

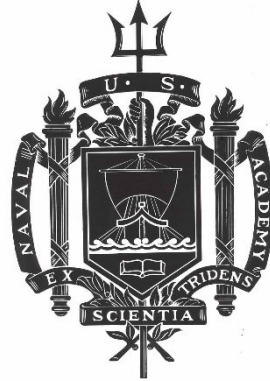
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NO. 514

Superradiance of Few Driven Two-Level Quantum Dot Emitters in the Bad Cavity Limit

by

Midshipman 1/C Joseph B. Wiedemann, USN



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IN THE BAD CAVITY LIMIT**

by

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Abstract

The pursuit of an integrated quantum optics system requires the ability to determine the effect of design parameters on the quantum electrodynamics regime. We develop a simple master equation for few, driven, two-level emitters in the bad cavity regime. Comparing the resulting photon-photon correlation function in the steady state with experimental data from InGaAs quantum dots coupled to a photonic crystal waveguide, we validate our model parameters for coupling strength, spontaneous decay rate and incoherent driving rate. Building upon this model, we explore the superradiant regime for InGaAs quantum dots coherently driven within a waveguide, creating a model that captures the collective behavior of two-level emitters in the bad cavity limit.

Keywords

Superradiance, Master Equation, Quantum Sensing, Quantum Dots

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Contents

1	Introduction	4
2	Background	4
3	Methods	6
3.1	Master Equation Derivation	7
3.2	Creating a Hamiltonian	11
3.3	Operators	11
3.3.1	Spin Operators	11
3.3.2	Ladder Operators	12
3.4	Hamiltonian Terms	13
4	Master Equation Development	14
4.1	Basic Master Equation	15
4.2	Including Basic Quantum Effect	16
4.3	Combining Effects	17
4.4	Reaching the steady-state	19
4.5	NRL Experiment	19
5	Second-Order Photon Correlation Measurement	21
5.1	Determining the Proper Parameters	22
6	Results	24
7	Conclusion	26
A	Coding the Master Equation	30
B	Expectation values of the density matrix	35
C	Coding photon-correlation	36

1 Introduction

Quantum mechanics is notorious for its weirdness. Richard Feynman, the greatest communicator in physics, famously remarked *"I think I can safely say no one understands quantum mechanics"* [8]. While the oddity and beauty of quantum behavior still evades complete understanding, there has been a recent transition in our relationship with quantum mechanics. The theory of quantum mechanics began in the early twentieth century, but our experimental prowess and collective understanding has evolved into the ability to engineer on scales where the laws of quantum mechanics dictate the dynamics. This has brought on a new industrial revolution; a quantum revolution in engineering and computing.

Current research is exploring how the signature oddities of quantum systems such as superposition and coupling can produce ways to surpass the classical limits in computing, communication, and sensing [5]. Quantum sensing is a particularly exciting field, rapidly expanding its use as advancements bring on new possibilities. It measures external fields, such as magnetic fields, through the change those fields impart on the energy gap within the quantum system. This property does not have a bias for the magnetic field source, so quantum sensing can be applied in navigation technology measuring the earth's magnetic field as much as it can be applied in medical technology measuring magnetic field produce by brain signals. To create this technology, there has to be further research into an ultra stable narrow frequency standard, which is the output of the quantum system used to measure the magnetic fields. Superradiance is a promising candidate to create scalable quantum sensing systems because it exploits the quantum coupling property to achieve frequency bandwidths narrower than classical limit and the standard quantum limit. In a superradiant quantum system, numerous atoms act collectively in a choral way - which in effect washes out some of the local variations between atoms when atoms act individually. The quantum system's output frequency is what is actually measured, so that ultra stable narrow frequency that the superradiant system produces is the necessary foundation to make precise measurements. Currently it has been shown that superradiance can be achieved with quantum dots [18] and in nitrogen vacancy diamond [1], but it remains an open question if superradiance can be achieved in the solid state with the more affordable and available material: silicon carbide.

Determining if silicon carbide is a viable superradiant source requires a theoretical exploration into its dynamics in order to determine the experimental parameters that could ultimate test its superradiant viability. Developing a modular Master Equation provides an avenue to explore different quantum systems in a series of predictions and vetting that robustly explores the quantum behavior. Modeling this behavior will provide depth to current publication and predict future experimental regimes, which can be confirmed and ultimately integrated into quantum sensing technology.

2 Background

Before exploring superradiance it is important to understand the fundamental differences between the classical and quantum descriptions of light. Classical physics dictates that energy falls on a continuous spectrum, in the way a baseball can be thrown at any speed. The fundamental difference in quantum mechanics is that energy comes in discrete packets, based on the difference between the specific energy levels the object can occupy. This would be like a baseball that can travel 5 and 10 MPH but not 6, 7, 8, or 9 MPH and does not pass through those speeds while changing from

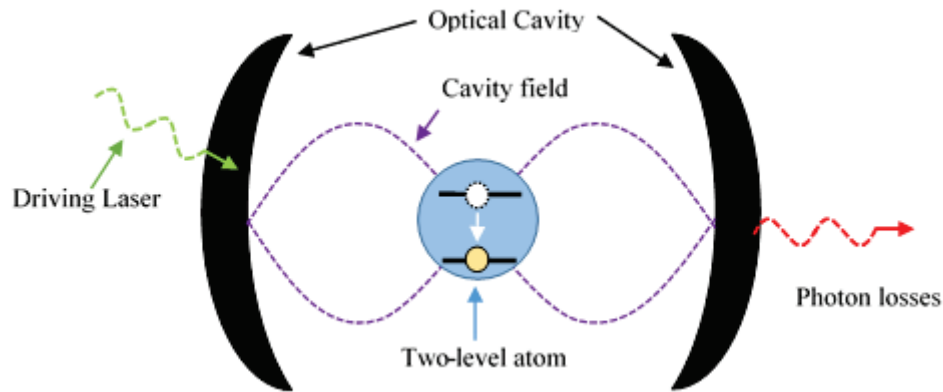


Figure 1: Components of a quantum electrodynamics system

5 to 10 MPH: rather absurd to believe. In short, baseballs do not behave this way because they are big, but photons are quite the opposite. In the quantum description of light, the unit of energy transfer of light, the photon, has quantized energy levels. These energy levels reflect countable "quanta" of energy in the different components of the system. We can identify the elements of the system: the emitters, the cavity, and driving source. Each constituent element has quantized energy levels, so the entire system has quantized energy as well. Its energy does not fall on a spectrum like the baseball, so we describe the energy of the system by the probability of the system being in its different quantized energy states.

As seen in figure 1, the emitters are atom-scale objects represented by the 2 or more energy levels they contain, or the two parallel lines when shown pictorially. The cavity is the localized area that contains the emitters and the electromagnetic energy build up. The physical construction of the cavity will support different electromagnetic modes, represented by the sinusoids with nodes at each end of the cavity. The sinusoidal pattern is just generally representative of how electromagnetic energy inside a cavity behaves. The amount of energy is related to the wavelength of that sinusoid. The emitter does not need to be at the node of this wave pattern; the wavelength of the electromagnetic energy just needs to correspond to a quanta of energy that matches, or nearly matches, the energy difference in the emitter. The cavity-emitter system exists in some larger environment, which interacts through energy entering the system in a driving form and energy leaving the system through the photon losses.

The photon losses are particularly important in superradiant systems because it prevents significant back action that disrupts the coupling effect. Coupling is what gives these systems their quantum signature, as the emitters act in a collective, choral way rather than individual emitters. When developing the theory of superradiance, Robert Dicke preposed that assuming that emitters acted individually was an error and that emitters directly influenced one another through their coupling to the common radiation mode [6]. In his original paper he analyzed the cavity-less system in order to isolate the quantum effects. Bad cavities are cavities that lose energy at a rate much, much higher than the rate they reflect energy so the bad cavity limit approximates a cavity-less system to an excellent degree. When energy is reflected the photons can interact directly with the emitters, which can disrupt the coupling to the common radiation mode. In the superradiant regime there is a non-linear enhancement in the quantity of light produced and the frequency distribution narrows. Non-linear enhancement is in stark contrast to the linear relationship between

the emitters and number of photons produced in a classical system. Dicke predicted a n^2 number of photons produced by a superradiant system, where n is the number of emitters in the system. There is always a distribution around a central frequency for electromagnetic energy, however the coupling effect makes the ensemble of emitters act like a single emitter which narrows the width of this distribution by a factor of \sqrt{n} .

These superradiant emission signatures have motivated the steady pursuit of a superradiant laser as a ultra-stable narrow frequency standard. There has been success in developing a superradiant laser in the gaseous state [3], laser cooled atomic ensembles [2], and success in achieving a superradiant emission from quantum dot [10] and diamond defect [1] systems. Solid state defects are specifically engineered systems where individual atoms in a lattice are removed or replaced in order to create an emitter. The lattice serves as a structural framework that aids in spacing the emitters properly, while the extra electron and absent atom act like a spin-active system. In diamond, two carbon atoms are removed and one is replaced with a nitrogen atom to form a nitrogen-vacancy or NV center diamond. Solid state defect systems would be the most desirable superradiant systems to integrate because of the relative ease on manufacture and structural stability, yet this remains underdeveloped due to the expense of diamonds. A similar spin-active defect can form in silicon carbide by removing a silicon atom to create a silicon vacancy, but the superradiant regime for silicon vacancy silicon carbide remains unexplored.

Creating the superradiant solid-state laser would create the frequency standard that approaches the quantum limit [19] or even surpasses it. Quantum sensitivity has expanded our ability to understand the world around us, from applications as broad as systems capable of detecting gravitational waves [4], improving atomic clocks [14], improving Nuclear Magnetic Resonance imaging [9], and even advancing bio-sensing capabilities [7]. The quantum revolution will continue to redefine what is possible, and exploring superradiance is a thrilling step in that process.

