NAVSEA WARFARE CENTERS DAHLGREN

On Missile Guidance For Intercept

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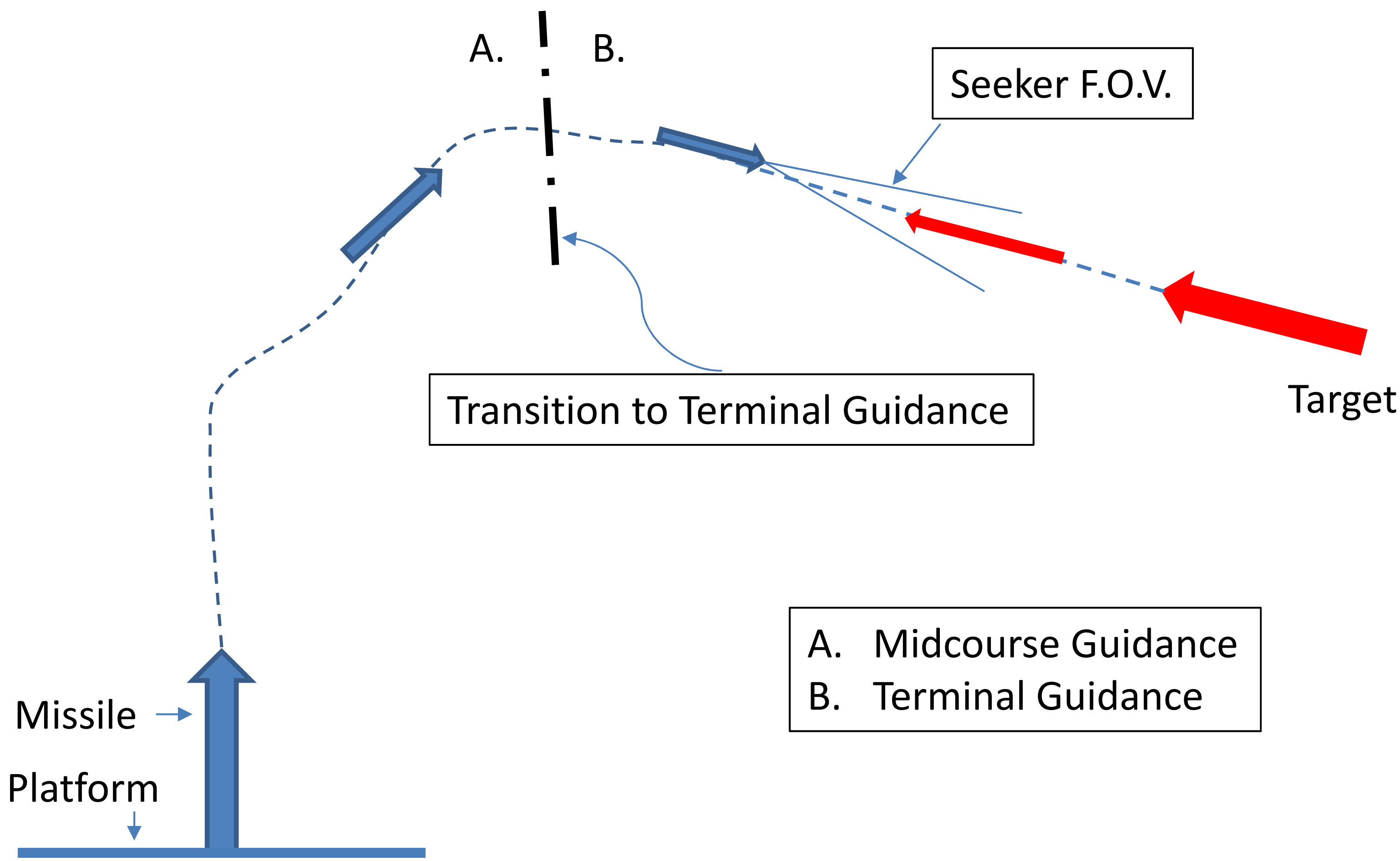
INTRODUCTION

