

Noise Characterization of Quantum Teleportation with Imperfectly Prepared States

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EXECUTIVE SUMMARY

The purpose of this document is to report the results and their significance for the NISE project "Noise characterization of quantum teleportation with imperfectly prepared states".

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NOISE CHARACTERIZATION OF QUANTUM TELEPORTATION WITH IMPERFECTLY PREPARED STATES

1. BACKGROUND

There are many ways to represent information with a quantum state; for example, one could encode a classical bit into the polarization state of photon. Additionally, one could let the state represent a piece of quantum information like entanglement. Each of these choices induce different noise dynamics. To completely characterize this noise, and to optimize the performance of the quantum information technologies, we build a model of the evolution of all possible states called a *quantum channel*. In general, quantum states are fragile and easily couple to their environment, causing them to decohere. Due to this fragility, quantum states are difficult to prepare and measure. Furthermore, imperfect preparation and measurements of states can inject noise into the system as well. In this chapter, we will introduce the mathematical and physical background necessary to describe the noisy evolution of a system induced by teleporting through an imperfectly prepared state.

1.1 Quantum Information

1.1.1 Quantum States and Evolution

In order to discuss the evolution of a quantum system under a teleportation protocol, a complete mathematical description of environments and unknown states are needed. There are various equivalent ways to axiomatically build quantum states and describe their evolution. In this work, we will study integral numbers of qubits, and therefore can restrict ourselves to finite dimensional systems. Let \mathcal{H}^{2^n} be a Hilbert space of dimension 2^n ; i.e., the state space of an n -qubit system.

Definition 1.1.1. A quantum state is described by a positive semi-definite, trace one, linear operator $\rho : \mathcal{H}^{2^n} \rightarrow \mathcal{H}^{2^n}$ called the *density operator*.

If ρ is in a *pure state*, there exists a unit vector $|\psi\rangle \in H$ such that $\rho = |\psi\rangle\langle\psi|$, meaning that ρ is a rank one projector. Otherwise, ρ is a statistical ensemble of pure states, referred to as a *mixed state*; in this case there are probabilities p_i and unit vectors $|\psi_i\rangle \in H$ such that $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$. From the spectral theorem, such a decomposition is always possible, and it is not possible to both decompose a density operator as a mixed state and a pure state. Additionally, for a pure state $\rho = |\psi\rangle\langle\psi|$, the state vector $|\psi\rangle$ completely characterizes the quantum system and two will be used interchangeably.

Pure states represent quantum systems with maximal information and minimal entropy. Therefore, these states are the ideal for quantum information purposes. Within the set of pure states, entangled states form a particularly useful class of quantum systems:

Definition 1.1.2. For a pure state $\rho = |\psi\rangle\langle\psi|$, if there exist $|\eta\rangle \in H^{2^r}$ and $|\nu\rangle \in H^{2^s}$ such that $\rho = |\eta\rangle\langle\eta| \otimes |\nu\rangle\langle\nu|$, then ρ is said to be *separable*. If no such decomposition into tensor products of smaller systems exists, then ρ is said to be an *entangled state*.

Entanglement is necessary for many complex quantum protocols spanning sensing, computing, and networking; however, generating high-quality entangled states on demand is still a challenge area in quantum information science. In this work, we will study how imperfect entanglement can affect the transfer of information while utilizing some of these communication protocols.

While we will need the density operator/state vector formulation to analyze certain quantum systems, in previous 6.1 base program projects, and their resulting publications [5,6,9], it is useful to expand a density matrix in terms of measurement bases, and analyze how evolution affects these basis elements. For the single qubit, the popular choice is the basis of Pauli spin operators:

$$\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \text{ and } \sigma_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (1)$$

For an n -qubit system, we will use the n -qubit spin operators:

Definition 1.1.3. Define $\Lambda^n = \left\{ \frac{1}{\sqrt{2^n-1}} \sigma_{j_1} \otimes \cdots \otimes \sigma_{j_n} \mid j_i \in \{1, 2, 3, 4\} \right\}$ to be the collection of n -qubit spin operators ordered by the dictionary order. Call the i -th element in the order λ_i .

By representing the last $4^n - 1$ elements in vector form $\lambda = [\lambda_2 \ \lambda_3 \ \cdots \ \lambda_4^n]$, we have a convenient method to represent an n -qubit quantum state:

$$\rho = \frac{I + cr \cdot \lambda}{2^n}, \quad (2)$$

where $r \in \mathbb{R}^{4^n}$ is called the *Euclidean/Bloch* vector and c is a constant ensuring pure states have a Bloch vector with norm 1. For the single qubit, the collection of Bloch vectors is the entire unit ball, with the pure states on the surface. For the systems larger than a qubit, the collection of Bloch vectors is a deformed retract of the unit sphere [7].

There are a many methods to transmit information using a quantum state.

Definition 1.1.4. An n -qubit *quantum channel* is a positive, trace preserving, linear map $\Phi : \mathcal{H}^{2^n} \rightarrow \mathcal{H}^{2^n}$.

Many sources include *complete positivity* in the definition of a quantum channel [10]. An n -qubit quantum channel is completely positive if and only if $I_{2^n} \otimes \Phi$ is also a positive map [8]; complete positivity requires the initial state and the environment it couples with to be in a pure product state. However, in this NISE report we are considering state teleportation, which necessitates initial entanglement between a state and its environment. We have shown in a previous base program project, that with relatively benign assumptions, dynamics need not be completely positive [9,12]. Unfortunately, there are very few tools to systemically analyze the information content of channels that are not completely positive. In this work, we will show that for two-qubit teleportation, the dynamics are in fact completely positive.

Similar to the Euclidean representation of states, there is a Euclidean representation for quantum channels as well. Linear operators are completely determined by their actions on a basis, so each quantum channel Φ induces a unique affine map $f(r) = Mr + b$, called the *Euclidean/Bloch* representation of the channel, on the Bloch vector so that

$$\Phi(\rho) = \frac{I + f(r) \cdot \lambda}{2^n}. \quad (3)$$

When a channel fixes the identity matrix it is called *unital* and its Bloch representation is linear. We have introduced the Euclidean representation of quantum information systems because it offers a concrete way to envision the dynamics in which all of the coefficients in the expansion arise as expectation values of a particular measurement scheme. Previous base program work has developed easier ways to analyze the informatics of quantum channels using the Euclidean representation, where using the traditional methods would be computationally difficult. A complete discussion of the Euclidean representation of a quantum system is outside the scope of this report, but for a complete discussion, see [5,6].

1.1.2 Quantum Networks

As quantum technologies mature, we will need to remotely communicate and transfer information between them in order to leverage their full potential. For example, an n -qubit quantum state lies in a 2^n -dimensional Hilbert space, and consequently, there are $4^n - 1$ real parameters that are needed to fully characterize the state. Therefore, classically transmitting the amplitudes necessary to reconstruct a 40-qubit state would require a minimum of 9 trillion terabytes of classical information; even if we do not assume an arbitrary degree of precession needed for many of the most sophisticated protocols. However, if we were to transmit the information quantum mechanically, it would be theoretically possible to do so with a 40-photon state. This transfer can be done with a *quantum network*.

A quantum network consists of various transmission media and protocols that allows for the transfer of entanglement between multiple nodes. While quantum networks possess many of the same engineering challenges of classical networks, they will need to overcome obstacles that are uniquely quantum mechanical in nature. For example, classical networks can utilize standard signal amplification techniques to boost a degrading signal. However, due to the no cloning theorem [2], these amplification techniques are not applicable to quantum mechanical systems. Furthermore, because of a variety of factors [3], including fiber loss, quantum signals can only propagate for approximately 100 km before the signal becomes too lossy to extract useful information [13]. To circumvent this fundamental limit of quantum state transmission, a *quantum repeater* is placed between Alice and Bob in order to minimize the effective distance and therefore signal degradation. The basic quantum repeater works as follows: two shared entangled states are established, one between Alice and a repeater, R, and one between Bob and R. At this point the repeater performs a Bell measurement on the two states at R resulting in the repeater's initial entangled state with Alice being teleported to Bob. At the end, this process establishes entanglement between Alice and Bob's states. Since the Bell measurement destroys the information shared between Alice and the repeater and between the repeater and Bob, this process does not violate the no cloning theorem.

While exotic sounding, quantum teleportation is the fundamental phenomenon utilized by a repeater. Not only does teleportation allow for a signal to be repeated, potentially an indefinite number of times [4], it also allows for information to be transferred without passing through physical space. In this case, Alice and Bob would not make use of a repeater necessarily, but instead use a similar protocol for Alice to transfer an "information qubit" to Bob. Specifically, Alice would interact an information qubit with her half of the entangled pair. Through the use of local operations and nominal classical communication, Alice can transfer her information qubit to Bob without the qubit traversing physical space, and therefore circumventing any interception by an eavesdropper.