3 Methods

Quantum mechanics is explored by developing an Hamiltonian and constructing the appropriate equation to describe how the system interacts with its environment. The Hamiltonian describes the energy mechanism of the system, and breaks down complex processes into its constituent fundamental actions. These actions correspond to base changes in quantum systems such as raising or lowering the energy level, and populating or annihilating a state in the system. More complex processes are described with different combinations of these raising, lowering, populating, and annihilating actions. We leverage this breakdown because we can express them mathematically through the operators that make up the basis of quantum mechanics. Once we have created our operators, we can determine what advanced processes we need to include in the Hamiltonian. The Hamiltonian can then be put into the equation of state used to evolve the system. A superradiant system is an open system because information escapes the system into the environment through photons, so we have to use an equation that describes this exchange. This entails using the Master Equation: the open system equivalent of the possibly more familiar Schrodinger equation. In our case we assume that the system-environment is such that each snapshot is independent of the former: a memory-less exchange. Formally, this is known as the Markov approximation and will be discussed further in the Master Equation derivation.

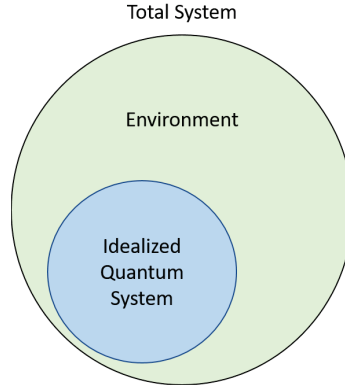


Figure 2: The most general system includes the quantum systems and its surrounding environment

3.1 Master Equation Derivation

As previously mentioned, in order to study quantum mechanics we must develop an equation that properly describes how a system evolves in time. The most general example we can consider would be one in which the total system includes the quantum system and its surrounding environment: shown in figure 2. This is the starting point to study any body that exists in nature, because there is always information that exchanges between the quantum system and its surrounding environment.

However, this is an impossible system to fully evolve in time because the "environment" extends to include the entire universe. This would require a description of every possible state of the universe, which requires a lot of computing power and a few Nobel prizes. This system can be simplified though, by looking at the information exchange dynamics across the open boundary. One of the most important and common assumptions to make is the quantum system connects to external baths via a Markovian interaction. This means that information escapes from the quantum system into the immediate surrounding baths so quickly that information can flow across the boundary with no dependence on whether or not information was exchanged immediately preceding. This is known as a "memoryless" interaction, because the baths have no memory of the last time information escaped the system into the environment. This assumption reduces the system we have to consider to just the quantum system and its immediately surrounding baths. This dramatically simplifies the total system and can be evolved in time with a Lindbladian Master Equation. The following derivation has been well established for many years, and this specific derivation is based on the publication in [15].

Before we can evolve this open system in time we have to briefly revisit closed systems and how they evolve in time. In physics, the *Hilbert Space* describes the set of states that a quantum system can exist in. This is the exhaustive list of all possible states the quantum system could exist in. In a closed system, the description of a specific quantum object within the Hilbert Space is given by its wavefunction: ψ . This wavefunction contains the probability density of the quantum object existing in each of the possible states within the Hilbert Space. The wavefunction for a closed system evolves in time with the Schrodinger equation:

$$\frac{d}{dt}|\psi(t)\rangle = \frac{-i}{\hbar}H|\psi(t)\rangle \quad (1)$$

Open systems still have a Hilbert Space, but cannot be fully described by a single wavefunction.

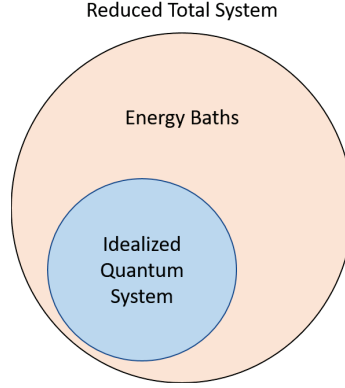


Figure 3: Applying the assumption of a Markovian interaction simplifies the total system we consider

Instead, we have to consider the product of wavefunctions that will contain both pure and mixed states. This results the *density matrix* and is expressed by:

$$\rho \equiv \sum_i |\psi_i\rangle\langle\psi_i| \quad (2)$$

As the product of two wavefunctions, the diagonal is the product of probability density for pure states, making it pure probability for the pure states. As such, the trace of the density matrix must be 1 and self-preserving as no probability can escape the Hilbert space. In order to understand how the density matrix evolves in time, we can combine the previous two equations.

$$\frac{d}{dt}|\psi_i\rangle\langle\psi_i| = \frac{-i}{\hbar} (H|\psi_i\rangle\langle\psi_i| - |\psi_i\rangle\langle\psi_i|H) \quad (3)$$

Which we can rewrite using the commutator expression: $[A, B] = AB - BA$

$$\dot{\rho} = \frac{-i}{\hbar} [H, \rho] \quad (4)$$

This is the Von-Neumann equation. This is how we would evolve the density matrix of the entire universe if we could. The first step in writing a Lindbladian master equation is to trace over the environment degrees of freedom in order to eliminate them. This prevents us from having to express the quantum states of the environment as part of our Hilbert Space. Now we separate the Hamiltonian into its effect on the system and the environment:

$$H_{Total} = H_{sys} \otimes \mathbb{I}_{env} + \mathbb{I}_{sys} \otimes H_{env} + \alpha H_I \quad (5)$$

This equation says that the total Hamiltonian can be expressed as the energy of the system, energy of the environment, and the interaction weighted by α between the two. The \mathbb{I} matrices of the environment and system represent identity matrices of their respective Hilbert spaces, so the Kronecker products are just preserving the total Hilbert Space. Since we have split up the Hamiltonian, we can step into the interaction picture where the density matrix evolves in time governed by the interaction Hamiltonian.

$$\frac{d\hat{\rho}_T}{dt} = \frac{-i}{\hbar} \alpha [\hat{H}_I, \hat{\rho}_T] \quad (6)$$

Which we can integrate:

$$\hat{\rho}_T(t) = \hat{\rho}_T(0) - i\alpha \int_0^t ds [\hat{H}_I, \hat{\rho}_T(s)] \quad (7)$$

In order to prevent integrating over all previous time, we replace $\hat{\rho}_T$ in equation 6 with the definition in equation 7.

$$\frac{\hat{\rho}_T(t)}{dt} = \frac{-i}{\hbar} \alpha [\hat{H}_I(t), \hat{\rho}_T(0)] - \alpha^2 \int_0^t ds [\hat{H}_I(t), [\hat{H}_I, \hat{\rho}_T(s)]] \quad (8)$$

The term $\hat{\rho}_T(s)$ still depends on the previous state of the system, which we are still looking to eliminate by integrating over. So we apply the same process one more time and get the equation:

$$\frac{\hat{\rho}_T(t)}{dt} = \frac{-i}{\hbar} \alpha [\hat{H}_I(t), \hat{\rho}_T(0)] - \alpha^2 \int_0^t ds [\hat{H}_I(t), [\hat{H}_I, \hat{\rho}_T(s)]] + O(\alpha^3) \quad (9)$$

The term at the end represents operators that depend on third order interactions between the system and its environment. Our first approximation is that the interaction between the system and the environment is small, so α^3 is negligible, and we can remove that term. Next, as mentioned above, we reduce the system by integrating over the environmental degrees of freedom: this will reduce the density matrix from the total system (ρ_T) to just the quantum system and surrounding baths ρ

$$\frac{\hat{\rho}_T(t)}{dt} = Tr_E \left[\frac{\hat{\rho}_T(t)}{dt} \right] = \frac{-i}{\hbar} \alpha Tr_E [[\hat{H}_I(t), \hat{\rho}_T(0)]] - \alpha^2 \int_0^t ds Tr_E [[\hat{H}_I(t), [\hat{H}_I, \hat{\rho}_T(s)]]] \quad (10)$$

This equation is not yet a closed time-evolution of just $\rho(t)$. In order to reduce it, we assume at $t = 0$ the system and environment can be expressed in the separable form $\rho_T(0) = \rho(0) \otimes \rho_E(0)$. This establishes that there are no correlations between the system and environment at the zero-time point. This is valid for Markovian systems as the correlation timescales are much smaller than any other system timescale. This, in short, allows us to eliminate the first term in equation 10 because it results in a energy shift that is unchanging in time and has no effect on system dynamics. It also allows us to decouple the density matrix of the quantum system and the surrounding baths, as the lack of correlation means the total density matrix can be represented by a product of the quantum system and environment.

$$\dot{\hat{\rho}}(t) = -\alpha^2 \int_0^\infty ds Tr_E [[\hat{H}_I(t), [\hat{H}_I(s-t), \hat{\rho}(t) \otimes \hat{\rho}_E(0)]]] \quad (11)$$

To rewrite the interaction Hamiltonian, we first need to expand it by saying that there are operators who are eigenvectors of the interaction superoperator. These operators can be expanded as:

$$C_k = \sum_w C_k(\omega) \quad (12)$$

Where the operators fulfill the condition:

$$[H, C_k(\omega)] = -\omega C_k(\omega) \quad (13)$$

Since ω is an eigenvalue of H , we can easily express the Hermitian conjugate:

$$[H, C_k^\dagger(\omega)] = \omega C_k^\dagger(\omega) \quad (14)$$

Now we can rewrite the interaction Hamiltonian:

$$H_I(t) = \sum_{k,\omega} e^{-i\omega t} C_k(\omega) \otimes E_k(t) = \sum_{k,\omega} e^{i\omega t} C_k^\dagger(\omega) \otimes E_k^\dagger(t) \quad (15)$$

If we put this back into equation 11, and complete some complex algebra, we arrive at the equation:

$$\dot{\hat{\rho}}(t) = \sum_{\omega,\omega',k,l} \left(e^{i(\omega'-\omega)t} \Gamma_{kl}(\omega) [C_l(\omega) \hat{\rho}(t), C_k^\dagger(\omega')] + e^{i(\omega-\omega')t} \Gamma_{lk}^*(\omega') [C_l(\omega), \hat{\rho}(t) C_k^\dagger(\omega')] \right) \quad (16)$$