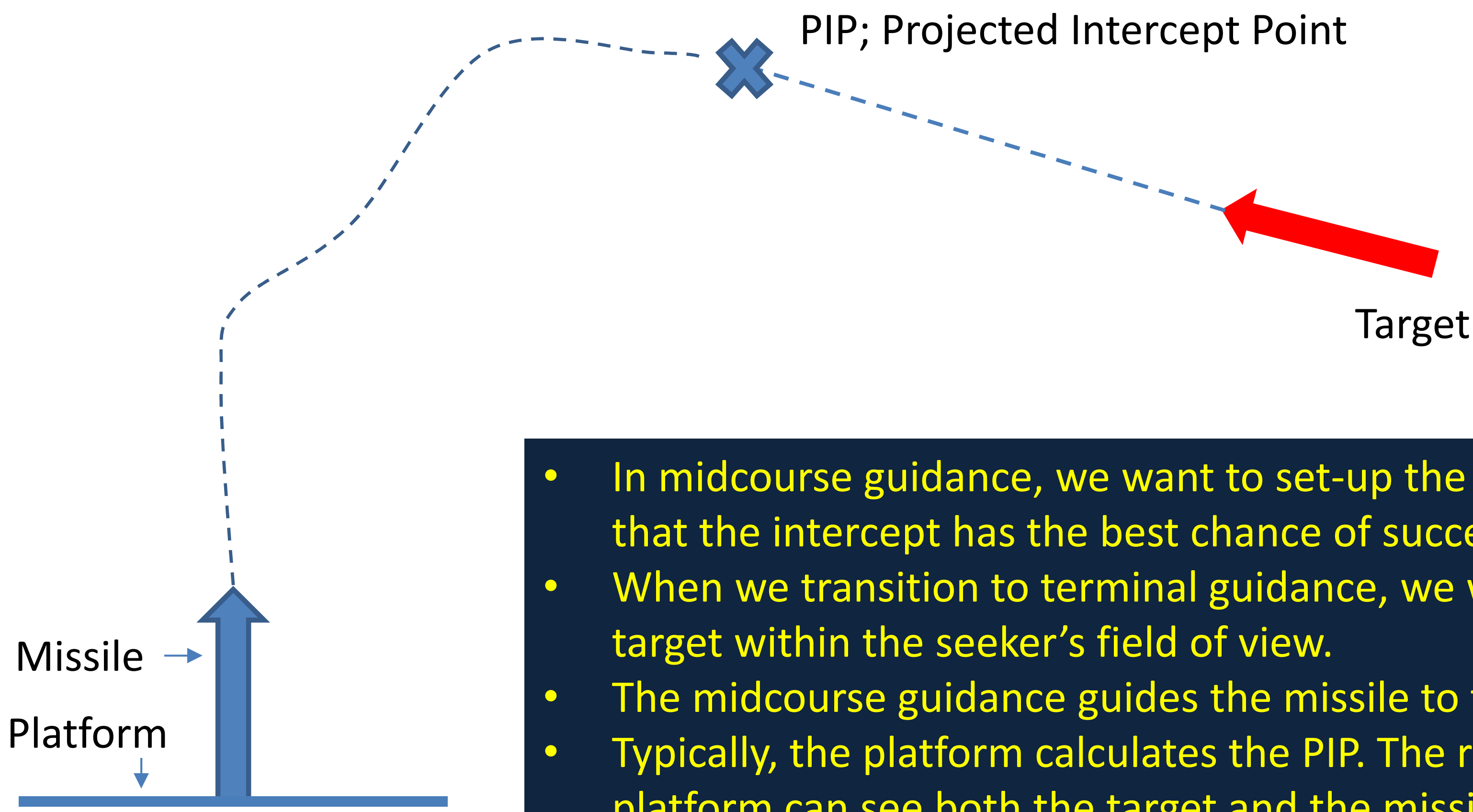


INTRODUCTION

- We want to intercept an airborne target, or a target on the surface with a homing missile.
- The missile has a seeker and an attitude control system.
- There is a platform, such as a ship, which has a radar to track the missile and the target. This platform possibly launches the missile.
- We will discuss two parts of the intercept guidance problem: midcourse guidance and terminal guidance.
- In midcourse guidance the target is not in the field of view of the missile, since they are too far apart.
- In terminal guidance, the target is in the field of view of the missile seeker.

MIDCOURSE GUIDANCE

MIDCOURSE GUIDANCE



- In midcourse guidance, we want to set-up the end game so that the intercept has the best chance of success.
- When we transition to terminal guidance, we want the target within the seeker's field of view.
- The midcourse guidance guides the missile to the PIP.
- Typically, the platform calculates the PIP. The radar on the platform can see both the target and the missile.
- As the missile and target fly, the PIP is continually updated.