With an entangled bell state

$$|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \quad (4)$$

the information qubit would be transferred without error. However, there is no known method capable of generating perfect Bell states on demand. The natural question then arises, "How does the imperfect preparation of an entangled state effect the transfer of information?". This noise is not a result of environmental factors that cause decoherence, but instead stem from faulty implementations of the protocol itself. While this question was answered for single qubit teleportation in [14], it has not been well studied for the larger systems which are needed for a fully functioning quantum network. As a first step, we have characterized the noise for two-qubit teleportation through an imperfectly prepared state. In the following, we will show that this noise is described by a completely positive map, which is not guaranteed since the teleportation protocol explicitly requires the state to become entangled with external qubits before it is partially measured. We then calculate its Bloch representation of this channel and provide lower bounds for the channels capacity. It is worth noting that since quantum states can make use of entanglement across the qubits, the capacity of an n -qubit channel cannot be calculated from the individual constituent qubit channels. Consequently, there is no closed form solution to calculating the capacity of this channel. Using the special structure of these channels, we will show how to perform these capacity calculations for certain communication bases.

1.2 Quantum Teleportation

In this section we provide a step-by-step discussion of quantum teleportation which provides a method for sending a qubit by making use of the maximally entangled state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. As previously stated, there does not currently exist any experimental method to generate this state in an exact and reliable manner. For this reason, the consequences that arise when using an imperfectly prepared entangled state are essential for the implementation of quantum networks. As the teleportation of a single qubit has been previously studied and its associated errors characterized [14], the contribution of this report we will be to extend these results to the teleportation of two qubits. To this end, we begin by discussing the teleportation of a single qubit and then outline an analogous protocol for two qubits. Let us first take a moment to discuss an integral operation for teleportation, the Bell measurement.

The Bell Measurement

The Bell measurement is a joint measurement of two qubits that determines which of the four Bell states a system is in. Mathematically speaking, a Bell measurement is a projection onto the Bell state basis, and therefore, an entangling operation. That is, even if the system is not in a Bell state initially, it is after a Bell measurement. On a computational note, it is significant that the Bell states form an orthonormal basis for the state space of a two qubit system. Explicitly, given an arbitrary two qubit state, $|\Phi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$, there exist $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \in \mathbb{C}$ such that

$$|\Phi\rangle = \tilde{a}|\Phi_1\rangle + \tilde{b}|\Phi_2\rangle + \tilde{c}|\Phi_3\rangle + \tilde{d}|\Phi_4\rangle, \quad (5)$$

where $|\Phi_i\rangle$ is the i^{th} Bell state and $|\tilde{a}|^2 + |\tilde{b}|^2 + |\tilde{c}|^2 + |\tilde{d}|^2 = 1$. Moreover, basic calculations show that

$$|\Phi_i\rangle = \text{CNOT}_{1,2}\text{H}|i\rangle, \quad (6)$$

where $|i\rangle$ is the i^{th} element of the standard computational basis for the state space of a two qubit system, H is the hadamard gate applied to the first qubit, and $\text{CNOT}_{1,2}$ is the CNOT gate where the control is the first qubit and the target is the second qubit. It then follows that Alice's measurement may instead be viewed as the application of $\text{CNOT}_{1,2}\text{H}$ followed by a measurement in the computational basis. While this might appear pedantic, it allows for far easier calculations as we will see shortly.

Single Qubit Teleportation

Single qubit teleportation allows a sender (Alice) to transmit a qubit $|\psi\rangle$ to a receiver (Bob) via the following steps:

1. Before the protocol begins, Alice and Bob each possess one qubit from the maximally entangled two qubit state

$$|\Phi_1\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle). \quad (7)$$

We denote the total three qubit system by $|\Psi\rangle = |\psi\rangle \otimes |\Phi\rangle$, where Alice has the first 2 qubits and Bob holds the last.

2. Alice performs a Bell measurement on the two qubits she possesses. Recall, this operation is equivalent to the application of $\text{CNOT}_{1,2}\text{H}$ followed by a measurement in the computational basis. Through this method, Alice obtains one of the four possible results: $m = |00\rangle$, $m = |01\rangle$, $m = |10\rangle$, or $m = |11\rangle$.
3. Since the Bell measurement is an entangling operation, the state of Bob's qubit will be determined by the outcome of Alice's measurement in the previous step. Explicitly, Bob's state is

$$\begin{cases} \alpha|0\rangle + \beta|1\rangle & \text{if } m = |00\rangle \\ \alpha|1\rangle + \beta|0\rangle & \text{if } m = |01\rangle \\ \alpha|0\rangle - \beta|1\rangle & \text{if } m = |10\rangle \\ \alpha|1\rangle - \beta|0\rangle & \text{if } m = |11\rangle \end{cases} \quad (8)$$

4. Alice finally sends the classical bit string $m = ij$ associated with her measurement to Bob. He then applies the operator $Z^i X^j$ to the qubit he holds, thereby completely recovering the original qubit $|\psi\rangle$. Note, the Z and X operators act on a qubit as follows: $Z|0\rangle = |0\rangle$, $Z|1\rangle = -|1\rangle$, $X|0\rangle = |1\rangle$, and $X|1\rangle = |0\rangle$.

Full recovery of the qubit $|\psi\rangle$ relies on the assumption that one can generate the maximally entangled state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ on demand. As there is currently no experimental scheme that can achieve this, Lanzagorta and Martin explored the errors that arise when teleporting through the imperfectly prepared state [14]:

$$|\Phi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle, \quad (9)$$

where $a, b, c, d \in \mathbb{C}$ and $|a|^2 + |b|^2 + |c|^2 + |d|^2 = 1$. In [14] it was shown that the set of diagonal qubit channels in the Bloch representation is exactly the set of 3 by 3 matrices that describe the errors associated with teleportation through imperfectly prepared states. To this end, we refer to these matrices as the *qubit teleportation channels*. The primary goal of this project is to investigate whether or not this result extends to the teleportation of two qubits.

The g-State Measurement

While the teleportation of $n \geq 2$ qubits follows similarly, including the sharing of a $2n$ entangled qubit state, we have multiple views of maximal entanglement when $n \geq 2$. Therefore, the analogous ‘‘Bell states’’ for four qubit systems are not as clearly defined. The entangled states used in this work, referred to as ‘‘g-states’’, are generated by extending the methods of construction for two qubit Bell states to a $2n$ qubit system. Explicitly, for the case of two qubit teleportation, or $n = 2$, we begin with the following standard computational basis of a four qubit system:

$$\begin{aligned}
|1\rangle &= |0000\rangle, & |2\rangle &= |0001\rangle, & |3\rangle &= |0010\rangle, & |4\rangle &= |0011\rangle, \\
|5\rangle &= |0100\rangle, & |6\rangle &= |0101\rangle, & |7\rangle &= |0110\rangle, & |8\rangle &= |0111\rangle, \\
|9\rangle &= |1000\rangle, & |10\rangle &= |1001\rangle, & |11\rangle &= |1010\rangle, & |12\rangle &= |1011\rangle, \\
|13\rangle &= |1100\rangle, & |14\rangle &= |1101\rangle, & |15\rangle &= |1110\rangle, & |16\rangle &= |1111\rangle.
\end{aligned} \tag{10}$$

From this basis we compute our 16 entangled states by applying the following operation: $\text{CNOT}_{1,3}\text{CNOT}_{2,4}\text{H}_1\text{H}_2$, where H_i is the hadamard gate on the i^{th} qubit and $\text{CNOT}_{i,j}$ is the CNOT gate with i as the control and j as the target qubit. Upon inspection, it is easy to see that this is simply an extension of the same process that generates Bell states for a two qubit system. That is, to generate the $2n$ qubit g-state basis we simply apply the following composition of quantum gates to the computational basis elements:

$$\prod_{i=1}^n \text{CNOT}_{i,n+i} \prod_{i=1}^n \text{H}_i = \prod_{i=1}^n \text{CNOT}_{i,n+i} \text{H}_i = \prod_{i=1}^n \text{G}_i, \tag{11}$$

where $\text{G}_i = \text{CNOT}_{i,n+i}\text{H}_i$. From this process we have that for a four qubit system, our 16 $|g\rangle$ -states are generated as follows:

$$\begin{aligned}
|1\rangle &= |0000\rangle \longrightarrow |g_1\rangle = \frac{1}{2}(|0000\rangle + |0101\rangle + |1010\rangle + |1111\rangle) \\
|2\rangle &= |0001\rangle \longrightarrow |g_2\rangle = \frac{1}{2}(|0001\rangle + |0100\rangle + |1011\rangle + |1110\rangle) \\
|3\rangle &= |0010\rangle \longrightarrow |g_5\rangle = \frac{1}{2}(|0010\rangle + |0111\rangle + |1000\rangle + |1101\rangle) \\
|4\rangle &= |0011\rangle \longrightarrow |g_6\rangle = \frac{1}{2}(|0011\rangle + |0110\rangle + |1001\rangle + |1100\rangle) \\
|5\rangle &= |0100\rangle \longrightarrow |g_4\rangle = \frac{1}{2}(|0000\rangle - |0101\rangle + |1010\rangle - |1111\rangle) \\
|6\rangle &= |0101\rangle \longrightarrow |g_3\rangle = \frac{1}{2}(|0001\rangle - |0100\rangle + |1011\rangle - |1110\rangle) \\
|7\rangle &= |0110\rangle \longrightarrow |g_8\rangle = \frac{1}{2}(|0010\rangle - |0111\rangle + |1000\rangle - |1101\rangle) \\
|8\rangle &= |0111\rangle \longrightarrow |g_7\rangle = \frac{1}{2}(|0011\rangle - |0110\rangle + |1001\rangle - |1100\rangle) \\
|9\rangle &= |1000\rangle \longrightarrow |g_{13}\rangle = \frac{1}{2}(|0000\rangle + |0101\rangle - |1010\rangle - |1111\rangle) \\
|10\rangle &= |1001\rangle \longrightarrow |g_{14}\rangle = \frac{1}{2}(|0001\rangle + |0100\rangle - |1011\rangle - |1110\rangle) \\
|11\rangle &= |1010\rangle \longrightarrow |g_9\rangle = \frac{1}{2}(|0010\rangle + |0111\rangle - |1000\rangle - |1101\rangle) \\
|12\rangle &= |1011\rangle \longrightarrow |g_{10}\rangle = \frac{1}{2}(|0011\rangle + |0110\rangle - |1001\rangle - |1100\rangle) \\
|13\rangle &= |1100\rangle \longrightarrow |g_{16}\rangle = \frac{1}{2}(|0000\rangle - |0101\rangle - |1010\rangle + |1111\rangle) \\
|14\rangle &= |1101\rangle \longrightarrow |g_{15}\rangle = \frac{1}{2}(|0001\rangle - |0100\rangle - |1011\rangle + |1110\rangle) \\
|15\rangle &= |1110\rangle \longrightarrow |g_{12}\rangle = \frac{1}{2}(|0010\rangle - |0111\rangle - |1000\rangle + |1101\rangle) \\
|16\rangle &= |1111\rangle \longrightarrow |g_{11}\rangle = \frac{1}{2}(|0011\rangle - |0110\rangle - |1001\rangle + |1100\rangle)
\end{aligned} \tag{12}$$

Remark. Initially, one would think that the g -states should be labelled such that $\prod G_j|i\rangle = |g_i\rangle$, however, the notation we have chosen will simplify calculations later on; specifically, in Eq. 32.

Similar to Bell measurements, it then follows that a measurement in the g -state basis is equivalent to applying the operation

$$\left[\prod_{i=1}^n G_i \right]^\dagger \quad (13)$$

followed by a measurement in the computational basis. Again, we have defined $G_i = \text{CNOT}_{i,n+i}H_i$. Note, this operation is unitary. In other words, it is simply a change of basis for the state space of a $2n$ qubit system. It then follows that given an arbitrary four-qubit state, $|\Phi\rangle = \sum_{i=1}^{16} a_i|i\rangle$, there exist $\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_{16} \in \mathbb{C}$ such that

$$|\Phi\rangle = \sum_{i=1}^{16} q_i|g_i\rangle, \quad (14)$$

where $|g_i\rangle$ are the g -states given by Equation 12 and $\sum |q_i|^2 = 1$.

Two-Qubit Teleportation

With the use of our g -states, we teleport the two-qubit state $|\psi\rangle = b_1|00\rangle + b_2|01\rangle + b_3|10\rangle + b_4|11\rangle$, via the following steps:

1. Before the protocol begins, Alice and Bob each possess two qubits from the entangled four-qubit state

$$|g_1\rangle = \frac{1}{2}(|0000\rangle + |0101\rangle + |1010\rangle + |1111\rangle) \quad (15)$$

We denote the total six-qubit system by $|\Psi\rangle = |\psi\rangle \otimes |g_1\rangle$, where Alice has the first 4 qubits and Bob holds the last 2.

2. Alice performs a g -state measurement on the four qubits she possesses. Recall, this operation is equivalent to the application of $\left[\prod G_i \right]^\dagger$ followed by a measurement in the computational basis. Through this method, Alice obtains one of the 16 possible computational basis elements (i.e. $m = |ijkl\rangle$).
3. Step 2 is an entangling operation and thus the two-qubit state Bob holds is determined by Alice's measurement outcome, similar to that of single qubit teleportation. We do not list Bob's possible outcomes here as the list is 16 states long and will be explicitly discussed in the following chapter where we will provide all calculations involved in the teleportation of a two-qubit state.
4. Alice finally sends the classical bit string $m = ijkl$ associated with her measurement to Bob. He then applies the operator $Z_1^i Z_2^j X_1^k X_2^l$ on his qubits, where the subindices indicate which qubit the gate acts on, thus completely recovering the original qubit $|\psi\rangle$.

Just like the single qubit case, this fully recovery hinges on the ability to reliably generate the state $|g_1\rangle = \frac{1}{2}(|0000\rangle + |0101\rangle + |1010\rangle + |1111\rangle)$ on demand, which is not currently achievable. Therefore, for

the remainder of this work, we will perform the calculations for the teleportation of a two-qubit state through the arbitrary four-qubit state

$$|\phi\rangle = \sum_{i=1}^{16} a_i |i\rangle, \quad (16)$$

where $a_i \in \mathbb{C}$ for all i and $\sum |a_i|^2 = 1$.

2. TWO QUBIT TELEPORTATION WITH IMPERFECTLY PREPARED STATES

In this chapter we will perform all calculations in two-qubit state teleportation through an imperfectly prepared state. We begin by discussing each step one by one and conclude by calculating the density operator of Bob's final state.

2.1 The Protocol

2.1.1 Step 1: Defining the Composite State

Before the protocol begins Alice and Bob hold the first and second halves of the following four-qubit state, respectively:

$$|\phi\rangle = \sum_{i=1}^{16} a_i |i\rangle, \quad (17)$$

where the $|i\rangle$'s are the elements of the standard computational basis given in Equation 10. Alice wishes to send Bob the two qubit state

$$|\psi\rangle = b_1|00\rangle + b_2|00\rangle + b_3|00\rangle + b_4|00\rangle. \quad (18)$$

We then have from the postulates of quantum mechanics that the total six-qubit composite system is

$$|\Psi\rangle = |\psi\rangle \otimes |\phi\rangle. \quad (19)$$

2.1.2 Step 2: The g -State Measurement

As we saw in the previous chapter, Alice first applies the operation

$$\left[\prod G_i \right]^\dagger = \prod_{i=1}^n H_i \text{CNOT}_{i,n+i} \quad (20)$$

on the total composite system $|\Psi\rangle$. Explicitly, we get the following:

$$\begin{aligned}
 |\Psi'\rangle &= \left[\prod_i G_i \right]^\dagger |\Psi\rangle = \prod_{i=1}^n H_i \text{CNOT}_{i,n+i} |\Psi\rangle \\
 &= \frac{1}{2} b_1 (|00\rangle + |01\rangle + |10\rangle + |11\rangle) \otimes |\phi\rangle \\
 &+ \frac{1}{2} b_2 (|00\rangle - |01\rangle + |10\rangle - |11\rangle) \otimes [a_1|5\rangle + a_2|6\rangle + a_3|7\rangle + a_4|8\rangle + a_5|1\rangle + a_6|2\rangle + a_7|3\rangle \\
 &\quad + a_8|4\rangle + a_9|13\rangle + a_{10}|14\rangle + a_{11}|15\rangle + a_{12}|16\rangle + a_{13}|9\rangle + a_{14}|10\rangle + a_{15}|11\rangle + a_{16}|12\rangle] \quad (21) \\
 &+ \frac{1}{2} b_3 (|00\rangle + |01\rangle - |10\rangle - |11\rangle) \otimes [a_1|9\rangle + a_2|10\rangle + a_3|11\rangle + a_4|12\rangle + a_5|13\rangle + a_6|14\rangle \\
 &\quad + a_7|15\rangle + a_8|16\rangle + a_9|1\rangle + a_{10}|2\rangle + a_{11}|3\rangle + a_{12}|4\rangle + a_{13}|5\rangle + a_{14}|6\rangle + a_{15}|7\rangle + a_{16}|8\rangle] \\
 &+ \frac{1}{2} b_4 (|00\rangle - |01\rangle - |10\rangle + |11\rangle) \otimes [a_1|13\rangle + a_2|14\rangle + a_3|15\rangle + a_4|16\rangle + a_5|9\rangle + a_6|10\rangle \\
 &\quad + a_7|11\rangle + a_8|12\rangle + a_9|5\rangle + a_{10}|6\rangle + a_{11}|7\rangle + a_{12}|8\rangle + a_{13}|1\rangle + a_{14}|2\rangle + a_{15}|3\rangle + a_{16}|4\rangle].
 \end{aligned}$$

Lastly, Alice performs a measurement in the computational basis.