We can simplify this equation using the rotating wave approximation, where we realize in the time dependency that terms with $|\omega - \omega'| \gg \alpha^2$ will oscillate much faster than the coupling timescale of the system. Thus, these terms do not contribute to the oscillatory behavior of the overall system. This reduces our equation to:

$$\dot{\hat{\rho}}(t) = \sum_{\omega,k,l} \left(\Gamma_{kl}(\omega) [C_l(\omega) \hat{\rho}(t), C_k^\dagger(\omega)] + \Gamma_{lk}^*(\omega) [C_l(\omega), \hat{\rho}(t) C_k^\dagger] \right) \quad (17)$$

Separating the Hermitian and non-Hermitian parts of the dynamics regrouping allow us to write the equation as:

$$\dot{\rho}(t) = \frac{-i}{\hbar} [H + H_{LS}, \rho(t)] + \sum_{\omega,k,l} \gamma_{kl}(\omega) \left(C_l(\omega) \rho(t) C_k^\dagger(\omega) - \frac{1}{2} C_k^\dagger C_l(\omega), \rho(t) \right) \quad (18)$$

The H_{LS} is called the *Lamb Shift* Hamiltonian and it is responsible for renormalizing the system energy levels due to the interaction to the environment. In our system this will take the form of connecting the emitters to the common radiation mode. The last reduction to make it realizing that the matrix formed by the γ_{kl} coefficients is positive and therefore can be diagonalised. Thus writing the master equation in its diagonal form will give us our Linblad Master Equation.

$$\dot{\rho}(t) = \frac{-i}{\hbar} [H, \rho(t)] + \sum_n \frac{1}{2} \left(2C_n \rho(t) C_n^\dagger - C_n^\dagger C_n \rho - \rho C_n^\dagger C_n \right) \quad (19)$$

This is the general form of the master equation we will use to describe our system. It captures the interaction between a quantum system and its surrounding baths, with the assumption that the system-bath correlation timescales are smaller than any other system timescales.

3.2 Creating a Hamiltonian

To create a Hamiltonian, it is important to consider broadly what components make up the quantum system and how they will interact with one another. Hamiltonian mechanics is incredibly useful in the way it establishes clarity in the cause and effect relationships between the quantum system components and the results. The superradiant system at its core will have atom-like emitters, cavity modes, and the interaction between the emitters and cavity. Calling the emitters "atom-like" just means that whatever the object is, it has two distinct energy levels to transition between. There are numerous system of interest that behave this way that are not technically individual atoms, so it is more appropriate to refer to them as emitters. The cavity modes is the energy stored as photons inside the optical cavity before they escape. The important quantum behavior is the cavity-mediated interactions between independent emitters. This is addressing the oversight of classical descriptions of light that Robert Dicke pointed out in his seminal paper on superradiance [6]. In essence it means that energy transitions in one emitter can simultaneously influence other emitters through the energy states in the cavity. This differs from the classical picture which dictates that emitters release their energy in a cascading, sequential manner and are 'unaware' of other emitters around them.

To describe these elements we need a mechanism to describe populating the cavity mode, changing between the two emitter energy levels, and coupling the emitters to the cavity mode. Therefore we know our Hamiltonian will have three components: cavity, emitter, and interaction. In order to write out these terms we have to develop our operators and understand how they function so that we can combine them to describe these three terms.

3.3 Operators

At first approach it would seem impossible to try and write down an operator for every action that can be done to a system of any size. Yet, the properties of linear algebra allow us to expand the operator written for the simplest system so that it can function on a system of any dimension. Furthermore, entire processes can be broken down into basic steps that we find we can describe with just five operators. Now we can vet the physics of five individual operators on their simplest application and have complete confidence that when woven into a complicated equation, all of the physics will be sound. In our quantum system, the 5 basic actions we need to describe is: going up a spin state, going down a spin state, changing between the atom's energy level, absorbing a photon, and releasing a photon.

3.3.1 Spin Operators

Spin in quantum mechanics is vaguely a description of energy and its orientation in a atomic scale system. The *spin-active* qualifier in my title is describing a quantum object with two possible energy orientations: the upper, spin up, or the lower, spin down. Unimpressed by the number of names already, physicists normally refer to these states as the *excited* "e" and *ground* "g" state respectively. The shorthand mathematical notation to describe these states is $|e\rangle$ and $|g\rangle$. To go between these two states we need an operator to go up, and an operator to go down. Astutely named, the operator to raise the energy is the spin up operator and it acts as follows:

$$\sigma^+ |g\rangle = |e\rangle \quad (20)$$

Similarly, to lower the energy state, the spin down operator acts as follows:

$$\sigma^- |e\rangle = |g\rangle \quad (21)$$

Both the spin up and spin down operator have the important property that when they act on a state already at its upper or lower limit, the operator annihilates the state. This means that applying the spin up operator on the excited state returns "0" rather than just returning state unchanged. As a result, when the system has multiple atoms, the operator has to be able to distinguish between the states that describe different atoms and be able act only on the desired atom. As an example, if the system has two atoms the raising operator would do the following:

$$\sigma_{1st\ atom}^+ |gg\rangle = |eg\rangle \quad (22)$$

$$\sigma_{2nd\ atom}^+ |eg\rangle = |ee\rangle \quad (23)$$

$$\sigma_{1st\ atom}^+ |ee\rangle = |0\rangle \quad (24)$$

These properties prevent the system from exploding to an unrealistic infinite population in the excited or ground state. This properly models the physical mechanism the emitters use to store energy: the potential energy based on the electrons orientation. As such, the spin operator only applies to the emitters because of this annihilation property. Since there are only two energy orientations, the total probability of the emitter's energy state must be conserved between the excited and ground state, so more probability cannot be added by absorbing energy. The spin up or down orientation has implications on whether that draws or contributes energy to the system, which is described by the spin-z operator.

The spin-z operator describes spin along the z-axis, which sets the 0 point for energy in between the two energy level in the emitter. It dictates that when the emitter is in the excited state energy is absorbed from the system and when in the ground state the energy is added to the system. The spin-z operator is the mechanism to describe how the energy stored inside the atom is behaving. Cavity energy states differ in that their mechanism to store energy is the number of photons that populate that energy state. Because photons are bosons, they can infinitely populate the same state at zero Kelvin.

3.3.2 Ladder Operators

Bosons filling the same state without limit, and being able to completely empty, presents a different challenge than swapping between energy levels in the emitter. Notably, each cavity population can exist for every possible emitter configuration. This makes the cavity population act as a "subspace" that linearly expands the number of possible configurations the system could be in. This modifies the notation with an exterior subscript n that denotes the number of photons in the cavity mode. If there is a system with one emitter and a maximum capacity of two photons the possible states are:

$$ground : |g\rangle_0, |g\rangle_1, |g\rangle_2 \quad (25)$$

$$excited : |e\rangle_0, |e\rangle_1, |e\rangle_2 \quad (26)$$

We create a maximum photon capacity to establish a sufficiently large, but finite, dimension to encapsulate the system. The operators to populate and annihilate these subspace states are called the creation and annihilation operators, sometimes called the ladder operators because they can go up and down a rung on the energy ladder. These operators do not act on the emitters so they do not have to be able to distinguish between atoms. Their base functions work as follows:

$$a^\dagger |e\rangle_n = \sqrt{n+1} |e\rangle_{n+1} \quad (27)$$

$$a |e\rangle_n = \sqrt{n} |e\rangle_{n-1} \quad (28)$$

These operators combine so frequently that their pairing is called the *number operator* because it multiplies the state by the number of photons it contains as seen below:

$$a^\dagger a |e\rangle_n = n |e\rangle_n \quad (29)$$

The creation and annihilation operators, similar to the spin up and down operators, kill states that try and go beyond the upper or lower limit of the photon subspace. This does not conflict with the boson property because the subspace accounts for the state without any photons, the vacuum state, and trying to annihilate energy from the vacuum state is nonphysical. Similarly, raising the population beyond the ceiling creates an unbounded model. As such the operators have the final property:

$$a^\dagger |e\rangle_{nmax} = |0\rangle \quad (30)$$

$$a |g\rangle_0 = |0\rangle \quad (31)$$

These five operators are the fundamental actions that allow us to explore superradiant quantum systems. With each operator working properly at the fundamental unit level, it can expand to any generalized dimension with Kronecker products. Each emitter in the system expands the the dimensions of the system by a factor of 2, so it grows exponentially. That growth is still linearly scaled by the photon subspace as mentioned above, and still incorporated in the dimensions of the operators. A functional set of operators will be able to covert any initial state vector into any other state vector, regardless of how many emitters or photons are in the system. With that in hand the operators could be strung together to form the components of the Hamiltonian.