MIDCOURSE GUIDANCE

$\vec{t}$ Tangent vector

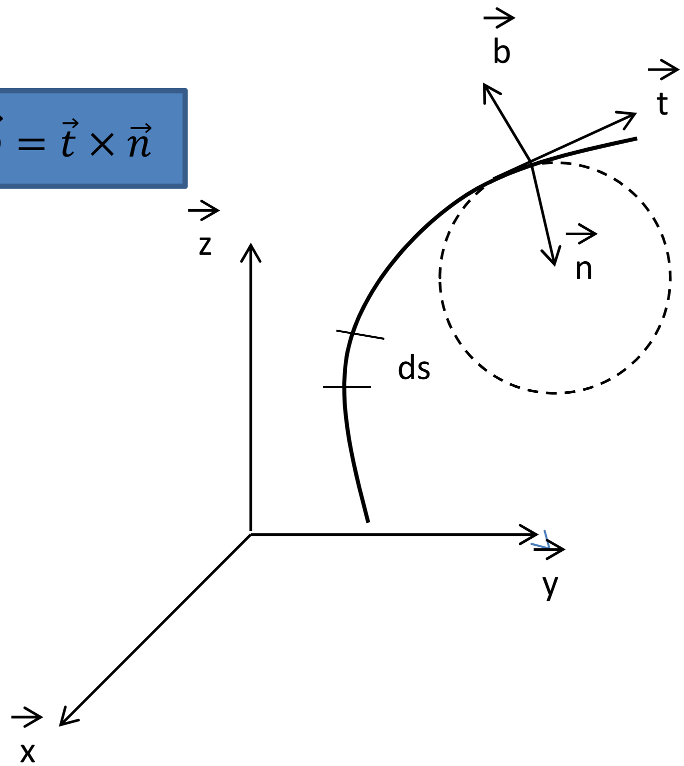
$\vec{n}$ Normal vector

$\vec{b}$ Binormal vector

κ = curvature; one over the radius of curvature.

τ = torsion; the rate of change of the binormal with s. In the normal direction.

$$\vec{b} = \vec{t} \times \vec{n}$$



Formulas of Frenet

$$\frac{d\vec{t}}{ds} = \kappa\vec{n}$$

$$\frac{d\vec{n}}{ds} = -\kappa\vec{t} + \tau\vec{b}$$

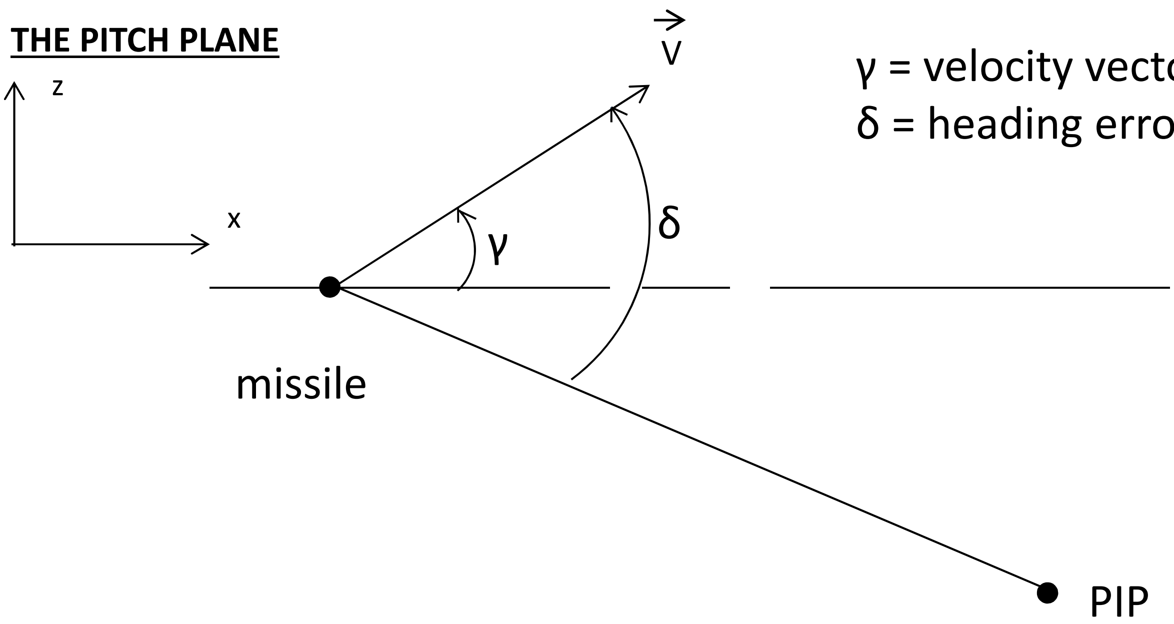
$$\frac{d\vec{b}}{ds} = -\tau\vec{n}$$

A good strategy is to fly to the PIP while minimizing the curvature. This is called kappa guidance.