2.1.3 Step 3: The Received State

Alice's measurement outcome at the end of step 2 determines the state that Bob receives. In particular, Bob will have one of the following 16 states, where $(|\psi_i\rangle)$ occurs when Alice measures $|i\rangle$:

$$\begin{aligned}
 |\psi_1\rangle &= (b_1 a_1 + b_2 a_5 + b_3 a_9 + b_4 a_{13})|00\rangle + (b_1 a_2 + b_2 a_6 + b_3 a_{10} + b_4 a_{14})|01\rangle \\
 &\quad + (b_1 a_3 + b_2 a_7 + b_3 a_{11} + b_4 a_{15})|10\rangle + (b_1 a_4 + b_2 a_8 + b_3 a_{12} + b_4 a_{16})|11\rangle \\
 |\psi_2\rangle &= (b_1 a_5 + b_2 a_1 + b_3 a_{13} + b_4 a_9)|00\rangle + (b_1 a_6 + b_2 a_2 + b_3 a_{14} + b_4 a_{10})|01\rangle \\
 &\quad + (b_1 a_7 + b_2 a_3 + b_3 a_{15} + b_4 a_{11})|10\rangle + (b_1 a_8 + b_2 a_4 + b_3 a_{16} + b_4 a_{12})|11\rangle \\
 |\psi_3\rangle &= (b_1 a_9 + b_2 a_{13} + b_3 a_1 + b_4 a_5)|00\rangle + (b_1 a_{10} + b_2 a_{14} + b_3 a_2 + b_4 a_6)|01\rangle \\
 &\quad + (b_1 a_{11} + b_2 a_{15} + b_3 a_3 + b_4 a_7)|10\rangle + (b_1 a_{12} + b_2 a_{16} + b_3 a_4 + b_4 a_8)|11\rangle \\
 |\psi_4\rangle &= (b_1 a_{13} + b_2 a_9 + b_3 a_5 + b_4 a_1)|00\rangle + (b_1 a_{14} + b_2 a_{10} + b_3 a_6 + b_4 a_2)|01\rangle \\
 &\quad + (b_1 a_{15} + b_2 a_{11} + b_3 a_7 + b_4 a_3)|10\rangle + (b_1 a_{16} + b_2 a_{12} + b_3 a_8 + b_4 a_4)|11\rangle \\
 |\psi_5\rangle &= (b_1 a_1 - b_2 a_5 + b_3 a_9 - b_4 a_{13})|00\rangle + (b_1 a_2 - b_2 a_6 + b_3 a_{10} - b_4 a_{14})|01\rangle \\
 &\quad + (b_1 a_3 - b_2 a_7 + b_3 a_{11} - b_4 a_{15})|10\rangle + (b_1 a_4 - b_2 a_8 + b_3 a_{12} - b_4 a_{16})|11\rangle \\
 |\psi_6\rangle &= (b_1 a_5 - b_2 a_1 + b_3 a_{13} - b_4 a_9)|00\rangle + (b_1 a_6 - b_2 a_2 + b_3 a_{14} - b_4 a_{10})|01\rangle \\
 &\quad + (b_1 a_7 - b_2 a_3 + b_3 a_{15} - b_4 a_{11})|10\rangle + (b_1 a_8 - b_2 a_4 + b_3 a_{16} - b_4 a_{12})|11\rangle \quad (22) \\
 |\psi_7\rangle &= (b_1 a_9 - b_2 a_{13} + b_3 a_1 - b_4 a_5)|00\rangle + (b_1 a_{10} - b_2 a_{14} + b_3 a_2 - b_4 a_6)|01\rangle \\
 &\quad + (b_1 a_{11} - b_2 a_{15} + b_3 a_3 - b_4 a_7)|10\rangle + (b_1 a_{12} - b_2 a_{16} + b_3 a_4 - b_4 a_8)|11\rangle \\
 |\psi_8\rangle &= (b_1 a_{13} - b_2 a_9 + b_3 a_5 - b_4 a_1)|00\rangle + (b_1 a_{14} - b_2 a_{10} + b_3 a_6 - b_4 a_2)|01\rangle \\
 &\quad + (b_1 a_{15} - b_2 a_{11} + b_3 a_7 - b_4 a_3)|10\rangle + (b_1 a_{16} - b_2 a_{12} + b_3 a_8 - b_4 a_4)|11\rangle \\
 |\psi_9\rangle &= (b_1 a_1 + b_2 a_5 - b_3 a_9 - b_4 a_{13})|00\rangle + (b_1 a_2 + b_2 a_6 - b_3 a_{10} - b_4 a_{14})|01\rangle \\
 &\quad + (b_1 a_3 + b_2 a_7 + b_3 a_{11} + b_4 a_{15})|10\rangle + (b_1 a_4 + b_2 a_8 - b_3 a_{12} - b_4 a_{16})|11\rangle \\
 |\psi_{10}\rangle &= (b_1 a_5 + b_2 a_1 - b_3 a_{13} - b_4 a_9)|00\rangle + (b_1 a_6 + b_2 a_2 + b_3 a_{14} + b_4 a_{10})|01\rangle \\
 &\quad + (b_1 a_7 + b_2 a_3 - b_3 a_{15} - b_4 a_{11})|10\rangle + (b_1 a_8 + b_2 a_4 - b_3 a_{16} - b_4 a_{12})|11\rangle \\
 |\psi_{11}\rangle &= (b_1 a_9 + b_2 a_{13} - b_3 a_1 - b_4 a_5)|00\rangle + (b_1 a_{10} + b_2 a_{14} - b_3 a_2 - b_4 a_6)|01\rangle \\
 &\quad + (b_1 a_{11} + b_2 a_{15} - b_3 a_3 - b_4 a_7)|10\rangle + (b_1 a_{12} + b_2 a_{16} - b_3 a_4 - b_4 a_8)|11\rangle
 \end{aligned}$$

$$\begin{aligned}
|\psi_{12}\rangle &= (b_1a_{13} + b_2a_9 - b_3a_5 - b_4a_1)|00\rangle + (b_1a_{14} + b_2a_{10} - b_3a_6 - b_4a_2)|01\rangle \\
&\quad + (b_1a_{15} + b_2a_{11} - b_3a_7 - b_4a_3)|10\rangle + (b_1a_{16} + b_2a_{12} - b_3a_8 - b_4a_4)|11\rangle \\
|\psi_{13}\rangle &= (b_1a_1 - b_2a_5 - b_3a_9 + b_4a_{13})|00\rangle + (b_1a_2 - b_2a_6 - b_3a_{10} + b_4a_{14})|01\rangle \\
&\quad + (b_1a_3 - b_2a_7 - b_3a_{11} + b_4a_{15})|10\rangle + (b_1a_4 - b_2a_8 - b_3a_{12} + b_4a_{16})|11\rangle \\
|\psi_{14}\rangle &= (b_1a_5 - b_2a_1 - b_3a_{13} + b_4a_9)|00\rangle + (b_1a_6 - b_2a_2 - b_3a_{14} + b_4a_{10})|01\rangle \\
&\quad + (b_1a_7 - b_2a_3 - b_3a_{15} + b_4a_{11})|10\rangle + (b_1a_8 - b_2a_4 - b_3a_{16} + b_4a_{12})|11\rangle \\
|\psi_{15}\rangle &= (b_1a_9 - b_2a_{13} - b_3a_1 + b_4a_5)|00\rangle + (b_1a_{10} - b_2a_{14} - b_3a_2 + b_4a_6)|01\rangle \\
&\quad + (b_1a_{11} - b_2a_{15} - b_3a_3 + b_4a_7)|10\rangle + (b_1a_{12} - b_2a_{16} - b_3a_4 + b_4a_8)|11\rangle \\
|\psi_{16}\rangle &= (b_1a_{13} - b_2a_9 - b_3a_5 + b_4a_1)|00\rangle + (b_1a_{14} - b_2a_{10} - b_3a_6 + b_4a_2)|01\rangle \\
&\quad + (b_1a_{15} - b_2a_{11} - b_3a_7 + b_4a_3)|10\rangle + (b_1a_{16} - b_2a_{12} - b_3a_8 + b_4a_4)|11\rangle.
\end{aligned} \tag{23}$$