3.4 Hamiltonian Terms

An important idea in quantum mechanics is that energy comes in quantized packets equal to the product between the reduced Planck constant and frequency of the carrying wave: $\hbar\omega$. From this principle, we recognize that the energy contribution from the cavity and emitter terms should be this energy packet multiplied by a proportionality factor based on the state population. For the cavity term we have already seen that we can create the number operator from the creation and annihilation operator, and the ability of bosons to stack in the same state mean that the total energy

should be just the cavity energy multiplied by the total number of photons. Therefore we can express the cavity term in the Hamiltonian as:

$$H_{cavity} = \hbar\omega_{cavity}a^\dagger a \quad (32)$$

Similarly for the emitter term, we know the spin-z term already expresses the energy of the atom based on its excited or ground state status. Because it defines the 0 energy point in-between the two energy levels, a factor of a half pops out so that the emitter term does not double the energy gap between the energy levels. This transition will have its own frequency and is expressed as:

$$H_{emitter} = \hbar\omega_{emitter}\sigma_z \quad (33)$$

Cavity mediated interactions initially appear complex, but if we adopt a sequential view of how energy stored in the excited state of one emitter could transfer to the excited state of a neighboring emitter it becomes more clear. The initial excited atom would have to transition from the excited state down to the ground state (σ^-), and this would spontaneously create a photon inside the cavity (a^\dagger). This photon would then be seen by the neighboring atom and absorbed. That would cause an the atom to transition from the ground state up to the excited state (σ^+), and by conservation of energy the cavity photon would have to be annihilated (a). To be in the proper units of a Hamiltonian this process needs to be in units of energy, so the frequency of this process is called the *Rabi* frequency: Ω_R . How rapidly this cavity-mediated interaction occurs is a sign of the strength of the quantum coupling so at times the Rabi frequency is called the g factor. Operators act from right to left, so stringing together this process together in the proper order becomes:

$$H_{interaction} = \hbar\Omega_R(a\sigma^+ + a^\dagger\sigma^-) \quad (34)$$

This is also called the Jaynes-Cummings term. It has been widely studied in the intersection of quantum mechanics and electromagnetism, known as cavity quantum-electrodynamics. The Jaynes-Cummings term represents the interaction term we noted in our derivation of the master equation above. Now we have our full Hamiltonian:

$$H = \hbar\omega_{cavity}a^\dagger a + \hbar\omega_{emitter}\sigma_z + \hbar\Omega_R(a\sigma^+ + a^\dagger\sigma^-) \quad (35)$$

Vetting the physics going into the Hamiltonian is the bedrock to a sound investigation into new physics, so although developing the Hamiltonian appears to be a simple process, it is laborious making sure each operator maintain fidelity to its properties. The Hamiltonian is a springboard into true exploration as we can incorporate it into the appropriate equation to evolve Hamiltonians.

4 Master Equation Development

The results of this semester are best demonstrated through the development of the Master Equation, as the end result of the semester is a functioning model that exhibits non-classical enhancement for a generalized quantum system. However, creating this model is the result of numerous important vetting results along the way.

4.1 Basic Master Equation

The master equation is a differential equation used in physics to describe an open system, which is a system that interacts with its environment through the transfer of information. This is different than equations studied in the classroom, such as Schrodinger's equation, because all of the information in that system is self contained. Thus, the system can be describe with a single vector, with each position in the vector representing one of the quantized states it could be in. In contrast, the systems evolved in the master equation require a density matrix, ρ , to describe the system. The density matrix is an incredibly powerful way to describe a system, as the diagonal elements are the pure probability of the system being in the quantized states. The off-diagonal elements represent the coherence between the pure states, which can reveal very interesting physics. As a whole, the density matrix represents everything that is known about the system and everything that *can* be known about the system. To reveal some of the information the physics starts to feel like an archaeological dig, but before that can be done we have to construct the master equation to find the steady state density matrix.

The general form of the Master equation is:

$$\dot{\rho}(t) = \frac{-i}{\hbar} [H(t), \rho(t)] + \sum_n \frac{1}{2} [2C_n \rho(t) C_n^\dagger - \rho(t) C_n^\dagger C_n - C_n^\dagger C_n \rho(t)] \quad (36)$$

Where the C_n terms are jump operators, which form the dissipating connection between the system and its environment. This occurs through actions like spontaneous decay and cavity loss, where photons are escaping the quantum system into the larger environment. Information also passes inward through diving, whether it be coherent or incoherent. These terms are the *Liouvillian* terms because the sum can be expressed as the sum over all the Liouvillian super-operators: $\mathcal{L}(C_n)$. This equation may look more complicated, but it just expands upon physics that we can vet with our own intuition - including Shrodinger mechanics.

The first system to evaluate with the Master Equation is a simple spontaneous decay, to ensure that the excited state exponentially decays and the ground state conversely exponentially grows. We understand this system's behavior with enough depth that we confirm whether or not the system is working to create a first foothold before increasing complexity. To create the simple spontaneous decay model, we need to remove the cavity mediated interactions because we do not want to see any quantum effects. In addition, the only connection with the environment should be through spontaneous decay, which is the Liouvillian of the spin down operator. Therefore the Master Equation for the spontaneous decay system is:

$$\dot{\rho}(t) = \frac{-i}{\hbar} [H(t), \rho(t)] + \frac{1}{2} [2\sigma^- \rho(t) \sigma^+ - \rho(t) \sigma^+ \sigma^- - \sigma^+ \sigma^- \rho(t)] \quad (37)$$

Where,

$$[H(t), \rho(t)] = \hbar \omega_{cavity} [a^\dagger a, \rho] + \frac{\hbar \omega_{atom}}{2} [\sigma_z, \rho] \quad (38)$$

If we initialize this system in the pure excited state and evolve it in time according to the previous equation, we produce the following plot of the probability of finding the system in either the pure excited or ground state. For depth on how we use MATLAB to evolve the master equation in time reference appendix A.

The excited and ground state have the proper exponential exchange, and subsequent plots properly demonstrated that the photon subspace was unaffected by this system as expected. Without

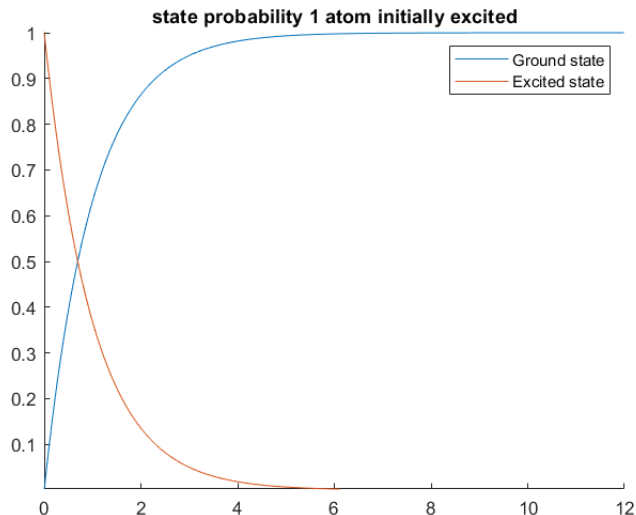


Figure 4: Spontaneous Emission with the Master Equation

anything driving the cavity modes and without cavity mediated interactions, the evolution of the emitter energy should have no holding on the photon population. This is one of the few plots where we are concerned with the actual state probability rather than trying to look at what we would physically measure, such as the photon number expectation value. When vetting the solver, we understand the physics in exchanging state probabilities rather than photon expectation values. However, superradiance can be revealed in the relative enhancement in the photon value rather, so the focus shifts away from state probabilities and into photon expectation values.

4.2 Including Basic Quantum Effect

The next thing to vet is whether or not the basic cavity-mediated relationship was working or not. When plotting the photon number, would we see the the energy release from the atoms into the cavity, and then escape into the environment. Of note, the photon number is not directly in the density matrix - we have to take an expectation value as the density matrix changes with time in order to get what we are looking for. An expectation value is in short a projection of the desired quantity on the system and it determines how likely it would be to find that quantity based on how much it overlaps with the system. For the photon number, we need the expectation value of the number operator to find out the probability of measure n photons in the cavity at time t . The expectation value of the number operator is:

$$\langle a^\dagger a \rangle = \text{tr}(\rho a^\dagger a) \quad (39)$$

Where tr is taking the trace of its argument. As previously stated, the diagonal elements of the density matrix are the pure probabilities, so the project of the photon number on each of these probabilities and summing them together gives the total probability of measuring all the photons all the states are producing. With a high escape rate the pure state probabilities will dominantly be in the 0-photon state which is why we achieve photon expectation values much less than 1.

To test this system, we isolated the coupling effect and included direct cavity loss - which is the Liouvillian term for the annihilation operator:

$$\dot{\rho} = -i\Omega_R \left[\left(a\sigma^+ + a^\dagger\sigma^- \right), \rho \right] + \frac{\hbar\kappa}{2} \left(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a \right) \quad (40)$$

Where κ is the cavity loss rate. In our model, this is the fastest time constant so that we can approximate a low quality cavity. Counter-intuitively, the more opportunities photons have to interact with atoms directly, the weaker the coupling between atoms. To test this system, the initial conditions have all the energy stored in the excited state in the atoms, and no photons inside the cavity. We should expect to see the photon number start at 0, rise to a maximum and decay back out into the environment. We should not see re-absorption in this system.

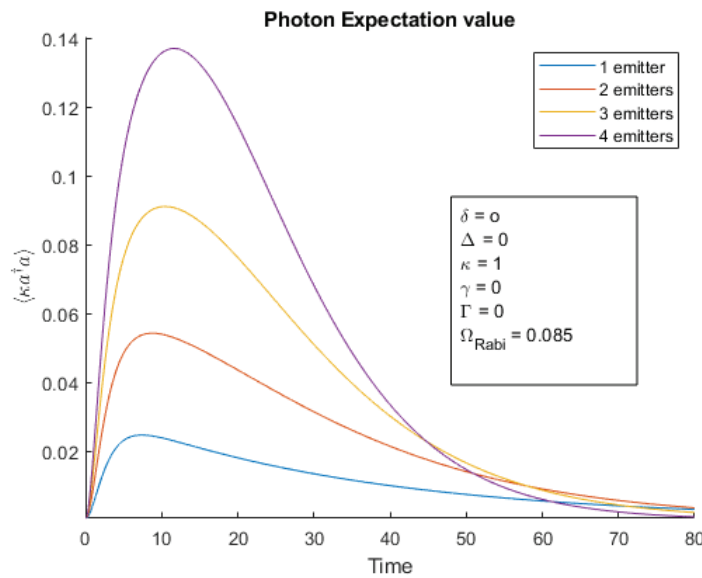


Figure 5: Testing Cavity Mediated Exchange

We see here even with a very weak Rabi frequency, we achieve the proper profile that we expect for this scenario. Furthermore, we see the expected increase in enhancement as a function of the number of emitters in the system. With the each element of the Hamiltonian functioning properly we have the footing to begin increasing the complexity.