MIDCOURSE GUIDANCE

- There are some smoothness conditions.
- With these conditions, to each smooth curve there corresponds a unique κ and τ ; and vise-versa.
- In particular,
  - Continuous $\kappa(s)$ and $\tau(s)$ give unique continuous and differentiable tangent, normal and binormal vectors (according to the formulas of Frenet) and a unique curve in space, up to a translation and vice-versa.

MIDCOURSE GUIDANCE



γ = velocity vector angle
δ = heading error angle

Formulas of Frenet

$$\frac{d\vec{t}}{ds} = \kappa\vec{n}$$

$$\frac{d\vec{n}}{ds} = -\kappa\vec{t} + \tau\vec{b}$$

$$\frac{d\vec{b}}{ds} = -\tau\vec{n}$$

$$\frac{d\gamma}{dR} = -\kappa\sec(\delta)$$

$$\frac{d\delta}{dR} = -\kappa\sec(\delta) - \frac{\tan(\delta)}{R}$$

$$= -\begin{bmatrix} 0 \\ \tan(\delta)/R \end{bmatrix} - \sec(\delta)\begin{bmatrix} 1 \\ 1 \end{bmatrix}\kappa$$

MIDCOURSE GUIDANCE

$$\frac{d\begin{bmatrix}\gamma\\ \delta\end{bmatrix}}{dP} = \begin{bmatrix}0\\ \tan(\delta)/(R_0 - P)\end{bmatrix} + \sec(\delta)\begin{bmatrix}1\\ 1\end{bmatrix}\kappa$$

$$J(\gamma, \delta, P, \kappa) = \int_0^{R_0} \left((\kappa^2/2) + \omega^2\right)\sec(\delta)\, dP$$

MIDCOURSE GUIDANCE

$$\frac{d\begin{bmatrix}\gamma\\ \delta\end{bmatrix}}{dP} = \begin{bmatrix}0\\ \tan(\delta)/(R_0 - P)\end{bmatrix} + \sec(\delta)\begin{bmatrix}1\\ 1\end{bmatrix}\kappa$$

$$J(\gamma, \delta, P, \kappa) = \int_0^{R_0} \left((\kappa^2/2) + \omega^2\right)\sec(\delta)\, dP$$

$$\begin{bmatrix}z_1\\ z_2\end{bmatrix} = \begin{bmatrix}\gamma - \delta\\ -\tan(\delta)/R\end{bmatrix}$$

$$\begin{bmatrix}\gamma\\ \delta\end{bmatrix} = \begin{bmatrix}z_1 - \tan^{-1}(Rz_2)\\ -\tan^{-1}(Rz_2)\end{bmatrix}$$

$$\kappa = \alpha + \beta u$$

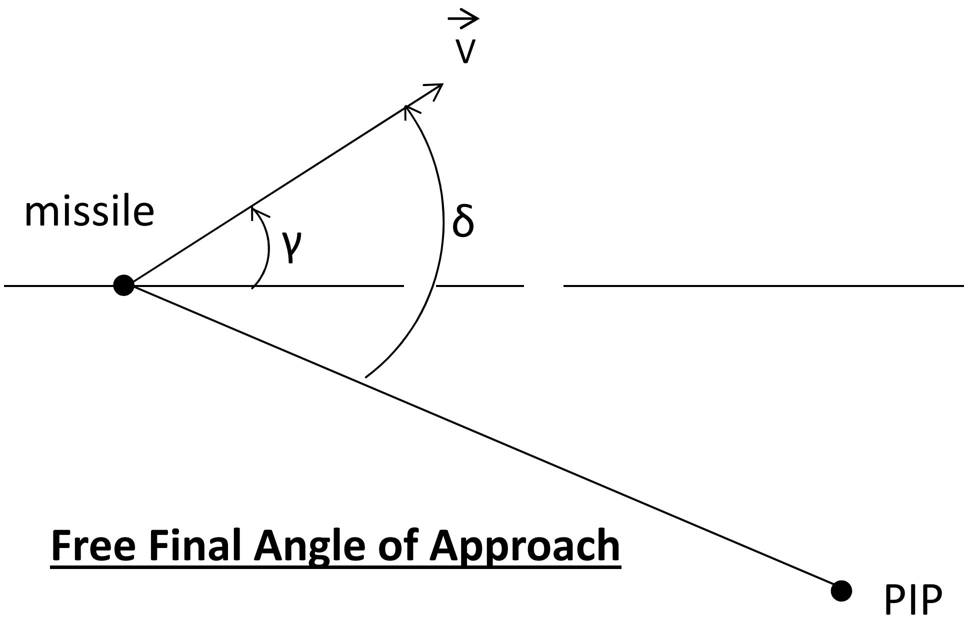
$$\alpha = -\frac{\sin(\delta)(\cos^2(\delta) + 1)}{R}$$