2.1.4 Step 4: Bob's Interaction

Note each $|i\rangle$ can be written as $|x y z w\rangle$, where $x, y, z, w \in \{0, 1\}$. Bob then applies $Z_1^x Z_2^y X_1^z X_2^w$ where the subscript denotes the qubit and $Z|0\rangle = |0\rangle$, $Z|1\rangle = -|1\rangle$, $X|0\rangle = |1\rangle$, and $X|1\rangle = |0\rangle$. After this operation, Bob then holds one of the 16 possible states:

$$\begin{aligned}
|\psi_1^B\rangle &= (b_1a_1 + b_2a_5 + b_3a_9 + b_4a_{13})|00\rangle + (b_1a_2 + b_2a_6 + b_3a_{10} + b_4a_{14})|01\rangle \\
&\quad + (b_1a_3 + b_2a_7 + b_3a_{11} + b_4a_{15})|10\rangle + (b_1a_4 + b_2a_8 + b_3a_{12} + b_4a_{16})|11\rangle \\
|\psi_2^B\rangle &= (b_1a_6 + b_2a_2 + b_3a_{14} + b_4a_{10})|00\rangle + (b_1a_5 + b_2a_1 + b_3a_{13} + b_4a_9)|01\rangle \\
&\quad + (b_1a_8 + b_2a_4 + b_3a_{16} + b_4a_{12})|10\rangle + (b_1a_7 + b_2a_3 + b_3a_{15} + b_4a_{11})|11\rangle \\
|\psi_3^B\rangle &= (b_1a_{11} + b_2a_{15} + b_3a_3 + b_4a_7)|00\rangle + (b_1a_{12} + b_2a_{16} + b_3a_4 + b_4a_8)|01\rangle \\
&\quad + (b_1a_9 + b_2a_{13} + b_3a_1 + b_4a_5)|10\rangle + (b_1a_{10} + b_2a_{14} + b_3a_2 + b_4a_6)|11\rangle \\
|\psi_4^B\rangle &= (b_1a_{16} + b_2a_{12} + b_3a_8 + b_4a_4)|00\rangle + (b_1a_{15} + b_2a_{11} + b_3a_7 + b_4a_3)|01\rangle \\
&\quad + (b_1a_{14} + b_2a_{10} + b_3a_6 + b_4a_2)|10\rangle + (b_1a_{13} + b_2a_9 + b_3a_5 + b_4a_1)|11\rangle \\
|\psi_5^B\rangle &= (b_1a_1 - b_2a_5 + b_3a_9 - b_4a_{13})|00\rangle - (b_1a_2 - b_2a_6 + b_3a_{10} - b_4a_{14})|01\rangle \\
&\quad + (b_1a_3 - b_2a_7 + b_3a_{11} - b_4a_{15})|10\rangle - (b_1a_4 - b_2a_8 + b_3a_{12} - b_4a_{16})|11\rangle \\
|\psi_6^B\rangle &= (b_1a_6 - b_2a_2 + b_3a_{14} - b_4a_{10})|00\rangle - (b_1a_5 - b_2a_1 + b_3a_{13} - b_4a_9)|01\rangle \\
&\quad + (b_1a_8 - b_2a_4 + b_3a_{16} - b_4a_{12})|10\rangle - (b_1a_7 - b_2a_3 + b_3a_{15} - b_4a_{11})|11\rangle \\
|\psi_7^B\rangle &= (b_1a_{11} - b_2a_{15} + b_3a_3 - b_4a_7)|00\rangle - (b_1a_{12} - b_2a_{16} + b_3a_4 - b_4a_8)|01\rangle \\
&\quad + (b_1a_9 - b_2a_{13} + b_3a_1 - b_4a_5)|10\rangle - (b_1a_{10} - b_2a_{14} + b_3a_2 - b_4a_6)|11\rangle \\
|\psi_8^B\rangle &= (b_1a_{16} - b_2a_{12} + b_3a_8 - b_4a_4)|00\rangle - (b_1a_{15} - b_2a_{11} + b_3a_7 - b_4a_3)|01\rangle \\
&\quad + (b_1a_{14} - b_2a_{10} + b_3a_6 - b_4a_2)|10\rangle - (b_1a_{13} - b_2a_9 + b_3a_5 - b_4a_1)|11\rangle \\
|\psi_9^B\rangle &= (b_1a_1 + b_2a_5 - b_3a_9 - b_4a_{13})|00\rangle + (b_1a_2 + b_2a_6 - b_3a_{10} - b_4a_{14})|01\rangle \\
&\quad - (b_1a_3 + b_2a_7 + b_3a_{11} + b_4a_{15})|10\rangle - (b_1a_4 + b_2a_8 - b_3a_{12} - b_4a_{16})|11\rangle \\
|\psi_{10}^B\rangle &= (b_1a_6 + b_2a_2 + b_3a_{14} + b_4a_{10})|00\rangle + (b_1a_5 + b_2a_1 - b_3a_{13} - b_4a_9)|01\rangle \\
&\quad - (b_1a_8 + b_2a_4 - b_3a_{16} - b_4a_{12})|10\rangle - (b_1a_7 + b_2a_3 - b_3a_{15} - b_4a_{11})|11\rangle \\
|\psi_{11}^B\rangle &= (b_1a_{11} + b_2a_{15} - b_3a_3 - b_4a_7)|00\rangle + (b_1a_{12} + b_2a_{16} - b_3a_4 - b_4a_8)|01\rangle \\
&\quad - (b_1a_9 + b_2a_{13} - b_3a_1 - b_4a_5)|10\rangle - (b_1a_{10} + b_2a_{14} - b_3a_2 - b_4a_6)|11\rangle
\end{aligned} \tag{24}$$

$$\begin{aligned}
 |\psi_{12}^B\rangle &= (b_1a_{16} + b_2a_{12} - b_3a_8 - b_4a_4)|00\rangle + (b_1a_{15} + b_2a_{11} - b_3a_7 - b_4a_3)|01\rangle \\
 &\quad - (b_1a_{14} + b_2a_{10} - b_3a_6 - b_4a_2)|10\rangle - (b_1a_{13} + b_2a_9 - b_3a_5 - b_4a_1)|11\rangle \\
 |\psi_{13}^B\rangle &= (b_1a_1 - b_2a_5 - b_3a_9 + b_4a_{13})|00\rangle - (b_1a_2 - b_2a_6 - b_3a_{10} + b_4a_{14})|01\rangle \\
 &\quad - (b_1a_3 - b_2a_7 - b_3a_{11} + b_4a_{15})|10\rangle + (b_1a_4 - b_2a_8 - b_3a_{12} + b_4a_{16})|11\rangle \\
 |\psi_{14}^B\rangle &= (b_1a_6 - b_2a_2 - b_3a_{14} + b_4a_{10})|00\rangle - (b_1a_5 - b_2a_1 - b_3a_{13} + b_4a_9)|01\rangle \\
 &\quad - (b_1a_8 - b_2a_4 - b_3a_{16} + b_4a_{12})|10\rangle + (b_1a_7 - b_2a_3 - b_3a_{15} + b_4a_{11})|11\rangle \\
 |\psi_{15}^B\rangle &= (b_1a_{11} - b_2a_{15} - b_3a_3 + b_4a_7)|00\rangle - (b_1a_{12} - b_2a_{16} - b_3a_4 + b_4a_8)|01\rangle \\
 &\quad - (b_1a_9 - b_2a_{13} - b_3a_1 + b_4a_5)|10\rangle + (b_1a_{10} - b_2a_{14} - b_3a_2 + b_4a_6)|11\rangle \\
 |\psi_{16}^B\rangle &= (b_1a_{16} - b_2a_{12} - b_3a_8 + b_4a_4)|00\rangle - (b_1a_{15} - b_2a_{11} - b_3a_7 + b_4a_3)|01\rangle \\
 &\quad - (b_1a_{14} - b_2a_{10} - b_3a_6 + b_4a_2)|10\rangle + (b_1a_{13} - b_2a_9 - b_3a_5 + b_4a_1)|11\rangle
 \end{aligned} \tag{25}$$

2.2 Calculating the Density Operator

We can now construct a density operator for each possible state $\rho_i = |\psi_i^B\rangle\langle\psi_i^B|$, each being obtained by Bob with probability equal to the reciprocal of its normalization constant. That is, the overall density operator Bob receives is the sum of ρ_i 's; $\rho = \sum \rho_i$. We then have that the final density operator obtained by Bob is given by