4.3 Combining Effects

The repeated process of releasing photons into the cavity and absorbing them back into the emitters is called Rabi oscillations. In an ideal superradiant system, they would not exist because the photons are escaping the cavity so fast that they cannot be re-absorbed. However, Rabi oscillations are a quantum effect and indicative of the right interaction framework to explore superradiance. Seeing Rabi oscillations indicate that the parameters should be re-balanced to explore superradiance, rather than scrap the process. Balancing these parameters becomes a much more complex problem as our Master Equation at this point is more complete than the previous two. The number of terms

obfuscates some of the cause and effect that was very easy to determine on reduced equations. Now the master equation is:

$$\begin{aligned} \dot{\rho} = \frac{-i}{\hbar} * \left((\hbar\omega_{cavity}) [a^\dagger a, \rho] + \frac{\hbar\omega_{atom}}{2} [\sigma_z, \rho] + \Omega_R [a\sigma^+ + a^\dagger\sigma^-] \right) \\ + \frac{\hbar\gamma}{2} (2\sigma^+ \rho \sigma^- - \sigma^- \sigma^+ \rho - \rho \sigma^- \sigma^+) \\ + \frac{\hbar\kappa}{2} (2a\rho a^\dagger - a^\dagger a \rho - \rho a^\dagger a) \end{aligned} \quad (41)$$

This system now includes the energy associated with the photons in the cavity, the energy stored in the atoms, the energy exchange between the cavity and atoms, along with spontaneous decay and cavity losses. This system does not have any driving to perpetuate the effect so we expect the steady state value to go to 0. Here we are starting to explore superradiance, so we continue to take the expectation value of the photon number as a function of time. After using the same initial conditions as the last system we achieve the plot:

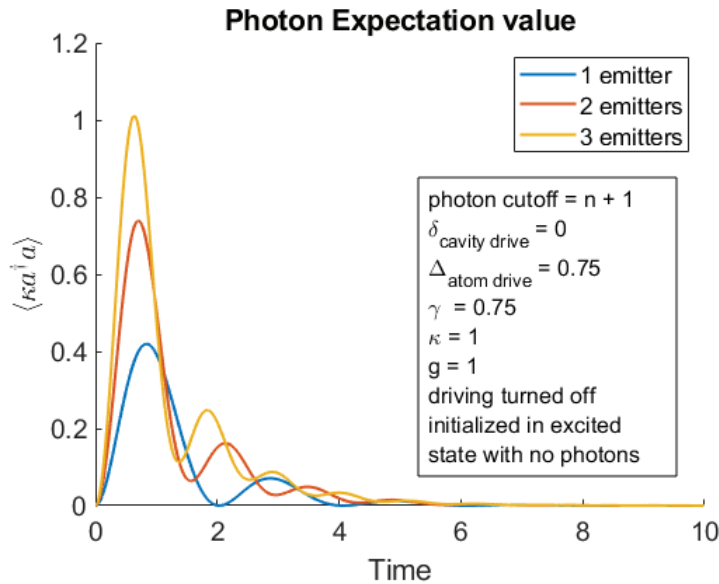


Figure 6: Quantum effects continue with more complex master equation

The important effects seen in figure 6 are the Rabi oscillations and the narrowing primary emission peak. The Rabi oscillations are indicating a strong quantum behavior, but not the proper cavity regime to explore superradiance. However, in this system the Rabi frequency is equivalent to the cavity loss rate which is a set up that would be incredibly unlikely to find superradiance. This result does demonstrate a narrowing primary peak, which is one of the hallmarks of quantum enhancement. Seeing multiple quantum behaviors in the result is an important result because it indicates that the physics contained in the model is consistent, even if the wrong regime is being tested.

4.4 Reaching the steady-state

In order to create a model that complements experiment, the model must be able to explore quantum behavior in the steady-state. There are numerous reasons that experimental work is done in the steady-state, one reason being the time scale for the single emission or steady-state ramp up is shorter than experimental resolution. Steady-state is only possible if there can be a balance of energy crossing from the environment into the quantum system, and from the system out into the environment. Up to this point we have had every system begin with energy stored in the quantum system, but no additional sources re-stimulating the emitters. The energy input can take numerous forms, however we have access to experimental superradiance data for a system of spin-active emitters incoherently driven. Incoherent driving can be approximated as a thermal distribution of energy that has some probability of entering the system, resulting in the spontaneous absorption of energy in the emitters at the rate Γ . Now putting the system in the same initial conditions, no photons inside the cavity and all emitters in the spin up state, and including the incoherent driving term should see an enhancement peak lead into a non-zero steady expectation value. We can conduct our analysis in natural units, where we set $\hbar = 1$ since it is an energy constant. As such, when propagating the equation 42, we find the results in figure 7:

$$\begin{aligned} \dot{\rho} = -i * & \left((\omega_{cavity}) [a^\dagger a, \rho] + \frac{1}{2} \omega_{atom} [\sigma_z, \rho] + \Omega_R [a\sigma^+ + a^\dagger\sigma^-] \right) \\ & + \frac{\gamma}{2} (2\sigma^+ \rho \sigma^- - \sigma^- \sigma^+ \rho - \rho \sigma^- \sigma^+) + \frac{\Gamma}{2} (2\sigma^- \rho \sigma^+ - \sigma^+ \sigma^- \rho - \rho \sigma^+ \sigma^-) \\ & + \frac{\kappa}{2} (2a\rho a^\dagger - a^\dagger a \rho - \rho a^\dagger a) \end{aligned} \quad (42)$$

Notably, this plot contains many of the signatures we are looking for in our quantum system. The peak photon expectation value occurs earlier with more emitters. In addition, the Rabi oscillations before the steady-state expectation value demonstrate that in addition to being incoherently driven, there are cavity-mediated interactions exchanging the energy between the emitters and cavity mode before escaping.

We now look to vet our master equation against experimental data, in order to demonstrate that our minimal equation can capture all the relevant system dynamics. If our model aligns with experiment with high fidelity, then our development from first principles creates an accurate master equation and we can extend our model to predict new regimes by including new terms from our first principles approach. Dr. Joel Grim and Dr. Sam Carter conducted an experiment at the Naval Research Laboratory demonstrating superradiance for up to three quantum dots in a waveguide and gave us their original data so that we could vet our master equation.

4.5 NRL Experiment

A natural question is how do we know that our model, which started from first principles, is a good candidate to model this specific experiment? We start by looking at each element of the experiment, and can compare that to our model. First looking at the emitters, they used Indium-Gallium-Arsenide (InGaAs) quantum dots. These are man-made nanoscale objects that behave like fabricated atoms. They are semi-conductors whose band gap behaves like its two energy levels

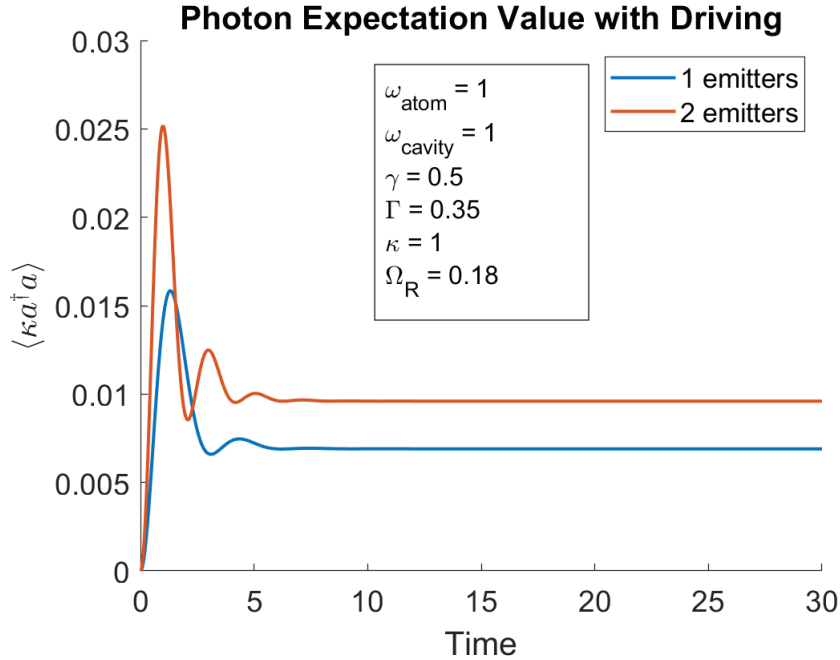


Figure 7: Photon Expectation Number for Incoherently Driven Quantum System

[17]. These emitters will have no solid-state interactions, such as phonon broadening, so they can be treated simply as two-level emitters. Our model represents these exactly, as any quantum object that has two energy levels will have energy associated with its spin state σ_z , spontaneously emit, and be able to be driven. The cavity in the NRL experiment was a waveguide, which is a directional lattice that allows the electromagnetic energy to flow along its longitudinal axis. Because there are no mirrors providing back-action, the energy escapes immediately which behaves like a very bad cavity. While it may seem that a waveguide should not be considered to be a cavity, the important property is having a common radiation mode that couples with the emitters. Even very bad cavities and waveguides have this common radiation mode, so we can model a waveguide as a cavity who has a very strong loss rate. The system is driven by a thermal distribution, which is incoherent. A coherently driven system would be the quantum dots pumped by resonant lasers. This means that we expect the NRL experiment to have associated with the cavity radiation mode, the spin state of the quantum dots, and the exchange between the quantum dots and cavity mode. Those are the three pieces of our Hamiltonian. The NRL system couples to its environment through incoherent driving, cavity loss, and the spontaneous decay of the quantum dots. Thus, the NRL experiment is a perfect candidate to vet our model because each element can be represented individually by the terms in our Hamiltonian or Liouvillian, and we have the proper interaction terms. Now that we have an experiment to compare our model to, we need to extract the measurements made during the experiment from our model.

