimes H_2$, then $y \in\mathsf{h}_2\, H_2^\bullet$ and hence $\langle x, y\rangle \in\mathsf{h}\, H_\otimes^\bullet$. This is a contradiction, hence $x \notin\mathsf{h}_1\, H_1^\bullet$. Letting $y \in\mathsf{h}_2\, H_2^\bullet$ is similar, whence $x \notin\mathsf{h}_1\, H_1^\bullet$ and $y \notin\mathsf{h}_2\, H_2^\bullet$.

■

3.5 Null Sum

Definition 3.5.1

$$\langle x, y\rangle \in\mathsf{h}\, H_1 \ovee H_2 \text{ iff } \langle x, y\rangle \in\mathsf{h}\, (H_1 \times H_2^\circ) \cup (H_1^\circ \times H_2).$$

We also have for Q_i well- and null-pointed,

$$\langle x, y\rangle \in\mathsf{h}\, Q_1 \ovee Q_2 \text{ iff } \langle x, y\rangle \in\mathsf{h}\, (Q_1 \times H_2^\circ) \cup (H_1^\circ \times Q_2).$$

Note that the definition does not say

$$\langle x, y\rangle \in_{\mathsf{h}} H_1 \circledvee H_2 \text{ iff } \langle x, y\rangle \in_{\mathsf{h}} ((H_1 - H_1^{\circ}) \times H_2^{\circ}) \cup (H_1^{\circ} \times (H_2 - H_2^{\circ})).$$

Since $\top_1^{\mathbb{H}} = H_1 - H_1^{\circ}$ and $\top_2^{\mathbb{H}} = H_2 - H_2^{\circ}$, this would define

$$\top_{\circledvee}^{\mathbb{H}} = \top_1^{\mathbb{H}} \circledvee \top_2^{\mathbb{H}}.$$

Since Q_1 and Q_2 are wn-pointed, $H_1^{\bullet} \times H_2^{\circ} \subseteq (Q_1 \times H_2^{\circ})$. Similarly, $H_1^{\circ} \times H_2^{\bullet} \subseteq H_1^{\circ} \times Q_2$. Also $(H_1^{\circ} \times H_2^{\circ}) \cap (Q_1 \circledvee Q_2) = \emptyset$ and hence $Q_1 \circledvee Q_2$ is wn-pointed.

When computing $H - (Q_1 \otimes Q_2)$ or $H - (Q_1 \circledvee Q_2)$, it is always assumed that $H = H_1 \otimes H_2$ and $H = H_1 \circledvee H_2$ respectively.

This representation resulted from the problems of getting $[f^{\star}\rangle$ to work well with lattice bounds and have closure under some kind of product representation. It is tempting to let the universe for UCLA propositions at location h be $\top^{\mathbb{H}}$ and evaluating

$$x \notin_{\mathsf{h}} Q \text{ iff } x \in_{\mathsf{h}} \top^{\mathbb{H}} - Q.$$

The assumption is that x is in the ambient universe H. The condition for the operator $[f^{\star}\rangle$ is

$$x \in_{\mathsf{h}} [f^{\star}\rangle Q \text{ iff } \exists y (y \notin_{\mathsf{k}} Q \text{ and } \neg \mathcal{F}^{\star} x y).$$

Without the null points, we could not now rely on a point not being in $\top^{\mathbb{K}}$ for $x \in_{\mathsf{h}} [f^{\star}\rangle \top^{\mathbb{K}}$. This becomes a problem when $Q = \top^{\mathbb{K}}$ because then $y \in_{\mathsf{k}} K - K^{\circ}$ and that $y \notin_{\mathsf{k}} \top^{\mathbb{K}}$, yet $K - K^{\circ} = \top^{\mathbb{K}}$. The clear implication is that the universe must contain K°.

Lemma 3.5.2

$$H_{\circledvee}^{\circ} \cap (H_1 \circledvee H_2) = H_{\circledvee}^{\circ} \qquad H_{\circledvee}^{\circ} \cap (Q_1 \circledvee Q_2) = \emptyset$$

Proof: These following directly from the definitions for $\circledvee$ since $H_{\circledvee}^{\circ} = (H_1 \circledvee H_2)^{\circ} = H_1^{\circ} \times H_2^{\circ}$ and $Q_1 \circledvee Q_2$ is null-pointed. This latter can only happen if ${H_1}^{\circ} \cap Q_1 = \emptyset = {H_2}^{\circ} \cap Q_2$. ■

Lemma 3.5.3 *If* $\langle x, y\rangle \in_{\mathsf{h}} H_1 \circledvee H_2$*, then*

$$x \in_{\mathsf{h}_1} H_1^{\bullet} \text{ or } y \in_{\mathsf{h}_2} H_2^{\bullet} \text{ implies } \langle x, y\rangle \in_{\mathsf{h}} H_{\circledvee}^{\bullet}.$$

Proof: Assume the premise. Let $x \in_{\mathsf{h}_1} H_1^{\bullet}$. Since $\langle x, y\rangle \in_{\mathsf{h}} H_1 \circledvee H_2$, then $y \in_{\mathsf{h}_2} H_2^{\circ}$. Hence $\langle x, y\rangle \in_{\mathsf{h}} H_{\circledvee}^{\bullet}$. The case for $y \in_{\mathsf{h}_2} H_2^{\bullet}$ is similar.

Just as an acid test, we prove the contraposition. Assume the premise, i.e. $\langle x, y\rangle \in_{\mathsf{h}} H_1 \circledvee H_2$ and let $\langle x, y\rangle \notin_{\mathsf{h}} H^{\bullet}_{\circledvee}$, then by definition

$$\langle x, y\rangle \notin_{\mathsf{h}} (H_1^{\bullet} \times {H_2}^{\circ}) \cup (H_1^{\circ} \times H_2^{\bullet}).$$

So

$$\langle x, y\rangle \notin_{\mathsf{h}} H_1^{\bullet} \times {H_2}^{\circ} \text{ and } \langle x, y\rangle \notin_{\mathsf{h}} H_1^{\circ} \times H_2^{\bullet}.$$

Assume again the premise, i.e., $\langle x, y\rangle \in_{\mathsf{h}} H_1 \circledvee H_2$. From the first conjunct, either $x \notin_{\mathsf{h}_1} H_1^{\bullet}$ or $y \notin_{\mathsf{h}_2} H_2^{\circ}$. Suppose $x \in_{\mathsf{h}_1} H_1^{\bullet}$, then $y \notin_{\mathsf{h}_2} H_2^{\circ}$. However, then $\langle x, y\rangle \notin_{\mathsf{h}} H_1 \circledvee H_2$, which is a contradiction. So $x \notin_{\mathsf{h}_1} H_1^{\bullet}$. The case of $y \notin_{\mathsf{h}_2} H_2^{\bullet}$ is similar.

■

Lemma 3.5.4

$$\langle x, y\rangle \in_{\mathsf{h}} H_1 \circledvee H_2 - Q_1 \circledvee Q_2 \text{ implies } (x \notin_{\mathsf{h}_1} Q_1 \text{ and } y \in_{\mathsf{h}_2} H_2^{\circ}) \text{ or } (x \in_{\mathsf{h}_1} H_1^{\circ} \text{ and } y \notin_{\mathsf{h}_2} Q_2).$$

Proof: Assume $\langle x, y\rangle \in_{\mathsf{h}} H_1 \circledvee H_2 - Q_1 \circledvee Q_2$, then since $H_1 \circledvee H_2 = (H_1 \times H_2^{\circ}) \cup (H_1^{\circ} \times H_2)$,

$$\langle x, y\rangle \in_{\mathsf{h}} (H_1 \times H_2^{\circ}) \cup (H_1^{\circ} \times H_2).$$

Hence either $x \in_{\mathsf{h}_1} H_1^{\circ}$ or $y \in_{\mathsf{h}_2} H_2^{\circ}$. Let $x \in_{\mathsf{h}_1} H_1^{\circ}$. If $y \in_{\mathsf{h}_2} Q_2$, then $\langle x, y\rangle \in_{\mathsf{h}} Q_1 \circledvee Q_2$. This is a contradiction, so $y \notin_{\mathsf{h}_2} Q_2$. Letting $y \in_{\mathsf{h}_2} H_2^{\circ}$ is similar resulting in $x \notin_{\mathsf{h}_1} Q_1$.

■

Lemma 3.5.5

$$\top^{\mathbb{H}}_{\circledvee} - (Q_1 \circledvee Q_2) = (\top^{\mathbb{H}}_1 - Q_1) \circledvee (\top^{\mathbb{H}}_2 - Q_2).$$

Proof:

1 $\langle b_1, b_2\rangle \in_{\mathsf{h}} \top^{\mathbb{H}}_{\circledvee} - (Q_1 \circledvee Q_2)$ . . . . assume

2 $\langle b_1, b_2\rangle \in_{\mathsf{h}} \top^{\mathbb{H}}_{\circledvee}$ and $\langle b_1, b_2\rangle \notin_{\mathsf{h}} Q_1 \circledvee Q_2$ . . . . set th., line 1

3 $\langle b_1, b_2\rangle \in_{\mathsf{h}} \top^{\mathbb{H}}_1 \times H^{\circ}_2$ or $\langle b_1, b_2\rangle \in_{\mathsf{h}} H^{\circ}_1 \times \top^{\mathbb{H}}_2$ . . . . def. $\top^{\mathbb{H}}_{\circledvee}$, line 2

4 $\langle b_1, b_2\rangle \in_{\mathsf{h}} \top^{\mathbb{H}}_1 \times H^{\circ}_2$ . . . . assume

5 $b_1 \in_{\mathsf{h}_1} \top^{\mathbb{H}}_1$ and $b_2 \in_{\mathsf{h}_2} H^{\circ}_2$ . . . . def. ×, line 4

6 $b_1 \in_{\mathsf{h}_1} Q_1$ . . . . assume

7 $\langle b_1, b_2\rangle \in_{\mathsf{h}} Q_1 \circledvee Q_2$ . . . . def. $\circledvee$, lines 5, 6

8 $\rightarrow\leftarrow$ . . . . Contradiction, lines 2, 7

9 $b_1 \notin_{\mathsf{h}_1} Q_1$ . . . . CL, line 6

10 $b_1 \in_{\mathsf{h}_1} \top^{\mathbb{H}}_1 - Q_1$ . . . . set th., lines 5, 9

11 $\langle b_1, b_2\rangle \in_{\mathsf{h}} (\top^{\mathbb{H}}_1 - Q_1) \times H^{\circ}_2$ . . . . def. ×, lines 5, 10

12 $\langle b_1, b_2\rangle \in_{\mathsf{h}} (\top^{\mathbb{H}}_1 - Q_1) \circledvee (\top^{\mathbb{H}}_2 - Q_2)$ . . . . def. $\circledvee$, line 11

13 $\langle b_1, b_2\rangle \in_{\mathsf{h}} H^{\circ}_1 \times \top^{\mathbb{H}}_2$ . . . . assume

14 subproof is similar to the previous subproof . . . . line xx

15 $\langle b_1, b_2\rangle \in_{\mathsf{h}} (\top^{\mathbb{H}}_1 - Q_1) \circledvee (\top^{\mathbb{H}}_2 - Q_2)$ . . . . line xx

16 $\langle b_1, b_2\rangle \in_{\mathsf{h}} (\top^{\mathbb{H}}_1 - Q_1) \circledvee (\top^{\mathbb{H}}_2 - Q_2)$ . . . . ∨-Elim, lines 3, 4, 13

and

1 $\langle b_1, b_2\rangle \in_{\mathsf{h}} (\top^{\mathbb{H}}_1 - Q_1) \circledvee (\top^{\mathbb{H}}_2 - Q_2)$ . . . . assume

2 $\langle b_1, b_2\rangle \in_{\mathsf{h}} (\top^{\mathbb{H}}_1 - Q_1) \times H^{\circ}_2$ or $\langle b_1, b_2\rangle \in_{\mathsf{h}} H^{\circ}_1 \times (\top^{\mathbb{H}}_2 - Q_2)$ . . . . set th., line 2

3 $\langle b_1, b_2\rangle \in_{\mathsf{h}} (\top^{\mathbb{H}}_1 - Q_1) \times H^{\circ}_2$ . . . . assume

4 $b_1 \in_{\mathsf{h}_1} \top^{\mathbb{H}}_1 - Q_1$ and $b_2 \in_{\mathsf{h}_2} H^{\circ}_2$ . . . . def. ×, line 3

5 $b_1 \in_{\mathsf{h}_1} \top^{\mathbb{H}}_1$ and $b_1 \notin_{\mathsf{h}_1} Q_1$ . . . . set th., line 4

6 $\langle b_1, b_2\rangle \in_{\mathsf{h}} \top^{\mathbb{H}}_1 \times H^{\circ}_2$ . . . . def. ×, lines 4, 5

7 $\langle b_1, b_2\rangle \in_{\mathsf{h}} \top^{\mathbb{H}}_{\circledvee}$ . . . . def. $\top^{\mathbb{H}}_{\circledvee}$, line 6

8 $\langle b_1, b_2\rangle \notin_{\mathsf{h}} Q_1 \times H^{\circ}_2$ . . . . def. ×, lines 4, 5

9 $\langle b_1, b_2\rangle \notin_{\mathsf{h}} H^{\circ}_1 \times Q_2$ . . . . $\top^{\mathbb{H}}_1$ is null-pointed, line 5

10 $\langle b_1, b_2\rangle \notin_{\mathsf{h}} Q_1 \circledvee Q_2$ . . . . def. $\circledvee$, lines 8, 9

11 $\langle b_1, b_2\rangle \in_{\mathsf{h}} \top^{\mathbb{H}}_{\circledvee} - (Q_1 \circledvee Q_2)$ . . . . set th., lines 7, 10

12 $\langle b_1, b_2\rangle \in_{\mathsf{h}} H^{\circ}_1 \times (\top^{\mathbb{H}}_2 - Q_2)$ . . . . assume

13 subproof is similar to the previous subproof . . . . lines xx

14 $\langle b_1, b_2\rangle \in_{\mathsf{h}} \top^{\mathbb{H}}_{\circledvee} - Q_1 \circledvee Q_2$ . . . . lines xx

15 $\langle b_1, b_2\rangle \in_{\mathsf{h}} \top^{\mathbb{H}}_{\circledvee} - (Q_1 \circledvee Q_2)$ . . . . ∨-Elim, lines 2, 3, 12

The proof in prose is as follows:

Let $\langle b_1, b_2\rangle \in_{\mathsf{h}} \big(\top^{\mathbb{H}}_{\circledvee} - (Q_1 \circledvee Q_2)\big)$. Hence $\langle b_1, b_2\rangle \in_{\mathsf{h}} \top^{\mathbb{H}}_{\circledvee}$ and $\langle b_1, b_2\rangle \notin_{\mathsf{h}} Q_1 \circledvee Q_2$. From the representation of $\top^{\mathbb{H}}_{\circledvee}$, either $\langle b_1, b_2\rangle \in_{\mathsf{h}} \top^{\mathbb{H}}_1 \times H^{\circ}_2$ or $\langle b_1, b_2\rangle \in_{\mathsf{h}} H^{\circ}_1 \times \top^{\mathbb{H}}_2$. Choose $\langle b_1, b_2\rangle \in_{\mathsf{h}} \top^{\mathbb{H}}_1 \times H^{\circ}_2$, thus $b_1 \in_{\mathsf{h}_1} \top^{\mathbb{H}}_1$. If $b_1 \in_{\mathsf{h}_1} Q_1$, then $\langle b_1, b_2\rangle \in_{\mathsf{h}} Q_1 \circledvee Q_2$ which is a contradiction. So $b_1 \notin_{\mathsf{h}} Q_1$, and $b_1 \in_{\mathsf{h}_1} \top^{\mathbb{H}}_1 - Q_1$ and $\langle b_1, b_2\rangle \in_{\mathsf{h}} (\top^{\mathbb{H}}_1 - Q_1) \circledvee (\top^{\mathbb{H}}_2 - Q_2)$. Choosing $\langle b_1, b_2\rangle \in_{\mathsf{h}} H^{\circ}_1 \times \top^{\mathbb{H}}_2$ is similar resulting in $\langle b_1, b_2\rangle \in_{\mathsf{h}} (\top^{\mathbb{H}}_1 - Q_1) \circledvee (\top^{\mathbb{H}}_2 - Q_2)$. Therefore

$$\top^{\mathbb{H}}_{\circledvee} - (Q_1 \circledvee Q_2) \subseteq_{\mathsf{h}} (\top^{\mathbb{H}}_1 - Q_1) \circledvee (\top^{\mathbb{H}}_2 - Q_2).$$

Next, let $\langle b_1, b_2\rangle \in_{\mathsf{h}} (\top^{\mathbb{H}}_1 - Q_1) \circledvee (\top^{\mathbb{H}}_2 - Q_2)$, then either $\langle b_1, b_2\rangle \in_{\mathsf{h}} (\top^{\mathbb{H}}_1 - Q_1) \times H^{\circ}_2$ or $\langle b_1, b_2\rangle \in_{\mathsf{h}} H^{\circ}_1 \times (\top^{\mathbb{H}}_2 - Q_2)$. Choose $\langle b_1, b_2\rangle \in_{\mathsf{h}} (\top^{\mathbb{H}}_1 - Q_1) \times H^{\circ}_2$. Hence $b_1 \in_{\mathsf{h}_1} \top^{\mathbb{H}}_1 - Q_1$ and $b_2 \in_{\mathsf{h}_2} H^{\circ}_2$, resulting in $b_1 \in_{\mathsf{h}_1} \top^{\mathbb{H}}_1$ and $b_1 \notin_{\mathsf{h}_1} Q_1$. So $b_1 \notin_{\mathsf{h}_1} Q_1$ and $b_1 \notin_{\mathsf{h}_1} H^{\circ}_1$. Thus $\langle b_1, b_2\rangle \in_{\mathsf{h}} \top^{\mathbb{H}}_1 \times H^{\circ}_2$ and $\langle b_1, b_2\rangle \notin_{\mathsf{h}} Q_1 \circledvee Q_2$. So $\langle b_1, b_2\rangle \in_{\mathsf{h}} \top^{\mathbb{H}}_{\circledvee} - (Q_1 \circledvee Q_2)$. Choosing $\langle b_1, b_2\rangle \in_{\mathsf{h}} H^{\circ}_1 \times (H_2 - (Q_2 \cup H^{\circ}_2))$ is similar. Therefore

$$(\top^{\mathbb{H}}_1 - Q_1) \circledvee (\top^{\mathbb{H}}_2 - Q_2) \subseteq_{\mathsf{h}} \top^{\mathbb{H}}_{\circledvee} - (Q_1 \circledvee Q_2).$$

■

Note that the above Lemma has no need to account for well-points. Hence the set complement of that Lemma is inapplicable for our purposes. The negation $\overset{\bullet\circ}{\neg}$ does account for well-points. By definition,

$$\overset{\bullet\circ}{\neg}(Q_1 \circledvee Q_2) \overset{def}{=} \top^{\mathbb{H}}_{\circledvee} \overset{\bullet\circ}{-} (Q_1 \circledvee Q_2) = \big(\top^{\mathbb{H}}_{\circledvee} - (Q_1 \circledvee Q_2)\big) \cup H^{\bullet}_{\circledvee}.$$

The $\overset{\bullet\circ}{-}$ has the nice property of distributing across $\circledvee$:

Lemma 3.5.6

$$\top^{\mathbb{H}}_{\circledvee} \overset{\bullet\circ}{-} (Q_1 \circledvee Q_2) = (\top^{\mathbb{H}}_1 \overset{\bullet\circ}{-} Q_1) \circledvee (\top^{\mathbb{H}}_2 \overset{\bullet\circ}{-} Q_2).$$

Proof:

1	$\langle b_1, b_2\rangle \in_{\mathsf{h}} \top^{\mathbb{H}}_{\ovee} \overset{\bullet\circ}{-} (Q_1 \ovee Q_2)$	assume
2	$\langle b_1, b_2\rangle \in_{\mathsf{h}} \left(\top^{\mathbb{H}}_{\ovee} - (Q_1 \ovee Q_2)\right) \cup H^{\bullet}_{\ovee}$	def. $\overset{\bullet\circ}{-}$, line 1
3	$\langle b_1, b_2\rangle \in_{\mathsf{h}} \top^{\mathbb{H}}_{\ovee} - (Q_1 \ovee Q_2)$ or $\langle b_1, b_2\rangle \in_{\mathsf{h}} H^{\bullet}_{\ovee}$	set th., line 2
4	$\langle b_1, b_2\rangle \in_{\mathsf{h}} \top^{\mathbb{H}}_{\ovee} - (Q_1 \ovee Q_2)$	assume
5	$\langle b_1, b_2\rangle \in_{\mathsf{h}} (\top^{\mathbb{H}}_1 - Q_1) \ovee (\top^{\mathbb{H}}_2 - Q_2)$	Lemma 3.5.5, line 4
6	$\langle b_1, b_2\rangle \in_{\mathsf{h}} (\top^{\mathbb{H}}_1 - Q_1) \times H^{\circ}_2$ or $\langle b_1, b_2\rangle \in_{\mathsf{h}} H^{\circ}_1 \times (\top^{\mathbb{H}}_2 - Q_2)$	def. $\ovee$, line 5
7	$\langle b_1, b_2\rangle \in_{\mathsf{h}} ((\top^{\mathbb{H}}_1 - Q_1) \cup H^{\bullet}_1) \times H^{\circ}_2$ or $\langle b_1, b_2\rangle \in_{\mathsf{h}} H^{\circ}_1 \times ((\top^{\mathbb{H}}_2 - Q_2) \cup H^{\bullet}_2)$	set th., line 6
8	$\langle b_1, b_2\rangle \in_{\mathsf{h}} (\top^{\mathbb{H}}_1 \overset{\bullet\circ}{-} Q_1) \times H^{\circ}_2$ or $\langle b_1, b_2\rangle \in_{\mathsf{h}} H^{\circ}_1 \times (\top^{\mathbb{H}}_2 \overset{\bullet\circ}{-} Q_2)$	def. $\overset{\bullet\circ}{\neg}$, line 7
9	$\langle b_1, b_2\rangle \in_{\mathsf{h}} H^{\bullet}_{\ovee}$	assume
10	$\langle b_1, b_2\rangle \in_{\mathsf{h}} (H^{\bullet}_1 \times H^{\circ}_2) \cup (H^{\circ}_1 \times H^{\bullet}_2)$	Definition 3.2.2 for $H^{\bullet}_{\ovee}$, line 9
11	$\langle b_1, b_2\rangle \in_{\mathsf{h}} (H^{\bullet}_1 \times H^{\circ}_2)$ or $\langle b_1, b_2\rangle \in_{\mathsf{h}} (H^{\circ}_1 \times H^{\bullet}_2)$	set th., line 10
12	$\langle b_1, b_2\rangle \in_{\mathsf{h}} H^{\bullet}_1 \times H^{\circ}_2$	assume
13	$b_1 \in_{\mathsf{h}_1} H^{\bullet}_1$ and $b_2 \in_{\mathsf{h}_2} H^{\circ}_2$	def. $\times$, line 12
14	$b_1 \in_{\mathsf{h}_1} (\top^{\mathbb{H}}_1 - Q_1) \cup H^{\bullet}_1$	set th., line 13
15	$b_1 \in_{\mathsf{h}_1} \top^{\mathbb{H}}_1 \overset{\bullet\circ}{-} Q_1$	def. $\overset{\bullet\circ}{\neg}$, line 14
16	$\langle b_1, b_2\rangle \in_{\mathsf{h}} (\top^{\mathbb{H}}_1 \overset{\bullet\circ}{-} Q_1) \times H^{\circ}_2$	def. $\times$, lines 13, 15
17	$\langle b_1, b_2\rangle \in_{\mathsf{h}} (\top^{\mathbb{H}}_1 \overset{\bullet\circ}{-} Q_1) \times H^{\circ}_2$ or $\langle b_1, b_2\rangle \in_{\mathsf{h}} H^{\circ}_1 \times (\top^{\mathbb{H}}_2 \overset{\bullet\circ}{-} Q_2)$	$\vee$-Intro, line 16
18	$\langle b_1, b_2\rangle \in_{\mathsf{h}} H^{\circ}_1 \times H^{\bullet}_2$	assume
19	$b_1 \in_{\mathsf{h}_1} H^{\circ}_1$ and $b_2 \in_{\mathsf{h}_2} H^{\bullet}_2$	def. $\times$, line 18
20	$b_2 \in_{\mathsf{h}_2} (\top^{\mathbb{H}}_2 - Q_2) \cup H^{\bullet}_2$	set th., line 19
21	$b_2 \in_{\mathsf{h}_2} \top^{\mathbb{H}}_2 \overset{\bullet\circ}{-} Q_2$	def. $\overset{\bullet\circ}{\neg}$, line 20
22	$\langle b_1, b_2\rangle \in_{\mathsf{h}} H^{\circ}_1 \times (\top^{\mathbb{H}}_2 \overset{\bullet\circ}{-} Q_2)$	def. $\times$, lines 19, 21
23	$\langle b_1, b_2\rangle \in_{\mathsf{h}} (\top^{\mathbb{H}}_1 \overset{\bullet\circ}{-} Q_1) \times H^{\circ}_2$ or $\langle b_1, b_2\rangle \in_{\mathsf{h}} H^{\circ}_1 \times (\top^{\mathbb{H}}_2 \overset{\bullet\circ}{-} Q_2)$	$\vee$-Intro, line 22
24	$\langle b_1, b_2\rangle \in_{\mathsf{h}} (\top^{\mathbb{H}}_1 \overset{\bullet\circ}{-} Q_1) \times H^{\circ}_2$ or $\langle b_1, b_2\rangle \in_{\mathsf{h}} H^{\circ}_1 \times (\top^{\mathbb{H}}_2 \overset{\bullet\circ}{-} Q_2)$	$\vee$-Elim, lines 11, 12, 18
25	$\langle b_1, b_2\rangle \in_{\mathsf{h}} (\top^{\mathbb{H}}_1 \overset{\bullet\circ}{-} Q_1) \times H^{\circ}_2$ or $\langle b_1, b_2\rangle \in_{\mathsf{h}} H^{\circ}_1 \times (\top^{\mathbb{H}}_2 \overset{\bullet\circ}{-} Q_2)$	$\vee$-Elim, lines 3, 4, 9
26	$\langle b_1, b_2\rangle \in_{\mathsf{h}} (\top^{\mathbb{H}}_1 \overset{\bullet\circ}{-} Q_1) \ovee (\top^{\mathbb{H}}_2 \overset{\bullet\circ}{-} Q_2)$	def. $\ovee$, line 25

and

1 $\langle b_1, b_2\rangle \in_{\mathsf{h}} (\top^{\mathbb{H}}_1 \overset{\bullet\circ}{-} Q_1) \ovee (\top^{\mathbb{H}}_2 \overset{\bullet\circ}{-} Q_2)$ . . . . assume
2 $\langle b_1, b_2\rangle \in_{\mathsf{h}} (\top^{\mathbb{H}}_1 \overset{\bullet\circ}{-} Q_1) \times H^{\circ}_2$ or $\langle b_1, b_2\rangle \in_{\mathsf{h}} H^{\circ}_1 \times (\top^{\mathbb{H}}_2 \overset{\bullet\circ}{-} Q_2)$ . . . . def. $\overset{\bullet\circ}{-}$, line 1
3 $\langle b_1, b_2\rangle \in_{\mathsf{h}} (\top^{\mathbb{H}}_1 \overset{\bullet\circ}{-} Q_1) \times H^{\circ}_2$ . . . . assume
4 $b_1 \in_{\mathsf{h}_1} \top^{\mathbb{H}}_1 \overset{\bullet\circ}{-} Q_1$ and $b_2 \in_{\mathsf{h}_2} H^{\circ}_2$ . . . . def. ×, line 3
5 $b_1 \in_{\mathsf{h}_1} (\top^{\mathbb{H}}_1 - Q_1) \cup H^{\bullet}_1$ . . . . def. $\overset{\bullet\circ}{-}$, line 4
6 $b_1 \in_{\mathsf{h}_1} \top^{\mathbb{H}}_1 - Q_1$ or $b_1 \in_{\mathsf{h}_1} H^{\bullet}_1$ . . . . set th., line 5
7 $b_1 \in_{\mathsf{h}_1} \top^{\mathbb{H}}_1 - Q_1$ . . . . assume
8 $b_1 \in_{\mathsf{h}_1} \top^{\mathbb{H}}_1$ and $b_1 \notin_{\mathsf{h}_1} Q_1$ . . . . set th., line 7
9 $\langle b_1, b_2\rangle \notin_{\mathsf{h}} Q_1 \times H^{\circ}_2$ . . . . def. ×, lines 4, 8
10 $b_1 \notin_{\mathsf{h}} H^{\circ}_1$ . . . . $\top^{\mathbb{H}}_1$ null-pointed, line 8
11 $\langle b_1, b_2\rangle \notin_{\mathsf{h}} H^{\circ}_1 \times Q_2$ . . . . def. ×, line 10
12 $\langle b_1, b_2\rangle \notin_{\mathsf{h}} Q_1 \ovee Q_2$ . . . . def. $\ovee$, lines 9, 11
13 $\langle b_1, b_2\rangle \in_{\mathsf{h}} \top^{\mathbb{H}}_1 \times H^{\circ}_2$ . . . . def. ×, lines 4, 8
14 $\langle b_1, b_2\rangle \in_{\mathsf{h}} (\top^{\mathbb{H}}_1 \times H^{\circ}_2) \cup (H^{\circ}_1 \times \top^{\mathbb{H}}_2)$ . . . . set th., line 13
15 $\langle b_1, b_2\rangle \in_{\mathsf{h}} \top^{\mathbb{H}}_{\ovee}$ . . . . def $\top^{\mathbb{H}}_{\ovee}$., line 14
16 $\langle b_1, b_2\rangle \in_{\mathsf{h}} \top^{\mathbb{H}}_{\ovee} - (Q_1 \ovee Q_2)$ . . . . set th., lines 12, 15
17 $\langle b_1, b_2\rangle \in_{\mathsf{h}} (\top^{\mathbb{H}}_{\ovee} - (Q_1 \ovee Q_2)) \cup H^{\bullet}_{\ovee}$ . . . . set th., line 16
18 $b_1 \in_{\mathsf{h}_1} H^{\bullet}_1$ . . . . assume
19 $\langle b_1, b_2\rangle \in_{\mathsf{h}} H^{\bullet}_1 \times H^{\circ}_2$ . . . . def. ×, lines 4, 18
20 $\langle b_1, b_2\rangle \in_{\mathsf{h}} (H^{\bullet}_1 \times H^{\circ}_2) \cup (H^{\circ}_1 \times H^{\bullet}_2)$ . . . . set th., line 19
21 $\langle b_1, b_2\rangle \in_{\mathsf{h}} H^{\bullet}_{\ovee}$ . . . . def. $H^{\bullet}_{\ovee}$, line 20
22 $\langle b_1, b_2\rangle \in_{\mathsf{h}} (\top^{\mathbb{H}}_{\ovee} - (Q_1 \ovee Q_2)) \cup H^{\bullet}_{\ovee}$ . . . . set th., line 21
23 $\langle b_1, b_2\rangle \in_{\mathsf{h}} (\top^{\mathbb{H}}_{\ovee} - (Q_1 \ovee Q_2)) \cup H^{\bullet}_{\ovee}$ . . . . ∨-Elim, lines 6, 7, 18
24 $\langle b_1, b_2\rangle \in_{\mathsf{h}} H^{\circ}_1 \times (\top^{\mathbb{H}}_2 \overset{\bullet\circ}{-} Q_2)$ . . . . assume
25 subproof is similar to previous subproof . . . . doh, lines xx
26 $\langle b_1, b_2\rangle \in_{\mathsf{h}} (\top^{\mathbb{H}}_{\ovee} - (Q_1 \ovee Q_2)) \cup H^{\bullet}_{\ovee}$ . . . . ∨-Elim, lines xx
27 $\langle b_1, b_2\rangle \in_{\mathsf{h}} (\top^{\mathbb{H}}_{\ovee} - (Q_1 \ovee Q_2)) \cup H^{\bullet}_{\ovee}$ . . . . ∨-Elim, lines 2, 3, 24
28 $\langle b_1, b_2\rangle \in_{\mathsf{h}} \top^{\mathbb{H}}_{\ovee} \overset{\bullet\circ}{-} (Q_1 \ovee Q_2)$ . . . . def. $\overset{\bullet\circ}{-}$, line 27

■

3.6 Infinite Null Sums

We need infinite null sums, elements of which we use brackets to denote, e.g., $\langle x\rangle$, which is a sequence of values in some ambient sets H_i, and we use x_i to denote the i-th term.

Definition 3.6.1 For $X_i \in \{H_i, Q_i\}$,

$$\varovee_i X_i = \bigcup_j \prod_i \begin{cases} X_j & i = j \\ H_i^\circ & \text{otherwise} \end{cases}$$

Let

$$\text{ıı}_j X_j = \prod_i \begin{cases} X_j & \text{if } i = j \\ H_i^\circ & \text{otherwise} \end{cases}$$

then we can rephrase the definition as

$$\varovee_i X_i = \bigcup_j \text{ıı}_j X_j .$$

Sometimes it is handy to use the element form:

$$\langle x \rangle \in\!\mathsf{h}\ \varovee_i X_i \text{ iff } \exists! j\ (\langle x \rangle \in\!\mathsf{h}\ \text{ıı}_j X_j) \text{ iff } \exists! j\ \forall i \left(x_i \in\!\mathsf{h}_i \begin{cases} X_j & i = j \\ H_i^\circ & \text{otherwise} \end{cases} \right)$$

In the sequel, we will denote $\varovee_i H_i$ with $H_{\varovee}$ and $\varovee_i Q_i$ with $Q_{\varovee}$ where in both cases an ambient index set is assumed.

Note that the definition says

$$\langle x \rangle \in\!\mathsf{h}\ H_{\varovee} \text{ iff } \exists j!\ (\langle x \rangle \in\!\mathsf{h}\ \text{ıı}_j H_j)$$

not

$$\langle x \rangle \in\!\mathsf{h}\ H_{\varovee} \text{ iff } \exists j!\ (\langle x \rangle \in\!\mathsf{h}\ \text{ıı}_i (H_i - H_i^\circ)).$$

Since $\top_i^{\mathbb{H}} = H_i - H_i^\circ$, this latter would define

$$H_{\varovee} = \top_{\varovee}^{\mathbb{H}} = \varovee_i \top_i^{\mathbb{H}} .$$

Since Q_i is wn-pointed, $H_i^\bullet \subseteq Q_i$. Thus, $\text{ıı}_j H_j^\bullet \subseteq \text{ıı}_j Q_j$. Also $H_{\varovee}^\circ \cap \varovee_i Q_i = \emptyset$. Hence $Q_{\varovee}$ is wn-pointed. Also,

$$\langle x \rangle \in\!\mathsf{h}\ Q_{\varovee} \text{ iff } \exists j!\ (\langle x \rangle \in\!\mathsf{h}_j\ \text{ıı}_j (Q_j - H_j^\circ))$$

since Q_j is assumed null-pointed.

Intuitively, $\langle x \rangle \in\!\mathsf{h}\ H_{\varovee}$ just when there is some j and $x_j \in\!\mathsf{h}_j\ H_j$ and x_i is a member of H_i° for every other index not equal to j. A similar statement holds for $\langle x \rangle \in\!\mathsf{h}\ Q_{\varovee}$ except notice that the j for which $x_j \in\!\mathsf{h}_j\ Q_j$ means that $x_j \notin\!\mathsf{h}_j\ H_j^\circ$, while for H_j, it is possible for $x_j \in\!\mathsf{h}_j\ H_j^\circ$.

Lemma 3.6.2

$$H^{\circ}_{\circledvee} \cap H_{\circledvee} = H^{\circ}_{\circledvee} \qquad H^{\circ}_{\circledvee} \cap Q_{\circledvee} = \emptyset$$

Proof: This follows directly from the definitions for $\circledvee$ since $H^{\circ}_{\circledvee} = \prod_i H^{\circ}_i$ and $Q_{\circledvee}$ is null-pointed. ■

Lemma 3.6.3 *If* $\langle x \rangle \in_{\mathsf{h}} H_{\circledvee}$*, then*

$$\exists j!(x_j \in_{\mathsf{h}_j} H^{\bullet}_j) \text{ implies } \langle x \rangle \in_{\mathsf{h}} H^{\bullet}_{\circledvee}.$$

Proof: Assume the premise. Let $x \in_{\mathsf{h}_j} H^{\bullet}_j$. Since $\langle x \rangle \in_{\mathsf{h}} H_{\circledvee}$, then for all $i \neq j$, $(x_i \in_{\mathsf{h}_i} H^{\circ}_i)$. Hence $\langle x \rangle \in_{\mathsf{h}} H^{\bullet}_{\circledvee}$. ■

Lemma 3.6.4

$$\langle x \rangle \in_{\mathsf{k}} H_{\circledvee} - Q_{\circledvee} \text{ implies } \exists j! \left(x_j \notin_{\mathsf{h}_j} Q_j \text{ and } \forall i \ (i \neq j \text{ implies } x_i \in_{\mathsf{h}_i} H^{\circ}_i)\right).$$

Proof: Recall that

$$H_{\circledvee} = \bigcup_i \amalg_i H_i. \qquad Q_{\circledvee} = \textstyle\bigcup_i \amalg_i Q_i.$$

Assume $\langle x \rangle \in_{\mathsf{h}} H_{\circledvee} - Q_{\circledvee}$, since $H_{\circledvee} = \bigcup_i \amalg_i H_i$, there is some j such that $x_j \in_{\mathsf{h}_j} H_j$ and for all $i \neq j$, $x_i \in_{\mathsf{h}_i} H^{\circ}_i$. Since $Q_{\circledvee} = \bigcup_i \amalg_i Q_i$ for all i, $x_i \notin_{\mathsf{h}_i} Q_i$. Hence for $i = j$, $x_j \notin_{\mathsf{h}_j} Q_j$.

■

Lemma 3.6.5

$$\top^{\mathbb{H}}_{\circledvee} - Q_{\circledvee} = \circledvee_i(\top^{\mathbb{H}}_i - Q_i).$$

Proof:

1	$\langle x\rangle \in\mathsf{h}\ \top^{\mathbb{H}}_{\circledvee} - Q_{\circledvee}$	assume
2	$\langle x\rangle \in\mathsf{h}\ \top^{\mathbb{H}}_{\circledvee}$ and $\langle x\rangle \notin\mathsf{h}\ Q_{\circledvee}$	set th., line 2
3	$\exists j!\ \big(\langle x\rangle \in\mathsf{h}\ \varkappa_j \top^{\mathbb{H}}_j\big)$	def. $\top^{\mathbb{K}}_{\circledvee}$, line 2
4	$\boxed{n}\quad \langle x\rangle \in\mathsf{h}\ \varkappa_n \top^{\mathbb{H}}_n$	assume
5	$x_n \in\mathsf{h}_n\ \top^{\mathbb{H}}_n$ and $\forall i\ (i \neq n$ implies $x_i \in\mathsf{h}_i\ H^{\circ}_i)$	def. $\varkappa_n$, line 3
6	$x_n \in\mathsf{h}_n\ Q_n$	assume
7	$\langle x\rangle \in\mathsf{h}\ Q_{\circledvee}$	def. $\circledvee$, lines 5, 6
8	$\rightarrowtail\!\leftarrowtail$	Contradiction, lines 2, 7
9	$x_n \notin\mathsf{h}_n\ Q_n$	CL, line 6
10	$x_n \in\mathsf{h}_n\ \top^{\mathbb{H}}_n - Q_n$	set th., lines 5, 9
11	$\langle x\rangle \in\mathsf{h}\ \circledvee_i(\top^{\mathbb{H}}_i - Q_i)$	def. $\circledvee$, lines 5, 10
12	$\langle x\rangle \in\mathsf{h}\ \circledvee_i(\top^{\mathbb{H}}_i - Q_i)$	$\exists$-Elim, lines 3, 4

and

1	$\langle x\rangle \in\mathsf{h}\ \circledvee_i(\top^{\mathbb{H}}_i - Q_i)$	assume
2	$\exists j!\ (\langle x\rangle \in\mathsf{h}\ \varkappa_j(\top^{\mathbb{H}}_j - Q_j))$	set th., line 2
3	$\exists j!\ (x_j \in\mathsf{h}_j\ \top^{\mathbb{H}}_j - Q_j$ and $\forall i\ (i \neq j$ implies $x_i \in\mathsf{h}_i\ H^{\circ}_i))$	set th., line 2
4	$\boxed{n}\quad x_n \in\mathsf{h}_n\ \top^{\mathbb{H}}_n - Q_n$ and $\forall i\ (i \neq n$ implies $x_i \in\mathsf{h}_i\ H^{\circ}_i)$	assume
5	$x_n \in\mathsf{h}_n\ \top^{\mathbb{H}}_n$ and $x_n \notin\mathsf{h}_n\ Q_n$	set th., line 4
6	$\langle x\rangle \in\mathsf{h}\ \top^{\mathbb{H}}_{\circledvee}$	def. $\circledvee$, lines 4, 5
7	$\langle x\rangle \notin\mathsf{h}\ Q_{\circledvee}$	def. $\circledvee$, lines 4, 5
8	$\langle x\rangle \in\mathsf{h}\ \top^{\mathbb{H}}_{\circledvee} - Q_{\circledvee}$	set th., lines 6, 7
9	$\langle x\rangle \in\mathsf{h}\ \top^{\mathbb{H}}_{\circledvee} - Q_{\circledvee}$	$\exists$-Elim, lines 3, 4

■

Lemma 3.6.6

$$\overset{\bullet\circ}{\neg}\, Q_{\circledvee} \overset{def}{=} \top^{\mathbb{H}}_{\circledvee} \overset{\bullet\circ}{-} Q_{\circledvee} = \circledvee_i(\top^{\mathbb{H}}_i \overset{\bullet\circ}{-} Q_i)$$

Proof:

1	$\langle x\rangle \in\mathsf{h}\ \top^{\mathbb{H}}_{\circledvee} \overset{\bullet\circ}{-} Q_{\circledvee}$	assume
2	$\langle x\rangle \in\mathsf{h}\ (\top^{\mathbb{H}}_{\circledvee} - Q_{\circledvee}) \cup H^{\bullet}_{\circledvee}$	def. $\overset{\bullet\circ}{-}$, line 1
3	$\langle x\rangle \in\mathsf{h}\ \top^{\mathbb{H}}_{\circledvee} - Q_{\circledvee}$ or $\langle x\rangle \in\mathsf{h}\ H^{\bullet}_{\circledvee}$	set th., line 2
4	$\langle x\rangle \in\mathsf{h}\ \top^{\mathbb{H}}_{\circledvee} - Q_{\circledvee}$	assume
5	$\langle x\rangle \in\mathsf{h}\ \circledvee_i(\top^{\mathbb{H}}_i - Q_i)$	Lemma 3.6.5, line 4
6	$\exists j!\ \big(\langle x\rangle \in\mathsf{h}\ \varkappa_j(\top^{\mathbb{H}}_j - Q_j)\big)$	def. $\circledvee$, line 5
7	$\exists j!\ \big(\langle x\rangle \in\mathsf{h}\ \varkappa_j((\top^{\mathbb{H}}_j - Q_j) \cup H^{\bullet}_j)\big)$	set th., line 6
8	$\exists j!\ \big(\langle x\rangle \in\mathsf{h}\ \varkappa_j(\top^{\mathbb{H}}_j \overset{\bullet\circ}{-} Q_j)\big)$	def. $\overset{\bullet\circ}{\neg}$, line 7
9	$\langle x\rangle \in\mathsf{h}\ H^{\bullet}_{\circledvee}$	assume
10	$\exists j!\ (\langle x\rangle \in\mathsf{h}\ \varkappa_j H^{\bullet}_j)$	Definition 3.2.3 for $H^{\bullet}_{\circledvee}$, line 9
11	$\boxed{n}\quad \langle x\rangle \in\mathsf{h}\ \varkappa_n H^{\bullet}_n$	assume
12	$x_n \in\mathsf{h}_n\ H^{\bullet}_n$	def. $\varkappa_n$, line 11
13	$x_n \in\mathsf{h}_n\ (\top^{\mathbb{H}}_n - Q_n) \cup H^{\bullet}_n$	set th., line 12
14	$x_n \in\mathsf{h}_n\ \top^{\mathbb{H}}_n \overset{\bullet\circ}{-} Q_n$	def. $\overset{\bullet\circ}{\neg}$, line 13
15	$\langle x\rangle \in\mathsf{h}\ \varkappa_n(\top^{\mathbb{H}}_n \overset{\bullet\circ}{-} Q_n)$	def. $\varkappa_n$, lines 11, 14
16	$\exists j!\ \big(\langle x\rangle \in\mathsf{h}\ \varkappa_j(\top^{\mathbb{H}}_j \overset{\bullet\circ}{-} Q_j)\big)$	$\exists$-Intro, line 15
17	$\exists j!\ \big(\langle x\rangle \in\mathsf{h}\ \varkappa_j(\top^{\mathbb{H}}_j \overset{\bullet\circ}{-} Q_j)\big)$	$\exists$-Elim, lines 10, 11
18	$\exists j!\ \big(\langle x\rangle \in\mathsf{h}\ \varkappa_j(\top^{\mathbb{H}}_j \overset{\bullet\circ}{-} Q_j)\big)$	$\vee$-Elim, lines 3, 4, 9
19	$\langle x\rangle \in\mathsf{h}\ \circledvee_i(\top^{\mathbb{H}}_i \overset{\bullet\circ}{-} Q_i)$	def. $\circledvee$, line 18

and

1	$\langle x\rangle \in_{\mathsf{h}} \circledvee_i(\top^{\mathbb{H}}_i \overset{\bullet\circ}{-} Q_i)$	assume
2	$\exists j!\ \big(\langle x\rangle \in_{\mathsf{h}} \underline{u}_j(\top^{\mathbb{H}}_j \overset{\bullet\circ}{-} Q_j)\big)$	def. $\overset{\bullet\circ}{-}$, line 1
3	$\boxed{n}\quad \langle x\rangle \in_{\mathsf{h}} \underline{u}_n(\top^{\mathbb{H}}_n \overset{\bullet\circ}{-} Q_n)$	assume
4	$x_n \in_{\mathsf{h}_n} \top^{\mathbb{H}}_n \overset{\bullet\circ}{-} Q_n$	def. $\underline{u}_n$, line 3
5	$x_n \in_{\mathsf{h}_n} (\top^{\mathbb{H}}_n - Q_n) \cup H^{\bullet}_n$	def. $\overset{\bullet\circ}{-}$, line 4
6	$x_n \in_{\mathsf{h}_n} \top^{\mathbb{H}}_n - Q_n$ or $x_n \in_{\mathsf{h}_n} H^{\bullet}_n$	set th., line 5
7	$x_n \in_{\mathsf{h}_n} \top^{\mathbb{H}}_n - Q_n$	assume
8	$x_n \in_{\mathsf{h}_n} \top^{\mathbb{H}}_n$ and $x_n \notin_{\mathsf{h}_n} Q_n$	set th., line 7
9	$\langle x\rangle \notin_{\mathsf{h}} \underline{u}_n Q_n$	def. $\underline{u}_n$, lines 3, 8
10	$x_n \notin_{\mathsf{h}_n} H^{\circ}_n$	$\top^{\mathbb{H}}_n$ is null-pointed, line 8
11	$\langle x\rangle \notin_{\mathsf{h}} Q_{\circledvee}$	def. $\circledvee$, lines 9, 10
12	$\langle x\rangle \in_{\mathsf{h}} \underline{u}_n \top^{\mathbb{H}}_n$	def $\underline{u}_n$, lines 3, 8
13	$\langle x\rangle \in_{\mathsf{h}} \top^{\mathbb{H}}_{\circledvee}$	def $\top^{\mathbb{H}}_{\circledvee}$, line 12
14	$\langle x\rangle \in_{\mathsf{h}} \top^{\mathbb{H}}_{\circledvee} - Q_{\circledvee}$	set th., lines 11, 13
15	$\langle x\rangle \in_{\mathsf{h}} (\top^{\mathbb{H}}_{\circledvee} - Q_{\circledvee}) \cup H^{\bullet}_{\circledvee}$	set th., line 14
16	$x_n \in_{\mathsf{h}_n} H^{\bullet}_n$	assume
17	$\langle x\rangle \in_{\mathsf{h}} \underline{u}_n H^{\bullet}_n$	def. $\underline{u}_n$, lines 3, 16
18	$\langle x\rangle \in_{\mathsf{h}} H^{\bullet}_{\circledvee}$	def. $H^{\bullet}_{\circledvee}$, line 17
19	$\langle x\rangle \in_{\mathsf{h}} (\top^{\mathbb{H}}_{\circledvee} - Q_{\circledvee}) \cup H^{\bullet}_{\circledvee}$	set th., line 18
20	$\langle x\rangle \in_{\mathsf{h}} (\top^{\mathbb{H}}_{\circledvee} - Q_{\circledvee}) \cup H^{\bullet}_{\circledvee}$	$\vee$-Elim, lines 6, 7, 16
21	$\langle x\rangle \in_{\mathsf{h}} (\top^{\mathbb{H}}_{\circledvee} - Q_{\circledvee}) \cup H^{\bullet}_{\circledvee}$	$\exists$-Elim, lines 2, 3
22	$\langle x\rangle \in_{\mathsf{h}} \top^{\mathbb{H}}_{\circledvee} \overset{\bullet\circ}{-} Q_{\circledvee}$	def. $\overset{\bullet\circ}{-}$, line 21

■

3.7 General Representation Function

The type general refers to either a filter or an ideal depending upon whether the type is either $\wedge$ or $\vee$ respectively. The lattice operator $\bowtie_i$ is $\wedge$ if i is $\wedge$ and $\vee$ if i is $\vee$.

Lemma 3.7.1 *If the operator τ has type is $i \mapsto j$ for $i, j \in \{\wedge, \vee\}$, then*

$$\lambda p\ .\ (\tau)^{-1} x_j .$$

is of type i, where for a set to be of type i means for the set to be a filter if $i = \wedge$ and an ideal if $i = \vee$ and x_j is designated similarly.

Proof: Let $i, j \in \{\wedge, \vee\}$:

$$\begin{aligned} a, b \in \lambda p \ . \ (\tau)^{-1} x_j &\text{ iff } \tau a, \tau b \in x_j \\ &\text{ iff } \tau a \bowtie_j \tau b \in x_j \\ &\text{ iff } \tau (a \bowtie_i b) \in x_j \\ &\text{ iff } a \bowtie_i b \in \lambda p \ . \ (\tau)^{-1} x_j \end{aligned}$$

■

4. CLOSURE PROPERTIES

In the sections below, any reference to *normality* is actually a reference to the type of an operator. The term “normality” is from modal logic.

4.1 Distribution with Smash Products and Null Sums

Only the forward operators are dealt with in this section, the backward operators are entirely analogous.

The ambient space for smssh product, which we denote as $\mathsf{h}_1 \otimes \mathsf{h}_2$, must be constructed. From the proofs of closure, there are the conditions in Table 4.1:

op	**con**	**rel**	**Menu**
$[-^{\perp}\rangle$	$\otimes$	$\mathcal{F}^{\perp}\langle a_1, a_2\rangle\langle b_1, b_2\rangle$ iff $\mathcal{F}_1^{\perp} a_1 b_1$ and $\mathcal{F}_2^{\perp} a_2 b_2$	C
$[-^{\diamond}\rangle$	$\otimes$	$\mathcal{F}^{\diamond}\langle a_1, a_2\rangle\langle b_1, b_2\rangle$ iff $\mathcal{F}_1^{\diamond} a_1 b_1$ and $\mathcal{F}_2^{\diamond} a_2 b_2$	C
$[-^{\sharp}\rangle$	$\otimes$	$\neg\mathcal{F}^{\sharp}\langle a_1, a_2\rangle\langle b_1, b_2\rangle$ iff $\neg\mathcal{F}_1^{\sharp} a_1 b_1$ and $\neg\mathcal{F}_2^{\sharp} a_2 b_2$	D
$[-^{!}\rangle$	$\otimes$	$\neg\mathcal{F}^{!}\langle a_1, a_2\rangle\langle b_1, b_2\rangle$ iff $\neg\mathcal{F}_1^{!} a_1 b_1$ and $\neg\mathcal{F}_2^{!} a_2 b_2$	D

Table 4.1: Relation Conditions for Closure under Smash Product

From the proofs of closure, there are the conditions in Table 4.2:

op	**con**	**rel**	**Menu**
$[-^{\flat}\rangle$	$\varovee$	$\mathcal{F}^{\flat}\langle a_1, a_2\rangle\langle b_1, b_2\rangle$ iff $\mathcal{F}_1^{\flat} a_1 b_1$ or $\mathcal{F}_2^{\flat} a_2 b_2$	C
$[-^{?}\rangle$	$\varovee$	$\mathcal{F}^{?}\langle a_1, a_2\rangle\langle b_1, b_2\rangle$ iff $\mathcal{F}_1^{?} a_1 b_1$ or $\mathcal{F}_2^{?} a_2 b_2$	C
$[-^{\square}\rangle$	$\varovee$	$\neg\mathcal{F}^{\square}\langle a_1, a_2\rangle\langle b_1, b_2\rangle$ iff $\neg\mathcal{F}_1^{\square} a_1 b_1$ or $\neg\mathcal{F}_2^{\square} a_2 b_2$	D
$[-^{\star}\rangle$	$\varovee$	$\neg\mathcal{F}^{\star}\langle a_1, a_2\rangle\langle b_1, b_2\rangle$ iff $\neg\mathcal{F}_1^{\star} a_1 b_1$ or $\neg\mathcal{F}_2^{\star} a_2 b_2$	D

Table 4.2: Relation Conditions for Closure under Null Sums

The following allows us to define the smash products and null sums for distributed frames:

$$\neg^{\rho}\mathcal{F}^{\rho} = \begin{cases} \neg^{\rho}\mathcal{F}^{\rho} = \mathcal{F}^{\rho} & \text{if } \rho \in \{^{\perp}, ^{\diamond}, ^{\flat}, ^{?}\} \\ \neg^{\rho}\mathcal{F}^{\rho} = \neg\mathcal{F}^{\rho} & \text{if } \rho \in \{^{\sharp}, ^{!}, ^{\square}, ^{\star}\} \end{cases}$$

Definition 4.1.1 Let $\mathsf{h}_1 = \langle H_1, \neg^{\rho}\mathcal{H}_1^{\rho}, \mathbb{H}_1\rangle$ and $\mathsf{h}_2 = \langle H_2, \neg^{\tau}\mathcal{H}_2^{\tau}, \mathbb{H}_2\rangle$, then

$$\mathsf{h} \overset{def}{=} \mathsf{h}_1 \otimes \mathsf{h}_2 \overset{def}{=} \langle H_1 \otimes H_2, \neg^{\rho}\mathcal{H}_1^{\rho} \times \neg^{\tau}\mathcal{H}^{\tau}, \mathbb{H}_1 \otimes \mathbb{H}_2\rangle.$$

where

$$Q \in_{\mathsf{h}} \mathbb{H}_1 \otimes \mathbb{H}_2 \overset{def}{=} \exists Q_1, Q_2(Q_1 \in_{\mathsf{h}_1} \mathbb{H}_1 \text{ and } Q_2 \in_{\mathsf{h}_2} \mathbb{H}_2 \text{ and } Q = Q_1 \otimes Q_2).$$

Note that $\neg^{\rho}\mathcal{H}_1^{\rho} \times \neg^{\tau}\mathcal{H}^{\tau}$ allows for $\mathcal{H}_1$ and $\mathcal{H}_2$ to be different kinds of relations. The use here of local relations is entirely parametric. This is not so for distributed relations because of the use they will be put through in constructing distributed smash products.

If $f_1 : \mathsf{h}_1 \rightharpoonup \mathsf{k}_1$, $f_2 : \mathsf{h}_2 \rightharpoonup \mathsf{k}_2$, $\mathsf{h} = \mathsf{h}_1 \otimes \mathsf{h}_2$, and $\mathsf{k} = \mathsf{k}_1 \otimes \mathsf{k}_2$, then $f : \mathsf{h} \rightharpoonup \mathsf{k}$ where

$$\neg^{\rho} f^{\rho} \overset{def}{=} \neg^{\rho} f_1^{\rho} \otimes \neg^{\rho} f_2^{\rho} : \mathsf{h}_1 \otimes \mathsf{h}_2 \rightharpoonup \mathsf{k}_1 \otimes \mathsf{k}_2,$$

and

$$\neg^{\rho}\mathcal{F}^{\rho} = \neg^{\rho}\mathcal{F}_1^{\rho} \times \neg^{\rho}\mathcal{F}_2^{\rho} \quad \mathcal{F}_1^{\rho} \subseteq H_1 \times K_1,\ \mathcal{F}_2^{\rho} \subseteq H_2 \times K_2.$$

Definition 4.1.2 Let $\mathsf{h}_1 = \langle H_1, \neg^{\rho}\mathcal{H}_1^{\rho}, \mathbb{H}_1\rangle$ and $\mathsf{h}_2 = \langle H_2, \neg^{\tau}\mathcal{H}_2^{\tau}, \mathbb{H}_2\rangle$, then

$$\mathsf{h} \overset{def}{=} \mathsf{h}_1 \ovee \mathsf{h}_2 \overset{def}{=} \langle H_1 \ovee H_2, \neg^{\rho}\mathcal{H}_1^{\rho} + \neg^{\tau}\mathcal{H}_2^{\tau}, \mathbb{H}_1 \ovee \mathbb{H}_2\rangle.$$

where + is the disjoint sum for relations and where

$$Q \in_{\mathsf{h}} \mathbb{H}_1 \ovee \mathbb{H}_2 \overset{def}{=} \exists Q_1, Q_2(Q_1 \in_{\mathsf{h}_1} \mathbb{H}_1 \text{ and } Q_2 \in_{\mathsf{h}_2} \mathbb{H}_2 \text{ and } Q = Q_1 \ovee Q_2).$$

The use here of local relations is again entirely parametric.

If $f_1 : \mathsf{h}_1 \rightharpoonup \mathsf{k}_1$, $f_2 : \mathsf{h}_2 \rightharpoonup \mathsf{k}_2$, $\mathsf{h} = \mathsf{h}_1 \ovee \mathsf{h}_2$, and $\mathsf{k} = \mathsf{k}_1 \ovee \mathsf{k}_2$, then $f : \mathsf{h} \rightharpoonup \mathsf{k}$ where

$$\neg^{\rho} f^{\rho} \overset{def}{=} \neg^{\rho} f_1^{\rho} \ovee \neg^{\rho} f_2^{\rho} : \mathsf{h}_1 \ovee \mathsf{h}_2 \rightharpoonup \mathsf{k}_1 \ovee \mathsf{k}_2,$$

and

$$\neg^{\rho}\mathcal{F}^{\rho} = \neg^{\rho}\mathcal{F}_1^{\rho} \times \neg^{\rho}\mathcal{F}_2^{\rho} \quad \mathcal{F}_1^{\rho} \subseteq H_1 \times K_1,\ \mathcal{F}_2^{\rho} \subseteq H_2 \times K_2.$$

4.2 $[-^{\perp}\rangle, [-^{\perp}\cdot\rangle$: Closure Under Smash Product

All proofs involving $[-^{\perp}\cdot\rangle$ are similar to their siblings for $[-^{\perp}\rangle$ and we elide them. The information necessary to construct them is in the following Section 4.3.

The type of $[-^{\perp}\rangle$ is $\vee\!\mapsto\!\wedge$.

Definition 4.2.1

$$x \in_{\mathsf{h}} [f^{\perp}\rangle Q \text{ iff } \forall y(y \notin_{\mathsf{k}} Q \text{ or } \mathcal{F}^{\perp}xy).$$

Definition 4.2.2 The canonical relation is

$$\mathcal{F}^{\perp}xy \text{ iff } \exists b(b \in_{\mathsf{k}} y \text{ and } [f^{\perp}\rangle b \in_{\mathsf{h}} x).$$

The negation of the canonical definition is

$$\neg\mathcal{F}^{\perp}xy \text{ iff } \forall b(b \notin_{\mathsf{k}} y \text{ or } [f^{\perp}\rangle b \notin_{\mathsf{h}} x).$$

Frame Condition 4.2.3 The null frame conditions for $\mathcal{F}^{\perp}$ are

$$(1) \quad \forall y(\neg\mathcal{F}^{\perp}H^{\circ}y) \qquad (2) \quad \forall x(\neg\mathcal{F}^{\perp}xK^{\circ})$$

Frame Condition 4.2.4 The well frame conditions for $\mathcal{F}^{\perp}$ are

$$(1) \quad \forall x(x \notin_{\mathsf{h}} H^{\circ} \text{ implies } \mathcal{F}^{\perp}xK^{\bullet}) \qquad (2) \quad \forall y(y \notin_{\mathsf{k}} K^{\circ} \text{ implies } \mathcal{F}^{\perp}H^{\bullet}y).$$

Lemma 4.2.5 *Canonically, the Frame Conditions hold.*

Proof: Frame Conditions 4.2.3 (1) and 4.2.3 (2): let $x = \circ_H$ or $y = \circ_K$, then

$$\forall b(b \notin_{\mathsf{k}} y \text{ or } [f^{\perp}\rangle b \notin_{\mathsf{h}} x)$$

holds since $\circ_H = \emptyset$ and $\circ_K = \emptyset$. So $\neg\mathcal{F}^{\perp}xy$ obtains in both cases.

Frame Condition 4.2.4 (1): choose an arbitrary x and let $x \neq \circ_H$. Let $y = \bullet_K$, then $\bot_{\mathsf{k}} \in_{\mathsf{k}} y$. From the type of $[-^{\perp}\rangle$, $[f^{\perp}\rangle\bot_{\mathsf{k}} = \top_{\mathsf{h}}$. Since $x \neq \circ_H$, then $\top_{\mathsf{h}} \in_{\mathsf{h}} x$. Hence $\mathcal{F}^{\perp}xy$ obtains.

Frame Condition 4.2.4 (2): choose an arbitrary y and let $y \neq \circ_K$. Let $x = \bullet_H$. For all $b \in_{\mathsf{k}} y$, it is the case that $[f^{\perp}\rangle b \in_{\mathsf{h}} x$ since $\bullet_H$ includes the entire carrier set. Since the choice of y was arbitrary, for all y, $\mathcal{F}xy$ obtains.

■

Lemma 4.2.6 *The Frame Conditions 4.2.3 and 4.2.4 are consistent with each other.*

Proof: Note that the well-point sets and null-point sets are always disjoint, then the quantifiers and antecedents of the conditionals of Frame Conditions 4.2.3 and 4.2.4 prevent any conflicts.

■

Lemma 4.2.7 $[f^{\perp}\rangle Q$ *is null-pointed if* Q *is well-pointed, and well-pointed if* Q *is null-pointed.*

Proof: By definition,

$$\begin{aligned} x \notin_{h} [f^{\perp}\rangle Q \text{ iff } & \neg\forall y(y \notin_{k} Q \text{ or } \mathcal{F}^{\perp}xy) \\ \text{iff } & \exists y(y \in_{k} Q \text{ and } \neg\mathcal{F}^{\perp}xy) \end{aligned}$$

Assume Q is well-pointed. Let $y \in_{k} K^{\bullet}$ and $x \in_{h} H^{\circ}$, then by assumption, $y \in_{k} Q$. From the Frame Condition 4.2.3 (1), $\neg\mathcal{F}^{\perp}H^{\circ}y$. Therefore, $x \notin_{h} [f^{\perp}\rangle Q$ and $[f^{\perp}\rangle Q$ is null-pointed.

Next, assume Q is null-pointed. Let $x \in_{h} H^{\bullet}$ and choose some arbitrary $y \in_{k} Q$. Since Q is null-pointed, $y \notin_{k} K^{\circ}$ and hence $\mathcal{F}^{\perp}H^{\bullet}y$ holds from the Frame Condition 4.2.4 (2). So for all y, $y \in_{k} Q$ implies $\mathcal{F}^{\perp}xy$. By definition, $x \in_{h} [f^{\perp}\rangle Q$ and $[f^{\perp}\rangle Q$ is well-pointed.

■

Lemma 4.2.8 $[f^{\perp}\rangle$ *preserves its bounds.*

Proof: The type of $[f^{\perp}\rangle$ is $\vee\mapsto\wedge$, hence we want $[f^{\perp}\rangle \bot^{\mathbb{K}} = \top^{\mathbb{H}}$. The definition is

$$x \in_{h} [f^{\perp}\rangle Q \text{ iff } \forall y(y \notin_{k} Q \text{ or } \mathcal{F}^{\perp}xy).$$

We have

$$x \in_{h} [f^{\perp}\rangle \top^{\mathbb{K}} \text{ iff } \forall y(y \notin_{k} \bot^{\mathbb{K}} \text{ or } \mathcal{F}^{\perp}xy).$$

From Lemma 4.2.7, $[f^{\perp}\rangle \bot^{\mathbb{K}}$ is well- and null-pointed. Since $\top^{\mathbb{H}}$ is the top of the lattice, $[f^{\perp}\rangle \bot^{\mathbb{K}} \subseteq_{h} \top^{\mathbb{H}}$.

Next, let $x \in_{h} \top^{\mathbb{H}}$. By definition $\top^{\mathbb{H}} \cap H^{\circ} = \emptyset$, so $x \notin_{h} H^{\circ}$. Choose some arbitrary y and assume $y \in_{k} \bot^{\mathbb{K}}$. Since $\bot^{\mathbb{K}} = K^{\bullet}$, then from the Frame Condition 4.2.4 (1), $\mathcal{F}^{\perp}xK^{\bullet}$ and so $\mathcal{F}^{\perp}xy$ obtains. Hence for any y, it is the case that $y \notin_{k} Q$ or $\mathcal{F}^{\perp}xy$.

■

Lemma 4.2.9

$$x \in_{h} [f^{\perp}\rangle \beta b \text{ iff } x \in_{h} \beta[f^{\perp}\rangle b.$$

Proof: The right to left is the easy direction. Assume $x \in_{\mathsf{h}} \beta[f^{\perp}\rangle\, b$ and choose an arbitrary y such that $\neg\mathcal{F}xy$. The canonical definition is

$$\begin{aligned} \neg\mathcal{F}xy &\text{ iff } \neg\exists a(a \in_{\mathsf{k}} y \text{ and } [f^{\perp}\rangle\, a \in_{\mathsf{h}} x) \\ &\text{ iff } \forall a(a \notin_{\mathsf{k}} y \text{ or } [f^{\perp}\rangle\, a \notin_{\mathsf{h}} x) \\ &\text{ iff } \forall a([f^{\perp}\rangle\, a \in_{\mathsf{h}} x \text{ implies } a \notin_{\mathsf{k}} y). \end{aligned}$$

Hence $b \notin_{\mathsf{k}} y$, whence $y \notin_{\mathsf{k}} \beta b$. So for all y, $\neg\mathcal{F}^{\perp}xy$ implies $y \notin_{\mathsf{k}} \beta b$. By definition, $x \in_{\mathsf{h}} [f^{\perp}\rangle\, b$.

We go in the other direction via contraposition. Assume $[f^{\perp}\rangle\, b \notin_{\mathsf{h}} x$. Let

$$\hat{y} = (\lambda p \,.\, [f^{\perp}\rangle\, p)^{-1}x.$$

Given the type, from Lemma 3.7.1 $\hat{y}$ is an ideal. Also, $b \notin_{\mathsf{k}} \hat{y}$ since $b \in_{\mathsf{k}} \hat{y}$ implies $[f^{\perp}\rangle\, b \in_{\mathsf{h}} x$, which is a contradiction. Construct y by using Zorn's Lemma to maximalize $\langle \uparrow b, \hat{y}\rangle$ to construct y, whence $b \in_{\mathsf{k}} y$. By construction, we have defined $\neg\mathcal{F}^{\perp}xy$, i.e.,

$$\forall a([f^{\perp}\rangle\, a \in_{\mathsf{h}} x \text{ implies } a \notin_{\mathsf{k}} y)$$

Therefore there exists a y, such that $b \in_{\mathsf{k}} y$ and so $y \notin_{\mathsf{k}} \beta b$. Thus, $y \notin_{\mathsf{k}} \beta b$ and $\neg\mathcal{F}^{\perp}xy$, whence $x \notin_{\mathsf{h}} [f^{\perp}\rangle \beta b$.