$$\begin{aligned}
 \rho_{12} &= b_1\bar{b}_2 2\text{Re}(a_1\bar{a}_6 + a_{11}\bar{a}_{16}) + b_2\bar{b}_1 2\text{Re}(a_2\bar{a}_5 + a_{12}\bar{a}_{15}) \\
 &\quad + b_3\bar{b}_4 2\text{Re}(a_3\bar{a}_8 + a_9\bar{a}_{14}) + b_4\bar{b}_3 2\text{Re}(a_4\bar{a}_7 + a_{10}\bar{a}_{13}) = \bar{\rho}_{21} \\
 \rho_{13} &= b_1\bar{b}_3 2\text{Re}(a_1\bar{a}_{11} + a_6\bar{a}_{16}) + b_3\bar{b}_1 2\text{Re}(a_3\bar{a}_9 + a_8\bar{a}_{14}) \\
 &\quad + b_2\bar{b}_4 2\text{Re}(a_2\bar{a}_{12} + a_5\bar{a}_{15}) + b_4\bar{b}_2 2\text{Re}(a_4\bar{a}_{10} + a_7\bar{a}_{13}) = \bar{\rho}_{31} \\
 \rho_{14} &= b_1\bar{b}_4 2\text{Re}(a_1\bar{a}_{16} + a_6\bar{a}_{11}) + b_4\bar{b}_1 2\text{Re}(a_4\bar{a}_{13} + a_7\bar{a}_{10}) \\
 &\quad + b_2\bar{b}_3 2\text{Re}(a_2\bar{a}_{15} + a_5\bar{a}_{12}) + b_3\bar{b}_2 2\text{Re}(a_3\bar{a}_{14} + a_8\bar{a}_9) = \bar{\rho}_{41} \\
 \rho_{23} &= b_2\bar{b}_3 2\text{Re}(a_1\bar{a}_{16} + a_6\bar{a}_{11}) + b_3\bar{b}_2 2\text{Re}(a_4\bar{a}_{13} + a_7\bar{a}_{10}) \\
 &\quad + b_1\bar{b}_4 2\text{Re}(a_2\bar{a}_{15} + a_5\bar{a}_{12}) + b_4\bar{b}_1 2\text{Re}(a_3\bar{a}_{14} + a_8\bar{a}_9) = \bar{\rho}_{32} \\
 \rho_{24} &= b_2\bar{b}_4 2\text{Re}(a_1\bar{a}_{11} + a_6\bar{a}_{16}) + b_4\bar{b}_2 2\text{Re}(a_3\bar{a}_9 + a_8\bar{a}_{14}) \\
 &\quad + b_1\bar{b}_3 2\text{Re}(a_2\bar{a}_{12} + a_5\bar{a}_{15}) + b_3\bar{b}_1 2\text{Re}(a_4\bar{a}_{10} + a_7\bar{a}_{13}) = \bar{\rho}_{42} \\
 \rho_{34} &= b_3\bar{b}_4 2\text{Re}(a_1\bar{a}_6 + a_{11}\bar{a}_{16}) + b_4\bar{b}_3 2\text{Re}(a_2\bar{a}_5 + a_{12}\bar{a}_{15}) \\
 &\quad + b_1\bar{b}_2 2\text{Re}(a_3\bar{a}_8 + a_9\bar{a}_{14}) + b_2\bar{b}_1 2\text{Re}(a_4\bar{a}_7 + a_{10}\bar{a}_{13}) = \bar{\rho}_{43} \\
 \rho_{11} &= |b_1|^2(|a_1|^2 + |a_6|^2 + |a_{11}|^2 + |a_{16}|^2) + |b_2|^2(|a_2|^2 + |a_5|^2 + |a_{12}|^2 + |a_{15}|^2) \\
 &\quad + |b_3|^2(|a_3|^2 + |a_8|^2 + |a_9|^2 + |a_{14}|^2) + |b_4|^2(|a_4|^2 + |a_7|^2 + |a_{10}|^2 + |a_{14}|^2) \\
 \rho_{22} &= |b_1|^2(|a_2|^2 + |a_5|^2 + |a_{12}|^2 + |a_{15}|^2) + |b_2|^2(|a_1|^2 + |a_6|^2 + |a_{11}|^2 + |a_{16}|^2) \\
 &\quad + |b_3|^2(|a_4|^2 + |a_7|^2 + |a_{10}|^2 + |a_{14}|^2) + |b_4|^2(|a_3|^2 + |a_8|^2 + |a_9|^2 + |a_{14}|^2) \\
 \rho_{33} &= |b_1|^2(|a_3|^2 + |a_8|^2 + |a_9|^2 + |a_{14}|^2) + |b_2|^2(|a_4|^2 + |a_7|^2 + |a_{10}|^2 + |a_{14}|^2) \\
 &\quad + |b_3|^2(|a_1|^2 + |a_6|^2 + |a_{11}|^2 + |a_{16}|^2) + |b_4|^2(|a_2|^2 + |a_5|^2 + |a_{12}|^2 + |a_{15}|^2) \\
 \rho_{44} &= |b_1|^2(|a_4|^2 + |a_7|^2 + |a_{10}|^2 + |a_{14}|^2) + |b_2|^2(|a_3|^2 + |a_8|^2 + |a_9|^2 + |a_{14}|^2) \\
 &\quad + |b_3|^2(|a_2|^2 + |a_5|^2 + |a_{12}|^2 + |a_{15}|^2) + |b_4|^2(|a_1|^2 + |a_6|^2 + |a_{11}|^2 + |a_{16}|^2)
 \end{aligned} \tag{26}$$

3. THE TWO QUBIT TELEPORTATION CHANNEL

3.1 Calculating the Bloch Vector

We know from [6] that the relationship between a two-qubit density operator and its Bloch vector is given by

$$\rho = \frac{\sqrt{3}}{4} \begin{pmatrix} \frac{1}{\sqrt{3}} + r_3 + r_{12} + r_{15} & r_1 + r_{13} - i(r_2 + r_{14}) & r_4 + r_7 - i(r_8 + r_{11}) & r_5 - r_{10} - i(r_6 + r_9) \\ r_1 + r_{13} + i(r_2 + r_{14}) & \frac{1}{\sqrt{3}} - r_3 + r_{12} - r_{15} & r_5 + r_{10} + i(r_6 - r_9) & r_4 - r_7 - i(r_8 - r_{11}) \\ r_4 + r_7 + i(r_8 + r_{11}) & r_5 + r_{10} - i(r_6 - r_9) & \frac{1}{\sqrt{3}} + r_3 - r_{12} - r_{15} & r_1 - r_{13} - i(r_2 - r_{14}) \\ r_5 - r_{10} + i(r_6 + r_9) & r_4 - r_7 + i(r_8 - r_{11}) & r_1 - r_{13} + i(r_2 - r_{14}) & \frac{1}{\sqrt{3}} - r_3 - r_{12} + r_{15} \end{pmatrix}. \quad (27)$$

From some basic arithmetic, we then have that the Bloch vector Alice wishes to teleport and what Bob receives are given by the following respectively:

$$\begin{aligned} r_1 &= \frac{1}{\sqrt{3}}(\rho_{12}^A + \rho_{21}^A + \rho_{34}^A + \rho_{43}^A) = \frac{2}{\sqrt{3}}\text{Re}(\alpha_1\alpha_2^* + \alpha_3\alpha_4^*), \\ r_2 &= \frac{i}{\sqrt{3}}(\rho_{12}^A - \rho_{21}^A + \rho_{34}^A - \rho_{43}^A) = \frac{2}{\sqrt{3}}\text{Im}(\alpha_1^*\alpha_2 + \alpha_3^*\alpha_4), \\ r_3 &= \frac{1}{\sqrt{3}}(\rho_{11}^A - \rho_{22}^A + \rho_{33}^A - \rho_{44}^A) = \frac{1}{\sqrt{3}}(|\alpha_1|^2 - |\alpha_2|^2 + |\alpha_3|^2 - |\alpha_4|^2), \\ r_4 &= \frac{1}{\sqrt{3}}(\rho_{13}^A + \rho_{31}^A + \rho_{24}^A + \rho_{42}^A) = \frac{2}{\sqrt{3}}\text{Re}(\alpha_1\alpha_3^* + \alpha_2\alpha_4^*), \\ r_5 &= \frac{1}{\sqrt{3}}(\rho_{14}^A + \rho_{41}^A + \rho_{23}^A + \rho_{32}^A) = \frac{2}{\sqrt{3}}\text{Re}(\alpha_1\alpha_4^* + \alpha_2\alpha_3^*), \\ r_6 &= \frac{i}{\sqrt{3}}(\rho_{41}^A - \rho_{14}^A - \rho_{32}^A + \rho_{23}^A) = \frac{2}{\sqrt{3}}\text{Im}(\alpha_1\alpha_4^* - \alpha_2\alpha_3^*), \\ r_7 &= \frac{1}{\sqrt{3}}(\rho_{13}^A + \rho_{31}^A - \rho_{24}^A - \rho_{42}^A) = \frac{2}{\sqrt{3}}\text{Re}(\alpha_1\alpha_3^* - \alpha_2\alpha_4^*), \\ r_8 &= \frac{i}{\sqrt{3}}(\rho_{13}^A - \rho_{31}^A + \rho_{24}^A - \rho_{42}^A) = \frac{2}{\sqrt{3}}\text{Im}(\alpha_1^*\alpha_3 + \alpha_2^*\alpha_4), \\ r_9 &= \frac{i}{\sqrt{3}}(\rho_{14}^A - \rho_{41}^A + \rho_{23}^A - \rho_{32}^A) = \frac{2}{\sqrt{3}}\text{Im}(\alpha_1^*\alpha_4 + \alpha_2^*\alpha_3), \\ r_{10} &= \frac{1}{\sqrt{3}}(\rho_{23}^A + \rho_{32}^A - \rho_{14}^A - \rho_{41}^A) = \frac{2}{\sqrt{3}}\text{Re}(-\alpha_1\alpha_4^* + \alpha_2\alpha_3^*), \\ r_{11} &= \frac{i}{\sqrt{3}}(\rho_{13}^A - \rho_{31}^A - \rho_{24}^A + \rho_{42}^A) = \frac{2}{\sqrt{3}}\text{Im}(\alpha_1^*\alpha_3 - \alpha_2^*\alpha_4), \\ r_{12} &= \frac{1}{\sqrt{3}}(\rho_{11}^A + \rho_{22}^A - \rho_{33}^A - \rho_{44}^A) = \frac{1}{\sqrt{3}}(|\alpha_1|^2 + |\alpha_2|^2 - |\alpha_3|^2 - |\alpha_4|^2), \\ r_{13} &= \frac{1}{\sqrt{3}}(\rho_{12}^A + \rho_{21}^A - \rho_{34}^A - \rho_{43}^A) = \frac{2}{\sqrt{3}}\text{Re}(\alpha_1\alpha_2^* - \alpha_3\alpha_4^*), \\ r_{14} &= \frac{i}{\sqrt{3}}(\rho_{12}^A - \rho_{21}^A - \rho_{34}^A + \rho_{43}^A) = \frac{2}{\sqrt{3}}\text{Im}(\alpha_1^*\alpha_2 - \alpha_3^*\alpha_4), \\ r_{15} &= \frac{1}{\sqrt{3}}(\rho_{11}^A - \rho_{22}^A - \rho_{33}^A + \rho_{44}^A) = \frac{1}{\sqrt{3}}(|\alpha_1|^2 - |\alpha_2|^2 - |\alpha_3|^2 + |\alpha_4|^2). \end{aligned} \quad (28)$$