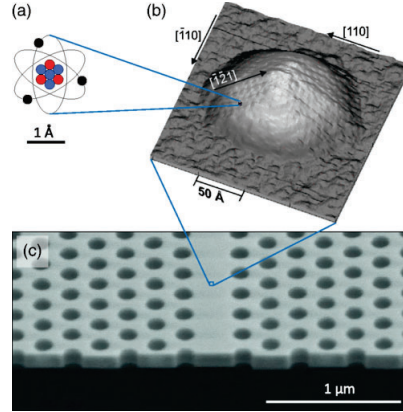


Figure 8: A close up of an InGaAs quantum dot within a waveguide [13]

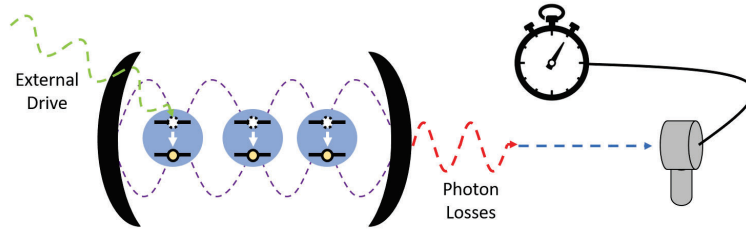


Figure 9: To measure the second-order photon correlation experimentally, time the space between measurements at the photon detector

5 Second-Order Photon Correlation Measurement

When constructing our master equation we look at properties like the photon expectation value inside the cavity because we have an intuitive understanding of what those plots reveal, and because they are much easier to express. Experiments with superradiant systems look at different quantity: the second-order photon correlation measurement or $g^{(2)}$. The quantity says if I experimentally measure a photon escaping the system at $\tau = 0$ what is the probability I will measure another photon at time τ later? This would be measured with a timer attached to a single-photon detector, such as that seen in figure 9.

In order to calculate this quantity we first need to let our system evolve to the steady state, which we have already established. When making a measurement in quantum mechanics, the system goes from a probabilistic distribution to the observed state. This is called collapsing the wavefunction. These observable, or measurable, quantities are represented by operators. To measure a photon we need to use operators to represent detecting a photon. This is done by combining the creation and annihilation operators into the number operator. We apply these to the steady-state density matrix to create new initial conditions.

$$Measure_{\tau=0} = a\rho_{steady-state}a^\dagger \quad (43)$$

We then revolve these initial conditions in time with the same master equation:

$$\begin{aligned}
\dot{\rho} = & -i * \left((\omega_{cavity}) [a^\dagger a, \rho] + \frac{1}{2} \omega_{atom} [\sigma_z, \rho] + \Omega_R [a\sigma^+ + a^\dagger\sigma^-] \right) \\
& + \frac{\gamma}{2} (2\sigma^+ \rho \sigma^- - \sigma^- \sigma^+ \rho - \rho \sigma^- \sigma^+) + \frac{\Gamma}{2} (2\sigma^- \rho \sigma^+ - \sigma^+ \sigma^- \rho - \rho \sigma^+ \sigma^-) \\
& + \frac{\kappa}{2} (2a\rho a^\dagger - a^\dagger a \rho - \rho a^\dagger a)
\end{aligned} \tag{44}$$

This time evolution will be represented by the superoperator $\hat{\mathcal{L}}\{\rho\}$. We use this shorthand so that we can neatly represent how we measure another photon at time τ later:

$$Measure_\tau = Tr \left[a^\dagger a \left(\hat{\mathcal{L}}\{a\rho_{steady-state}a^\dagger\} \right) \right] \tag{45}$$

In order to make this a probability, we normalize the quantity by dividing by the square of the initial measurement

$$g^{(2)}(\tau) = \frac{Tr \left[a^\dagger a \left(\hat{\mathcal{L}}\{a\rho_{steady-state}a^\dagger\} \right) \right]}{Tr (a\rho_{steady-state}a^\dagger)^2} \tag{46}$$

This was how we numerically calculated the second-order photon correlation. In relevant literature it is more common to see the quantity represented in standard quantum mechanics expectation value notation:

$$g^{(2)}(\tau) = \frac{\langle a^\dagger(t)a^\dagger(t+\tau)a(t+\tau)a(t) \rangle}{\langle a^\dagger(t)a(t) \rangle^2} \tag{47}$$

With this equation we now have a quantity that we extract from our model that compares directly to the experimental data taken at NRL. The important test of the model now is to see if we can get a high fidelity fit by adjusting the coefficients for the various terms in our master equation.

5.1 Determining the Proper Parameters

Finding the proper parameters to use is a substantial challenge because there are 6 different variables to balance. There are some parameters we can extract from the experimental set-up, but the actual weights of these coefficients may differ as there can be effective interactions that influence the behavior. As an example, the spontaneous decay term contains all of the dephasing for our system, so other effects that contribute to this dephasing may make this coefficient be different than the excited state lifespan. As such, we first refer back to literature to get a baseline for the relative scaling for our parameters. A study conducting master equation analysis comparing single-mode cavity and free-space superradiance found that the couple strength, Ω_R , should be close to, or less than a tenth of the cavity loss rate, κ [16]. Notably this work looked at coherently driven atoms, so we must consider that our incoherent driving may push us to have a slightly different weighting. Interestingly, they found that the pumping rate did not determine the radiation behavior if it was less than the loss rate. This establishes that our first foothold should be determining what the cavity loss rate is. Another work which looked at cold atomic ensembles with a high optical depth found that the cavity enhanced optical depth was determined by the atomic behavior, namely spontaneous

emission, more than it was driven by the Rabi frequency [11]. For their system they found that $\kappa = 9\text{MHz}$, $\gamma = 5.75\text{MHz}$, and $\Omega_R = 0.25\text{MHz}$. Exploring cavity-enhanced optical depth is a different effect than superradiance, but their system is in the bad-cavity limit and the weak coupling regime. This shows that the spontaneous decay term can possibly be weighted quite heavily, and the coupling strength can be even less than one-tenth of the cavity loss rate. Looking further into the difference between the weak and strong coupling regime, a study exploring interactions of single emitters in various cavity limits found that a system had to have the coupling strength less than the cavity loss rate and spontaneous decay rate in order for the system to be in the weak coupling regime [12]. In order for the system to be in the strong coupling regime, the coupling strength needed to be greater than both the cavity loss rate and the spontaneous decay rate. The important note for our system was ensuring that we were remaining true to the weak coupling regime and not standing in-between the weak and strong coupling regime.

The published paper for the NRL experiment had the lifetime of the quantum dot as 0.7 nanoseconds, and from some additional parameters we could determine the cavity loss rate. We were able to obtain the white-light transmission data for the waveguide, which means we had the data for the transmissibility across all relevant light wavelengths. We found that the effectively quality factor was 430, which is indicative of a bad cavity. From this we could calculate the lifetime of a photon inside the cavity by taking the distance the photon traveled and dividing that by the velocity inside the medium.

$$lifetime = \frac{d_{travel}}{v_{medium}} \quad (48)$$

The distance traveled is going to be the product of twice the length of the cavity, the average number of round-trips the photon takes within the cavity (Q-factor):

$$d_{travel} = 2 * Q * \ell \quad (49)$$

The velocity in the medium is the speed of light divided by the index of refraction, so the lifetime is:

$$lifetime = \frac{2 * Q * \ell * n}{c} \quad (50)$$

Which resulted in a cavity loss rate of approximately 6.35 GHz. With these two parameters as starting points and the relative scaling from the literature, we ran our master equation over numerous sets of parameters in order to determine how each parameter would effect our $g^{(2)}$ fit. We were first looking for the hallmarks of superradiance in $g^{(2)}$. The quantum behavior means that for more than 1 emitter, there is an enhanced probability of measuring a photon at $\tau = 0$ as the coupling effect draws out photons synchronously. There should also be a suppression for some relaxation period as the system resets, until it gets back to its steady state $g^{(2)}$ probability of 1. In the single emitter case, we should just see this suppression immediately, which is called antibunching.

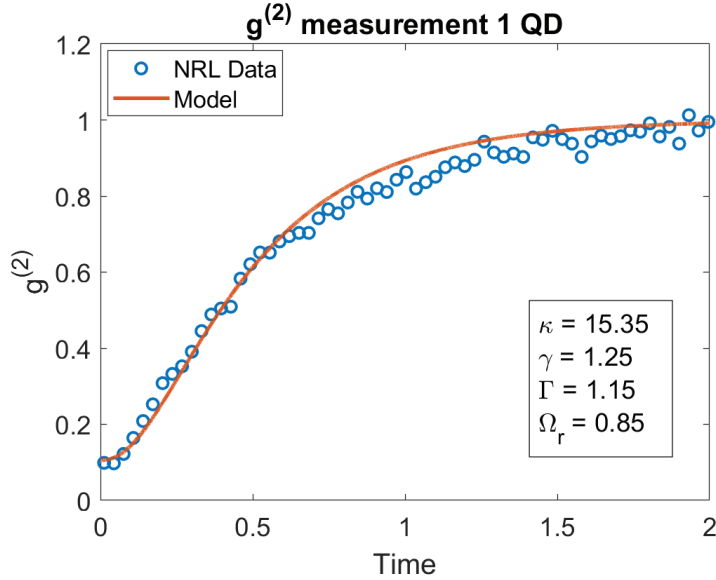


Figure 10: The comparison between our model and the experimental data for the single emitter

6 Results

Our initial results are incredibly exciting, particularly for the single emitter case. Our model fits the experimental data incredibly well, demonstrating that our model has the ability to describe the dynamics governing this quantum system.

Our extracted parameters fit within the relative scaling that we noted from the literature, and the weightings are substantially different from the starting values. The cavity loss rate as high as it is reflects a shorter lifespan, likely due to an overestimate of the Q-factor of the cavity. We expected the waveguide to have a very fast loss rate, and the cavity loss rate gave us the initial steepness in our curve. The challenge with the single emitter case is it is not as sensitive to the coupling strength or dephasing, so when we scale the system to two emitters we expect that we have to rescale some of the coefficients.