■

We assume $f = f_1 \otimes f_2$, then

$$\mathcal{F}^{\perp}\langle a_1, a_2\rangle\langle b_1, b_2\rangle \text{ iff } \mathcal{F}_1^{\perp}a_1b_1 \text{ and } \mathcal{F}^{\perp}a_2b_2.$$

Lemma 4.2.10 *Let $f = f_1 \otimes f_2$, then*

$$[f^{\perp}\rangle\,(Q_1 \otimes Q_2) = [f_1^{\perp}\rangle\, Q_1 \otimes [f_2^{\perp}\rangle\, Q_2 \qquad [f^{\perp}\cdot\rangle(P_1 \otimes P_2) = [f_1^{\perp}\cdot\rangle P_1 \otimes [f_2^{\perp}\cdot\rangle P_2.$$

Proof:

1	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f^{\perp}\rangle (Q_1 \otimes Q_2)$	assume
2	$(\langle a_1, a_2\rangle \in_{\mathsf{h}} [f^{\perp}\rangle (Q_1 \otimes Q_2) - H^{\bullet}_{\otimes})$ or $\langle a_1, a_2\rangle \in_{\mathsf{h}} H^{\bullet}_{\otimes}$	Lemma 4.2.7, $[-^{\perp}\rangle$ well-pointed, line 1
3	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f^{\perp}\rangle (Q_1 \otimes Q_2) - H^{\bullet}_{\otimes}$	assume
4	$\forall\langle y, v\rangle(\langle y, v\rangle \notin_{\mathsf{k}} Q_1 \otimes Q_2$ or $\mathcal{F}\langle a_1, a_2\rangle\langle y, v\rangle)$	def. $[-^{\perp}\rangle$, line 3
5	$\boxed{b_1}$ $b_1 \in_{\mathsf{k}_1} Q_1$	assume
6	$b_1 \in_{\mathsf{k}_1} Q_1 - K^{\bullet}_1$ or $b_1 \in_{\mathsf{k}_1} K^{\bullet}_1$	Q_1 is well-pointed, line 5
7	$b_1 \in_{\mathsf{k}_1} Q_1 - K^{\bullet}_1$	assume
8	$\exists u(u \in_{\mathsf{k}_2} K^{\bullet}_2)$	$K^{\bullet}_2 \neq \emptyset$
9	$\boxed{b_2}$ $b_2 \in_{\mathsf{k}_2} K^{\bullet}_2$	assume
10	$\langle b_1, b_2\rangle \notin_{\mathsf{k}} Q_1 \otimes Q_2$ or $\mathcal{F}^{\perp}\langle a_1, a_2\rangle\langle b_1, b_2\rangle$	$\forall$-Elim, line 4
11	$\langle b_1, b_2\rangle \notin_{\mathsf{k}} Q_1 \otimes Q_2$	assume
12	$b_1 \notin_{\mathsf{k}_1} Q_1$ or $b_2 \notin_{\mathsf{k}_2} Q_2$	Lemma 3.4.2, line 11
13	$b_1 \notin_{\mathsf{k}_1} Q_1$	CL, Q_2 is well-pointed, lines 9, 12
14	$\rightarrow\leftarrow$	Contradiction, lines 7, 13
15	$\mathcal{F}^{\perp}\langle a_1, a_2\rangle\langle b_1, b_2\rangle$	CL, lines 10, 11
16	$\mathcal{F}^{\perp}_1 a_1 b_1$	def. $\mathcal{F}^{\perp}$, line 15
17	$\mathcal{F}^{\perp}_1 a_1 b_1$	$\exists$-Elim, lines 8, 9
18	$b_1 \in_{\mathsf{k}_1} K^{\bullet}_1$	assume
19	$a_1 \in_{\mathsf{h}_1} H^{\circ}_1$	assume
20	$\langle a_1, a_2\rangle \in_{\mathsf{h}} H^{\circ}_{\otimes}$	Definition 3.2.2, line 19
21	$\rightarrow\leftarrow$	Contradiction, $[-^{\perp}\rangle$ is null-pointed, Lemma 4.2.7, line 20
22	$a_1 \notin_{\mathsf{h}_1} H^{\circ}_1$	CL, line 18
23	$\mathcal{F}^{\perp}_1 a_1 b_1$	Frame Condition 4.2.4 (1), lines 18, 22
24	$\mathcal{F}^{\perp}_1 a_1 b_1$	$\vee$-Elim, lines 6, 7, 18
25	$\forall y(y \notin_{\mathsf{k}_1} Q_1$ or $\mathcal{F}^{\perp}_1 a_1 y)$	$\forall$-Intro, CL, line 5
26	$a_1 \in_{\mathsf{h}_1} [f_1^{\perp}\rangle Q_1$	def. $[-^{\perp}\rangle$, line 25
27	$\boxed{b_2}$ $b_2 \in_{\mathsf{k}_2} Q_2$	assume
28	Similar to previous subproof	lines xx
29	$\mathcal{F}^{\perp}_2 a_2 b_2$	doh, lines xx
30	$\forall v(v \notin_{\mathsf{k}_2} Q_2$ or $\mathcal{F}^{\perp}_2 a_2 v)$	CL, line 27
31	$a_2 \in_{\mathsf{h}_2} [f_2^{\perp}\rangle Q_2$	def. $[-^{\perp}\rangle$, line 30
32	$\langle a_1, a_2\rangle \in_{\mathsf{h}} H_1 \otimes H_2$	$f = f_1 \otimes f_2$, line 1
33	$\langle a_1, a_2\rangle \notin_{\mathsf{h}} H^{\bullet}_{\otimes}$ and $\langle a_1, a_2\rangle \in_{\mathsf{h}} H_1 \otimes H_2$	$\wedge$-Intro, lines 3, 32
34	$a_1 \notin_{\mathsf{h}_1} H^{\bullet}_1$ and $a_2 \notin_{\mathsf{h}_2} H^{\bullet}_2$	Lemma 3.4.9, line 33
35	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f_1^{\perp}\rangle Q_1 \otimes [f_2^{\perp}\rangle Q_2$	def. $\otimes$, line 34
36	$\langle a_1, a_2\rangle \in_{\mathsf{h}} H^{\bullet}_{\otimes}$	assume
37	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f_1^{\perp}\rangle Q_1 \otimes [f_2^{\perp}\rangle Q_2$	Lemma 4.2.7, $[-^{\perp}\rangle$ well-pointed, line 36
38	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f_1^{\perp}\rangle Q_1 \otimes [f_2^{\perp}\rangle Q_2$	$\vee$-Elim, lines 2, 3, 36

To fill in the similarity above

1	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f^{\perp}\rangle (Q_1 \otimes Q_2)$	assume
2	$(\langle a_1, a_2\rangle \in_{\mathsf{h}} [f^{\perp}\rangle (Q_1 \otimes Q_2) - H^{\bullet}_{\otimes})$ or $\langle a_1, a_2\rangle \in_{\mathsf{h}} H^{\bullet}_{\otimes}$	Lemma 4.2.7, $[-^{\perp}\rangle$ well-pointed, line 1
3	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f^{\perp}\rangle (Q_1 \otimes Q_2) - H^{\bullet}_{\otimes}$	assume
4	$\forall\langle y, v\rangle(\langle y, v\rangle \notin_{\mathsf{k}} Q_1 \otimes Q_2$ or $\mathcal{F}\langle a_1, a_2\rangle\langle y, v\rangle)$	def. $[-^{\perp}\rangle$, line 3
5	$\boxed{b_2}$ $b_2 \in_{\mathsf{k}_2} Q_1$	assume
6	$b_2 \in_{\mathsf{k}_2} Q_2 - K^{\bullet}_2$ or $b_2 \in_{\mathsf{k}_2} K^{\bullet}_2$	Q_2 is well-pointed, line 5
7	$b_2 \in_{\mathsf{k}_2} Q_2 - K^{\bullet}_2$	assume
8	$\exists u(u \in_{\mathsf{k}_1} K^{\bullet}_1)$	$K^{\bullet}_1 \neq \emptyset$
9	$\boxed{b_1}$ $b_1 \in_{\mathsf{k}_1} K^{\bullet}_1$	assume
10	$\langle b_1, b_2\rangle \notin_{\mathsf{k}} Q_1 \otimes Q_2$ or $\mathcal{F}\langle a_1, a_2\rangle\langle b_1, b_2\rangle$	∀-Elim, line 4
11	$\langle b_1, b_2\rangle \notin_{\mathsf{k}} Q_1 \otimes Q_2$	assume
12	$b_1 \notin_{\mathsf{k}_1} Q_1$ or $b_2 \notin_{\mathsf{k}_2} Q_2$	Lemma 3.4.2, line 22
13	$b_2 \notin_{\mathsf{k}_2} Q_2$	CL, Q_2 is well-pointed, lines 9, 12
14	$\rightarrow\!\leftarrow$	Contradiction, lines 7, 13
15	$\mathcal{F}^{\perp}\langle a_1, a_2\rangle\langle b_1, b_2\rangle$	CL, lines 10, 11
16	$\mathcal{F}^{\perp}_2 a_2 b_2$	def. $\mathcal{F}^{\perp}$, line 15
17	$\mathcal{F}^{\perp}_2 a_2 b_2$	∃-Elim, lines 8, 9
18	$b_2 \in_{\mathsf{k}_2} K^{\bullet}_2$	assume
19	$a_2 \in_{\mathsf{h}_2} H^{\circ}_2$	assume
20	$\langle a_1, a_2\rangle \in_{\mathsf{h}} H^{\circ}_{\otimes}$	Definition 3.2.2, line 19
21	$\rightarrow\!\leftarrow$	Contradiction, $[-^{\perp}\rangle$ is null-pointed, Lemma 4.2.7, line 20
22	$a_2 \notin_{\mathsf{h}_2} H^{\circ}_2$	CL line 19
23	$\mathcal{F}^{\perp}_2 a_2 b_2$	Frame Condition 4.2.4 (1), lines 18, 22
24	$\mathcal{F}^{\perp}_2 a_2 b_2$	∨-Elim, lines 6, 7, 18
25	$\forall v(v \notin_{\mathsf{k}_2} Q_2$ or $\mathcal{F}^{\perp}_2 a_2 v)$	∀-Intro, CL, line 5

Now for the other direction.

1	$\mid \underline{\langle a_1, a_2\rangle \in_{h} [f_1^{\perp}\rangle Q_1 \otimes [f_2^{\perp}\rangle Q_2}$	assume
2	$\mid \langle a_1, a_2\rangle \in_{h} ([f_1^{\perp}\rangle Q_1 - H_1^{\bullet}) \times ([f_2^{\perp}\rangle Q_2 - H_2^{\bullet})$ or $\langle a_1, a_2\rangle \in_{h} H_{\otimes}^{\bullet}$	def. ⊗, line 1
3	$\mid\mid \underline{\langle a_1, a_2\rangle \in_{h} ([f_1^{\perp}\rangle Q_1 - H_1^{\bullet}) \times ([f_2^{\perp}\rangle Q_2 - H_2^{\bullet})}$	assume
4	$\mid\mid\mid \underline{\boxed{b_1, b_2} \quad \langle b_1, b_2\rangle \in_{k} Q_1 \otimes Q_2}$	assume
5	$\mid\mid\mid \langle b_1, b_2\rangle \in_{k} (Q_1 - K_1^{\bullet}) \times (Q_2 - K_2^{\bullet})$ or $\langle b_1, b_2\rangle \in_{k} K_{\otimes}^{\bullet}$	def. ⊗, line 4
6	$\mid\mid\mid\mid \underline{\langle b_1, b_2\rangle \in_{k} (Q_1 - K_1^{\bullet}) \times (Q_2 - K_2^{\bullet})}$	assume
7	$\mid\mid\mid\mid b_1 \in_{k_1} Q_1 - K_1^{\bullet}$ and $b_2 \in_{k_2} Q_2 - K_2^{\bullet}$	def. ×, line 6
8	$\mid\mid\mid\mid b_1 \in_{k_1} Q_1$ and $b_2 \in_{k_2} Q_2$	set th., line 7
9	$\mid\mid\mid\mid a_1 \in_{h_1} [f_1^{\perp}\rangle Q_1 - H_1^{\bullet}$ and $a_2 \in_{h_2} [f_2^{\perp}\rangle Q_2 - H_2^{\bullet}$	def. ×, line 3
10	$\mid\mid\mid\mid a_1 \in_{h_1} [f_1^{\perp}\rangle Q_1$ and $a_2 \in_{h_2} [f_2^{\perp}\rangle Q_2$	set th., line 9
11	$\mid\mid\mid\mid \forall y (y \notin_{k_1} Q_1$ or $\mathcal{F}_1^{\perp} a_1 y)$	def. $[f_1^{\perp}\rangle$, line 10
12	$\mid\mid\mid\mid \forall v (v \notin_{k_2} Q_2$ or $\mathcal{F}_2^{\perp} a_2 v)$	def. $[f_2^{\perp}\rangle$, line 10
13	$\mid\mid\mid\mid b_1 \notin_{k_1} Q_1$ or $\mathcal{F}_1^{\perp} a_1 b_1$	def. ∀-Elim, line 11
14	$\mid\mid\mid\mid b_2 \notin_{k_2} Q_2$ or $\mathcal{F}_2^{\perp} a_2 b_2$	def. ∀-Elim, line 12
15	$\mid\mid\mid\mid \mathcal{F}_1^{\perp} a_1 b_1$ and $\mathcal{F}_2^{\perp} a_2 b_2$	CL, lines 8, 13, 14
16	$\mid\mid\mid\mid \mathcal{F}^{\perp} \langle a_1, a_2\rangle \langle b_1, b_2\rangle$	def. $\mathcal{F}^{\perp}$, line 15
17	$\mid\mid\mid\mid \underline{\langle b_1, b_2\rangle = K_{\otimes}^{\bullet}}$	assume
18	$\mid\mid\mid\mid a_1 \notin_{h_1} H_1^{\circ}$ and $a_2 \notin_{h_2} H_2^{\circ}$	$[-^{\perp}\rangle$ null-pointed, Lemma 4.2.7, line 3
19	$\mid\mid\mid\mid \mathcal{F}_1^{\perp} a_1 b_1$ and $\mathcal{F}_2^{\perp} a_2 b_2$	Frame Condition 4.2.4 (1), lines 17, 18
20	$\mid\mid\mid\mid \mathcal{F}^{\perp} \langle a_1, a_2\rangle \langle b_1, b_2\rangle$	def. $\mathcal{F}^{\perp}$, line 19
21	$\mid\mid\mid \mathcal{F}^{\perp} \langle a_1, a_2\rangle \langle b_1, b_2\rangle$	∨-Elim, lines 5, 6, 17
22	$\mid\mid \forall \langle y, v\rangle (\langle y, v\rangle \notin_{k} Q_1 \otimes Q_1$ or $\mathcal{F}^{\perp} \langle a_1, a_2\rangle \langle y, v\rangle)$	∀-Intro, line 4
23	$\mid\mid \underline{\langle a_1, a_2\rangle \in_{h} H_{\otimes}^{\bullet}}$	assume
24	$\mid\mid \langle a_1, a_2\rangle \in_{h} [f^{\perp}\rangle (Q_1 \otimes Q_2)$	$[-^{\perp}\rangle$ well-pointed, Lemma 4.2.7, line 23
25	$\mid \langle a_1, a_2\rangle \in_{h} [f^{\perp}\rangle (Q_1 \otimes Q_2)$	∨-Elim, lines 2, 3, 23

The proofs for $[f^{\perp}\cdot\rangle$ are similar. Let us try to shorten that second proof and remove the use of the Frame Condition 4.2.4.

1	$\mid \underline{\langle a_1, a_2\rangle \in_{h} [f_1^{\perp}\rangle Q_1 \otimes [f_2^{\perp}\rangle Q_2}$	assume
2	$\mid \langle a_1, a_2\rangle \in_{h} [f_1^{\perp}\rangle Q_1 \times [f_2^{\perp}\rangle Q_2$	Lemma 3.4.2, line 1
3	$\mid\mid \underline{\boxed{b_1, b_2} \quad \langle b_1, b_2\rangle \in_{k} Q_1 \otimes Q_2}$	assume
4	$\mid\mid a_1 \in_{h_1} [f_1^{\perp}\rangle Q_1$ and $a_2 \in_{h_2} [f_2^{\perp}\rangle Q_2$	def. ×, line 2
5	$\mid\mid b_1 \in_{k_1} Q_1$ and $b_2 \in_{k_2} Q_2$	Lemma 3.4.2, def. ×, line 3
6	$\mid\mid \mathcal{F}_1^{\perp} a_1 b_1$ and $\mathcal{F}_2^{\perp} a_2 b_2$	def. $[f_1^{\perp}\rangle, [f_2^{\perp}\rangle$, lines 4, 5
7	$\mid\mid \mathcal{F}^{\perp} \langle a_1, a_2\rangle \langle b_1, b_2\rangle$	def. $\mathcal{F}^{\perp}$, line 6
8	$\mid \langle a_1, a_2\rangle \in_{h} [f^{\perp}\rangle (Q_1 \otimes Q_2)$	def. $[f^{\perp}\rangle$, line 3

■

4.3 Info for $[-^{\perp\cdot}\rangle$

$[-^{\perp\cdot}\rangle$ is of type $\vee\!\mapsto\!\wedge$.

Definition 4.3.1

$$y \in_{\mathsf{k}} [f^{\perp\cdot}\rangle P \text{ iff } \forall x(x \notin_{\mathsf{h}} P \text{ or } \mathcal{F}^{\perp\cdot}xy).$$

Definition 4.3.2 The canonical relation is

$$\mathcal{F}^{\perp\cdot}xy \text{ iff } \exists a(a \in_{\mathsf{h}} x \text{ and } [f^{\perp\cdot}\rangle a \in_{\mathsf{k}} y).$$

The negation of the canonical definition is

$$\neg\mathcal{F}^{\perp\cdot}xy \text{ iff } \forall a(a \notin_{\mathsf{h}} x \text{ or } [f^{\perp\cdot}\rangle a \notin_{\mathsf{k}} y).$$

Frame Condition 4.3.3 The null frame conditions for $\mathcal{F}^{\perp\cdot}$ are

$$(1) \quad \forall y(\neg\mathcal{F}^{\perp\cdot}H^{\circ}y) \qquad (2) \quad \forall x(\neg\mathcal{F}^{\perp\cdot}xK^{\circ})$$

Frame Condition 4.3.4 The well frame conditions for $\mathcal{F}^{\perp\cdot}$ are

$$(1) \quad \forall x(x \notin_{\mathsf{h}} H^{\circ} \text{ implies } \mathcal{F}^{\perp\cdot}xK^{\bullet}) \qquad (2) \quad \forall y(y \notin_{\mathsf{k}} K^{\circ} \text{ implies } \mathcal{F}^{\perp\cdot}H^{\bullet}y).$$

Lemma 4.3.5 *Canonically, the Frame Conditions hold.*

Proof: The Frame Conditions 4.3.3 (1) and 4.3.3 (2): let $x = \circ_H$ or $y = \circ_K$, then

$$\forall a(a \notin_{\mathsf{h}} x \text{ or } [f^{\perp\cdot}\rangle a \notin_{\mathsf{k}} y)$$

holds since $\circ_H = \emptyset$ and $\circ_K = \emptyset$. So $\neg\mathcal{F}^{\perp\cdot}xy$ in both cases.

The Frame Condition 4.3.4 (1): let $x \neq \circ_H$ and $y = \bullet_K$, then $\top_{\mathsf{h}} \in_{\mathsf{h}} x$ and $[f^{\perp\cdot}\rangle\top_{\mathsf{h}} \in_{\mathsf{k}} y$. Hence $\mathcal{F}^{\perp\cdot}x\bullet_K$ obtains.

The Frame Condition 4.3.4 (2): Let $y \neq \circ_K$ and $x = \bullet_H$, then $\perp_{\mathsf{h}} \in_{\mathsf{h}} x$. From normality, $[f^{\perp\cdot}\rangle\perp_{\mathsf{h}} \in_{\mathsf{k}} y$. Hence $\mathcal{F}^{\perp\cdot}H^{\circ}y$ obtains.

■

Lemma 4.3.6 *The Frame Conditions 4.3.3 and 4.3.4 are consistent with each other.*

Proof: The well-point sets and null-point sets are always disjoint. This disjointness and quantifiers and antecedents of the conditionals prevent any conflicts in the Frame Conditions 4.3.3 and 4.3.4.

■

4.4 $[-^{\star}\rangle, [-^{\star}\cdot\rangle$: Closure Under Null Sum

All proofs involving $[-^{\star}\cdot\rangle$ are similar to their siblings for $[-^{\star}\rangle$ and we elide them. The information necessary to construct them is in the following Section 4.5.

$[f^{\star}\rangle$ is of type $\wedge{\mapsto}\vee$:

Definition 4.4.1

$$x \in_{\mathsf{h}} [f^{\star}\rangle Q \text{ iff } \exists y(y \notin_{\mathsf{k}} Q \text{ and } \neg\mathcal{F}^{\star}xy).$$

The canonical relation is

$$\mathcal{F}^{\star}xy \text{ iff } \exists b(b \notin_{\mathsf{k}} y \text{ and } [f^{\star}\rangle\, b \notin_{\mathsf{h}} x)$$

The negation of the canonical relation is

$$\neg\mathcal{F}^{\star}xy \text{ iff } \forall b(b \in_{\mathsf{k}} y \text{ or } [f^{\star}\rangle\, b \in_{\mathsf{h}} x)$$

Frame Condition 4.4.2 The null frame conditions for $\mathcal{F}^{\star}$, which are the same as the backward conditions, are

$$(1) \quad \forall x(x \notin_{\mathsf{h}} H^{\bullet} \text{ implies } \mathcal{F}^{\star}xK^{\circ}) \qquad \text{and} \qquad (2) \quad \forall y(y \notin_{\mathsf{k}} K^{\bullet} \text{ implies } \mathcal{F}^{\star}H^{\circ}y).$$

Frame Condition 4.4.3 The well frame conditions for $\mathcal{F}^{\star}$, which are the same as the backward conditions, are

$$(1) \quad \forall y(\neg\mathcal{F}^{\star}H^{\bullet}y) \qquad \text{and} \qquad (2) \quad \forall x(\neg\mathcal{F}^{\star}xK^{\bullet}).$$

Lemma 4.4.4 *Canonically, the Frame Conditions hold.*

Proof: Frame Condition 4.4.2 (1): choose any x such that $x \neq \bullet_H$ and let $b = \top_{\mathsf{k}}$. Let $y = \circ_K$. Since $y = \emptyset$, then $b \notin_{\mathsf{k}} y$. From normality, $[f^{\star}\rangle \top_{\mathsf{k}} = \bot_{\mathsf{h}}$ and $\bot_{\mathsf{h}} \notin_{\mathsf{h}} x$. Hence $[f^{\star}\rangle\, b \notin_{\mathsf{h}} x$. Hence there is some b such that $b \notin_{\mathsf{k}} y$ and $[f^{\star}\rangle\, b \notin_{\mathsf{h}} x$, whence $\mathcal{F}^{\star}xy$ obtains.

Frame Condition 4.4.2 (2): choose any $y \neq \bullet_K$ and let $b = \bot_{\mathsf{k}}$. Let $x = \circ_H$. Since y is a proper filter, $b \notin_{\mathsf{k}} y$. From $\circ_H = \emptyset$, it follows that $[f^\star\rangle\, b \notin_{\mathsf{h}} x$. Hence there is some b such that $b \notin_{\mathsf{k}} y$ and $[f^\star\rangle\, b \notin_{\mathsf{h}} x$, whence $\mathcal{F}^\star xy$ obtains.

Frame Conditions 4.4.3 (1) and 4.4.3 (2): for all b, $b \in_{\mathsf{k}} \bullet_K$ and $[f^\star\rangle\, b \in_{\mathsf{h}} \bullet_H$. Hence, given the canonical definition of $\neg\mathcal{F}^\star$, for any x and y, the two statements hold. ■

Lemma 4.4.5 *The Frame Conditions 4.4.2 and 4.4.3 are consistent with each other.*

Proof: Note that the well-point sets and null-point sets are always disjoint, then the quantifiers and antecedents of the conditionals Frame Conditions 4.4.2 and 4.4.3 prevent any conflicts.

■

Lemma 4.4.6 *For any* $Q \in_{\mathsf{k}} \mathbb{K}$, $[f^\star\rangle\, Q$ *is well-pointed if* Q *is null-pointed, and always null-pointed.*

Proof: Assume Q is null-pointed and let $y \in_{\mathsf{k}} K^\circ$. Choose some arbitrary $x \in_{\mathsf{h}} H^\bullet$. Since Q is null-pointed, $y \notin_{\mathsf{k}} Q$. From the Frame Condition 4.4.3 (1), $\neg\mathcal{F}^\star H^\bullet y$ and so $\neg\mathcal{F}^\star xy$. So there is some y such that $y \notin_{\mathsf{k}} Q$ and $\neg\mathcal{F}^\star xy$, whence, by definition, $x \in_{\mathsf{h}} [f^\star\rangle\, Q$, and thus $H^\bullet \subseteq_{\mathsf{h}} [f^\star\rangle\, Q$.

Next, assume towards a reductio that $x \in_{\mathsf{h}} H^\circ$ and that $x \in_{\mathsf{h}} [f^\star\rangle\, Q$. From the definition of $[f^\star\rangle\, Q$ there is some y such that $y \notin_{\mathsf{k}} Q$ and $\neg\mathcal{F}^\star xy$. Since Q is well-pointed, $y \notin_{\mathsf{k}} K^\bullet$. From the Frame Condition 4.4.2 (2), $\mathcal{F}^\star H^\circ y$, whence $\mathcal{F}^\star xy$, which is a contradiction. Hence $x \notin_{\mathsf{h}} [f^\star\rangle\, Q$ and $[f^\star\rangle\, Q$ is null-pointed.

■

Lemma 4.4.7 $[f^\star\rangle$ *preserves its bounds.*

Proof: The type of $[f^\star\rangle$ is $\wedge\!\mapsto\!\!\!\rightarrow\!\vee$, hence we want $[f^\star\rangle\, \top^{\mathbb{K}} = \bot^{\mathbb{H}}$. The definition is

$$x \in_{\mathsf{h}} [f^\star\rangle\, Q \text{ iff } \exists y (y \notin_{\mathsf{k}} Q \text{ and } \neg\mathcal{F}^\star xy).$$

We have

$$x \in_{\mathsf{h}} [f^\star\rangle\, \top^{\mathbb{K}} \text{ iff } \exists y (y \notin_{\mathsf{k}} \top^{\mathbb{K}} \text{ and } \neg\mathcal{F}^\star xy).$$

From Lemma 4.4.6, $[f^\star\rangle\, \top^{\mathbb{K}}$ is well- and null-pointed. Since $\bot^{\mathbb{H}}$ is the bottom of the lattice, $\bot^{\mathbb{H}} \subseteq [f^\star\rangle\, \top^{\mathbb{K}}$.

Next, either $x \notin_{\mathsf{h}} H^\bullet$ or $x \in_{\mathsf{h}} H^\bullet$. Let $x \notin_{\mathsf{h}} H^\bullet$ and note that $\bot^{\mathbb{H}} = H^\bullet$. We prove a contraposition to show $[f^\star\rangle\, \top^{\mathbb{K}} \subseteq \bot^{\mathbb{H}}$. To prove $x \notin_{\mathsf{h}} [f^\star\rangle\, \top^{\mathbb{K}}$, we must show

$$\forall y (y \in_{\mathsf{k}} \top^{\mathbb{K}} \text{ or } \mathcal{F}^\star xy).$$

We show this in the form

$$\forall y (y \notin_{\mathsf{k}} \top^{\mathbb{K}} \text{ implies } \mathcal{F}^\star xy).$$

The only points not in $\top^{\mathbb{K}}$ are those in K°. So let $y \in_{\mathsf{k}} K^\circ$. From the Frame Condition 4.4.2 (1), $\mathcal{F}^\star x K^\circ$, and so $\mathcal{F}^\star xy$ obtains, and $x \notin_{\mathsf{h}} [f^\star\rangle\, \top^{\mathbb{K}}$. Unwinding the contraposition, for $x \notin_{\mathsf{h}} H^\bullet$, we have $[f^\star\rangle\, \top^{\mathbb{K}} \subseteq \bot^{\mathbb{H}}$.

Now let $x \in_{\mathsf{h}} H^{\bullet}$. From Lemma 4.4.6, $x \in_{\mathsf{h}} [f^{\star}\rangle \top^{\mathbb{K}}$. Since $H^{\bullet} = \bot^{\mathbb{H}}$, then $[f^{\star}\rangle Q \subseteq_{\mathsf{h}} \bot^{\mathbb{H}}$. ■

Lemma 4.4.8

$$x \in_{\mathsf{h}} [f^{\star}\rangle \beta b \text{ iff } x \in_{\mathsf{h}} \beta[f^{\star}\rangle b.$$

Proof: The left to right is the easy direction. Assume $x \in_{\mathsf{h}} [f^{\star}\rangle \beta b$, then there is some y such that $y \notin_{\mathsf{k}} \beta b$ and $\neg\mathcal{F}^{\star}xy$. Using the canonical definition,

$$\neg\mathcal{F}^{\star}xy \text{ iff } \forall a(a \notin_{\mathsf{h}} y \text{ implies } [f^{\star}\rangle\, a \in_{\mathsf{h}} x).$$

Since $b \notin_{\mathsf{k}} y$, then $[f^{\star}\rangle\, b \in_{\mathsf{h}} x$. Hence $x \in_{\mathsf{h}} \beta[f^{\star}\rangle\, b$.

To go in the other direction. Assume $x \in_{\mathsf{h}} \beta[f^{\star}\rangle\, b$, so that $[f^{\star}\rangle\, b \in_{\mathsf{h}} x$. Let

$$\hat{y} = (\lambda p\ .\ [f^{\star}\rangle\, p)^{-1}\overline{x}.$$

From Lemma 3.7.1, $\hat{y}$ is a filter. Also, $b \notin_{\mathsf{k}} \hat{y}$ since $b \in_{\mathsf{k}} \hat{y}$ implies $[f^{\star}\rangle\, b \in_{\mathsf{h}} \overline{x}$, which is a contradiction. Construct y by using Zorn's Lemma to maximalize $\langle \hat{y}, \downarrow b\rangle$. Let $a \notin_{\mathsf{k}} y$ and assume $[f^{\star}\rangle\, a \in_{\mathsf{h}} \overline{x}$. Whence $a \in_{\mathsf{k}} y$, which is a contradiction. Therefore $[f^{\star}\rangle\, a \in_{\mathsf{h}} x$ and we have shown that

$$\forall a(a \notin_{\mathsf{k}} y \text{ implies } [f^{\star}\rangle\, a \in_{\mathsf{h}} x)$$

This latter is $\neg\mathcal{F}^{\star}xy$:

$$\begin{aligned} \neg\exists a(a \notin_{\mathsf{k}} y \text{ and } [f^{\star}\rangle\, a \notin_{\mathsf{h}} x) &\text{ iff } \forall a(a \in_{\mathsf{k}} y \text{ or } [f^{\star}\rangle\, a \in_{\mathsf{h}} x) \\ &\text{ iff } \forall a(a \notin_{\mathsf{k}} y \text{ implies } [f^{\star}\rangle\, a \in_{\mathsf{h}} x) \end{aligned}$$

Therefore there exists a y, such that $b \notin_{\mathsf{k}} y$ and $\neg\mathcal{F}^{\star}xy$, whence $x \in_{\mathsf{h}} [f^{\star}\rangle \beta b$.

■

We assume $\neg f = \neg f_1 \ovee \neg f_2$, then

$$\neg\mathcal{F}^{\star}\langle a_1, a_2\rangle\langle b_1, b_2\rangle \text{ iff } \neg\mathcal{F}_1^{\star}a_1b_1 \text{ or } \neg\mathcal{F}_2^{\star}a_2b_2.$$

Lemma 4.4.9 *Let* $\neg f = \neg f_1 \ovee \neg f_2$, *then*

$$[f^{\star}\rangle\,(Q_1 \ovee Q_2) = [f_1^{\star}\rangle Q_1 \ovee [f_2^{\star}\rangle Q_2 \qquad [f^{\star}\cdot\rangle(P_1 \ovee P_2) = [f_1^{\star}\rangle P_1 \ovee [f_2^{\star}\rangle P_2$$

Proof:

1	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f^\star\rangle(Q_1 \circledvee Q_2)$	assume
2	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f^\star\rangle(Q_1 \circledvee Q_2) - H^\bullet_{\circledvee}$ or $\langle a_1, a_2\rangle \in_{\mathsf{h}} H^\bullet_{\circledvee}$	Lemma 4.4.6, line 1
3	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f^\star\rangle(Q_1 \circledvee Q_2) - H^\bullet_{\circledvee}$	assume
4	$\langle a_1, a_2\rangle \notin_{\mathsf{h}} H^\bullet_{\circledvee}$	set th., line 3
5	$a_1 \notin_{\mathsf{h}_1} H^\bullet_1$ and $a_2 \notin_{\mathsf{h}_2} H^\bullet_2$	Lemma 3.5.3, line 4
6	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f^\star\rangle(Q_1 \circledvee Q_2)$	set th., line 3
7	$\exists\langle y, v\rangle(\langle y, v\rangle \notin_{\mathsf{k}} Q_1 \circledvee Q_2$ and $\neg\mathcal{F}^\star\langle a_1, a_2\rangle\langle y, v\rangle)$	def. $[-^\star\rangle$, line 6
8	$\boxed{b_1, b_2}$ $\langle b_1, b_2\rangle \notin_{\mathsf{k}} Q_1 \circledvee Q_2$ and $\neg\mathcal{F}^\star\langle a_1, a_2\rangle\langle b_1, b_2\rangle$	assume
9	$\neg\mathcal{F}^\star_1 a_1 b_1$ or $\neg\mathcal{F}^\star_2 a_2 b_2$	def. $\neg\mathcal{F}^\star$, line 7
10	$\langle b_1, b_2\rangle \in_{\mathsf{k}} K_1 \circledvee K_2 - Q_1 \circledvee Q_2$	$K_1 \circledvee K_2$ is the universe, line 8
11	$(b_1 \notin_{\mathsf{k}_1} Q_1$ and $b_2 \in_{\mathsf{k}_2} K^\circ_2)$ or $(b_1 \in_{\mathsf{k}_1} K^\circ_1$ and $b_2 \notin_{\mathsf{k}_2} Q_2)$	Lemma 3.5.4, line 10
12	$b_1 \in_{\mathsf{k}_1} K^\circ_1$ and $b_2 \notin_{\mathsf{k}_2} Q_2$	assume
13	$\mathcal{F}^\star_1 a_1 b_1$	Frame Condition 4.4.2 (1), lines 5, 12
14	$\neg\mathcal{F}^\star_2 a_2 b_2$	CL, lines 9, 13
15	$b_2 \notin_{\mathsf{k}_2} Q_2$ and $\neg\mathcal{F}^\star_2 a_2 b_2$	$\wedge$-Elim, $\wedge$-Intro lines 12, 14
16	$\exists v(v \notin_{\mathsf{k}_2} Q_2$ and $\neg\mathcal{F}^\star_2 a_2 v)$	$\exists$-Intro line 15
17	$a_2 \in_{\mathsf{h}_2} [f_2^\star\rangle Q_2$	def. $[-^\star\rangle$ line 16
18	$a_1 \in_{\mathsf{h}_1} H^\circ_1$ or $a_2 \in_{\mathsf{h}_2} H^\circ_2$	$H_1 \circledvee H_2$ is the universe, line 1
19	$a_2 \notin_{\mathsf{h}_2} H^\circ_2$	$[-^\star\rangle$ null-pointed, Lemma 4.4.6, line 17
20	$a_1 \in_{\mathsf{h}_1} H^\circ_1$	CL, lines 18, 19
21	$\langle a_1, a_2\rangle \in_{\mathsf{h}} H^\circ_1 \times [f_2^\star\rangle Q_2$	def. ×, lines 17, 20
22	$\langle a_1, a_2\rangle \in_{\mathsf{h}} ([f_1^\star\rangle Q_1 \times H^\circ_2) \cup (H^\circ_1 \times [f_2^\star\rangle Q_2)$	set th., line 21
23	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f_1^\star\rangle Q_1 \circledvee [f_2^\star\rangle Q_2$	def. $\circledvee$, line 22
24	$b_1 \notin_{\mathsf{k}_1} Q_1$ and $b_2 \in_{\mathsf{k}_2} K^\circ_2$	assume
25	subproof is similar to previous subproof	lines xx
26	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f_1^\star\rangle Q_1 \circledvee [f_2^\star\rangle Q_2$	def. $\circledvee$, lines xx
27	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f_1^\star\rangle Q_1 \circledvee [f_2^\star\rangle Q_2$	$\vee$-Elim, lines 11, 12, 24
28	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f_1^\star\rangle Q_1 \circledvee [f_2^\star\rangle Q_2$	$\exists$-Elim, lines 7, 8
29	$\langle a_1, a_2\rangle \in_{\mathsf{h}} H^\bullet_{\circledvee}$	assume
30	$\langle a_1, a_2\rangle \in_{\mathsf{h}} H^\bullet_1 \times H^\circ_2$ or $\langle a_1, a_2\rangle \in_{\mathsf{h}} H^\circ_1 \times H^\bullet_2$	def. $\circledvee$, line 29
31	$a_1 \in_{\mathsf{h}_1} [f_1^\star\rangle Q_1$ or $a_2 \in_{\mathsf{h}_2} [f^\star\rangle Q_2$	Lemma 4.4.6, $[f_1^\star\rangle Q_1, [f_2^\star\rangle Q_2$ well-pointed, line 30
32	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f_1^\star\rangle Q_1 \circledvee [f_2^\star\rangle Q_2$	def. $\circledvee$, lines 30, 31
33	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f_1^\star\rangle Q_1 \circledvee [f_2^\star\rangle Q_2$	$\vee$-Elim, lines 2, 3, 29

To fill in the similar subproof above:

1	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f^\star\rangle (Q_1 \varovee Q_2)$	assume
2	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f^\star\rangle (Q_1 \varovee Q_2) - H^\bullet_{\varovee}$ or $\langle a_1, a_2\rangle \in_{\mathsf{h}} H^\bullet_{\varovee}$	Lemma 4.4.6, line 1
3	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f^\star\rangle (Q_1 \varovee Q_2) - H^\bullet_{\varovee}$	assume
4	$\langle a_1, a_2\rangle \notin_{\mathsf{h}} H^\bullet_{\varovee}$	set th., line 3
5	$a_1 \notin_{\mathsf{h}_1} H^\bullet_1$ and $a_2 \notin_{\mathsf{h}_2} H^\bullet_2$	Lemma 3.5.3, line 4
6	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f^\star\rangle (Q_1 \varovee Q_2)$	set th., line 3
7	$\exists\langle y, v\rangle(\langle y, v\rangle \notin_{\mathsf{k}} Q_1 \varovee Q_2$ and $\neg\mathcal{F}^\star\langle a_1, a_2\rangle\langle y, v\rangle)$	def. $[-^\star\rangle$, line 6
8	$\boxed{b_1, b_2}$ $\langle b_1, b_2\rangle \notin_{\mathsf{k}} Q_1 \varovee Q_2$ and $\neg\mathcal{F}^\star\langle a_1, a_2\rangle\langle b_1, b_2\rangle$	assume
9	$\neg\mathcal{F}^\star_1 a_1 b_1$ or $\neg\mathcal{F}^\star_2 a_2 b_2$	def. $\mathcal{F}^\star$, line 7
10	$\langle b_1, b_2\rangle \in_{\mathsf{k}} K_1 \varovee K_2 - Q_1 \varovee Q_2$	$K_1 \varovee K_2$ is the universe, line 8
11	$(b_1 \notin_{\mathsf{k}_1} Q_1$ and $b_2 \in_{\mathsf{k}_2} K^\circ_2)$ or $(b_1 \in_{\mathsf{k}_1} K^\circ_1$ and $b_2 \notin_{\mathsf{k}_2} Q_2)$	Lemma 3.5.4, line 10
12	$b_1 \notin_{\mathsf{k}_1} Q_1$ and $b_2 \in_{\mathsf{k}_2} K^\circ_2$	assume
13	$\mathcal{F}^\star_2 a_2 b_2$	Frame Condition 4.4.2 (1), lines 5, 12
14	$\neg\mathcal{F}^\star_1 a_1 b_1$	CL, lines 9, 13
15	$b_1 \notin_{\mathsf{k}_1} Q_1$ and $\neg\mathcal{F}^\star_1 a_1 b_1$	$\wedge$-Elim, $\wedge$-Intro lines 12, 14
16	$\exists y(y \notin_{\mathsf{k}_1} Q_1$ and $\neg\mathcal{F}^\star_1 a_1 y)$	$\exists$-Intro line 15
17	$a_1 \in_{\mathsf{h}_1} [f_1^\star\rangle Q_1$	def. $[-^\star\rangle$ line 16
18	$a_1 \in_{\mathsf{h}_1} H^\circ_1$ or $a_2 \in_{\mathsf{h}_2} H^\circ_2$	$H_1 \varovee H_2$ is the universe, line 1
19	$a_1 \notin_{\mathsf{h}_1} H^\circ_1$	$[-^\star\rangle$ null-pointed, 4.4.6, line 17
20	$a_2 \in_{\mathsf{h}_2} H^\circ_2$	CL, lines 18, 19
21	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f_1^\star\rangle Q_1 \times H^\circ_2$	def. $\times$, lines 17, 20
22	$\langle a_1, a_2\rangle \in_{\mathsf{h}} ([f_1^\star\rangle Q_1 \times H^\circ_2) \cup (H^\circ_1 \times [f_2^\star\rangle Q_2)$	set th., line 21
23	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f_1^\star\rangle Q_1 \varovee [f_2^\star\rangle Q_2$	def. $\varovee$, line 22

Now the other direction

1	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f_1^{\star}\rangle Q_1 \circledvee [f_2^{\star}\rangle Q_2$	assume
2	$(a_1 \in_{\mathsf{h}_1} [f_1^{\star}\rangle Q_1$ and $a_2 \in_{\mathsf{h}_2} H_2^{\circ})$ or $(a_1 \in_{\mathsf{h}_1} H_1^{\circ}$ and $a_2 \in_{\mathsf{h}_2} [f_2^{\star}\rangle Q_2)$	def. $\circledvee$, line 1
3	$a_1 \in_{\mathsf{h}_1} [f_1^{\star}\rangle Q_1$ and $a_2 \in_{\mathsf{h}_2} H_2^{\circ}$	assume
4	$\exists y(y \notin_{\mathsf{k}_1} Q_1$ and $\neg\mathcal{F}_1^{\star} a_1 y)$	def. $[-^{\star}\rangle$, line 3
5	$\boxed{b_1}$ $b_1 \notin_{\mathsf{k}_1} Q_1$ and $\neg\mathcal{F}_1^{\star} a_1 b_1$	assume
6	$\exists v(v \in_{\mathsf{k}_2} K_2^{\circ})$	$K_2^{\circ} \neq \emptyset$
7	$\boxed{c}$ $c \in_{\mathsf{k}_2} K_2^{\circ}$	assume
8	$\langle b_1, c\rangle \notin_{\mathsf{k}} Q_1 \times K_2^{\circ}$	def $\times$, lines 5, 7
9	$c \notin_{\mathsf{k}_2} Q_2$	Q_2 is null-pointed, line 7
10	$\langle b_1, c\rangle \notin_{\mathsf{k}} K_1^{\circ} \times Q_2$	def. $\times$, line 9
11	$\langle b_1, c\rangle \notin_{\mathsf{k}} (Q_1 \times K_2^{\circ}) \cup (K_1^{\circ} \times Q_2)$	set th., lines 8, 10
12	$\langle b_1, c\rangle \notin_{\mathsf{k}} Q_1 \circledvee Q_2$	def. $\circledvee$, line 11
13	$\neg\mathcal{F}_1^{\star} a_1 b_1$ or $\neg\mathcal{F}_2^{\star} a_2 c$	$\wedge$-Elim, $\vee$-Intro, line 5
14	$\neg\mathcal{F}^{\star}\langle a_1, a_2\rangle\langle b_1, c\rangle$	def. $\neg\mathcal{F}^{\star}$, line 13
15	$\langle b_1, c\rangle \notin_{\mathsf{k}} Q_1 \circledvee Q_2$ and $\neg\mathcal{F}^{\star}\langle a_1, a_2\rangle\langle b_1, c\rangle$	$\wedge$-Intro, lines 12, 14
16	$\exists\langle y, v\rangle(\langle y, v\rangle \notin_{\mathsf{k}} Q_1 \circledvee Q_2$ and $\neg\mathcal{F}^{\star}\langle a_1, a_2\rangle\langle y, v\rangle)$	$\exists$-Intro, line 15
17	$\langle a_1, a_2\rangle \in_{\mathsf{h}} \top_1^{\mathbb{H}} \times H_2^{\circ}$	set th., line 3
18	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f^{\star}\rangle(Q_1 \circledvee Q_2)$	def. $[-^{\star}\rangle$, lines 16, 17
19	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f^{\star}\rangle(Q_1 \circledvee Q_2)$	$\exists$-Elim, lines 6, 7
20	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f^{\star}\rangle(Q_1 \circledvee Q_2)$	$\exists$-Elim, lines 4, 5
21	$a_2 \in_{\mathsf{h}_2} [f_2^{\star}\rangle Q_2$ and $a_1 \in_{\mathsf{h}_1} H_1^{\circ}$	assume
22	subproof is similar to previous subproof	lines xx
23	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f^{\star}\rangle(Q_1 \circledvee Q_2)$	def. $[-^{\star}\rangle$, line xx
24	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f^{\star}\rangle(Q_1 \circledvee Q_2)$	$\vee$-Elim, lines 2, 3, 21

To fill in the similar subproof above:

1	$\langle a_1, a_2 \rangle \in_{\mathsf{h}} [f_1^\star\rangle Q_1 \varovee [f_2^\star\rangle Q_2$	assume
2	$(a_1 \in_{\mathsf{h}_1} [f_1^\star\rangle Q_1$ and $a_2 \in_{\mathsf{h}_2} H_2^\circ)$ or $(a_1 \in_{\mathsf{h}_1} H_1^\circ$ and $a_2 \in_{\mathsf{h}_2} [f_2^\star\rangle Q_2)$	def. $\varovee$, line 1
3	$\quad a_2 \in_{\mathsf{h}_2} [f_2^\star\rangle Q_2$ and $a_1 \in_{\mathsf{h}_1} H_1^\circ$	assume
4	$\quad \exists v(v \notin_{\mathsf{k}_2} Q_2$ and $\neg \mathcal{F}_2^\star a_2 v)$	def. $[-^\star\rangle$, line 3
5	$\quad\quad \boxed{b_2}\ \ b_2 \notin_{\mathsf{k}_2} Q_2$ and $\neg \mathcal{F}_2^\star a_2 b_2$	assume
6	$\quad\quad \exists y(y \in_{\mathsf{k}_1} K_1^\circ)$	$K_1^\circ \neq \emptyset$
7	$\quad\quad\quad \boxed{c}\ \ c \in_{\mathsf{k}_1} K_1^\circ$	assume
8	$\quad\quad\quad \langle c, b_2 \rangle \notin_{\mathsf{k}} K_1^\circ \times Q_2$	def $\times$, lines 5, 7
9	$\quad\quad\quad c \notin_{\mathsf{k}_1} Q_1$	Q_1 is null-pointed, line 7
10	$\quad\quad\quad \langle c, b_2 \rangle \notin_{\mathsf{k}} Q_1 \times K_2^\circ$	def. $\times$, line 9
11	$\quad\quad\quad \langle c, b_2 \rangle \notin_{\mathsf{k}} (Q_1 \times K_2^\circ) \cup (K_1^\circ \times Q_2)$	set th., lines 8, 10
12	$\quad\quad\quad \langle c, b_2 \rangle \notin_{\mathsf{k}} Q_1 \varovee Q_2$	def. $\varovee$, line 11
13	$\quad\quad\quad \neg \mathcal{F}_1^\star a_1 c$ or $\neg \mathcal{F}_2^\star a_2 b_2$	$\wedge$-Elim, $\vee$-Intro, line 5
14	$\quad\quad\quad \neg \mathcal{F}^\star \langle a_1, a_2 \rangle \langle c, b_2 \rangle$	def. $\neg \mathcal{F}^\star$, line 13
15	$\quad\quad\quad \langle c, b_2 \rangle \notin_{\mathsf{k}} Q_1 \varovee Q_2$ and $\neg \mathcal{F}^\star \langle a_1, a_2 \rangle \langle c, b_2 \rangle$	$\wedge$-Intro, lines 12, 14
16	$\quad\quad\quad \exists \langle y, v \rangle (\langle y, v \rangle \notin_{\mathsf{k}} Q_1 \varovee Q_2$ and $\neg \mathcal{F}^\star \langle a_1, a_2 \rangle \langle y, v \rangle)$	$\exists$-Intro, line 15
17	$\quad\quad\quad \langle a_1, a_2 \rangle \in_{\mathsf{h}} H_1^\circ \times \top_2^{\mathbb{H}}$	set th., line 3
18	$\quad\quad\quad \langle a_1, a_2 \rangle \in_{\mathsf{h}} [f^\star\rangle (Q_1 \varovee Q_2)$	def. $[-^\star\rangle$, lines 16, 17
19	$\quad\quad \langle a_1, a_2 \rangle \in_{\mathsf{h}} [f^\star\rangle (Q_1 \varovee Q_2)$	$\exists$-Elim, lines 6, 7
20	$\quad \langle a_1, a_2 \rangle \in_{\mathsf{h}} [f^\star\rangle (Q_1 \varovee Q_2)$	$\exists$-Elim, lines 4, 5

■

Incidentally, trying to use merely well-pointed models, and not the well- and null-pointed models, fails for the obvious reason that if we pick $Q = \top^{\mathbb{K}}$, then there is no way to satisfy

$$x \in_{\mathsf{h}} [f^\star\rangle \top^{\mathbb{K}} \text{ iff } \exists y(y \notin_{\mathsf{k}} \top^{\mathbb{K}} \text{ and } \neg \mathcal{F}^\star xy)$$

because

$$\forall y(y \in_{\mathsf{k}} \top^{\mathbb{K}})$$

in such models.

Setting $Q_2 = \top_{\mathbb{H}_2}$ causes the case at line 12 in the first proof to close. The rest works as is.

4.5 Info for $[-^\star\cdot\rangle$

$[f^\star\rangle$ is of type $\wedge\!\mapsto\!\vee$:

Definition 4.5.1

$$y \in_{\mathsf{k}} [f^{\star}\cdot\rangle P \text{ iff } \exists x(x \notin_{\mathsf{h}} P \text{ and } \neg\mathcal{F}^{\star\cdot}xy).$$

The canonical relation is

$$\mathcal{F}^{\star\cdot}xy \text{ iff } \exists a(a \notin_{\mathsf{h}} x \text{ and } [f^{\star}\cdot\rangle a \notin_{\mathsf{k}} y)$$

The negation of the canonical relation is

$$\neg\mathcal{F}^{\star\cdot}xy \text{ iff } \forall a(a \in_{\mathsf{h}} x \text{ or } [f^{\star}\cdot\rangle a \in_{\mathsf{k}} y)$$

Frame Condition 4.5.2 The null frame conditions for $\mathcal{F}^{\star\cdot}$ are

$$(1) \quad \forall x(x \notin_{\mathsf{h}} H^{\bullet} \text{ implies } \mathcal{F}^{\star\cdot}xK^{\circ}) \qquad \text{and} \qquad (2) \quad \forall y(y \notin_{\mathsf{h}} K^{\bullet} \text{ implies } \mathcal{F}^{\star\cdot}H^{\circ}y).$$

Frame Condition 4.5.3 The well frame conditions for $\mathcal{F}^{\star\cdot}$ are

$$(1) \quad \forall y(\neg\mathcal{F}^{\star\cdot}H^{\bullet}y) \qquad \text{and} \qquad (2) \quad \forall x(\neg\mathcal{F}^{\star\cdot}xK^{\bullet}).$$

Lemma 4.5.4 *Canonically, the Frame Conditions hold.*

Proof: Frame Condition 4.5.2 (1): assume $x \neq \bullet_H$ and let $y = \circ_K$, then there is some $a \notin_{\mathsf{h}} x$ and $[f^{\star}\cdot\rangle a \notin_{\mathsf{k}} y$. So $\mathcal{F}^{\star\cdot}xy$ obtains.

Frame Condition 4.5.2 (2): assume $y \neq \bullet_K$ and let $x = \circ_H$, then $\top_{\mathsf{h}} \notin_{\mathsf{h}} x$. From normality, $[f^{\star}\cdot\rangle\top_{\mathsf{h}} = \bot_{\mathsf{k}}$ and hence $[f^{\star}\cdot\rangle\top_{\mathsf{h}} \notin_{\mathsf{k}} y$. So there is some a such that $a \notin_{\mathsf{h}} x$ and $[f^{\star}\cdot\rangle a \notin_{\mathsf{k}} y$, whence $\mathcal{F}^{\star\cdot}xy$ obtains.

Frame Conditions 4.5.3 (1) and 4.5.3 (2): for all a, $a \in_{\mathsf{h}} \bullet_H$ and $[f^{\star}\cdot\rangle a \in_{\mathsf{k}} \bullet_K$. Hence, for any y, $\neg\mathcal{F}^{\star\cdot} \bullet_H y$, and for any x, $\neg\mathcal{F}^{\star\cdot}x\bullet_K$ obtains. ∎

Lemma 4.5.5 *The Frame Conditions 4.5.2 and 4.5.3 are consistent with each other.*

Proof: The well-point sets and null-point sets are always disjoint. This disjointness and quantifiers and antecedents of the conditionals prevent any conflicts in the Frame Conditions 4.5.2 and 4.5.3.

∎

4.6 $[-^{\star}\rangle\,, [-^{\star}\cdot\rangle$: Closure Under Infinite Null Sum

The representation we will ultimately use is

$$\vec{x} \mathrel{\in\mathsf{h}} \Pi_i H_i, \text{ such that } \exists! j \forall i (i \neq j \text{ implies } \vec{x}_i \mathrel{\in\mathsf{h}_i} H_i^{\circ}).$$

This leaves the possibility that for $i = j$, $\vec{x}_j \neq H_j^{\circ}$.

We let $\langle a \rangle$ and $\langle b \rangle$ refer to vectors from here on where

$$\exists j! \; \forall i \; \left(\pi_i \langle x \rangle \mathrel{\in\mathsf{h}_i} \begin{cases} H_j & i = j \\ H_i^{\circ} & \text{otherwise} \end{cases} \right)$$

A similar statement holds for $\langle b \rangle$ with respect to k. The reasoning is mundane, using $\vec{a}$ just looks bad in the typeset.

We assume $f = \ovee_i f_i$, then

$$\begin{aligned} \neg\mathcal{F}^{\star}\langle a \rangle \langle b \rangle &\text{ iff } \bigvee \neg\mathcal{F}_i^{\star} a_i b_i \\ &\text{ iff } \exists i (\neg\mathcal{F}_i^{\star} a_i b_i.) \end{aligned}$$

Another way of stating this is

$$\neg\mathcal{F}^{\star} = \amalg_i \neg\mathcal{F}_i^{\star}.$$

Lemma 4.6.1 *Let $f = \ovee_i f_i$, then*

$$[f^{\star}\rangle\,(\ovee_i Q_i) = \ovee_i [f_i^{\star}\rangle\, Q_i \qquad [f^{\star}\cdot\rangle(\ovee_i Q_i) = \ovee_i [f_i^{\star}\cdot\rangle Q_i$$

Proof:

1	$\langle a\rangle \in_{\mathsf{h}} [f^\star\rangle(\ovee_i Q_i)$	assume
2	$\langle a\rangle \in_{\mathsf{h}} [f^\star\rangle(\ovee_i Q_i) - H^\bullet_{\ovee}$ or $\langle a\rangle \in_{\mathsf{h}} H^\bullet_{\ovee}$	Lemma 4.4.6, line 1
3	$\langle a\rangle \in_{\mathsf{h}} [f^\star\rangle(\ovee_i Q_i) - H^\bullet_{\ovee}$	assume
4	$\langle a\rangle \notin_{\mathsf{h}} H^\bullet_{\ovee}$	set th., line 3
5	$\forall i(a_i \notin_{\mathsf{h}_i} H^\bullet_i)$	Lemma 3.6.3, line 4
6	$\langle a\rangle \in_{\mathsf{h}} [f^\star\rangle(\ovee_i Q_i)$	set th., line 3
7	$\exists\langle y\rangle(\langle b\rangle \notin_{\mathsf{k}} \ovee_i Q_i$ and $\neg\mathcal{F}^\star\langle a\rangle\langle y\rangle)$	def. $[-^\star\rangle$, line 6
8	$\boxed{\langle b\rangle}$ $\langle b\rangle \notin_{\mathsf{k}} \ovee_i Q_i$ and $\neg\mathcal{F}^\star\langle a\rangle\langle b\rangle$	assume
9	$\bigvee \neg\mathcal{F}^\star_i a_i b_i$	def. $\neg\mathcal{F}^\star$, line 7
10	$\langle b\rangle \in_{\mathsf{k}} \ovee_i K - \ovee_i Q_i$	$\ovee_i K$ is the universe, line 8
11	$\exists n!\forall i(b_n \notin_{\mathsf{k}_n} Q_n$ and $(i \neq n$ implies $b_i \in_{\mathsf{k}_i} K^\circ_i))$	Lemma 3.6.4, line 10
12	$\boxed{j}$ $\forall i(b_j \notin_{\mathsf{k}_j} Q_j$ and $(i \neq j$ implies $b_i \in_{\mathsf{k}_i} K^\circ_i))$	assume
13	$\forall i(i \neq j$ implies $\mathcal{F}^\star_i a_i b_i)$	Frame Condition 4.4.2 (1), lines 5, 12
14	$\neg\mathcal{F}^\star_j a_j b_j$	CL, lines 9, 13
15	$b_j \notin_{\mathsf{k}_j} Q_j$ and $\neg\mathcal{F}^\star_j a_j b_j$	∧-Elim, ∧-Intro lines 12, 14
16	$\exists v(v \notin_{\mathsf{k}_j} Q_j$ and $\neg\mathcal{F}^\star_j a_j v)$	∃-Intro line 15
17	$a_j \in_{\mathsf{h}_j} [f_j^\star\rangle Q_j$	def. $[-^\star\rangle$ line 16
18	$a_j \notin_{\mathsf{h}_j} H^\circ_j$	Lemma 4.4.6 $[-^\star\rangle$ null-pointed, line 17
19	$\forall i(i \neq j$ implies $a_i \in_{\mathsf{h}_i} H^\circ_i)$	$\ovee H_i$ is the universe, line 1
20	$\langle a\rangle \in_{\mathsf{h}} \ovee_i [f_i^\star\rangle Q_i$	def. $\ovee$, lines 17, 19
21	$\langle a\rangle \in_{\mathsf{h}} \ovee_i [f_i^\star\rangle Q_i$	∃-Elim, lines 11, 12
22	$\langle a\rangle \in_{\mathsf{h}} \ovee_i [f_i^\star\rangle Q_i$	∃-Elim, lines 7, 8
23	$\langle a\rangle = H^\bullet_{\ovee}$	assume
24	$\exists n\forall i(a_n \in_{\mathsf{h}} H^\bullet_n$ and $(i \neq n$ implies $a_i \in_{\mathsf{h}} H^\circ_i))$	def. $H^\bullet_{\ovee}$, Definition 3.2.3, line 23
25	$\exists n\forall i(a_n \in_{\mathsf{h}} [f_n^\star\rangle Q_n$ and $(i \neq n$ implies $a_i \in_{\mathsf{h}} H^\circ_i))$	Lemma 4.4.6, $[-^\star\rangle$ well-pointed, line 24
26	$\langle a\rangle \in_{\mathsf{h}} \ovee_i [f_i^\star\rangle Q_i$	def. $\ovee_i$, Definition 3.6.1, line 25
27	$\langle a\rangle \in_{\mathsf{h}} \ovee_i [f_i^\star\rangle Q_i$	∨-Elim, lines 2, 3, 23

Now the other direction

1	$\langle a\rangle \in_{\mathsf{h}} \ovee_i [f_i^{\star}\rangle Q_i$	assume
2	$\exists n \forall i(a_n \in_{\mathsf{h}_n} [f_n^{\star}\rangle Q_n \text{ and } (i \neq n \text{ implies } a_i \in_{\mathsf{h}_i} H_i^{\circ}))$	def. $\ovee_i$, Definition 3.6.1, line 1
3	$\boxed{j} \quad \forall i(a_j \in_{\mathsf{h}_j} [f_j^{\star}\rangle Q_j \text{ and } (i \neq j \text{ implies } a_i \in_{\mathsf{h}_i} H_i^{\circ}))$	assume
4	$\exists y(y \notin_{\mathsf{k}_j} Q_j \text{ and } \neg\mathcal{F}_j^{\star} a_j y)$	def. $[-^{\star}\rangle$, line 3
5	$\forall l\, \exists v_l(l \neq j \text{ implies } v_l \in_{\mathsf{k}_l} K_l^{\circ})$	for all i, $K_l^{\circ} \neq \emptyset$
6	$\boxed{b_j} \quad b_j \notin_{\mathsf{k}_j} Q_j \text{ and } \neg\mathcal{F}_j^{\star} a_j b_j$	assume
7	$\boxed{l}$	assume
8	$\exists v_l(l \neq j \text{ implies } v_l \in_{\mathsf{k}_l} K_l^{\circ})$	$\forall$-Elim, line 5
9	$\boxed{b_l} \quad l \neq j \text{ implies } b_l \in_{\mathsf{k}_l} K_l^{\circ}$	assume
10	$l \neq j \text{ implies } b_l \notin_{\mathsf{k}_l} Q_l$	Q_l is null-pointed, line 9
11	$l = j \text{ implies } b_l \notin_{\mathsf{k}_l} Q_l$	CL, line 6
12	$b_l \notin_{\mathsf{k}_l} Q_l$	CL, lines 10, 11
13	$\neg\mathcal{F}_j^{\star} a_j b_j \text{ or } \neg\mathcal{F}_l^{\star} a_l b_l$	$\wedge$-Elim, $\vee$-Intro, line 6
14	$b_l \notin_{\mathsf{k}_l} Q_l \text{ and } (\neg\mathcal{F}_j^{\star} a_j b_j \text{ or } \neg\mathcal{F}_l^{\star} a_l b_l)$	$\wedge$-Intro, lines 12, 13
15	$\exists v_l\big(v_l \notin_{\mathsf{k}_l} Q_l \text{ and } (\neg\mathcal{F}_j^{\star} a_j b_j \text{ or } \neg\mathcal{F}_l^{\star} a_l v_l)\big)$	$\exists$-Intro, line 14
16	$\exists v_l\big(v_l \notin_{\mathsf{k}_l} Q_l \text{ and } (\neg\mathcal{F}_j^{\star} a_j b_j \text{ or } \neg\mathcal{F}_l^{\star} a_l v_l)\big)$	$\exists$-Elim, lines 8, 9
17	$\forall l\, \exists v_l\big(v_l \notin_{\mathsf{k}_l} Q_l \text{ and } (\neg\mathcal{F}^{\star} a_j b_j \text{ or } \neg\mathcal{F}^{\star} a_l v_l)\big)$	$\forall$-Intro, line 7
18	$\exists\langle y\rangle(\langle y\rangle \notin_{\mathsf{k}} \ovee_i Q_i \text{ and } \neg\mathcal{F}^{\star}\langle a\rangle\langle y\rangle)$	def. $\ovee_i$, Definition 3.6.1, def. $\mathcal{F}^{\star}$, line 17
19	$\exists\langle y\rangle(\langle y\rangle \notin_{\mathsf{k}} \ovee_i Q_i \text{ and } \neg\mathcal{F}^{\star}\langle a\rangle\langle y\rangle)$	$\exists$-Elim, lines 4, 6
20	$\langle a\rangle \in_{\mathsf{h}} [f^{\star}\rangle(\ovee_i Q_i)$	def. $[-^{\star}\rangle$, line 19
21	$\langle a\rangle \in_{\mathsf{h}} [f^{\star}\rangle(\ovee_i Q_i)$	$\exists$-Elim, lines 2, 3

■

4.7 $[-^{!}\rangle, [-^{!}\cdot\rangle$: Closure Under Smash Product

All proofs involving $[-^{!}\cdot\rangle$ are similar to their siblings for $[-^{!}\rangle$ and we elide them. The information necessary to construct them is in the following Section 4.8.

The type of $[f^{!}\rangle$ is $\vee\mapsto\wedge$.

Definition 4.7.1

$$x \in_{\mathsf{h}} [f^{!}\rangle Q \text{ iff } \forall y(y \notin_{\mathsf{k}} Q \text{ or } \neg\mathcal{F}^{!}xy).$$

Definition 4.7.2 The canonical relation and its complement are

$$\mathcal{F}^{!}xy \text{ iff } \forall b(b \notin_{\mathsf{k}} y \text{ or } [f^{!}\rangle b \notin_{\mathsf{h}} x) \qquad \neg\mathcal{F}^{!}xy \text{ iff } \exists b(b \in_{\mathsf{k}} y \text{ and } [f^{!}\rangle b \in_{\mathsf{h}} x).$$

Frame Condition 4.7.3

$$(1) \quad \forall y(\mathcal{F}^{!}H^{\circ}y) \qquad (2) \quad \forall x(\mathcal{F}^{!}xK^{\circ})$$

Frame Condition 4.7.4

$$(1)\quad \forall x(x \notin_{\mathsf{h}} H^\circ \text{ implies } \neg\mathcal{F}^{!}xK^\bullet) \qquad (2)\quad \forall y(y \notin_{\mathsf{k}} K^\circ \text{ implies } \neg\mathcal{F}^{!}H^\bullet y)$$

Lemma 4.7.5 *Canonically, the Frame Conditions hold.*

Proof: For Frame Conditions 4.7.3, for all b, $b \notin_{\mathsf{k}} \circ_K$ and $[f^{!}\rangle\, b \notin_{\mathsf{h}} \circ_H$. Hence both $\forall y(\mathcal{F}^{!} \circ_H y)$ and $\forall x(\mathcal{F}^{!}x\circ_K)$ hold.

For Frame Conditions 4.7.4 (1), $\bot_{\mathsf{k}} \in_{\mathsf{k}} \bullet_K$ and $[f^{!}\rangle\, \bot_{\mathsf{k}} = \top_{\mathsf{h}}$. Choose some arbitrary x such that $x \neq \circ_H$ then $\top_{\mathsf{h}} \in_{\mathsf{h}} x$. So there exists some b such that $b \in_{\mathsf{k}} \bullet_K$ and $[f^{!}\rangle\, b \in_{\mathsf{h}} x$, the formula $\forall x(\neg\mathcal{F}^{!}x\bullet_K)$ is satisfied.

For Frame Conditions 4.7.4 (2), choose some arbitrary $y \notin_{\mathsf{k}} \circ_K$. Since $\top_{\mathsf{k}} \in_{\mathsf{k}} y$ and $[f^{!}\rangle\, \top_{\mathsf{k}} \in_{\mathsf{h}} \bullet_H$, then $\forall y(\neg\mathcal{F}^{!} \bullet_H y)$ is satisfied.

■

Lemma 4.7.6 *The Frame Conditions 4.7.3 and 4.7.4 are consistent with each other.*

Proof: Note that the well-point sets and null-point sets are always disjoint, then the quantifiers and antecedents of the conditionals of Frame Conditions 4.7.4 and 4.7.3 prevent any conflicts.

■

Lemma 4.7.7 *$[f^{!}\rangle\, Q$ is well pointed if Q is null-pointed. $[f^{!}\rangle\, Q$ is null-pointed if Q is well-pointed.*

Proof: Assume Q is null-pointed. By definition,

$$x \in_{\mathsf{h}} [f^{!}\rangle\, Q \text{ iff } \forall y(y \notin_{\mathsf{k}} Q \text{ or } \neg\mathcal{F}^{!}xy).$$

Let $x \in_{\mathsf{h}} H^\bullet$. Since Q is null-pointed, $K^\circ \cap Q = \emptyset$. Hence if $y \in_{\mathsf{k}} Q$, then $y \notin_{\mathsf{k}} K^\circ$. From the Frame Condition 4.7.4 (2), the second disjunct is satisfied. Hence $[f^{!}\rangle\, Q$ is well-pointed.

Assume Q is well-pointed. By definition,

$$x \notin_{\mathsf{h}} [f^{!}\rangle\, Q \text{ iff } \exists y(y \in_{\mathsf{k}} Q \text{ and } \mathcal{F}^{!}xy).$$

Let $x \in_{\mathsf{h}} H^\circ$. From the assumption that Q is well-pointed, $K^\bullet \subseteq_{\mathsf{k}} Q$ and $K^\bullet \neq \emptyset$, so there is some $y \in_{\mathsf{k}} K^\bullet$ such that $y \in_{\mathsf{k}} Q$. From the Frame Condition 4.7.3 (1), $\mathcal{F}^{!}H^\circ y$. Hence there is some y such that $y \in_{\mathsf{k}} Q$ and $\mathcal{F}^{!}xy$, whence $x \notin_{\mathsf{h}} [f^{!}\rangle\, Q$.