$$\begin{aligned} r'_1 &= 2\text{Re}(a_1a_6^* + a_{11}a_{16}^* + a_2a_5^* + a_{12}a_{15}^* + a_9a_{14}^* + a_3a_8^* + a_{10}a_{13}^* + a_4a_7^*) \times r_1 \\ r'_2 &= 2\text{Re}(a_1a_6^* + a_{11}a_{16}^* - a_2a_5^* - a_{12}a_{15}^* + a_9a_{14}^* + a_3a_8^* - a_{10}a_{13}^* - a_4a_7^*) \times r_2 \\ r'_3 &= (|a_1|^2 + |a_6|^2 + |a_{11}|^2 + |a_{16}|^2 - |a_2|^2 - |a_5|^2 - |a_{12}|^2 - |a_{15}|^2 \\ &\quad + |a_3|^2 + |a_9|^2 + |a_8|^2 + |a_{14}|^2 - |a_4|^2 - |a_7|^2 - |a_{10}|^2 - |a_{13}|^2) \times r_3 \\ r'_4 &= 2\text{Re}(a_1a_{11}^* + a_6a_{16}^* + a_2a_{12}^* + a_5a_{15}^* + a_3a_9^* + a_8a_{14}^* + a_4a_{10}^* + a_7a_{13}^*) \times r_4 \\ r'_5 &= 2\text{Re}(a_1a_{16}^* + a_6a_{11}^* + a_2a_{15}^* + a_5a_{12}^* + a_3a_{14}^* + a_8a_9^* + a_4a_{13}^* + a_7a_{10}^*) \times r_5 \end{aligned} \quad (29)$$

$$\begin{aligned}
 r'_6 &= 2\text{Re}(a_1a_{16}^* + a_6a_{11}^* - a_2a_{15}^* - a_5a_{12}^* + a_3a_{14}^* + a_8a_9^* - a_4a_{13}^* - a_7a_{10}^*) \times r_6 \\
 r'_7 &= 2\text{Re}(a_1a_{11}^* + a_6a_{16}^* + a_3a_9^* + a_8a_{14}^* - a_2a_{12}^* - a_5a_{15}^* - a_4a_{10}^* - a_7a_{13}^*) \times r_7 \\
 r'_8 &= 2\text{Re}(-a_4a_{10}^* - a_7a_{13}^* - a_3a_9^* - a_8a_{14}^* + a_1a_{11}^* + a_6a_{16}^* + a_2a_{12}^* + a_5a_{15}^*) \times r_8 \\
 r'_9 &= 2\text{Re}(a_1a_{16}^* + a_6a_{11}^* - a_4a_{13}^* - a_7a_{10}^* + a_2a_{15}^* + a_5a_{12}^* - a_3a_{14}^* - a_8a_9^*) \times r_9 \\
 r'_{10} &= 2\text{Re}(a_1a_{16}^* + a_6a_{11}^* - a_2a_{15}^* - a_5a_{12}^* - a_3a_{14}^* - a_8a_9^* + a_4a_{13}^* + a_7a_{10}^*) \times r_{10} \\
 r'_{11} &= 2\text{Re}(a_1a_{11}^* + a_6a_{16}^* - a_2a_{12}^* - a_5a_{15}^* - a_3a_9^* - a_8a_{14}^* + a_4a_{10}^* + a_7a_{13}^*) \times r_{11} \\
 r'_{12} &= (|a_1|^2 + |a_6|^2 + |a_{11}|^2 + |a_{16}|^2 + |a_2|^2 + |a_5|^2 + |a_{12}|^2 + |a_{15}|^2 \\
 &\quad - |a_3|^2 - |a_9|^2 - |a_8|^2 - |a_{14}|^2 - |a_4|^2 - |a_7|^2 - |a_{10}|^2 - |a_{13}|^2) \times r_{12} \\
 r'_{13} &= 2\text{Re}(-a_3a_8^* - a_9a_{14}^* - a_4a_7^* - a_{10}a_{13}^* + a_1a_6^* + a_{11}a_{16}^* + a_2a_5^* + a_{12}a_{15}^*) \times r_{13} \\
 r'_{14} &= 2\text{Re}(a_1a_6^* + a_{11}a_{16}^* - a_2a_5^* - a_{12}a_{15}^* - a_3a_8^* - a_9a_{14}^* + a_4a_7^* + a_{10}a_{13}^*) \times r_{14} \\
 r'_{15} &= (|a_1|^2 + |a_6|^2 + |a_{11}|^2 + |a_{16}|^2 - |a_2|^2 - |a_5|^2 - |a_{12}|^2 - |a_{15}|^2 \\
 &\quad - |a_3|^2 - |a_9|^2 - |a_8|^2 - |a_{14}|^2 + |a_4|^2 + |a_7|^2 + |a_{10}|^2 + |a_{13}|^2) \times r_{15}
 \end{aligned} \tag{30}$$

It then follows that the effects of teleporting a two-qubit state via an imperfectly prepared state is described by a diagonal 15×15 matrix in the Bloch representation. In the next section we will show that not only is this diagonal matrix a two-qubit quantum channel, but that any diagonal two-qubit quantum channel can be described by the teleportation through an imperfectly prepared four-qubit entangled state.

3.2 The Spin Channels

From [6] we know that every diagonal two-qubit channel can be written as a convex sum of the following “two-qubit spin channels”:

$$\begin{aligned}
 \lambda_0 &= s_0 \otimes s_0, & \lambda_1 &= s_0 \otimes s_1, & \lambda_2 &= s_0 \otimes s_2, & \lambda_3 &= s_0 \otimes s_3, \\
 \lambda_4 &= s_1 \otimes s_0, & \lambda_5 &= s_1 \otimes s_1, & \lambda_6 &= s_1 \otimes s_2, & \lambda_7 &= s_1 \otimes s_3, \\
 \lambda_8 &= s_2 \otimes s_0, & \lambda_9 &= s_2 \otimes s_1, & \lambda_{10} &= s_2 \otimes s_2, & \lambda_{11} &= s_2 \otimes s_3, \\
 \lambda_{12} &= s_3 \otimes s_0, & \lambda_{13} &= s_3 \otimes s_1, & \lambda_{14} &= s_3 \otimes s_2, & \lambda_{15} &= s_3 \otimes s_3,
 \end{aligned} \tag{31}$$

where $s_0, s_x, s_y,$ and s_z are the Bloch representations of the spin channels given in Eq. 1. We now show that an arbitrary four-qubit state can be written as a sum of states each of which generate a unique two-qubit spin channel. Furthermore, we will show that the channel induced by the overall state can then be written as a convex sum of these spin channels.