We see in figure 11 that the agreement is not nearly as strong as the single emitter case. We kept all the same parameters to see how well they governed the entire system, and introduced detuning between the emitters themselves. This δ represents the difference in energy levels between the first and second emitter in the system. We could use this parameter to control the initial enhancement at $\tau = 0$. Our model still has the right qualitative signatures for the two emitter system. There is an enhancement at $\tau = 0$ and a suppression immediately after that lasts through a relaxation period. The experimental data notably has a much more drastic suppression and longer relaxation period. In discussing this behavior with Dr. Grim, we determined that the strong suppression is most likely due to strong dephasing behaviors than we accounted for with our initial parameter set. The incredibly strong agreement for the single emitter and qualitative agreement for the two emitter system indicate that we could have much high fidelity by tuning the parameters specifically to this system. As previously mentioned, having multiple emitters in the system makes it more sensitive to many of the interactions we are trying to fit. We initially fit the parameters to the single emitter system because it was the quickest to compute.

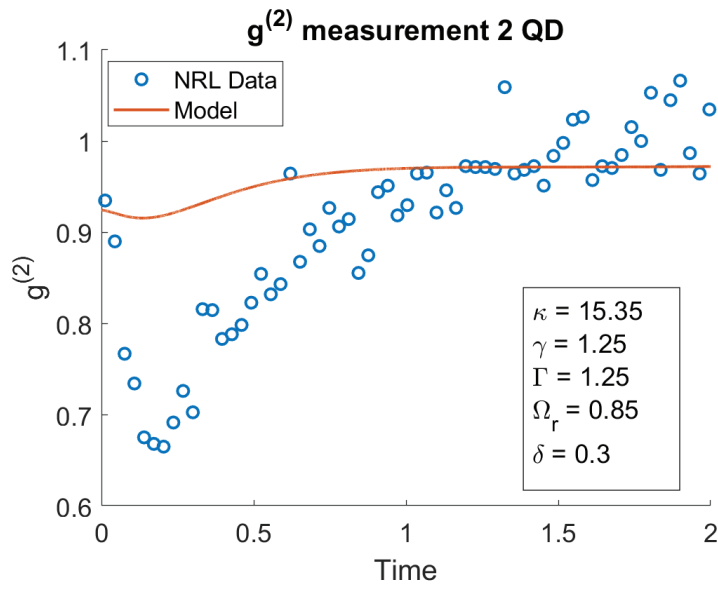


Figure 11: The two emitter second-order photon correlation comparison

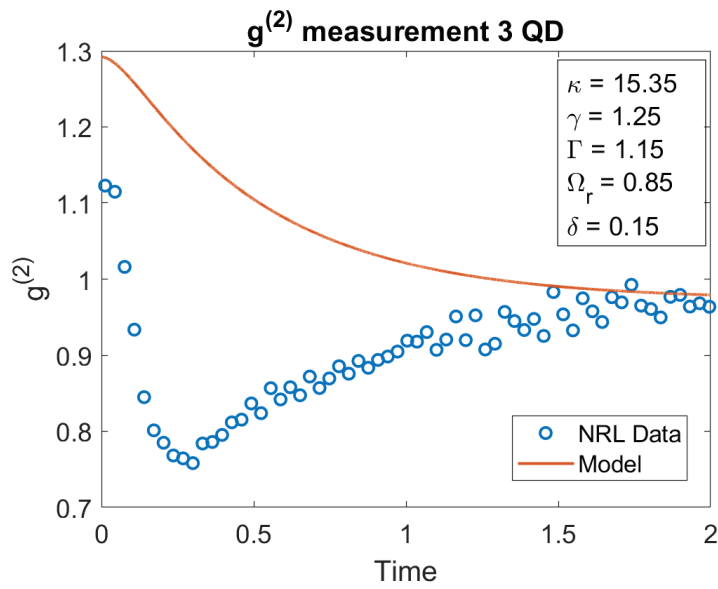


Figure 12: Second-order photon correlation comparison for 3 emitters

The issues we saw in the two emitter plot extend to the 3 emitter system as well. Our model predicts too strong of an enhancement and not enough dephasing dynamics to capture the suppression for the system. The lack of suppression points to the system likely being over driven, in that there may energy flowing into the system too quickly and the emitters re-emit together without a relaxation period. Optimizing the fit for this regime remains a focus for this model.

7 Conclusion

This project has produced a master equation built from first principles that can describe superradiant in open quantum systems. It may appear that the second-order photon correlation plots were the primary focus of our results, but they are just a means of testing what we have created. The master equation itself is what we set out to create because it is an incredibly powerful tool. We started from first principles of physics and made only few, but well understood, assumptions in order to create a system we could describe and propagate. We preserved an enormous amount of generality and expressed dynamics in a minimal model. The high fidelity prediction our model had for the single emitter second-order photon correlation measurement demonstrates that our master equation is an appropriate means to investigate these cavity quantum electrodynamics systems. We needed to start by recreating a well established experiment because that data should be able to draw out from modeling the system. Since we are able to accomplish that, we know that we can model n two-level emitters incoherently driven in the bad cavity limit by the following master equation:

$$\begin{aligned} \dot{\rho} = \sum_n \left(-i \left(\delta [\sigma_z, \rho] + \Delta [a^\dagger a, \rho] + \Omega_r [a\sigma^+ + a^\dagger\sigma^-, \rho] \right) + \frac{\gamma}{2} (2\sigma^- \rho \sigma^+ - \sigma^+ \sigma^- \rho - \rho \sigma^+ \sigma^-) \right. \\ \left. + \frac{\Gamma}{2} (2\sigma^+ \rho \sigma^- - \sigma^- \sigma^+ \rho - \rho \sigma^- \sigma^+) \right) + \frac{\kappa}{2} (2a\rho a^\dagger - a^\dagger a \rho - \rho a^\dagger a) \end{aligned} \quad (51)$$

Now if we want to predict new experimental regimes or explore different systems, we have a framework that we can build from and modify that appropriately describes the dynamics. Describing different experimental set ups are a matter of using different operators in either the coupling Liouvillian terms or in the Hamiltonian describing the systems internal energy. Those operators are well understood and come from the same first principles that we used to develop this specific master equation. This means we can explore quantum dots driven by a laser by exchanging incoherent driving for coherent driving by switching out the spontaneous absorption term for directly populating cavity states. We can explore solid state materials by including phonon broadening terms that contribute to the Hamiltonian, and we can expand our Hilbert space to include greater than two-level systems. We can use our model to find the superradiant regime for all these different types of systems, looking at the second-order photon correlation function and extracting the microscopic parameters to set up experimental design. Our master equation is an incredibly useful tool to explore quantum systems of interest, and our work vetting it against the NRL data gives us confidence to explore novel systems.

Future work will look at continuing to optimize the parameter space for the two and three emitter regimes to prove that our model contains the dynamics to capture that behavior. There are

numerous ways to do this, in varying levels of computational expense. Once the parameters have been optimized for these systems, we are most interested in applying our master equation to predict the superradiant regime for spin-active solid state defects. Spin-active defects are intentional variations in a crystalline structure to create an abnormality that acts like a multi-level emitter. One of the best characterized defects is the nitrogen vacancy in diamond. Diamond is just a carbon lattice, and if two adjacent carbons are removed with one replaced with a nitrogen, there will be an extra valence electron paired with the hole in the lattice. This behaves exactly like a two-level emitter and is the best on-ramp for our master equation to describe superradiance in the solid state. One of the main concerns with nitrogen vacancies is the fact that diamond is such an expensive material to work with. Silicon vacancy silicon carbide is another possible superradiant candidate, because very similar to nitrogen vacancy diamond, the missing silicon in the crystalline structure acts as a multi-level emitter. Silicon carbide is a ubiquitous, cheap material already widely manufactured for computing and its energy transition is within the telecommunications band. Using our master equation we can determine the baseline experimental parameters to begin testing the superradiant regime for silicon vacancy silicon carbide, leading to important experimental advancements.

This project, developing a master equation to describe few driven two-level emitters in the bad cavity limit, has produced an incredibly useful tool to analyze collective behavior of emitters in cavity systems. Exploring new materials, systems, and regimes will help discover new possibilities for superradiant lasing systems, which can be ultra-stable frequency standards for quantum sensing systems. These future sensors will go beyond the standard classical limit and hold new capabilities to detect magnetic and electric fields that escape modern technology. This could bring forward advances in medical, navigation, and telecommunications technology to aid in solving confounding problems, and surpass problems we have yet to even discover.

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A Coding the Master Equation

Differential equations can either be solved analytically or numerically. The former is more familiar, because it is problems where the solution is a function we can write down. A quick example is the simple harmonic oscillator differential equation:

$$\ddot{x} = -k^2x$$

We know that the second derivative of sinusoidal functions return themselves with a negative coefficient so the analytical general solution to this differential equation is:

$$x(t) = A \cos(kt + \phi) + B \sin(kt + \phi)$$

This is integrating by writing an anti-derivative. A numerical solution integrates through numerical integration, which means we do not write down a function that solves the differential equation. This is usually because we can't! There are many systems worth studying where we cannot write an equation that says that will be in this state at this time, but we can write a differential equation that says how the system will change in infinitesimal time steps. Since we know how the system changes, we can track those small changes and save the values at every time step. We can do this over many many steps to find out what the system will look like at our desired time, or we can plot these values to see how it is changing over a time span. The key idea is a numerical solution tracks the tiny changes from iteration to iteration and the output is the numbers at every time step.

If our equation is a function of time, then numerical integration puts in tiny time steps into the differential equation to see how the system changes in that small step. That change is added to the current state and repeated for the next tiny step. One of these steps mathematically would be;

$$x(t + \epsilon) = x(t) + \dot{x}(t + \epsilon)$$

This is the approach we have to take to evolve our Master Equation. The challenge is now to write our differential equation in a general way so that it will produce the proper system of equations for whatever parameters we use. As discussed in the main body of the text, this means creating the proper *Hilbert Space*. Because quantum mechanical systems occupy discrete states, everything can be described with matrices and the *Hilbert Space* is the dimension of the matrices. Thus, we combine some foundations quantum mechanics and linear algebra to write our terms.