■

Lemma 4.7.8 $[f^{!}\rangle$ *preserves its bounds.*

Proof: The type of $[f^{!}\rangle$ is $\vee\!\mapsto\!\wedge$, hence we want $[f^{!}\rangle\,\bot^{\mathbb{K}} = \top^{\mathbb{H}}$. The definition is

$$x \in\mathsf{h} [f^{!}\rangle\,\bot^{\mathbb{K}} \text{ iff } \forall y(y \notin\mathsf{k} \bot^{\mathbb{K}} \text{ or } \neg\mathcal{F}^{!}xy).$$

From Lemma 4.7.7, $[f^{!}\rangle\,\bot^{\mathbb{K}}$ is wn-pointed. Since $\top^{\mathbb{H}}$ is the top of the lattice, $[f^{!}\rangle\,\bot^{\mathbb{K}} \subseteq\mathsf{h} \top^{\mathbb{H}}$.

Next, let $x \in\mathsf{h} \top^{\mathbb{H}}$. By definition $\top^{\mathbb{H}} \cap H^{\circ} = \emptyset$, so $x \notin\mathsf{h} H^{\circ}$. Choose some arbitrary y and assume $y \in\mathsf{k} \bot^{\mathbb{K}}$. Since $\bot^{\mathbb{K}} = K^{\bullet}$, then from the Frame Condition 4.7.4 (1), $\neg\mathcal{F}^{!}xy$ obtains. Hence for any y, it is the case that $y \notin\mathsf{k} \bot^{\mathbb{K}}$ or $\neg\mathcal{F}^{!}xy$.

∎

Lemma 4.7.9

$$x \in\mathsf{h} [f^{!}\rangle\beta b \text{ iff } x \in\mathsf{h} \beta[f^{!}\rangle\,b.$$

Proof: The right to left is the easy direction. Assume $x \in\mathsf{h} \beta[f^{!}\rangle\,b$ so $[f^{!}\rangle\,b \in\mathsf{h} x$. Choose some arbitrary y such that $\mathcal{F}^{!}xy$. Using the canonical definition,

$$\mathcal{F}^{!}xy \text{ iff } \forall a(a \notin\mathsf{k} y \text{ or } [f^{!}\rangle\,a \notin\mathsf{h} x),$$

we have $b \notin\mathsf{k} y$ and so $y \notin\mathsf{k} \beta b$. So we have shown $\forall y(\mathcal{F}^{!}xy$ implies $y \notin\mathsf{k} \beta b)$, whence $x \in\mathsf{h} [f^{!}\rangle\beta b$.

We go in the other direction via contraposition. Assume $[f^{\star}\rangle\,b \notin\mathsf{h} x$. Let

$$\hat{y} = (\lambda p\ .\ [f^{!}\rangle\,p)^{-1}x.$$

From Lemma 3.7.1, $\hat{y}$ is an ideal. If $b \in\mathsf{k} \hat{y}$, then $[f^{!}\rangle\,b \in\mathsf{h} x$, which is a contradiction; whence $b \notin\mathsf{k} \hat{y}$. Construct y by using Zorn's Lemma to maximalize $\langle\uparrow b, \hat{y}\rangle$. By construction, $[f^{!}\rangle\,a \in\mathsf{h} x$ implies $a \notin\mathsf{k} y$. We have shown the contrapositive of

$$\forall a(a \in\mathsf{k} y \text{ implies } [f^{!}\rangle\,a \notin\mathsf{h} x).$$

This latter is $\mathcal{F}^{!}xy$. Also by construction $b \in\mathsf{k} y$, whence $y \in\mathsf{k} \beta b$. Therefore there exists some y, such that $y \in\mathsf{k} \beta b$ and $\mathcal{F}^{!}xy$, whence $x \notin\mathsf{h} [f^{!}\rangle\beta b$.

∎

Lemma 4.7.10 *Let* $\neg f = \neg f_1 \otimes \neg f_2$, *then* $\neg\mathcal{F}^{!}\langle a_1, a_2\rangle\langle b_1, b_2\rangle$ iff $\neg\mathcal{F}^{!}_{1}a_1b_1$ and $\neg\mathcal{F}^{!}_{2}a_2b_2$ *and*

$$[f^{!}\rangle\,(Q_1 \otimes Q_2) = [f_1^{!}\rangle\,Q_1 \otimes [f_2^{!}\rangle\,Q_2 \qquad [f^{!\cdot}\rangle(P_1 \otimes P_2) = [f_1^{!\cdot}\rangle P_1 \otimes [f_2^{!\cdot}\rangle P_2$$

Proof:

1	$\langle a_1, a_2\rangle \in_{h} [f^{!}\rangle (Q_1 \otimes Q_2)$	assume
2	$(\langle a_1, a_2\rangle \in_{h} [f^{!}\rangle (Q_1 \otimes Q_2) - H^{\bullet}_{\otimes})$ or $\langle a_1, a_2\rangle \in_{h} H^{\bullet}_{\otimes}$	Lemma 4.7.7, $[-^{!}\rangle$ well-pointed, line 1
3	$\langle a_1, a_2\rangle \in_{h} [f^{!}\rangle (Q_1 \otimes Q_2) - H^{\bullet}_{\otimes}$	assume
4	$\forall\langle y, v\rangle(\langle y, v\rangle \notin_{k} Q_1 \otimes Q_2$ or $\neg\mathcal{F}\langle a_1, a_2\rangle\langle y, v\rangle)$	def. $[-^{!}\rangle$, line 3
5	$\boxed{b_1}$ $b_1 \in_{k_1} Q_1$	assume
6	$b_1 \in_{k_1} Q_1 - K^{\bullet}_1$ or $b_1 \in_{k_1} K^{\bullet}_1$	Q_1 is well-pointed, line 5
7	$b_1 \in_{k_1} Q_1 - K^{\bullet}_1$	assume
8	$\exists u(u \in_{k_2} K^{\bullet}_2)$	$K^{\bullet}_2 \neq \emptyset$
9	$\boxed{b_2}$ $b_2 \in_{k_2} K^{\bullet}_2$	assume
10	$\langle b_1, b_2\rangle \notin_{k} Q_1 \otimes Q_2$ or $\neg\mathcal{F}^{!}\langle a_1, a_2\rangle\langle b_1, b_2\rangle$	$\forall$-Elim, line 4
11	$\langle b_1, b_2\rangle \notin_{k} Q_1 \otimes Q_2$	assume
12	$b_1 \notin_{k_1} Q_1$ or $b_2 \notin_{h_2} Q_2$	Lemma 3.4.2, line 11
13	$b_1 \notin_{k_1} Q_1$	CL, Q_2 is well-pointed, lines 9, 12
14	$\rightarrow\leftarrow$	Contradiction, lines 5, 11
15	$\neg\mathcal{F}^{!}\langle a_1, a_2\rangle\langle b_1, b_2\rangle$	CL, lines 10, 11
16	$\neg\mathcal{F}^{!}_1 a_1 b_1$	def. $\mathcal{F}^{!}$, line 15
17	$\neg\mathcal{F}^{!}_1 a_1 b_1$	$\exists$-Elim, lines 8, 9
18	$b_1 \in_{k_1} K^{\bullet}_1$	assume
19	$a_1 \in_{h_1} H^{\circ}_1$	assume
20	$\langle a_1, a_2\rangle \in_{h} H^{\circ}_{\otimes}$	Definition 3.2.2, line 19
21	$\rightarrow\leftarrow$	Contradiction, $[-^{!}\rangle$ is null-pointed, Lemma 4.7.7, line 20
22	$a_1 \notin_{h_1} H^{\circ}_1$	CL, line 19
23	$\neg\mathcal{F}^{!}_1 a_1 b_1$	Frame Condition 4.7.4 (1), lines 18, 22
24	$\neg\mathcal{F}^{!}_1 a_1 b_1$	$\vee$-Elim, lines 6, 7, 18
25	$\forall y(y \notin_{k_1} Q_1$ or $\neg\mathcal{F}^{!}_1 a_1 y)$	$\forall$-Intro, CL, line 5
26	$a_1 \in_{h_1} [f^{!}_1\rangle Q_1$	def. $[-^{!}\rangle$, line 25
27	$\boxed{b_2}$ $b_2 \in_{k_2} Q_2$	assume
28	subproof is similar to previous subproof	lines xx
29	$\neg\mathcal{F}^{!}_2 a_2 b_2$	lines xx
30	$\forall v(v \notin_{k_2} Q_2$ or $\neg\mathcal{F}^{!}_2 a_2 v)$	CL, line 27
31	$a_2 \in_{h_2} [f^{!}_2\rangle Q_2$	def. $[-^{!}\rangle$, line 30
32	$\langle a_1, a_2\rangle \in_{h} H_1 \otimes H_2$	$f = f_1 \otimes f_2$, line 1
33	$\langle a_1, a_2\rangle \notin_{h} H^{\bullet}_{\otimes}$ and $\langle a_1, a_2\rangle \in_{h} H_1 \otimes H_2$	$\wedge$-Intro, lines 3, 32
34	$a_1 \notin_{h_1} H^{\bullet}_1$ and $a_2 \notin_{h_2} H^{\bullet}_2$	Lemma 3.4.9, line 33
35	$\langle a_1, a_2\rangle \in_{h} [f^{!}_1\rangle Q_1 \otimes [f^{!}_2\rangle Q_2$	def. $\otimes$, line 34
36	$\langle a_1, a_2\rangle \in_{h} H^{\bullet}_{\otimes}$	assume
37	$\langle a_1, a_2\rangle \in_{h} [f^{!}_1\rangle Q_1 \otimes [f^{!}_2\rangle Q_2$	Lemma 4.7.7, $[-^{!}\rangle$ well-pointed, line 36
38	$\langle a_1, a_2\rangle \in_{h} [f^{!}_1\rangle Q_1 \otimes [f^{!}_2\rangle Q_2$	$\vee$-Elim, lines 2, 3, 36

To fill in the similar subproof above

1	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f^{!}\rangle (Q_1 \otimes Q_2)$	assume
2	$(\langle a_1, a_2\rangle \in_{\mathsf{h}} [f^{!}\rangle (Q_1 \otimes Q_2) - H^{\bullet}_{\otimes})$ or $\langle a_1, a_2\rangle \in_{\mathsf{h}} H^{\bullet}_{\otimes}$	Lemma 4.7.7, $[-^{!}\rangle$ well-pointed, line 1
3	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f^{!}\rangle (Q_1 \otimes Q_2) - H^{\bullet}_{\otimes}$	assume
4	$\forall\langle y, v\rangle(\langle y, v\rangle \notin_{\mathsf{k}} Q_1 \otimes Q_2$ or $\neg\mathcal{F}\langle a_1, a_2\rangle\langle y, v\rangle)$	def. $[-^{!}\rangle$
5	$\boxed{b_2}$ $b_2 \in_{\mathsf{k}_2} Q_2$	assume
6	$b_2 \in_{\mathsf{k}_2} Q_2 - K^{\bullet}_2$ or $b_2 \in_{\mathsf{k}_2} K^{\bullet}_2$	Q_2 is well-pointed, line 5
7	$b_2 \in_{\mathsf{k}_2} Q_2 - K^{\bullet}_2$	assume
8	$\exists u(u \in_{\mathsf{k}_1} K^{\bullet}_1)$	$K^{\bullet}_1 \neq \emptyset$
9	$\boxed{b_1}$ $b_1 \in_{\mathsf{k}_1} K^{\bullet}_1$	assume
10	$\langle b_1, b_2\rangle \notin_{\mathsf{k}} Q_1 \otimes Q_2$ or $\neg\mathcal{F}\langle a_1, a_2\rangle\langle b_1, b_2\rangle$	$\forall$-Elim, line 4
11	$\langle b_1, b_2\rangle \notin_{\mathsf{k}} Q_1 \otimes Q_2$	assume
12	$b_1 \notin_{\mathsf{k}_1} Q_1$ or $b_2 \notin_{\mathsf{k}_2} Q_2$	Lemma 3.4.2, line 11
13	$b_2 \notin_{\mathsf{k}_2} Q_2$	CL, Q_1 is well-pointed, lines 9, 12
14	$\rightarrow\leftarrow$	Contradiction, lines 7, 13
15	$\neg\mathcal{F}^{!}\langle a_1, a_2\rangle\langle b_1, b_2\rangle$	CL, lines 10, 11
16	$\neg\mathcal{F}^{!}_2 a_2 b_2$	def. $\mathcal{F}^{!}$, line 13
17	$\neg\mathcal{F}^{!}_2 a_2 b_2$	$\exists$-Elim, lines 8, 9
18	$b_2 \in_{\mathsf{k}_2} K^{\bullet}_2$	assume
19	$a_2 \in_{\mathsf{h}_2} H^{\circ}_2$	assume
20	$\langle a_1, a_2\rangle \in_{\mathsf{h}} H^{\circ}_{\otimes}$	Definition 3.2.2, line 19
21	$\rightarrow\leftarrow$	Contradiction, $[-^{\perp}\rangle$ is null-pointed, Lemma 4.7.7, line 20
22	$a_2 \notin_{\mathsf{h}_2} H^{\circ}_2$	CL, line 19
23	$\neg\mathcal{F}^{!}_2 a_2 b_2$	Frame Condition 4.7.4 (1), line 18, 22
24	$\neg\mathcal{F}^{!}_2 a_2 b_2$	$\vee$-Elim, lines 6, 7, 18
25	$\forall v(v \notin_{\mathsf{k}_2} Q_2$ or $\neg\mathcal{F}^{!}_2 a_2 v)$	$\forall$-Intro, CL, line 5
26	$a_2 \in_{\mathsf{h}_2} [f_2^{!}\rangle Q_2$	def. $[-^{!}\rangle$, line 25

Now for the other direction.

1 | $\langle a_1, a_2\rangle \in_{\mathsf{h}} [f_1^{!}\rangle Q_1 \otimes [f_2^{!}\rangle Q_2$ . . . . assume

2 | $\langle a_1, a_2\rangle \in_{\mathsf{h}} ([f_1^{!}\rangle Q_1 - H_1^\bullet) \times ([f_2^{!}\rangle Q_2 - H_2^\bullet)$ or $\langle a_1, a_2\rangle \in_{\mathsf{h}} H_\otimes^\bullet$ . . . . . . def. ⊗, line 1

3 | | $\langle a_1, a_2\rangle \in_{\mathsf{h}} ([f_1^{!}\rangle Q_1 - H_1^\bullet) \times ([f_2^{!}\rangle Q_2 - H_2^\bullet)$ . . . . assume

4 | | | $\boxed{b_1, b_2}$ $\langle b_1, b_2\rangle \in_{\mathsf{k}} Q_1 \otimes Q_2$ . . . . assume

5 | | | $\langle b_1, b_2\rangle \in_{\mathsf{k}} (Q_1 - K_1^\bullet) \times (Q_2 - K_2^\bullet)$ or $\langle b_1, b_2\rangle \in_{\mathsf{k}} K_\otimes^\bullet$ . . . . def. ⊗, line 4

6 | | | | $\langle b_1, b_2\rangle \in_{\mathsf{k}} (Q_1 - K_1^\bullet) \times (Q_2 - K_2^\bullet)$ . . . . assume

7 | | | | $b_1 \in_{\mathsf{k}_1} Q_1 - K_1^\bullet$ and $b_2 \in_{\mathsf{k}_2} Q_2 - K_2^\bullet$ . . . . def. ×, line 6

8 | | | | $b_1 \in_{\mathsf{k}_1} Q_1$ and $b_2 \in_{\mathsf{k}_2} Q_2$ . . . . def. −, line 7

9 | | | | $a_1 \in_{\mathsf{h}_1} [f_1^{!}\rangle Q_1 - H_1^\bullet$ and $a_2 \in_{\mathsf{h}_2} [f_2^{!}\rangle Q_2 - H_2^\bullet$ . . . . def. ×, line 3

10 | | | | $a_1 \in_{\mathsf{h}_1} [f_1^{!}\rangle Q_1$ and $a_2 \in_{\mathsf{h}_2} [f_2^{!}\rangle Q_2$ . . . . set th., line 9

11 | | | | $\forall y(y \notin_{\mathsf{k}_1} Q_1$ or $\neg\mathcal{F}_1^{!} a_1 y)$ . . . . def. $[f_1^{!}\rangle$, line 10

12 | | | | $\forall v(v \notin_{\mathsf{k}_2} Q_2$ or $\neg\mathcal{F}_2^{!} a_2 v)$ . . . . def. $[f_2^{!}\rangle$, line 10

13 | | | | $b_1 \notin_{\mathsf{k}_1} Q_1$ or $\neg\mathcal{F}_1^{!} a_1 b_1$ . . . . def. ∀-Elim, line 11

14 | | | | $b_2 \notin_{\mathsf{k}_2} Q_2$ or $\neg\mathcal{F}_2^{!} a_2 b_2$ . . . . def. ∀-Elim, line 12

15 | | | | $\neg\mathcal{F}_1^{!} a_1 b_1$ and $\neg\mathcal{F}_2^{!} a_2 b_2$ . . . . CL, lines 8, 13, 14

16 | | | | $\neg\mathcal{F}^{!}\langle a_1, a_2\rangle\langle b_1, b_2\rangle$ . . . . def. $\mathcal{F}^{!}$, line 15

17 | | | | $\langle b_1, b_2\rangle = K_\otimes^\bullet$ . . . . assume

18 | | | | $a_1 \notin_{\mathsf{h}_1} H_1^\circ$ and $a_2 \notin_{\mathsf{h}_2} H_2^\circ$ . . . . $[-^{!}\rangle$ null-pointed, Lemma 4.7.7, line 3

19 | | | | $\neg\mathcal{F}_1^{!} a_1 b_1$ and $\neg\mathcal{F}_2^{!} a_2 b_2$ . . . . Frame Condition 4.7.4 (1), lines 17, 18

20 | | | | $\neg\mathcal{F}^{!}\langle a_1, a_2\rangle\langle b_1, b_2\rangle$ . . . . def. $\neg\mathcal{F}^{!}$, line 19

21 | | | $\neg\mathcal{F}^{!}\langle a_1, a_2\rangle\langle b_1, b_2\rangle$ . . . . ∨-Elim, lines 5, 6, 17

22 | | $\forall\langle y, v\rangle(\langle y, v\rangle \notin_{\mathsf{k}} Q_1 \otimes Q_1$ or $\neg\mathcal{F}^{!}\langle a_1, a_2\rangle\langle y, v\rangle)$ . . . . ∀-Intro, line 4

23 | | $\langle a_1, a_2\rangle \in_{\mathsf{h}} H_\otimes^\bullet$ . . . . assume

24 | | $\langle a_1, a_2\rangle \in_{\mathsf{h}} [f^{!}\rangle(Q_1 \otimes Q_2)$ . . . . $[-^{!}\rangle$ well-pointed, Lemma 4.7.7, line 23

25 | $\langle a_1, a_2\rangle \in_{\mathsf{h}} [f^{!}\rangle(Q_1 \otimes Q_2)$ . . . . ∨-Elim, lines 2, 3, 23

The proofs for $[f^{!}\rangle$ are similar. We shorten that second proof by removing use of the Frame Condition 4.7.4.

1 | $\langle a_1, a_2\rangle \in_{\mathsf{h}} [f_1^{!}\rangle Q_1 \otimes [f_2^{!}\rangle Q_2$ . . . . assume

2 | $\langle a_1, a_2\rangle \in_{\mathsf{h}} [f_1^{!}\rangle Q_1 \times [f_2^{!}\rangle Q_2$ . . . . Lemma 3.4.2, line 1

3 | | $\boxed{b_1, b_2}$ $\langle b_1, b_2\rangle \in_{\mathsf{k}} Q_1 \otimes Q_2$ . . . . assume

4 | | $a_1 \in_{\mathsf{h}_1} [f_1^{!}\rangle Q_1$ and $a_2 \in_{\mathsf{h}_2} [f_2^{!}\rangle Q_2$ . . . . def. ×, line 2

5 | | $b_1 \in_{\mathsf{k}_1} Q_1$ and $b_2 \in_{\mathsf{k}_2} Q_2$ . . . . Lemma 3.4.2, def. ×, line 3

6 | | $\neg\mathcal{F}_1^{!} a_1 b_1$ and $\neg\mathcal{F}_2^{!} a_2 b_2$ . . . . def. $[f_1^{!}\rangle$, $[f_2^{!}\rangle$, lines 4, 5

7 | | $\neg\mathcal{F}^{!}\langle a_1, a_2\rangle\langle b_1, b_2\rangle$ . . . . def. $\mathcal{F}^{!}$, line 6

8 | $\langle a_1, a_2\rangle \in_{\mathsf{h}} [f^{!}\rangle(Q_1 \otimes Q_2)$ . . . . def. $[f^{!}\rangle$, line 3

4.8 Info for $[-^{!\cdot}\rangle$

The type of $[f^{!\cdot}\rangle$ is $\vee\!\mapsto\!\wedge$.

Definition 4.8.1

$$y \in_{\mathsf{k}} [f^{!\cdot}\rangle P \text{ iff } \forall x(x \notin_{\mathsf{h}} P \text{ or } \neg\mathcal{F}^{!\cdot}xy).$$

Definition 4.8.2 The canonical relation is

$$\mathcal{F}^{!\cdot}xy \text{ iff } \forall a(a \notin_{\mathsf{h}} x \text{ or } [f^{!\cdot}\rangle a \notin_{\mathsf{k}} y).$$

The complement of the canonical relation is

$$\neg\mathcal{F}^{!\cdot}xy \text{ iff } \exists a(a \in_{\mathsf{h}} x \text{ and } [f^{!\cdot}\rangle a \in_{\mathsf{k}} y).$$

Frame Condition 4.8.3 The null frame conditions for $\mathcal{F}^{!\cdot}$ are

$$(1) \quad \forall y(\mathcal{F}^{!\cdot}H^{\circ}y) \qquad (2) \quad \forall x(\mathcal{F}^{!\cdot}xK^{\circ})$$

Frame Condition 4.8.4 The well frame conditions for $\mathcal{F}^{!\cdot}$ are

$$(1) \quad \forall x(x \notin_{\mathsf{h}} H^{\circ} \text{ implies } \neg\mathcal{F}^{!\cdot}xK^{\bullet}) \qquad (2) \quad \forall y(y \notin_{\mathsf{k}} K^{\circ} \text{ implies } \neg\mathcal{F}^{!\cdot}H^{\bullet}y)$$

Lemma 4.8.5 *Canonically, the Frame Conditions hold.*

Proof: Frame Conditions 4.8.3: let $x = \circ_H$ or $y = \circ_K$, then

$$\mathcal{F}^{!\cdot}xy \text{ iff } \forall a(a \notin_{\mathsf{h}} x \text{ or } [f^{!\cdot}\rangle a \notin_{\mathsf{k}} y)$$

holds since $\circ_H = \emptyset$ and $\circ_K = \emptyset$. So $\mathcal{F}^{!\cdot}xy$ obtains in both cases..

Frame Condition 4.8.4 (1): let $x \neq \circ_H$ and $y = \bullet_K$, then $\top_{\mathsf{h}} \in_{\mathsf{h}} x$ and $[f^{!\cdot}\rangle\top_{\mathsf{h}} \in_{\mathsf{k}} y$. So so there is some a such that $a \in_{\mathsf{h}} x$ and $[f^{!\cdot}\rangle a \in_{\mathsf{k}} y$, whence $\mathcal{F}^{!\cdot}xy$ obtains.

Frame Condition 4.8.4 (2): let $y \neq \circ_K$ and $x = \bullet_H$, then $\bot_{\mathsf{h}} \in_{\mathsf{h}} x$ and, from normality, $[f^{!\cdot}\rangle\bot_{\mathsf{h}} \in_{\mathsf{k}} y$. So so there is some a such that $a \in_{\mathsf{h}} x$ and $[f^{!\cdot}\rangle a \in_{\mathsf{k}} y$, whence $\mathcal{F}^{!\cdot}xy$ obtains.

■

Lemma 4.8.6 *The Frame Conditions 4.8.4 and 4.8.3 are consistent with each other.*

Proof: The well-point sets and null-point sets are always disjoint. This disjointness and quantifiers and antecedents of the conditionals prevent any conflicts in the Frame Conditions 4.8.4 and 4.8.3.

■

4.9 $[-^{?}\rangle, [-^{?}\cdot\rangle$: Closure Under Null Sum

All proofs involving $[-^{?}\cdot\rangle$ are similar to their siblings for $[-^{?}\rangle$ and we elide them. The information necessary to construct them is in the following Section 4.10.

The type of $[f^{?}\rangle$ is $\wedge\mapsto\vee$.

Definition 4.9.1

$$x \in_{\mathsf{h}} [f^{?}\rangle Q \text{ iff } \exists y(y \notin_{\mathsf{k}} Q \text{ and } \mathcal{F}^{?}xy).$$

Definition 4.9.2 The canonical relation for $\mathcal{F}^{?}$ is

$$\mathcal{F}^{?}xy \text{ iff } \forall b(b \in_{\mathsf{k}} y \text{ or } [f^{?}\rangle b \in_{\mathsf{h}} x)$$

The negation of the canonical definition is

$$\neg\mathcal{F}^{?}xy \text{ iff } \exists b(b \notin_{\mathsf{k}} y \text{ and } [f^{?}\rangle b \notin_{\mathsf{h}} x)$$

Frame Condition 4.9.3 The null frame conditions for $\mathcal{F}^{?}$ are

$$(1) \quad \forall x(x \notin_{\mathsf{h}} H^{\bullet} \text{ implies } \neg\mathcal{F}^{?}xK^{\circ}) \qquad (2) \quad \forall y(y \notin_{\mathsf{k}} K^{\bullet} \text{ implies } \neg\mathcal{F}^{?}H^{\circ}y)$$

Frame Condition 4.9.4 The well frame conditions for $\mathcal{F}^{?}$ are

$$(1) \quad \forall y(\mathcal{F}^{?}H^{\bullet}y) \qquad (2) \quad \forall x(\mathcal{F}^{?}xK^{\bullet})$$

Lemma 4.9.5 *The Frame Conditions hold canonically.*

Proof: Frame Condition 4.9.3 (1): choose an arbitrary x such that $x \neq \bullet_H$. Let $y = \circ_K$, then $\top_{\mathsf{k}} \notin_{\mathsf{k}} y$. Since $[f^{?}\rangle \top_{\mathsf{k}} = \bot_{\mathsf{h}}$, then $[f^{?}\rangle \top_{\mathsf{k}} \notin_{\mathsf{h}} x$. Hence there is some b such that $b \notin_{\mathsf{k}} y$ and $[f^{?}\rangle b \notin_{\mathsf{h}} x$, whence $\neg\mathcal{F}^{?}xy$ obtains.

Frame Condition 4.9.3 (2): choose an arbitrary y such that $y \neq \bullet_K$, then $\bot_{\mathsf{k}} \notin_{\mathsf{k}} y$. Let $x = \circ_H$, then $[f^{?}\rangle \bot_{\mathsf{k}} \notin_{\mathsf{h}} \circ_H$. Hence there is some b such that $b \notin_{\mathsf{k}} y$ and $[f^{?}\rangle b \notin_{\mathsf{h}} x$, whence $\neg\mathcal{F}^{?}xy$ obtains.

Frame Conditions 4.9.4 (1) and 4.9.4 (2): for all b, $b \in_{\mathsf{k}} \bullet_K$ and $[f^{?}\rangle b \in_{\mathsf{h}} \bullet_H$, hence both $\forall y(\mathcal{F}^{?} \bullet_H y)$ and $\forall x(\mathcal{F}^{?}x\bullet_K)$ hold.

■

Lemma 4.9.6 *The Frame Conditions 4.9.3 and 4.9.4 are consistent with each other.*

Proof: Note that the well-point sets and null-point sets are always disjoint, then the quantifiers and antecedents of the conditionals of Frame Conditions 4.9.4 and 4.9.3 prevent any conflicts.

■

Lemma 4.9.7 $[f^{?}\rangle Q$ *is null-pointed if* Q *is well-pointed, and well-pointed if* Q *is null pointed.*

Proof: By definition,

$$\begin{aligned} x \notin_{\mathsf{h}} [f^{?}\rangle Q \text{ iff } & \neg\exists y(y \notin_{\mathsf{k}} Q \text{ and } \mathcal{F}^{?}xy) \\ \text{iff } & \forall y(y \in_{\mathsf{k}} Q \text{ or } \neg\mathcal{F}^{?}xy) \end{aligned}$$

Assume Q is well-pointed and $x \in_{\mathsf{h}} H^{\circ}$. If $y \in_{\mathsf{k}} K^{\bullet}$, then by assumption, $y \in_{\mathsf{k}} Q$. If $y \notin_{\mathsf{k}} K^{\bullet}$ then $\neg\mathcal{F}^{?}xy$ obtains from the Frame Condition 4.9.3 (2). Hence for all y, either $y \in_{\mathsf{k}} Q$ or $\neg\mathcal{F}^{?}xy$. Therefore $H^{\circ} \cap [f^{?}\rangle Q = \emptyset$.

Next, assume Q is null-pointed and, since $K^{\circ} \neq \emptyset$, that $y \in_{\mathsf{k}} K^{\circ}$. Let $x \in_{\mathsf{h}} H^{\bullet}$. By assumption, $y \notin_{\mathsf{k}} Q$. From Frame Condition 4.9.4 (1), $\mathcal{F}^{?}xy$. By definition, $x \in_{\mathsf{h}} [f^{?}\rangle Q$.

■

Lemma 4.9.8 $[f^{?}\rangle$ *preserves its bounds.*

Proof: The type of $[f^{?}\rangle$ is $\wedge\!\mapsto\!\vee$, hence we want $[f^{?}\rangle \top^{\mathbb{K}} = \bot^{\mathbb{H}}$. The definition is

$$x \in_{\mathsf{h}} [f^{?}\rangle Q \text{ iff } \exists y(y \notin_{\mathsf{k}} Q \text{ and } \mathcal{F}^{?}xy).$$

We have

$$x \in_{\mathsf{h}} [f^{?}\rangle \top^{\mathbb{K}} \text{ iff } \exists y(y \notin_{\mathsf{k}} \top^{\mathbb{K}} \text{ and } \mathcal{F}^{?}xy).$$

Hence

$$x \notin_{\mathsf{h}} [f^{?}\rangle \top^{\mathbb{K}} \text{ iff } \forall y(y \notin_{\mathsf{k}} \top^{\mathbb{K}} \text{ implies } \neg\mathcal{F}^{?}xy).$$

From Lemma 4.9.7, $[f^{?}\rangle \top^{\mathbb{K}}$ is well- and null-pointed. Since $\bot^{\mathbb{H}}$ is the bottom of the lattice, $\bot^{\mathbb{H}} \subseteq_{\mathsf{h}} [f^{?}\rangle \top^{\mathbb{K}}$.

Next, either $x \notin_{\mathsf{h}} H^{\bullet}$ or $x \in_{\mathsf{h}} H^{\bullet}$. Let $x \notin_{\mathsf{h}} H^{\bullet}$. We show a contraposition. The only points not in $\top^{\mathbb{K}}$ are those in K°. Since $K^{\circ} \neq \emptyset$ and $K^{\circ} \cap Q = \emptyset$, choose some $y \in_{\mathsf{k}} K^{\circ}$. From Frame Condition 4.9.3 (1), $\neg\mathcal{F}^{?}xy$ obtains and so $x \notin_{\mathsf{h}} [f^{?}\rangle\, \top^{\mathbb{K}}$.

Now let $x \in_{\mathsf{h}} H^{\bullet}$. Since $[f^{?}\rangle$ is well-pointed (Lemma 4.9.7), $x \in_{\mathsf{h}} [f^{?}\rangle\, \top^{\mathbb{K}}$. Since $H^{\bullet} = \bot^{\mathbb{H}}$, then $[f^{?}\rangle\, Q \subseteq_{\mathsf{h}} \bot^{\mathbb{H}}$. ■

Lemma 4.9.9

$$x \in_{\mathsf{h}} [f^{?}\rangle \beta b \text{ iff } x \in_{\mathsf{h}} \beta[f^{?}\rangle\, b.$$

Proof: Left to right is the easy direction, assume $x \in_{\mathsf{h}} [f^{?}\rangle \beta b$, by definition there is some $y \notin_{\mathsf{k}} \beta b$ and $\mathcal{F}^{?}xy$. Since $y \notin_{\mathsf{k}} \beta b$, then $b \notin_{\mathsf{k}} y$. Using the canonical definition of $\mathcal{F}^{?}xy$, $b \notin_{\mathsf{k}} y$ implies $[f^{?}\rangle\, b \in_{\mathsf{h}} x$, and hence $x \in_{\mathsf{h}} \beta[f^{?}\rangle\, b$.

Right to left is the harder direction. Let $x \in_{\mathsf{h}} \beta[f^{?}\rangle\, b$, so $[f^{?}\rangle\, b \in_{\mathsf{h}} x$. Construct $\hat{y} = (\lambda p\ .\ [f^{?}\rangle\, p)^{-1}\overline{x}$. From Lemma 3.7.1, $\hat{y}$ is a filter. Assume $b \in_{\mathsf{k}} \hat{y}$, then $[f^{?}\rangle\, b \in_{\mathsf{h}} \overline{x}$ which contradicts the premise. Hence $\langle \hat{y}, \downarrow b\rangle$ is a disjoint pair that we maximalize to yield y. By construction $b \notin_{\mathsf{k}} y$ so $y \notin_{\mathsf{k}} \beta b$. Also by construction, if $[f^{?}\rangle\, a \notin_{\mathsf{h}} x$ then $a \in_{\mathsf{k}} y$ and so by the canonical definition, $\mathcal{F}^{?}xy$ holds. Therefore $x \in_{\mathsf{h}} [f^{?}\rangle \beta b$.

■

We assume $f = f_1 \varovee f_2$, then

$$\mathcal{F}^{?}\langle a_1, a_2\rangle\langle b_1, b_2\rangle \text{ iff } \mathcal{F}_1^{?}a_1 b_1 \text{ or } \mathcal{F}^{?}a_2 b_2.$$

Lemma 4.9.10 *Let $f = f_1 \varovee f_2$, then*

$$[f^{?}\rangle\, (Q_1 \varovee Q_2) = [f^{?}\rangle Q_1 \varovee [f^{?}\rangle Q_2 \qquad [f^{?}\cdot\rangle(P_1 \varovee P_2) = [f_1^{?}\cdot\rangle P_1 \varovee [f_2^{?}\cdot\rangle P_2.$$

Proof:

1	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f^{?}\rangle(Q_1 \ovee Q_2)$	assume
2	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f^{?}\rangle(Q_1 \ovee Q_2) - H^{\bullet}_{\ovee}$ or $\langle a_1, a_2\rangle \in_{\mathsf{h}} H^{\bullet}_{\ovee}$	Lemma 4.9.7, line 1
3	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f^{?}\rangle(Q_1 \ovee Q_2) - H^{\bullet}_{\ovee}$	assume
4	$\langle a_1, a_2\rangle \notin_{\mathsf{h}} H^{\bullet}_{\ovee}$	set th., line 3
5	$a_1 \notin_{\mathsf{h}_1} H^{\bullet}_1$ and $a_2 \notin_{\mathsf{h}_2} H^{\bullet}_2$	Lemma 3.5.3, line 4
6	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f^{?}\rangle(Q_1 \ovee Q_2)$	set th., line 3
7	$\exists\langle y, v\rangle(\langle y, v\rangle \notin_{\mathsf{k}} Q_1 \ovee Q_2$ and $\mathcal{F}^{?}\langle a_1, a_2\rangle\langle y, v\rangle)$	def. $[-^{?}\rangle$, line 6
8	$\boxed{b_1, b_2}$ $\langle b_1, b_2\rangle \notin_{\mathsf{k}} Q_1 \ovee Q_2$ and $\mathcal{F}^{?}\langle a_1, a_2\rangle\langle b_1, b_2\rangle$	assume
9	$\mathcal{F}^{?}_1 a_1 b_1$ or $\mathcal{F}^{?}_2 a_2 b_2$	def. $\mathcal{F}^{?}$, line 7
10	$\langle b_1, b_2\rangle \in_{\mathsf{k}} K_1 \ovee K_2 - Q_1 \ovee Q_2$	$K_1 \ovee K_2$ is the universe, line 8
11	$(b_1 \notin_{\mathsf{k}_1} Q_1$ and $b_2 \in_{\mathsf{k}_2} K^{\circ}_2)$ or $(b_1 \in_{\mathsf{k}_1} K^{\circ}_1$ and $b_2 \notin_{\mathsf{k}_2} Q_2)$	Lemma 3.5.4, line 10
12	$b_1 \in_{\mathsf{k}_1} K^{\circ}_1$ and $b_2 \notin_{\mathsf{k}_2} Q_2$	assume
13	$\neg\mathcal{F}^{?}_1 a_1 b_1$	Frame Condition 4.9.3 (1), lines 5, 12
14	$\mathcal{F}^{?}_2 a_2 b_2$	CL, lines 9, 13
15	$b_2 \notin_{\mathsf{k}_2} Q_2$ and $\mathcal{F}^{?}_2 a_2 b_2$	$\wedge$-Elim, $\wedge$-Intro lines 12, 14
16	$\exists v(v \notin_{\mathsf{k}_2} Q_2$ and $\mathcal{F}^{?}_2 a_2 v)$	$\exists$-Intro line 15
17	$a_2 \in_{\mathsf{h}_2} [f_2^{?}\rangle Q_2$	def. $[-^{?}\rangle$ line 16
18	$a_1 \in_{\mathsf{h}_1} H^{\circ}_1$ or $a_2 \in_{\mathsf{h}_2} H^{\circ}_2$	$H_1 \ovee H_2$ is the universe, line 1
19	$a_2 \notin_{\mathsf{h}_2} H^{\circ}_2$	$[-^{\star}\rangle$ null-pointed, Lemma 4.9.7, line 17
20	$a_1 \in_{\mathsf{h}_1} H^{\circ}_1$	CL, lines 18, 19
21	$\langle a_1, a_2\rangle \in_{\mathsf{h}} H^{\circ}_1 \times [f_2^{?}\rangle Q_2$	def. $\times$, lines 17, 20
22	$\langle a_1, a_2\rangle \in_{\mathsf{h}} ([f_1^{?}\rangle Q_1 \times H^{\circ}_2) \cup (H^{\circ}_1 \times [f_2^{?}\rangle Q_2)$	set th., line 21
23	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f_1^{?}\rangle Q_1 \ovee [f_2^{?}\rangle Q_2$	def. $\ovee$, line 22
24	$b_1 \notin_{\mathsf{k}_1} Q_1$ and $b_2 \in_{\mathsf{k}_2} K^{\circ}_2$	assume
25	subproof is similar to previous subproof	lines xx
26	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f_1^{?}\rangle Q_1 \ovee [f_2^{?}\rangle Q_2$	def. $\ovee$, lines xx
27	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f_1^{?}\rangle Q_1 \ovee [f_2^{?}\rangle Q_2$	$\vee$-Elim, lines 11, 12, 24
28	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f_1^{?}\rangle Q_1 \ovee [f_2^{?}\rangle Q_2$	$\exists$-Elim, lines 7, 8
29	$\langle a_1, a_2\rangle \in_{\mathsf{h}} H^{\bullet}_{\ovee}$	assume
30	$\langle a_1, a_2\rangle \in_{\mathsf{h}} H^{\bullet}_1 \times H^{\circ}_2$ or $\langle a_1, a_2\rangle \in_{\mathsf{h}} H^{\circ}_1 \times H^{\bullet}_2$	def. $\ovee$, line 29
31	$a_1 \in_{\mathsf{h}_1} [f_1^{?}\rangle Q_1$ or $a_2 \in_{\mathsf{h}_2} [f^{?}\rangle Q_2$	Lemma 4.9.7, $[f_1^{?}\rangle Q_1$, $[f_2^{?}\rangle Q_2$ well-pointed, line 30
32	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f_1^{?}\rangle Q_1 \ovee [f_2^{?}\rangle Q_2$	def. $\ovee$, lines 30, 31
33	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f_1^{?}\rangle Q_1 \ovee [f_2^{?}\rangle Q_2$	$\vee$-Elim, lines 2, 3, 29

To fill in the similar subproof above:

1	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f^{?}\rangle (Q_1 \ovee Q_2)$	assume
2	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f^{?}\rangle (Q_1 \ovee Q_2) - H^{\bullet}_{\ovee}$ or $\langle a_1, a_2\rangle \in_{\mathsf{h}} H^{\bullet}_{\ovee}$	Lemma 4.9.7, line 1
3	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f^{?}\rangle (Q_1 \ovee Q_2) - H^{\bullet}_{\ovee}$	assume
4	$\langle a_1, a_2\rangle \notin_{\mathsf{h}} H^{\bullet}_{\ovee}$	set th., line 3
5	$a_1 \notin_{\mathsf{h}_1} H^{\bullet}_1$ and $a_2 \notin_{\mathsf{h}_2} H^{\bullet}_2$	Lemma 3.5.3, line 4
6	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f^{?}\rangle (Q_1 \ovee Q_2)$	set th., line 3
7	$\exists\langle y, v\rangle(\langle y, v\rangle \notin_{\mathsf{k}} Q_1 \ovee Q_2$ and $\mathcal{F}^{?}\langle a_1, a_2\rangle\langle y, v\rangle)$	def. $[-^{\star}\rangle$, line 6
8	$\boxed{b_1, b_2}$ $\langle b_1, b_2\rangle \notin_{\mathsf{k}} Q_1 \ovee Q_2$ and $\mathcal{F}^{?}\langle a_1, a_2\rangle\langle b_1, b_2\rangle$	assume
9	$\mathcal{F}^{?}_1 a_1 b_1$ or $\mathcal{F}^{?}_2 a_2 b_2$	def. $\mathcal{F}^{?}$, line 8
10	$\langle b_1, b_2\rangle \in_{\mathsf{k}} K_1 \ovee K_2 - Q_1 \ovee Q_2$	$K_1 \ovee K_2$ is the universe, line 8
11	$(b_1 \notin_{\mathsf{k}_1} Q_1$ and $b_2 \in_{\mathsf{k}_2} K^{\circ}_2)$ or $(b_1 \in_{\mathsf{k}_1} K^{\circ}_1$ and $b_2 \notin_{\mathsf{k}_2} Q_2)$	Lemma 3.5.4, line 10
12	$b_1 \notin_{\mathsf{k}_1} Q_1$ and $b_2 \in_{\mathsf{k}_2} K^{\circ}_2$	assume
13	$\neg\mathcal{F}^{?}_2 a_2 b_2$	Frame Condition 4.9.3 (1), lines 5, 12
14	$\mathcal{F}^{?}_1 a_1 b_1$	CL, lines 9, 13
15	$b_1 \notin_{\mathsf{k}_1} Q_1$ and $\mathcal{F}^{?}_1 a_1 b_1$	$\wedge$-Elim, $\wedge$-Intro lines 12, 14
16	$\exists y(y \notin_{\mathsf{k}_1} Q_1$ and $\mathcal{F}^{?}_1 a_1 y)$	$\exists$-Intro line 15
17	$a_1 \in_{\mathsf{h}_1} [f_1^{?}\rangle Q_1$	def. $[-^{?}\rangle$ line 16
18	$a_1 \in_{\mathsf{h}_1} H^{\circ}_1$ or $a_2 \in_{\mathsf{h}_2} H^{\circ}_2$	$H_1 \ovee H_2$ is the universe, line 1
19	$a_1 \notin_{\mathsf{h}_1} H^{\circ}_1$	$[-^{\star}\rangle$ null-pointed, 4.4.6, line 17
20	$a_2 \in_{\mathsf{h}_2} H^{\circ}_2$	CL, lines 18, 19
21	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f_1^{?}\rangle Q_1 \times H^{\circ}_2$	def. $\times$, lines 17, 20
22	$\langle a_1, a_2\rangle \in_{\mathsf{h}} ([f_1^{?}\rangle Q_1 \times H^{\circ}_2) \cup (H^{\circ}_1 \times [f_2^{?}\rangle Q_2)$	set th., line 21
23	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f_1^{?}\rangle Q_1 \ovee [f_2^{?}\rangle Q_2$	def. $\ovee$, line 22

Now the other direction

1	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f_1^{?}\rangle Q_1 \ovee [f_2^{?}\rangle Q_2$	assume
2	$(a_1 \in_{\mathsf{h}_1} [f_1^{?}\rangle Q_1$ and $a_2 \in_{\mathsf{h}_2} H_2^{\circ})$ or $(a_1 \in_{\mathsf{h}_1} H_1^{\circ}$ and $a_2 \in_{\mathsf{h}_2} [f_2^{?}\rangle Q_2)$	def. $\ovee$, line 1
3	$a_1 \in_{\mathsf{h}_1} [f_1^{?}\rangle Q_1$ and $a_2 \in_{\mathsf{h}_2} H_2^{\circ}$	assume
4	$\exists y(y \notin_{\mathsf{k}_1} Q_1$ and $\mathcal{F}_1^{?} a_1 y)$	def. $[-^{\star}\rangle$, line 3
5	$\boxed{b_1}$ $b_1 \notin_{\mathsf{k}_1} Q_1$ and $\mathcal{F}_1^{?} a_1 b_1$	assume
6	$\exists v(v \in_{\mathsf{k}_2} K_2^{\circ})$	$K_2^{\circ} \neq \emptyset$
7	$\boxed{c}$ $c \in_{\mathsf{k}_2} K_2^{\circ}$	assume
8	$\langle b_1, c\rangle \notin_{\mathsf{k}} Q_1 \times K_2^{\circ}$	def ×, lines 5, 7
9	$c \notin_{\mathsf{k}_2} Q_2$	Q_2 is null-pointed, line 7
10	$\langle b_1, c\rangle \notin_{\mathsf{k}} K_1^{\circ} \times Q_2$	def. ×, line 9
11	$\langle b_1, c\rangle \notin_{\mathsf{k}} (Q_1 \times K_2^{\circ}) \cup (K_1^{\circ} \times Q_2)$	set th., lines 8, 10
12	$\langle b_1, c\rangle \notin_{\mathsf{k}} Q_1 \ovee Q_2$	def. $\ovee$, line 11
13	$\mathcal{F}_1^{?} a_1 b_1$ or $\mathcal{F}_2^{?} a_2 c$	∧-Elim, ∨-Intro, line 5
14	$\mathcal{F}^{?}\langle a_1, a_2\rangle\langle b_1, c\rangle$	def. $\mathcal{F}^{?}$, line 13
15	$\langle b_1, c\rangle \notin_{\mathsf{k}} Q_1 \ovee Q_2$ and $\mathcal{F}^{?}\langle a_1, a_2\rangle\langle b_1, c\rangle$	∧-Intro, lines 12, 14
16	$\exists\langle y, v\rangle(\langle y, v\rangle \notin_{\mathsf{k}} Q_1 \ovee Q_2$ and $\mathcal{F}^{?}\langle a_1, a_2\rangle\langle y, v\rangle)\rangle$	∃-Intro, line 15
17	$\langle a_1, a_2\rangle \in_{\mathsf{h}} \top_1^{\mathbb{H}} \times H_2^{\circ}$	set th., line 3
18	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f^{?}\rangle (Q_1 \ovee Q_2)$	def. $[-^{?}\rangle$, lines 16, 17
19	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f^{?}\rangle (Q_1 \ovee Q_2)$	∃-Elim, lines 6, 7
20	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f^{?}\rangle (Q_1 \ovee Q_2)$	∃-Elim, lines 4, 5
21	$a_2 \in_{\mathsf{h}_2} [f_2^{\star}\rangle Q_2$ and $a_1 \in_{\mathsf{h}_1} H_1^{\circ}$	assume
22	subproof is similar to previous subproof	lines xx
23	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f^{?}\rangle (Q_1 \ovee Q_2)$	def. $[-^{?}\rangle$, line xx
24	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f^{?}\rangle (Q_1 \ovee Q_2)$	∨-Elim, lines 2, 3, 21

To fill in the similar subproof above:

1	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f_1^{?}\rangle Q_1 \ovee [f_2^{?}\rangle Q_2$	assume
2	$(a_1 \in_{\mathsf{h_1}} [f_1^{?}\rangle Q_1$ and $a_2 \in_{\mathsf{h_2}} H_2^{\circ})$ or $(a_1 \in_{\mathsf{h_1}} H_1^{\circ}$ and $a_2 \in_{\mathsf{h_2}} [f_2^{?}\rangle Q_2)$	def. $\ovee$, line 1
3	$a_1 \in_{\mathsf{h_1}} H_1^{\circ}$ and $a_2 \in_{\mathsf{h_2}} [f_2^{?}\rangle Q_2$	assume
4	$\exists v(v \notin_{\mathsf{k_2}} Q_2$ and $\mathcal{F}_2^{?} a_2 v)$	def. $[-^{?}\rangle$, line 3
5	$\boxed{b_2}$ $b_2 \notin_{\mathsf{k_2}} Q_2$ and $\mathcal{F}_2^{?} a_2 b_2$	assume
6	$\exists y(y \in_{\mathsf{k_1}} K_1^{\circ})$	$K_1^{\circ} \neq \emptyset$
7	$\boxed{c}$ $c \in_{\mathsf{k_1}} K_1^{\circ}$	assume
8	$\langle c, b_2\rangle \notin_{\mathsf{k}} K_1^{\circ} \times Q_2$	def $\times$, lines 5, 7
9	$c \notin_{\mathsf{k_1}} Q_1$	Q_1 is null-pointed, line 7
10	$\langle c, b_2\rangle \notin_{\mathsf{k}} Q_1 \times K_2^{\circ}$	def. $\times$, line 9
11	$\langle c, b_2\rangle \notin_{\mathsf{k}} (Q_1 \times K_2^{\circ}) \cup (K_1^{\circ} \times Q_2)$	set th., lines 8, 10
12	$\langle c, b_2\rangle \notin_{\mathsf{k}} Q_1 \ovee Q_2$	def. $\ovee$, line 11
13	$\mathcal{F}_1^{?} a_1 c$ or $\mathcal{F}_2^{?} a_2 b_2$	$\wedge$-Elim, $\vee$-Intro, line 5
14	$\mathcal{F}^{?}\langle a_1, a_2\rangle\langle c, b_2\rangle$	def. $\mathcal{F}^{?}$, line 13
15	$\langle c, b_2\rangle \notin_{\mathsf{k}} Q_1 \ovee Q_2$ and $\mathcal{F}^{?}\langle a_1, a_2\rangle\langle c, b_2\rangle$	$\wedge$-Intro, lines 12, 14
16	$\exists\langle y, v\rangle(\langle y, v\rangle \notin_{\mathsf{k}} Q_1 \ovee Q_2$ and $\mathcal{F}^{?}\langle a_1, a_2\rangle\langle y, v\rangle)$	$\exists$-Intro, line 15
17	$\langle a_1, a_2\rangle \in_{\mathsf{h}} H_1^{\circ} \times \top_2^{\mathbb{H}}$	set th., line 3
18	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f^{?}\rangle(Q_1 \ovee Q_2)$	def. $[-^{?}\rangle$, lines 16, 17
19	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f^{?}\rangle(Q_1 \ovee Q_2)$	$\exists$-Elim, lines 6, 7
20	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f^{?}\rangle(Q_1 \ovee Q_2)$	$\exists$-Elim, lines 4, 5

∎

4.10 Info for $[-^{?}\cdot\rangle$

The type of $[f^{?}\cdot\rangle$ is $\wedge\!\mapsto\!\vee$.

Definition 4.10.1

$$y \in_{\mathsf{k}} [f^{?}\cdot\rangle P \text{ iff } \exists x(x \notin_{\mathsf{h}} P \text{ and } \mathcal{F}^{?\cdot} xy).$$

Definition 4.10.2 The canonical relation for $\mathcal{F}^{?\cdot}$ is

$$\mathcal{F}^{?\cdot} xy \text{ iff } \forall a(a \in_{\mathsf{h}} x \text{ or } [f^{?}\cdot\rangle a \in_{\mathsf{k}} y)$$

or

$$\neg\mathcal{F}^{?\cdot} xy \text{ iff } \exists a(a \notin_{\mathsf{h}} x \text{ and } [f^{?}\cdot\rangle a \notin_{\mathsf{k}} y)$$

Frame Condition 4.10.3 The null frame conditions for $\mathcal{F}^{?\cdot}$ are

$$(1)\quad \forall x(x \notin_{\mathsf{h}} H^{\bullet} \text{ implies } \neg\mathcal{F}^{?\cdot}xK^{\circ}) \qquad (2)\quad \forall y(y \notin_{\mathsf{k}} K^{\bullet} \text{ implies } \neg\mathcal{F}^{?\cdot}H^{\circ}y)$$

Frame Condition 4.10.4 The well frame conditions for $\mathcal{F}^{?\cdot}$ are

$$(1)\quad \forall y(\mathcal{F}^{?\cdot}H^{\bullet}y) \qquad (2)\quad \forall x(\mathcal{F}^{?\cdot}xK^{\bullet})$$

Lemma 4.10.5 *The Frame Conditions hold canonically.*

Proof: Frame Condition 4.10.3 (1): choose an arbitrary x such that $x \neq \bullet_H$, then $\bot_{\mathsf{h}} \notin_{\mathsf{h}} x$. Let $y = \circ_K$, then $[f^{?}\cdot\rangle\bot_{\mathsf{h}} \notin_{\mathsf{k}} \circ_K$. Hence there is some a such that $a \notin_{\mathsf{h}} x$ and $[f^{?}\cdot\rangle a \notin_{\mathsf{k}} y$, whence $\neg\mathcal{F}^{?\cdot}xy$ obtains.

Frame Condition 4.10.3 (2): choose an arbitrary y such that $y \neq \bullet_K$. Let $x = \circ_H$, then $\top_{\mathsf{h}} \notin_{\mathsf{h}} x$. Since $[f^{?}\cdot\rangle\top_{\mathsf{k}} = \bot_{\mathsf{k}}$, then $[f^{?}\cdot\rangle\top_{\mathsf{h}} \notin_{\mathsf{k}} y$. Hence there is some a such that $a \notin_{\mathsf{h}} x$ and $[f^{?}\cdot\rangle a \notin_{\mathsf{k}} y$, whence $\neg\mathcal{F}^{?\cdot}xy$ obtains.

Frame Conditions 4.10.4 (1) and 4.10.4 (2): for all a, $a \in_{\mathsf{h}} \bullet_H$ and $[f^{?}\cdot\rangle a \in_{\mathsf{k}} \bullet_K$, hence both $\forall y(\mathcal{F}^{?\cdot}\bullet_H y)$ and $\forall x(\mathcal{F}^{?\cdot}x\bullet_K)$ hold.

■

Lemma 4.10.6 *The Frame Conditions 4.10.3 and 4.10.4 are consistent with each other.*

Proof: The well-point sets and null-point sets are always disjoint. This disjointness and quantifiers and antecedents of the conditionals prevent any conflicts in the Frame Conditions 4.10.3 and 4.10.4.

■

4.11 $[-^{\flat}\rangle$, $[-^{\flat}\cdot\rangle$: Closure Under Null Sum

All proofs involving $[-^{\flat}\cdot\rangle$ are similar to their siblings for $[-^{\flat}\rangle$ and we elide them. The information necessary to construct them is in the following Section 4.12.

The type of $[f^{\flat}\rangle$ is $\wedge\mapsto\wedge$.

Definition 4.11.1

$$x \in_{\mathsf{h}} [f^{\flat}\rangle Q \text{ iff } \forall y(y \in_{\mathsf{k}} Q \text{ or } \mathcal{F}^{\flat}xy).$$

Definition 4.11.2 The canonical relation is defined as

$$\mathcal{F}^{\flat}xy \text{ iff } \exists b(b \notin_{\mathsf{k}} y \text{ and } [f^{\flat}\rangle\, b \in_{\mathsf{h}} x).$$

The negated relation is

$$\neg\mathcal{F}^{\flat}xy \text{ iff } \forall b(b \in_{\mathsf{k}} y \text{ or } [f^{\flat}\rangle\, b \notin_{\mathsf{h}} x).$$

Frame Condition 4.11.3 The null frame conditions for $\mathcal{F}^{\flat}$ are

$$(1) \quad \forall x(x \notin_{\mathsf{h}} H^{\circ} \text{ implies } \mathcal{F}^{\flat}xK^{\circ}) \qquad (2) \quad \forall y(\neg\mathcal{F}^{\flat}H^{\circ}y).$$

Frame Condition 4.11.4 The well frame conditions for $\mathcal{F}^{\flat}$ are

$$(1) \quad \forall y(y \notin_{\mathsf{k}} K^{\bullet} \text{ implies } \mathcal{F}^{\flat}H^{\bullet}y) \qquad (2) \quad \forall x(\neg\mathcal{F}^{\flat}xK^{\bullet}).$$

Lemma 4.11.5 *Canonically, the Frame Conditions hold.*

Proof: Frame Condition 4.11.3 (1): choose any x such that $x \neq \circ_{H}$. Let $y = \circ_{K}$. Hence $\top_{\mathsf{k}} \notin_{\mathsf{k}} y$ and from the type of $[-^{\flat}\rangle$, $[f^{\flat}\rangle \top_{\mathsf{k}} = \top_{\mathsf{h}}$ and $\top_{\mathsf{h}} \in_{\mathsf{h}} x$. So there is some b such that $b \notin_{\mathsf{k}} y$ and $[f^{\flat}\rangle\, b \in_{\mathsf{h}} x$, whence $\mathcal{F}^{\flat}xy$ obtains.

Frame Condition 4.11.3 (2): choose some arbitrary y and let $x = \circ_{H}$. For all b, it is the case that $[f^{\flat}\rangle\, b \notin_{\mathsf{h}} \circ_{H}$. Hence, $\neg\mathcal{F}^{\flat}xy$.

Frame Condition 4.11.4 (1): choose any y such that $y \neq \bullet_{K}$. Let $x = \bullet_{H}$. Hence $\bot_{\mathsf{k}} \notin_{\mathsf{k}} y$. Since $\bullet_{H}$ is the entire carrier set, $[f^{\flat}\rangle \bot_{\mathsf{k}} \in_{\mathsf{h}} x$. So there is some b such that $b \notin_{\mathsf{k}} y$ and $[f^{\flat}\rangle\, b \in_{\mathsf{h}} x$, whence $\mathcal{F}^{\flat}xy$ obtains.

Frame Condition 4.11.4 (2): choose any x and let $y = \bullet_{K}$ then for all b, $b \in_{\mathsf{k}} y$, so $\neg\mathcal{F}^{\flat}xy$ obtains.

■

Lemma 4.11.6 *The Frame Conditions 4.11.3 and 4.11.4 are consistent.*

Proof: The well-point sets and null-point sets are always disjoint. The disjointedness of the well- and null-point sets and antecedents of the conditionals of Frame Conditions 4.11.3 and 4.11.4 prevent any conflicts.

■

Lemma 4.11.7 *$[f^\flat\rangle Q$ is null-pointed if Q is null-pointed, and well-pointed if Q is well-pointed.*

Proof: By definition,

$$\begin{aligned} x \notin_h [f^\flat\rangle Q &\text{ iff } \neg\forall y(y \in_k Q \text{ or } \mathcal{F}^\flat xy) \\ &\text{ iff } \exists y(y \notin_k Q \text{ and } \neg\mathcal{F}^\flat xy) \end{aligned}$$

Let $x \in_h H^\circ$ and $y \in_k K^\circ$, then by assumption, $y \notin_k Q$. From the Frame Condition 4.11.3 (2), $\neg\mathcal{F}^\flat xy$. So there is some y such that $y \notin_k Q$ and $\neg\mathcal{F}^\flat xy$, whence $x \notin_h [f^\flat\rangle Q$, and $[f^\flat\rangle Q$ is null-pointed.

To show $[f^\flat\rangle Q$ is well-pointed, we need to show

$$x \in_h H^\bullet \text{ implies } \forall y(y \notin_k Q \text{ implies } \mathcal{F}^\flat xy)$$

Let $x \in_h H^\bullet$ and choose some arbitrary $y \notin_k Q$. Since Q is well-pointed, $K^\bullet \subseteq_k Q$ and $y \notin_k K^\bullet$. Therefore $\mathcal{F}^\flat xy$ holds from the Frame Condition 4.11.4 (1). So for all y, $y \notin_k Q$ implies $\mathcal{F}^\flat xy$. By definition, $x \in_h [f^\flat\rangle Q$ and $[f^\flat\rangle Q$ is well-pointed.

■

Lemma 4.11.8 *$[f^\flat\rangle$ preserves its bounds.*

Proof: The type of $[f^\flat\rangle$ is $\wedge\mapsto\wedge$, hence we want $[f^\flat\rangle \top^{\mathbb{K}} = \top^{\mathbb{H}}$. The definition is

$$x \in_h [f^\flat\rangle Q \text{ iff } \forall y(y \in_k Q \text{ or } \mathcal{F}^\flat xy).$$

We have

$$x \in_h [f^\flat\rangle \top^{\mathbb{K}} \text{ iff } \forall y(y \in_k \top^{\mathbb{K}} \text{ or } \mathcal{F}^\flat xy).$$

From Lemma 4.11.7, $[f^\flat\rangle \top^{\mathbb{K}}$ is well- and null-pointed. Since $\top^{\mathbb{H}}$ is the top of the lattice, $[f^\flat\rangle \top^{\mathbb{K}} \subseteq_h \top^{\mathbb{H}}$.

Next, let $x \in_h \top^{\mathbb{H}}$. Choose some arbitrary y. Either $y \in_k K^\circ$ or $y \notin_k K^\circ$. Let $y \in_k K^\circ$. Since $x \in_h \top^{\mathbb{H}}$, then $x \notin_h H^\circ$, and from the Frame Condition 4.11.3 (1), $\mathcal{F}^\flat xy$ holds. If $y \notin_k K^\circ$, then $y \in_k \top^{\mathbb{K}}$. Hence, for any y, either $y \in_k \top^{\mathbb{K}}$ or $\mathcal{F}^\flat xy$. Therefore, $x \in_h [f^\flat\rangle \top^{\mathbb{K}}$.

■

The function β is the Stone representation function.

Lemma 4.11.9

$$x \in_h [f^\flat\rangle \beta b \text{ iff } x \in_h \beta[f^\flat\rangle b.$$

Proof: The right to left is the easy direction. Assume $x \in\mathsf{h} \beta[f^{\flat}\rangle\, b$, then $[f^{\flat}\rangle\, b \in\mathsf{h} x$. Choose some arbitrary y and let $y \notin\mathsf{k} \beta b$, then $b \notin\mathsf{k} y$. By the canonical definition, $\mathcal{F}^{\flat}xy$. Since the choice of y was arbitrary, we have

$$\forall y(b \notin\mathsf{k} y \text{ implies } \mathcal{F}^{\flat}xy)$$

and hence

$$\forall y(y \in\mathsf{k} \beta b \text{ or } \mathcal{F}^{\flat}xy).$$

So $x \in\mathsf{h} [f^{\flat}\rangle \beta b$.

To go in the other direction, we prove the usual contrapositive. Assume $x \notin\mathsf{h} \beta[f^{\flat}\rangle\, b$, hence $[f^{\flat}\rangle \notin\mathsf{h} x$. Construct

$$\hat{y} = (\lambda p\ .\ [f^{\flat}\rangle\, p)^{-1}x.$$

y is a filter because of the type of $[f^{\flat}\rangle$. Suppose $b \in\mathsf{k} \hat{y}$, then $[f^{\flat}\rangle\, b \in\mathsf{h} x$, which is a contradiction. So $b \notin\mathsf{k} \hat{y}$ and $\downarrow b \cap \hat{y} = \emptyset$. Maximalize $\hat{y}$ against $\downarrow b$ using Zorn's Lemma to yield the maximal filter y. By construction we have

$$\forall a([f^{\flat}\rangle\, a \in\mathsf{h} x \text{ implies } a \in\mathsf{k} y)$$

Since

$$\forall a([f^{\flat}\rangle\, a \in\mathsf{h} x \text{ implies } a \in\mathsf{k} y) \text{ iff } \neg\exists a([f^{\flat}\rangle\, a \in\mathsf{h} x \text{ and } a \notin\mathsf{k} y),$$

we have constructed $\neg\mathcal{F}^{\flat}xy$. Hence we have

$$\exists y(y \notin\mathsf{k} \beta b \text{ and } \neg\mathcal{F}^{\flat}xy).$$

Since

$$\neg\exists y(y \notin\mathsf{k} \beta b \text{ and } \neg\mathcal{F}^{\flat}xy) \text{ iff } \forall y(y \in\mathsf{k} \beta b \text{ or } \mathcal{F}^{\flat}xy),$$

the contraposition has been proven.