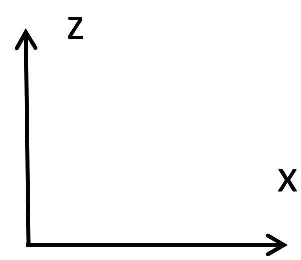
$$\beta = -R\cos^3(\delta)$$

$$\frac{d}{dP}\begin{bmatrix}z_1\\ z_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}z_1\\ z_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u$$

$$\bar{J} = \int_0^{R_0}\left(\left(2 + \frac{R^2\omega^2}{2}\right) - 2Rz_2u + \frac{R^2}{2}u^2\right)dP + H.O.T.$$

MIDCOURSE GUIDANCE





Free Final Angle of Approach

$$\kappa = F(\omega, R, \gamma, \delta\ )$$

$$\gamma = \gamma(\omega, R, R_0, \gamma_0, \delta_0)$$

$$x(s + \Delta s) = x(s) + \Delta s \cdot \cos(\gamma)$$

$$z(s + \Delta s) = z(s) + \Delta s \cdot \sin(\gamma)$$

Fixed Final Angle of Approach

$$\kappa = F\left(\omega, R, \gamma, \delta, \gamma_f\right)$$

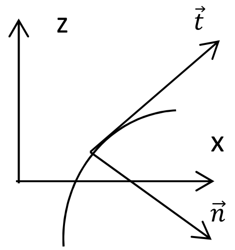
$$\gamma = \gamma\left(\omega, R, R_0, \gamma_0, \delta_0, \gamma_f\right)$$

$$x(s + \Delta s) = x(s) + \Delta s \cdot \cos(\gamma)$$

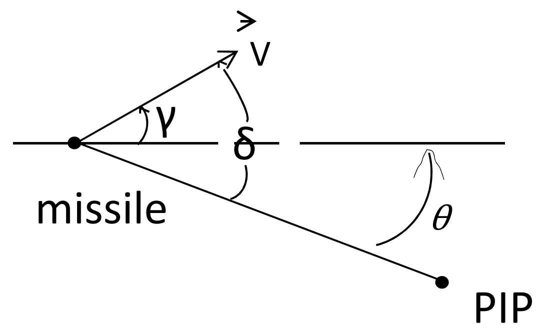
$$z(s + \Delta s) = z(s) + \Delta s \cdot \sin(\gamma)$$

MIDCOURSE GUIDANCE

THE PITCH PLANE



$$t_z = n_x;\ t_x = -n_z$$



Formulas of Frenet

$$\frac{d\vec{t}}{ds} = \kappa\vec{n}$$

$$\frac{d\vec{n}}{ds} = -\kappa\vec{t} + \tau\vec{b}$$

$$\frac{d\vec{b}}{ds} = -\tau\vec{n}$$

$$\frac{d\gamma}{ds} = \frac{d}{ds}\sin^{-1}(t_z) = \frac{\frac{dt_z}{ds}}{\sqrt{1-(t_z)^2}} = \frac{\kappa n_z}{t_x} = -\kappa$$