The Hilbert Space is defined by the macro-parameters: the number of emitters we have in the system and the number of photons that can exist in the cavity at one point. In the bad cavity limit photons escape very quickly so we can set the photon limits relatively low without losing any accuracy. We consider two-level systems, so each emitter can either be in the excited or ground state. We also consider each emitter to be distinguishable, colloquially unique, so different combinations of two excited and one in the ground state all represent different states. This set of states just models binary counting, so the subspace due to the emitters scales as:

$$\mathbb{I}_{emitters} = \mathbb{I}(2^n)$$

Where \mathbb{I} is the identity matrix and n is the number of emitters in the system. To further draw this connection we can compare binary counting from zero to three with the different combinations for two emitters

$$\begin{array}{c} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{array} = \begin{array}{c} |gg\rangle \\ |ge\rangle \\ |eg\rangle \\ |ee\rangle \end{array}$$

The photon sub-states are easier. The photons are bosons, so any photon sub-state can occur at any emitter state. If we allow there to be m photons, the system could have 0, 1, 2 up to m photons. This means our possible states, or Hilbert Space, expands by a factor of $m + 1$ where m is the number of photons.

So if we consider a system with one emitter and up to 1 photon in the cavity, the diagonal elements of the density matrix will represent:

$$\mathbb{I}_{HilbertSpace}(2^n(m+1)) = \begin{bmatrix} |g\rangle_0 & 0 & 0 & 0 \\ 0 & |g\rangle_1 & 0 & 0 \\ 0 & 0 & |e\rangle_0 & 0 \\ 0 & 0 & 0 & |e\rangle_1 \end{bmatrix}$$

This is how the system dimensions will expand with larger and larger systems. The computational cost increases dramatically due to the exponential scaling from the distinguishable emitters. By this formula we see that if we consider a system of 3 emitters with a 3 photon cutoff we evolve a 32 x 32 matrix at every step.

Now that we have the base dimensions we need to generalize the operators discussed in the main body of the text. Each of these operators have a base matrix that completes their function. Additionally we can expand these operators using Kronecker products, which operate as:

If

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

then

$$A \otimes B = \begin{bmatrix} 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \\ 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \end{bmatrix}$$

This operation multiplies the matrix \mathbb{B} by the scalar value of \mathbb{A} and puts that matrix product where the scalar value was. The resulting matrix size will be the product of the size of the constituent matrices. This function is built into MATLAB. To look at how we scale our spin operators we can take a closer look at the spin down operator. We start with its constituent matrix for a two-level system:

$$\sigma^- = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Since this base matrix is a 2x2 matrix and we are going to expand it with a Kronecker product, we know we will Kronecker this base matrix with a identity matrix smaller than the total Hilbert Space by a factor of 2. However, we cannot simply Kronecker the base matrix with an identity matrix. We need to ensure that states can only decay into physically possible transitions. In essence, if we have two emitters we cannot have the second emitter decay into the first. The operator must behave such that it acts on a specific emitter and makes that emitter go from its excited to ground state. We accomplish this by making this operator a function of not only the macro-parameters, but the emitter index as well. Indexing the emitters allows us to separate them mathematically and create a sum to address each emitter individually. This means that the function for the spin-down operator behaves by the following. Let the atom index = j

If $j = 1$:

$$\mathbb{D} = \mathbb{I}(2^{n-1}) \otimes \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\hat{\sigma}^- = \mathbb{I}(m+1) \otimes \mathbb{D}$$

elseif $j = n$:

$$\mathbb{D} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \otimes \mathbb{I}(2^{n-1})$$

$$\hat{\sigma}^- = \mathbb{I}(m+1) \otimes \mathbb{D}$$

else

$$\mathbb{D} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \otimes \mathbb{I}(2^{n-1})$$

$$\mathbb{D}' = \mathbb{I}(2^{n-j}) \otimes \mathbb{D}$$

$$\hat{\sigma}^- = \mathbb{I}(m+1) \otimes \mathbb{D}'$$

This creates a generalized operator that can act on any system of n emitters and m photons, specifically decaying the emitter selected with j . We test this matrix on simple wavefunctions to ensure it behaves properly. We do the same thing for the spin-up and spin-z operators as well, only changing the base matrix.

The creation and annihilation operators only affect the photon subspace so they are easier to code. They do not depend on the emitter number so they are only a function of the macro-parameters n and m . We create a vector from 1 to m photons representing the possible sub-states:

$$\vec{A} = [1 : m]$$

$$\vec{A}' = \sqrt{\vec{A}}$$

$$\mathbb{A} = \text{diag}(A', 1)$$

This creates the matrix

$$\mathbb{A} = \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & 0 & \sqrt{n} \end{bmatrix}$$

Lastly, we need to expand this to the emitter subspace which we achieve with a Kronecker product:

$$\hat{a} = \mathbb{A} \otimes \mathbb{I}(2^n)$$

The creation operator is just the Hermitian conjugate of the annihilation operator so we can write that operator using the built-in dagger function.

Now we have established our Hilbert Space and constructed all of our constituent operators in a general form. We combine them to create the Hamiltonian and Liouvillian terms as outlined in the main body of the paper to create our master equation:

$$\dot{\rho} = \sum_n \left(-i \left(\delta [\sigma_z, \rho] + \Delta [a^\dagger a, \rho] + \Omega_r [a\sigma^+ + a^\dagger \sigma^-, \rho] \right) + \frac{\gamma}{2} (2\sigma^- \rho \sigma^+ - \sigma^+ \sigma^- \rho - \rho \sigma^+ \sigma^-) \right. \\ \left. + \frac{\Gamma}{2} (2\sigma^+ \rho \sigma^- - \sigma^- \sigma^+ \rho - \rho \sigma^- \sigma^+) \right) + \frac{\kappa}{2} (2a\rho a^\dagger - a^\dagger a \rho - \rho a^\dagger a)$$

The sum is over the number of emitters because those terms require the emitter index as previously mentioned. The entire right hand side of this differential equation is a single matrix of size $\mathbb{I}(2^n(m+1))$ that is entirely numbers describing how the probability distribution of the system changes slightly during the tiny time step. This means we have a set of $2^n(m+1)$ coupled ordinary first-order differential equations.

MATLAB has several built in numerical integration functions. We use *ODE45* because it employs Runge-Kutta integration. Runge-Kutta integration is a broad family of iterative solving methods that find the value of a function based on the immediately preceding value and the infinitesimal change. In MATLAB, the *ODE45* uses the explicit Runge-Kutta(4,5) form which finds the next value of ρ from its differential equation $f(t, \rho)$ through the following:

$$K_1 = h f(t_n, \rho_n)$$

$$K_2 = h f\left(t_n + \frac{h}{2}, \rho_n + \frac{K_1}{2}\right)$$

$$K_3 = h f\left(t_n + \frac{h}{2}, \rho_n + \frac{K_2}{2}\right)$$

$$K_4 = h f(t_n + h, \rho_n + K_3)$$

$$\rho_{n+1} = \rho_t + \frac{K_1}{6} + \frac{K_2}{3} + \frac{K_3}{4} + \frac{K_4}{6} + O(h^5)$$

So now we have a method of writing our operators for any Hilbert Space, we can add them together to create the Master Equation, and we can evolve that set of differential equations in time. In order to create our data we establish initial conditions, which is a density matrix of size $2^n(m+1)$. We specify the conditions of the system by assigning the probability values for the states. For simplicity we selected to put the system in the state where all emitters are excited and there are no photons inside the cavity. This selection is not particularly important because all initial conditions will evolve to the same steady-state for a the same set of parameters (here we mean all parameters, including $(\delta, \Delta, \gamma, \Gamma, \Omega_r, \kappa)$). The steady-state is when the density matrix no longer changes from iteration to iteration, so it has reached an equilibrium. To allow this system to evolve to steady-state we evolve the density matrix and save the density matrix at each time step as another page in the matrix. We then subtract the previous density matrix from the current time step and take the trace of the resulting matrix. That trace shows the total change in the probability distribution of the states. Once that change goes within our tolerance the system is at the steady-state. We set our change tolerance to $1e-5$ which is a 0.001% change. We set the tolerance above zero because numerical methods ultimately truncate and to go completely to 0 may not occur, so we save significant computation time without losing any accuracy. Additionally we could let the system run and visually see the steady state occur if the photon expectation value flat-lined.

Typically to get to the steady state we have to evolve the system for 3 nanoseconds, our time step is $1e-4 * 1 \text{ ns}$ so we iterate the system approximately 30,000 times. This means the output data we save is a matrix that whose dimensions are:

$$\rho = (2^n(m+1)) \times (2^n(m+1)) \times 30,000$$

B Expectation values of the density matrix

As a quick recall, expectation values are what we plot in nearly all of our figures. Expectation values are operators acting on the density matrix at each time step, where each time stamp is represented by one of the 30,000 pages in our density matrix. For an individual time step the expectation value of an operator is defined as:

$$\langle Q \rangle = Tr [\rho \hat{Q}]$$

Which is the trace of the product of our density matrix with the operator matrix for operator \hat{Q} . We can find expectation values as a function of time by creating a new vector which stores the value of the trace for each page in the matrix and plot that against our stored time vector.

C Coding photon-correlation

To code the second-order photon correlation measurement we just expand the process outlined in appendix A. As seen in equation 44, the initial conditions of the measurement are the creation and annihilation operators acting on the steady-state density matrix. These are all square matrices which are the size of the Hilbert Space, so the output is a new density matrix which is the same dimensions of everything we have been analyzing. Thus, we can put this matrix in our exact same ODE45 solver as before. Because the $g^{(2)}$ is a two-photon correlation, we just take the new pages of the evolved $g^{(2)}$ density matrix and find the expectation value of finding a photon for that time step. This is exactly what equation 46 specifies. Lastly we divide that value by the initial measurement in order to normalize the function, which produces equation 47.