■

We assume $f = f_1 \varovee f_2$, then

$$\mathcal{F}^{\flat}\langle a_1, a_2\rangle\langle b_1, b_2\rangle \text{ iff } \mathcal{F}_1^{\flat}a_1b_1 \text{ or } \mathcal{F}_2^{\flat}a_2b_2.$$

Lemma 4.11.10 *Let $f = f_1 \varovee f_2$, then*

$$[f^{\flat}\rangle\,(Q_1 \varovee Q_2) = [f_1^{\flat}\rangle\, Q_1 \varovee [f_2^{\flat}\rangle\, Q_2 \qquad [f^{\flat}\cdot\rangle(P_1 \varovee P_2) = [f_1^{\flat}\cdot\rangle P_1 \varovee [f_2^{\flat}\cdot\rangle P_2$$

Proof:

1	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f^{\flat}\rangle(Q_1 \varovee Q_2)$	assume
2	$\forall\langle y, v\rangle\big(\langle y, v\rangle \in_{\mathsf{k}} Q_1 \varovee Q_2$ or $\mathcal{F}^{\flat}\langle a_1, a_2\rangle\langle y, v\rangle\big)$	def. $[f^{\flat}\rangle$, line 1
3	$(a_1 \in_{\mathsf{h}_1} H_1$ and $a_2 \in_{\mathsf{h}_2} H_2^{\circ})$ or $(a_1 \in_{\mathsf{h}_1} H_1^{\circ}$ and $a_2 \in_{\mathsf{h}_2} H_2)$	def. $\varovee$ universe, line 1
4	$a_1 \in_{\mathsf{h}_1} H_1$ and $a_2 \in_{\mathsf{h}_2} H_2^{\circ}$	assume
5	$\boxed{b_1}\quad b_1 \notin_{\mathsf{k}_1} Q_1$	assume
6	$\exists u(u \in_{\mathsf{k}_2} K_2^{\circ})$	$K_2^{\circ} \neq \emptyset$
7	$\boxed{b_2}\quad b_2 \in_{\mathsf{k}_2} K_2^{\circ}$	assume
8	$\langle b_1, b_2\rangle \notin_{\mathsf{k}} Q_1 \times K_2^{\circ}$ and $\langle b_1, b_2\rangle \notin_{\mathsf{k}} K_1^{\circ} \times Q_2$	Q_2 is null-pointed, set th., lines 5, 7
9	$\langle b_1, b_2\rangle \notin_{\mathsf{k}} Q_1 \varovee Q_2$	def. $\varovee$, line 8
10	$\langle b_1, b_2\rangle \notin_{\mathsf{k}} Q_1 \varovee Q_2$ implies $\mathcal{F}^{\flat}\langle a_1, a_2\rangle\langle b_1, b_2\rangle$	$\forall$-Elim, line 2
11	$\mathcal{F}^{\flat}\langle a_1, a_2\rangle\langle b_1, b_2\rangle$	$\supset$-Elim, lines 9, 10
12	$\mathcal{F}_1^{\flat} a_1 b_1$ or $\mathcal{F}_2^{\flat} a_2 b_2$	def. $\mathcal{F}^{\flat}$, line 11
13	$\neg\mathcal{F}_2^{\flat} a_2 b_2$	Frame Condition 4.11.3 (2), line 4
14	$\mathcal{F}_1^{\flat} a_1 b_1$	CL, lines 12, 13
15	$\mathcal{F}_1^{\flat} a_1 b_1$	$\exists$-Elim, lines 6, 7
16	$\forall y(y \notin_{\mathsf{k}_1} Q_1$ implies $\mathcal{F}_1^{\flat} a_1 y)$	$\forall$-Intro, line 5
17	$a_1 \in_{\mathsf{h}_1} [f_1^{\flat}\rangle Q_1$	def. $[-^{\flat}\rangle$, line 16
18	$a_1 \in_{\mathsf{h}_1} [f_1^{\flat}\rangle Q_1$ and $a_2 \in_{\mathsf{h}_2} H_2^{\circ}$	$\wedge$-Elim, $\wedge$-Intro, lines 4, 17
19	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f_1^{\flat}\rangle Q_1 \varovee [f_2^{\flat}\rangle Q_2$	def. $\varovee$, line 18
20	$a_1 \in_{\mathsf{h}_1} H_1^{\circ}$ and $a_2 \in_{\mathsf{h}_2} H_2$	assume
21	subproof is similar to the previous subproof	lines xx
22	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f_1^{\flat}\rangle Q_1 \varovee [f_2^{\flat}\rangle Q_2$	def. $\varovee$, line xx
23	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f_1^{\flat}\rangle Q_1 \varovee [f_2^{\flat}\rangle Q_2$	$\vee$-Elim, lines 3, 4, 20

To fill in the similar subproof above:

1	$\langle a_1, a_2\rangle \in\mathsf{h} [f^{\flat}\rangle(Q_1 \ovee Q_2)$	assume
2	$\forall\langle y, v\rangle\big(\langle y, v\rangle \in\mathsf{k} Q_1 \ovee Q_2$ or $\mathcal{F}^{\flat}\langle a_1, a_2\rangle\langle y, v\rangle\big)$	def. $[f^{\flat}\rangle$, line 1
3	$(a_1 \in\mathsf{h}_1 H_1$ and $a_2 \in\mathsf{h}_2 H_2^{\circ})$ or $(a_1 \in\mathsf{h}_1 H_1^{\circ}$ and $a_2 \in\mathsf{h}_2 H_2)$	def. $\ovee$ universe, line 1
4	$a_1 \in\mathsf{h}_1 H_1^{\circ}$ and $a_2 \in\mathsf{h}_2 H_2$	assume
5	$\boxed{b_2}\quad b_2 \notin\mathsf{k}_2 Q_2$	assume
6	$\exists u(u \in\mathsf{k}_1 K_1^{\circ})$	$K_1^{\circ} \neq \emptyset$
7	$\boxed{b_1}\quad b_1 \in\mathsf{k}_1 K_1^{\circ}$	assume
8	$\langle b_1, b_2\rangle \notin\mathsf{k} Q_1 \times K_2^{\circ}$ and $\langle b_1, b_2\rangle \notin\mathsf{k} K_1^{\circ} \times Q_2$	Q_1 is null-pointed, set th., lines 5, 7
9	$\langle b_1, b_2\rangle \notin\mathsf{k} Q_1 \ovee Q_2$	def. $\ovee$, line 8
10	$\langle b_1, b_2\rangle \notin\mathsf{k} Q_1 \ovee Q_2$ implies $\mathcal{F}^{\flat}\langle a_1, a_2\rangle\langle b_1, b_2\rangle$	$\forall$-Elim, line 2
11	$\mathcal{F}^{\flat}\langle a_1, a_2\rangle\langle b_1, b_2\rangle$	$\supset$-Elim, lines 9, 10
12	$\mathcal{F}_1^{\flat}a_1b_1$ or $\mathcal{F}_2^{\flat}a_2b_2$	def. $\mathcal{F}^{\flat}$, line 11
13	$\neg\mathcal{F}_1^{\flat}a_1b_1$	Frame Condition 4.11.3 (2), line 4
14	$\mathcal{F}_2^{\flat}a_2b_2$	CL, lines 12, 13
15	$\mathcal{F}_2^{\flat}a_2b_2$	$\exists$-Elim, lines 6, 7
16	$\forall v(v \notin\mathsf{k}_2 Q_2$ implies $\mathcal{F}_2^{\flat}a_2v)$	$\forall$-Intro, line 5
17	$a_2 \in\mathsf{h}_2 [f_2^{\flat}\rangle Q_2$	def. $[-^{\flat}\rangle$, line 16
18	$a_1 \in\mathsf{h}_1 H_1^{\circ}$ and $a_2 \in\mathsf{h}_2 [f_2^{\flat}\rangle Q_2$	$\wedge$-Elim, $\wedge$-Intro, lines 4, 17
19	$\langle a_1, a_2\rangle \in\mathsf{h} [f_1^{\flat}\rangle Q_1 \ovee [f_2^{\flat}\rangle Q_2$	def. $\ovee$, line 18

The other direction:

1 $\langle a_1, a_2\rangle \in_{\mathsf{h}} [f_1^{\flat}\rangle Q_1 \ovee [f_2^{\flat}\rangle Q_2$ . . . . . assume

2 $(a_1 \in_{\mathsf{h}_1} [f_1^{\flat}\rangle Q_1$ and $a_2 \in_{\mathsf{h}_2} H_2^{\circ})$ or $(a_1 \in_{\mathsf{h}_1} H_1^{\circ}$ and $a_2 \in_{\mathsf{h}_2} [f_2^{\flat}\rangle Q_2)$ . . . . . def. $\ovee$, line 1

3 $a_1 \in_{\mathsf{h}_1} [f_1^{\flat}\rangle Q_1$ and $a_2 \in_{\mathsf{h}_2} H_2^{\circ}$ . . . . . assume

4 $\boxed{b_1, b_2}$ $\langle b_1, b_2\rangle \notin_{\mathsf{k}} Q_1 \ovee Q_2$ . . . . . assume

5 $(b_1 \in_{\mathsf{k}_1} K_1^{\circ}$ and $b_2 \notin_{\mathsf{k}_2} Q_2)$ or $(b_1 \notin_{\mathsf{k}_1} Q_1$ and $b_2 \in_{\mathsf{k}_2} K_2^{\circ})$ . . . . . Lemma 3.5.4, line 4

6 $b_1 \in_{\mathsf{k}_1} K_1^{\circ}$ and $b_2 \notin_{\mathsf{k}_2} Q_2$ . . . . . assume

7 $a_1 \notin_{\mathsf{h}_1} H_1^{\circ}$ . . . . . Lemma 4.11.7, $[f_1^{\flat}\rangle Q_1$ is null-pointed, line 3

8 $\mathcal{F}_1^{\flat} a_1 b_1$ . . . . . Frame Condition 4.11.3 (1), lines 6, 7

9 $\mathcal{F}^{\flat}\langle a_1, a_2\rangle\langle b_1, b_2\rangle$ . . . . . def. $\mathcal{F}^{\flat}$, line 8

10 $b_1 \notin_{\mathsf{k}_1} Q_1$ and $b_2 \in_{\mathsf{k}_2} K_2^{\circ}$ . . . . . assume

11 $\forall y(y \notin_{\mathsf{k}_1} Q_1$ implies $\mathcal{F}_1^{\flat} a_1 y)$ . . . . . def. $[f_1^{\flat}\rangle$, line 3

12 $b_1 \notin_{\mathsf{k}_1} Q_1$ implies $\mathcal{F}_1^{\flat} a_1 b_1$ . . . . . $\forall$-Elim, line 11

13 $\mathcal{F}_1^{\flat} a_1 b_1$ . . . . . $\supset$-Elim, lines 10, 12

14 $\mathcal{F}_1^{\flat} a_1 b_1$ or $\mathcal{F}_2^{\flat} a_2 b_2$ . . . . . $\vee$-Intro, line 13

15 $\mathcal{F}^{\flat}\langle a_1, a_2\rangle\langle b_1, b_2\rangle$ . . . . . def. $\mathcal{F}^{\flat}$, line 14

16 $\mathcal{F}^{\flat}\langle a_1, a_2\rangle\langle b_1, b_2\rangle$ . . . . . $\vee$-Elim, lines 5, 6, 10

17 $\forall\langle y, v\rangle(\langle y, v\rangle \notin_{\mathsf{k}} Q_1 \ovee Q_2$ implies $\mathcal{F}^{\flat}\langle a_1, a_2\rangle\langle y, v\rangle)$ . . . . . $\forall$-Intro, line 4

18 $\langle a_1, a_2\rangle \in_{\mathsf{h}} [f^{\flat}\rangle (Q_1 \ovee Q_2)$ . . . . . def. $[-^{\flat}\rangle$, line 17

19 $a_1 \in_{\mathsf{h}_1} H_1^{\circ}$ and $a_2 \in_{\mathsf{h}_2} [f_2^{\flat}\rangle Q_2$ . . . . . assume

20 subproof similar to previous subproof . . . . . lines xx

21 $\langle a_1, a_2\rangle \in_{\mathsf{h}} [f^{\flat}\rangle (Q_1 \ovee Q_2)$ . . . . . def. $[-^{\flat}\rangle$, line xx

22 $\langle a_1, a_2\rangle \in_{\mathsf{h}} [f^{\flat}\rangle (Q_1 \ovee Q_2)$ . . . . . $\vee$-Elim, lines 2, 3, 19

To fill in the similar subproof above:

1	$\langle a_1, a_2\rangle \in_{h} [f_1^{\flat}\rangle Q_1 \ovee [f_2^{\flat}\rangle Q_2$	assume
2	$(a_1 \in_{h_1} [f_1^{\flat}\rangle Q_1$ and $a_2 \in_{h_2} H_2^{\circ})$ or $(a_1 \in_{h_1} H_1^{\circ}$ and $a_2 \in_{h_2} [f_2^{\flat}\rangle Q_2)$	def. $\ovee$, line 1
3	$a_1 \in_{h_1} H_1^{\circ}$ and $a_2 \in_{h_2} [f_2^{\flat}\rangle Q_2$	assume
4	$\boxed{b_1, b_2}$ $\langle b_1, b_2\rangle \notin_{k} Q_1 \ovee Q_2$	assume
5	$(b_1 \in_{k_1} K_1^{\circ}$ and $b_2 \notin_{k_2} Q_2)$ or $(b_1 \notin_{k_1} Q_1$ and $b_2 \in_{k_2} K_2^{\circ})$	Lemma 3.5.4, line 4
6	$b_1 \notin_{k_1} Q_1$ and $b_2 \in_{k_2} K_2^{\circ}$	assume
7	$a_2 \notin_{h_2} H_2^{\circ}$	Lemma 4.11.7, $[f_2^{\flat}\rangle Q_2$ is null-pointed, line 3
8	$\mathcal{F}_2^{\flat} a_2 b_2$	Frame Condition 4.11.3 (1), line 6, 7
9	$\mathcal{F}^{\flat}\langle a_1, a_2\rangle\langle b_1, b_2\rangle$	def. $\mathcal{F}^{\flat}$, line 8
10	$b_1 \in_{k_1} K_1^{\circ}$ and $b_2 \notin_{k_2} Q_2$	assume
11	$\forall v(v \notin_{k_2} Q_2$ implies $\mathcal{F}_2^{\flat} a_2 v)$	def. $[f_2^{\flat}\rangle$, line 3
12	$b_2 \notin_{k_2} Q_2$ implies $\mathcal{F}_2^{\flat} a_2 b_2$	$\forall$-Elim, line 11
13	$\mathcal{F}_2^{\flat} a_2 b_2$	$\supset$-Elim, lines 10, 12
14	$\mathcal{F}_1^{\flat} a_1 b_1$ or $\mathcal{F}_2^{\flat} a_2 b_2$	$\vee$-Intro, line 13
15	$\mathcal{F}^{\flat}\langle a_1, a_2\rangle\langle b_1, b_2\rangle$	def. $\mathcal{F}^{\flat}$, line 14
16	$\mathcal{F}^{\flat}\langle a_1, a_2\rangle\langle b_1, b_2\rangle$	$\vee$-Elim, lines 5, 6, 10
17	$\forall\langle y, v\rangle(\langle y, v\rangle \notin_{k} Q_1 \ovee Q_2$ implies $\mathcal{F}^{\flat}\langle a_1, a_2\rangle\langle y, v\rangle)$	$\forall$-Intro, line 4
18	$\langle a_1, a_2\rangle \in_{h} [f^{\flat}\rangle (Q_1 \ovee Q_2)$	def. $[-^{\flat}\rangle$, line 17

■

4.12 Info for $[-^{\flat}\cdot\rangle$

The type of $[-^{\flat}\cdot\rangle$ is $\wedge\mapsto\wedge$.

Definition 4.12.1

$$y \in_{k} [f^{\flat}\cdot\rangle P \text{ iff } \forall x(x \in_{h} P \text{ or } \mathcal{F}^{\flat\cdot} xy).$$

Definition 4.12.2 The canonical relation is defined as

$$\mathcal{F}^{\flat\cdot} xy \text{ iff } \exists a(a \notin_{h} x \text{ and } [f^{\flat}\cdot\rangle a \in_{k} y).$$

The negated relation is

$$\neg\mathcal{F}^{\flat\cdot} xy \text{ iff } \forall a(a \in_{h} x \text{ or } [f^{\flat}\cdot\rangle a \notin_{k} y).$$

Frame Condition 4.12.3 The null frame conditions for $\mathcal{F}^{\flat\cdot}$ are

$$(1) \quad \forall y(y \notin_{k} K^{\circ} \text{ implies } \mathcal{F}^{\flat\cdot} H^{\circ} y) \qquad (2) \quad \forall x(\neg\mathcal{F}^{\flat\cdot} x K^{\circ}).$$

Frame Condition 4.12.4 The well frame conditions for $\mathcal{F}^{\flat\cdot}$ are

$$(1)\quad \forall x(x \notin_{\mathsf{h}} H^{\bullet} \text{ implies } \mathcal{F}^{\flat\cdot}xK^{\bullet}) \qquad (2)\quad \forall y(\neg\mathcal{F}^{\flat\cdot}H^{\bullet}y).$$

Lemma 4.12.5 *Canonically, the Frame Conditions hold.*

Proof: Frame Condition 4.12.3 (1): choose any y such that $y \neq \circ_K$. Let $x = \circ_H$. Hence $\top_{\mathsf{h}} \notin_{\mathsf{h}} x$ and from the type of $[-^{\flat}\cdot\rangle$, $[f^{\flat}\cdot\rangle\top_{\mathsf{h}} = \top_{\mathsf{k}}$ and $\top_{\mathsf{k}} \in_{\mathsf{k}} y$. So there is some a such that $a \notin_{\mathsf{h}} x$ and $[f^{\flat}\cdot\rangle a \in_{\mathsf{k}} y$, whence $\mathcal{F}^{\flat\cdot}xy$ obtains.

Frame Condition 4.12.3 (2): choose some arbitrary x and let $y = \circ_K$. For all a, it is the case that $[f^{\flat}\cdot\rangle a \notin_{\mathsf{k}} \circ_K$. Hence, $\neg\mathcal{F}^{\flat\cdot}xy$.

Frame Condition 4.12.4 (1): choose any x such that $x \neq \bullet_H$. Let $y = \bullet_K$. Hence $\bot_{\mathsf{h}} \notin_{\mathsf{h}} x$. Since $\bullet_K$ is the entire carrier set, $[f^{\flat}\cdot\rangle\bot_{\mathsf{h}} \in_{\mathsf{k}} y$. So there is some a such that $a \notin_{\mathsf{h}} x$ and $[f^{\flat}\cdot\rangle a \in_{\mathsf{k}} y$, whence $\mathcal{F}^{\flat\cdot}xy$ obtains.

Frame Condition 4.12.4 (2): choose any y and let $x = \bullet_H$ then for all a, $a \in_{\mathsf{h}} x$, so $\neg\mathcal{F}^{\flat\cdot}xy$ obtains. ■

Lemma 4.12.6 *The Frame Conditions 4.12.3 and 4.12.4 are consistent with each other.*

Proof: The well-point sets and null-point sets are always disjoint. This disjointness and quantifiers and antecedents of the conditionals prevent any conflicts in the Frame Conditions 4.12.3 and 4.12.4.

■

4.13 $[-^{\sharp}\rangle$, $[-^{\sharp}\cdot\rangle$: Closure Under Smash Product

All proofs involving $[-^{\sharp}\cdot\rangle$ are similar to their siblings for $[-^{\sharp}\rangle$ and we elide them. The information necessary to construct them is in the following Section 4.14.

$[-^{\sharp}\rangle$ is of type $\vee\!\mapsto\!\vee$.

Definition 4.13.1

$$x \in_{\mathsf{h}} [f^{\sharp}\rangle Q \text{ iff } \exists y(y \in_{\mathsf{k}} Q \text{ and } \neg\mathcal{F}^{\sharp}xy).$$

Definition 4.13.2 The canonical relation and its negation are defined as,

$$\mathcal{F}^{\sharp}xy \text{ iff } \exists b(b \in_{\mathsf{k}} y \text{ and } [f^{\sharp}\rangle\, b \notin_{\mathsf{h}} x) \qquad \neg\mathcal{F}^{\sharp}xy \text{ iff } \forall b(b \notin_{\mathsf{k}} y \text{ or } [f^{\sharp}\rangle\, b \in_{\mathsf{h}} x).$$

Frame Condition 4.13.3 The null frame conditions for $\mathcal{F}^\sharp$ are

$$(1) \quad \forall y(y \notin_{\mathsf{k}} K^\circ \text{ implies } \mathcal{F}^\sharp H^\circ y) \qquad (2) \quad \forall x(\neg\mathcal{F}^\sharp x K^\circ).$$

Frame Condition 4.13.4 The well frame conditions for $\mathcal{F}^\sharp$ are

$$(1) \quad \forall x(x \notin_{\mathsf{k}} H^\bullet \text{ implies } \mathcal{F}^\sharp x K^\bullet). \qquad (2) \quad \forall y(\neg\mathcal{F}^\sharp H^\bullet y)$$

Lemma 4.13.5 *Canonically, the Frame Conditions hold.*

Proof: Frame Condition 4.13.3 (1): Choose some arbitrary y and assume $y \neq \circ_K$. Let $x = \circ_H$. Then $y \neq \emptyset$ so there is some $b \in_{\mathsf{k}} y$. Since $x = \emptyset$, then $[f^\sharp\rangle\, b \notin_{\mathsf{h}} x$. By definition, $\mathcal{F}^\sharp xy$.

Frame Condition 4.13.3 (2): Choose some arbitrary x. Let $y = \circ_K$, then for all b, $b \notin_{\mathsf{k}} y$ and $\neg\mathcal{F}^\sharp xy$ holds.

Frame Condition 4.13.4 (1): Choose $x \neq \bullet_H$ and assume $y = \bullet_K$. Let $b = \bot_{\mathsf{k}}$ and hence $b \in_{\mathsf{k}} y$. From the type of $[-^\sharp\rangle$, $[f^\sharp\rangle\, b = \bot_{\mathsf{h}}$. Since $x \neq \bullet_H$, $[f^\sharp\rangle\, b \notin_{\mathsf{h}} x$. So there is some b such that $b \in_{\mathsf{k}} y$ and $[f^\sharp\rangle\, b \notin_{\mathsf{h}} x$, whence $\mathcal{F}^\sharp xy$ obtains.

Frame Condition 4.13.4 (2): Choose some arbitrary y and let $x = \bullet_H$. For all b, $[f^\sharp\rangle\, b \in_{\mathsf{h}} x$, whence $\neg\mathcal{F}^\sharp xy$ obtains.

■

Lemma 4.13.6 *The Frame Conditions 4.13.3 and 4.13.4 are consistent with each other.*

Proof: Note that the well-point sets and null-point sets are always disjoint, then the quantifiers and antecedents of the conditionals of Frame Conditions 4.13.3 and 4.13.4 prevent any conflicts.

■

Lemma 4.13.7 $[f^\sharp\rangle\, Q$ *is null-pointed if* Q *is null-pointed, and well-pointed if* Q *is well-pointed.*

Proof: Our conditions are

$$x \in_{\mathsf{h}} [f^\sharp\rangle\, Q \text{ iff } \exists y(y \in_{\mathsf{k}} Q \text{ and } \neg\mathcal{F}^\sharp xy). \qquad x \notin_{\mathsf{h}} [f^\sharp\rangle\, Q \text{ iff } \forall y(y \notin_{\mathsf{k}} Q \text{ or } \mathcal{F}^\sharp xy).$$

Assume Q is null-pointed and let $x \in_{\mathsf{h}} H^\circ$, then we must show that for all y, $y \in_{\mathsf{k}} Q$ implies $\mathcal{F}^\sharp xy$. Choose some arbitrary y such that $y \in_{\mathsf{k}} Q$. Since $K^\circ \cap Q = \emptyset$, for all $y \in_{\mathsf{k}} Q$, $y \notin_{\mathsf{k}} K^\circ$. From Frame Condition 4.13.3 (1), $\mathcal{F}^\sharp xy$. Since the choice of y was arbitrary, the condition for $x \notin_{\mathsf{h}} [f^\sharp\rangle\, Q$ holds and $[f^\sharp\rangle\, Q$ is null-pointed.

To show $[f^{\sharp}\rangle\, Q$ is well-pointed, we need to show

$$x \in_{h} H^{\bullet} \text{ implies } \exists y(y \in_{k} Q \text{ and } \neg\mathcal{F}^{\sharp}xy).$$

Assume Q is well-pointed, hence $K^{\bullet} \subseteq_{k} Q$. Let $x \in_{h} H^{\bullet}$. Since $K^{\bullet} \neq \emptyset$, there is some $y \in_{k} Q$ from $K^{\bullet}$. From Frame Condition 4.13.4 (2), $\neg\mathcal{F}^{\sharp}xy$ obtains and hence $x \in_{h} [f^{\sharp}\rangle\, Q$. So $H^{\bullet} \subseteq_{h} [f^{\sharp}\rangle\, Q$ and $[f^{\sharp}\rangle\, Q$ is well-pointed. ■

Lemma 4.13.8 $[f^{\sharp}\rangle$ *preserves its bounds.*

Proof: The type of $[f^{\sharp}\rangle$ is $\vee\mapsto\vee$, hence we want $[f^{\sharp}\rangle \perp^{\mathbb{K}} = \perp^{\mathbb{H}}$. The definition is

$$x \in_{h} [f^{\sharp}\rangle\, Q \text{ iff } \exists y(y \in_{k} Q \text{ and } \neg\mathcal{F}^{\sharp}xy).$$

Since $\perp^{\mathbb{H}} = H^{\bullet}$, then since all sets in $\mathbb{H}$ are well-pointed, $\perp^{\mathbb{H}} \subseteq_{k} [f^{\sharp}\rangle \perp^{\mathbb{K}}$.

The other direction is by contraposition. We have

$$x \notin_{h} [f^{\sharp}\rangle \perp^{\mathbb{K}} \text{ iff } \forall y(y \notin_{k} \perp^{\mathbb{K}} \text{ or } \mathcal{F}^{\sharp}xy).$$

and wish to show $[f^{\sharp}\rangle \perp^{\mathbb{K}} \subseteq_{h} \perp^{\mathbb{H}}$ by showing that $x \notin_{h} \perp^{\mathbb{H}}$ implies $x \notin_{h} [f^{\sharp}\rangle \perp^{\mathbb{K}}$. Assume $x \notin_{h} \perp^{\mathbb{H}}$. Since $\perp^{\mathbb{H}} = H^{\bullet}$, then $x \notin_{h} H^{\bullet}$. Choose any $y \in_{k} \perp^{\mathbb{K}}$, hence $y \in_{k} K^{\bullet}$. From Frame Condition 4.13.4 (1), $\mathcal{F}^{\sharp}xy$ holds. Hence $x \notin_{h} [f^{\sharp}\rangle \perp^{\mathbb{K}}$.

■

Lemma 4.13.9

$$x \in_{h} [f^{\sharp}\rangle \beta b \text{ iff } x \in_{h} \beta[f^{\sharp}\rangle\, b.$$

Proof: The left to right is the easy direction. Assume $x \in_{h} [f^{\sharp}\rangle \beta b$, then there is some y such that $y \in_{k} \beta b$ and $\neg\mathcal{F}^{\sharp}xy$. Using the canonical definition, we have

$$\neg\mathcal{F}^{\sharp}xy \text{ iff } \forall a(a \in_{k} y \text{ implies } [f^{\sharp}\rangle\, a \in_{h} x).$$

Since $b \in_{k} y$, then $[f^{\sharp}\rangle\, b \in_{h} x$. Hence $x \in_{h} \beta[f^{\sharp}\rangle\, b$.

To go in the other direction. Assume $[f^{\sharp}\rangle\, b \in_{h} x$. Let

$$\hat{y} = (\lambda p\ .\ [f^{\sharp}\rangle\, p)^{-1}\overline{x}.$$

Given the type and Lemma 3.7.1, $\hat{y}$ is an ideal. Assume $b \in_{k} \hat{y}$. Since $b \in_{k} \hat{y}$ implies $[f^{\sharp}\rangle\, b \in_{h} \overline{x}$, we have $[f^{\sharp}\rangle\, b \in_{h} \overline{x}$, which is a contradiction. So $b \notin_{k} \hat{y}$ Construct y by using Zorn's Lemma to maximalize $\langle \uparrow b, \hat{y}\rangle$. By construction,

$$\forall a([f^{\sharp}\rangle\, a \notin_{h} x \text{ implies } a \notin_{k} y)$$

which is equivalent to the canonical definition of $\neg\mathcal{F}^{\#}xy$. Also by construction, $b \in_{k} y$ and so $y \in_{k} \beta b$. Therefore there exists some y, such that $y \in_{k} \beta b$ and $\neg\mathcal{F}^{\#}xy$, whence $x \in_{h} [f^{\#}\rangle\beta b$.

■

Lemma 4.13.10 *Let* $\neg f = \neg f_1 \otimes \neg f_2$*, then* $\neg\mathcal{F}^{\#}\langle a_1, a_2\rangle\langle b_1, b_2\rangle$ iff $\neg\mathcal{F}^{\#}_{1}a_1b_1$ and $\neg\mathcal{F}^{\#}a_2b_2$ *and*

$$[f^{\#}\rangle Q_1 \otimes Q_2 = [f_1^{\#}\rangle Q_1 \otimes [f_2^{\#}\rangle Q_2 \qquad [f^{\#}\cdot\rangle P_1 \otimes P_2 = [f_1^{\#}\cdot\rangle P_1 \otimes [f_2^{\#}\cdot\rangle P_2$$

Proof: Note this proof is nearly identical to that for $[-^{\circ}\rangle$, only the composition of relations has changed to accommodate the negation in front of $\mathcal{F}^{\#}$ in the UCLA definition:

1	$\langle a_1, a_2\rangle \in_{h} [f^{\#}\rangle(Q_1 \otimes Q_2)$	assume
2	$(\langle a_1, a_2\rangle \in_{h} [f^{\#}\rangle(Q_1 \otimes Q_2) - H^{\bullet}_{\otimes})$ or $\langle a_1, a_2\rangle \in_{h} H^{\bullet}_{\otimes}$	$[-^{\#}\rangle$ is well-pointed, Lemma 4.13.7, line 1
3	$\langle a_1, a_2\rangle \in_{h} [f^{\#}\rangle(Q_1 \otimes Q_2) - H^{\bullet}_{\otimes}$	assume
4	$\langle a_1, a_2\rangle \in_{h} H_1 \otimes H_2$	$f = f_1 \otimes f_2$, line 1
5	$\langle a_1, a_2\rangle \notin_{h} H^{\bullet}_{\otimes}$ and $\langle a_1, a_2\rangle \in_{h} H_1 \otimes H_2$	$\wedge$-Intro, lines 3, 4
6	$a_1 \notin_{h_1} H^{\bullet}_1$ and $a_2 \notin_{h_2} H^{\bullet}_2$	Lemma 3.4.9, line 5
7	$\exists\langle y, v\rangle(\langle y, v\rangle \in_{k} Q_1 \otimes Q_2$ and $\neg\mathcal{F}^{\#}\langle a_1, a_2\rangle\langle y, v\rangle)$	def. $[-^{\#}\rangle$, line 3
8	$\boxed{b_1, b_2}$ $\langle b_1, b_2\rangle \in_{k} Q_1 \otimes Q_2$ and $\neg\mathcal{F}^{\#}\langle a_1, a_2\rangle\langle b_1, b_2\rangle$	assume
9	$((\langle b_1, b_2\rangle \in_{k} (Q_1 - K^{\bullet}_1) \times (Q_2 - K^{\bullet}_2))$ or $\langle b_1, b_2\rangle \in_{k} K^{\bullet}_{\otimes}$	def. $\otimes$, line 8
10	$\langle b_1, b_2\rangle \in_{k} (Q_1 - K^{\bullet}_1) \times (Q_2 - K^{\bullet}_2)$	assume
11	$\neg\mathcal{F}^{\#}_{1}a_1b_1$ and $\neg\mathcal{F}^{\#}_{2}a_2b_2$	def. $\neg\mathcal{F}^{\#}$, line 8
12	$b_1 \in_{k_1} Q_1$ and $b_2 \in_{k_2} Q_2$	set th., line 10
13	$a_1 \in_{h_1} [f_1^{\#}\rangle Q_1$ and $a_2 \in_{h_2} [f_2^{\#}\rangle Q_2$	def. $[-^{\#}\rangle$, lines 11, 12
14	$a_1 \in_{h_1} [f_1^{\#}\rangle Q_1 - H^{\bullet}_1$ and $a_2 \in_{h_2} [f_2^{\#}\rangle Q_2 - H^{\bullet}_2$	set th., lines 6, 13
15	$\langle b_1, b_2\rangle \in_{k} K^{\bullet}_{\otimes}$	assume
16	$b_1 \in_{k_1} K^{\bullet}_1$ and $b_2 \in_{k_2} K^{\bullet}_2$	def. $K^{\bullet}_{\otimes}$, line 15
17	$\mathcal{F}^{\#}_{1}a_1b_1$ and $\mathcal{F}^{\#}_{2}a_2b_2$	Frame Condition 4.13.4 (1), lines 6, 16
18	$\rightarrow\leftarrow$	Contradiction, lines 11, 17
19	$a_1 \in_{h_1} [f_1^{\#}\rangle Q_1 - H^{\bullet}_1$ and $a_2 \in_{h_2} [f_2^{\#}\rangle Q_2 - H^{\bullet}_2$	CL, lines 9, 10, 15
20	$a_1 \in_{h_1} [f_1^{\#}\rangle Q_1 - H^{\bullet}_1$ and $a_2 \in_{h_2} [f_2^{\#}\rangle Q_2 - H^{\bullet}_2$	$\exists$-Elim, lines 7, 8
21	$\langle a_1, a_2\rangle \in_{h} ([f_1^{\#}\rangle Q_1 - H^{\bullet}_1) \times ([f_2^{\#}\rangle Q_2 - H^{\bullet}_2)$	def. $\times$, line 20
22	$\langle a_1, a_2\rangle \in_{h} \big(([f_1^{\#}\rangle Q_1 - H^{\bullet}_1) \times ([f_2^{\#}\rangle Q_2 - H^{\bullet}_2)\big) \cup H^{\bullet}_{\otimes}$	set th., line 21
23	$\langle a_1, a_2\rangle \in_{h} H^{\bullet}_{\otimes}$	assume
24	$\langle a_1, a_2\rangle \in_{h} \big(([f_1^{\#}\rangle Q_1 - H^{\bullet}_1) \times ([f_2^{\#}\rangle Q_2 - H^{\bullet}_2)\big) \cup H^{\bullet}_{\otimes}$	set th., line 23
25	$\langle a_1, a_2\rangle \in_{h} \big(([f_1^{\#}\rangle Q_1 - H^{\bullet}_1) \times ([f_2^{\#}\rangle Q_2 - H^{\bullet}_2)\big) \cup H^{\bullet}_{\otimes}$	$\vee$-Elim, lines 2, 3, 23
26	$\langle a_1, a_2\rangle \in_{h} [f_1^{\#}\rangle Q_1 \otimes [f_2^{\#}\rangle Q_2$	def. $\otimes$, line 25

Now the other direction:

1	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f_1^\sharp\rangle Q_1 \otimes [f_2^\sharp\rangle Q_2$	assume
2	$\langle a_1, a_2\rangle \in_{\mathsf{h}} \big(([f_1^\sharp\rangle Q_1 - H_1^\bullet) \times ([f_2^\sharp\rangle Q_2 - H_2^\bullet)\big) \cup H_\otimes^\bullet$	def. ⊗, line 1
3	$\langle a_1, a_2\rangle \in_{\mathsf{h}} ([f_1^\sharp\rangle Q_1 - H_1^\bullet) \times ([f_2^\sharp\rangle Q_2 - H_2^\bullet)$	assume
4	$a_1 \in_{\mathsf{h}_1} [f_1^\sharp\rangle Q_1$ and $a_1 \notin_{\mathsf{h}_1} H_1^\bullet$ and $a_2 \in_{\mathsf{h}_2} [f_2^\sharp\rangle Q_2$ and $a_2 \notin_{\mathsf{h}_2} H_2^\bullet$	set th., line 3
5	$\exists y(y \in_{\mathsf{k}_1} Q_1$ and $\neg\mathcal{F}_1^\sharp a_1 y)$	def. $[-^\sharp\rangle$, line 4
6	$\exists v(v \in_{\mathsf{k}_2} Q_2$ and $\neg\mathcal{F}_2^\sharp a_2 v)$	def. $[-^\sharp\rangle$, line 4
7	$\boxed{b_1}$ $b_1 \in_{\mathsf{k}_1} Q_1$ and $\neg\mathcal{F}_1^\sharp a_1 b_1$	assume
8	$\boxed{b_2}$ $b_2 \in_{\mathsf{k}_2} Q_2$ and $\neg\mathcal{F}_2^\sharp a_2 b_2$	assume
9	$b_1 \in_{\mathsf{k}_1} Q_1 - K_1^\bullet$ or $b_1 \in_{\mathsf{k}_1} K_1^\bullet$	Q_1 is well-pointed, line 7
10	$b_2 \in_{\mathsf{k}_2} Q_2 - K_2^\bullet$ or $b_2 \in_{\mathsf{k}_2} K_2^\bullet$	Q_2 is well-pointed, line 8
11	$b_1 \in_{\mathsf{k}_1} K_1^\bullet$	assume
12	$\mathcal{F}_1^\sharp a_1 b_1$	Frame Condition 4.13.4 (1) lines 4, 11
13	$\rightarrow\leftarrow$	Contradiction lines 7, 12
14	$b_2 \in_{\mathsf{k}_2} K_2^\bullet$	assume
15	$\mathcal{F}_2^\sharp a_2 b_2$	Frame Condition 4.13.4 (1) lines 4, 14
16	$\rightarrow\leftarrow$	Contradiction lines 8, 15
17	$b_1 \in_{\mathsf{k}_1} Q_1 - K_1^\bullet$ and $b_2 \in_{\mathsf{k}_2} Q_2 - K_2^\bullet$	CL, lines 9. 10, 11, 14
18	$\langle b_1, b_2\rangle \in_{\mathsf{k}} Q_1 \otimes Q_2$	def. ⊗, line 17
19	$\neg\mathcal{F}_1^\sharp a_1 b_1$ and $\neg\mathcal{F}_2^\sharp a_2 b_2$	∧-Elim, ∧-Intro, lines 7, 8
20	$\neg\mathcal{F}^\sharp\langle a_1, a_2\rangle\langle b_1, b_2\rangle$	def. $\neg\mathcal{F}^\sharp$, line 19
21	$\langle b_1, b_2\rangle \in_{\mathsf{k}} Q_1 \otimes Q_2$ and $\neg\mathcal{F}^\sharp\langle a_1, a_2\rangle\langle b_1, b_2\rangle$	∧-Intro, lines 18, 20
22	$\exists\langle y, v\rangle(\langle y, v\rangle \in_{\mathsf{k}} Q_1 \otimes Q_2$ and $\neg\mathcal{F}^\sharp\langle a_1, a_2\rangle\langle y, v\rangle)$	∃-Intro, line 21
23	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f^\sharp\rangle (Q_1 \otimes Q_2)$	def. $[-^\sharp\rangle$, line 22
24	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f^\sharp\rangle (Q_1 \otimes Q_2)$	∃-Elim, lines 6, 8
25	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f^\sharp\rangle (Q_1 \otimes Q_2)$	∃-Elim, lines 5, 7
26	$\langle a_1, a_2\rangle \in_{\mathsf{h}} H_\otimes^\bullet$	assume
27	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f^\sharp\rangle (Q_1 \otimes Q_2)$	$[-^\sharp\rangle$ is well-pointed, Lemma 4.13.7, line 24
28	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f^\sharp\rangle (Q_1 \otimes Q_2)$	∨-Elim, lines 2, 3, 26

■

4.14 Info for $[-^{\sharp}\cdot\rangle$

The type of $[f^{\sharp}\cdot\rangle$ is $\vee\!\mapsto\!\vee$.

Definition 4.14.1

$$y \in\!\mathsf{k}\ [f^{\sharp}\cdot\rangle P \text{ iff } \exists x(x \in\!\mathsf{h}\ P \text{ and } \neg\mathcal{F}^{\sharp\cdot}xy).$$

Definition 4.14.2 The canonical relation is defined as

$$\mathcal{F}^{\sharp\cdot}xy \text{ iff } \exists a(a \in\!\mathsf{h}\ x \text{ and } [f^{\sharp}\cdot\rangle a \notin\!\mathsf{k}\ y).$$

The negated relation is

$$\neg\mathcal{F}^{\sharp\cdot}xy \text{ iff } \forall a(a \notin\!\mathsf{h}\ x \text{ or } [f^{\sharp}\cdot\rangle a \in\!\mathsf{k}\ y).$$

Frame Condition 4.14.3 The null frame conditions for $\mathcal{F}^{\sharp\cdot}$ are

$$(1)\quad \forall x(x \notin\!\mathsf{h}\ H^{\circ} \text{ implies } \mathcal{F}^{\sharp\cdot}xK^{\circ}) \qquad (2)\quad \forall y(\neg\mathcal{F}^{\sharp\cdot}H^{\circ}y).$$

Frame Condition 4.14.4 The well frame conditions for $\mathcal{F}^{\sharp\cdot}$ are

$$(1)\quad \forall y(y \notin\!\mathsf{k}\ K^{\bullet} \text{ implies } \mathcal{F}^{\sharp\cdot}H^{\bullet}y) \qquad (2)\quad \forall x(\neg\mathcal{F}^{\sharp\cdot}xK^{\bullet}).$$

Lemma 4.14.5 *Canonically, the Frame Conditions hold.*

Proof: Frame Condition 4.13.3 (1): Choose some arbitrary x and assume $x \neq \circ_H$. Let $y = \circ_K$. Then $x \neq \emptyset$ so there is some $a \in\!\mathsf{h}\ x$. Since $y = \emptyset$, then $[f^{\sharp}\cdot\rangle a \notin\!\mathsf{k}\ y$. By definition, $\mathcal{F}^{\sharp\cdot}xy$.

Frame Condition 4.13.3 (2): Choose some arbitrary y. Let $x = \circ_H$, then for all a, $a \notin\!\mathsf{h}\ x$ and $\neg\mathcal{F}^{\sharp\cdot}xy$ holds.

Frame Condition 4.13.4 (1): Choose $y \neq \bullet_K$ and assume $x = \bullet_H$. Let $a = \bot_{\mathsf{h}}$ and hence $a \in\!\mathsf{h}\ x$. From the type of $[-^{\sharp}\cdot\rangle$, $[f^{\sharp}\cdot\rangle a = \bot_{\mathsf{k}}$. Since $y \neq \bullet_K$, $[f^{\sharp}\cdot\rangle a \notin\!\mathsf{k}\ y$. So there is some a such that $a \in\!\mathsf{h}\ x$ and $[f^{\sharp}\cdot\rangle a \notin\!\mathsf{k}\ y$, whence $\mathcal{F}^{\sharp\cdot}xy$ obtains.

Frame Condition 4.13.4 (2): Choose some arbitrary x and let $y = \bullet_K$. For all a, $[f^{\sharp}\cdot\rangle a \in\!\mathsf{k}\ y$, whence $\neg\mathcal{F}^{\sharp\cdot}xy$ obtains.

■

Lemma 4.14.6 *The Frame Conditions 4.14.3 and 4.14.4 are consistent.*

Proof: The well-point sets and null-point sets are always disjoint. This disjointness and quantifiers and antecedents of the conditionals prevent any conflicts in the Frame Conditions 4.14.3 and 4.14.4.

■

4.15 $[-^{\diamond}\rangle$, $[-^{\diamond}\cdot\rangle$: Closure Under Smash Product

All proofs involving $[-^{\diamond}\cdot\rangle$ are similar to their siblings for $[-^{\diamond}\rangle$ and we elide them. The information necessary to construct them is in the following Section 4.16.

$[-^{\diamond}\rangle$ is of type $\vee \mapsto \vee$.

Definition 4.15.1

$$x \in_{\mathsf{h}} [f^{\diamond}\rangle Q \text{ iff } \exists y(y \in_{\mathsf{k}} Q \text{ and } \mathcal{F}^{\diamond}xy).$$

Definition 4.15.2 The canonical relation and its negation are

$$\mathcal{F}^{\diamond}xy \text{ iff } \forall b(b \notin_{\mathsf{k}} y \text{ or } [f^{\diamond}\rangle\, b \in_{\mathsf{h}} x) \qquad \neg\mathcal{F}^{\diamond}xy \text{ iff } \exists b(b \in_{\mathsf{k}} y \text{ and } [f^{\diamond}\rangle\, b \notin_{\mathsf{h}} x).$$

Frame Condition 4.15.3 The null frame conditions for $\mathcal{F}^{\diamond}$ are

$$(1) \quad \forall y(y \notin_{\mathsf{k}} K^{\circ} \text{ implies } \neg\mathcal{F}^{\diamond}H^{\circ}y) \qquad (2) \quad \forall x(\mathcal{F}^{\diamond}xK^{\circ})$$

Frame Condition 4.15.4 The well frame conditions for $\mathcal{F}^{\diamond}$ are

$$(1) \quad \forall x(x \notin_{\mathsf{h}} H^{\bullet} \text{ implies } \neg\mathcal{F}^{\diamond}xK^{\bullet}) \qquad (2) \quad \forall y(\mathcal{F}^{\diamond}H^{\bullet}y).$$

Lemma 4.15.5 *Canonically, the Frame Conditions hold.*

Proof: Frame Condition 4.15.3 (1): Choose some arbitrary y and assume $y \neq \circ_K$. Let $x = \circ_H$. Then $y \neq \emptyset$ so there is some $b \in_{\mathsf{k}} y$. Since $x = \emptyset$, then $[f^{\diamond}\rangle\, b \notin_{\mathsf{h}} x$. By definition, $\neg\mathcal{F}^{\diamond}xy$.

Frame Condition 4.15.3 (2): Choose some arbitrary x. Let $y = \circ_K$, then for all b, $b \notin_{\mathsf{k}} y$ and $\mathcal{F}^{\diamond}xy$ holds.

Frame Condition 4.15.4 (1): Choose $x \neq \bullet_H$ and assume $y = \bullet_K$. Let $b = \bot_{\mathsf{k}}$ and hence $b \in_{\mathsf{k}} y$. From the type of $[-^{\diamond}\rangle$, $[f^{\diamond}\rangle\, b = \bot_{\mathsf{h}}$. Since $x \neq \bullet_H$, $[f^{\diamond}\rangle\, b \notin_{\mathsf{h}} x$. So there is some b such that $b \in_{\mathsf{k}} y$ and $[f^{\diamond}\rangle\, b \notin_{\mathsf{h}} x$. So $\neg\mathcal{F}^{\diamond}xy$ holds.

Frame Condition 4.15.4 (2): Choose some arbitrary y and let $x = \bullet_H$. For all b, $[f^{\diamond}\rangle\, b \in_{\mathsf{h}} x$ and hence $\mathcal{F}^{\diamond}xy$ holds.

■

Lemma 4.15.6 *The Frame Conditions 4.15.3 and 4.15.4 are consistent with each other.*

Proof: Note that the well-point sets and null-point sets are always disjoint, then the quantifiers and antecedents of the conditionals of Frame Conditions 4.15.3 and 4.15.4 prevent any conflicts.

■

Lemma 4.15.7 *$[f^\circ\rangle Q$ is null-pointed if Q is null-pointed, and well-pointed if Q is well-pointed.*

Proof: Our conditions are

$$x \in_{\mathsf{h}} [f^\circ\rangle Q \text{ iff } \exists y (y \in_{\mathsf{k}} Q \text{ and } \mathcal{F}^\circ xy). \qquad x \notin_{\mathsf{h}} [f^\circ\rangle Q \text{ iff } \forall y (y \notin_{\mathsf{k}} Q \text{ or } \neg\mathcal{F}^\circ xy).$$

Assume Q is null-pointed and let $x \in_{\mathsf{h}} H^\circ$. Since Q is null-pointed, $K^\circ \cap Q = \emptyset$. Choose some $y \in_{\mathsf{k}} Q$, then $y \notin_{\mathsf{k}} K^\circ$. From Frame Condition 4.15.3 (1), $\neg\mathcal{F}^\circ xy$ holds. Hence for all y, either $y \notin_{\mathsf{k}} Q$ or $\neg\mathcal{F}^\circ xy$, whence $x \notin_{\mathsf{h}} [f^\circ\rangle Q$.

Assume Q is well-pointed. Let $x \in_{\mathsf{h}} H^\bullet$ and let $y \in_{\mathsf{k}} K^\bullet$. Hence $y \in_{\mathsf{k}} Q$ and from Frame Condition 4.15.4 (2), $\mathcal{F}^\circ H^\bullet y$ holds. Hence there is some y such that $y \in_{\mathsf{k}} Q$ and $\mathcal{F}^\circ xy$, whence $H^\bullet \subseteq_{\mathsf{h}} [f^\circ\rangle Q$.

■

Lemma 4.15.8 *$[f^\circ\rangle$ preserves its bounds.*

Proof: The type of $[f^\circ\rangle$ is $\vee\mapsto\vee$, hence we want $[f^\circ\rangle \perp^{\mathbb{K}} = \perp^{\mathbb{H}}$. $\perp^{\mathbb{H}} \subseteq_{\mathsf{h}} [f^\circ\rangle \perp^{\mathbb{K}}$ since $\perp^{\mathbb{H}}$ is the bottom element of $\mathbb{H}$.

Next, by definition, $\perp^{\mathbb{K}} = K^\bullet$. Assume $x \in_{\mathsf{h}} [f^\circ\rangle \perp^{\mathbb{K}}$, then there is some y such that $y \in_{\mathsf{k}} K^\bullet$ and $\mathcal{F}^\circ xy$. If $x \notin_{\mathsf{h}} H^\bullet$, then from Frame Condition 4.15.4 (1), $\neg\mathcal{F}^\circ xy$. This is contradiction, hence $x \in_{\mathsf{h}} H^\bullet$ and $[f^\circ\rangle \perp^{\mathbb{K}} = H^\bullet$.

■

Lemma 4.15.9

$$x \in_{\mathsf{h}} [f^\circ\rangle \beta b \text{ iff } x \in_{\mathsf{h}} \beta [f^\circ\rangle b.$$

Proof: The left to right is the easy direction. Assume $x \in_{\mathsf{h}} [f^\circ\rangle \beta b$, hence there is some y such that $y \in_{\mathsf{k}} \beta b$ and $\mathcal{F}^\circ xy$. From the canonical definition for $\mathcal{F}^\circ$, for all $a \in_{\mathsf{k}} y$ then $[f^\circ\rangle a \in_{\mathsf{h}} x$, hence $[f^\circ\rangle b \in_{\mathsf{h}} x$ and $x \in_{\mathsf{h}} \beta [f^\circ\rangle b$.

To go in the other direction, assume $x \in_{\mathsf{h}} \beta [f^\circ\rangle b$, and so $[f^\circ\rangle b \in_{\mathsf{h}} x$. Let

$$\hat{y} = (\lambda p \,.\, [f^\circ\rangle p)^{-1} \overline{x}.$$

Given the type, from Lemma 3.7.1 $\hat{y}$ is an ideal. Construct y by using Zorn's Lemma to maximalize $\langle \uparrow b, \hat{y} \rangle$ to construct y, whence $b \in_{\mathsf{k}} y$. By construction $\forall a ([f^\circ\rangle a \notin_{\mathsf{h}} x \text{ implies } a \notin_{\mathsf{k}} y)$. Hence we have defined $\mathcal{F}^\circ xy$. Therefore there exists a y, such that $b \in_{\mathsf{k}} y$ and $\mathcal{F}^\circ xy$, whence $x \in_{\mathsf{h}} [f^\circ\rangle \beta b$.

■

Lemma 4.15.10 *Let* $f = f_1 \otimes f_2$, *then* $\mathcal{F}^{\diamond}\langle a_1, a_2\rangle\langle b_1, b_2\rangle$ iff $\mathcal{F}_1^{\diamond} a_1 b_1$ and $\mathcal{F}_2^{\diamond} a_2 b_2$ *and*

$$[f^{\diamond}\rangle Q_1 \otimes Q_2 = [f^{\diamond}\rangle Q_1 \otimes [f^{\diamond}\rangle Q_2 \qquad [f^{\diamond}\cdot\rangle P_1 \otimes P_2 = [f^{\diamond}\cdot\rangle P_1 \otimes [f^{\diamond}\cdot\rangle P_2$$

Proof:

1	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f^{\diamond}\rangle (Q_1 \otimes Q_2)$	assume
2	$(\langle a_1, a_2\rangle \in_{\mathsf{h}} [f^{\diamond}\rangle (Q_1 \otimes Q_2) - H^{\bullet}_{\otimes})$ or $\langle a_1, a_2\rangle \in_{\mathsf{h}} H^{\bullet}_{\otimes}$	$[-^{\diamond}\rangle$ is well-pointed, Lemma 4.15.7, line 1
3	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f^{\diamond}\rangle (Q_1 \otimes Q_2) - H^{\bullet}_{\otimes}$	assume
4	$\langle a_1, a_2\rangle \in_{\mathsf{h}} H_1 \otimes H_2$	$f = f_1 \otimes f_2$, line 1
5	$\langle a_1, a_2\rangle \notin_{\mathsf{h}} H^{\bullet}_{\otimes}$ and $\langle a_1, a_2\rangle \in_{\mathsf{h}} H_1 \otimes H_2$	$\wedge$-Intro, lines 3, 4
6	$a_1 \notin_{\mathsf{h}_1} H^{\bullet}_1$ and $a_2 \notin_{\mathsf{h}_2} H^{\bullet}_2$	Lemma 3.4.9, line 5
7	$\exists\langle y, v\rangle(\langle y, v\rangle \in_{\mathsf{k}} Q_1 \otimes Q_2$ and $\mathcal{F}^{\diamond}\langle a_1, a_2\rangle\langle y, v\rangle)$	def. $[-^{\diamond}\rangle$, line 3
8	$\boxed{b_1, b_2}$ $\langle b_1, b_2\rangle \in_{\mathsf{k}} Q_1 \otimes Q_2$ and $\mathcal{F}^{\diamond}\langle a_1, a_2\rangle\langle b_1, b_2\rangle$	assume
9	$(\langle b_1, b_2\rangle \in_{\mathsf{k}} (Q_1 - K^{\bullet}_1) \times (Q_2 - K^{\bullet}_2))$ or $\langle b_1, b_2\rangle \in_{\mathsf{k}} K^{\bullet}_{\otimes}$	def. $\otimes$, line 8
10	$\langle b_1, b_2\rangle \in_{\mathsf{k}} (Q_1 - K^{\bullet}_1) \times (Q_2 - K^{\bullet}_2)$	assume
11	$\mathcal{F}_1^{\diamond} a_1 b_1$ and $\mathcal{F}_2^{\diamond} a_2 b_2$	def. $\mathcal{F}^{\diamond}$, line 8
12	$b_1 \in_{\mathsf{k}_1} Q_1$ and $b_2 \in_{\mathsf{k}_2} Q_2$	set th., line 10
13	$a_1 \in_{\mathsf{h}_1} [f_1^{\diamond}\rangle Q_1$ and $a_2 \in_{\mathsf{h}_2} [f_2^{\diamond}\rangle Q_2$	def. $[-^{\diamond}\rangle$, lines 11, 12
14	$a_1 \in_{\mathsf{h}_1} [f_1^{\diamond}\rangle Q_1 - H^{\bullet}_1$ and $a_2 \in_{\mathsf{h}_2} [f_2^{\diamond}\rangle Q_2 - H^{\bullet}_2$	set th., lines 6, 13
15	$\langle b_1, b_2\rangle \in_{\mathsf{k}} K^{\bullet}_{\otimes}$	assume
16	$b_1 \in_{\mathsf{k}_1} K^{\bullet}_1$ and $b_2 \in_{\mathsf{k}_2} K^{\bullet}_2$	def. $K^{\bullet}_{\otimes}$, line 15
17	$\neg\mathcal{F}_1^{\diamond} a_1 b_1$ and $\neg\mathcal{F}_2^{\diamond} a_2 b_2$	Frame Condition 4.15.4 (1), lines 6, 16
18	$\rightarrow\!\leftarrow$	Contradiction, lines 11, 17
19	$a_1 \in_{\mathsf{h}_1} [f_1^{\diamond}\rangle Q_1 - H^{\bullet}_1$ and $a_2 \in_{\mathsf{h}_2} [f_2^{\diamond}\rangle Q_2 - H^{\bullet}_2$	CL, lines 9, 10, 15
20	$a_1 \in_{\mathsf{h}_1} [f_1^{\diamond}\rangle Q_1 - H^{\bullet}_1$ and $a_2 \in_{\mathsf{h}_2} [f_2^{\diamond}\rangle Q_2 - H^{\bullet}_2$	$\exists$-Elim, lines 7, 8
21	$\langle a_1, a_2\rangle \in_{\mathsf{h}} ([f_1^{\diamond}\rangle Q_1 - H^{\bullet}_1) \times ([f_2^{\diamond}\rangle Q_2 - H^{\bullet}_2)$	def. $\times$, line 20
22	$\langle a_1, a_2\rangle \in_{\mathsf{h}} \big(([f_1^{\diamond}\rangle Q_1 - H^{\bullet}_1) \times ([f_2^{\diamond}\rangle Q_2 - H^{\bullet}_2)\big) \cup H^{\bullet}_{\otimes}$	set th., line 21
23	$\langle a_1, a_2\rangle \in_{\mathsf{h}} H^{\bullet}_{\otimes}$	assume
24	$\langle a_1, a_2\rangle \in_{\mathsf{h}} \big(([f_1^{\diamond}\rangle Q_1 - H^{\bullet}_1) \times ([f_2^{\diamond}\rangle Q_2 - H^{\bullet}_2)\big) \cup H^{\bullet}_{\otimes}$	set th., line 23
25	$\langle a_1, a_2\rangle \in_{\mathsf{h}} \big(([f_1^{\diamond}\rangle Q_1 - H^{\bullet}_1) \times ([f_2^{\diamond}\rangle Q_2 - H^{\bullet}_2)\big) \cup H^{\bullet}_{\otimes}$	$\vee$-Elim, lines 2, 3, 23
26	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f_1^{\diamond}\rangle Q_1 \otimes [f_2^{\diamond}\rangle Q_2$	def. $\otimes$, line 25

Now the other direction:

1	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f_1^{\circ}\rangle Q_1 \otimes [f_2^{\circ}\rangle Q_2$	assume
2	$\langle a_1, a_2\rangle \in_{\mathsf{h}} \big(([f_1^{\circ}\rangle Q_1 - H_1^{\bullet}) \times ([f_2^{\circ}\rangle Q_2 - H_2^{\bullet})\big) \cup H_{\otimes}^{\bullet}$	def. ⊗, line 1
3	$\langle a_1, a_2\rangle \in_{\mathsf{h}} ([f_1^{\circ}\rangle Q_1 - H_1^{\bullet}) \times ([f_2^{\circ}\rangle Q_2 - H_2^{\bullet})$	assume
4	$a_1 \in_{\mathsf{h}_1} [f_1^{\circ}\rangle Q_1$ and $a_1 \notin_{\mathsf{h}_1} H_1^{\bullet}$ and $a_2 \in_{\mathsf{h}_2} [f_2^{\circ}\rangle Q_2$ and $a_2 \notin_{\mathsf{h}_2} H_2^{\bullet}$	set th., line 3
5	$\exists y(y \in_{\mathsf{k}_1} Q_1$ and $\mathcal{F}_1^{\circ} a_1 y)$	def. $[-^{\circ}\rangle$, line 4
6	$\exists v(v \in_{\mathsf{k}_2} Q_2$ and $\mathcal{F}_2^{\circ} a_2 v)$	def. $[-^{\circ}\rangle$, line 4
7	$\boxed{b_1}$ $\quad b_1 \in_{\mathsf{k}_1} Q_1$ and $\mathcal{F}_1^{\circ} a_1 b_1$	assume
8	$\boxed{b_2}$ $\quad b_2 \in_{\mathsf{k}_2} Q_2$ and $\mathcal{F}_2^{\circ} a_2 b_2$	assume
9	$b_1 \in_{\mathsf{k}_1} Q_1 - K_1^{\bullet}$ or $b_1 \in_{\mathsf{k}_1} K_1^{\bullet}$	Q_1 is well-pointed, line 7
10	$b_2 \in_{\mathsf{k}_2} Q_2 - K_2^{\bullet}$ or $b_2 \in_{\mathsf{k}_2} K_2^{\bullet}$	Q_2 is well-pointed, line 8
11	$b_1 \in_{\mathsf{k}_1} K_1^{\bullet}$	assume
12	$\neg\mathcal{F}_1^{\circ} a_1 b_1$	Frame Condition 4.15.4 (1) lines 4, 11
13	$\rightarrow\leftarrow$	Contradiction lines 7, 12
14	$b_2 \in_{\mathsf{k}_2} K_2^{\bullet}$	assume
15	$\neg\mathcal{F}_2^{\circ} a_2 b_2$	Frame Condition 4.15.4 (1) lines 4, 14
16	$\rightarrow\leftarrow$	Contradiction lines 8, 15
17	$b_1 \in_{\mathsf{k}_1} Q_1 - K_1^{\bullet}$ and $b_2 \in_{\mathsf{k}_2} Q_2 - K_2^{\bullet}$	CL, lines 9. 10, 11, 14
18	$\langle b_1, b_2\rangle \in_{\mathsf{k}} Q_1 \otimes Q_2$	def. ⊗, line 17
19	$\mathcal{F}_1^{\circ} a_1 b_1$ and $\mathcal{F}_2^{\circ} a_2 b_2$	∧-Elim, ∧-Intro, lines 7, 8
20	$\mathcal{F}^{\circ}\langle a_1, a_2\rangle\langle b_1, b_2\rangle$	def. $\mathcal{F}^{\circ}$, line 19
21	$\langle b_1, b_2\rangle \in_{\mathsf{k}} Q_1 \otimes Q_2$ and $\mathcal{F}^{\circ}\langle a_1, a_2\rangle\langle b_1, b_2\rangle$	∧-Intro, lines 18, 20
22	$\exists\langle y, v\rangle(\langle y, v\rangle \in_{\mathsf{k}} Q_1 \otimes Q_2$ and $\mathcal{F}^{\circ}\langle a_1, a_2\rangle\langle y, v\rangle)$	∃-Intro, line 21
23	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f^{\circ}\rangle(Q_1 \otimes Q_2)$	def. $[-^{\circ}\rangle$, line 22
24	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f^{\circ}\rangle(Q_1 \otimes Q_2)$	∃-Elim, lines 6, 8
25	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f^{\circ}\rangle(Q_1 \otimes Q_2)$	∃-Elim, lines 5, 7
26	$\langle a_1, a_2\rangle \in_{\mathsf{h}} H_{\otimes}^{\bullet}$	assume
27	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f^{\circ}\rangle(Q_1 \otimes Q_2)$	$[-^{\circ}\rangle$ is well-pointed, Lemma 4.15.7, line 24
28	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f^{\circ}\rangle(Q_1 \otimes Q_2)$	∨-Elim, lines 2, 3, 26

■

4.16 Info for $[-^{\circ}\cdot\rangle$

$[-^{\circ}\cdot\rangle$ is of type $\vee\mapsto\vee$.

Definition 4.16.1

$$y \in_{\mathsf{k}} [f^{\circ}\cdot\rangle P \text{ iff } \exists x(x \in_{\mathsf{h}} P \text{ and } \mathcal{F}^{\circ\cdot} xy).$$

Definition 4.16.2 The canonical relation is

$$\mathcal{F}^{\diamond\cdot}xy \text{ iff } \forall a(a \notin_{\mathsf{h}} x \text{ or } [f^{\diamond}\cdot\rangle a \in_{\mathsf{k}} y).$$

The negation of the canonical definition is

$$\neg\mathcal{F}^{\diamond\cdot}xy \text{ iff } \exists a(a \in_{\mathsf{h}} x \text{ and } [f^{\diamond}\cdot\rangle a \notin_{\mathsf{k}} y).$$

Frame Condition 4.16.3 The null frame conditions for $\mathcal{F}^{\diamond\cdot}$ are

$$(1) \quad \forall x(x \notin_{\mathsf{h}} H^{\circ} \text{ implies } \neg\mathcal{F}^{\diamond\cdot}xK^{\circ}) \qquad (2) \quad \forall y(\mathcal{F}^{\diamond\cdot}H^{\circ}y)$$

Frame Condition 4.16.4 The well frame conditions for $\mathcal{F}^{\diamond\cdot}$ are

$$(1) \quad \forall y(y \notin_{\mathsf{k}} K^{\bullet} \text{ implies } \neg\mathcal{F}^{\diamond\cdot}H^{\bullet}y) \qquad (2) \quad \forall x(\mathcal{F}^{\diamond\cdot}xK^{\bullet}).$$

Lemma 4.16.5 *Canonically, the Frame Conditions 4.16.3 and 4.16.4 hold.*

Proof: Frame Condition 4.16.3 (1): Choose some arbitrary x and assume $x \neq \circ_H$, then $x \neq \emptyset$; so there is some $a \in_{\mathsf{h}} x$. Let $y = \circ_K$. Since $y = \emptyset$, then $[f^{\diamond}\cdot\rangle a \notin_{\mathsf{k}} y$. By definition, $\neg\mathcal{F}^{\diamond\cdot}xy$.

Frame Condition 4.16.3 (2): Choose some arbitrary y. Let $x = \circ_H$, then for all a, $a \notin_{\mathsf{h}} x$ and, by definition, $\mathcal{F}^{\diamond\cdot}xy$ holds.

Frame Condition 4.16.4 (1): Choose $y \neq \bullet_K$ and assume $x = \bullet_H$. Let $a = \bot_{\mathsf{h}}$ and hence $a \in_{\mathsf{h}} x$. From the type of $[-^{\diamond}\cdot\rangle$, $[f^{\diamond}\cdot\rangle a = \bot_{\mathsf{k}}$. Since $y \neq \bullet_K$, $[f^{\diamond}\cdot\rangle a \notin_{\mathsf{k}} y$. So there is some a such that $a \in_{\mathsf{h}} x$ and $[f^{\diamond}\cdot\rangle a \notin_{\mathsf{k}} y$; so $\neg\mathcal{F}^{\diamond\cdot}xy$ holds.

Frame Condition 4.16.4 (2): Choose some arbitrary x and let $y = \bullet_K$. For all a, $[f^{\diamond}\cdot\rangle a \in_{\mathsf{k}} y$ and hence $\mathcal{F}^{\diamond\cdot}xy$ holds.

■

Lemma 4.16.6 *The Frame Conditions 4.16.3 and 4.16.4 are consistent with each other.*

Proof: The well-point sets and null-point sets are always disjoint. This disjointness and quantifiers and antecedents of the conditionals prevent any conflicts in the Frame Conditions 4.16.3 and 4.16.4.

■

4.17 $[-^{\square}\rangle$, $[-^{\square}\cdot\rangle$ Closure Under Null Sum

All proofs involving $[-^{\square}\cdot\rangle$ are similar to their siblings for $[-^{\square}\rangle$ and we elide them. The information necessary to construct them is in the following Section 4.18.

$[-^{\square}\rangle$ is of type $\wedge\!\mapsto\!\wedge$.

Definition 4.17.1

$$x \in_{\mathsf{h}} [f^{\square}\rangle Q \text{ iff } \forall y(y \in_{\mathsf{k}} Q \text{ or } \neg\mathcal{F}^{\square}xy).$$

Definition 4.17.2 The canonical relation is

$$\mathcal{F}^{\square}xy \text{ iff } \forall b(b \in_{\mathsf{k}} y \text{ or } [f^{\square}\rangle b \notin_{\mathsf{h}} x).$$

The negation of the canonical definition is

$$\neg\mathcal{F}^{\square}xy \text{ iff } \exists b(b \notin_{\mathsf{k}} y \text{ and } [f^{\square}\rangle b \in_{\mathsf{h}} x).$$

Frame Condition 4.17.3 The null frame conditions for $\mathcal{F}^{\square}$ are

$$(1) \quad \forall x(x \notin_{\mathsf{h}} H^{\circ} \text{ implies } \neg\mathcal{F}^{\square}xK^{\circ}) \qquad (2) \quad \forall y(\mathcal{F}^{\square}H^{\circ}y).$$

Frame Condition 4.17.4 The well frame conditions for $\mathcal{F}^{\square}$ are

$$(1) \quad \forall y(y \notin_{\mathsf{k}} K^{\bullet} \text{ implies } \neg\mathcal{F}^{\square}H^{\bullet}y) \qquad (2) \quad \forall x(\mathcal{F}^{\square}xK^{\bullet})$$

Lemma 4.17.5 *Canonically, the Frame Conditions hold.*

Proof: Frame Condition 4.17.3 (1): Choose any x such that $x \neq \circ_H$ and let $y = \circ_K$. Let $b = \top_{\mathsf{k}}$ and hence $b \notin_{\mathsf{k}} y$. From the type of $[-^{\square}\rangle$, $[f^{\square}\rangle b = \top_{\mathsf{h}}$. Since $x \neq \circ_H$, $[f^{\square}\rangle b \in_{\mathsf{h}} x$. So there is some b such that $b \notin_{\mathsf{k}} y$ and $[f^{\square}\rangle b \in_{\mathsf{h}} x$, whence $\neg\mathcal{F}^{\square}xy$ obtains.

Frame Condition 4.17.3 (2): Choose any y and let $x = \circ_H$. For all b, $[f^{\square}\rangle b \notin_{\mathsf{h}} x$ and hence $\mathcal{F}^{\square}xy$ obtains.

Frame Condition 4.17.4 (1): Choose any y and let $y \neq \bullet_K$. Let $x = \bullet_H$. Then $y \neq {\uparrow}\bot_{\mathsf{k}}$ so there is some $b \notin_{\mathsf{k}} y$. Since $x = {\uparrow}\bot_{\mathsf{h}}$, then $[f^{\square}\rangle b \in_{\mathsf{h}} x$. By definition, $\neg\mathcal{F}^{\square}xy$.

Frame Condition 4.17.4 (2): Choose any x. Let $y = \bullet_K$, then for all b, $b \in_{\mathsf{k}} y$, whence $\mathcal{F}^{\square}xy$ obtains.

■

Lemma 4.17.6 *The Frame Conditions 4.17.3 and 4.17.4 are consistent with each other.*

Proof: Note that the well-point sets and null-point sets are always disjoint, then the quantifiers and antecedents of the conditionals of Frame Conditions 4.17.3 and 4.17.4 prevent any conflicts.

■

Lemma 4.17.7 *$[f^{\square}\rangle Q$ is null-pointed if Q is null-pointed, and well-pointed if Q is well-pointed.*

Proof: Our conditions are

$$x \in_{\mathsf{h}} [f^{\square}\rangle Q \text{ iff } \forall y(y \in_{\mathsf{k}} Q \text{ or } \neg\mathcal{F}^{\square}xy) \qquad x \notin_{\mathsf{h}} [f^{\square}\rangle Q \text{ iff } \exists y(y \notin_{\mathsf{k}} Q \text{ and } \mathcal{F}^{\square}xy).$$

Assume Q is null-pointed and let $x \in_{\mathsf{h}} H^{\circ}$. Since Q is null-pointed, $K^{\circ} \cap Q = \emptyset$. Since $K^{\circ} \neq \emptyset$, choose some $y \in_{\mathsf{k}} K^{\circ}$, then $y \notin_{\mathsf{k}} Q$. From Frame Condition 4.17.3 (2), $\mathcal{F}^{\square}H^{\circ}y$ obtains. So there is some y, $y \notin_{\mathsf{k}} Q$ and $\mathcal{F}^{\square}xy$, whence $x \notin_{\mathsf{h}} [f^{\square}\rangle Q$.

Let $x \in_{\mathsf{h}} H^{\bullet}$. Assume Q is well-pointed and choose any y such that $y \notin_{\mathsf{k}} Q$. Since $K^{\bullet} \subseteq_{\mathsf{k}} Q$, then $y \notin_{\mathsf{k}} K^{\bullet}$. From Frame Condition 4.17.4 (1), $\neg\mathcal{F}^{\square}H^{\bullet}y$ obtains. So for all y, $y \notin_{\mathsf{k}} Q$ implies $\neg\mathcal{F}^{\square}xy$, whence $H^{\bullet} \subseteq_{\mathsf{h}} [f^{\square}\rangle Q$.

■

Lemma 4.17.8 *$[f^{\square}\rangle$ preserves its bounds.*

Proof: The type of $[f^{\square}\rangle$ is $\wedge\!\mapsto\!\!\!\rightarrow\!\wedge$, hence we want $[f^{\square}\rangle \top^{\mathbb{K}} = \top^{\mathbb{H}}$. Since $\top^{\mathbb{H}}$ is the top of the lattice of sets $\mathbb{H}$, then $[f^{\square}\rangle \top^{\mathbb{K}} \subseteq_{\mathsf{h}} \top^{\mathbb{H}}$.