3.2.1 The g -states

Plugging into Equation 29 we have that the following g -states generate the two-qubit spin channels:

$$\begin{aligned}
|g_1\rangle &= \frac{1}{2}(|0000\rangle + |0101\rangle + |1010\rangle + |1111\rangle) \rightarrow |\lambda_1\rangle \\
|g_2\rangle &= \frac{1}{2}(|0001\rangle + |0100\rangle + |1011\rangle + |1110\rangle) \rightarrow |\lambda_2\rangle \\
|g_3\rangle &= \frac{1}{2}(|0001\rangle - |0100\rangle + |1011\rangle - |1110\rangle) \rightarrow |\lambda_3\rangle \\
|g_4\rangle &= \frac{1}{2}(|0000\rangle - |0101\rangle + |1010\rangle - |1111\rangle) \rightarrow |\lambda_4\rangle \\
|g_5\rangle &= \frac{1}{2}(|0010\rangle + |0111\rangle + |1000\rangle + |1101\rangle) \rightarrow |\lambda_5\rangle \\
|g_6\rangle &= \frac{1}{2}(|0011\rangle + |0110\rangle + |1001\rangle + |1100\rangle) \rightarrow |\lambda_6\rangle \\
|g_7\rangle &= \frac{1}{2}(|0011\rangle - |0110\rangle + |1001\rangle - |1100\rangle) \rightarrow |\lambda_7\rangle \\
|g_8\rangle &= \frac{1}{2}(|0010\rangle - |0111\rangle + |1000\rangle - |1101\rangle) \rightarrow |\lambda_8\rangle \\
|g_9\rangle &= \frac{1}{2}(|0010\rangle + |0111\rangle - |1000\rangle - |1101\rangle) \rightarrow |\lambda_9\rangle \\
|g_{10}\rangle &= \frac{1}{2}(|0011\rangle + |0110\rangle - |1001\rangle - |1100\rangle) \rightarrow |\lambda_{10}\rangle \\
|g_{11}\rangle &= \frac{1}{2}(|0011\rangle - |0110\rangle - |1001\rangle + |1100\rangle) \rightarrow |\lambda_{11}\rangle \\
|g_{12}\rangle &= \frac{1}{2}(|0010\rangle - |0111\rangle - |1000\rangle + |1101\rangle) \rightarrow |\lambda_{12}\rangle \\
|g_{13}\rangle &= \frac{1}{2}(|0000\rangle + |0101\rangle - |1010\rangle - |1111\rangle) \rightarrow |\lambda_{13}\rangle \\
|g_{14}\rangle &= \frac{1}{2}(|0001\rangle + |0100\rangle - |1011\rangle - |1110\rangle) \rightarrow |\lambda_{14}\rangle \\
|g_{15}\rangle &= \frac{1}{2}(|0001\rangle - |0100\rangle - |1011\rangle + |1110\rangle) \rightarrow |\lambda_{15}\rangle \\
|g_{16}\rangle &= \frac{1}{2}(|0000\rangle - |0101\rangle - |1010\rangle + |1111\rangle) \rightarrow |\lambda_{16}\rangle
\end{aligned} \tag{32}$$

Recalling, that for every $|g_i\rangle$, there exists a computation basis element $|j\rangle$ such that $|g_i\rangle = \text{CNOTH}|j\rangle$, where CNOTH is the composition of the same CNOT and Hadamard gates Alice applies at the beginning of the teleportation protocol. This is significant because CNOTH is a unitary matrix, in other words CNOTH is a change of basis. It then follows that every four-qubit quantum state can be written as a sum of our $|g_i\rangle$ states, where the coefficients are unique. In particular, if an arbitrary four-qubit state $|\phi\rangle$ is written in the standard computational basis, then the following q_i 's are the coefficients of $|\phi\rangle$ in the $|g\rangle$ basis:

$$\begin{aligned}
q_1 &= \frac{1}{2\sqrt{|a_1|^2+|a_6|^2+|a_{11}|^2+|a_{16}|^2}}(a_1 + a_6 + a_{11} + a_{16}), & q_2 &= \frac{1}{2\sqrt{|a_2|^2+|a_5|^2+|a_{12}|^2+|a_{15}|^2}}(a_2 + a_5 + a_{12} + a_{15}), \\
q_3 &= \frac{1}{2\sqrt{|a_2|^2+|a_5|^2+|a_{12}|^2+|a_{15}|^2}}(a_2 - a_5 + a_{12} - a_{15}), & q_4 &= \frac{1}{2\sqrt{|a_1|^2+|a_6|^2+|a_{11}|^2+|a_{16}|^2}}(a_1 - a_6 + a_{11} - a_{16}), \\
q_5 &= \frac{1}{2\sqrt{|a_3|^2+|a_8|^2+|a_9|^2+|a_{14}|^2}}(a_3 + a_8 + a_9 + a_{14}), & q_6 &= \frac{1}{2\sqrt{|a_4|^2+|a_7|^2+|a_{10}|^2+|a_{13}|^2}}(a_4 + a_7 + a_{10} + a_{13}), \\
q_7 &= \frac{1}{2\sqrt{|a_4|^2+|a_7|^2+|a_{10}|^2+|a_{13}|^2}}(a_4 - a_7 + a_{10} - a_{13}), & q_8 &= \frac{1}{2\sqrt{|a_3|^2+|a_8|^2+|a_9|^2+|a_{14}|^2}}(a_3 - a_8 + a_9 - a_{14}), \\
q_9 &= \frac{1}{2\sqrt{|a_3|^2+|a_8|^2+|a_9|^2+|a_{14}|^2}}(a_3 + a_8 - a_9 - a_{14}), & q_{10} &= \frac{1}{2\sqrt{|a_4|^2+|a_7|^2+|a_{10}|^2+|a_{13}|^2}}(a_4 + a_7 - a_{10} - a_{13}), \\
q_{11} &= \frac{1}{2\sqrt{|a_4|^2+|a_7|^2+|a_{10}|^2+|a_{13}|^2}}(a_4 - a_7 - a_{10} + a_{13}), & q_{12} &= \frac{1}{2\sqrt{|a_3|^2+|a_8|^2+|a_9|^2+|a_{14}|^2}}(a_3 - a_8 - a_9 + a_{14}), \\
q_{13} &= \frac{1}{2\sqrt{|a_1|^2+|a_6|^2+|a_{11}|^2+|a_{16}|^2}}(a_1 + a_6 - a_{11} - a_{16}), & q_{14} &= \frac{1}{2\sqrt{|a_2|^2+|a_5|^2+|a_{12}|^2+|a_{15}|^2}}(a_2 + a_5 - a_{12} - a_{15}), \\
q_{15} &= \frac{1}{2\sqrt{|a_2|^2+|a_5|^2+|a_{12}|^2+|a_{15}|^2}}(a_2 - a_5 - a_{12} + a_{15}), & q_{16} &= \frac{1}{2\sqrt{|a_1|^2+|a_6|^2+|a_{11}|^2+|a_{16}|^2}}(a_1 - a_6 - a_{11} + a_{16}).
\end{aligned}$$

Upon quick calculation, one can show that $\sum_{i=1}^{16} |q_i|^2 = 1$. This means that

$$|\phi\rangle = \sum_{i=1}^{16} q_i |g_i\rangle.$$

From Eq. 29, it then follows that letting $p_i = |q_i|^2$, the transformation induced by our state is a convex sum of spin channels with coefficients p_i . That is, we have shown that an arbitrary four-qubit state $\sum a_i |i\rangle$ can be rewritten as a sum of g -states, $\sum q_i |g_i\rangle$, whose effect on two-qubit teleportation can be described by a convex sum of 2-qubit spin channels. Explicitly,

$$\sum_{i=1}^{16} a_i |i\rangle \xrightarrow{\text{CNOTH}} \sum_{i=1}^{16} q_i |g_i\rangle \xrightarrow{\text{T}} \sum_{i=1}^{16} |q_i|^2 \lambda_i, \quad (33)$$

and we therefore have the following theorem.

Theorem 3.2.1. *The mapping $F : \Omega^4 \rightarrow \Lambda_2$ is bijective up to a local phase, where Ω^4 is the set of 4-qubit states and Λ_2 is the set of diagonal two-qubit channels.*

Since we considered an arbitrary four-qubit state we have that the transformation of any two-qubit state under imperfect teleportation is a diagonal quantum channel. Furthermore, for any diagonal two-qubit channel f , there exists a four-qubit state $|\phi\rangle$ that induces f . That is, the set of *two-qubit teleportation channels* is the set of diagonal two-qubit channels.

4. OPEN QUESTIONS AND FUTURE RESEARCH

The results of this work, while conclusive for the teleportation of two-qubit states, also holds the potential to extend to n -qubit states. The fact that our model for two-qubit teleportation is a direct extension of the single qubit case lends to our conjecture that it will provide a description of teleportation for an arbitrary number of qubits. Explicitly, we plan to apply inductive methods to our results thus far in order to determine if the following bijection is valid for n -qubit teleportation

$$\sum_{i=1}^{n^2} a_i |i\rangle \xrightarrow{\text{CNOTH}} \sum_{i=1}^{n^2} q_i |g_i\rangle \xrightarrow{\text{T}} \sum_{i=1}^{n^2} |q_i|^2 \lambda_i, \quad (34)$$

which would imply a generalized version of Theorem 3.2.1

Conjecture 4.0.1. *The mapping $F : \Omega^{2n} \rightarrow \Lambda_n$ is bijective up to a local phase, where Λ_n is the set of diagonal n -qubit channels.*

Upon success of characterizing the teleportation channels for multiple qubits, we then plan to investigate the implementation of adaptive schemes which aim to mitigate errors in the transmission of qubits through

quantum channels. In particular, the first of such methods would include a natural extension of Code 5547's patented process Adaptive Quantum Information Processing [11] similar to that given in [1]. Through these deliverables, we will be able to both account for the noise associated with teleportation through imperfectly prepared states and provide a means to study the mitigation of errors that naturally occur in quantum networks that rely on quantum repeaters.

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