For the other direction, assume $x \in_{\mathsf{h}} \top^{\mathbb{H}}$ and choose some arbitrary y. Let $y \notin_{\mathsf{k}} \top^{\mathbb{K}}$. Since $y \in_{\mathsf{k}} K$ and $\top^{\mathbb{K}} = K - K^{\circ}$, then $y \in_{\mathsf{k}} K^{\circ}$. Since $\top^{\mathsf{h}} \cap H^{\circ} = \emptyset$, then $x \notin_{\mathsf{h}} H^{\circ}$. From Frame Condition 4.17.3 (1), $\neg\mathcal{F}^{\square}xy$. Since the choice of y was arbitrary, for any y, $y \notin_{\mathsf{k}} \top^{\mathbb{K}}$ implies $\neg\mathcal{F}^{\square}xy$, whence $[f^{\square}\rangle \top^{\mathbb{K}} = \top^{\mathbb{H}}$.

■

Lemma 4.17.9

$$x \in_{\mathsf{h}} [f^{\square}\rangle \beta b \text{ iff } x \in_{\mathsf{h}} \beta[f^{\square}\rangle b.$$

Proof: The right to left is the easy direction. Assume $x \in_{\mathsf{h}} \beta[f^{\square}\rangle b$ so that $[f^{\square}\rangle b \in_{\mathsf{h}} x$, and choose some arbitrary y such that $\mathcal{F}^{\square}xy$. From the canonical definition for $\mathcal{F}^{\square}$, for all $[f^{\square}\rangle a \in_{\mathsf{h}} x$ implies $a \in_{\mathsf{k}} y$, hence $b \in_{\mathsf{k}} y$. Since the choice of y was arbitrary, by definition $x \in_{\mathsf{h}} [f^{\square}\rangle \beta b$.

The other direction is a contraposition. Assume that $x \notin_{\mathsf{h}} \beta[f^{\square}\rangle b$, so that $[f^{\square}\rangle b \notin_{\mathsf{h}} x$. Let

$$\hat{y} = (\lambda p \,.\, [f^{\square}\rangle p)^{-1} x.$$

Given the type, from Lemma 3.7.1 $\hat{y}$ is a filter. If $b \in_{\mathsf{k}} \hat{y}$, then $[f^{\square}\rangle b \in_{\mathsf{h}} x$, which is a contradiction. So $b \notin_{\mathsf{k}} \hat{y}$. Construct y by using Zorn's Lemma to maximalize $\langle \hat{y}, \downarrow b\rangle$ to construct y. By construction

$$\forall a([f^{\square}\rangle\, a \in_{\mathsf{h}} x \text{ implies } a \in_{\mathsf{k}} y).$$

By definition, $\mathcal{F}^{\square}xy$ holds. By construction, $b \notin_{\mathsf{k}} y$ and so $y \notin_{\mathsf{k}} \beta b$. Therefore there is some y such that $y \notin_{\mathsf{k}} \beta b$ and $\mathcal{F}^{\square}xy$. By definition, $x \notin_{\mathsf{h}} [f^{\square}\rangle \beta b$.

■

We assume $\neg f = \neg f_1 \ovee \neg f_2$, then

$$\neg\mathcal{F}^{\square}\langle a_1, a_2\rangle\langle b_1, b_2\rangle \text{ iff } \neg\mathcal{F}_1^{\square}a_1b_1 \text{ or } \neg\mathcal{F}_2^{\square}a_2b_2.$$

Lemma 4.17.10 *Let $\neg f = \neg f_1 \ovee \neg f_2$, then*

$$[f^{\square}\rangle\,(Q_1 \ovee Q_2) = [f_1^{\square}\rangle\, Q_1 \ovee [f_2^{\square}\rangle\, Q_2 \qquad [f^{\square}\rangle\,(P_1 \ovee P_2) = [f_1^{\square}\rangle\, P_1 \ovee [f_2^{\square}\rangle\, P_2$$

Proof:

1	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f^{\square}\rangle\,(Q_1 \ovee Q_2)$	assume
2	$\forall\langle y, v\rangle\big(\langle y, v\rangle \in_{\mathsf{k}} Q_1 \ovee Q_2 \text{ or } \neg\mathcal{F}^{\square}\langle a_1, a_2\rangle\langle y, v\rangle\big)$	def. $[f^{\square}\rangle$, line 1
3	$(a_1 \in_{\mathsf{h}_1} H_1 \text{ and } a_2 \in_{\mathsf{h}_2} H_2^{\circ}) \text{ or } (a_1 \in_{\mathsf{h}_1} H_1^{\circ} \text{ and } a_2 \in_{\mathsf{h}_2} H_2)$	def. $\ovee$ universe, line 1
4	$a_1 \in_{\mathsf{h}_1} H_1 \text{ and } a_2 \in_{\mathsf{h}_2} H_2^{\circ}$	assume
5	$\boxed{b_1} \quad b_1 \notin_{\mathsf{k}_1} Q_1$	assume
6	$\exists u(u \in_{\mathsf{k}_2} K_2^{\circ})$	$K_2^{\circ} \neq \emptyset$
7	$\boxed{b_2} \quad b_2 \in_{\mathsf{k}_2} K_2^{\circ}$	assume
8	$\langle b_1, b_2\rangle \notin_{\mathsf{k}} Q_1 \times K_2^{\circ} \text{ and } \langle b_1, b_2\rangle \notin_{\mathsf{k}} K_1^{\circ} \times Q_2$	Q_2 is null-pointed, set th., lines 5, 7
9	$\langle b_1, b_2\rangle \notin_{\mathsf{k}} Q_1 \ovee Q_2$	def. $\ovee$, line 8
10	$\langle b_1, b_2\rangle \notin_{\mathsf{k}} Q_1 \ovee Q_2 \text{ implies } \neg\mathcal{F}^{\square}\langle a_1, a_2\rangle\langle b_1, b_2\rangle$	$\forall$-Elim, line 2
11	$\neg\mathcal{F}^{\square}\langle a_1, a_2\rangle\langle b_1, b_2\rangle$	$\supset$-Elim, lines 9, 10
12	$\neg\mathcal{F}_1^{\square}a_1b_1 \text{ or } \neg\mathcal{F}_2^{\square}a_2b_2$	def. $\neg\mathcal{F}^{\square}$, line 11
13	$\mathcal{F}_2^{\square}a_2b_2$	Frame Condition 4.17.3 (2), line 4
14	$\neg\mathcal{F}_1^{\square}a_1b_1$	CL, lines 12, 13
15	$\neg\mathcal{F}_1^{\square}a_1b_1$	$\exists$-Elim, lines 6, 7
16	$\forall y(y \notin_{\mathsf{k}_1} Q_1 \text{ implies } \neg\mathcal{F}_1^{\square}a_1y)$	$\forall$-Intro, line 5
17	$a_1 \in_{\mathsf{h}_1} [f_1^{\square}\rangle\, Q_1$	def. $[-^{\square}\rangle$, line 16
18	$a_1 \in_{\mathsf{h}_1} [f_1^{\square}\rangle\, Q_1 \text{ and } a_2 \in_{\mathsf{h}_2} H_2^{\circ}$	$\wedge$-Elim, $\wedge$-Intro, lines 4, 17
19	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f_1^{\square}\rangle\, Q_1 \ovee [f_2^{\square}\rangle\, Q_2$	def. $\ovee$, line 18
20	$a_1 \in_{\mathsf{h}_1} H_1^{\circ} \text{ and } a_2 \in_{\mathsf{h}_2} H_2$	assume
21	subproof is similar to previous subproof	lines xx
22	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f_1^{\square}\rangle\, Q_1 \ovee [f_2^{\square}\rangle\, Q_2$	def. $\ovee$, line xx
23	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f_1^{\square}\rangle\, Q_1 \ovee [f_2^{\square}\rangle\, Q_2$	$\vee$-Elim, lines 3, 4, 20

To fill in the similar subproof above:

1	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f^{\square}\rangle (Q_1 \ovee Q_2)$	assume
2	$\forall\langle y, v\rangle\big(\langle y, v\rangle \in_{\mathsf{k}} Q_1 \ovee Q_2 \text{ or } \neg\mathcal{F}^{\square}\langle a_1, a_2\rangle\langle y, v\rangle\big)$	def. $[f^{\flat}\rangle$, line 1
3	$(a_1 \in_{\mathsf{h}_1} H_1 \text{ and } a_2 \in_{\mathsf{h}_2} H_2^{\circ}) \text{ or } (a_1 \in_{\mathsf{h}_1} H_1^{\circ} \text{ and } a_2 \in_{\mathsf{h}_2} H_2)$	def. $\ovee$ universe, line 1
4	$a_1 \in_{\mathsf{h}_1} H_1^{\circ} \text{ and } a_2 \in_{\mathsf{h}_2} H_2$	assume
5	$\boxed{b_2} \quad b_2 \notin_{\mathsf{k}_2} Q_2$	assume
6	$\exists u(u \in_{\mathsf{k}_1} K_1^{\circ})$	$K_1^{\circ} \neq \emptyset$
7	$\boxed{b_1} \quad b_1 \in_{\mathsf{k}_1} K_1^{\circ}$	assume
8	$\langle b_1, b_2\rangle \notin_{\mathsf{k}} Q_1 \times K_2^{\circ} \text{ and } \langle b_1, b_2\rangle \notin_{\mathsf{k}} K_1^{\circ} \times Q_2$	Q_1 is null-pointed, set th., lines 5, 7
9	$\langle b_1, b_2\rangle \notin_{\mathsf{k}} Q_1 \ovee Q_2$	def. $\ovee$, line 8
10	$\langle b_1, b_2\rangle \notin_{\mathsf{k}} Q_1 \ovee Q_2 \text{ implies } \neg\mathcal{F}^{\square}\langle a_1, a_2\rangle\langle b_1, b_2\rangle$	$\forall$-Elim, line 2
11	$\neg\mathcal{F}^{\square}\langle a_1, a_2\rangle\langle b_1, b_2\rangle$	$\supset$-Elim, lines 9, 10
12	$\neg\mathcal{F}_1^{\square} a_1 b_1 \text{ or } \neg\mathcal{F}_2^{\square} a_2 b_2$	def. $\mathcal{F}^{\square}$, line 11
13	$\mathcal{F}^{\square} a_1 b_1$	Frame Condition 4.17.3 (2), line 4
14	$\neg\mathcal{F}_2^{\square} a_2 b_2$	CL, lines 12, 13
15	$\neg\mathcal{F}_2^{\square} a_2 b_2$	$\exists$-Elim, lines 6, 7
16	$\forall v(v \notin_{\mathsf{k}_2} Q_2 \text{ implies } \neg\mathcal{F}_2^{\square} a_2 v)$	$\forall$-Intro, line 5
17	$a_2 \in_{\mathsf{h}_2} [f_2^{\square}\rangle Q_2$	def. $[-^{\square}\rangle$, line 16
18	$a_1 \in_{\mathsf{h}_1} H_1^{\circ} \text{ and } a_2 \in_{\mathsf{h}_2} [f_2^{\square}\rangle Q_2$	$\wedge$-Elim, $\wedge$-Intro, lines 4, 17
19	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f_1^{\square}\rangle Q_1 \ovee [f_2^{\square}\rangle Q_2$	def. $\ovee$, line 18

Now the other direction:

1	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f_1^{\square}\rangle Q_1 \varovee [f_2^{\flat}\rangle Q_2$	assume
2	$(a_1 \in_{\mathsf{h}_1} [f_1^{\square}\rangle Q_1$ and $a_2 \in_{\mathsf{h}_2} H_2^{\circ})$ or $(a_1 \in_{\mathsf{h}_1} H_1^{\circ}$ and $a_2 \in_{\mathsf{h}_2} [f_2^{\square}\rangle Q_2)$	def. $\varovee$, line 1
3	$a_1 \in_{\mathsf{h}_1} [f_1^{\flat}\rangle Q_1$ and $a_2 \in_{\mathsf{h}_2} H_2^{\circ}$	assume
4	$\boxed{b_1, b_2}$ $\langle b_1, b_2\rangle \notin_{\mathsf{k}} Q_1 \varovee Q_2$	assume
5	$(b_1 \in_{\mathsf{k}_1} K_1^{\circ}$ and $b_2 \notin_{\mathsf{k}_2} Q_2)$ or $(b_1 \notin_{\mathsf{k}_1} Q_1$ and $b_2 \in_{\mathsf{k}_2} K_2^{\circ})$	Lemma 3.5.4, line 4
6	$b_1 \in_{\mathsf{k}_1} K_1^{\circ}$ and $b_2 \notin_{\mathsf{k}_2} Q_2$	assume
7	$a_1 \notin_{\mathsf{h}_1} H_1^{\circ}$	Lemma 4.17.7, $[f_1^{\square}\rangle Q_1$ is null-pointed, line 3
8	$\neg\mathcal{F}_1^{\square} a_1 b_1$	Frame Condition 4.17.3 (1), lines 6, 7
9	$\neg\mathcal{F}^{\square}\langle a_1, a_2\rangle\langle b_1, b_2\rangle$	def. $\neg\mathcal{F}^{\square}$, line 8
10	$b_1 \notin_{\mathsf{k}_1} Q_1$ and $b_2 \in_{\mathsf{k}_2} K_2^{\circ}$	assume
11	$\forall y(y \notin_{\mathsf{k}_1} Q_1$ implies $\neg\mathcal{F}_1^{\square} a_1 y)$	def. $[f_1^{\square}\rangle$, line 3
12	$b_1 \notin_{\mathsf{k}_1} Q_1$ implies $\neg\mathcal{F}_1^{\square} a_1 b_1$	$\forall$-Elim, line 11
13	$\neg\mathcal{F}_1^{\square} a_1 b_1$	$\supset$-Elim, lines 10, 12
14	$\neg\mathcal{F}_1^{\square} a_1 b_1$ or $\neg\mathcal{F}_2^{\square} a_2 b_2$	$\vee$-Intro, line 13
15	$\neg\mathcal{F}^{\square}\langle a_1, a_2\rangle\langle b_1, b_2\rangle$	def. $\neg\mathcal{F}^{\square}$, line 14
16	$\neg\mathcal{F}^{\square}\langle a_1, a_2\rangle\langle b_1, b_2\rangle$	$\vee$-Elim, lines 5, 6, 10
17	$\forall\langle y, v\rangle(\langle y, v\rangle \notin_{\mathsf{k}} Q_1 \varovee Q_2$ implies $\neg\mathcal{F}^{\square}\langle a_1, a_2\rangle\langle y, v\rangle)$	$\forall$-Intro, line 4
18	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f^{\square}\rangle (Q_1 \varovee Q_2)$	def. $[-^{\square}\rangle$, line 17
19	$a_1 \in_{\mathsf{h}_1} H_1^{\circ}$ and $a_2 \in_{\mathsf{h}_2} [f_2^{\square}\rangle Q_2$	assume
20	subproof is similar to previous subproof	lines xx
21	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f^{\square}\rangle (Q_1 \varovee Q_2)$	def. $[-^{\square}\rangle$, line xx
22	$\langle a_1, a_2\rangle \in_{\mathsf{h}} [f^{\square}\rangle (Q_1 \varovee Q_2)$	$\vee$-Elim, lines 2, 3, 19

To fill in the similar subproof above:

1	$\mid\ \langle a_1, a_2\rangle \in_{\mathsf{h}} [f_1^{\square}\rangle Q_1 \varovee [f_2^{\square}\rangle Q_2$	assume
2	$\mid\ (a_1 \in_{\mathsf{h}_1} [f_1^{\flat}\rangle Q_1 \text{ and } a_2 \in_{\mathsf{h}_2} H_2^{\circ}) \text{ or } (a_1 \in_{\mathsf{h}_1} H_1^{\circ} \text{ and } a_2 \in_{\mathsf{h}_2} [f_2^{\flat}\rangle Q_2)$	def. $\varovee$, line 1
3	$\mid\ \mid\ a_1 \in_{\mathsf{h}_1} H_1^{\circ} \text{ and } a_2 \in_{\mathsf{h}_2} [f_2^{\square}\rangle Q_2$	assume
4	$\mid\ \mid\ \mid\ \boxed{b_1, b_2}\quad \langle b_1, b_2\rangle \notin_{\mathsf{k}} Q_1 \varovee Q_2$	assume
5	$\mid\ \mid\ \mid\ (b_1 \in_{\mathsf{k}_1} K_1^{\circ} \text{ and } b_2 \notin_{\mathsf{k}_2} Q_2) \text{ or } (b_1 \notin_{\mathsf{k}_1} Q_1 \text{ and } b_2 \in_{\mathsf{k}_2} K_2^{\circ})$	Lemma 3.5.4, line 4
6	$\mid\ \mid\ \mid\ \mid\ b_1 \notin_{\mathsf{k}_1} Q_1 \text{ and } b_2 \in_{\mathsf{k}_2} K_2^{\circ}$	assume
7	$\mid\ \mid\ \mid\ \mid\ a_2 \notin_{\mathsf{h}_2} H_2^{\circ}$	Lemma 4.17.7, $[f_2^{\square}\rangle Q_2$ is null-pointed, line 3
8	$\mid\ \mid\ \mid\ \mid\ \neg\mathcal{F}_2^{\square} a_2 b_2$	Frame Condition 4.17.3 (1), lines 6, 7
9	$\mid\ \mid\ \mid\ \mid\ \neg\mathcal{F}^{\square}\langle a_1, a_2\rangle\langle b_1, b_2\rangle$	def. $\neg\mathcal{F}^{\square}$, line 8
10	$\mid\ \mid\ \mid\ \mid\ b_1 \in_{\mathsf{k}_1} K_1^{\circ} \text{ and } b_2 \notin_{\mathsf{k}_2} Q_2$	assume
11	$\mid\ \mid\ \mid\ \mid\ \forall v(v \notin_{\mathsf{k}_2} Q_2 \text{ implies } \neg\mathcal{F}_2^{\square} a_2 v)$	def. $[f_2^{\square}\rangle$, line 3
12	$\mid\ \mid\ \mid\ \mid\ b_2 \notin_{\mathsf{k}_2} Q_2 \text{ implies } \neg\mathcal{F}_2^{\square} a_2 b_2$	$\forall$-Elim, line 11
13	$\mid\ \mid\ \mid\ \mid\ \neg\mathcal{F}_2^{\square} a_2 b_2$	$\supset$-Elim, lines 10, 12
14	$\mid\ \mid\ \mid\ \mid\ \neg\mathcal{F}_1^{\square} a_1 b_1 \text{ or } \neg\mathcal{F}_2^{\square} a_2 b_2$	$\vee$-Intro, line 13
15	$\mid\ \mid\ \mid\ \mid\ \neg\mathcal{F}^{\square}\langle a_1, a_2\rangle\langle b_1, b_2\rangle$	def. $\neg\mathcal{F}^{\square}$, line 14
16	$\mid\ \mid\ \mid\ \neg\mathcal{F}^{\square}\langle a_1, a_2\rangle\langle b_1, b_2\rangle$	$\vee$-Elim, lines 5, 6, 10
17	$\mid\ \mid\ \forall\langle y, v\rangle(\langle y, v\rangle \notin_{\mathsf{k}} Q_1 \varovee Q_2 \text{ implies } \neg\mathcal{F}^{\square}\langle a_1, a_2\rangle\langle y, v\rangle)$	$\forall$-Intro, line 4
18	$\mid\ \mid\ \langle a_1, a_2\rangle \in_{\mathsf{h}} [f^{\square}\rangle (Q_1 \varovee Q_2)$	def. $[-^{\square}\rangle$, line 17

■

4.18 Info for $[-^{\square\cdot}\rangle$

$[-^{\square\cdot}\rangle$ is of type $\wedge\!\mapsto\!\wedge$.

Definition 4.18.1

$$y \in_{\mathsf{k}} [f^{\square\cdot}\rangle P \text{ iff } \forall x(x \in_{\mathsf{h}} P \text{ or } \neg\mathcal{F}^{\square\cdot} xy).$$

Definition 4.18.2 The canonical relation is

$$\mathcal{F}^{\square\cdot} xy \text{ iff } \forall a(a \in_{\mathsf{h}} x \text{ or } [f^{\square\cdot}\rangle a \notin_{\mathsf{k}} y).$$

The negation of the canonical definition is

$$\neg\mathcal{F}^{\square\cdot} xy \text{ iff } \exists a(a \notin_{\mathsf{h}} x \text{ and } [f^{\square\cdot}\rangle a \in_{\mathsf{k}} y).$$

Frame Condition 4.18.3 The null frame conditions for $\mathcal{F}^{\square\cdot}$ are

$$(1)\quad \forall y(y \notin_{\mathsf{k}} K^{\circ} \text{ implies } \neg\mathcal{F}^{\square\cdot} H^{\circ} y) \qquad (2)\quad \forall x(\mathcal{F}^{\square\cdot} x K^{\circ}).$$

Frame Condition 4.18.4 The well frame conditions for $\mathcal{F}^{\square\cdot}$ are

$$(1)\quad \forall x(x \notin_{\mathsf{h}} H^{\bullet} \text{ implies } \neg\mathcal{F}^{\square\cdot}xK^{\bullet}) \qquad (2)\quad \forall y(\mathcal{F}^{\square\cdot}H^{\bullet}y)$$

Lemma 4.18.5 *Canonically, the Frame Conditions hold.*

Proof: Frame Condition 4.17.3 (1): Choose any y such that $y \neq \circ_K$ and let $x = \circ_H$. Let $a = \top_{\mathsf{h}}$ and hence $a \notin_{\mathsf{h}} x$. From the type of $[-^{\square\cdot}\rangle$, $[f^{\square\cdot}\rangle a = \top_{\mathsf{k}}$. Since $y \neq \circ_K$, $[f^{\square\cdot}\rangle a \in_{\mathsf{k}} y$. So there is some a such that $a \notin_{\mathsf{h}} x$ and $[f^{\square\cdot}\rangle a \in_{\mathsf{k}} y$, whence $\neg\mathcal{F}^{\square\cdot}xy$ obtains.

Frame Condition 4.17.3 (2): Choose any x and let $y = \circ_K$. For all a, $[f^{\square\cdot}\rangle a \notin_{\mathsf{k}} y$ and hence $\mathcal{F}^{\square\cdot}xy$ obtains.

Frame Condition 4.17.4 (1): Choose any x and let $x \neq \bullet_H$. Let $y = \bullet_K$. Then $x \neq \uparrow\perp_{\mathsf{h}}$ so there is some $a \notin_{\mathsf{h}} x$. Since $y = \uparrow\perp_{\mathsf{k}}$, then $[f^{\square\cdot}\rangle a \in_{\mathsf{k}} y$. By definition, $\neg\mathcal{F}^{\square\cdot}xy$.

Frame Condition 4.17.4 (2): Choose any y. Let $x = \bullet_H$, then for all a, $a \in_{\mathsf{h}} x$, whence $\mathcal{F}^{\square\cdot}xy$ obtains.

■

Lemma 4.18.6 *The Frame Conditions 4.18.3 and 4.18.4 are consistent with each other.*

Proof: The well-point sets and null-point sets are always disjoint. This disjointness and quantifiers and antecedents of the conditionals prevent any conflicts in the Frame Conditions 4.18.3 and 4.18.4.

■

5. COMPLEMENTS

This section supports closure under null sums for frames and is very much dependent on pairs of $\overset{\bullet\circ}{\neg}$ connected operators. The original Galois operators in [5] have similar properties using classical negation. They did not address closure properties.

5.1 Complements for the Galois Operators

In [3]. Goldblatt shows closure under disjoint sums. However, [2] has the best explanation of the Goldblatt-Thomason for why this is important. Their rendition says each frame can be decomposed into a collection of point-generated frames where the points index the collection. By that they mean that for each frame, pick a point and follow the relation from that point including all the visited points. This gives a collection of point-generated frames for each frame. Those get collected into a disjoint sum. The point-generated frames are the main reason to use disjoint sums and infinite sums at that. For us, we will use a null sum. The representation we will ultimately use is

$$\vec{x} \in\mathsf{h} \Pi_i H_i, \text{ such that } \exists! j \forall i (i \neq j \text{ implies } \vec{x}_i \in\mathsf{h}_i H_i^{\circ}).$$

This leaves the possibility that for $i = j, \vec{x}_j \notin\mathsf{h}_i H_j^{\circ}$.

Consider the proofs for $\overset{\bullet\circ}{\neg}\,[f^{\perp}\rangle Q = [f^{\star}\rangle \overset{\bullet\circ}{\neg}\, Q$ (see sequel). The main supporting Lemma says that $\mathcal{F}^{\perp}$ and $\mathcal{F}^{\star}$ are the same relation as long as none of the special points are involved. If we do include those points, then the two relations are not the same.

It might be best to consider $\overset{\bullet\circ}{\neg}$ to be a morphism between frames, one with $\mathcal{F}^{\perp}$ and the other with $\mathcal{F}^{\star}$. We note that $\overset{\bullet\circ}{\neg}$ does not preserve those relations on the nose due to the special points. Also, when thought of as a frame morphism, $\overset{\bullet\circ}{\neg}$ is the identity mapping points. It performs most of its work on the set algebras. More technically,

$$\overset{\bullet\circ}{\neg} : ((H, \mathcal{H}, \mathbb{H}), \mathcal{F}^{\perp}, (K, \mathcal{K}, \mathbb{K})) \twoheadrightarrow ((H, \mathcal{H}, \mathbb{H}), \mathcal{G}^{\star}, (K, \mathcal{K}, \mathbb{K}))$$

where for all the standard points, $\mathcal{F}^{\perp} = \mathcal{G}^{\star}$, but for the non-standard points, the relations are determined by the Frame Conditions involving the special points. In this sense, $\overset{\bullet\circ}{\neg}$ is not a relation morphism at the level of points. It is rather swapping one relation for another.

The situation is still not resolved, however. It appears we can morph $\mathcal{F}^{\perp}$ into $\mathcal{G}^{\star}$ by ignoring the frame conditions for $\mathcal{F}^{\perp}$, which requires throwing all those tuples involving special points out of the relation and substituting the tuples demanded of the frame conditions for $\mathcal{G}^{\star}$.

The result is that we can pair up the frames according to which are related by $\overset{\bullet\circ}{\neg}$. This is no different than the original Goldblatt-Thomason theorem that only had one kind of frame to consider if you ignore the set algebras. Technically, the set algebras only needed to support a single modal operator. The $[-^{\square}\rangle$ operator is related by Boolean negation to $[-^{\diamond}\rangle$ in that the negation distributes across $\vee$ and $[-^{\square}\rangle$. This closed the $[-^{\square}\rangle$ frames with respect to disjoint sums.

We can define the following sets:

$$\mathcal{F}_{\circ}^{\perp} \overset{def}{=} \{\langle x, y\rangle \mid x \in\mathsf{h} H^{\circ}\} \cup \{\langle x, y\rangle \mid y \in\mathsf{k} K^{\circ}\}.$$

and

$$\mathcal{F}_{\bullet}^{\perp} \overset{def}{=} \{\langle x, y\rangle \mid x \notin\mathsf{h} H^{\circ} \text{ and } y \in\mathsf{k} K^{\bullet}\} \cup \{\langle x, y\rangle \mid y \notin\mathsf{k} K^{\circ} \text{ and } x \in\mathsf{h} H^{\bullet}\}.$$

Regarding the first set of the second condition, this picks out points that satisfy Frame Condition 4.2.4 (1). It helps to re-express that condition as

$$\forall x, y (x \notin\mathsf{h} H^{\circ} \text{ and } y \in\mathsf{k} K^{\bullet} \text{ implies } \mathcal{F}^{\perp} xy).$$

Since the universal quantifier is out in front and at the top level of the semantics, this can be rewritten as

$$x \notin_{\mathsf{h}} H^{\circ} \text{ and } y \in_{\mathsf{k}} K^{\bullet} \text{ implies } \mathcal{F}^{\perp}xy.$$

Hence the set uses the condition in its definition.

Let

$$\mathrm{FC}^{\perp} \overset{def}{=} \mathcal{F}_{\circ}^{\perp} \cup \mathcal{F}_{\bullet}^{\perp} \qquad \mathrm{FC}^{\star} \overset{def}{=} \mathcal{F}_{\circ}^{\star} \cup \mathcal{F}_{\bullet}^{\star}$$

where $\mathcal{F}_{\circ}^{\star}$ and $\mathcal{F}_{\bullet}^{\star}$ are defined similarly to $\mathcal{F}_{\circ}^{\perp}$ and $\mathcal{F}_{\bullet}^{\perp}$. The relationship between the associated relations is now clear:

$$(\mathcal{F}^{\perp} - \mathrm{FC}^{\perp}) \cup \mathrm{FC}^{\star} = \mathcal{F}^{\star} \qquad (\mathcal{F}^{\star} - \mathrm{FC}^{\star}) \cup \mathrm{FC}^{\perp} = \mathcal{F}^{\perp}.$$

In effect, we can transform a frame using $\mathcal{F}^{\perp}$ into a frame using $\mathcal{F}^{\star}$ by throwing out the tuples associated with the special points in the $\mathcal{F}^{\perp}$ frame and replacing them with the tuples associated with the special points in the $\mathcal{F}^{\star}$ frame.

The conclusion is that we can transform $\mathcal{F}^{\perp}$ frames into $\mathcal{F}^{\star}$ frames and thus bypass the issue that $\mathcal{F}^{\perp}$ frames are not closed under null sums.

5.2 Traditional (but Distributed) Necessity and Possibility: $[f^{\square}\rangle, [f^{\diamond}\rangle$

One might have expected the canonical relations for $[f^{\square}\rangle$ and $[f^{\diamond}\rangle$ would be related by using the duality between maximal filters and maximal ideals. However, that turned out not to be the case. We derive the relation $\mathcal{F}^{\diamond}$ from $\mathcal{F}^{\square}$:

$$\begin{array}{rll}
\mathcal{F}^{\square}xy & \text{iff } \forall b\big([f^{\square}\rangle\, b \in_{\mathsf{h}} x \text{ implies } b \in_{\mathsf{k}} y\big) & \text{def. of } \mathcal{F}^{\square} \\
 & \text{iff } \forall b\big(b \notin_{\mathsf{k}} y \text{ implies } [f^{\square}\rangle\, b \notin_{\mathsf{h}} x\big) & \text{contraposition} \\
 & \text{iff } \forall b\big(\neg b \in_{\mathsf{k}} y \text{ implies } \neg[f^{\square}\rangle\, b \in_{\mathsf{h}} x\big) & \text{evaluation condition} \\
 & \text{iff } \forall b\big(\neg b \in_{\mathsf{k}} y \text{ implies } [f^{\diamond}\rangle\, \neg b \in_{\mathsf{h}} x\big) & [f^{\square}\rangle \text{ and } [f^{\diamond}\rangle \text{ are Boolean negations of each other} \\
 & \text{iff } \forall c\big(c \in_{\mathsf{k}} y \text{ implies } [f^{\diamond}\rangle\, c \in_{\mathsf{h}} x\big) & \text{Boolean negation is period two} \\
 & \text{iff } \mathcal{F}^{\diamond}xy & \text{def. } \mathcal{F}^{\diamond}
\end{array}$$

where the third to the second last line follows because of Boolean negation being a period two operator.

The canonical relations are

$$\mathcal{F}^{\square}xy \text{ iff } \forall b\big([f^{\square}\rangle\, a \in_{\mathsf{h}} x \text{ implies } a \in_{\mathsf{k}} y\big) \qquad \mathcal{F}^{\diamond}xy \text{ iff } \forall c\big(c \in_{\mathsf{k}} y \text{ implies } [f^{\diamond}\rangle\, c \in_{\mathsf{h}} x\big)$$

5.3 $[f^{\perp}\rangle, [f^{\star}\rangle$: Complements of Each Other

Lemma 5.3.1 *Canonically, assume $x \notin_{\mathsf{h}} \{\bullet_H, \circ_H\}$ and $y \notin_{\mathsf{k}} \{\bullet_K, \circ_K\}$. If for all b, $\neg[f^{\perp}\rangle b = [f^{\star}\rangle \neg b$, then*

$$\mathcal{F}^{\perp} = \mathcal{F}^{\star}.$$

Proof: Assume $x \notin_{\mathsf{h}} \{\bullet_H, \circ_H\}$ and $y \notin_{\mathsf{k}} \{\bullet_K, \circ_K\}$, hence x and y are maximal filters:

$$\begin{array}{rll}
\mathcal{F}^{\star}xy & \text{iff } \exists b(b \notin_{\mathsf{k}} y \text{ and } [f^{\star}\rangle b \notin_{\mathsf{h}} x) & \text{def. } \mathcal{F}^{\star} \\
 & \text{iff } \exists c(\neg c \notin_{\mathsf{k}} y \text{ and } [f^{\star}\rangle \neg c \notin_{\mathsf{h}} x) & \neg \text{ is period-two} \\
 & \text{iff } \exists c(\neg c \notin_{\mathsf{k}} y \text{ and } \neg[f^{\perp}\rangle c \notin_{\mathsf{h}} x) & [f^{\perp}\rangle c = \neg[f^{\star}\rangle \neg c \\
 & \text{iff } \exists c(c \in_{\mathsf{k}} y \text{ and } [f^{\perp}\rangle c \in_{\mathsf{h}} x) & \text{evaluation condition} \\
 & \text{iff } \mathcal{F}^{\perp}xy & \text{def. } \mathcal{F}^{\perp}
\end{array}$$

■

Frame Condition 5.3.2 As an axiom on frames, we assume

$$x \notin_{\mathsf{h}} H^{\bullet} \cup H^{\circ} \text{ and } y \notin_{\mathsf{k}} K^{\bullet} \cup K^{\circ} \text{ implies } \mathcal{F}^{\perp}xy \text{ iff } \mathcal{F}^{\star}xy.$$

Lemma 5.3.3

$$\overset{\bullet\circ}{\neg}\,[f^{\perp}\rangle Q = [f^{\star}\rangle\,\overset{\bullet\circ}{\neg}\,Q.$$

Proof:

1	$a \in\mathsf{h} \overset{\bullet\circ}{\neg} [f^{\perp}\rangle Q$	assume
2	$a \in\mathsf{h} (\top^{\mathbb{H}} - [f^{\perp}\rangle Q)$ or $a \in\mathsf{h} H^{\bullet}$	def. $\overset{\bullet\circ}{\neg}$, line 1
3	$a \in\mathsf{h} \top^{\mathbb{H}} - [f^{\perp}\rangle Q$	assume
4	$a \in\mathsf{h} \top^{\mathbb{H}}$ and $a \notin\mathsf{h} [f^{\perp}\rangle Q$	set th., line 3
5	$\neg\forall y(y \notin\mathsf{k} Q$ or $\mathcal{F}^{\perp}ay)$	def. $[f^{\perp}\rangle$, line 4
6	$\exists y(y \in\mathsf{k} Q$ and $\neg\mathcal{F}^{\perp}ay)$	CL, line 5
7	$\boxed{b}$ $b \in\mathsf{k} Q$ and $\neg\mathcal{F}^{\perp}ab$	assume
8	$b \in\mathsf{k} K^{\bullet}$ or $b \notin\mathsf{k} K^{\bullet}$	CL
9	$b \in\mathsf{k} K^{\bullet}$	assume
10	$a \notin\mathsf{h} H^{\circ}$	$\top^{\mathbb{H}}$ null-pointed, line 4
11	$\mathcal{F}^{\perp}ab$	Frame Condition 4.2.4 (1), lines 9, 10
12	$\rightarrow\leftarrow$	Contradiction, lines 7, 11
13	$b \notin\mathsf{k} K^{\bullet}$	CL, lines 8, 9
14	$a \notin\mathsf{h} H^{\bullet}$ and $a \notin\mathsf{h} H^{\circ}$	$\top^{\mathbb{H}}$ null-pointed, $[f^{\perp}\rangle Q$ well-pointed, lines 3, 4
15	$b \notin\mathsf{k} K^{\bullet}$ and $b \notin\mathsf{k} K^{\circ}$	$\wedge$-Intro, Q null-pointed, lines 7, 13
16	$\neg\mathcal{F}^{\perp}ab$	$\wedge$-Elim, line 7
17	$\neg\mathcal{F}^{\star}ab$	Frame Condition 5.3.2, lines 14, 15, 16
18	$b \in\mathsf{k} K^{\circ}$ or $b \in\mathsf{k} Q$	$\wedge$-Elim, $\vee$-Intro, line 7
19	$(b \in\mathsf{k} K^{\circ}$ or $b \in\mathsf{k} Q)$ and $b \notin\mathsf{k} K^{\bullet}$	$\wedge$-Intro, lines 13, 18
20	$b \notin\mathsf{k} \overset{\bullet\circ}{\neg} Q$	Lemma 3.3.5, line 19
21	$b \notin\mathsf{k} \overset{\bullet\circ}{\neg} Q$ and $\neg\mathcal{F}^{\star}ab$	$\wedge$-Intro, lines 17, 20
22	$\exists y(y \notin\mathsf{k} \overset{\bullet\circ}{\neg} Q$ and $\neg\mathcal{F}^{\star}ay)$	$\exists$-Intro, line 21
23	$a \in\mathsf{h} [f^{\star}\rangle \overset{\bullet\circ}{\neg} Q$	def. $[-^{\star}\rangle$, line 22
24	$a \in\mathsf{h} [f^{\star}\rangle \overset{\bullet\circ}{\neg} Q$	$\exists$-Elim, lines 6, 7
25	$a \in\mathsf{h} H^{\bullet}$	assume
26	$\exists y(y \in\mathsf{k} K^{\circ})$	$K^{\circ} \neq \emptyset$
27	$\boxed{b}$ $b \in\mathsf{k} K^{\circ}$	assume
28	$\neg\mathcal{F}^{\star}ab$	Frame Condition 4.4.3 (1), line 25
29	$b \notin\mathsf{k} K^{\bullet}$	$K^{\circ} \cap K^{\bullet} = \emptyset$, line 27
30	$b \in\mathsf{k} K^{\circ}$ or $b \in\mathsf{k} Q$	$\vee$-Intro, line 27
31	$(b \in\mathsf{k} K^{\circ}$ or $b \in\mathsf{k} Q)$ and $b \notin\mathsf{k} K^{\bullet}$	$\wedge$-Intro, lines 29, 30
32	$b \notin\mathsf{k} \overset{\bullet\circ}{\neg} Q$	Lemma 3.3.5, line 31
33	$b \notin\mathsf{k} \overset{\bullet\circ}{\neg} Q$ and $\neg\mathcal{F}^{\star}ab$	$\wedge$-Intro, lines 28, 32
34	$\exists y(y \notin\mathsf{k} \overset{\bullet\circ}{\neg} Q$ and $\neg\mathcal{F}^{\star}ay)$	$\exists$-Intro, line 33
35	$a \in\mathsf{h} [f^{\star}\rangle \overset{\bullet\circ}{\neg} Q$	def. $[-^{\star}\rangle$, line 34
36	$a \in\mathsf{h} [f^{\star}\rangle \overset{\bullet\circ}{\neg} Q$	$\exists$-Elim, line 26, 27
37	$a \in\mathsf{h} [f^{\star}\rangle \overset{\bullet\circ}{\neg} Q$	$\vee$-Elim, line 2, 3, 25

Now for the other direction:

1	$a \in_{\mathsf{h}} [f^{\star}\rangle \overset{\bullet\circ}{\neg} Q$	assume
2	$\exists y(y \notin_{\mathsf{k}} \overset{\bullet\circ}{\neg} Q$ and $\neg \mathcal{F}^{\star} ay)$	def. $[-^{\star}\rangle$, line 1
3	$a \in_{\mathsf{h}} H^{\bullet}$ or $a \notin_{\mathsf{h}} H^{\bullet}$	CL
4	$a \notin_{\mathsf{h}} H^{\bullet}$	assume
5	$\boxed{b}$ $b \notin_{\mathsf{k}} \overset{\bullet\circ}{\neg} Q$ and $\neg \mathcal{F}^{\star} ab$	assume
6	$(b \in_{\mathsf{k}} K^{\circ}$ or $b \in_{\mathsf{k}} Q)$ and $b \notin_{\mathsf{k}} K^{\bullet}$	Lemma 3.3.5, line 5
7	$b \in_{\mathsf{k}} K^{\circ}$	assume
8	$\mathcal{F}^{\star} ab$	Frame Condition 4.4.2 (1), lines 4, 7
9	$\rightarrow\leftarrow$	Contradiction, lines 5, 8
10	$b \notin_{\mathsf{k}} K^{\bullet}$ and $b \in_{\mathsf{k}} Q$	CL, lines 6, 7
11	$a \notin_{\mathsf{h}} H^{\bullet}$ and $a \notin_{\mathsf{h}} H^{\circ}$	Lemma 4.2.7, $[-^{\perp}\rangle$ null-pointed, lines 1, 4
12	$b \notin_{\mathsf{k}} K^{\bullet}$ and $b \notin_{\mathsf{k}} K^{\circ}$	Q is null-pointed, line 10
13	$\neg \mathcal{F}^{\star} ab$	$\wedge$-Elim, line 5
14	$\neg \mathcal{F}^{\perp} ab$	Frame Condition 5.3.2, lines 11, 12, 13
15	$b \in_{\mathsf{k}} Q$ and $\neg \mathcal{F}^{\perp} ab$	$\wedge$-Elim, $\wedge$-Intro, lines 10, 14
16	$\exists y(y \in_{\mathsf{k}} Q$ and $\neg \mathcal{F}^{\perp} ay)$	$\exists$-Intro, line 15
17	$\neg \forall y(y \notin_{\mathsf{k}} Q$ or $\mathcal{F}^{\perp} ay)$	CL, line 16
18	$a \notin_{\mathsf{h}} [f^{\perp}\rangle Q$	def. $[-^{\perp}\rangle$, line 17
19	$a \in_{\mathsf{h}} H - H^{\circ}$	H is the universe, set th., line 11
20	$a \in_{\mathsf{h}} \top^{\mathbb{H}}$	def. $\top^{\mathbb{H}}$, line 19
21	$a \in_{\mathsf{h}} \top^{\mathbb{H}} - [f^{\perp}\rangle Q$	set th., lines 18, 20
22	$a \in_{\mathsf{h}} \top^{\mathbb{H}} - [f^{\perp}\rangle Q$ or $a \in_{\mathsf{h}} H^{\bullet}$	$\vee$-Intro, line 21
23	$a \in_{\mathsf{h}} \overset{\bullet\circ}{\neg} [f^{\perp}\rangle Q$	def. $\overset{\bullet\circ}{\neg}$, line 22
24	$a \in_{\mathsf{h}} \overset{\bullet\circ}{\neg} [f^{\perp}\rangle Q$	$\exists$-Elim, lines 2, 5
25	$a \in_{\mathsf{h}} H^{\bullet}$	assume
26	$a \in_{\mathsf{h}} (\top^{\mathbb{H}} - [f^{\perp}\rangle Q)$ or $a \in_{\mathsf{h}} H^{\bullet}$	$\vee$-Intro, line 25
27	$a \in_{\mathsf{h}} \overset{\bullet\circ}{\neg} [f^{\perp}\rangle Q$	def. $\overset{\bullet\circ}{\neg}$, line 26
28	$a \in_{\mathsf{h}} \overset{\bullet\circ}{\neg} [f^{\perp}\rangle Q$	$\vee$-Elim, line 3, 4, 25

■

Corollary 5.3.4 *The following equations are valid and equivalent:*

$$\overset{\bullet\circ}{\neg} [f^{\perp}\rangle Q = [f^{\star}\rangle \overset{\bullet\circ}{\neg} Q \qquad [f^{\perp}\rangle Q = \overset{\bullet\circ}{\neg} [f^{\star}\rangle \overset{\bullet\circ}{\neg} Q \qquad \overset{\bullet\circ}{\neg} [f^{\star}\rangle Q = [f^{\perp}\rangle \overset{\bullet\circ}{\neg} Q \qquad [f^{\star}\rangle Q = \overset{\bullet\circ}{\neg} [f^{\perp}\rangle \overset{\bullet\circ}{\neg} Q.$$

Proof: $\overset{\bullet\circ}{\neg}\,[f^{\perp}\rangle Q = [f^{\star}\rangle\,\overset{\bullet\circ}{\neg}\,Q$ holds from Lemma 5.3.3. The following chain of equalities shows the formulas are also valid and equivalent.

$$\begin{array}{lll}
\overset{\bullet\circ}{\neg}\,[f^{\perp}\rangle Q = [f^{\star}\rangle\,\overset{\bullet\circ}{\neg}\,Q & \text{implies } \overset{\bullet\circ}{\neg}\overset{\bullet\circ}{\neg}\,[f^{\perp}\rangle Q = \overset{\bullet\circ}{\neg}\,[f^{\star}\rangle\,\overset{\bullet\circ}{\neg}\,Q & \overset{\bullet\circ}{\neg} \text{ is a function} \\
 & \text{implies } [f^{\perp}\rangle Q = \overset{\bullet\circ}{\neg}\,[f^{\star}\rangle\,\overset{\bullet\circ}{\neg}\,Q & \text{Lemma 3.3.3} \\
 & \text{implies } [f^{\perp}\rangle\,\overset{\bullet\circ}{\neg}\,Q = \overset{\bullet\circ}{\neg}\,[f^{\star}\rangle\,\overset{\bullet\circ}{\neg}\overset{\bullet\circ}{\neg}\,Q & \text{substitution} \\
 & \text{implies } [f^{\perp}\rangle\,\overset{\bullet\circ}{\neg}\,Q = \overset{\bullet\circ}{\neg}\,[f^{\star}\rangle Q & \text{Lemma 3.3.3} \\
 & \text{implies } \overset{\bullet\circ}{\neg}\,[f^{\perp}\rangle\,\overset{\bullet\circ}{\neg}\,Q = \overset{\bullet\circ}{\neg}\overset{\bullet\circ}{\neg}\,[f^{\star}\rangle Q & \overset{\bullet\circ}{\neg} \text{ is a function} \\
 & \text{implies } \overset{\bullet\circ}{\neg}\,[f^{\perp}\rangle\,\overset{\bullet\circ}{\neg}\,Q = [f^{\star}\rangle Q & \text{Lemma 3.3.3} \\
 & \text{implies } \overset{\bullet\circ}{\neg}\,[f^{\perp}\rangle\,\overset{\bullet\circ}{\neg}\overset{\bullet\circ}{\neg}\,Q = [f^{\star}\rangle\,\overset{\bullet\circ}{\neg}\,Q & \text{substitution} \\
 & \text{implies } \overset{\bullet\circ}{\neg}\,[f^{\perp}\rangle Q = [f^{\star}\rangle\,\overset{\bullet\circ}{\neg}\,Q & \text{Lemma 3.3.3}
\end{array}$$

■

5.4 $[f^{!}\rangle, [f^{?}\rangle$: Complements of Each Other

Lemma 5.4.1 *Canonically, assume* $x \notin_{\mathsf{h}} \{\bullet_H, \circ_H\}$ *and* $y \notin_{\mathsf{k}} \{\bullet_K, \circ_K\}$*. If for all b,* $\neg[f^{!}\rangle b = [f^{?}\rangle\neg b$*, then*

$$\mathcal{F}^{!} = \mathcal{F}^{?}.$$

Proof: Assume $x \notin_{\mathsf{h}} \{\bullet_H, \circ_H\}$ and $y \notin_{\mathsf{k}} \{\bullet_K, \circ_K\}$, hence x and y are maximal filters:.

$$\begin{array}{lll}
\neg\mathcal{F}^{?}xy & \text{iff } \exists b(b \notin_{\mathsf{k}} y \text{ and } [f^{?}\rangle b \notin_{\mathsf{h}} x) & \text{def. } \mathcal{F}^{?} \\
 & \text{iff } \exists c(\neg c \notin_{\mathsf{k}} y \text{ and } [f^{?}\rangle\neg c \notin_{\mathsf{h}} x) & \neg \text{ is period-two} \\
 & \text{iff } \exists c(\neg c \notin_{\mathsf{k}} y \text{ and } \neg[f^{!}\rangle c \notin_{\mathsf{h}} x) & [f^{!}\rangle c = \neg[f^{?}\rangle\neg c \\
 & \text{iff } \exists c(c \in_{\mathsf{k}} y \text{ and } [f^{!}\rangle c \in_{\mathsf{h}} x) & \text{evaluation condition} \\
 & \text{iff } \neg\mathcal{F}^{!}xy & \text{def. } \mathcal{F}^{!}
\end{array}$$

■

Frame Condition 5.4.2 As an axiom on frames, we assume

$$x \notin_{\mathsf{h}} H^{\bullet} \cup H^{\circ} \text{ and } y \notin_{\mathsf{k}} K^{\bullet} \cup K^{\circ} \text{ implies } \mathcal{F}^{!}xy \text{ iff } \mathcal{F}^{?}xy.$$

Lemma 5.4.3

$$\overset{\bullet\circ}{\neg}\,[f^{!}\rangle Q = [f^{?}\rangle\,\overset{\bullet\circ}{\neg}\,Q.$$

Proof:

1	$a \in_h \overset{\bullet\circ}{\neg}[f^{!}\rangle Q$	assume
2	$a \in_h (\top^{\mathbb{H}} - [f^{!}\rangle Q)$ or $a \in_h H^{\bullet}$	def. $\overset{\bullet\circ}{\neg}$, line 1
3	$a \in_h \top^{\mathbb{H}} - [f^{!}\rangle Q$	assume
4	$a \in_h \top^{\mathbb{H}}$ and $a \notin_h [f^{!}\rangle Q$	set th., line 3
5	$\neg\forall y(y \notin_k Q$ or $\neg\mathcal{F}^{!}ay)$	def. $[f^{!}\rangle$, line 4
6	$\exists y(y \in_k Q$ and $\mathcal{F}^{!}ay)$	CL, line 5
7	$\boxed{b}\quad b \in_k Q$ and $\mathcal{F}^{!}ab$	assume
8	$b \in_k K^{\bullet}$ or $b \notin_k K^{\bullet}$	CL
9	$b \in_k K^{\bullet}$	assume
10	$a \notin_h H^{\circ}$	$\top^{\mathbb{H}}$ null-pointed, line 4
11	$\neg\mathcal{F}^{!}ab$	Frame Condition 4.7.4 (1), lines 9, 10
12	$\rightarrow\leftarrow$	Contradiction, lines 7, 11
13	$b \notin_k K^{\bullet}$	CL, lines 8, 9
14	$a \notin_h H^{\bullet}$ and $a \notin_h H^{\circ}$	$\top^{\mathbb{H}}$ null-pointed, $[f^{!}\rangle Q$ well-pointed, line 4
15	$b \notin_k K^{\bullet}$ and $b \notin_k K^{\circ}$	Q null-pointed, lines 7, 13
16	$\mathcal{F}^{!}ab$	$\wedge$-Elim, line 7
17	$\mathcal{F}^{?}ab$	Frame Condition 5.4.2, lines 14, 15, 16
18	$b \in_k K^{\circ}$ or $b \in_k Q$	$\vee$-Intro, line 7
19	$(b \in_k K^{\circ}$ or $b \in_k Q)$ and $b \notin_k K^{\bullet}$	$\wedge$-Intro, lines 13, 18
20	$b \notin_k \overset{\bullet\circ}{\neg}Q$	Lemma 3.3.5, line 19
21	$b \notin_k \overset{\bullet\circ}{\neg}Q$ and $\mathcal{F}^{?}ab$	$\wedge$-Intro, lines 17, 20
22	$\exists y(y \notin_k \overset{\bullet\circ}{\neg}Q$ and $\mathcal{F}^{?}ay)$	$\exists$-Intro, line 21
23	$a \in_h [f^{?}\rangle \overset{\bullet\circ}{\neg}Q$	def. $[-^{?}\rangle$, line 22
24	$a \in_h [f^{?}\rangle \overset{\bullet\circ}{\neg}Q$	$\exists$-Elim, lines 6, 7
25	$a \in_h H^{\bullet}$	assume
26	$\exists y(y \in_k K^{\circ})$	$K^{\circ} \neq \emptyset$
27	$\boxed{b}\quad b \in_k K^{\circ}$	assume
28	$\mathcal{F}^{?}ab$	Frame Condition 4.9.4 (1), line 25
29	$b \notin_k K^{\bullet}$	$K^{\circ} \cap K^{\bullet} = \emptyset$, line 27
30	$b \in_k K^{\circ}$ or $b \in_k Q$	$\vee$-Intro, line 27
31	$(b \in_k K^{\circ}$ or $b \in_k Q)$ and $b \notin_k K^{\bullet}$	$\wedge$-Intro, lines 29, 30
32	$b \notin_k \overset{\bullet\circ}{\neg}Q$	Lemma 3.3.5, line 31
33	$b \notin_k \overset{\bullet\circ}{\neg}Q$ and $\mathcal{F}^{?}ab$	$\wedge$-Intro, lines 28, 32
34	$\exists y(y \notin_k \overset{\bullet\circ}{\neg}Q$ and $\mathcal{F}^{?}ay)$	$\exists$-Intro, line 33
35	$a \in_h [f^{?}\rangle \overset{\bullet\circ}{\neg}Q$	def. $[-^{?}\rangle$, line 34
36	$a \in_h [f^{?}\rangle \overset{\bullet\circ}{\neg}Q$	$\exists$-Elim, line 26, 27
37	$a \in_h [f^{?}\rangle \overset{\bullet\circ}{\neg}Q$	$\vee$-Elim, line 2, 3, 25

Now for the other direction:

1	$\mid a \in_{\mathsf{h}} [f^{?}\rangle \overset{\bullet\circ}{\neg} Q$	assume
2	$\mid \exists y(y \notin_{\mathsf{k}} \overset{\bullet\circ}{\neg} Q \text{ and } \mathcal{F}^{?}ay)$	def. $[-^{?}\rangle$, line 1
3	$\mid a \in_{\mathsf{h}} H^{\bullet} \text{ or } a \notin_{\mathsf{h}} H^{\bullet}$	CL
4	$\mid\mid a \notin_{\mathsf{h}} H^{\bullet}$	assume
5	$\mid\mid\mid \boxed{b}\quad b \notin_{\mathsf{h}} \overset{\bullet\circ}{\neg} Q \text{ and } \mathcal{F}^{?}ab$	assume
6	$\mid\mid\mid (b \in_{\mathsf{k}} K^{\circ} \text{ or } b \in_{\mathsf{k}} Q) \text{ and } b \notin_{\mathsf{k}} K^{\bullet}$	Lemma 3.3.5, line 5
7	$\mid\mid\mid\mid b \in_{\mathsf{k}} K^{\circ}$	assume
8	$\mid\mid\mid\mid \neg\mathcal{F}^{?}ab$	Frame Condition 4.9.3 (1), lines 4, 7
9	$\mid\mid\mid\mid \rightarrow\!\leftarrow$	Contradiction, linse 5, 8
10	$\mid\mid\mid b \notin_{\mathsf{k}} K^{\bullet} \text{ and } b \in_{\mathsf{k}} Q$	CL, lines 6, 7
11	$\mid\mid\mid a \notin_{\mathsf{h}} H^{\bullet} \text{ and } a \notin_{\mathsf{h}} H^{\circ}$	Lemma 4.9.7, $[-^{?}\rangle$ null-pointed, lines 1, 4
12	$\mid\mid\mid b \notin_{\mathsf{k}} K^{\bullet} \text{ and } b \notin_{\mathsf{k}} K^{\circ}$	Q is null-pointed, line 10
13	$\mid\mid\mid \mathcal{F}^{?}ab$	$\wedge$-Elim, line 8
14	$\mid\mid\mid \mathcal{F}^{!}ab$	Frame Condition 5.4.2, lines 11, 12, 13
15	$\mid\mid\mid b \in_{\mathsf{k}} Q \text{ and } \mathcal{F}^{!}ab$	$\wedge$-Elim, $\wedge$-Intro, lines 10, 14
16	$\mid\mid\mid \exists y(y \in_{\mathsf{k}} Q \text{ and } \mathcal{F}^{!}ay)$	$\exists$-Intro, line 15
17	$\mid\mid\mid \neg\forall y(y \notin_{\mathsf{k}} Q \text{ or } \neg\mathcal{F}^{!}ay)$	CL, line 16
18	$\mid\mid\mid a \notin_{\mathsf{h}} [f^{!}\rangle Q$	def. $[-^{!}\rangle$, line 17
19	$\mid\mid\mid a \in_{\mathsf{h}} H - H^{\circ}$	H is the universe, set th., line 11
20	$\mid\mid\mid a \in_{\mathsf{h}} \top^{\mathbb{H}}$	def. $\top^{\mathbb{H}}$, line 19
21	$\mid\mid\mid a \in_{\mathsf{h}} \top^{\mathbb{H}} - [f^{!}\rangle Q$	set th., line 18, 20
22	$\mid\mid\mid a \in_{\mathsf{h}} \top^{\mathbb{H}} - [f^{\perp}\rangle Q \text{ or } a \in_{\mathsf{h}} H^{\bullet}$	$\vee$-Intro, line 21
23	$\mid\mid\mid a \in_{\mathsf{h}} \overset{\bullet\circ}{\neg} [f^{!}\rangle Q$	def. $\overset{\bullet\circ}{\neg}$, line 22
24	$\mid\mid a \in_{\mathsf{h}} \overset{\bullet\circ}{\neg} [f^{!}\rangle Q$	$\exists$-Elim, lines 2, 5
25	$\mid\mid a \in_{\mathsf{h}} H^{\bullet}$	assume
26	$\mid\mid a \in_{\mathsf{h}} (\top^{\mathbb{H}} - [f^{!}\rangle Q) \text{ or } a \in_{\mathsf{h}} H^{\bullet}$	$\vee$-Intro, line 25
27	$\mid\mid a \in_{\mathsf{h}} \overset{\bullet\circ}{\neg} [f^{!}\rangle Q$	def. $\overset{\bullet\circ}{\neg}$, line 26
28	$\mid a \in_{\mathsf{h}} \overset{\bullet\circ}{\neg} [f^{!}\rangle Q$	$\vee$-Elim, line 3, 4, 25

■

Corollary 5.4.4 *The following equations are valid and equivalent:*

$$\overset{\bullet\circ}{\neg} [f^{!}\rangle Q = [f^{?}\rangle \overset{\bullet\circ}{\neg} Q \qquad [f^{!}\rangle Q = \overset{\bullet\circ}{\neg} [f^{?}\rangle \overset{\bullet\circ}{\neg} Q \qquad \overset{\bullet\circ}{\neg} [f^{?}\rangle Q = [f^{!}\rangle \overset{\bullet\circ}{\neg} Q \qquad [f^{?}\rangle Q = \overset{\bullet\circ}{\neg} [f^{!}\rangle \overset{\bullet\circ}{\neg} Q.$$

Proof: $\overset{\bullet\circ}{\neg}\,[f^{!}\rangle Q = [f^{?}\rangle\,\overset{\bullet\circ}{\neg}\,Q$ holds from Lemma 5.4.3. The following chain of equalities shows the formulas are also valid and equivalent.

$$\begin{aligned}
\overset{\bullet\circ}{\neg}\,[f^{!}\rangle Q = [f^{?}\rangle\,\overset{\bullet\circ}{\neg}\,Q \text{ implies } & \overset{\bullet\circ}{\neg}\,\overset{\bullet\circ}{\neg}\,[f^{!}\rangle Q = \overset{\bullet\circ}{\neg}\,[f^{?}\rangle\,\overset{\bullet\circ}{\neg}\,Q && \overset{\bullet\circ}{\neg} \text{ is a function}\\
\text{implies } & [f^{!}\rangle Q = \overset{\bullet\circ}{\neg}\,[f^{?}\rangle\,\overset{\bullet\circ}{\neg}\,Q && \text{Lemma 3.3.3}\\
\text{implies } & [f^{!}\rangle\,\overset{\bullet\circ}{\neg}\,Q = \overset{\bullet\circ}{\neg}\,[f^{?}\rangle\,\overset{\bullet\circ}{\neg}\,\overset{\bullet\circ}{\neg}\,Q && \text{substitution}\\
\text{implies } & [f^{!}\rangle\,\overset{\bullet\circ}{\neg}\,Q = \overset{\bullet\circ}{\neg}\,[f^{?}\rangle Q && \text{Lemma 3.3.3}\\
\text{implies } & \overset{\bullet\circ}{\neg}\,[f^{!}\rangle\,\overset{\bullet\circ}{\neg}\,Q = \overset{\bullet\circ}{\neg}\,\overset{\bullet\circ}{\neg}\,[f^{?}\rangle Q && \overset{\bullet\circ}{\neg} \text{ is a function}\\
\text{implies } & \overset{\bullet\circ}{\neg}\,[f^{!}\rangle\,\overset{\bullet\circ}{\neg}\,Q = [f^{?}\rangle Q && \text{Lemma 3.3.3}\\
\text{implies } & \overset{\bullet\circ}{\neg}\,[f^{!}\rangle\,\overset{\bullet\circ}{\neg}\,\overset{\bullet\circ}{\neg}\,Q = [f^{?}\rangle\,\overset{\bullet\circ}{\neg}\,Q && \text{substitution}\\
\text{implies } & \overset{\bullet\circ}{\neg}\,[f^{!}\rangle Q = [f^{?}\rangle\,\overset{\bullet\circ}{\neg}\,Q && \text{Lemma 3.3.3}
\end{aligned}$$

■

5.5 $[f^{\flat}\rangle, [f^{\sharp}\rangle$: Complements of Each Other

Lemma 5.5.1 *Canonically, assume* $x \notin_{\mathsf{h}} \{\bullet_H, \circ_H\}$ *and* $y \notin_{\mathsf{k}} \{\bullet_K, \circ_K\}$. *If for all* b, $\neg[f^{\flat}\rangle b = [f^{\sharp}\rangle\neg b$, *then*

$$\mathcal{F}^{\flat} = \mathcal{F}^{\sharp}.$$

Proof: Assume $x \notin_{\mathsf{h}} \{\bullet_H, \circ_H\}$ and $y \notin_{\mathsf{k}} \{\bullet_K, \circ_K\}$, hence x and y are maximal filters:.

$$\begin{aligned}
\mathcal{F}^{\sharp}xy \text{ iff } & \exists b(b \in_{\mathsf{k}} y \text{ and } [f^{\sharp}\rangle b \notin_{\mathsf{h}} x) && \text{def. } \mathcal{F}^{\sharp}\\
\text{iff } & \exists c(\neg c \in_{\mathsf{k}} y \text{ and } [f^{\sharp}\rangle\neg c \notin_{\mathsf{h}} x) && \neg \text{ is period-two}\\
\text{iff } & \exists c(\neg c \in_{\mathsf{k}} y \text{ and } \neg[f^{\flat}\rangle c \notin_{\mathsf{h}} x) && \neg[f^{\flat}\rangle c = [f^{\sharp}\rangle\neg c\\
\text{iff } & \exists c(c \notin_{\mathsf{k}} y \text{ and } [f^{\flat}\rangle c \in_{\mathsf{h}} x) && \neg \text{ evaluation condition}\\
\text{iff } & \mathcal{F}^{\flat}xy && \text{def. } \mathcal{F}^{\flat}
\end{aligned}$$

■

Frame Condition 5.5.2 As an axiom on frames, we assume

$$x \notin_{\mathsf{h}} H^{\bullet} \cup H^{\circ} \text{ and } y \notin_{\mathsf{k}} K^{\bullet} \cup K^{\circ} \text{ implies } \mathcal{F}^{\flat}xy \text{ iff } \mathcal{F}^{\sharp}xy.$$

Lemma 5.5.3

$$\overset{\bullet\circ}{\neg}\,[f^{\flat}\rangle Q = [f^{\sharp}\rangle\,\overset{\bullet\circ}{\neg}\,Q.$$

Proof:

1	$a \in_{\mathsf{h}} \overset{\bullet\circ}{\neg} [f^{\flat}\rangle Q$	assume
2	$a \in_{\mathsf{h}} (\top^{\mathbb{H}} - [f^{\flat}\rangle Q)$ or $a \in_{\mathsf{h}} H^{\bullet}$	def. $\overset{\bullet\circ}{\neg}$, line 1
3	$a \in_{\mathsf{h}} \top^{\mathbb{H}} - [f^{\flat}\rangle Q$	assume
4	$a \in_{\mathsf{h}} \top^{\mathbb{H}}$ and $a \notin_{\mathsf{h}} [f^{\flat}\rangle Q$	set th., line 3
5	$\neg\forall y(y \in_{\mathsf{k}} Q$ or $\mathcal{F}^{\flat}ay)$	def. $[f^{\flat}\rangle$, line 4
6	$\exists y(y \notin_{\mathsf{k}} Q$ and $\neg\mathcal{F}^{\flat}ay)$	CL, line 5
7	$\boxed{b}\quad b \notin_{\mathsf{k}} Q$ and $\neg\mathcal{F}^{\flat}ab$	assume
8	$b \in_{\mathsf{k}} K^{\circ}$ or $b \notin_{\mathsf{k}} K^{\circ}$	CL
9	$b \in_{\mathsf{k}} K^{\circ}$	assume
10	$a \notin_{\mathsf{h}} H^{\circ}$	$\top^{\mathbb{H}}$ null-pointed, line 4
11	$\mathcal{F}^{\flat}ab$	Frame Condition 4.11.3 (1), lines 9, 10
12	$\rightarrow\!\leftarrow$	Contradiction, lines 7, 11
13	$b \notin_{\mathsf{k}} K^{\circ}$	CL, lines 8, 9
14	$a \notin_{\mathsf{h}} H^{\bullet}$ and $a \notin_{\mathsf{h}} H^{\circ}$	$\top^{\mathbb{H}}$ null-pointed, $[f^{\flat}\rangle Q$ well-pointed, lines 4
15	$b \notin_{\mathsf{k}} K^{\bullet}$ and $b \notin_{\mathsf{k}} K^{\circ}$	Q well-pointed, lines 7, 13
16	$\neg\mathcal{F}^{\flat}ab$	$\wedge$-Elim, line 7
17	$\neg\mathcal{F}^{\sharp}ab$	Frame Condition 5.5.2, lines 14, 15, 16
18	$b \in_{\mathsf{k}} \top^{\mathbb{K}} - Q$	set th., lines 7, 13
19	$(b \in_{\mathsf{k}} \top^{\mathbb{K}} - Q)$ or $b \in_{\mathsf{k}} K^{\bullet}$	$\vee$-Intro, line 18
20	$b \in_{\mathsf{k}} \overset{\bullet\circ}{\neg} Q$	def. $\overset{\bullet\circ}{\neg}$, line 19
21	$b \in_{\mathsf{k}} \overset{\bullet\circ}{\neg} Q$ and $\neg\mathcal{F}^{\sharp}ab$	$\wedge$-Intro, lines 17, 20
22	$\exists y(y \in_{\mathsf{k}} \overset{\bullet\circ}{\neg} Q$ and $\neg\mathcal{F}^{\sharp}ay)$	$\exists$-Intro, line 21
23	$a \in_{\mathsf{h}} [f^{\sharp}\rangle \overset{\bullet\circ}{\neg} Q$	def. $[-^{\sharp}\rangle$, line 22
24	$a \in_{\mathsf{h}} [f^{\sharp}\rangle \overset{\bullet\circ}{\neg} Q$	$\exists$-Elim, lines 6, 7
25	$a \in_{\mathsf{h}} H^{\bullet}$	assume
26	$\exists y(y \in_{\mathsf{k}} K^{\bullet})$	$K^{\bullet} \neq \emptyset$
27	$\boxed{b}\quad b \in_{\mathsf{k}} K^{\bullet}$	assume
28	$\neg\mathcal{F}^{\sharp}ab$	Frame Condition 4.13.4 (2), line 25
29	$(b \notin_{\mathsf{k}} K^{\circ}$ and $b \notin_{\mathsf{k}} Q)$ or $b \in_{\mathsf{k}} K^{\bullet}$	$\vee$-Intro, line 27
30	$b \in_{\mathsf{k}} \overset{\bullet\circ}{\neg} Q$	def. $\overset{\bullet\circ}{\neg}$, line 29
31	$b \in_{\mathsf{k}} \overset{\bullet\circ}{\neg} Q$ and $\neg\mathcal{F}^{\sharp}ab$	$\wedge$-Intro, lines 28, 30
32	$\exists y(y \in_{\mathsf{k}} \overset{\bullet\circ}{\neg} Q$ and $\neg\mathcal{F}^{\sharp}ay)$	$\exists$-Intro, line 31
33	$a \in_{\mathsf{h}} [f^{\sharp}\rangle \overset{\bullet\circ}{\neg} Q$	def. $[-^{\sharp}\rangle$, line 32
34	$a \in_{\mathsf{h}} [f^{\sharp}\rangle \overset{\bullet\circ}{\neg} Q$	$\exists$-Elim, line 26, 27
35	$a \in_{\mathsf{h}} [f^{\sharp}\rangle \overset{\bullet\circ}{\neg} Q$	$\vee$-Elim, line 2, 3, 25

Now for the other direction:

1	$a \in_{\mathsf{h}} [f^{\sharp}\rangle \overset{\bullet\circ}{\neg} Q$	assume
2	$\exists y(y \in_{\mathsf{k}} \overset{\bullet\circ}{\neg} Q \text{ and } \neg\mathcal{F}^{\sharp}ay)$	def. $[-^{\sharp}\rangle$, line 1
3	$a \in_{\mathsf{h}} H^{\bullet}$ or $a \notin_{\mathsf{h}} H^{\bullet}$	CL
4	$a \notin_{\mathsf{h}} H^{\bullet}$	assume
5	$\boxed{b}$ $b \in_{\mathsf{k}} \overset{\bullet\circ}{\neg} Q \text{ and } \neg\mathcal{F}^{\sharp}ab$	assume
6	$(b \notin_{\mathsf{k}} K^{\circ} \text{ and } b \notin_{\mathsf{k}} Q) \text{ or } b \in_{\mathsf{k}} K^{\bullet}$	def. $\overset{\bullet\circ}{\neg}$, line 5
7	$b \in_{\mathsf{k}} K^{\bullet}$	assume
8	$\mathcal{F}^{\sharp}ab$	Frame Condition 4.13.4 (1), lines 4, 7
9	$\rightarrow\leftarrow$	Contradiction, linse 5, 8
10	$b \notin_{\mathsf{k}} K^{\circ} \text{ and } b \notin_{\mathsf{k}} Q$	CL, lines 6, 7
11	$a \notin_{\mathsf{h}} H^{\bullet} \text{ and } a \notin_{\mathsf{h}} H^{\circ}$	Lemma 4.13.7, $[-^{\sharp}\rangle$ null-pointed, lines 1, 4
12	$b \notin_{\mathsf{k}} K^{\bullet} \text{ and } b \notin_{\mathsf{k}} K^{\circ}$	Q is well-pointed, line 10
13	$\neg\mathcal{F}^{\sharp}ab$	$\wedge$-Elim, line 5
14	$\neg\mathcal{F}^{\flat}ab$	Frame Condition 5.5.2, lines 11, 12, 13
15	$b \notin_{\mathsf{k}} Q \text{ and } \neg\mathcal{F}^{\flat}ab$	$\wedge$-Elim, $\wedge$-Intro, lines 10, 14
16	$\exists y(y \notin_{\mathsf{k}} Q \text{ and } \neg\mathcal{F}^{\flat}ay)$	$\exists$-Intro, line 15
17	$\neg\forall y(y \in_{\mathsf{k}} Q \text{ or } \mathcal{F}^{\flat}ay)$	CL, line 16
18	$a \notin_{\mathsf{h}} [f^{\flat}\rangle Q$	def. $[-^{\flat}\rangle$, line 17
19	$a \in_{\mathsf{h}} \top^{\mathbb{H}} - [f^{\flat}\rangle Q$	set th., lines 11, 18
20	$a \in_{\mathsf{h}} \top^{\mathbb{H}} - [f^{\flat}\rangle Q \text{ or } a \in_{\mathsf{h}} H^{\bullet}$	$\vee$-Intro, line 19
21	$a \in_{\mathsf{h}} \overset{\bullet\circ}{\neg} [f^{\flat}\rangle Q$	def. $\overset{\bullet\circ}{\neg}$, line 20
22	$a \in_{\mathsf{h}} \overset{\bullet\circ}{\neg} [f^{\flat}\rangle Q$	$\exists$-Elim, lines 2, 5
23	$a \in_{\mathsf{h}} H^{\bullet}$	assume
24	$a \in_{\mathsf{h}} (\top^{\mathbb{H}} - [f^{\flat}\rangle Q) \text{ or } a \in_{\mathsf{h}} H^{\bullet}$	$\vee$-Intro, line 23
25	$a \in_{\mathsf{h}} \overset{\bullet\circ}{\neg} [f^{\flat}\rangle Q$	def. $\overset{\bullet\circ}{\neg}$, line 24
26	$a \in_{\mathsf{h}} \overset{\bullet\circ}{\neg} [f^{\flat}\rangle Q$	$\vee$-Elim, line 3, 4, 23

■

Corollary 5.5.4 *The following equations are valid and equivalent:*

$$\overset{\bullet\circ}{\neg} [f^{\flat}\rangle Q = [f^{\sharp}\rangle \overset{\bullet\circ}{\neg} Q \qquad [f^{\flat}\rangle Q = \overset{\bullet\circ}{\neg} [f^{\sharp}\rangle \overset{\bullet\circ}{\neg} Q \qquad \overset{\bullet\circ}{\neg} [f^{\sharp}\rangle Q = [f^{\flat}\rangle \overset{\bullet\circ}{\neg} Q \qquad [f^{\sharp}\rangle Q = \overset{\bullet\circ}{\neg} [f^{\flat}\rangle \overset{\bullet\circ}{\neg} Q.$$

Proof: $\overset{\bullet\circ}{\neg}\,[f^{\flat}\rangle Q = [f^{\sharp}\rangle\,\overset{\bullet\circ}{\neg}\,Q$ holds from Lemma 5.5.3. The following chain of equalities shows the formulas are also valid and equivalent.

$$\begin{array}{lll}
\overset{\bullet\circ}{\neg}\,[f^{\flat}\rangle Q = [f^{\sharp}\rangle\,\overset{\bullet\circ}{\neg}\,Q & \text{implies } \overset{\bullet\circ}{\neg}\,\overset{\bullet\circ}{\neg}\,[f^{\flat}\rangle Q = \overset{\bullet\circ}{\neg}\,[f^{\sharp}\rangle\,\overset{\bullet\circ}{\neg}\,Q & \overset{\bullet\circ}{\neg} \text{ is a function} \\
 & \text{implies } [f^{\flat}\rangle Q = \overset{\bullet\circ}{\neg}\,[f^{\sharp}\rangle\,\overset{\bullet\circ}{\neg}\,Q & \text{Lemma 3.3.3} \\
 & \text{implies } [f^{\flat}\rangle\,\overset{\bullet\circ}{\neg}\,Q = \overset{\bullet\circ}{\neg}\,[f^{\sharp}\rangle\,\overset{\bullet\circ}{\neg}\,\overset{\bullet\circ}{\neg}\,Q & \text{substitution} \\
 & \text{implies } [f^{\flat}\rangle\,\overset{\bullet\circ}{\neg}\,Q = \overset{\bullet\circ}{\neg}\,[f^{\sharp}\rangle Q & \text{Lemma 3.3.3} \\
 & \text{implies } \overset{\bullet\circ}{\neg}\,[f^{\flat}\rangle\,\overset{\bullet\circ}{\neg}\,Q = \overset{\bullet\circ}{\neg}\,\overset{\bullet\circ}{\neg}\,[f^{\sharp}\rangle Q & \overset{\bullet\circ}{\neg} \text{ is a function} \\
 & \text{implies } \overset{\bullet\circ}{\neg}\,[f^{\flat}\rangle\,\overset{\bullet\circ}{\neg}\,Q = [f^{\sharp}\rangle Q & \text{Lemma 3.3.3} \\
 & \text{implies } \overset{\bullet\circ}{\neg}\,[f^{\flat}\rangle\,\overset{\bullet\circ}{\neg}\,\overset{\bullet\circ}{\neg}\,Q = [f^{\sharp}\rangle\,\overset{\bullet\circ}{\neg}\,Q & \text{substitution} \\
 & \text{implies } \overset{\bullet\circ}{\neg}\,[f^{\flat}\rangle Q = [f^{\sharp}\rangle\,\overset{\bullet\circ}{\neg}\,Q & \text{Lemma 3.3.3}
\end{array}$$

■

5.6 $[f^{\Box}\rangle, [f^{\diamond}\rangle$: Complements of Each Other

Lemma 5.6.1 *Canonically, assume* $x \notin_{\mathsf{h}} \{\bullet_H, \circ_H\}$ *and* $y \notin_{\mathsf{k}} \{\bullet_K, \circ_K\}$. *If for all* b, $\neg[f^{\Box}\rangle b = [f^{\diamond}\rangle\neg b$, *then*

$$\mathcal{F}^{\Box} = \mathcal{F}^{\diamond}.$$

Proof: Assume $x \notin_{\mathsf{h}} \{\bullet_H, \circ_H\}$ and $y \notin_{\mathsf{k}} \{\bullet_K, \circ_K\}$, hence x and y are maximal filters:.

$$\begin{array}{lll}
\mathcal{F}^{\diamond}xy & \text{iff } \forall b(b \notin_{\mathsf{k}} y \text{ or } [f^{\diamond}\rangle b \in_{\mathsf{h}} x) & \text{def. } \mathcal{F}^{\diamond} \\
 & \text{iff } \forall c(\neg c \notin_{\mathsf{k}} y \text{ or } [f^{\diamond}\rangle\neg c \in_{\mathsf{h}} x) & \neg \text{ is period-two} \\
 & \text{iff } \forall c(\neg c \notin_{\mathsf{k}} y \text{ or } \neg[f^{\Box}\rangle c \in_{\mathsf{h}} x) & \neg[f^{\Box}\rangle c = [f^{\diamond}\rangle\neg c \\
 & \text{iff } \forall c(c \in_{\mathsf{k}} y \text{ or } [f^{\Box}\rangle c \notin_{\mathsf{h}} x) & \neg \text{ evaluation condition} \\
 & \text{iff } \mathcal{F}^{\Box}xy & \text{def. } \mathcal{F}^{\Box}
\end{array}$$

■

Frame Condition 5.6.2 As an axiom on frames, we assume

$$x \notin_{\mathsf{h}} H^{\bullet} \cup H^{\circ} \text{ and } y \notin_{\mathsf{k}} K^{\bullet} \cup K^{\circ} \text{ implies } \mathcal{F}^{\Box}xy \text{ iff } \mathcal{F}^{\diamond}xy.$$

Lemma 5.6.3

$$\overset{\bullet\circ}{\neg}\,[f^{\Box}\rangle Q = [f^{\diamond}\rangle\,\overset{\bullet\circ}{\neg}\,Q.$$

Proof:

1	$a \in_{\mathsf{h}} \overset{\bullet\circ}{\neg}\, [f^{\Box}\rangle Q$	assume
2	$a \in_{\mathsf{h}} (\top^{\mathbb{H}} - [f^{\Box}\rangle Q)$ or $a \in_{\mathsf{h}} H^{\bullet}$	def. $\overset{\bullet\circ}{\neg}$, line 1
3	$a \in_{\mathsf{h}} \top^{\mathbb{H}} - [f^{\Box}\rangle\, [Q^{\Box}\rangle$	assume
4	$a \in_{\mathsf{h}} \top^{\mathbb{H}}$ and $a \notin_{\mathsf{h}} [f^{\Box}\rangle Q$	set th., line 3
5	$\neg\forall y(y \in_{\mathsf{k}} Q$ or $\neg\mathcal{F}^{\Box}ay)$	def. $[f^{\Box}\rangle$, line 4
6	$\exists y(y \notin_{\mathsf{k}} Q$ and $\mathcal{F}^{\flat}ay)$	CL, line 5
7	$\boxed{b}$ $b \notin_{\mathsf{k}} Q$ and $\mathcal{F}^{\Box}ab$	assume
8	$b \in_{\mathsf{k}} K^{\circ}$ or $b \notin_{\mathsf{k}} K^{\circ}$	CL
9	$b \in_{\mathsf{k}} K^{\circ}$	assume
10	$a \notin_{\mathsf{h}} H^{\circ}$	$\top^{\mathbb{H}}$ null-pointed, line 4
11	$\neg\mathcal{F}^{\Box}ab$	Frame Condition 4.17.3 (1), lines 9, 10
12	$\rightarrow\!\leftarrow$	Contradiction, lines 7, 11
13	$b \notin_{\mathsf{k}} K^{\circ}$	CL, lines 8, 9
14	$a \notin_{\mathsf{h}} H^{\bullet}$ and $a \notin_{\mathsf{h}} H^{\circ}$	$[f^{\Box}\rangle Q$ well-pointed, $\top^{\mathbb{H}}$ null-pointed, line 4
15	$b \notin_{\mathsf{k}} K^{\bullet}$ and $b \notin_{\mathsf{k}} K^{\circ}$	Q well-pointed, lines 7, 13
16	$\mathcal{F}^{\Box}ab$	$\wedge$-Elim, line 7
17	$\mathcal{F}^{\diamond}ab$	Frame Condition 5.6.2, lines 14, 15, 16
18	$b \in_{\mathsf{k}} \top^{\mathbb{K}} - Q$	set th., lines 7, 13
19	$(b \in_{\mathsf{k}} \top^{\mathbb{K}} - Q)$ or $b \in_{\mathsf{k}} K^{\bullet}$	$\vee$-Intro, line 18
20	$b \in_{\mathsf{k}} \overset{\bullet\circ}{\neg} Q$	def. $\overset{\bullet\circ}{\neg}$, line 19
21	$b \in_{\mathsf{k}} \overset{\bullet\circ}{\neg} Q$ and $\mathcal{F}^{\diamond}ab$	$\wedge$-Intro, lines 17, 20
22	$\exists y(y \in_{\mathsf{k}} \overset{\bullet\circ}{\neg} Q$ and $\mathcal{F}^{\diamond}ay)$	$\exists$-Intro, line 21
23	$a \in_{\mathsf{h}} [f^{\diamond}\rangle \overset{\bullet\circ}{\neg} Q$	def. $[-^{\diamond}\rangle$, line 22
24	$a \in_{\mathsf{h}} [f^{\diamond}\rangle \overset{\bullet\circ}{\neg} Q$	$\exists$-Elim, lines 6, 7
25	$a \in_{\mathsf{h}} H^{\bullet}$	assume
26	$\exists y(y \in_{\mathsf{k}} K^{\bullet})$	$K^{\bullet} \neq \emptyset$
27	$\boxed{b}$ $b \in_{\mathsf{k}} K^{\bullet}$	assume
28	$\mathcal{F}^{\diamond}ab$	Frame Condition 4.15.4 (2), line 25
29	$(b \notin_{\mathsf{k}} K^{\circ}$ and $b \notin_{\mathsf{k}} Q)$ or $b \in_{\mathsf{k}} K^{\bullet}$	$\vee$-Intro, line 27
30	$b \in_{\mathsf{k}} \overset{\bullet\circ}{\neg} Q$	def. $\overset{\bullet\circ}{\neg}$, line 29
31	$b \in_{\mathsf{k}} \overset{\bullet\circ}{\neg} Q$ and $\mathcal{F}^{\diamond}ab$	$\wedge$-Intro, lines 28, 30
32	$\exists y(y \in_{\mathsf{k}} \overset{\bullet\circ}{\neg} Q$ and $\mathcal{F}^{\diamond}ay)$	$\exists$-Intro, line 31
33	$a \in_{\mathsf{h}} [f^{\diamond}\rangle \overset{\bullet\circ}{\neg} Q$	def. $[-^{\diamond}\rangle$, line 32
34	$a \in_{\mathsf{h}} [f^{\diamond}\rangle \overset{\bullet\circ}{\neg} Q$	$\exists$-Elim, line 26, 27
35	$a \in_{\mathsf{h}} [f^{\diamond}\rangle \overset{\bullet\circ}{\neg} Q$	$\vee$-Elim, line 2, 3, 25

Now for the other direction:

1	$a \in_{\mathsf{h}} [f^{\diamond}\rangle \overset{\bullet\circ}{\neg} Q$	assume
2	$\exists y(y \in_{\mathsf{k}} \overset{\bullet\circ}{\neg} Q$ and $\mathcal{F}^{\diamond}ay)$	def. $[-^{\diamond}\rangle$, line 1
3	$a \in_{\mathsf{h}} H^{\bullet}$ or $a \notin_{\mathsf{h}} H^{\bullet}$	CL
4	$a \notin_{\mathsf{h}} H^{\bullet}$	assume
5	$\boxed{b}$ $b \in_{\mathsf{k}} \overset{\bullet\circ}{\neg} Q$ and $\mathcal{F}^{\diamond}ab$	assume
6	$(b \notin_{\mathsf{k}} K^{\circ}$ and $b \notin_{\mathsf{k}} Q)$ or $b \in_{\mathsf{k}} K^{\bullet}$	def. $\overset{\bullet\circ}{\neg}$, line 5
7	$b \in_{\mathsf{k}} K^{\bullet}$	assume
8	$\neg\mathcal{F}^{\diamond}ab$	Frame Condition 4.15.4 (1), lines 4, 7
9	$\rightarrow\leftarrow$	Contradiction, linse 5, 8
10	$b \notin_{\mathsf{k}} K^{\circ}$ and $b \notin_{\mathsf{k}} Q$	CL, lines 6, 7
11	$a \notin_{\mathsf{h}} H^{\bullet}$ and $a \notin_{\mathsf{h}} H^{\circ}$	Lemma 4.15.7, $[-^{\diamond}\rangle$ null-pointed, lines 1, 4
12	$b \notin_{\mathsf{k}} K^{\bullet}$ and $b \notin_{\mathsf{k}} K^{\circ}$	Q is well-pointed, line 10
13	$\mathcal{F}^{\diamond}ab$	$\wedge$-Elim, line 5
14	$\mathcal{F}^{\Box}ab$	Frame Condition 5.6.2, lines 11, 12, 13
15	$b \notin_{\mathsf{k}} Q$ and $\mathcal{F}^{\Box}ab$	$\wedge$-Elim, $\wedge$-Intro, lines 10, 14
16	$\exists y(y \notin_{\mathsf{k}} Q$ and $\mathcal{F}^{\Box}ay)$	$\exists$-Intro, line 15
17	$\neg\forall y(y \in_{\mathsf{k}} Q$ or $\neg\mathcal{F}^{\Box}ay)$	CL, line 16
18	$a \notin_{\mathsf{h}} [f^{\Box}\rangle Q$	def. $[-^{\Box}\rangle$, line 17
19	$a \in_{\mathsf{h}} \top^{\mathbb{H}} - [f^{\Box}\rangle Q$	set th., lines 11, 18
20	$a \in_{\mathsf{h}} \top^{\mathbb{H}} - [f^{\Box}\rangle Q$ or $a \in_{\mathsf{h}} H^{\bullet}$	$\vee$-Intro, line 19
21	$a \in_{\mathsf{h}} \overset{\bullet\circ}{\neg} [f^{\Box}\rangle Q$	def. $\overset{\bullet\circ}{\neg}$, line 20
22	$a \in_{\mathsf{h}} \overset{\bullet\circ}{\neg} [f^{\Box}\rangle Q$	$\exists$-Elim, lines 2, 5
23	$a \in_{\mathsf{h}} H^{\bullet}$	assume
24	$a \in_{\mathsf{h}} (\top^{\mathbb{H}} - [f^{\Box}\rangle Q)$ or $a \in_{\mathsf{h}} H^{\bullet}$	$\vee$-Intro, line 23
25	$a \in_{\mathsf{h}} \overset{\bullet\circ}{\neg} [f^{\Box}\rangle Q$	def. $\overset{\bullet\circ}{\neg}$, line 24
26	$a \in_{\mathsf{h}} \overset{\bullet\circ}{\neg} [f^{\Box}\rangle Q$	$\vee$-Elim, line 3, 4, 23

■

Corollary 5.6.4 *The following equations are valid and equivalent:*

$$\overset{\bullet\circ}{\neg} [f^{\Box}\rangle Q = [f^{\diamond}\rangle \overset{\bullet\circ}{\neg} Q \qquad [f^{\Box}\rangle Q = \overset{\bullet\circ}{\neg} [f^{\diamond}\rangle \overset{\bullet\circ}{\neg} Q \qquad \overset{\bullet\circ}{\neg} [f^{\diamond}\rangle Q = [f^{\Box}\rangle \overset{\bullet\circ}{\neg} Q \qquad [f^{\diamond}\rangle Q = \overset{\bullet\circ}{\neg} [f^{\Box}\rangle \overset{\bullet\circ}{\neg} Q.$$

Proof: $\overset{\bullet\circ}{\neg}\,[f^{\square}\rangle Q = [f^{\diamond}\rangle\,\overset{\bullet\circ}{\neg}\,Q$ holds from Lemma 5.6.3. The following chain of equalities shows the formulas are also valid and equivalent.

$$\begin{array}{lll}
\overset{\bullet\circ}{\neg}\,[f^{\square}\rangle Q = [f^{\diamond}\rangle\,\overset{\bullet\circ}{\neg}\,Q & \text{implies } \overset{\bullet\circ}{\neg}\overset{\bullet\circ}{\neg}\,[f^{\square}\rangle Q = \overset{\bullet\circ}{\neg}\,[f^{\diamond}\rangle\,\overset{\bullet\circ}{\neg}\,Q & \overset{\bullet\circ}{\neg}\text{ is a function}\\
 & \text{implies } [f^{\square}\rangle Q = \overset{\bullet\circ}{\neg}\,[f^{\diamond}\rangle\,\overset{\bullet\circ}{\neg}\,Q & \text{Lemma 3.3.3}\\
 & \text{implies } [f^{\square}\rangle\,\overset{\bullet\circ}{\neg}\,Q = \overset{\bullet\circ}{\neg}\,[f^{\diamond}\rangle\,\overset{\bullet\circ}{\neg}\overset{\bullet\circ}{\neg}\,Q & \text{substitution}\\
 & \text{implies } [f^{\square}\rangle\,\overset{\bullet\circ}{\neg}\,Q = \overset{\bullet\circ}{\neg}\,[f^{\diamond}\rangle Q & \text{Lemma 3.3.3}\\
 & \text{implies } \overset{\bullet\circ}{\neg}\,[f^{\square}\rangle\,\overset{\bullet\circ}{\neg}\,Q = \overset{\bullet\circ}{\neg}\overset{\bullet\circ}{\neg}\,[f^{\diamond}\rangle Q & \overset{\bullet\circ}{\neg}\text{ is a function}\\
 & \text{implies } \overset{\bullet\circ}{\neg}\,[f^{\square}\rangle\,\overset{\bullet\circ}{\neg}\,Q = [f^{\diamond}\rangle Q & \text{Lemma 3.3.3}\\
 & \text{implies } \overset{\bullet\circ}{\neg}\,[f^{\square}\rangle\,\overset{\bullet\circ}{\neg}\overset{\bullet\circ}{\neg}\,Q = [f^{\diamond}\rangle\,\overset{\bullet\circ}{\neg}\,Q & \text{substitution}\\
 & \text{implies } \overset{\bullet\circ}{\neg}\,[f^{\square}\rangle Q = [f^{\diamond}\rangle\,\overset{\bullet\circ}{\neg}\,Q & \text{Lemma 3.3.3}
\end{array}$$

■

5.7 Complements Applied to Closures

This is a brief example of using the new Boolean negation to show that $[-^{\perp}\rangle$ is closed under null sums. The frame is not directly closed under null sums. However, if we are allowed to make a detour to the frames that support $[-^{\star}\rangle$, then we can show closure. First, we need , which says

$$\big((H_1 \ovee H_2) \overset{\bullet\circ}{-} (Q_1 \ovee Q_2)\big) = (H_1 \overset{\bullet\circ}{-} Q_1) \ovee (H_2 \overset{\bullet\circ}{-} Q_2).$$

Given the definition of $\overset{\bullet\circ}{-}$, this Lemma is the equivalent of

$$\overset{\bullet\circ}{\neg}(Q_1 \ovee Q_2) = \overset{\bullet\circ}{\neg}\,Q_1 \ovee \overset{\bullet\circ}{\neg}\,Q_2.$$

We use this Lemma in the following argument:

$$\begin{array}{lll}
[f^{\perp}\rangle(Q_1 \ovee Q_2) & = \overset{\bullet\circ}{\neg}\overset{\bullet\circ}{\neg}\,[f^{\perp}\rangle(Q_1 \ovee Q_2) & \text{Lemma 3.3.3}\\
 & = \overset{\bullet\circ}{\neg}\,[f^{\star}\rangle\,\overset{\bullet\circ}{\neg}(Q_1 \ovee Q_2) & \text{Lemma 5.3.3}\\
 & = \overset{\bullet\circ}{\neg}\,[f^{\star}\rangle(\overset{\bullet\circ}{\neg}\,Q_1 \ovee \overset{\bullet\circ}{\neg}\,Q_2) & \text{Lemma 3.5.6}\\
 & = \overset{\bullet\circ}{\neg}([f_1^{\star}\rangle\,\overset{\bullet\circ}{\neg}\,Q_1 \ovee [f_2^{\star}\rangle\,\overset{\bullet\circ}{\neg}\,Q_2) & \text{Lemma 4.4.9}\\
 & = \overset{\bullet\circ}{\neg}(\overset{\bullet\circ}{\neg}\,[f_1^{\perp}\rangle Q_1 \ovee \overset{\bullet\circ}{\neg}\,[f_2^{\perp}\rangle Q_2) & \text{Lemma 5.3.3}\\
 & = \overset{\bullet\circ}{\neg}\overset{\bullet\circ}{\neg}([f_1^{\perp}\rangle Q_1 \ovee [f_2^{\perp}\rangle Q_2) & \text{Lemma 3.5.6}\\
 & = [f_1^{\perp}\rangle Q_1 \ovee [f_2^{\perp}\rangle Q_2 & \text{Lemma 3.3.3}
\end{array}$$

6. RESIDUATION

The Galois operators may also be paired up according to their residuation properties as opposed to their Boolean negation properties.

6.1 Galois Operator Residuation Properties

The Galois operators are either monotonic or antitonic:

Definition 6.1.1 An operator $[f\rangle$ for $f : \mathsf{h} \rightharpoonup \mathsf{k}$ is *monotonic* if

$$a \le_{\mathsf{k}} b \text{ implies } [f\rangle a \le_{\mathsf{h}} [f\rangle b,$$

and *antitonic* if

$$a \le_{\mathsf{k}} b \text{ implies } [f\rangle b \le_{\mathsf{h}} [f\rangle a,$$

The operators and their tonicity properties are in the following Table 6.1 where the first four lines show monotone operators and the second four lines show antitone operators.

Tonicity Property

Forward Operator	**Backward Operator**
$a \le_{\mathsf{k}} b$ implies $[f^{\square}\rangle a \le_{\mathsf{h}} [f^{\square}\rangle b$	$a \le_{\mathsf{h}} b$ implies $[f^{\square}\cdot\rangle a \le_{\mathsf{k}} [f^{\square}\cdot\rangle b$
$a \le_{\mathsf{k}} b$ implies $[f^{\diamond}\rangle a \le_{\mathsf{h}} [f^{\diamond}\rangle b$	$a \le_{\mathsf{h}} b$ implies $[f^{\diamond}\cdot\rangle a \le_{\mathsf{k}} [f^{\diamond}\cdot\rangle b$
$a \le_{\mathsf{k}} b$ implies $[f^{\sharp}\rangle a \le_{\mathsf{h}} [f^{\sharp}\rangle b$	$a \le_{\mathsf{h}} b$ implies $[f^{\sharp}\cdot\rangle a \le_{\mathsf{k}} [f^{\sharp}\cdot\rangle b$
$a \le_{\mathsf{k}} b$ implies $[f^{\flat}\rangle a \le_{\mathsf{h}} [f^{\flat}\rangle b$	$a \le_{\mathsf{h}} b$ implies $[f^{\flat}\cdot\rangle a \le_{\mathsf{k}} [f^{\flat}\cdot\rangle b$
$a \le_{\mathsf{k}} b$ implies $[f^{!}\rangle b \le_{\mathsf{h}} [f^{!}\rangle a$	$a \le_{\mathsf{h}} b$ implies $[f^{!}\cdot\rangle b \le_{\mathsf{k}} [f^{!}\cdot\rangle a$
$a \le_{\mathsf{k}} b$ implies $[f^{?}\rangle b \le_{\mathsf{h}} [f^{?}\rangle a$	$a \le_{\mathsf{h}} b$ implies $[f^{?}\cdot\rangle b \le_{\mathsf{k}} [f^{?}\cdot\rangle a$
$a \le_{\mathsf{k}} b$ implies $[f^{\perp}\rangle b \le_{\mathsf{h}} [f^{\perp}\rangle a$	$a \le_{\mathsf{h}} b$ implies $[f^{\perp}\cdot\rangle b \le_{\mathsf{k}} [f^{\perp}\cdot\rangle a$
$a \le_{\mathsf{k}} b$ implies $[f^{\star}\rangle b \le_{\mathsf{h}} [f^{\star}\rangle a$	$a \le_{\mathsf{h}} b$ implies $[f^{\star}\cdot\rangle b \le_{\mathsf{k}} [f^{\star}\cdot\rangle a$

Table 6.1

Theorem 6.1.2 *In the context of a monotonic and antitonic operators, the residuation properties on the left of Table 6.2 are equivalent to the inequalities on the right:*

Res. Ops	***Residuation Property***	***Inequalities***
$[-^{\square}\rangle, [-^{\diamond}\cdot\rangle$	$[f^{\diamond}\cdot\rangle a \le_{\mathsf{k}} b$ iff $a \le_{\mathsf{h}} [f^{\square}\rangle b$	$a \le_{\mathsf{h}} [f^{\square}\rangle [f^{\diamond}\cdot\rangle a$ and $[f^{\diamond}\cdot\rangle [f^{\square}\rangle b \le_{\mathsf{k}} b$
$[-^{\square}\cdot\rangle, [-^{\diamond}\rangle$	$[f^{\diamond}\rangle b \le_{\mathsf{h}} a$ iff $b \le_{\mathsf{k}} [f^{\square}\cdot\rangle a$	$b \le_{\mathsf{k}} [f^{\square}\cdot\rangle [f^{\diamond}\rangle b$ and $[f^{\diamond}\rangle [f^{\square}\cdot\rangle a \le_{\mathsf{h}} a$
$[-^{\flat}\rangle, [-^{\sharp}\cdot\rangle$	$[f^{\sharp}\cdot\rangle a \le_{\mathsf{k}} b$ iff $a \le_{\mathsf{h}} [f^{\flat}\rangle b$	$a \le_{\mathsf{h}} [f^{\flat}\rangle [f^{\sharp}\cdot\rangle a$ and $[f^{\sharp}\cdot\rangle [f^{\flat}\rangle b \le_{\mathsf{k}} b$
$[-^{\flat}\cdot\rangle, [-^{\sharp}\rangle$	$[f^{\sharp}\rangle b \le_{\mathsf{h}} a$ iff $b \le_{\mathsf{k}} [f^{\flat}\cdot\rangle a$	$b \le_{\mathsf{k}} [f^{\flat}\cdot\rangle [f^{\sharp}\rangle b$ and $[f^{\sharp}\rangle [f^{\flat}\cdot\rangle a \le_{\mathsf{h}} a$
$[-^{?}\rangle, [-^{?}\cdot\rangle$	$[f^{?}\cdot\rangle a \le_{\mathsf{k}} b$ iff $[f^{?}\rangle b \le_{\mathsf{h}} a$	$[f^{?}\rangle [f^{?}\cdot\rangle a \le_{\mathsf{h}} a$ and $[f^{?}\cdot\rangle [f^{?}\rangle b \le_{\mathsf{k}} b$
$[-^{\star}\rangle, [-^{\star}\cdot\rangle$	$[f^{\star}\cdot\rangle a \le_{\mathsf{k}} b$ iff $[f^{\star}\rangle b \le_{\mathsf{h}} a$	$[f^{\star}\rangle [f^{\star}\cdot\rangle a \le_{\mathsf{h}} a$ and $[f^{\star}\cdot\rangle [f^{\star}\rangle b \le_{\mathsf{k}} b$
$[-^{\perp}\rangle, [-^{\perp}\cdot\rangle$	$b \le_{\mathsf{k}} [f^{\perp}\cdot\rangle a$ iff $a \le_{\mathsf{h}} [f^{\perp}\rangle b$	$a \le_{\mathsf{h}} [f^{\perp}\rangle [f^{\perp}\cdot\rangle a$ and $b \le_{\mathsf{k}} [f^{\perp}\cdot\rangle [f^{\perp}\rangle b$
$[-^{!}\rangle, [-^{!}\cdot\rangle$	$b \le_{\mathsf{k}} [f^{!}\cdot\rangle a$ iff $a \le_{\mathsf{h}} [f^{!}\rangle b$	$a \le_{\mathsf{h}} [f^{!}\rangle [f^{!}\cdot\rangle a$ and $b \le_{\mathsf{k}} [f^{!}\cdot\rangle [f^{!}\rangle b$

Table 6.2: Operator Residuation Properties

Residuation is intimately connected to the type:

Theorem 6.1.3 *For lattices, the tonicity properties and residuation implies the operator type as shown in Table 6.3:*

Type	***Property***	***Type***	***Property***
$\vee\mapsto\vee$	$[f^{\diamond}\cdot\rangle(a \vee b) \overset{\mathsf{k}}{=} [f^{\diamond}\cdot\rangle a \vee [f^{\diamond}\cdot\rangle b$	$\vee\mapsto\vee$	$[f^{\diamond}\rangle(a \vee b) \overset{\mathsf{h}}{=} [f^{\diamond}\rangle a \vee [f^{\diamond}\rangle b$
$\wedge\mapsto\wedge$	$[f^{\square}\rangle(a \wedge b) \overset{\mathsf{h}}{=} [f^{\square}\rangle a \wedge [f^{\square}\rangle b$	$\wedge\mapsto\wedge$	$[f^{\square}\cdot\rangle(a \wedge b) \overset{\mathsf{k}}{=} [f^{\square}\cdot\rangle a \wedge [f^{\square}\cdot\rangle b$
$\wedge\mapsto\wedge$	$[f^{\flat}\cdot\rangle(a \wedge b) \overset{\mathsf{k}}{=} [f^{\flat}\cdot\rangle a \wedge [f^{\flat}\cdot\rangle b$	$\wedge\mapsto\wedge$	$[f^{\flat}\rangle(a \wedge b) \overset{\mathsf{h}}{=} [f^{\flat}\rangle a \wedge [f^{\flat}\rangle b$
$\vee\mapsto\vee$	$[f^{\sharp}\rangle(a \vee b) \overset{\mathsf{h}}{=} [f^{\sharp}\rangle a \vee [f^{\sharp}\rangle b$	$\vee\mapsto\vee$	$[f^{\sharp}\cdot\rangle(a \vee b) \overset{\mathsf{k}}{=} [f^{\sharp}\cdot\rangle a \vee [f^{\sharp}\cdot\rangle b$
$\vee\mapsto\wedge$	$[f^{\perp}\cdot\rangle(a \vee b) \overset{\mathsf{k}}{=} [f^{\perp}\cdot\rangle a \wedge [f^{\perp}\cdot\rangle b$	$\wedge\mapsto\vee$	$[f^{\star}\rangle(a \wedge b) \overset{\mathsf{h}}{=} [f^{\star}\rangle a \vee [f^{\star}\rangle b$
$\vee\mapsto\wedge$	$[f^{\perp}\rangle(a \vee b) \overset{\mathsf{h}}{=} [f^{\perp}\rangle a \wedge [f^{\perp}\rangle b$	$\wedge\mapsto\vee$	$[f^{\star}\cdot\rangle(a \wedge b) \overset{\mathsf{k}}{=} [f^{\star}\cdot\rangle a \vee [f^{\star}\cdot\rangle b$
$\vee\mapsto\wedge$	$[f^{!}\cdot\rangle(a \vee b) \overset{\mathsf{k}}{=} [f^{!}\cdot\rangle a \wedge [f^{!}\cdot\rangle b$	$\wedge\mapsto\vee$	$[f^{?}\rangle(a \wedge b) \overset{\mathsf{h}}{=} [f^{?}\rangle a \vee [f^{?}\rangle b$
$\vee\mapsto\wedge$	$[f^{!}\rangle(a \vee b) \overset{\mathsf{h}}{=} [f^{!}\rangle a \wedge [f^{!}\rangle b$	$\wedge\mapsto\vee$	$[f^{?}\cdot\rangle(a \wedge b) \overset{\mathsf{k}}{=} [f^{?}\cdot\rangle a \vee [f^{?}\cdot\rangle b$

Table 6.3

Proof: We show a few of the proofs.

Assume residuation for $[f^{\square}\cdot\rangle$ and $[f^{\diamond}\rangle$. To show the type of $[f^{\square}\cdot\rangle$ we must show

$$[f^{\square}\cdot\rangle(a \wedge c) \overset{\mathsf{k}}{=} [f^{\square}\cdot\rangle a \wedge [f^{\square}\cdot\rangle c.$$

Since $a \wedge c \leq a, c$, from monotonicity we have $[f^{\square}\cdot\rangle(a \wedge c) \leq_{\mathsf{k}} [f^{\square}\cdot\rangle a, [f^{\square}\cdot\rangle c$ and hence $[f^{\square}\cdot\rangle(a \wedge c) \leq_{\mathsf{k}} [f^{\square}\cdot\rangle a \wedge [f^{\square}\cdot\rangle c$. To go in the other direction,

$$\begin{aligned}
[f^{\square}\cdot\rangle a \wedge [f^{\square}\cdot\rangle c &\leq_{\mathsf{k}} [f^{\square}\cdot\rangle a \\
[f^{\square}\cdot\rangle a \wedge [f^{\square}\cdot\rangle c &\leq_{\mathsf{k}} [f^{\square}\cdot\rangle c \\
[f^{\diamond}\rangle([f^{\square}\cdot\rangle a \wedge [f^{\square}\cdot\rangle c) &\leq_{\mathsf{h}} a \\
[f^{\diamond}\rangle([f^{\square}\cdot\rangle a \wedge [f^{\square}\cdot\rangle c) &\leq_{\mathsf{h}} c \\
[f^{\diamond}\rangle([f^{\square}\cdot\rangle a \wedge [f^{\square}\cdot\rangle c) &\leq_{\mathsf{h}} a \wedge c \\
[f^{\square}\cdot\rangle a \wedge [f^{\square}\cdot\rangle c &\leq_{\mathsf{k}} [f^{\square}\cdot\rangle(a \wedge c)
\end{aligned}$$

To show the type of $[f^{\diamond}\rangle$ we must show

$$[f^{\diamond}\rangle(a \vee c) \overset{\mathsf{h}}{=} [f^{\diamond}\rangle a \vee [f^{\diamond}\rangle c.$$

Since $a, c \leq a \vee c$, from monotonicity we have $[f^{\circ}\rangle\, a, [f^{\circ}\rangle\, c \leq_{\mathsf{h}} [f^{\circ}\rangle\, (a \vee c)$ and hence $[f^{\circ}\rangle\, a \vee [f^{\circ}\rangle\, c \leq_{\mathsf{h}} [f^{\circ}\rangle\, (a \vee c)$. To go in the other direction,

$$\begin{aligned}
[f^{\circ}\rangle\, a &\leq_{\mathsf{h}} [f^{\circ}\rangle\, a \vee [f^{\circ}\rangle\, c \\
[f^{\circ}\rangle\, c &\leq_{\mathsf{h}} [f^{\circ}\rangle\, a \vee [f^{\circ}\rangle\, c \\
a &\leq_{\mathsf{k}} [f^{\Box}\cdot\rangle([f^{\circ}\rangle\, a \vee [f^{\circ}\rangle\, c) \\
b &\leq_{\mathsf{k}} [f^{\Box}\cdot\rangle([f^{\circ}\rangle\, a \vee [f^{\circ}\rangle\, c) \\
a \vee b &\leq_{\mathsf{k}} [f^{\Box}\cdot\rangle([f^{\circ}\rangle\, a \vee [f^{\circ}\rangle\, c) \\
[f^{\circ}\rangle\, (a \vee b) &\leq_{\mathsf{h}} [f^{\circ}\rangle\, a \vee [f^{\circ}\rangle\, c
\end{aligned}$$

Assume residuation for $[f^{\flat}\rangle$ and $[f^{\sharp}\cdot\rangle$. To show the type of $[f^{\flat}\rangle$ we must show

$$[f^{\flat}\rangle\, (a \wedge c) \overset{\mathsf{h}}{=} [f^{\flat}\rangle\, a \wedge [f^{\flat}\rangle\, c.$$

Since $a \wedge c \leq a, c$, from monotonicity we have $[f^{\flat}\rangle\, (a \wedge c) \leq_{\mathsf{h}} [f^{\flat}\rangle\, a, [f^{\flat}\rangle\, c$ and hence $[f^{\flat}\rangle\, (a \wedge c) \leq_{\mathsf{h}} [f^{\flat}\rangle\, a \wedge [f^{\flat}\rangle\, c$. To go in the other direction,

$$\begin{aligned}
[f^{\flat}\rangle\, a \wedge [f^{\flat}\rangle\, c &\leq_{\mathsf{h}} [f^{\flat}\rangle\, a \\
[f^{\flat}\rangle\, a \wedge [f^{\flat}\rangle\, c &\leq_{\mathsf{h}} [f^{\flat}\rangle\, c \\
[f^{\sharp}\cdot\rangle([f^{\flat}\rangle\, a \wedge [f^{\flat}\rangle\, c) &\leq_{\mathsf{k}} a \\
[f^{\sharp}\cdot\rangle([f^{\flat}\rangle\, a \wedge [f^{\flat}\rangle\, c) &\leq_{\mathsf{k}} c \\
[f^{\sharp}\cdot\rangle([f^{\flat}\rangle\, a \wedge [f^{\flat}\rangle\, c) &\leq_{\mathsf{k}} a \wedge c \\
[f^{\flat}\rangle\, a \wedge [f^{\Box}\cdot\rangle c &\leq_{\mathsf{h}} [f^{\flat}\rangle\, (a \wedge c)
\end{aligned}$$

To show the type of $[f^{\sharp}\cdot\rangle$ we must show

$$[f^{\sharp}\cdot\rangle(a \vee c) \overset{\mathsf{k}}{=} [f^{\sharp}\cdot\rangle a \vee [f^{\sharp}\cdot\rangle c.$$

Since $a, c \leq a \vee c$, from monotonicity we have $[f^{\sharp}\cdot\rangle a, [f^{\sharp}\cdot\rangle c \leq_{\mathsf{k}} [f^{\sharp}\cdot\rangle(a \vee c)$ and hence $[f^{\sharp}\cdot\rangle a \vee [f^{\sharp}\cdot\rangle c \leq_{\mathsf{k}} [f^{\sharp}\cdot\rangle(a \vee c)$.To go in the other direction,

$$\begin{aligned}
[f^{\sharp}\cdot\rangle a &\leq_{\mathsf{k}} [f^{\sharp}\cdot\rangle a \vee [f^{\sharp}\cdot\rangle c \\
[f^{\sharp}\cdot\rangle c &\leq_{\mathsf{k}} [f^{\sharp}\cdot\rangle a \vee [f^{\sharp}\cdot\rangle c \\
a &\leq_{\mathsf{h}} [f^{\flat}\rangle\, ([f^{\sharp}\cdot\rangle a \vee [f^{\sharp}\cdot\rangle c) \\
b &\leq_{\mathsf{h}} [f^{\flat}\rangle\, ([f^{\sharp}\cdot\rangle a \vee [f^{\sharp}\cdot\rangle c) \\
a \vee b &\leq_{\mathsf{h}} [f^{\flat}\rangle\, ([f^{\sharp}\cdot\rangle a \vee [f^{\sharp}\cdot\rangle c) \\
[f^{\sharp}\cdot\rangle(a \vee b) &\leq_{\mathsf{k}} [f^{\sharp}\cdot\rangle a \vee [f^{\sharp}\cdot\rangle c
\end{aligned}$$

■

6.2 Residuation and the Set Theoretic Properties

Lemma 6.2.1 *Assuming the tonicity properties of the Galois operators, the Table 6.4 relational identities hold canonically*

Relational Identities	
$\mathcal{F}^{\Box} = \mathcal{F}^{\diamond\cdot}$	$\mathcal{F}^{?} = \mathcal{F}^{?\cdot}$
$\mathcal{F}^{\Box\cdot} = \mathcal{F}^{\diamond}$	$\mathcal{F}^{\star} = \mathcal{F}^{\star\cdot}$
$\mathcal{F}^{\flat} = \mathcal{F}^{\sharp\cdot}$	$\mathcal{F}^{\perp} = \mathcal{F}^{\perp\cdot}$
$\mathcal{F}^{\flat\cdot} = \mathcal{F}^{\sharp}$	$\mathcal{F}^{!} = \mathcal{F}^{!\cdot}$

Table 6.4

Proof: For all cases, both sets of Frame Conditions are the same, so we need only check the cases where the special points are not involved.

$\mathcal{F}^{\Box} = \mathcal{F}^{\diamond\cdot}$: The two definitions are

$$\mathcal{F}^{\Box}xy \text{ iff } \forall b(b \in_k y \text{ or } [f^{\Box}\rangle\, b \notin_h x) \qquad \mathcal{F}^{\diamond\cdot}xy \text{ iff } \forall a(a \notin_h x \text{ or } [f^{\diamond}\cdot\rangle a \in_k y).$$

Let $\mathcal{F}^{\Box}xy$. Choose some $a \in_h x$. From residuation, $a \leq_h [f^{\Box}\rangle\,[f^{\diamond}\cdot\rangle a$ and hence $[f^{\Box}\rangle\,[f^{\diamond}\cdot\rangle a \in_h x$. Let $b = [f^{\diamond}\cdot\rangle a$. From the definition of $\mathcal{F}^{\Box}$, we have $b \in_k y$. Therefore $\forall a(a \notin_h x$ or $[f^{\diamond}\cdot\rangle a \in_k y)$, and $\mathcal{F}^{\diamond\cdot}xy$ obtains.

To go in the other direction, let $\mathcal{F}^{\diamond\cdot}xy$. Choose some b such that $b \notin_k y$. From the residuation properties, $[f^{\diamond}\cdot\rangle\,[f^{\Box}\rangle\, b \notin_k y$. Let $a = [f^{\Box}\rangle\, b$. From the definition of $\mathcal{F}^{\diamond\cdot}$, we have $a \notin_h x$. Therefore $\forall b(b \in_k y$ or $[f^{\Box}\rangle\, b \notin_h x)$, and $\mathcal{F}^{\Box}xy$ obtains.

$\mathcal{F}^{\Box\cdot} = \mathcal{F}^{\diamond}$: The two definitions are

$$\mathcal{F}^{\Box\cdot}xy \text{ iff } \forall a(a \in_h x \text{ or } [f^{\Box}\cdot\rangle a \notin_k y) \qquad \mathcal{F}^{\diamond}xy \text{ iff } \forall b(b \notin_k y \text{ or } [f^{\diamond}\rangle\, b \in_h x).$$

Let $\mathcal{F}^{\Box\cdot}xy$. Choose some $b \in_k y$. From residuation, $b \leq_k [f^{\Box}\cdot\rangle\,[f^{\diamond}\rangle\, b$ and hence $[f^{\Box}\cdot\rangle\,[f^{\diamond}\rangle\, b \in_k y$. Let $a = [f^{\diamond}\rangle\, b$. From the definition of $\mathcal{F}^{\Box\cdot}$, we have $a \in_h x$. Therefore $\forall b(b \notin_k y$ or $[f^{\diamond}\rangle\, b \in_h x)$, and $\mathcal{F}^{\diamond}xy$ obtains.

To go in the other direction, let $\mathcal{F}^{\diamond}xy$. Choose some a such that $a \notin_h x$. From the residuation properties, $[f^{\diamond}\rangle\,[f^{\Box}\cdot\rangle a \notin_h x$. Let $b = [f^{\Box}\cdot\rangle a$. From the definition of $\mathcal{F}^{\diamond}$, we have $b \notin_k y$. Therefore $\forall a(a \in_h x$ or $[f^{\Box}\cdot\rangle a \notin_k y)$, and $\mathcal{F}^{\Box\cdot}xy$ obtains.

$\mathcal{F}^{\flat} = \mathcal{F}^{\sharp\cdot}$: The two definitions are

$$\mathcal{F}^{\flat}xy \text{ iff } \exists b(b \notin_k y \text{ and } [f^{\flat}\rangle\, b \in_h x) \qquad \mathcal{F}^{\sharp\cdot}xy \text{ iff } \exists a(a \in_h x \text{ and } [f^{\sharp}\cdot\rangle a \notin_k y).$$

Let $\mathcal{F}^{\flat}xy$. There is some b such that $b \notin_{\mathsf{k}} y$ and $[f^{\flat}\rangle\, b \in_{\mathsf{h}} x$. From the residuation properties, $[f^{\sharp}\cdot\rangle\, [f^{\flat}\rangle\, b \notin_{\mathsf{k}} y$. Let $a = [f^{\flat}\rangle\, b$, then $a \in_{\mathsf{h}} x$ and $[f^{\sharp}\cdot\rangle a \notin_{\mathsf{k}} y$. Therefore $\exists a(a \in_{\mathsf{h}} x$ and $[f^{\sharp}\cdot\rangle a \notin_{\mathsf{k}} y)$, and $\mathcal{F}^{\sharp\cdot}xy$ obtains.

To go in the other direction, let $\mathcal{F}^{\sharp\cdot}xy$ so there is some $a \in_{\mathsf{h}} x$ and $[f^{\sharp}\cdot\rangle a \notin_{\mathsf{k}} y$. From residuation, $a \leq [f^{\flat}\rangle\, [f^{\sharp}\cdot\rangle a$ and hence $[f^{\flat}\rangle\, [f^{\sharp}\cdot\rangle a \in_{\mathsf{h}} x$. Let $b = [f^{\sharp}\cdot\rangle a$. Therefore $\exists b(b \notin_{\mathsf{k}} y$ and $[f^{\flat}\rangle\, b \in_{\mathsf{h}} x)$, and $\mathcal{F}^{\flat}xy$ obtains.

$\mathcal{F}^{\flat\cdot} = \mathcal{F}^{\sharp}$: The two definitions are

$$\mathcal{F}^{\flat\cdot}xy \text{ iff } \exists a(a \notin_{\mathsf{h}} x \text{ and } [f^{\flat}\cdot\rangle a \in_{\mathsf{k}} y) \qquad \mathcal{F}^{\sharp}xy \text{ iff } \exists b(b \in_{\mathsf{k}} y \text{ and } [f^{\sharp}\rangle\, b \notin_{\mathsf{h}} x).$$

Let $\mathcal{F}^{\flat\cdot}xy$. There is some a such that $a \notin_{\mathsf{h}} x$ and $[f^{\flat}\cdot\rangle a \in_{\mathsf{k}} y$. From the residuation properties, $[f^{\sharp}\rangle\, [f^{\flat}\cdot\rangle a \notin_{\mathsf{h}} x$. Let $b = [f^{\flat}\cdot\rangle a$, then $b \in_{\mathsf{k}} y$ and $[f^{\sharp}\rangle\, b \notin_{\mathsf{h}} x$. Therefore $\exists b(b \in_{\mathsf{k}} y$ and $[f^{\sharp}\rangle\, b \notin_{\mathsf{h}} x)$, and $\mathcal{F}^{\sharp}xy$ obtains.

To go in the other direction, let $\mathcal{F}^{\sharp}xy$ so there is some $b \in_{\mathsf{k}} y$ and $[f^{\sharp}\rangle\, b \notin_{\mathsf{h}} x$. From residuation, $b \leq_{\mathsf{k}} [f^{\flat}\cdot\rangle\, [f^{\sharp}\rangle\, b$ and hence $[f^{\flat}\cdot\rangle\, [f^{\sharp}\rangle\, b \in_{\mathsf{k}} y$. Let $a = [f^{\sharp}\rangle\, b$. Therefore $\exists a(a \notin_{\mathsf{h}} x$ and $[f^{\flat}\cdot\rangle a \in_{\mathsf{k}} y)$, and $\mathcal{F}^{\flat\cdot}xy$ obtains.

$\mathcal{F}^{?} = \mathcal{F}^{?\cdot}$: The two definitions are

$$\mathcal{F}^{?}xy \text{ iff } \forall b(b \in_{\mathsf{k}} y \text{ or } [f^{?}\rangle\, b \in_{\mathsf{h}} x) \qquad \mathcal{F}^{?\cdot}xy \text{ iff } \forall a(a \in_{\mathsf{h}} x \text{ or } [f^{?}\cdot\rangle a \in_{\mathsf{k}} y).$$

Let $\mathcal{F}^{?}xy$. Choose some $a \notin_{\mathsf{h}} x$. From residuation, $[f^{?}\rangle\, [f^{?}\cdot\rangle a \leq_{\mathsf{h}} a$ and hence $[f^{?}\rangle\, [f^{?}\cdot\rangle a \notin_{\mathsf{h}} x$. Let $b = [f^{?}\cdot\rangle a$, then $[f^{?}\rangle\, b \notin_{\mathsf{h}} x$. From the definition of $\mathcal{F}^{?}xy$, we have $b \in_{\mathsf{k}} y$. Therefore $[f^{?}\cdot\rangle a \notin_{\mathsf{k}} y$ and thus $\forall a(a \in_{\mathsf{h}} x$ or $[f^{?}\cdot\rangle a \in_{\mathsf{k}} y)$. So $\mathcal{F}^{?\cdot}xy$ obtains.

To go in the other direction, let $\mathcal{F}^{?\cdot}xy$. Choose some b such that $b \notin_{\mathsf{k}} y$. From the residuation properties, $[f^{?}\cdot\rangle\, [f^{?}\rangle\, b \notin_{\mathsf{k}} y$. Let $a = [f^{?}\rangle\, b$, then $[f^{?}\cdot\rangle a \notin_{\mathsf{k}} y$. From the definition of $\mathcal{F}^{?\cdot}xy$, we have $a \in_{\mathsf{h}} x$. Therefore $[f^{?}\rangle\, b \in_{\mathsf{h}} x$ and thus $\forall b(b \in_{\mathsf{k}} y$ or $[f^{?}\rangle\, b \in_{\mathsf{h}} x)$ and $\mathcal{F}^{?}xy$ obtains.

$\mathcal{F}^{\star} = \mathcal{F}^{\star\cdot}$: The two definitions are

$$\mathcal{F}^{\star}xy \text{ iff } \exists b(b \notin_{\mathsf{k}} y \text{ and } [f^{\star}\rangle\, b \notin_{\mathsf{h}} x) \qquad \mathcal{F}^{\star\cdot}xy \text{ iff } \exists a(a \notin_{\mathsf{h}} x \text{ and } [f^{\star}\cdot\rangle a \notin_{\mathsf{k}} y).$$

Let $\mathcal{F}^{\star}xy$, so there is some $b \notin_{\mathsf{k}} y$ and $[f^{\star}\rangle\, b \notin_{\mathsf{h}} x$. From residuation, $[f^{\star}\cdot\rangle\, [f^{\star}\rangle\, b \leq_{\mathsf{k}} b$, hence $[f^{\star}\cdot\rangle\, [f^{\star}\rangle\, b \notin_{\mathsf{k}} y$. Let $a = [f^{\star}\rangle\, b$, then $a \notin_{\mathsf{h}} x$ and $[f^{\star}\cdot\rangle a \notin_{\mathsf{k}} y$. Therefore $\exists a(a \notin_{\mathsf{h}} x$ and $[f^{\star}\cdot\rangle a \notin_{\mathsf{k}} y)$, holds and $\mathcal{F}^{\star\cdot}xy$ obtains.

To go in the other direction, let $\mathcal{F}^{\star\cdot}xy$. There is some a such that $a \notin_{\mathsf{h}} x$ and $[f^{\star}\cdot\rangle a \notin_{\mathsf{k}} y$. From residuation, $[f^{\star}\rangle\, [f^{\star}\cdot\rangle a \leq_{\mathsf{h}} a$, hence $[f^{\star}\rangle\, [f^{\star}\cdot\rangle a \notin_{\mathsf{h}} x$. Let $b = [f^{\star}\cdot\rangle a$, then $b \notin_{\mathsf{k}} y$ and $[f^{\star}\rangle\, b \notin_{\mathsf{h}} x$. Therefore $\exists b(b \notin_{\mathsf{k}} y$ and $[f^{\star}\rangle\, b \notin_{\mathsf{h}} x)$, and $\mathcal{F}^{\star}xy$ obtains.

$\mathcal{F}^{\perp} = \mathcal{F}^{\perp\cdot}$: The two definitions are

$$\mathcal{F}^{\perp}xy \text{ iff } \exists b(b \in_{\mathsf{k}} y \text{ and } [f^{\perp}\rangle\, b \in_{\mathsf{h}} x) \qquad \mathcal{F}^{\perp\cdot}xy \text{ iff } \exists a(a \in_{\mathsf{h}} x \text{ and } [f^{\perp}\cdot\rangle a \in_{\mathsf{k}} y).$$

Let $\mathcal{F}^{\perp}xy$, so there is some $b \in_{\mathsf{k}} y$ and $[f^{\perp}\rangle\, b \in_{\mathsf{h}} x$. From residuation, $b \leq_{\mathsf{k}} [f^{\perp}\cdot\rangle\,[f^{\perp}\rangle\, b$, hence $[f^{\perp}\cdot\rangle\,[f^{\perp}\rangle\, b \in_{\mathsf{k}} y$. Let $a = [f^{\perp}\rangle\, b$, then $a \in_{\mathsf{h}} x$ and $[f^{\perp}\cdot\rangle a \in_{\mathsf{k}} y$. Therefore $\exists a(a \in_{\mathsf{h}} x$ and $[f^{\perp}\cdot\rangle a \in_{\mathsf{k}} y)$, holds and $\mathcal{F}^{\perp\cdot}xy$ obtains.

To go in the other direction, let $\mathcal{F}^{\perp\cdot}xy$. There is some a such that $a \in_{\mathsf{h}} x$ and $[f^{\perp}\cdot\rangle a \in_{\mathsf{k}} y$. From residuation, $a \leq_{\mathsf{h}} [f^{\perp}\rangle\,[f^{\perp}\cdot\rangle a$, hence $[f^{\perp}\rangle\,[f^{\perp}\cdot\rangle a \in_{\mathsf{h}} x$. Let $b = [f^{\perp}\cdot\rangle a$, then $b \in_{\mathsf{k}} y$ and $[f^{\perp}\rangle\, b \in_{\mathsf{h}} x$. Therefore $\exists b(b \in_{\mathsf{k}} y$ and $[f^{\perp}\rangle\, b \in_{\mathsf{h}} x)$, holds and $\mathcal{F}^{\perp}xy$ obtains.

$\mathcal{F}^{!} = \mathcal{F}^{!\cdot}$: The two definitions are

$$\mathcal{F}^{!}xy \text{ iff } \forall b(b \notin_{\mathsf{k}} y \text{ or } [f^{!}\rangle\, b \notin_{\mathsf{h}} x) \qquad \mathcal{F}^{!\cdot}xy \text{ iff } \forall a(a \notin_{\mathsf{h}} x \text{ or } [f^{!}\cdot\rangle a \notin_{\mathsf{k}} y).$$

Let $\mathcal{F}^{!}xy$. Choose some $a \in_{\mathsf{h}} x$. From residuation, $a \leq_{\mathsf{h}} [f^{!}\rangle\,[f^{!}\cdot\rangle a$ and hence $[f^{!}\rangle\,[f^{!}\cdot\rangle a \in_{\mathsf{h}} x$. Let $b = [f^{!}\cdot\rangle a$, then $[f^{!}\rangle\, b \in_{\mathsf{h}} x$. From the definition of $\mathcal{F}^{!}xy$, we have $b \notin_{\mathsf{k}} y$. Therefore $[f^{!}\cdot\rangle a \notin_{\mathsf{k}} y$ and thus $\forall a(a \notin_{\mathsf{h}} x$ or $[f^{!}\cdot\rangle a \notin_{\mathsf{k}} y)$ holds and $\mathcal{F}^{!\cdot}xy$ obtains.

To go in the other direction, let $\mathcal{F}^{!\cdot}xy$. Choose some b such that $b \in_{\mathsf{k}} y$. From residuation, $b \leq_{\mathsf{k}} [f^{!}\cdot\rangle\,[f^{!}\rangle\, b$, hence $[f^{!}\cdot\rangle\,[f^{!}\rangle\, b \in_{\mathsf{k}} y$. Let $a = [f^{!}\rangle\, b$, then $[f^{!}\cdot\rangle a \in_{\mathsf{k}} y$. From the definition of $\mathcal{F}^{!\cdot}xy$, we have $a \notin_{\mathsf{h}} x$. Therefore $[f^{!}\rangle\, b \notin_{\mathsf{h}} x$ and thus $\forall b(b \notin_{\mathsf{k}} y$ or $[f^{!}\rangle\, b \notin_{\mathsf{h}} x)$ and $\mathcal{F}^{!}xy$ obtains.

■

The following theorem shows that the set theoretic operators satisfy their required residuation properties:

Theorem 6.2.2 *Assume the relational identities of Table 6.4 hold generally, i.e., not just canonically. The conditions in the Table 6.5 hold. Each operator is in a column is paired with its residual. Each residual pair has a forward and a backward operator. Notice that the last four rows indicate each forward operator of those rows is paired with its backward version.*

Res. Ops	***Property***	***Res. Ops***	***Property***
$[-^{\square}\rangle, [-^{\diamond}\cdot\rangle$	$[f^{\diamond}\cdot\rangle P \subseteq_{\mathsf{k}} Q$ iff $P \subseteq_{\mathsf{h}} [f^{\square}\rangle Q$	$[-^{\flat}\cdot\rangle, [-^{\sharp}\rangle$	$[f^{\sharp}\rangle Q \subseteq_{\mathsf{h}} P$ iff $Q \subseteq_{\mathsf{k}} [f^{\flat}\cdot\rangle P$
$[-^{\square}\cdot\rangle, [-^{\diamond}\rangle$	$[f^{\diamond}\rangle Q \subseteq_{\mathsf{h}} P$ iff $Q \subseteq_{\mathsf{k}} [f^{\square}\cdot\rangle P$	$[-^{\flat}\rangle, [-^{\sharp}\cdot\rangle$	$[f^{\sharp}\cdot\rangle P \subseteq_{\mathsf{k}} Q$ iff $P \subseteq_{\mathsf{h}} [f^{\flat}\rangle Q$
$[-^{?}\rangle, [-^{?}\cdot\rangle$	$[f^{?}\cdot\rangle P \subseteq_{\mathsf{k}} Q$ iff $[f^{?}\rangle Q \subseteq_{\mathsf{h}} P$	$[-^{\star}\rangle, [-^{\star}\cdot\rangle$	$[f^{\star}\cdot\rangle P \subseteq_{\mathsf{k}} Q$ iff $[f^{\star}\rangle Q \subseteq_{\mathsf{h}} P$
$[-^{\perp}\rangle, [-^{\perp}\cdot\rangle$	$Q \subseteq_{\mathsf{k}} [f^{\perp}\cdot\rangle P$ iff $P \subseteq_{\mathsf{h}} [f^{\perp}\rangle Q$	$[-^{!}\rangle, [-^{!}\cdot\rangle$	$Q \subseteq_{\mathsf{k}} [f^{!}\cdot\rangle P$ iff $P \subseteq_{\mathsf{h}} [f^{!}\rangle Q$

Table 6.5

Proof: Assume the relational identities of Table 6.4 hold generally.

$[-^{\square}\rangle, [-^{\diamond}\cdot\rangle$: assume $P \subseteq_{\mathsf{h}} [f^{\square}\rangle Q$ and let $y \in_{\mathsf{k}} [f^{\diamond}\cdot\rangle P$. By definition, there is some x such that $x \in_{\mathsf{h}} P$ and $\mathcal{F}^{\diamond\cdot}xy$. So $x \in_{\mathsf{h}} [f^{\square}\rangle Q$. By definition, $\forall y(\mathcal{F}^{\square}xy$ implies $y \in_{\mathsf{k}} Q)$. From Table 6.4, $\mathcal{F}^{\square} = \mathcal{F}^{\diamond\cdot}$, we have $\forall y(\mathcal{F}^{\diamond\cdot}xy$ implies $y \in_{\mathsf{k}} Q)$. Since $\mathcal{F}^{\diamond\cdot}xy$, then $y \in_{\mathsf{k}} Q$; whence $[f^{\diamond}\cdot\rangle P \subseteq_{\mathsf{k}} Q$.

To go in the other direction, assume $[f^{\diamond}\cdot\rangle P \subseteq_{\mathsf{k}} Q$ and let $x \notin_{\mathsf{h}} [f^{\square}\rangle Q$. By definition, there is some y such that $y \notin_{\mathsf{k}} Q$ and $\mathcal{F}^{\square}xy$. So $y \notin_{\mathsf{k}} [f^{\diamond}\cdot\rangle P$ and by definition, $\forall y(\mathcal{F}^{\diamond\cdot}xy$ implies $x \notin_{\mathsf{h}} P)$. From Table 6.4 $\mathcal{F}^{\square} = \mathcal{F}^{\diamond\cdot}$, we have $\forall y(\mathcal{F}^{\square}xy$ implies $x \notin_{\mathsf{h}} P)$. Thus, $x \notin_{\mathsf{h}} P$. From contraposition, $P \subseteq_{\mathsf{h}} [f^{\square}\rangle Q$.

$[-^{\square}\cdot\rangle, [-^{\diamond}\rangle$: This case is similar to $[-^{\square}\rangle, [-^{\diamond}\cdot\rangle$.

$$x \in_{\mathsf{h}} [f^{\flat}\rangle Q \text{ iff } \forall y(y \in_{\mathsf{k}} Q \text{ or } \mathcal{F}^{\flat}xy) \qquad y \in_{\mathsf{k}} [f^{\sharp}\cdot\rangle P \text{ iff } \exists x(x \in_{\mathsf{h}} P \text{ and } \neg\mathcal{F}^{\sharp\cdot}xy)$$

Want

$$[f^{\sharp}\cdot\rangle P \subseteq_{\mathsf{k}} Q \text{ iff } P \subseteq_{\mathsf{h}} [f^{\flat}\rangle Q$$

$[-^{\flat}\rangle, [-^{\sharp}\cdot\rangle$: assume $[f^{\sharp}\cdot\rangle P \subseteq_{\mathsf{k}} Q$ and let $x \notin_{\mathsf{h}} [f^{\flat}\rangle Q$. By definition, there is some y such that $y \notin_{\mathsf{k}} Q$ and $\neg\mathcal{F}^{\flat}xy$. Hence $y \notin_{\mathsf{k}} [f^{\sharp}\cdot\rangle P$. So $\forall x(x \notin_{\mathsf{h}} P \text{ or } \mathcal{F}^{\sharp\cdot}xy)$. From Table 6.4, $\mathcal{F}^{\flat} = \mathcal{F}^{\sharp\cdot}$, so we have $\forall x(x \notin_{\mathsf{h}} P \text{ or } \mathcal{F}^{\flat}xy)$; whence $x \notin_{\mathsf{h}} P$. Thus $P \subseteq_{\mathsf{h}} [f^{\flat}\rangle Q$.

To go in the other direction, assume $P \subseteq_{\mathsf{h}} [f^{\flat}\rangle Q$ and let $y \in_{\mathsf{k}} [f^{\sharp}\cdot\rangle P$. So there is some x such that $x \in_{\mathsf{h}} P$ and $\neg\mathcal{F}^{\sharp\cdot}xy$. Hence $x \in_{\mathsf{h}} [f^{\flat}\rangle Q$. By definition, $\forall y(y \in_{\mathsf{k}} Q \text{ or } \mathcal{F}^{\flat}xy)$. From Table 6.4, $\mathcal{F}^{\flat} = \mathcal{F}^{\sharp\cdot}$, so $\neg\mathcal{F}^{\flat}xy$; whence $y \in_{\mathsf{k}} Q$. Thus $[f^{\sharp}\cdot\rangle P \subseteq_{\mathsf{k}} Q$.

$[-^{\flat}\cdot\rangle, [-^{\sharp}\rangle$: This case is similar to $[-^{\flat}\rangle, [-^{\sharp}\cdot\rangle$.

$$y \in_{\mathsf{k}} [f^{?}\cdot\rangle P \text{ iff } \exists x(x \notin_{\mathsf{h}} P \text{ and } \mathcal{F}^{?\cdot}xy) \qquad x \in_{\mathsf{h}} [f^{?}\rangle Q \text{ iff } \exists y(y \notin_{\mathsf{k}} Q \text{ and } \mathcal{F}^{?}xy)$$

Want

$$[f^{?}\cdot\rangle P \subseteq_{\mathsf{k}} Q \text{ iff } [f^{?}\rangle Q \subseteq_{\mathsf{h}} P$$

$[-^{?}\rangle, [-^{?}\cdot\rangle$: assume $[f^{?}\rangle Q \subseteq_{\mathsf{h}} P$ and let $y \in_{\mathsf{k}} [f^{?}\cdot\rangle P$. By definition, there is some x such that $x \notin_{\mathsf{h}} P$ and $\mathcal{F}^{?\cdot}xy$. So $x \notin_{\mathsf{h}} [f^{?}\rangle Q$. By definition, $\forall y(\mathcal{F}^{?}xy \text{ implies } y \in_{\mathsf{k}} Q)$. From Table 6.4, $\mathcal{F}^{?} = \mathcal{F}^{?\cdot}$, and so we have $\forall y(\mathcal{F}^{?\cdot}xy \text{ implies } y \in_{\mathsf{k}} Q)$, whence $y \in_{\mathsf{k}} Q$. Thus $[f^{?}\cdot\rangle P \subseteq_{\mathsf{k}} Q$.

To go in the other direction, assume $[f^{?}\cdot\rangle P \subseteq_{\mathsf{k}} Q$ and let $x \in_{\mathsf{h}} [f^{?}\rangle Q$. By definition, there is some y such that $y \notin_{\mathsf{k}} Q$ and $\mathcal{F}^{?}xy$. So $y \notin_{\mathsf{k}} [f^{?}\cdot\rangle P$. By definition, $\forall x(\mathcal{F}^{?\cdot}xy \text{ implies } x \in_{\mathsf{h}} P)$. From Table 6.4, $\mathcal{F}^{?} = \mathcal{F}^{?\cdot}$, and so we have $\forall x(\mathcal{F}^{?}xy \text{ implies } x \in_{\mathsf{h}} P)$; whence $x \in_{\mathsf{h}} P$. Thus $[f^{?}\rangle Q \subseteq_{\mathsf{h}} P$.

$[-^{\star}\rangle, [-^{\star}\cdot\rangle$: This case is similar to $[-^{?}\rangle, [-^{?}\cdot\rangle$.

$$y \in_{\mathsf{k}} [f^{!}\cdot\rangle P \text{ iff } \forall x(x \notin_{\mathsf{h}} P \text{ or } \neg\mathcal{F}^{!\cdot}xy) \qquad x \in_{\mathsf{h}} [f^{!}\rangle Q \text{ iff } \forall y(y \notin_{\mathsf{k}} Q \text{ or } \neg\mathcal{F}^{!}xy).$$

Want

$$Q \subseteq_{\mathsf{k}} [f^{!}\cdot\rangle P \text{ iff } P \subseteq_{\mathsf{h}} [f^{!}\rangle Q$$

$[-^{!}\rangle, [-^{!}\cdot\rangle$: assume $Q \subseteq_{\mathsf{k}} [f^{!}\cdot\rangle P$ and let $x \notin_{\mathsf{h}} [f^{!}\rangle Q$. By definition, there is some y such that $y \in_{\mathsf{k}} Q$ and $\mathcal{F}^{!}xy$. So $y \in_{\mathsf{k}} [f^{!}\cdot\rangle P$. By definition, $\forall x(x \notin_{\mathsf{h}} P$ or $\neg\mathcal{F}^{!\cdot}xy)$. From Table 6.4, $\mathcal{F}^{!} = \mathcal{F}^{!\cdot}$, and so we have $\forall x(x \notin_{\mathsf{h}} P$ or $\neg\mathcal{F}^{!}xy)$, whence $x \notin_{\mathsf{h}} P$. Thus $P \subseteq_{\mathsf{h}} [f^{!}\rangle Q$.

To go in the other direction, assume $P \subseteq_{\mathsf{h}} [f^{!}\rangle Q$ and let $y \notin_{\mathsf{k}} [f^{!}\cdot\rangle P$. By definition, there is some x such that $x \in_{\mathsf{h}} P$ and $\mathcal{F}^{!\cdot}xy$. So $x \in_{\mathsf{h}} [f^{!}\rangle Q$. By definition, $\forall y(y \notin_{\mathsf{k}} Q$ or $\neg\mathcal{F}^{!}xy)$. From Table 6.4, $\mathcal{F}^{!} = \mathcal{F}^{!\cdot}$, and so we have $\forall y(y \notin_{\mathsf{k}} Q$ or $\neg\mathcal{F}^{!\cdot}xy)$, whence $y \notin_{\mathsf{k}} Q$. Thus $Q \subseteq_{\mathsf{k}} [f^{!}\cdot\rangle P$.

$[-^{\perp}\rangle, [-^{\perp}\cdot\rangle$: This case is similar to $[-^{!}\rangle, [-^{!}\cdot\rangle$. ■

Theorem 6.2.3 *The set theoretic Galois operators satisfy the following tonicity properties of Table 6.6, which are the same as the algebraic tonicity properties of Table 6.1:*

Tonicity Property

Forward Operator	***Backward Operator***
$Q_1 \subseteq_{\mathsf{k}} Q_2$ implies $[f^{\square}\rangle Q_1 \subseteq_{\mathsf{h}} [f^{\square}\rangle Q_2$	$P_1 \subseteq_{\mathsf{h}} P_2$ implies $[f^{\square}\cdot\rangle P_1 \subseteq_{\mathsf{k}} [f^{\square}\cdot\rangle P_2$
$Q_1 \subseteq_{\mathsf{k}} Q_2$ implies $[f^{\diamond}\rangle Q_1 \subseteq_{\mathsf{h}} [f^{\diamond}\rangle Q_2$	$P_1 \subseteq_{\mathsf{h}} P_2$ implies $[f^{\diamond}\cdot\rangle P_1 \subseteq_{\mathsf{k}} [f^{\diamond}\cdot\rangle P_2$
$Q_1 \subseteq_{\mathsf{k}} Q_2$ implies $[f^{\sharp}\rangle Q_1 \subseteq_{\mathsf{h}} [f^{\sharp}\rangle Q_2$	$P_1 \subseteq_{\mathsf{h}} P_2$ implies $[f^{\sharp}\cdot\rangle P_1 \subseteq_{\mathsf{k}} [f^{\sharp}\cdot\rangle P_2$
$Q_1 \subseteq_{\mathsf{k}} Q_2$ implies $[f^{\flat}\rangle Q_1 \subseteq_{\mathsf{h}} [f^{\flat}\rangle Q_2$	$P_1 \subseteq_{\mathsf{h}} P_2$ implies $[f^{\flat}\cdot\rangle P_1 \subseteq_{\mathsf{k}} [f^{\flat}\cdot\rangle P_2$
$Q_1 \leq_{\mathsf{k}} Q_2$ implies $[f^{!}\rangle Q_2 \subseteq_{\mathsf{h}} [f^{!}\rangle Q_1$	$P_1 \subseteq_{\mathsf{h}} P_2$ implies $[f^{!}\cdot\rangle P_2 \subseteq_{\mathsf{k}} [f^{!}\cdot\rangle P_1$
$Q_1 \subseteq_{\mathsf{k}} Q_2$ implies $[f^{?}\rangle Q_2 \subseteq_{\mathsf{h}} [f^{?}\rangle Q_1$	$P_1 \subseteq_{\mathsf{h}} P_2$ implies $[f^{?}\cdot\rangle P_2 \subseteq_{\mathsf{k}} [f^{?}\cdot\rangle P_1$
$Q_1 \subseteq_{\mathsf{k}} Q_2$ implies $[f^{\perp}\rangle Q_2 \subseteq_{\mathsf{h}} [f^{\perp}\rangle Q_1$	$P_1 \subseteq_{\mathsf{h}} P_2$ implies $[f^{\perp}\cdot\rangle P_2 \subseteq_{\mathsf{k}} [f^{\perp}\cdot\rangle P_1$
$Q_1 \subseteq_{\mathsf{k}} Q_2$ implies $[f^{\star}\rangle Q_2 \subseteq_{\mathsf{h}} [f^{\star}\rangle Q_1$	$P_1 \subseteq_{\mathsf{h}} P_2$ implies $[f^{\star}\cdot\rangle P_2 \subseteq_{\mathsf{k}} [f^{\star}\cdot\rangle P_1$

Table 6.6

Proof: All of the backward operator proofs are similar to their forward operator siblings and we elide them.

$\underline{Q_1 \subseteq_{\mathsf{k}} Q_2 \text{ implies } [f^{\square}\rangle Q_1 \subseteq_{\mathsf{h}} [f^{\square}\rangle Q_2}$: the relevant definition is

$$x \in_{\mathsf{h}} [f^{\square}\rangle Q \text{ iff } \forall y(y \in_{\mathsf{k}} Q \text{ or } \neg\mathcal{F}^{\square}xy)$$

Assume $Q_1 \subseteq_{\mathsf{k}} Q_2$ and let $x \notin_{\mathsf{h}} [f^{\square}\rangle Q_2$. Hence there is some $y \notin_{\mathsf{k}} Q_2$ and $\mathcal{F}^{\square}xy$. From the assumption, $y \notin_{\mathsf{k}} Q_1$. By definition, $x \notin_{\mathsf{h}} [f^{\square}\rangle Q_1$. Thus $[f^{\square}\rangle Q_1 \subseteq_{\mathsf{h}} [f^{\square}\rangle Q_2$.

$\underline{Q_1 \subseteq_{\mathsf{k}} Q_2 \text{ implies } [f^{\diamond}\rangle Q_1 \leq_{\mathsf{h}} [f^{\diamond}\rangle Q_2}$: the relevant definition is

$$x \in_{\mathsf{h}} [f^{\diamond}\rangle Q \text{ iff } \exists y(y \in_{\mathsf{k}} Q \text{ and } \mathcal{F}^{\diamond}xy)$$

Assume $Q_1 \subseteq_{\mathsf{k}} Q_2$ and let $x \in_{\mathsf{h}} [f^{\diamond}\rangle Q_1$. By definition, there is some $y \in_{\mathsf{k}} Q_1$ and $\mathcal{F}^{\diamond}xy$. Since $Q_1 \subseteq_{\mathsf{k}} Q_2$, then $y \in_{\mathsf{k}} Q_2$. Thus$[f^{\diamond}\rangle Q_1 \subseteq_{\mathsf{h}} [f^{\diamond}\rangle Q_2$.

$\underline{Q_1 \subseteq_{\mathsf{k}} Q_2 \text{ implies } [f^{\sharp}\rangle Q_1 \leq_{\mathsf{h}} [f^{\sharp}\rangle Q_2}$: the relevant definition is

$$x \in_{\mathsf{h}} [f^{\sharp}\rangle Q \text{ iff } \exists y (y \in_{\mathsf{k}} Q \text{ and } \neg \mathcal{F}^{\sharp} xy)$$

Assume $Q_1 \subseteq_{\mathsf{k}} Q_2$ and let $x \in_{\mathsf{h}} [f^{\sharp}\rangle Q_1$. By definition, there is some $y \in_{\mathsf{k}} Q_1$ and $\neg \mathcal{F}^{\sharp} xy$. Since $Q_1 \subseteq_{\mathsf{k}} Q_2$, then $y \in_{\mathsf{k}} Q_2$. Thus $[f^{\sharp}\rangle Q_1 \leq_{\mathsf{h}} [f^{\sharp}\rangle Q_2$.

$\underline{Q_1 \subseteq_{\mathsf{k}} Q_2 \text{ implies } [f^{\flat}\rangle Q_1 \subseteq_{\mathsf{h}} [f^{\flat}\rangle Q_2}$: the relevant definition is

$$x \in_{\mathsf{h}} [f^{\flat}\rangle Q \text{ iff } \forall y (y \in_{\mathsf{k}} Q \text{ or } \mathcal{F}^{\flat} xy)$$

Assume $Q_1 \subseteq_{\mathsf{k}} Q_2$ and let $x \notin_{\mathsf{h}} [f^{\flat}\rangle Q_2$. Hence there is some $y \notin_{\mathsf{k}} Q_2$ and $\neg \mathcal{F}^{\flat} xy$. From the assumption, $y \notin_{\mathsf{k}} Q_1$. By definition, $x \notin_{\mathsf{h}} [f^{\flat}\rangle Q_1$. Thus $[f^{\flat}\rangle Q_1 \subseteq_{\mathsf{h}} [f^{\flat}\rangle Q_2$.

$\underline{Q_1 \leq_{\mathsf{k}} Q_2 \text{ implies } [f^{!}\rangle Q_2 \subseteq_{\mathsf{h}} [f^{!}\rangle Q_1}$: the relevant definition is

$$x \in_{\mathsf{h}} [f^{!}\rangle Q \text{ iff } \forall y (y \notin_{\mathsf{k}} Q \text{ or } \neg \mathcal{F}^{!} xy)$$

Assume $Q_1 \subseteq_{\mathsf{k}} Q_2$ and let $x \notin_{\mathsf{h}} [f^{!}\rangle Q_1$. Hence there is some $y \in_{\mathsf{k}} Q_1$ and $\mathcal{F}^{!} xy$. From the assumption, $y \in_{\mathsf{k}} Q_2$. By definition, $x \notin_{\mathsf{h}} [f^{!}\rangle Q_2$. Thus $[f^{!}\rangle Q_2 \subseteq_{\mathsf{h}} [f^{!}\rangle Q_1$.

$\underline{Q_1 \subseteq_{\mathsf{k}} Q_2 \text{ implies } [f^{?}\rangle Q_2 \subseteq_{\mathsf{h}} [f^{?}\rangle Q_1}$: the relevant definition is

$$x \in_{\mathsf{h}} [f^{?}\rangle Q \text{ iff } \exists y (y \notin_{\mathsf{k}} Q \text{ and } \mathcal{F}^{?} xy)$$

Assume $Q_1 \subseteq_{\mathsf{k}} Q_2$ and let $x \in_{\mathsf{h}} [f^{?}\rangle Q_2$. By definition, there is some $y \notin_{\mathsf{k}} Q_2$ and $\mathcal{F}^{?} xy$. Since $Q_1 \subseteq_{\mathsf{k}} Q_2$, then $y \notin_{\mathsf{k}} Q_1$. Thus $[f^{?}\rangle Q_2 \subseteq_{\mathsf{h}} [f^{?}\rangle Q_1$.

$\underline{Q_1 \subseteq_{\mathsf{k}} Q_2 \text{ implies } [f^{\perp}\rangle Q_2 \subseteq_{\mathsf{h}} [f^{\perp}\rangle Q_1}$: the relevant definition is

$$x \in_{\mathsf{h}} [f^{\perp}\rangle Q \text{ iff } \forall y (y \notin_{\mathsf{k}} Q \text{ or } \mathcal{F}^{\perp} xy)$$

Assume $Q_1 \subseteq_{\mathsf{k}} Q_2$ and let $x \notin_{\mathsf{h}} [f^{\perp}\rangle Q_1$. Hence there is some $y \in_{\mathsf{k}} Q_1$ and $\neg \mathcal{F}^{\perp} xy$. From the assumption, $y \in_{\mathsf{k}} Q_2$. By definition, $x \notin_{\mathsf{h}} [f^{\perp}\rangle Q_2$. Thus $[f^{\perp}\rangle Q_2 \subseteq_{\mathsf{h}} [f^{\perp}\rangle Q_1$.

$\underline{Q_1 \subseteq_{\mathsf{k}} Q_2 \text{ implies } [f^{\star}\rangle Q_2 \subseteq_{\mathsf{h}} [f^{\star}\rangle Q_1}$: the relevant definition is

$$x \in_{\mathsf{h}} [f^{\star}\rangle Q \text{ iff } \exists y (y \notin_{\mathsf{k}} Q \text{ and } \neg \mathcal{F}^{\star} xy)$$

Assume $Q_1 \subseteq_{\mathsf{k}} Q_2$ and let $x \in_{\mathsf{h}} [f^{\star}\rangle Q_2$. By definition, there is some $y \notin_{\mathsf{k}} Q_2$ and $\neg \mathcal{F}^{\star} xy$. Since $Q_1 \subseteq_{\mathsf{k}} Q_2$, then $y \notin_{\mathsf{k}} Q_1$. Thus $[f^{\star}\rangle Q_2 \subseteq_{\mathsf{h}} [f^{\star}\rangle Q_1$.

■

Theorem 6.2.4 *For set lattices, the tonicity properties and residuation implies the operator type as shown in Table 6.7:*

Type	***Property***	***Type***	***Property***
$\vee\mapsto\vee$	$[f^{\diamond}\cdot\rangle(P \vee Q) = [f^{\diamond}\cdot\rangle P \vee [f^{\diamond}\cdot\rangle Q$	$\vee\mapsto\vee$	$[f^{\diamond}\rangle(P \vee Q) = [f^{\diamond}\rangle P \vee [f^{\diamond}\rangle Q$
$\wedge\mapsto\wedge$	$[f^{\square}\rangle(P \wedge Q) = [f^{\square}\rangle P \wedge [f^{\square}\rangle Q$	$\wedge\mapsto\wedge$	$[f^{\square}\cdot\rangle(P \wedge Q) = [f^{\square}\cdot\rangle P \wedge [f^{\square}\cdot\rangle Q$
$\wedge\mapsto\wedge$	$[f^{\flat}\cdot\rangle(P \wedge Q) = [f^{\flat}\cdot\rangle P \wedge [f^{\flat}\cdot\rangle Q$	$\wedge\mapsto\wedge$	$[f^{\flat}\rangle(P \wedge Q) = [f^{\flat}\rangle P \wedge [f^{\flat}\rangle Q$
$\vee\mapsto\vee$	$[f^{\sharp}\rangle(P \vee Q) = [f^{\sharp}\rangle P \vee [f^{\sharp}\rangle Q$	$\vee\mapsto\vee$	$[f^{\sharp}\cdot\rangle(P \vee Q) = [f^{\sharp}\cdot\rangle P \vee [f^{\sharp}\cdot\rangle Q$
$\vee\mapsto\wedge$	$[f^{\perp}\cdot\rangle(P \vee Q) = [f^{\perp}\cdot\rangle P \wedge [f^{\perp}\cdot\rangle Q$	$\wedge\mapsto\vee$	$[f^{\star}\rangle(P \wedge Q) = [f^{\star}\rangle P \vee [f^{\star}\rangle Q$
$\vee\mapsto\wedge$	$[f^{\perp}\rangle(P \vee Q) = [f^{\perp}\rangle P \wedge [f^{\perp}\rangle Q$	$\wedge\mapsto\vee$	$[f^{\star}\cdot\rangle(P \wedge Q) = [f^{\star}\cdot\rangle P \vee [f^{\star}\cdot\rangle Q$
$\vee\mapsto\wedge$	$[f^{!}\cdot\rangle(P \vee Q) = [f^{!}\cdot\rangle P \wedge [f^{!}\cdot\rangle Q$	$\wedge\mapsto\vee$	$[f^{?}\rangle(P \wedge Q) = [f^{?}\rangle P \vee [f^{?}\rangle Q$
$\vee\mapsto\wedge$	$[f^{!}\rangle(P \vee Q) = [f^{!}\rangle P \wedge [f^{!}\rangle Q$	$\wedge\mapsto\vee$	$[f^{?}\cdot\rangle(P \wedge Q) = [f^{?}\cdot\rangle P \vee [f^{?}\cdot\rangle Q$

Table 6.7

Proof: The proof Theorem 6.1.3 is directly applicable here since there is no set theoretical nature to the algebraic proof. ■

7. GOLDBLATT-THOMASON THEOREM

Standing back a bit, it would appear all that is necessary is that the dual algebra of a frame is a modal algebra. Goldblatt could have stopped after his Theorem 1.6.2 because at point, he already knows that the frame is a modal frame in that it contains a modal algebra of sets. The tensor product of Boolean algebras is identified as the topological product of their dual Stone spaces [6] in the finite case and a somewhat more general topological product in the infinite case. Our Stone spaces, as frames, come with special points, i.e., $H^{\bullet}$ and H° as well as relations, and hence the topological product is a bit different. It may well be that $H_1 \otimes H_2$ as frames have their dual algebras $\mathbb{H}_1$ and $\mathbb{H}_2$ and these make $\mathbb{H}_1 \otimes \mathbb{H}_2$ a tensor product of algebras. The required commutative laws would need to be proven and that would take us outside the bounds of this paper.

7.1 Local Subframes

This section is nearly verbatim from [3] starting at page pp. 19. This we need to fix the notion of subframe and change the notation to serve our purposes in this paper.

Definition 7.1.1 If $\mathcal{H} \subseteq H^2$, then $M \subseteq_{\mathsf{h}} H$ is ***$\mathcal{H}$-hereditary*** iff $x \in_{\mathsf{h}} M$ and $\mathcal{H}xy$, then $y \in_{\mathsf{h}} M$.

This says that if one has $M \subseteq_{\mathsf{h}} H$, then starting at x and stepping to y via $\mathcal{H}$ requires that also that $y \in_{\mathsf{h}} M$.

Definition 7.1.2 If $\mathsf{h} = \langle H, \mathcal{H}, \mathbb{H}\rangle$ and $\mathsf{m} = \langle M, \mathcal{M}, \mathbb{M}\rangle$ are frames, then m is a ***subframe*** of h (written $\mathsf{m} \subseteq \mathsf{h}$) iff

(i) M is an $\mathcal{H}$-hereditary subset of H,

(ii) $\mathcal{M} = \mathcal{H} \cap (M \times M)$,

(iii) $\mathbb{M} = \{M \cap S : S \in \mathbb{H}\}$.

Theorem 7.1.3 *If M is an $\mathcal{H}$-hereditary subset of H, and $\mathcal{M} = \mathcal{H} \cap (M \times M)$, then*

(i) $M - (M \cap S) = M \cap (H - S)$,

(ii) $(M \cap S) \cap (M \cap Q) = M \cap (S \cap Q)$,

(iii) $[m^{\diamond}\rangle\,(M \cap S) = M \cap [h^{\diamond}\rangle\,S$,

(iv) $[m^{\Box}\rangle\,(M \cap S) = M \cap [h^{\Box}\rangle\,S$.

Proof:

(i) Let $x \in_{\mathsf{m}} M - (M \cap S)$, then $x \in_{\mathsf{m}} M$ and $x \notin_{\mathsf{m}} M \cap S$. Hence either $x \notin_{\mathsf{m}} M$ or $x \notin_{\mathsf{h}} S$. Since H is the ambient universe, $x \in_{\mathsf{h}} H$. Collecting our conditions $x \in_{\mathsf{m}} M$ and $x \in_{\mathsf{h}} H - S$, so $x \in_{\mathsf{h}} M \cap (H - S)$. To go the other way, let $x \in_{\mathsf{m}} M \cap (H - S)$. So $x \in_{\mathsf{m}} M$, $x \in_{\mathsf{h}} H$, and $x \notin_{\mathsf{h}} S$. Hence $x \notin_{\mathsf{m}} M \cap S$, and $x \in_{\mathsf{m}} M - (M \cap S)$.

(ii) This follows from set theory.

(iii) Let $x \in_{\mathsf{m}} [m^{\diamond}\rangle\,(M \cap S)$, then there is some y such that $y \in_{\mathsf{m}} M \cap S$ and $\mathcal{M}xy$. Hence $y \in_{\mathsf{m}} M$ and $y \in_{\mathsf{h}} S$. The universe for x is M, so $x \in_{\mathsf{m}} M$. Thus, $x, y \in_{\mathsf{h}} H$. By the assumption

$$\mathcal{M} \subseteq \mathcal{H} \cap (M \times M),$$

$x \in_{\mathsf{h}} [h^{\diamond}\rangle\,S$ and therefore $x \in_{\mathsf{m}} M \cap [h^{\diamond}\rangle\,S$.

To go in the other direction, let $x \in_{\mathsf{m}} M \cap [h^{\diamond}\rangle\,S$. There is then some y such that $y \in_{\mathsf{h}} S$ and $\mathcal{H}xy$. Since M is H hereditary and $x \in_{\mathsf{m}} M$, then $y \in_{\mathsf{m}} M$ and hence $y \in_{\mathsf{m}} M \cap S$ and $\langle x, y\rangle \in M \times M$. From the assumption

$$\mathcal{H} \cap (M \times M) \subseteq \mathcal{M},$$

$\mathcal{M}xy$ obtains. By definition, $x \in_{\mathsf{m}} [m^{\diamond}\rangle\,M \cap S$.

(iv) (iv) follows from (i), (iii), and the fact that $[m^{\Box}\rangle\,S = \neg[m^{\diamond}\rangle\,\neg S$ and $[h^{\Box}\rangle\,S = \neg[h^{\diamond}\rangle\,\neg S$.

■

Remark 7.1.4 *It is precisely the condition that $M \cap [h^{\diamond}\rangle\,S \subseteq [m^{\diamond}\rangle\,(M \cap S)$ that requires that M be H-hereditary. Namely this is because any y witnessing $x \in_{\mathsf{h}} [h^{\diamond}\rangle\,S$ under $\mathcal{H}xy$ could be anywhere including in $\neg M$. The hereditary condition and $x \in_{\mathsf{h}} M$ prevents this from occurring.*

Remark 7.1.5 *Regarding a frame* $\mathsf{h} = \langle H, \mathcal{H}, \mathbb{H}\rangle$*, there is nothing preventing* $\mathcal{H}$ *from being the identity relation. This allows frames with essentially no local Galois operators. Of course if there are local Galois operators, then their interpreting relations must be present in the frame and adjustments made to the Definitions, Lemmas, and Theorems of the sequel. It would simply require more machinery but of the form presented in the sequel.*

7.2 Local Heredity

The following Lemma is not in Goldblatt [3] but is new and although trivial, will set the stage for distributed heredity:

Lemma 7.2.1 *The commutative diagram of Diagram 7.1 where* $\mathcal{I}_M$ *is the inclusion relation of* M *into* H*, i.e.,* $\mathcal{I}_M$ *is the diagonal relation of* M *and so* $\mathcal{I}_M \subseteq M \times M$*, and the relations* $\mathcal{I}_M$*,* $\mathcal{M}$*, and* $\mathcal{H}$ *are considered as morphisms*

$$\begin{tikzcd}
M \arrow[r, "\mathcal{M}"] \arrow[d, tail, "\mathcal{I}_M"'] & M \arrow[d, tail, "\mathcal{I}_M"] \\
H \arrow[r, "\mathcal{H}"'] & H
\end{tikzcd}$$

Diagram 7.1: Local Subframe

is equivalent to $\mathcal{M} = \mathcal{H} \cap (M \times M)$ *and* M *being* $\mathcal{H}$*-hereditary. In particular,* $\mathcal{H} \circ \mathcal{I}_M \subseteq \mathcal{I}_M \circ \mathcal{M}$ *is equivalent to* $\mathcal{M} \supseteq \mathcal{H} \cap (M \times M)$ *and* M *being* $\mathcal{H}$*-hereditary, and* $\mathcal{I}_M \circ \mathcal{M} \subseteq \mathcal{H} \circ \mathcal{I}_M$ *is equivalent to* $\mathcal{M} \subseteq \mathcal{H} \cap (M \times M)$*.*

Proof: Note that $\mathcal{H} \circ \mathcal{I}_M$ the is morphism composition of relations which is defined as the relational composition $\mathcal{I}_M \cdot \mathcal{H}$, i.e., $\langle x, y\rangle \in \mathcal{I}_M \cdot \mathcal{H}$ iff $\exists z(\mathcal{I}_M xz$ and $\mathcal{H}zy)$. The composition $\mathcal{I}_M \circ \mathcal{H}$ is similar.

To show $\mathcal{H} \circ \mathcal{I}_M \subseteq \mathcal{I}_M \circ \mathcal{M}$ (from the diagram) implies $\mathcal{H}$-heredity, let $x \in_{\mathsf{m}} M$ and $\mathcal{H}xy$ Hence, $\langle x, y\rangle \in_{\mathsf{h}} \mathcal{H} \circ \mathcal{I}_M$. From the diagram, $\langle x, y\rangle \in_{\mathsf{m}} \mathcal{I}_M \circ \mathcal{M}$, and since $\mathcal{M} \subseteq M \times M$, then $y \in_{\mathsf{m}} M$. So M is $\mathcal{H}$-hereditary.

To show $\mathcal{H} \circ \mathcal{I}_M \subseteq \mathcal{I}_M \circ \mathcal{M}$ implies $\mathcal{H} \cap (M \times M) \subseteq \mathcal{M}$, let $\langle x, y\rangle \in_{\mathsf{m}} \mathcal{H} \cap (M \times M)$. So $\langle x, y\rangle \in_{\mathsf{h}} \mathcal{H}$ and $x, y \in_{\mathsf{m}} M$, and hence $\langle x, y\rangle \in_{\mathsf{h}} \mathcal{H} \circ \mathcal{I}_M$. Since the diagram commutes, $\langle x, y\rangle \in_{\mathsf{h}} \mathcal{I}_M \circ \mathcal{M}$. This latter is equivalent set theoretically to $\mathcal{M}$, so $\langle x, y\rangle \in_{\mathsf{m}} \mathcal{M}$ obtains and $\mathcal{M} \supseteq \mathcal{H} \cap (M \times M)$.

To show $\mathcal{I}_M \circ \mathcal{M} \subseteq \mathcal{H} \circ \mathcal{I}_M$ implies $\mathcal{M} \subseteq \mathcal{H} \cap (M \times M)$. The premise implies, set theoretically, that $\mathcal{M} \subseteq \mathcal{H}$. From this and the fact that $\mathcal{M} \subseteq M \times M$, it follows that $\mathcal{M} \subseteq \mathcal{H} \cap (M \times M)$.

To show that (M is $\mathcal{H}$-hereditary and $\mathcal{M} \supseteq \mathcal{H} \cap (M \times M)$) implies $\mathcal{H} \circ \mathcal{I}_M \subseteq \mathcal{I}_M \circ \mathcal{M}$, let $\langle x, y\rangle \in_{\mathsf{h}} \mathcal{H} \circ \mathcal{I}_M$. Hence $x \in_{\mathsf{m}} M$ and $\mathcal{H}xy$. From M being $\mathcal{H}$-hereditary, $y \in_{\mathsf{m}} M$. This implies $\langle x, y\rangle \in_{\mathsf{m}} \mathcal{H} \cap (M \times M)$ and so $\langle x, y\rangle \in_{\mathsf{m}} \mathcal{M}$. Since $\mathcal{M} = \mathcal{I}_M \circ \mathcal{M}$, we have that $\mathcal{H} \circ \mathcal{I}_M \subseteq \mathcal{I}_M \circ \mathcal{M}$.

To show $\mathcal{M} \subseteq \mathcal{H} \cap (M \times M)$ implies $\mathcal{I}_M \circ \mathcal{M} \subseteq \mathcal{H} \circ \mathcal{I}_M$, let $\langle x, y\rangle \in \mathcal{I}_M \circ \mathcal{M}$, so that $\langle x, y\rangle \in_{\mathsf{m}} \mathcal{M}$, and thus $\langle x, y\rangle \in \mathcal{H} \cap (M \times M)$. This implies $\langle x, y\rangle \in \mathcal{H}$ and $x \in_{\mathsf{m}} M$ and hence $\langle x, y\rangle \in \mathcal{H} \circ \mathcal{I}_M$.

■

The consequence is that a local subframe can now defined thus:

Definition 7.2.2 $\mathbb{M} = \langle M, \mathcal{M}, \mathbb{M}\rangle$ is a local subframe of $\mathcal{H} = \langle H, \mathcal{H}, \mathbb{H}\rangle$ if it satisfies the commutative conditions of Diagram 7.1.

Next, we need the notion of a local complement:

Definition 7.2.3 The complement of a local frame $\mathsf{h} = (H, \mathcal{H}, \mathbb{H})$, now denoted $\neg\mathsf{h}$ or $\neg(H, \mathcal{H}, \mathbb{H})$, is such that $\neg\mathcal{H} = (H \times H) - \mathcal{H}$.

Lemma 7.2.4 *If* m *be a subframe of* h, *then* $\neg\mathsf{m} \subseteq \neg\mathsf{h}$ *and* $\neg\mathcal{M} \supseteq \neg\mathcal{H} \cap (M \times M)$.

Proof: Let m be a subframe of h, and assume $\langle x, y\rangle \in \neg\mathcal{M}$. Note that $\langle x, y\rangle \in M \times M$. If $\langle x, y\rangle \in \mathcal{H}$, then $\langle x, y\rangle \in \mathcal{H} \circ \mathcal{I}_M$. From Diagram 7.1, $\langle x, y\rangle \in \mathcal{I}_M \circ \mathcal{M}$, and hence $\langle x, y\rangle \in \mathcal{M}$, which is a contradiction. So $\langle x, y\rangle \in \neg\mathcal{H}$ and $\neg\mathcal{M} \subseteq \neg\mathcal{H}$. Or one can use the weaker $\mathcal{M} \supseteq \mathcal{H} \cap (M \times M)$.

For the other condition, let $\langle x, y\rangle \in \neg\mathcal{H} \cap (M \times M)$ so that $\langle x, y\rangle \in \neg\mathcal{H}$. Towards a reductio, assume $\langle x, y\rangle \in \mathcal{M}$, therefore $\langle x, y\rangle \in \mathcal{I}_M \circ \mathcal{M}$. From Diagram 7.1, $\langle x, y\rangle \in \mathcal{H} \circ \mathcal{I}_M$ and thus, $\langle x, y\rangle \in \mathcal{H}$, which is a contradiction. So $\langle x, y\rangle \in \neg\mathcal{M}$. Or one can use that $\mathcal{M} \subseteq \mathcal{H}$. ■

It does not appear to be possible for M being H hereditary (or even satisfying the stronger conditions of Diagram 7.1) to imply $\neg M$ be $\neg H$ hereditary. This appears to also be unnecessary for closure under subframes. Still, the following Corollary obtains:

Corollary 7.2.5 *If the local frames* M *and* H *satisfy*

$$\mathcal{M} \subseteq \mathcal{H} \qquad \mathcal{H} \cap (M \times M) \subseteq \mathcal{M}$$

and $\neg M$ *is* $\neg H$*-hereditary, then* $\neg M$ *and* $\neg H$ *satisfy Diagram 7.1 with* $\neg\mathcal{H}$ *for* $\mathcal{H}$ *and* $\neg\mathcal{M}$ *for* $\mathcal{M}$.

We must augment the notion of subframe to handle well and null-points:

Definition 7.2.6 If $H = \langle H, \mathcal{H}, \mathbb{H}\rangle$ and $M = \langle M, \mathcal{M}, \mathbb{M}\rangle$ are frames, then M is a ***local pointed-subframe*** of H, again written $\mathsf{m} \subseteq \mathsf{h}$ iff

(i) The frames satisfy Diagram 7.1 for the local relations $\mathcal{H}$ and $\mathcal{M}$ as shown in Lemma 7.2.1.

(ii) $M^{\circ} = H^{\circ} \cap M \neq \emptyset$ and $M^{\bullet} = H^{\bullet} \cap M \neq \emptyset$.

(iii) $\mathbb{M} = \{M \cap S : S \in \mathbb{H}\}$.

It is not the case that the lattice $\mathbb{M}$ is a sublattice of the lattice $\mathbb{H}$. In the following theorem, we use $[{\scriptstyle -^{\star}}\rangle$ rather than Goldblatt's use of $[{\scriptstyle -^{\circ}}\rangle$ to test out at least one of new operators on pointed subframes.

Theorem 7.2.7 *Let* $(M, \neg\mathcal{M}^{\star}, \mathbb{M})$ *and* $(H, \neg\mathcal{H}^{\star}, \mathbb{H})$ *satisfy Diagram 7.1, then*

(i) $M - (M \cap P) = M \cap (H - P)$

(ii) $(M \cap S) \cap (M \cap P) = M \cap (S \cap P)$,

(iii) $[{\scriptstyle m^{\star}}\rangle\,(M \cap Q) = M \cap [{\scriptstyle h^{\star}}\rangle\,Q$ *and* $[{\scriptstyle m^{\star}\cdot}\rangle(M \cap P) = M \cap [{\scriptstyle h^{\star}\cdot}\rangle P$

Proof: Items (i) and (ii) follow directly from Theorem 7.1.3.

Let $x \in_{\mathsf{m}} [{\scriptstyle m^{\star}}\rangle\,(M \cap Q)$. By definition, $x \in_{\mathsf{m}} M$ and there is some y such that $y \notin_{\mathsf{m}} M \cap Q$ and $\neg\mathcal{M}^{\star}xy$. Hence either $y \notin_{\mathsf{m}} M$ or $y \notin_{\mathsf{h}} Q$. Since by definition, $\neg\mathcal{M} \subseteq M \times M$, then $y \in_{\mathsf{m}} M$, thus $y \notin_{\mathsf{m}} Q$. Since Diagram 7.1 commutes, $\neg\mathcal{M}^{\star} \subseteq \neg\mathcal{H}^{\star}$ and $y \in_{\mathsf{h}} H$. and so $\neg\mathcal{H}^{\star}xy$. Hence there is some y such that $y \notin_{\mathsf{h}} Q$ and $\neg\mathcal{H}^{\star}xy$. By definition, $x \in_{\mathsf{m}} M \cap [{\scriptstyle h^{\star}}\rangle\,Q$.

Next, let $x \in_{\mathsf{m}} M \cap [{\scriptstyle h^{\star}}\rangle\,Q$, so $x \in_{\mathsf{m}} M$ and $x \in_{\mathsf{h}} [{\scriptstyle h^{\star}}\rangle\,Q$. By definition, there is some y such that $y \notin_{\mathsf{h}} Q$ and $\neg\mathcal{H}^{\star}xy$. Since M is $\neg\mathcal{H}$-hereditary and $x \in_{\mathsf{m}} M$, then from Diagram 7.1, $y \in_{\mathsf{m}} M$ and $\neg\mathcal{M}^{\star}xy$. Since $y \notin_{\mathsf{h}} Q$, then $y \notin_{\mathsf{m}} M \cap Q$. By definition, $x \in_{\mathsf{m}} [{\scriptstyle m^{\star}}\rangle\,(M \cap Q)$.

■

7.3 Distributed and Pointed Subframes

We first define distributed heredity.

Definition 7.3.1 Let $g : \mathsf{m} \rightharpoonup \mathsf{n}$ and $f : \mathsf{h} \rightharpoonup \mathsf{k}$. If $\mathcal{F} \subseteq H \times K$, then $\langle M, N, \mathcal{G}\rangle \subseteq \langle H, K, \mathcal{F}\rangle$ is ***$\mathcal{F}$-hereditary*** when the following condition is satisfied:

$$\mathcal{F}xy \text{ and } x \in_{\mathsf{m}} M \text{ implies } y \in_{\mathsf{n}} N.$$

Lemma 7.3.2 *Let* $f : \mathsf{h} \rightharpoonup \mathsf{k}$. *The hereditary condition is equivalent to*

$$\mathcal{F} \cap (M \times K) \subseteq \mathcal{F} \cap (M \times N).$$

Proof: Assume the hereditary condition and let $\langle x, y\rangle \in \mathcal{F} \cap (M \times K)$. Hence $\langle x, y\rangle \in \mathcal{F}$, $x \in_m M$, and $y \in_k K$. From the hereditary condition, $y \in_n N$, thus $\langle x, y\rangle \in \mathcal{F} \cap (M \times N)$. To go in the other direction, assume $\mathcal{F} \cap (M \times K) \subseteq \mathcal{F} \cap (M \times N)$ and let $\langle x, y\rangle \in \mathcal{F}$ and $x \in_m M$. Hence $y \in_k K$ and $\langle x, y\rangle \in \mathcal{F} \cap (M \times K)$. Thus, $\langle x, y\rangle \in \mathcal{F} \cap (M \times N)$ and $y \in_n N$.

■

We can capture Goldblatt's hereditary condition and subset condition for $\mathcal{M}$ again in a commutative diagram. The diagram here is generalized over Diagram 7.1. Here we cannot rely upon an implicit relation $\mathcal{H}$ as in local heredity since the condition must span four localities.

Lemma 7.3.3 *The two conditions*

$$\mathcal{G} \subseteq \mathcal{F} \qquad \mathcal{F}xy \text{ and } x \in_m M \text{ implies } y \in_n N \text{ and } \mathcal{G}xy$$

are equivalent to the commutative diagram Diagram 7.2 of relations

$$\begin{tikzcd}
M \arrow[r, "\mathcal{G}"] \arrow[d, tail, "\mathcal{I}_M"'] & N \arrow[d, tail, "\mathcal{I}_N"] \\
H \arrow[r, "\mathcal{F}"'] & K
\end{tikzcd}$$

Diagram 7.2: Distributed Subframe

where $\mathcal{I}_M$ and $\mathcal{I}_N$ are inclusion maps as relations.

Proof: Assume the two conditions and let $x \in_m M$ and $\mathcal{F}xy$. From the second condition, $y \in_n N$ and $\mathcal{G}xy$. This shows $\mathcal{F} \circ \mathcal{I}_M \subseteq \mathcal{I}_N \circ \mathcal{G}$. Next, let $x \in_m M$ and $\mathcal{G}xy$. Since $\mathcal{G} \subseteq \mathcal{F}$, then $\mathcal{F}xy$ and $y \in_k K$. This shows that $\mathcal{I}_N \circ \mathcal{G} \subseteq \mathcal{F} \circ \mathcal{I}_M$.

Now assume the diagram commutes, and let $\mathcal{G}xy$ so that $x \in_m M$ and $y \in_n N$, then from set theory, $x \in_h H$ and $y \in_k K$. From the diagram, $\mathcal{F}xy$, so that $\mathcal{G} \subseteq \mathcal{F}$. Next, let $\mathcal{F}xy$ and $x \in_m M$, then $\langle x, y\rangle \in \mathcal{F} \circ \mathcal{I}_M$. From the diagram $(\mathcal{F} \circ \mathcal{I}_M) \subseteq (\mathcal{I}_N \circ \mathcal{G})$. Therefore $y \in_k \mathcal{I}_N \circ \mathcal{G}\{x\}$ and $y \in_n \mathcal{G}\{x\}$ (where $\mathcal{G}\{x\} = \{z \mid \mathcal{G}xz\}$), i.e., $\mathcal{G}xy$ and $y \in_n N$. ■

Lemma 7.3.4 *The commutative Diagram 7.2 is equivalent to $\mathcal{G} = \mathcal{F} \cap (M \times N)$ and (M, N) being (H, K)-hereditary. In particular, $\mathcal{F} \circ \mathcal{I}_M \subseteq \mathcal{I}_N \circ \mathcal{G}$ is equivalent to $\mathcal{G} \supseteq \mathcal{F} \cap (M \times N)$ and (M, N) being (H, K)-hereditary, and $\mathcal{I}_N \circ \mathcal{G} \subseteq \mathcal{F} \circ \mathcal{I}_M$ is equivalent to $\mathcal{G} \subseteq \mathcal{F} \cap (M \times N)$.*

Proof: To show $\mathcal{F} \circ \mathcal{I}_M \subseteq \mathcal{I}_N \circ \mathcal{G}$ implies $\mathcal{F} \cap (M \times N) \subseteq \mathcal{G}$, let $\langle x, y\rangle \in \mathcal{F} \cap (M \times N)$. So $\langle x, y\rangle \in \mathcal{F}$ and $x \in_m M, y \in_n N$, and hence $\langle x, y\rangle \in \mathcal{F} \circ \mathcal{I}_M$. Since the diagram commutes, $\langle x, y\rangle \in \mathcal{I}_N \circ \mathcal{G}$. This latter is equivalent set theoretically to $\mathcal{G}$, so $\langle x, y\rangle \in \mathcal{G}$ obtains and $\mathcal{G} \supseteq \mathcal{F} \cap (M \times N)$.

To show $\mathcal{F} \circ \mathcal{I}_M \subseteq \mathcal{I}_N \circ \mathcal{G}$ implies $(H, K, \mathcal{F})$-heredity, let $x \in_{\mathsf{m}} M$ and $\mathcal{F}xy$. Hence, $\langle x, y\rangle \in \mathcal{F} \circ \mathcal{I}_M$. From the diagram, $\langle x, y\rangle \in \mathcal{I}_N \circ \mathcal{G}$, and since $\mathcal{G} \subseteq M \times N$, then $y \in_{\mathsf{n}} N$. So $(M, N, \mathcal{G})$ is $(H, K, \mathcal{F})$-hereditary.

To show $(M, N, \mathcal{G})$ is $(H, K, \mathcal{F})$-hereditary and $\mathcal{G} \supseteq \mathcal{F} \cap (M \times N)$ together imply $\mathcal{F} \circ \mathcal{I}_M \subseteq \mathcal{I}_N \circ \mathcal{G}$, let $\langle x, y\rangle \in \mathcal{F} \circ \mathcal{I}_M$. Hence $x \in_{\mathsf{m}} M$ and $\langle x, y\rangle \in \mathcal{F}$. From $(M, N, \mathcal{G})$ being $(H, K, \mathcal{F})$-hereditary, $y \in_{\mathsf{n}} N$. This implies $\langle x, y\rangle \in \mathcal{F} \cap (M \times N)$ and so $\langle x, y\rangle \in \mathcal{G}$. Since $\mathcal{G} = \mathcal{I}_N \circ \mathcal{G}$, we have that $\mathcal{F} \circ \mathcal{I}_M \subseteq \mathcal{I}_N \circ \mathcal{G}$.

To show $\mathcal{I}_N \circ \mathcal{G} \subseteq \mathcal{F} \circ \mathcal{I}_M$ implies $\mathcal{G} \subseteq \mathcal{F} \cap (M \times N)$. The premise implies, set theoretically, that $\mathcal{G} \subseteq \mathcal{F}$. From this and the fact that $\mathcal{G} \subseteq M \times N$, it follows that $\mathcal{G} \subseteq \mathcal{F} \cap (M \times N)$.

To show $\mathcal{G} \subseteq \mathcal{F} \cap (M \times N)$ implies $\mathcal{I}_N \circ \mathcal{G} \subseteq \mathcal{F} \circ \mathcal{I}_M$, let $\langle x, y\rangle \in \mathcal{I}_N \circ \mathcal{G}$, so that $\langle x, y\rangle \in \mathcal{G}$, and thus $\langle x, y\rangle \in \mathcal{F} \cap (M \times N)$. This implies $\langle x, y\rangle \in \mathcal{F}$ and $x \in_{\mathsf{m}} M$ and hence $\langle x, y\rangle \in \mathcal{F} \circ \mathcal{I}_M$.

■

Note that the previous Lemma and its proof show that $\mathcal{G} \subseteq \mathcal{F}$ can immediately be read off the diagram by taking the top arrow $\mathcal{G}$ and seeing that is source and target are included in the source and target of $\mathcal{F}$, and by seeing that the diagram commutes.

The consequence of the previous Lemma is that a distributed subframe can now defined thus:

Definition 7.3.5 $\langle \mathsf{m}, \mathsf{n}, g : \mathsf{m} \rightharpoonup \mathsf{n}\rangle$ is a ***distributed subframe*** of $\langle \mathsf{h}, \mathsf{k}, f : \mathsf{h} \rightharpoonup \mathsf{k}\rangle$ if it satisfies the commutative conditions of Diagram 7.2.

Next, we need the notion of a local complement:

Definition 7.3.6 The complement of a distributed frame $\langle \mathsf{h}, \mathsf{k}, f : \mathsf{h} \rightharpoonup \mathsf{k}\rangle$, now denoted $\neg\langle \mathsf{h}, \mathsf{k}, f : \mathsf{h} \rightharpoonup \mathsf{k}\rangle$, is such that $\neg\mathcal{F} = (H \times K) - \mathcal{F}$.

Lemma 7.3.7 *If* $\langle \mathsf{m}, \mathsf{n}, g : \mathsf{m} \rightharpoonup \mathsf{n}\rangle$ *is a distributed subframe of* $\langle \mathsf{h}, \mathsf{k}, f : \mathsf{h} \rightharpoonup \mathsf{k}\rangle$ *, then* $\neg\mathcal{G} \subseteq \neg\mathcal{F}$ *and* $\neg\mathcal{G} \supseteq \neg\mathcal{F} \cap (M \times N)$*, where* $\neg\mathcal{F} = (H \times K) - \mathcal{F}$*.*

Proof: Let$\langle \mathsf{m}, \mathsf{n}, g : \mathsf{m} \rightharpoonup \mathsf{n}\rangle$ is a distributed subframe of $\langle \mathsf{h}, \mathsf{k}, f : \mathsf{h} \rightharpoonup \mathsf{k}\rangle$, and assume $\langle x, y\rangle \in \neg\mathcal{G}$. Note that $\langle x, y\rangle \in M \times N$. If $\langle x, y\rangle \in \mathcal{F}$, then $\langle x, y\rangle \in \mathcal{F} \circ \mathcal{I}_M$. From Diagram 7.2, $\langle x, y\rangle \in \mathcal{I}_N \circ \mathcal{G}$, and hence $\langle x, y\rangle \in \mathcal{G}$, which is a contradiction. So $\langle x, y\rangle \in \neg\mathcal{F}$ and $\neg\mathcal{G} \subseteq \neg\mathcal{F}$. Or one can use the weaker $\mathcal{G} \supseteq \mathcal{F} \cap (M \times N)$.

For the other condition, let $\langle x, y\rangle \in \neg\mathcal{F} \cap (M \times N)$. Towards a reductio, assume $\langle x, y\rangle \in \mathcal{G}$, therefore $\langle x, y\rangle \in \mathcal{I}_N \circ \mathcal{G}$. From Diagram 7.2, $\langle x, y\rangle \in \mathcal{F} \circ \mathcal{I}_M$ and thus, $\langle x, y\rangle \in \mathcal{F}$, which is a contradiction. So $\langle x, y\rangle \in \neg\mathcal{G}$. Or one can use that $\mathcal{G} \subseteq \mathcal{F}$. ■

It does not appear to be possible for $(M, N, \mathcal{G})$ being $(H, K, \mathcal{F})$ hereditary (or even satisfying the stronger conditions of Diagram 7.2) to imply $\neg(M, N, \mathcal{G})$ be $\neg(H, K, \mathcal{F})$ hereditary. This appears to also be unnecessary for closure under subframes. Still, the following Corollary obtains:

Corollary 7.3.8 *If the distributed frames $\langle \mathsf{m}, \mathsf{n}, g : \mathsf{m} \rightharpoonup \mathsf{n}\rangle$ and $\langle \mathsf{h}, \mathsf{k}, f : \mathsf{h} \rightharpoonup \mathsf{k}\rangle$ satisfy*

$$\mathcal{G} \subseteq \mathcal{F} \qquad \mathcal{F} \cap (M \times N) \subseteq \mathcal{G}$$

and $\neg(M, N, \mathcal{G})$ is $\neg(H, K, \mathcal{F})$-hereditary, then $\neg(M, N, \mathcal{G})$ and $\neg(H, K, \mathcal{F})$ satisfy Diagram 7.2 with $\neg\mathcal{F}$ for $\mathcal{F}$ and $\neg\mathcal{G}$ for $\mathcal{G}$.

We must augment the notion of subframe to handle well and null-points:

Definition 7.3.9 $\langle \mathsf{m}, \mathsf{n}, g : \mathsf{m} \rightharpoonup \mathsf{n}\rangle$ is a ***distributed pointed-subframe*** of $\langle \mathsf{h}, \mathsf{k}, f : \mathsf{h} \rightharpoonup \mathsf{k}\rangle$ iff

(i) The frames satisfy Diagram 7.2 for the distributed relations $\mathcal{F}$ and $\mathcal{G}$.

(ii) $M^{\circ} = H^{\circ} \cap M \neq \emptyset$, $M^{\bullet} = H^{\bullet} \cap M \neq \emptyset$, $N^{\circ} = K^{\circ} \cap N \neq \emptyset$, and $N^{\bullet} = K^{\bullet} \cap N \neq \emptyset$.

(iii) $\mathbb{M} = \{M \cap S : S \in \mathbb{H}\}$ and $\mathbb{N} = \{N \cap S : S \in \mathbb{K}\}$.

Lemma 7.3.10 *Let $\neg\langle \mathsf{m}, \mathsf{n}, g : \mathsf{m} \rightharpoonup \mathsf{n}\rangle$ and $\neg\langle \mathsf{h}, \mathsf{k}, f : \mathsf{h} \rightharpoonup \mathsf{k}\rangle$ satisfy Diagram 7.2 with the appropriate substitutions of $\neg\mathcal{F}$ and $\neg\mathcal{G}$ made for $\mathcal{F}$ and $\mathcal{G}$ respectively, then*

(i) $M - (M \cap S) = M \cap (H - S)$ *and* $N - (N \cap R) = N \cap (K - R)$

(ii) $(M \cap S) \cap (M \cap P) = M \cap (S \cap P)$ *and* $(N \cap R) \cap (N \cap Q) = N \cap (R \cap Q)$,

(iii) $[g^{\star}\rangle\,(N \cap Q) = M \cap [f^{\star}\rangle Q$ *and* $[g^{\star}\cdot\rangle(M \cap P) = N \cap [f^{\star}\cdot\rangle P$

Proof: Items (i) and (ii) follow directly from Theorem 7.1.3.

Let $x \in_{\mathsf{m}} [g^{\star}\rangle\,(N \cap Q)$. By definition, $x \in_{\mathsf{m}} M$ and there is some y such that $y \notin_{\mathsf{n}} N \cap Q$ and $\neg\mathcal{G}^{\star}xy$. Hence either $y \notin_{\mathsf{n}} N$ or $y \notin_{\mathsf{k}} Q$. Since by definition, $\neg\mathcal{G}^{\star} \subseteq M \times N$, then $y \in_{\mathsf{n}} N$, thus $y \notin_{\mathsf{k}} Q$. Since the Diagram 7.2 commutes, $\neg\mathcal{G} \subseteq \neg\mathcal{F}$ and $y \in_{\mathsf{k}} K$. and so $\neg\mathcal{F}^{\star}xy$. Hence there is some y such that $y \notin_{\mathsf{k}} Q$ and $\neg\mathcal{F}^{\star}xy$. By definition, $x \in_{\mathsf{m}} M \cap [f^{\star}\rangle Q$.

Next, let $x \in_{\mathsf{m}} M \cap [f^{\star}\rangle Q$, so $x \in_{\mathsf{m}} M$ and $x \in_{\mathsf{h}} [f^{\star}\rangle Q$. By definition, there is some y such that $y \notin_{\mathsf{k}} Q$ and $\neg\mathcal{F}^{\star}xy$. Since $\neg M$ is $\neg H$-hereditary and $x \in_{\mathsf{m}} M$, then from Diagram 7.2, $y \in_{\mathsf{m}} N$ and $\neg\mathcal{G}^{\star}xy$. Since $y \notin_{\mathsf{k}} Q$, then $y \notin_{\mathsf{n}} N \cap Q$. By definition, $x \in_{\mathsf{m}} [g^{\star}\rangle\,(N \cap Q)$.

■

8. APPENDIX 1

8.1 Disjoint Sum: Goldblatt [3]

We analyze the finite case of the disjoint sum construction for possibility. The infinite disjoint sums pose no difficulty. We are interested mainly in explicating Goldblatt's construction.

For the first part of this section, we will assume all domains are disjoint and hence will elide the injections $\iota\!\iota_i$. Also, Goldblatt's possibility operator is written $\boldsymbol{m}$. So we will switch to that while copying his work in this section.

Goldblatt also uses H to refer to a frame that we would put as $\mathsf{h} = (H, \mathcal{H}, \mathbb{H})$. For this section, we assume H can refer to either the entire frame or to the collection of frame points, allowing context to disambiguate use. His notation will be augmented to clear up remaining ambiguities.

First his main definition, and his notation had to be altered to protect the notation of this paper.

Definition 8.1.1 Let $\{H_i \mid i \in I\}$ be a collection of pairwise disjoint frames, i.e., $H_i \cap H_j = \emptyset$ for $i \neq j$. The ***disjoint union*** of the H_i's is the frame

$$H = \sum_{i \in I} H_i = (H_i, \mathcal{H}_i, \mathbb{H}_i),$$

where

(i) $H = \bigcup_{i \in I} H_i$,

(ii) $\mathcal{H} = \bigcup_{i \in I} \mathcal{H}_i$,

(iii) $\mathbb{H} = \{S \subseteq H : S \cap H_i \in \mathbb{H}_i, \quad \text{all } i \in I\}$.

Since the H_i's are disjoint, (iii) is equivalent to

(iv) $S \in \mathbb{H}$ iff there exists $S_i \in \mathbb{H}_i$, all $i \in I$, such that $S = \bigcup_{i \in I} S_i$.

Goldblatt's notation H^+ refers to the algebra of sets buried in the definition of a frame. We use $\mathbb{H}$ for this.

Theorem 8.1.2 *Let $S = \bigcup S_i$, $T = \bigcup T_i$, where $S_i, T_i \in \mathbb{H}_i$, all $i \in I$. Then*

(1) $H - S = \bigcup_{i \in I}(H_i - S_i)$,

(2) $S \cap T = \bigcup_{i \in I}(S_i \cap T_i)$,

(3) $\boldsymbol{m}_{\mathcal{R}}(S) = \bigcup_{i \in I}(\boldsymbol{m}_{\mathcal{R}_i}(S_i))$.

where $\mathcal{R}_i$ is the modal relation of the i-th frame and $\mathcal{R}$ is the union of all the $\mathcal{R}_i$.

The next theorem is out of the sequential order of Goldblatt.

Theorem 8.1.3 $(\Sigma H_i)^+$ *is isomorphic to the direct product of the family* $(H_i^+ : i \in I)$ *of MA's.*

In the proof, $\mathbb{H} = (H_i^+ : i \in I)$.

Proof: Define $\alpha : \mathbb{H} \rightarrowtail \prod_{i \in I} \mathbb{H}_i$, the Cartesian product of the $\mathbb{H}_i$'s, by

$$\alpha(S)(i) = S_i = S \cap W_i.$$

By Definition 8.1.1 (iii), we indeed have $\alpha(S) \in \prod \mathbb{H}_i$. By the uniqueness of the expression $S = \bigcup S_i$, α is readily shown to be a bijection. The MA-homomorphism (Modal Algebra-homomorphism) properties of α, with resect to the usual "point-wise" definition of operations on $\prod \mathbb{H}_i$, follow from 8.1.2, e.g.

$$\alpha(\boldsymbol{m}_{\mathcal{R}}(S))(i) = \boldsymbol{m}_{\mathcal{R}}(S) \cap H_i = \boldsymbol{m}_{\mathcal{R}_i}(S_i) = \boldsymbol{m}_{\mathcal{R}_i}(S_i) = \boldsymbol{m}_{\mathcal{R}_i}(\alpha(S)(i)) = \boldsymbol{m}(\alpha(S))(i),$$

so $\alpha(\boldsymbol{m}_{\mathcal{R}}(S)) = \boldsymbol{m}(\alpha(S))$.

■

Goldblatt writes $\langle Q_1, Q_2 \rangle$ for $Q_1 + Q_2$. The notation $\overline{Q_1}$ refers to the set theoretic complement of Q_1 taken in an ambient universe that is assumed in context.

Lemma 8.1.4 *If* $Q_1 + Q_2 \subseteq H_1 + H_2$ and $H_1 \cap H_2 = \emptyset$, *then*

$$\neg Q_1 = H_1 \cap \overline{Q_1} \qquad \neg Q_2 = H_2 \cap \overline{Q_2}.$$

Proof: Note that $a \in_{\mathsf{h}_1} \neg Q \overset{def}{=} H_1 - Q$. If $a \in_{\mathsf{h}_1} \neg Q$ then by definition $a \in_{\mathsf{h}_1} H_1 - Q$. This implies that $a \in_{h_1} H_1$ and $a \notin_{\mathsf{h}} Q$, whence $a \in_{\mathsf{h}} H_1 \cap \overline{Q}$.

To go the other direction, let $a \in_{\mathsf{h}} H_1 \cap \overline{Q}$, then $a \in_{h_1} H_1$ and $a \in_{\mathsf{h}} \overline{Q}$. So $a \in_{\mathsf{h}} H_1$ and $a \notin_{\mathsf{h}} Q$. Therefore, $a \in_{\mathsf{h}} H_1 - Q$, whence $a \in_{\mathsf{h}} \neg Q_1$. ■

Lemma 8.1.5 *If* $Q_1 + Q_2 \subseteq H_1 + H_2$ and $H_1 \cap H_2 = \emptyset$, *then*

$$\neg(Q_1 + Q_2) = \neg Q_1 + \neg Q_2.$$

Proof: Assume

$$Q_1 + Q_2 \subseteq H_1 + H_2 = \emptyset.$$

The injections are specified to make the proof clearer:

$$\begin{aligned}
\neg(Q_1 + Q_2) &= (\varkappa_1 H_1 \cup \varkappa_2 H_2) - (\varkappa_1 Q_1 \cup \varkappa_2 Q_2) && \text{def. } \neg \\
&= \varkappa_1 (H_1 - Q_1) \cup \varkappa_2 (H_2 - Q_2) && Q_1 \subseteq H_1, Q_2 \subseteq H_2, H_1 \cap H_2 = \emptyset \\
&= \varkappa_1 \neg Q_1 \cup \varkappa_2 \neg Q_2 && \text{def. } \neg \\
&= \neg Q_1 + \neg Q_2 && \text{def. } +
\end{aligned}$$

■

We will now show $[f^{\diamond}\rangle$ closure under finite disjoint sum. We will handle the distributed case. Just set the locality graph to a single node to capture the non-distributed case. For the rest of this section, we will switch to our notation. Also for rest of this section, we insert the injections to be absolutely clear all the conditions are properly met.

We assume

$$\mathcal{F}^{\diamond}ab \text{ iff } \mathcal{F}_1^{\diamond}ab \text{ or } \mathcal{F}_2^{\diamond}ab.$$

This better represented as

$$\mathcal{F}^{\diamond}_{\underline{\amalg}} = \amalg_1\mathcal{F}_1^{\diamond} \cup \amalg_2\mathcal{F}_2^{\diamond}$$

where $\amalg_i$ is an injection and (here) is applied pointwise to elements of its argument set, so that $\amalg_i\mathcal{F}_i^{\diamond} \subseteq (H_1 + H_2) \times (K_1 + K_2)$. We wish to prove

$$[f^{\diamond}\rangle\,(Q_1 + Q_2) = [f_1^{\diamond}\rangle\,Q_1 + \langle f_2\rangle Q_2.$$

However, there is a subtle shift in domains from $[f^{\diamond}\rangle\,(Q_1 + Q_2)$ to $[f_1^{\diamond}\rangle\,Q_1 + \langle f_2\rangle Q_2$ where the former uses $\mathcal{F}^{\diamond}_{\underline{\amalg}}$ but the latter seems to use

$$\mathcal{F}_1^{\diamond} \subseteq H_1 \times K_1 \qquad \mathcal{F}_2^{\diamond} \subseteq H_2 \times K_2$$

More technically, the situation is the following

$$\mathcal{F}^{\diamond}_{\underline{\amalg}} = \amalg_1\mathcal{F}_1^{\diamond} \cup \amalg_2\mathcal{F}_2^{\diamond} \subseteq (\amalg_1 H_1 \cup \amalg_2 H_2) \times (\amalg_1 K_1 \cup \amalg_2 K_2).$$

We need to define $[-_i^{\diamond}\rangle$ in the sum:

$$([\amalg_i f_i^{\diamond}\rangle)\,\amalg_i Q \overset{def}{=} \{x \in_{\mathsf{h}_i} \amalg_i H_i \mid \exists y(y \in_{\mathsf{k}_i} \amalg_i Q_i \text{ and } (\amalg_i\mathcal{F}_i^{\diamond})(\amalg_i x)(\amalg_i y))\}$$

i.., $[\amalg_i f_i^{\diamond}\rangle$ is the operator that uses $\amalg_i\mathcal{F}_i^{\diamond}$ in its evaluation in a frame. We can then prove

Technically, the h_i in $\in_{\mathsf{h}_i}$ should be $\amalg_i\mathsf{h}_i$ and similarly for k_i. However, at some point the notation becomes a hinderance and is best left implied.

Lemma 8.1.6

$$(\amalg_i[f_i^{\diamond}\rangle Q) = [\amalg_i f_i^{\diamond}\rangle\,\amalg_i Q.$$

where the left hand side first computes $[f_i^{\diamond}\rangle Q$ and then injects the result into H_i while the right hand side first injects the set Q into $\amalg_i K$ and computes using the relation $\amalg_i\mathcal{F}_i^{\diamond}$ where $\amalg_i\mathcal{F}_i^{\diamond} \subseteq \amalg_i H_i \times \amalg_i K_i$.

Proof: Let $\langle i, x\rangle \in_{\mathsf{h}_i} \amalg_i[f_i^{\diamond}\rangle Q$, then as $\amalg_i$ is an injection, $x \in_{\mathsf{h}_i} [f_i^{\diamond}\rangle Q$. So there is some y such that $y \in_{\mathsf{k}_i} Q$ and $\mathcal{F}_i^{\diamond}$. This gives us $\langle i, y\rangle \in_{\mathsf{k}_i} \amalg_i Q$ and $(\amalg_i\mathcal{F}_i^{\diamond})(\amalg_i x)(\amalg_i y)$. Thus $\langle i, x\rangle \in_{\mathsf{h}} [\amalg_i f_i^{\diamond}\rangle\,\amalg_i Q$.

To go in the other direction, let $\langle i,x\rangle \in_{\mathsf{h}_i} [\iota\!\iota_i f_i^\diamond\rangle\ \iota\!\iota_i Q$, then there is some $\langle i,y\rangle$ such that $\langle i,y\rangle \in_{\mathsf{k}_i} \iota\!\iota_i Q$ and $(\iota\!\iota_i \mathcal{F}_i^\diamond)(\iota\!\iota_i x)(\iota\!\iota_i y)$. Since $\iota\!\iota_i$ is an injection, $y \in_{\mathsf{k}_i}$ and $\mathcal{F}_i^\diamond xy$. So $x \in_{\mathsf{h}_i} [f_i^\diamond\rangle Q$ and $\langle i,x\rangle \in_{\mathsf{h}_i} \iota\!\iota_i [f_i^\diamond\rangle Q$.

This works because

$$\langle i,y\rangle \in_{\mathsf{k}_i} \iota\!\iota_i Q \text{ iff } y \in_{\mathsf{k}_i} Q \qquad \iota\!\iota_i \circ \mathcal{F}_i^\diamond = (\iota\!\iota_i \mathcal{F}_i^\diamond) \circ \iota\!\iota_i,$$

where the type of the first $\mathcal{F}_i^\diamond$ is $H_i \times K_i$ and the second is $(\iota\!\iota_1 H_1 \cup \iota\!\iota_2 H_2) \times (\iota\!\iota_1 K_1 \cup \iota\!\iota_2 K_2)$.

■

Diagram 8.1 shows the relations and injections (now in their relational form) as morphisms The middle two diagrams are saying the same things as the outer two diagrams but for the binary case. The two diagrams on the right uses relations as sets and functions as morphisms; it declares that $\mathcal{F}_{\iota\!\iota}^\diamond$ is a pushout, i.e., in the binary case, $\mathcal{F}_{\iota\!\iota}^\diamond = \iota\!\iota_1 \mathcal{F}_1^\diamond + \iota\!\iota_2 \mathcal{F}_2^\diamond$.

$$\begin{tikzcd}
\amalg_i H_i \arrow[d, "\amalg_i \mathcal{F}_i^\diamond"'] & H_i \arrow[l, "\iota\!\iota_i"'] \arrow[d, "\mathcal{F}_i^\diamond"] \\
\amalg_i K_i & K_i \arrow[l, "\iota\!\iota_i"]
\end{tikzcd}
\qquad
\begin{tikzcd}
H_1 \arrow[r, "\iota\!\iota_1"] \arrow[d, "\mathcal{F}_1^\diamond"'] & H_1 + H_2 \arrow[d, "\mathcal{F}_{\iota\!\iota}^\diamond"'] & H_2 \arrow[l, "\iota\!\iota_2"'] \arrow[d, "\mathcal{F}_2^\diamond"] \\
K_1 \arrow[r, "\iota\!\iota_1"'] & K_1 + K_2 & K_2 \arrow[l, "\iota\!\iota_2"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{F}_1^\diamond \arrow[r, "\iota\!\iota_1"] \arrow[dr, "f"'] & \mathcal{F}_{\iota\!\iota}^\diamond \arrow[d, dotted, "\langle f + g\rangle"] & \mathcal{F}_i^\diamond \arrow[l, "\iota\!\iota_2"'] \arrow[dl, "g"] \\
& \mathcal{G} &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{F}_{\iota\!\iota}^\diamond \arrow[d, dotted, "f_{\iota\!\iota}"'] & \mathcal{F}_2^\diamond \arrow[l, "\iota\!\iota_2"'] \arrow[dl, "g"] \\
\mathcal{G} &
\end{tikzcd}$$

Diagram 8.1: Commutativity of Sums

The consequence is that

Lemma 8.1.7

$$[f_{\iota\!\iota}^\diamond\rangle\,(\iota\!\iota_1 Q_1 + \iota\!\iota_2 Q_2) = ([\iota\!\iota_1 f_1^\diamond\rangle)\,\iota\!\iota_1 Q_1 + ([\iota\!\iota_2 f_2^\diamond\rangle)\,\iota\!\iota_2 Q_2$$

Proof:

1	$a \in_h [f^\diamond_{\varkappa}\rangle\, (Q_1 + Q_2)$	assume
2	$\exists y(y \in_k Q_1 + Q_2$ and $\mathcal{F}^\diamond ay)$	def. $[f^\diamond\rangle$, line 1
3	$\boxed{b}$ $b \in_k Q_1 + Q_2$ and $\mathcal{F}^\diamond ab$	assume
4	$b \in_k Q_1 + Q_2$	$\wedge$-Elim, line 3
5	$(\varkappa_1 \mathcal{F}_1^\diamond)ab$ or $(\varkappa_2 \mathcal{F}_2^\diamond)ab$	def. $\mathcal{F}$, line 3
6	$b \in_{k_1} \varkappa_1 Q_1$	assume
7	$b \in_{k_1} \varkappa_1 K_1$	set th., line 6
8	$b \notin_{k_2} \varkappa_2 K_2$	def. disjoint frames, line 7
9	$(\neg\, \varkappa_2 \mathcal{F}_2^\diamond)ab$	def. $\neg(\varkappa_2 \mathcal{F}_2^\diamond)$, disjoint frames, line 8
10	$(\varkappa_1 \mathcal{F}_1^\diamond)ab$	CL, lines 5, 9
11	$b \in_{k_1} \varkappa_1 Q_1$ and $(\varkappa_1 \mathcal{F}_1^\diamond)ab$	$\wedge$-Intro, lines 6, 10
12	$a \in_{h_1} [\varkappa_1 f_1^\diamond\rangle\, \varkappa_1 Q_1$	def $[\varkappa_1 f_1^\diamond\rangle$, line 11
13	$a \in_h [\varkappa_1 f_1^\diamond\rangle\, \varkappa_1 Q_1 + [\varkappa_2 f_2^\diamond\rangle\, \varkappa_2 Q_2$	$\vee$-Intro, line 12
14	$b \in_{k_2} \varkappa_2 Q_2$	assume
15	$b \in_{k_2} \varkappa_2 K_2$	set th., line 14
16	$b \notin_{h_1} \varkappa_1 K_1$	def. disjoint frames, line 15
17	$\neg(\varkappa_1 \mathcal{F}_1^\diamond)ab$	def. $\neg(\varkappa_1 \mathcal{F}_1^\diamond)$, disjoint frames, line 16
18	$(\varkappa_2 \mathcal{F}_2^\diamond)ab$	CL, lines 5, 17
19	$b \in_{k_2} \varkappa_2 Q_2$ and $(\varkappa_2 \mathcal{F}_2^\diamond)ab$	$\wedge$-Intro, lines 14, 18
20	$a \in_{h_2} [\varkappa_2 f_2^\diamond\rangle\, \varkappa_2 Q_2$	def $[\varkappa_2 f_2^\diamond\rangle$, line 19
21	$a \in_h [\varkappa_1 f_1^\diamond\rangle\, \varkappa_1 Q_1 + [\varkappa_2 f_2^\diamond\rangle\, \varkappa_2 Q_2$	$\vee$-Intro, line 20
22	$a \in_h [\varkappa_1 f_1^\diamond\rangle\, \varkappa_1 Q_1 + [\varkappa_2 f_2^\diamond\rangle\, \varkappa_2 Q_2$	$\vee$-Elim, lines 4, 5, 14
23	$a \in_h [\varkappa_1 f_1^\diamond\rangle\, \varkappa_1 Q_1 + [\varkappa_2 f_2^\diamond\rangle\, \varkappa_2 Q_2$	$\exists$-Elim, lines 2, 3

and

1	$a \in_{\mathsf{h}} [\varkappa_1 f_1^{\diamond}\rangle\ \varkappa_1 Q_1 + [\varkappa_2 f_2^{\diamond}\rangle\ \varkappa_2 Q_2$	assume
2	$a \in_{\mathsf{h}_1} [\varkappa_1 f_1^{\diamond}\rangle\ \varkappa_1 Q_1$	assume
3	$\exists y(y \in_{\mathsf{k}_1} \varkappa_1 Q_1$ and $(\varkappa_1 \mathcal{F}_1^{\diamond})ay)$	def. $[-^{\diamond}\rangle$, line 2
4	$\boxed{b}\quad b \in_{\mathsf{k}_1} \varkappa_1 Q_1$ and $(\varkappa_1 \mathcal{F}_1^{\diamond})ab$	assume
5	$b \in Q_1 + Q_2$ and $((\varkappa_1 \mathcal{F}_1^{\diamond})ab$ or $(\varkappa_2 \mathcal{F}_2^{\diamond})ab)$	set th., $\vee$-Intro, line 4
6	$b \in_{\mathsf{k}} Q_1 + Q_2$ and $\mathcal{F}_{\varkappa}^{\diamond}ab$	def. $\mathcal{F}_{\varkappa}^{\diamond}$, line 5
7	$\exists y(y \in_{\mathsf{k}} Q_1 + Q_2$ and $\mathcal{F}_{\varkappa}^{\diamond}ay)$	def. $\exists$-Intro, line 6
8	$a \in_{\mathsf{h}} [f_{\varkappa}^{\diamond}\rangle\,(Q_1 + Q_2)$	def. $[-^{\diamond}\rangle$, line 7
9	$a \in_{\mathsf{h}} [f_{\varkappa}^{\diamond}\rangle\,(Q_1 + Q_2)$	$\exists$-Elim, lines 3, 4
10	$a \in_{\mathsf{h}_2} [\varkappa_2 f_2^{\diamond}\rangle\ \varkappa_2 Q_2$	assume
11	$\exists v(v \in_{\mathsf{k}_2} \varkappa_2 Q_2$ and $(\varkappa_2 \mathcal{F}_2^{\diamond})av)$	def. $[-^{\diamond}\rangle$, line 10
12	$\boxed{b}\quad b \in_{\mathsf{k}_2} \varkappa_2 Q_2$ and $(\varkappa_2 \mathcal{F}_2^{\diamond})ab$	assume
13	$b \in_{\mathsf{k}} Q_1 + Q_2$ and $((\varkappa_1 \mathcal{F}_1^{\diamond})ab$ or $(\varkappa_2 \mathcal{F}_2^{\diamond})ab)$	set th., $\vee$-Intro, line 11
14	$b \in_{\mathsf{k}} Q_1 + Q_2$ and $\mathcal{F}_{\varkappa}^{\diamond}ab$	def. $\mathcal{F}_{\varkappa}^{\diamond}$, line 13
15	$\exists y(y \in_{\mathsf{k}} Q_1 + Q_2$ and $\mathcal{F}_{\varkappa}^{\diamond}ay)$	$\exists$-Intro, line 14
16	$a \in_{\mathsf{h}} [f_{\varkappa}^{\diamond}\rangle\,(Q_1 + Q_2)$	def. $[-^{\diamond}\rangle$, line 15
17	$a \in_{\mathsf{h}} [f_{\varkappa}^{\diamond}\rangle\,(Q_1 + Q_2)$	$\exists$-Elim, lines 11, 12
18	$a \in [f_{\varkappa}^{\diamond}\rangle\,(Q_1 + Q_2)$	$\vee$-Elim, lines 1, 2, 10

The proofs for $[f^{\diamond}\cdot\rangle$ are similar.

■

8.2 Counter-Example $[f^{\perp}\rangle$: Fails to Distribute Across Disjoint Sum

We assume non-pointed domains to show failure in the traditional framework of modal logic expanded to distributed domains. The same example could also falsify the formulas used if there was only a single domain, i.e., the traditional modal logic case. The caveat here is that there is no way to expand the quantification of $\neg\mathcal{F}_1^{\perp}$ from $H_1 \times K_1$ to $H_1 \times (K_1 \cup K_2)$ and $\neg\mathcal{F}_2^{\perp}$ from $H_2 \times K_2$ to $(H_1 \cup H_2) \times K_2$.

We assume

$$\mathcal{F}^{\perp}ab \text{ iff } \mathcal{F}_1^{\perp}ab \text{ or } \mathcal{F}_2^{\perp}ab.$$

Recall that

$$x \in_{\mathsf{h}} [f^{\perp}\rangle\,Q \text{ iff } \forall y(y \notin_{\mathsf{k}} Q \text{ or } \mathcal{F}^{\perp}xy)$$

and hence

$$x \notin_{\mathsf{h}} [f^{\perp}\rangle\,Q \text{ iff } \exists y(y \in_{\mathsf{k}} Q \text{ and } \neg\mathcal{F}^{\perp}xy)$$

Lemma 8.2.1

$$[f_1^{\perp}\rangle\, Q_1 + [f_2^{\perp}\rangle\, Q_2 \not\subseteq [f^{\perp}\rangle\, (Q_1 + Q_2).$$

Proof: We will construct a distributed frame that satisfies $[f_1^{\perp}\rangle\, Q_1 + [f_2^{\perp}\rangle\, Q_2$ but fails to satisfy $[f^{\perp}\rangle\, (Q_1 + Q_2)$.

We will assume all domains are disjoint so we need not be concerned with tags. Assume $K_1 = Q_1 = \{b_1\}$, $K_2 = Q_2 = \{b_2\}$, and $\mathcal{F}_1^{\perp} a b_1$. Note that $a \in_{h_1} H_1$ from $\mathcal{F}_1^{\perp} a b_1$. Therefore $a \in_{h_1} [f_1^{\perp}\rangle\, Q_1$, i.e.,

$$a \in_{h_1} [f_1^{\perp}\rangle\, Q_1 \text{ iff } \forall y \in_{k_1} K_1 (y \notin_{k_1} Q_1 \text{ or } \mathcal{F}_1^{\perp} a y).$$

Therefore

$$a \in_{h} [f_1^{\perp}\rangle\, Q_1 + [f_2^{\perp}\rangle\, Q_2.$$

Note that $\neg\mathcal{F}_2^{\perp} a b_2$ since $a \in_{h_1} H_1$ and $\mathcal{F}_2^{\perp} \subseteq H_2 \times K_2$. Hence $Q_1 + Q_2 = \{b_1\} + \{b_2\}$. We will fail to satisfy the following formula because the quantification spans $Q_1 + Q_2$:

$$a \in_{h} [f^{\perp}\rangle\, (Q_1 + Q_2) \text{ iff } \forall y \in_{k} K_1 \cup K_2 \big(y \notin_{k} Q_1 + Q_2 \text{ or } (\mathcal{F}_1^{\perp} a y \text{ or } \mathcal{F}_2^{\perp} a y)\big).$$

That is, we will satisfy

$$a \notin_{h} [f^{\perp}\rangle\, (Q_1 + Q_2) \text{ iff } \exists y \in_{k} K_1 \cup K_2 \big(y \in_{k} Q_1 + Q_2 \text{ and } (\neg\mathcal{F}_1^{\perp} a y \text{ and } \neg\mathcal{F}_2^{\perp} a y)\big).$$

Now we evaluate $a \in_{h} [f^{\perp}\rangle\, (Q_1 + Q_2)$. Let $y \in_{k_2} Q_2$. There is only one point, $y = b_2$, thus $\neg\mathcal{F}_2^{\perp} a b_2$ since $\mathcal{F}_2^{\perp} \subseteq H_2 \times K_2$ and $a \in_{h_1} H_1$ so that $a \notin_{h_2} H_2$. It is also the case that $\neg\mathcal{F}_1^{\perp} a b_2$ since $b_2 \notin_{k_1} K_1$ and $\mathcal{F}_1^{\perp} \subseteq H_1 \times K_1$. So we have satisfied

$$b_2 \in_{k} Q_1 + Q_2 \text{ and } \neg\mathcal{F}_1^{\perp} a b_2 \text{ and } \neg\mathcal{F}_2^{\perp} a b_2,$$

i.e.,

$$\exists y (y \in_{k} Q_1 + Q_2 \text{ and } \neg\mathcal{F}_1^{\perp} a y \text{ and } \neg\mathcal{F}_2^{\perp} a y)$$

Hence $a \notin_{h} [f^{\perp}\rangle\, (Q_1 + Q_2)$.

■

We can see the failure a bit more abstractly as the failure of the following proof at line 14. A bit more deeply, line 8 does not appear to make sense because $\mathcal{F}_2^{\perp} a b_1$ does not make sense. The reason is that $a \in_{h_1} H_1$ and $b_1 \in_{k_1} Q_1$ and $\mathcal{F}_2^{\perp} \subseteq H_2 \times K_2$. There is also a problem with line 9 where mysteriously we can expand the domain for $\neg\mathcal{F}_2^{\perp}$ from $H_2 \times K_2$ to $(H_1 \cup H_2) \times K_2$.

In the other direction, it appears we have a failed proof. Suppose $a \in_{h_1} [f_1^{\perp}\rangle Q_1$. The evaluation condition is, where I have put in the explicit quantification domain:

$$\forall y \in_{k_1} K_1 (y \in_{k_1} Q_1 \text{ implies } \mathcal{F}_1^{\perp} ay).$$

This is equivalent to

$$\forall y \in_{k_1} K_1 (y \notin_{k_1} Q_1 \text{ or } \mathcal{F}_1^{\perp} ay).$$

There is no way to expand the quantification from K_1 to $K_1 \cup K_2$. Yet, we are asked to accept that $\neg\mathcal{F}_2^{\perp} \subseteq (H_1 \cup H_2) \times (K_1 \cup K_2)$ even though all we have is $\mathcal{F}_2^{\perp} \subseteq H_2 \times K_2$.

This is no problem with the existentials because if there is some $y \in_{k_1} K_1$, then there is some $y \in_{k} K_1 \cup K_2$.

Proof:

1	$\vert\ \underline{a \in_{h} [f^{\perp}\rangle (Q_1 + Q_2)}$	assume
2	$\vert\ \forall y \in_{k} K_1 \cup K_2 \big(y \in_{k} Q_1 + Q_2 \text{ implies } (\mathcal{F}_1^{\perp} ay \text{ or } \mathcal{F}_2^{\perp} ay)\big)$	def. $[-^{\perp}\rangle$, line 1
3	$\vert\ a \in_{h_1} H_1 \text{ or } a \in_{h_2} H_2$	set th., $H_1 \cup H_2$ is the universe, line 1
4	$\vert\ \vert\ \underline{a \in_{h_1} H_1}$	assume
5	$\vert\ \vert\ \vert\ \underline{\boxed{b_1}\ \ b_1 \in_{k_1} K_1 \text{ and } b_1 \in_{k_1} Q_1}$	assume
6	$\vert\ \vert\ \vert\ b_1 \in_{k} Q_1 + Q_2$	set th., line 5
7	$\vert\ \vert\ \vert\ \mathcal{F}_1^{\perp} ab_1 \text{ or } \mathcal{F}_2^{\perp} ab_1$	$\forall$-Elim, $\supset$-Elim, lines 4, 6
8	$\vert\ \vert\ \vert\ \neg\mathcal{F}_2^{\perp} ab_1$	$\mathcal{F}_2^{\perp} \subseteq H_2 \times K_2$, line 5
9	$\vert\ \vert\ \vert\ \mathcal{F}_1^{\perp} ab_1$	CL, lines 7, 8
10	$\vert\ \vert\ \forall y \in_{k_1} K_1 (y \in_{k_1} Q_1 \text{ implies } \mathcal{F}_1^{\perp} ay)$	$\forall$-Intro, line 5
11	$\vert\ \vert\ a \in_{h_1} [f_1^{\perp}\rangle Q_1$	def. $[-^{\perp}\rangle$, line 10
12	$\vert\ \vert\ a \in_{h} [f_1^{\perp}\rangle Q_1 + [f_2^{\perp}\rangle Q_2$	def. +, line 11
13	$\vert\ \vert\ \underline{a \in_{h_2} H_2}$	assume
14	$\vert\ \vert$ blah	doh, line xx
15	$\vert\ \vert\ a \in_{h} [f_1^{\perp}\rangle Q_1 + [f_2^{\perp}\rangle Q_2$	def. +, line xx
16	$\vert\ a \in_{h} [f_1^{\perp}\rangle Q_1 + [f_2^{\perp}\rangle Q_2$	$\vee$-Elim, lines 3, 4, 13

and the failure at line 11 because there is no way to proceed past 8:

1	$a \in_{h} [f_1^{\perp}\rangle Q_1 + [f_2^{\perp}\rangle Q_2$	assume
2	$a \in_{h_1} [f_1^{\perp}\rangle Q_1$ or $a \in_{h_2} [f_2^{\perp}\rangle Q_2$	def. +, line 1
3	│ $a \in_{h_1} [f_1^{\perp}\rangle Q_1$	assume
4	│ $\forall y(y \in_{k_1} Q_1$ implies $\mathcal{F}_1^{\perp} ay)$	def. $[-^{\perp}\rangle$, line 3
5	│ │ $\boxed{b_1}$ $b \in_{k_1} Q_1 + Q_2$	assume
6	│ │ $b \in_{k_1} Q_1$ or $b \in_{k_2} Q_2$	def. +, line 5
7	│ │ │ $b \in_{k_2} Q_2$	assume
8	│ │ │ failure	line xx
9	│ │ │ $\mathcal{F}_1^{\perp} ab$ or $\mathcal{F}_2^{\perp} ab$	doh, lines xx
10	│ │ │ $\neg\mathcal{F}_1^{\perp} ab$ and $\neg\mathcal{F}_2^{\perp} ab$	$a \in_{h_1} H_1$, $b \in_{k_2} K_2$, lines 3, 7
11	│ ·	doh, line xx
12	│ $a \in_{h_2} [f_2^{\perp}\rangle Q_2$	assume
13	│ ·	doh, line xx
14	$a \in_{h} [f^{\perp}\rangle (Q_1 + Q_2)$	$\vee$-Elim, lines 3, 4, 12

■

8.3 Counter-Example $[f^{\star}\rangle$: Fails to Distribute Across Disjoint Sum

The same problem of needing enlarged domains arises here. We assume

$$\mathcal{F}^{\star} ab \text{ iff } \mathcal{F}_1^{\star} ab \text{ or } \mathcal{F}_2^{\star} ab.$$

Lemma 8.3.1

$$[f^{\star}\rangle (Q_1 + Q_2) = [f_1^{\star}\rangle Q_1 + [f_2^{\star}\rangle Q_2.$$

The problem with the proof below is at line 10 where $b \in_{k_1} K_1$ and $a \in_{h_2} H_2$. Hence $\neg\mathcal{R}_1^{\star} ab_1$ obtains and so $a \in_{h_1} [f_1^{\star}\rangle Q_1$ which is impossible. All domains and the relations $\mathcal{F}_1^{\star}$ and $\mathcal{F}_2^{\star}$ are considered disjoint so we need not be concerned with tagging.

Proof: We first ahow

$$[f^{\star}\rangle (Q_1 + Q_2) \subseteq [f_1^{\star}\rangle Q_1 + [f_2^{\star}\rangle Q_2.$$

Notably, the proof below uses the fact that $K_1 \cup K_2$ is the universe. The notation $\neg\mathcal{F}^{\star}$ will refer to $\neg(\mathcal{F}_1^{\star} \cup \mathcal{F}_2^{\star})$. Note that

$$\mathcal{F}^{\star} = \mathcal{F}_1^{\star} \cup \mathcal{F}_2^{\star}.$$

Consequently,

$$\mathcal{F}_1^\star \subseteq \mathcal{F}^\star \qquad \mathcal{F}^\star \subseteq \neg\mathcal{F}_1^\star.$$

This underwrites the step at 10.

1	$a \in_{\mathsf{h}} [f^\star\rangle (Q_1 + Q_2)$	assume
2	$\exists y(y \notin_{\mathsf{k}} Q_1 + Q_2$ and $(\neg\mathcal{F}^\star)ay)$	def. $[-^\star\rangle$, line 1
3	$a \in_{\mathsf{h}_1} H_1$ or $a \in_{\mathsf{h}_2} H_2$	$H_1 + H_2$ is the universe, line 1
4	$\boxed{b}$ $b \notin_{\mathsf{k}} Q_1 + Q_2$ and $(\neg\mathcal{F}^\star)ab$	assume
5	$b \notin_{\mathsf{k}_1} Q_1$ and $b \notin_{\mathsf{k}_2} Q_2$	def. +, line 4
6	$b \in_{\mathsf{k}_1} K_1$ or $b \in_{\mathsf{k}_2} K_2$	$K_1 \cup K_2$ is the universe, line 4
7	$b \in_{\mathsf{k}_1} K_1$	assume
8	$b \in_{\mathsf{k}_1} K_1 - Q_1$	set th., lines 4, 7
9	$a \in_{\mathsf{h}_2} H_2$	assume
10	$b \in_{\mathsf{k}_1} K_1 - Q_1$ and $\neg\mathcal{F}_1^\star ab$	$\wedge$-Elim, $\wedge$-Intro, $\neg\mathcal{F}^\star \subseteq \neg\mathcal{F}_1^\star$, lines 4, 8
11	$\exists y(y \notin_{\mathsf{k}_1} Q_1$ and $\neg\mathcal{F}_1^\star ay)$	$\exists$-Intro, line 10
12	$a \in_{\mathsf{h}_1} [f_1^\star\rangle Q_1$	def. $[-_1^\star\rangle$, line 11
13	$a \in_{\mathsf{h}_1} H_1$	$[-_1^\star\rangle$, line 12
14	$\rightarrow\!\leftarrow$	Contradiction, lines 9, 13
15	failure	lines xx
16	$a \in_{\mathsf{h}} [f_1^\star\rangle Q_1 + [f_2^\star\rangle Q_2$	$\exists$-Elim, lines 2, 3

Let us try the other direction,

$$[f_1^\star\rangle Q_1 + [f_2^\star\rangle Q_2 \subseteq [f^\star\rangle (Q_1 + Q_2).$$

This proof goes through:

1	$\mid\ \underline{a \in_{h} [f_1^{\star}\rangle Q_1 + [f_2^{\star}\rangle Q_2}$	assume
2	$\mid\ a \in_{h} [f_1^{\star}\rangle Q_1$ or $a \in_{h} [f_2^{\star}\rangle Q_2$	def. +, line 1
3	$\mid\ \mid\ \underline{a \in_{h_1} [f_1^{\star}\rangle Q_1}$	assume
4	$\mid\ \mid\ \exists y(y \notin_{k_1} Q_1$ and $\neg\mathcal{F}_1^{\star} a y)$	def. $[-^{\star}\rangle$-Elim, line 3
5	$\mid\ \mid\ \mid\ \underline{\boxed{b}\quad b \in_{k_1} K_1 - Q_1 \text{ and } \neg\mathcal{F}_1^{\star} a b}$	assume
6	$\mid\ \mid\ \mid\ b \in_{k_1} K_1$ and $b \notin_{k_1} Q_1$	set th., line 5
7	$\mid\ \mid\ \mid\ b \notin_{k_2} K_2$	set th., $K_1 \cap K_2 = \emptyset$, line 6
8	$\mid\ \mid\ \mid\ b \notin_{k_2} Q_2$	$Q_2 \subseteq K_2$, line 7
9	$\mid\ \mid\ \mid\ b \in_{k} K_1 \cup K_2$	set th., line 6
10	$\mid\ \mid\ \mid\ b \notin_{k} Q_1 + Q_2$	set th., lines 6, 8
11	$\mid\ \mid\ \mid\ b \notin_{k} Q_1 + Q_2$ and $\neg\mathcal{F}^{\star} a b$	$\wedge$-Elim, $\wedge$-Intro, lines 5, 10
12	$\mid\ \mid\ \mid\ \exists y(y \notin_{k} Q_1 + Q_2$ and $\neg\mathcal{F}^{\star} a y)$	$\exists$-Intro, line 11
13	$\mid\ \mid\ \exists y(y \notin_{k} Q_1 + Q_2$ and $\neg\mathcal{F}^{\star} a y)$	$\exists$-Elim, lines 4, 5
14	$\mid\ \mid\ \underline{a \in_{h_1} [f_2^{\star}\rangle Q_2}$	assume
15	$\mid\ \mid$ subproof is similar to previous subproof	lines xx
16	$\mid\ \mid\ \exists y(y \notin_{k} Q_1 + Q_2$ and $\neg\mathcal{F}^{\star} a y)$	$\exists$-Elim, lines xx
17	$\mid\ \exists y(y \notin_{k} Q_1 + Q_2$ and $\neg\mathcal{F}^{\star} a y)$	$\vee$-Elim, lines 2, 3, 14
18	$\mid\ a \in_{h} [f^{\star}\rangle (Q_1 + Q_2)$	$\exists$-Intro, line 11

■

We construct a counter-example working from the first proof. Let $a \in_{h} [f^{\star}\rangle (Q_1 + Q_2)$. Technically in this counter-example, we should be using $\varkappa_1 Q_1$ and $\varkappa_2 Q_2$ rather that Q_1 and Q_2 respectively, but then the notation is not providing any additional clarity. The main point is that $\mathcal{F}_1^{\star}$ and $\mathcal{F}_2^{\star}$ are injected into $(H_1 + H_2) \times (K_1 + K_2)$ and the complements of these relations must be computed in $(H_1 + H_2) \times (K_1 + K_2)$. In the case of $[f^{\circ}\rangle$, the proof of closure under + only used the complements to reduce the disjoint sum $\mathcal{F}_1^{\circ} + \mathcal{F}_2^{\circ}$ to either $\mathcal{F}_1^{\circ}$ or $\mathcal{F}_2^{\circ}$. Here, the difference is that the complement of the Kripke relation is used in defining $[-^{\star}\rangle$. That complement must occur in $(H_1 + H_2) \times (K_1 + K_2)$ if $[f^{\star}\rangle (Q_1 + Q_2) = [f_1^{\star}\rangle Q_1 + [f_2^{\star}\rangle Q_2$ is to have any meaning.

The following must be satisfied but in a way that satisfies $a \in_{h} [f_1^{\star}\rangle Q_1 + [f_2^{\star}\rangle Q_2$ incorrectly:

$$\exists y(y \notin_{k} Q_1 + Q_2 \text{ and } \neg\mathcal{F}_1^{\star} a y \text{ and } \neg\mathcal{F}_2^{\star} a y).$$

Let $H_1 = \{a_1\}$, $H_2 = \{a_2\}$, $K_1 = \{b_1, b_1'\}$, $Q_1 = \{b_1'\}$, $K_2 = \{b_2, b_2'\}$, $Q_2 = \{b_2'\}$, $\mathcal{F}_1^{\star} a_1 b_1$, $\mathcal{F}_2^{\star} a_2 b_2$, and $\mathcal{F}_2^{\star} a_2 b_2'$. Hence $\neg Q_1 = \{b_1\}$, $\neg\mathcal{F}_1^{\star} a_2 b_1$, and $\neg\mathcal{F}_2^{\star} a_2 b_1$. Since $b_1 \notin_{k_1} Q_1$ and $b_1 \notin_{k_2} Q_2$, then $b_1 \notin_{k} Q_1 + Q_2$. The following is satisfied for $y = b_1$:

$$\exists y(y \notin_{k} Q_1 + Q_2 \text{ and } \neg\mathcal{F}_1^{\star} a_2 y \text{ and } \neg\mathcal{F}_2^{\star} a_2 y),$$

So $a_2 \in_{\mathsf{h}_2} [f^\star\rangle\,(Q_1 + Q_2)$. Also, the following is satisfied for $y = b_1$:

$$\exists y(y \notin_{\mathsf{k}_1} Q_1 \text{ and } \neg\mathcal{F}_1^\star a_2 b_1).$$

By definition, $a_2 \in_{\mathsf{h}_2} [f_1^\star\rangle\,Q_1$ indicating that $a_2 \in_{\mathsf{h}_1} H_1$, which is a contradiction since $a_2 \in_{\mathsf{h}_2} H_2$. In addition, $a_2 \in_{\mathsf{h}_1} [f_2^\star\rangle\,Q_2$ but it was put there illicitly because the witness for the existential in the definition of $[f_2^\star\rangle\,Q_2$ is actually $b_1 \in_{\mathsf{k}_1} Q_1$. Note there is no witness for $a_2 \in_{\mathsf{h}_2} [f_2^\star\rangle\,Q_2$ for any $b \notin_{\mathsf{k}_2} Q_2$ because for any such b, i.e., $b = b_2$, $\neg\mathcal{F}_2^\star a_2 b$ cannot be satisfied.

It is also the case that $[f_i^\star\rangle$ does not respect the bounds. The type of $[-^\star\rangle$ is $\wedge \mapsto \vee$. Assuming non-empty domains, and we always do, then there is some b such that $b \notin_{\mathsf{k}_2} \top_2^{\mathbb{K}}$ and $\neg\mathcal{F}_2^\star ab$. Hence $a \in_{\mathsf{h}_2} [f_2^\star\rangle\,\top_2^{\mathbb{K}}$. Just pick some $b \in_{\mathsf{k}_1} K_1$. Since $K_1 \cap K_2 = \emptyset$ by assumption (or forcing the issue with tags) then $b \notin_{\mathsf{k}_2} \top_2^{\mathbb{K}}$. Since $\mathcal{F}_2^\star \subseteq H_2 \times K_2$, then $\neg\mathcal{F}_2^\star ab$. However, $[f_2^\star\rangle\,\top_2^{\mathbb{K}} = \emptyset$. Consequently, we have a simpler counter-example:

$$[f^\star\rangle\,(Q_1 + \top_2^{\mathbb{K}}) \neq [f_2^\star\rangle\,Q_1 + [f_2^\star\rangle\,\top_2^{\mathbb{K}}.$$

8.4 Counter-Example $[-^\perp\rangle$: Fails to Distribute Across Null Sum

We assume

$$\mathcal{F}^\perp\langle a_1, a_2\rangle\langle b_1, b_2\rangle \text{ iff } \mathcal{F}_1^\perp a_1 b_1 \text{ or } \mathcal{F}^\perp a_2 b_2.$$

which causes the right to left direction below to fail. We try to prove the following Lemma. Surprisingly left to right holds. However, right to left fails.

Lemma 8.4.1

$$[f^\perp\rangle\,(Q_1 \ovee Q_2) = [f^\perp\rangle\,Q_1 \ovee [f^\perp\rangle\,Q_2.$$

Proof:

1	$\langle a_1, a_2\rangle \in_h [f^\perp\rangle (Q_1 \circledvee Q_2)$	assume
2	$(\langle a_1, a_2\rangle \in_h H_1 \times H_2^\circ)$ or $(\langle a_1, a_2\rangle \in_h H_1^\circ \times H_2)$	$H_1 \circledvee H_2$ is the universe, line 1
3	$\langle a_1, a_2\rangle \in_h H_1 \times H_2^\circ$	assume
4	$a_1 \in_{h_1} H_1$ and $a_2 \in_{h_2} H_2^\circ$	set th., line 3
5	$\forall\langle y, v\rangle(\langle y, v\rangle \notin_k Q_1 \circledvee Q_2$ or $\mathcal{F}^\perp\langle a_1, a_2\rangle\langle y, v\rangle)$	def. $[-^\perp\rangle$, line 1
6	$\boxed{b_1}$ $b_1 \in_k Q_1$	assume
7	$b_2 \in_k K^\circ$	for some b_2, $K^\circ \neq \emptyset$
8	$\langle b_1, b_2\rangle \in_k Q_1 \times K_2^\circ$	set th., lines 6, 7
9	$\langle b_1, b_2\rangle \in_k Q_1 \circledvee Q_2$	def. $\circledvee$, line 8
10	$\mathcal{F}^\perp\langle a_1, a_2\rangle\langle b_1, b_2\rangle$	$\forall$-Elim, $\supset$-Elim, lines 5, 9
11	$\mathcal{F}_1^\perp a_1, b_2$ or $\mathcal{F}_2^\perp a_2 b_2$	def. $\mathcal{F}^\perp$, line 10
12	$\neg\mathcal{F}_2^\perp a_2 b_2$	Frame Condition 4.2.3 (1), line 4
13	$\mathcal{F}_1^\perp a_1 b_1$	CL, lines 11, 12
14	$\forall y(y \in_{k_1} Q_1$ implies $\mathcal{F}_1^\perp a_1 y)$	$\forall$-Intro, line 6
15	$a_1 \in_{h_1} [f^\perp\rangle Q_1$	def. $[-^\perp\rangle$, line 14
16	$\langle a_1, a_2\rangle \in_h [f^\perp\rangle Q_1 \times H_2^\circ$	set th., lines 4, 15
17	$\langle a_1, a_2\rangle \in_h [f^\perp\rangle Q_1 \circledvee [f^\perp\rangle Q_2$	def. $\circledvee$, line 16
18	$\langle a_1, a_2\rangle \in_h H_1^\circ \times H_2$	assume
19	subproof is similar to previous subproof	line xx
20	$\langle a_1, a_2\rangle \in_h [f^\perp\rangle Q_1 \circledvee [f^\perp\rangle Q_2$	def. $\circledvee$, line xx
21	$\langle a_1, a_2\rangle \in_h [f_1^\perp\rangle Q_1 \circledvee [f_2^\perp\rangle Q_2$	$\vee$-Elim, lines 2, 3, 18

Now the other direction, this is the direction that fails:

1	$\langle a_1,a_2\rangle \in_{\mathsf{h}} [f_1^{\perp}\rangle Q_1 \circledvee [f_2^{\perp}\rangle Q_2$	assume
2	$\langle a_1,a_2\rangle \in_{\mathsf{h}} [f_1^{\perp}\rangle Q_1 \times H_2^{\circ}$ or $\langle a_1,a_2\rangle \in_{\mathsf{h}} H_1^{\circ} \times [f_2^{\perp}\rangle Q_2$	def. $\circledvee$, line 1
3	$\boxed{b_1,b_2}$ $\langle b_1,b_2\rangle \in_{\mathsf{k}} Q_1 \circledvee Q_2$	assume
4	$(b_1 \in_{\mathsf{k}_1} Q_1$ and $b_2 \in_{\mathsf{k}_2} K_2^{\circ})$ or $(b_1 \in_{\mathsf{k}_1} K_1^{\circ}$ and $b_2 \in_{\mathsf{k}_2} Q_2)$	def. $\circledvee$, line 3
5	$b_1 \in_{\mathsf{k}_1} Q_1$ and $b_2 \in_{\mathsf{k}_2} K_2^{\circ}$	assume
6	$\langle a_1,a_2\rangle \in_{\mathsf{h}} [f_1^{\perp}\rangle Q_1 \times H_2^{\circ}$	assume
7	$a_1 \in_{\mathsf{h}_1} [f_1^{\perp}\rangle Q_1$ and $a_2 \in_{\mathsf{h}_2} H_2^{\circ}$	def. $\times$, line 6
8	$\forall y(y \in_{\mathsf{k}_1} Q_1$ implies $\mathcal{F}_1^{\perp} a_1 y)$	def. $[-^{\perp}\rangle$, line 7
9	$\mathcal{F}_1^{\perp} a_1 b_1$	$\forall$-Elim, $\supset$-Elim, lines 5, 8
10	$\mathcal{F}_1^{\perp} a_1 b_1$ or $\mathcal{F}_2^{\perp} a_2 b_2$	$\vee$-Intro, line 9
11	$\langle a_1,a_2\rangle \in_{\mathsf{h}} H_1^{\circ} \times [f_2^{\perp}\rangle Q_2$	assume
12	$a_1 \in_{\mathsf{h}_1} H_1^{\circ}$ and $a_2 \in_{\mathsf{h}_2} [f_2^{\perp}\rangle Q_2$	def. $\times$, line 6
13	$\forall v(v \in_{\mathsf{h}_2} Q_2$ implies $\mathcal{F}^{\perp} a_2 v)$	def. $[-^{\perp}\rangle$, line 12
14	$\neg\mathcal{F}_2^{\perp} a_2 b_2$ who ordered this	Frame Condition 4.2.3 (2), line 5
15	$\neg\mathcal{F}_1^{\perp} a_1 b_1$ who ordered this	Frame Condition 4.2.3 (1), line 12
16	$\mathcal{F}_1^{\perp} a_1 b_1$ or $\mathcal{F}_2^{\perp} a_2 b_2$ no way to get this	$\vee$-Elim, lines 2, 6, 11
17	$b_1 \in_{\mathsf{k}_1} K_1^{\circ}$ and $b_2 \in_{\mathsf{k}_2} Q_2$	assume
18	·	lines xx
19	$\mathcal{F}_1^{\perp} a_1 b_1$ or $\mathcal{F}_2^{\perp} a_2 b_2$ no way to get this	$\vee$-Elim, lines 2, xx
20	$\mathcal{F}_1^{\perp} a_1 b_1$ or $\mathcal{F}_2^{\perp} a_2 b_2$ no way to get this	$\vee$-Elim, lines 4, 5, 17
21	$\mathcal{F}^{\perp}\langle a_1,a_2\rangle\langle b_1,b_2\rangle$ no way to get this	def. $\mathcal{F}^{\perp}$, line 20
22	$\forall\langle y,v\rangle(\langle y,v\rangle \notin_{\mathsf{k}} Q_1 \circledvee Q_2$ or $\mathcal{F}\langle a_1,a_2\rangle\langle y,v\rangle)$ no way	def. $\forall$-Intro, line 3
23	$\langle a_1,a_2\rangle \in_{\mathsf{h}} [f^{\perp}\rangle(Q_1 \circledvee Q_2)$ no way	def. $[-^{\perp}\rangle$, line 22

■

Lemma 8.4.2

$$[f^{\perp}\rangle Q_1 \circledvee [f^{\perp}\rangle Q_2 \not\subseteq_{\mathsf{h}} [f^{\perp}\rangle(Q_1 \circledvee Q_2).$$

Proof: Since we are now using pointed domains, all the special points must be included. We try to read off the failed proof the necessary other points. We simply assume all the well and null points are present in the domains. We try to find an $\langle a_1,a_2\rangle$ to satisfy

$$\exists\langle y,v\rangle(\langle y,v\rangle \in_{\mathsf{k}} Q_1 \circledvee Q_2 \text{ and } \neg\mathcal{F}^{\perp}\langle a_1,a_2\rangle\langle y,v\rangle$$

and see if this satisfies

$$\langle a_1,a_2\rangle \in_{\mathsf{h}} [f_1^{\perp}\rangle Q_1 \times H_2^{\circ} \text{ or } \langle a_1,a_2\rangle \in_{\mathsf{h}} H_1^{\circ} \times [f_2^{\perp}\rangle Q_2.$$

Since $\neg\mathcal{F}^{\perp}\langle a_1, a_2\rangle\langle b_1, b_2\rangle$ as $\neg\mathcal{F}_1^{\perp} a_1 b_1$ and $\neg\mathcal{F}_2^{\perp} a_2 b_2$ occurs at line 15, we use the points labeled at that place in the failed proof; namely let $a_1 \in_{\mathsf{h}_1} H_1^{\circ}$, $a_2 \in_{\mathsf{h}_2} [f_2^{\perp}\rangle Q_2$, $b_1 \in_{\mathsf{h}_1} Q_1$, $b_2 \in_{\mathsf{k}_2} K_2^{\circ}$. Hence we have

$$\neg\mathcal{F}_1^{\perp} a_1 b_1 \text{ and } \neg\mathcal{F}_2^{\perp} a_2 b_2$$

from the frame conditions listed in the failed proof.

From the definition of $\varovee$, we have $\langle a_1, a_2\rangle \in_{\mathsf{h}} [f_1^{\perp}\rangle Q_1 \varovee [f_2^{\perp}\rangle Q_2$, i.e., $\langle a_1, a_2\rangle \in_{\mathsf{h}} H_1^{\circ} \times [f_2^{\perp}\rangle Q_2$. Since $\neg\mathcal{F}^{\perp}\langle a_1, a_2\rangle\langle b_1, b_2\rangle$ holds, and $\langle b_1, b_2\rangle \in_{\mathsf{k}} Q_1 \times H_2^{\circ}$, we have satisfied $\langle a_1, a_2\rangle \notin_{\mathsf{h}} [f^{\perp}\rangle (Q_1 \varovee Q_2)$.

Now we use ground domains and let for $X \in \{H_1, H_2, K_1, K_2\}$,

$$X^{\bullet} = \{\bullet_X\} \qquad X^{\circ} = \{\circ_X\},$$

and

$$Q_1 = K_1^{\bullet} \cup \{b_1\} \quad Q_2 = K_2^{\bullet} \cup \{d_2\}$$

where $b_2 \neq d_2$, and

$$H_1 = H_1^{\bullet} \cup H_1^{\circ} \cup \{c_1\} \quad K_1 = K_1^{\bullet} \cup K_1^{\circ} \cup \{d_1\} \qquad H_2 = H_2^{\bullet} \cup H_2^{\circ} \cup \{a_2\} \quad K_2 = K_2^{\bullet} \cup K_2^{\circ}.$$

where $c_1 \neq a_1$ and

$$\begin{aligned} \mathcal{F}_1^{\perp} &= \{\langle c_1, b_1\rangle, \langle \bullet_{H_1}, \bullet_{K_1}\rangle, \langle c_1, \bullet_{K_1}\rangle, \langle \bullet_{H_1}, d_1\rangle\} \\ \mathcal{F}_2^{\perp} &= \{\langle \bullet_{H_2}, \bullet_{K_2}\rangle, \langle a_2, \bullet_{K_2}\rangle, \langle \bullet_{H_2}, d_2\rangle\} \end{aligned}$$

We need to know that

$$a_2 \in_{\mathsf{h}_2} [f_2^{\perp}\rangle Q_2 \text{ iff } \forall v(v \notin_{\mathsf{k}_2} Q_2 \text{ or } \mathcal{F}_2^{\perp} a_2 v)$$

is satisfied. This means we must pick Q_2. We try $Q_2 = K_2^{\bullet}$. In this case, $Q_2 = \bot_2^{\mathbb{K}}$ and $[f_2^{\perp}\rangle Q_2 = \top_2^{\mathbb{H}} = H_2^{\bullet} \cup \{a_2\}$. It appears a_2 is unnecessary as a point not in $H_2^{\bullet}$. We will leave it in. Stretching out the universal quantifier and specifying the reasons why a line holds yields

$$\begin{array}{lll} a_2 \in_{\mathsf{h}_2} [f_2^{\perp}\rangle Q_2 & \text{iff } \circ_{K_2} \notin_{\mathsf{k}_2} Q_2 \text{ or } \mathcal{F}_2^{\perp} a_2 \circ_{K_2} & Q_2 \text{ is null-pointed} \\ & \text{and } \bullet_{K_2} \notin_{\mathsf{k}_2} Q_2 \text{ or } \mathcal{F}_2^{\perp} a_2 \bullet_{K_2} & \text{Frame Condition 4.2.4 (1)} \\ & \text{and } d_2 \notin_{\mathsf{k}_2} Q_2 \text{ or } \mathcal{F}_2^{\perp} a_2 d_2 & Q_2 = \bullet_{K_2} \end{array}$$

Therefore $a_2 \in_{\mathsf{h}_2} [f_2^{\perp}\rangle Q_2$.

Our conditions, $a_1 \in_{\mathsf{h}_1} H_1^{\circ}$, $a_2 \in_{\mathsf{h}_2} [f_2^{\perp}\rangle Q_2$, $b_1 \in_{\mathsf{h}_1} Q_1$, $b_2 \in_{\mathsf{k}_2} K_2^{\circ}$, guarantee that $\langle a_1, a_2\rangle \notin_{\mathsf{h}} [f^{\perp}\rangle (Q_1 \varovee Q_2)$ since $\langle b_1, b_2\rangle \in_{\mathsf{k}} Q_1 \varovee Q_2$ and $\neg\mathcal{F}^{\perp}\langle a_1, a_{\rangle}\langle b_1, b_2\rangle$. Thus the right hand side of the following obtains:

$$\begin{array}{lll} \langle a_1, a_2\rangle \notin_{\mathsf{h}} [f^{\perp}\rangle (Q_1 \varovee Q_2) & \text{iff } \exists\langle y, v\rangle(\langle y, v\rangle \in_{\mathsf{k}} Q_1 \varovee Q_2 \text{ and } \neg\mathcal{F}^{\perp}\langle a_1, a_2\rangle\langle y, v\rangle) & \text{def. } [-^{\perp}\rangle \\ & \text{if } \langle b_1, b_2\rangle \in_{\mathsf{k}} Q_1 \varovee Q_2 & b_1 \in_{\mathsf{k}_1} Q_1 \text{ and } b_2 \in_{\mathsf{k}_2} Q_2 \\ & \text{and } \neg\mathcal{F}^{\perp} a_1 b_1 & \text{Frame Condition 4.2.3 (1)} \\ & \text{and } \neg\mathcal{F}^{\perp} a_2 b_2 & \text{Frame Condition 4.2.3 (2)} \end{array}$$

■

9. APPENDIX 2

9.1 Additional Smash Product Lemmas

Both this Lemma and its Corollary are direct implications of the definition for ⊗, not used in five, four, and minimally in three for another Lemma which we have already replaced above. They can hence be dispensed with.

Lemma 9.1.1

$$\langle x, y\rangle \in_{\mathsf{h}} Q_1 \times Q_2 \text{ and } x \notin_{\mathsf{h}_1} H_1^{\bullet} \text{ and } y \notin_{\mathsf{h}_2} H_2^{\bullet} \text{ implies } \langle x, y\rangle \in_{\mathsf{h}} Q_1 \otimes Q_2.$$

or

$$(Q_1 - H_1^{\bullet}) \times (Q_2 - H_2^{\bullet}) \subseteq Q_1 \otimes Q_2.$$

Corollary 9.1.2

$$\langle x, y\rangle \notin_{\mathsf{h}} Q_1 \otimes Q_2 \text{ implies } x \notin_{\mathsf{h}_1} Q_1 \text{ or } y \notin_{\mathsf{h}_2} Q_2 \text{ or } x \notin_{\mathsf{h}} H_1^{\bullet} \text{ or } y \notin_{\mathsf{h}_2} H_2^{\bullet}.$$

Lemma 9.1.3

$$(Q_1 - H_1^{\bullet}) \times (Q_2 - H_2^{\bullet}) \subseteq (Q_1 \times Q_2) - H_{\otimes}^{\bullet}.$$

Proof:

$$\begin{aligned}
\langle x, y\rangle \in_{\mathsf{h}} (Q_1 - H_1^{\bullet}) \times (Q_2 - H_2^{\bullet}) &\text{ iff } x \in_{\mathsf{h}_1} Q_1 - H_1^{\bullet} \text{ and } y \in_{\mathsf{h}_2} Q_2 - H_2^{\bullet} \\
&\text{ implies } \langle x, y\rangle \in_{\mathsf{h}} Q_1 \times Q_2 \text{ and } \langle x, y\rangle \notin_{\mathsf{h}} H_1^{\bullet} \times H_2^{\bullet} \\
&\text{ iff } \langle x, y\rangle \in_{\mathsf{h}} (Q_1 \times Q_2) - (H_1^{\bullet} \times H_2^{\bullet}) \\
&\text{ iff } \langle x, y\rangle \in_{\mathsf{h}} (Q_1 \times Q_2) - H_{\otimes}^{\bullet}
\end{aligned}$$

■

Lemma 9.1.4

$$(H_1 \otimes H_2) \mathrel{\overset{\bullet\circ}{-}} (Q_1 \otimes Q_2) = ((H_1 \otimes H_2) - (Q_1 \otimes Q_2)) \cup H_{\otimes}^{\bullet}.$$

and

$$(H_1 \otimes H_2) \mathrel{\overset{\bullet\circ}{-}} (Q_1 \otimes Q_2) = (H_1 \otimes H_2) \mathrel{\overset{\bullet\circ}{-}} ((Q_1 - H_1^{\bullet}) \times (Q_2 - H_2^{\bullet}))$$

Proof: Starting from the definition,

$$\begin{aligned}
(H_1 \otimes H_2) \overset{\bullet\circ}{-} (Q_1 \otimes Q_2) &\overset{def}{=} \big((H_1 \otimes H_2) - ((Q_1 \otimes Q_2) \cup H^{\circ}_{\otimes})\big) \cup H^{\bullet}_{\otimes} && \text{definition of } \overset{\bullet\circ}{-} \\
&= \big(((H_1 \otimes H_2) - H^{\circ}_{\otimes}) - (Q_1 \otimes Q_2)\big) \cup H^{\bullet}_{\otimes}. && \text{set theory} \\
&= \big((H_1 \otimes H_2) - (Q_1 \otimes Q_2)\big) \cup H^{\bullet}_{\otimes}. && (H_1 \otimes H_2) \cap H^{\circ}_{\otimes} = \emptyset
\end{aligned}$$

and

$$\begin{aligned}
\big((H_1 \otimes H_2) - (Q_1 \otimes Q_2)\big) \cup H^{\bullet}_{\otimes} &= \big((H_1 \otimes H_2) - (((Q_1 - H^{\bullet}_1) \times (Q_2 - H^{\bullet}_2)) \cup H^{\bullet}_{\otimes})\big) \cup H^{\bullet}_{\otimes} \\
&= \big(((H_1 \otimes H_2) - H^{\bullet}_{\otimes}) - ((Q_1 - H^{\bullet}_1) \times (Q_2 - H^{\bullet}_2))\big) \cup H^{\bullet}_{\otimes} \\
&= (H_1 \otimes H_2) - ((Q_1 - H^{\bullet}_1) \times (Q_2 - H^{\bullet}_2))
\end{aligned}$$

■

Lemma 9.1.5

$$\begin{aligned}
&\Big((x \notin_{\mathsf{h}_1} Q_1 \text{ and } y \notin_{\mathsf{h}_2} H^{\bullet}_2) \text{ or} \\
&(y \notin_{\mathsf{h}_2} Q_2 \text{ and } x \notin_{\mathsf{h}_1} H^{\bullet}_1) \text{ or} \\
&\qquad (\langle x, y\rangle \in_{\mathsf{h}} H^{\bullet}_{\otimes})\Big) \text{ implies } \langle x, y\rangle \in_{\mathsf{h}} (H_1 \otimes H_2) \overset{\bullet\circ}{-} (Q_1 \otimes Q_2).
\end{aligned}$$

Proof: Assume $x \notin_{\mathsf{h}_1} Q_1$ and $y \notin_{\mathsf{h}_2} H^{\bullet}_2$, so $\langle x, y\rangle \notin_{\mathsf{h}} Q_1 \times Q_2$. From Lemma 3.4.2, $\langle x, y\rangle \notin_{\mathsf{h}} Q_1 \otimes Q_2$. Since $x \notin_{\mathsf{h}_1} Q_1$ then $x \notin_{\mathsf{h}_1} H^{\bullet}_1$. From Lemma 9.1.4, $\langle x, y\rangle \in_{\mathsf{h}} (H_1 \otimes H_2) - (Q_1 \otimes Q_2)$. The case $y \notin_{\mathsf{h}_2} Q_2$ and $x \neq \bullet_{H_1}$ is argued similarly.

If $\langle x, y\rangle \in_{\mathsf{h}} H^{\bullet}_{\otimes}$ then by definition $\langle x, y\rangle \in_{\mathsf{h}} H_1 \otimes H_2 \overset{\bullet\circ}{-} Q_1 \otimes Q_2$. ■

Lemma 9.1.6 $\langle x, y\rangle \in_{\mathsf{h}} (H_1 \otimes H_2) - (Q_1 \otimes Q_2)$ *implies*

$$(x \notin_{\mathsf{h}_1} Q_1 \text{ and } y \in_{\mathsf{h}_2} Q_2) \text{ or } (x \in_{\mathsf{h}_1} Q_1 \text{ and } y \notin_{\mathsf{h}_2} Q_2) \text{ or } (x \notin_{\mathsf{h}_1} Q_1 \text{ and } y \notin_{\mathsf{h}_2} Q_2).$$

Proof: Let $\langle x, y\rangle \in_{\mathsf{h}} (H_1 \otimes H_2) - (Q_1 \otimes Q_2)$. From Lemma 3.4.7, $x \notin_{\mathsf{h}_1} H^{\bullet}_1$ and $y \notin_{\mathsf{h}_2} H^{\bullet}_2$. Also, $\langle x, y\rangle \in_{\mathsf{h}} (H_1 \otimes H_2)$ and $\langle x, y\rangle \notin_{\mathsf{h}} (Q_1 \otimes Q_2)$. Suppose $x \in_{\mathsf{h}_1} Q_1$ or $y \in_{\mathsf{h}_2} Q_2$. If $x \in_{\mathsf{h}_1} Q_1$, then $x \in_{\mathsf{h}_1} Q_1$ and $y \notin_{\mathsf{h}_2} Q_2$ otherwise $\langle x, y\rangle \in_{\mathsf{h}} Q_1 \otimes Q_2$, which is a contradiction. Similarly, if $y \in_{\mathsf{h}_2} Q_2$ then $x \notin_{\mathsf{h}_1} Q_1$ and $y \in_{\mathsf{h}_2} Q_2$. Therefore

$$(x \notin_{\mathsf{h}_1} Q_1 \text{ and } y \in_{\mathsf{h}_2} Q_2) \text{ or } (x \in_{\mathsf{h}_1} Q_1 \text{ and } y \notin_{\mathsf{h}_2} Q_2) \text{ or } (x \notin_{\mathsf{h}_1} Q_1 \text{ and } y \notin_{\mathsf{h}_2} Q_2).$$

■

Lemma 9.1.7

$$(x \in_{\mathsf{h}_1} H_1^{\bullet} \text{ and } \langle x, y\rangle \in_{\mathsf{h}} Q_1 \otimes Q_2) \text{ implies } (y \in_{\mathsf{h}_2} H_2^{\bullet} \text{ and } \langle x, y\rangle \in_{\mathsf{h}} H_{\otimes}^{\bullet}).$$

and

$$(y \in_{\mathsf{h}_2} H_2^{\bullet} \text{ and } \langle x, y\rangle \in_{\mathsf{h}} Q_1 \otimes Q_2) \text{ implies } (x \in_{\mathsf{h}_1} H_1^{\bullet} \text{ and } \langle x, y\rangle \in_{\mathsf{h}} H_{\otimes}^{\bullet}).$$

Proof: Assume $x \in_{\mathsf{h}_1} H_1^{\bullet}$ and $\langle x, y\rangle \in_{\mathsf{h}} Q_1 \otimes Q_2$. By definition of $Q_1 \otimes Q_2$, then for any y, $\langle x, y\rangle \in_{\mathsf{h}} H_1^{\bullet} \times H_2^{\bullet}$ and so $y \in_{\mathsf{h}_2} H_2^{\bullet}$ and $\langle x, y\rangle \in_{\mathsf{h}} H_{\otimes}^{\bullet}$. The other statement is similar.

■

Lemma 9.1.8

$$\langle x, y\rangle \in_{\mathsf{h}} \overset{\bullet\circ}{\neg}(Q_1 \otimes Q_2) \text{ iff } \langle x, y\rangle \in_{\mathsf{h}} (Q_1 \otimes \overset{\bullet\circ}{\neg} Q_2) \cup (\overset{\bullet\circ}{\neg} Q_1 \otimes Q_2) \cup (\overset{\bullet\circ}{\neg} Q_1 \otimes \overset{\bullet\circ}{\neg} Q_2).$$

Proof:

1	$\langle x, y\rangle \in_h \overset{\bullet\circ}{\neg}(Q_1 \otimes Q_2)$	assume
2	$\langle x, y\rangle \in_h \big(H_1 \otimes H_2 - ((Q_1 \otimes Q_2) \cup H^\circ_\otimes)\big) \cup H^\bullet_\otimes$	def. $\overset{\bullet\circ}{\neg}$, line 1
3	$\langle x, y\rangle \in_h \big(H_1 \otimes H_2 - ((Q_1 \otimes Q_2) \cup H^\circ_\otimes)\big)$ or $\langle x, y\rangle \in_h H^\bullet_\otimes$	set th., line 2
4	$\langle x, y\rangle \in_h H^\bullet_\otimes$	assume
5	$\langle x, y\rangle \in_h H^\bullet_1 \times H^\bullet_2$	def. $H^\bullet_\otimes$, Definition 3.2.2, line 4
6	$\langle x, y\rangle \in_h \overset{\bullet\circ}{\neg} Q_1 \otimes \overset{\bullet\circ}{\neg} Q_2$	def. $\otimes$, Definition 3.4.1, line 5
7	$\langle x, y\rangle \in_h (Q_1 \otimes \overset{\bullet\circ}{\neg} Q_2) \cup (\overset{\bullet\circ}{\neg} Q_1 \otimes Q_2) \cup (\overset{\bullet\circ}{\neg} Q_1 \otimes \overset{\bullet\circ}{\neg} Q_2)$	set th., line 6
8	$\langle x, y\rangle \in_h H_1 \otimes H_2 - ((Q_1 \otimes Q_2) \cup H^\circ_\otimes)$	assume
9	$\langle x, y\rangle \in_h H_1 \otimes H_2$	set th.,, line 8
10	$x \notin_{h_1} H^\bullet_1$ and $y \notin_{h_2} H^\bullet_2$	Lemma 3.4.7, line 9
11	$\langle x, y\rangle \notin_h (Q_1 \otimes Q_2) \cup H^\circ_\otimes$	set th., line 8
12	$\langle x, y\rangle \notin_h Q_1 \otimes Q_2$	set th., line 12
13	$(x \notin_{h_1} Q_1$ and $y \in_{h_2} Q_2)$ or $(x \in_{h_1} Q_1$ and $y \notin_{h_2} Q_2)$ or $(x \notin_{h_1} Q_1$ and $y \notin_{h_2} Q_2)$	Lemma 9.1.6, line 13
14	$x \notin_{h_1} Q_1$ and $y \in_{h_2} Q_2$	assume
15	$x \notin_{h_1} H^\circ_1$	Corollary 3.4.4, line 9
16	$x \in_{h_1} (H_1 - (Q_1 \cup H^\circ_1)) \cup H^\bullet_1$	set th., lines 14, 15
17	$x \in_{h_1} \overset{\bullet\circ}{\neg} Q_1 - H^\bullet_1$	def. $\overset{\bullet\circ}{\neg}$, set th., lines 10, 16
18	$y \in_{h_2} Q_2 - H^\bullet_2$	set th., lines 10, 14
19	$\langle x, y\rangle \in_h \overset{\bullet\circ}{\neg} Q_1 \otimes Q_2$	def. $\otimes$, lines 17, 18
20	$\langle x, y\rangle \in_h (Q_1 \otimes \overset{\bullet\circ}{\neg} Q_2) \cup (\overset{\bullet\circ}{\neg} Q_1 \otimes Q_2) \cup (\overset{\bullet\circ}{\neg} Q_1 \otimes \overset{\bullet\circ}{\neg} Q_2)$	set th., line 19
21	$x \in_{h_1} Q_1$ and $y \notin_{h_2} Q_2$	assume
22	$y \notin_{h_2} H^\circ_2$	Corollary 3.4.4, line 9
23	$y \in_{h_2} (H_2 - (Q_2 \cup H^\circ_2)) \cup H^\bullet_2$	set th., lines 21, 22
24	$y \in_{h_2} \overset{\bullet\circ}{\neg} Q_2 - H^\bullet_2$	def. $\overset{\bullet\circ}{\neg}$, set th., lines 10, 23
25	$x \in_{h_1} Q_1 - H^\bullet_1$	set th., lines 10, 21
26	$\langle x, y\rangle \in_h Q_1 \otimes \overset{\bullet\circ}{\neg} Q_2$	def. $\otimes$, lines 24, 25
27	$\langle x, y\rangle \in_h (Q_1 \otimes \overset{\bullet\circ}{\neg} Q_2) \cup (\overset{\bullet\circ}{\neg} Q_1 \otimes Q_2) \cup (\overset{\bullet\circ}{\neg} Q_1 \otimes \overset{\bullet\circ}{\neg} Q_2)$	set th., line 26
28	$x \notin_{h_1} Q_1$ and $y \notin_{h_2} Q_2$	assume
29	$x \notin_{h_1} H^\circ_1$ and $y \notin_{h_2} H^\circ_2$	Corollary 3.4.4, line 9
30	$x \in_{h_1} (H_1 - (Q_1 \cup H^\circ_1))$ and $y \in_{h_2} (H_2 - (Q_2 \cup H^\circ_2))$	set th., lines 28, 29
31	$x \in_{h_1} (H_1 - (Q_1 \cup H^\circ_1)) \cup H^\bullet_1$ and $y \in_{h_2} (H_2 - (Q_2 \cup H^\circ_2)) \cup H^\bullet_2$	set th., line 30
32	$x \in_{h_1} (\overset{\bullet\circ}{\neg} Q_1 - H^\bullet_1)$ and $y \in_{h_2} (\overset{\bullet\circ}{\neg} Q_2 - H^\bullet_2)$	def. $\overset{\bullet\circ}{\neg}$, lines 10, 31
33	$\langle x, y\rangle \in_h \overset{\bullet\circ}{\neg} Q_1 \otimes \overset{\bullet\circ}{\neg} Q_2$	def. $\otimes$, line 32
34	$\langle x, y\rangle \in_h (Q_1 \otimes \overset{\bullet\circ}{\neg} Q_2) \cup (\overset{\bullet\circ}{\neg} Q_1 \otimes Q_2) \cup (\overset{\bullet\circ}{\neg} Q_1 \otimes \overset{\bullet\circ}{\neg} Q_2)$	set th., line 33
35	$\langle x, y\rangle \in_h (Q_1 \otimes \overset{\bullet\circ}{\neg} Q_2) \cup (\overset{\bullet\circ}{\neg} Q_1 \otimes Q_2) \cup (\overset{\bullet\circ}{\neg} Q_1 \otimes \overset{\bullet\circ}{\neg} Q_2)$	$\vee$-Elim, lines 13, 14, 21, 28
36	$\langle x, y\rangle \in_h (Q_1 \otimes \overset{\bullet\circ}{\neg} Q_2) \cup (\overset{\bullet\circ}{\neg} Q_1 \otimes Q_2) \cup (\overset{\bullet\circ}{\neg} Q_1 \otimes \overset{\bullet\circ}{\neg} Q_2)$	$\vee$-Elim, lines 3, 4, 8

1	│ $\langle x, y\rangle \in_{h} (Q_1 \otimes \overset{\bullet\circ}{\neg} Q_2) \cup (\overset{\bullet\circ}{\neg} Q_1 \otimes Q_2) \cup (\overset{\bullet\circ}{\neg} Q_1 \otimes \overset{\bullet\circ}{\neg} Q_2)$	assume
2	│ │ $\langle x, y\rangle \in_{h} Q_1 \otimes \overset{\bullet\circ}{\neg} Q_2$	assume
3	│ │ $x \in_{h_1} Q_1$ and $y \in_{h_2} \overset{\bullet\circ}{\neg} Q_2$	Lemma 3.4.2, line 2
4	│ │ $(y \notin_{h_2} Q_2$ and $y \notin_{h_2} H_2^{\circ})$ or $y \in_{h_2} H_2^{\bullet}$	def. $\overset{\bullet\circ}{\neg}$, line 3
5	│ │ │ $y \notin_{h_2} Q_2$ and $y \notin_{h_2} H_2^{\circ}$	assume
6	│ │ │ $\langle x, y\rangle \notin_{h} Q_1 \otimes Q_2$	Lemma 3.4.6, line 5
7	│ │ │ $\langle x, y\rangle \notin_{h} H_{\otimes}^{\circ}$	def. $H_{\otimes}^{\circ}$, line 5
8	│ │ │ $\langle x, y\rangle \notin_{h} (Q_1 \otimes Q_2) \cup H_{\otimes}^{\circ}$	set th., line 7
9	│ │ │ $\langle x, y\rangle \notin_{h} (Q_1 \otimes Q_2) \cup H_{\otimes}^{\circ}$ or $\langle x, y\rangle \in_{h} H_{\otimes}^{\bullet}$	$\vee$-Intro, line 8
10	│ │ │ $\langle x, y\rangle \in_{h} \overset{\bullet\circ}{\neg}(Q_1 \otimes Q_2)$	def. $\overset{\bullet\circ}{\neg}$, line 9
11	│ │ │ $y \in_{h_2} H_2^{\bullet}$	assume
12	│ │ │ $\langle x, y\rangle \in_{h} H_{\otimes}^{\bullet}$	Lemma 9.1.7, line 11
13	│ │ │ $\langle x, y\rangle \in_{h} \overset{\bullet\circ}{\neg}(Q_1 \otimes Q_2)$	def. $\overset{\bullet\circ}{\neg}$, line 12
14	│ │ $\langle x, y\rangle \in_{h} \overset{\bullet\circ}{\neg}(Q_1 \otimes Q_2)$	$\vee$-Elim, lines 4, 5, 11
15	│ │ $\langle x, y\rangle \in_{h} \overset{\bullet\circ}{\neg} Q_1 \otimes Q_2$	assume
16	│ │ $x \in_{h_1} \overset{\bullet\circ}{\neg} Q_1$ and $y \in_{h_2} Q_2$	Lemma 3.4.2, line 15
17	│ │ $(x \notin_{h_1} Q_1$ and $x \notin_{h_1} H_1^{\circ})$ or $x \in_{h_1} H_1^{\bullet}$	def. $\overset{\bullet\circ}{\neg}$, line 16
18	│ │ │ $x \notin_{h_1} Q_1$ and $x \notin_{h_1} H_1^{\circ}$	assume
19	│ │ │ $\langle x, y\rangle \notin_{h} Q_1 \otimes Q_2$	Lemma 3.4.6, line 18
20	│ │ │ $\langle x, y\rangle \notin_{h} H_{\otimes}^{\circ}$	def. $H_{\otimes}^{\circ}$, line 18
21	│ │ │ $\langle x, y\rangle \notin_{h} (Q_1 \otimes Q_2) \cup H_{\otimes}^{\circ}$	set th., line 20
22	│ │ │ $\langle x, y\rangle \notin_{h} (Q_1 \otimes Q_2) \cup H_{\otimes}^{\circ}$ or $\langle x, y\rangle \in_{h} H_{\otimes}^{\bullet}$	$\vee$-Intro, line 21
23	│ │ │ $\langle x, y\rangle \in_{h} \overset{\bullet\circ}{\neg}(Q_1 \otimes Q_2)$	def. $\overset{\bullet\circ}{\neg}$, line 22
24	│ │ │ $x \in_{h_1} H_1^{\bullet}$	assume
25	│ │ │ $\langle x, y\rangle \in_{h} H_{\otimes}^{\bullet}$	Lemma 9.1.7, line 24
26	│ │ │ $\langle x, y\rangle \in_{h} \overset{\bullet\circ}{\neg}(Q_1 \otimes Q_2)$	def. $\overset{\bullet\circ}{\neg}$, line 25
27	│ │ $\langle x, y\rangle \in_{h} \overset{\bullet\circ}{\neg}(Q_1 \otimes Q_2)$	$\vee$-Elim, lines 17, 18, 24
28	│ │ $\langle x, y\rangle \in_{h} \overset{\bullet\circ}{\neg} Q_1 \otimes \overset{\bullet\circ}{\neg} Q_2$	assume
29	│ │ $\langle x, y\rangle \in_{h} (\overset{\bullet\circ}{\neg} Q_1 - H_1^{\bullet}) \times (\overset{\bullet\circ}{\neg} Q_2 - H_2^{\bullet})$ or $\langle x, y\rangle \in_{h} H_{\otimes}^{\bullet}$	def. $\otimes$, line 28
30	│ │ │ $\langle x, y\rangle \in_{h} (\overset{\bullet\circ}{\neg} Q_1 - H_1^{\bullet}) \times (\overset{\bullet\circ}{\neg} Q_2 - H_2^{\bullet})$	assume
31	│ │ │ $x \in_{h_1} \overset{\bullet\circ}{\neg} Q_1$ and $y \in_{h_2} \overset{\bullet\circ}{\neg} Q_2$ and $x \notin_{h_1} H_1^{\bullet}$ and $y \notin_{h} H_2^{\bullet}$	def. $\times$, set th., line 30
32	│ │ │ $x \notin_{h_1} Q_1$ and $y \notin_{h_2} Q_2$ and $x \notin_{h_1} H_1^{\circ}$ and $y \notin_{h_2} H_2^{\circ}$	def. $\overset{\bullet\circ}{\neg}$, set th., line 31
33	│ │ │ $\langle x, y\rangle \notin_{h} Q_1 \otimes Q_2$	Lemma 3.4.2, line 32
34	│ │ │ $\langle x, y\rangle \notin_{h} H_{\otimes}^{\circ}$	def.. $H_{\otimes}^{\circ}$, line 32
35	│ │ │ $\langle x, y\rangle \notin_{h} Q_1 \otimes Q_2 \cup H_{\otimes}^{\circ}$	set th., lines 33, 34
36	│ │ │ $(\langle x, y\rangle \notin_{h} Q_1 \otimes Q_2 \cup H_{\otimes}^{\circ})$ or $\langle x, y\rangle \in_{h} H_{\otimes}^{\bullet}$	$\vee$-Intro, line 35
37	│ │ │ $\langle x, y\rangle \in_{h} \overset{\bullet\circ}{\neg}(Q_1 \otimes Q_2)$	def. $\overset{\bullet\circ}{\neg}$, line 36
38	│ │ │ $\langle x, y\rangle \in_{h} H_{\otimes}^{\bullet}$	assume
39	│ │ │ $\langle x, y\rangle \in_{h} \overset{\bullet\circ}{\neg}(Q_1 \otimes Q_2)$	def. $\overset{\bullet\circ}{\neg}$, line 38
40	│ │ $\langle x, y\rangle \in_{h} \overset{\bullet\circ}{\neg}(Q_1 \otimes Q_2)$	$\vee$-Elim, lines 29, 30, 38
41	│ $\langle x, y\rangle \in_{h} \overset{\bullet\circ}{\neg}(Q_1 \otimes Q_2)$	$\vee$-Elim, lines 1, 2, 15, 28

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