So, $$\frac{d\gamma}{dR} = \frac{d\gamma}{ds}\frac{ds}{dt}\frac{1}{dR/dt} = -\kappa \cdot V \cdot \frac{1}{V\cos\delta} = -\kappa \cdot \sec\delta$$

$$\theta = \tan^{-1}\left(\frac{z}{x}\right) = \tan^{-1}(u) \qquad \frac{du}{dt} = \frac{1}{x^2}\left(x\frac{dz}{dt} - \frac{dx}{dt}z\right)$$

$$\frac{d\theta}{dt} = x^2\frac{\frac{du}{dt}}{R^2} = \frac{x^2}{R^2}\left(\frac{1}{x^2}\left(x\frac{dz}{dt} - \frac{dx}{dt}z\right)\right) = \frac{1}{R^2}\left(R\cos(\delta-\gamma)\cdot V\sin\gamma - V\cos\gamma\cdot(-1)R\sin(\delta-\gamma)\right) = \frac{V}{R}\sin\delta$$

$$\frac{d\theta}{dR} = \frac{d\theta}{dt}\frac{1}{dR/dt} = \frac{V}{R}\sin\delta\cdot\frac{1}{-V\cos\delta} = -\frac{1}{R}\tan\delta$$

$$\frac{d\delta}{dR} = -\frac{1}{R}\tan\delta - \kappa\cdot\sec\delta$$

MIDCOURSE GUIDANCE

Formulas of Frenet

$$\frac{d\vec{t}}{ds} = \kappa\vec{n}$$

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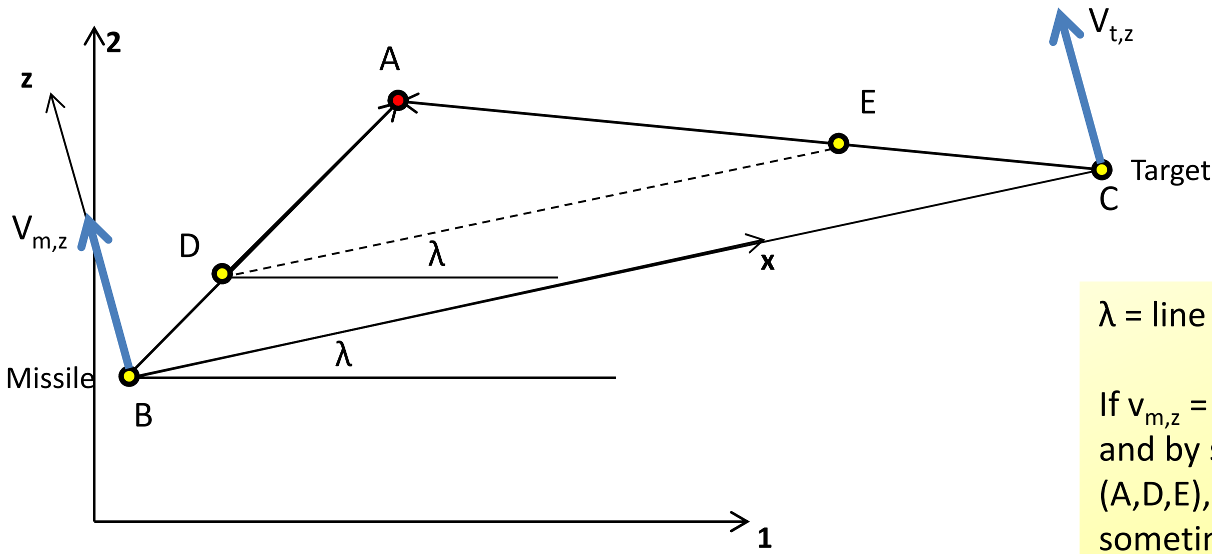
$$\frac{d\vec{b}}{ds} = -\tau\vec{n}$$

3D Kappa Guidance

- In 3D, we use the Formulas of Frenet for the plant equations. Omit $\vec{b}$ since $\vec{b} = \vec{t} \times \vec{n}$.
- There is a lot more bookkeeping. We must use quaternions.
- We don't have a solution to the optimal control problem in 3D at this time.
- As a first cut, we use the 2D optimal solution for the curvature and proportional control for the torsion.

TERMINAL GUIDANCE

TERMINAL GUIDANCE



λ = line of sight angle

If $v_{m,z} = v_{t,z}$ then λ remains a constant and by similar triangles: (A,B,C) and (A,D,E), a collision will occur. This is sometimes called the ”collision triangle.”

Proportional Navigation is defined by

$$a_{m,z} = N \cdot V_C \; \dot{\lambda}$$

where

$a_{m,z}$ is the missile lateral acceleration

N is a unitless constant, typically in the 3 to 5 range

V_C is the closing velocity

Proportional Navigation is equivalent to

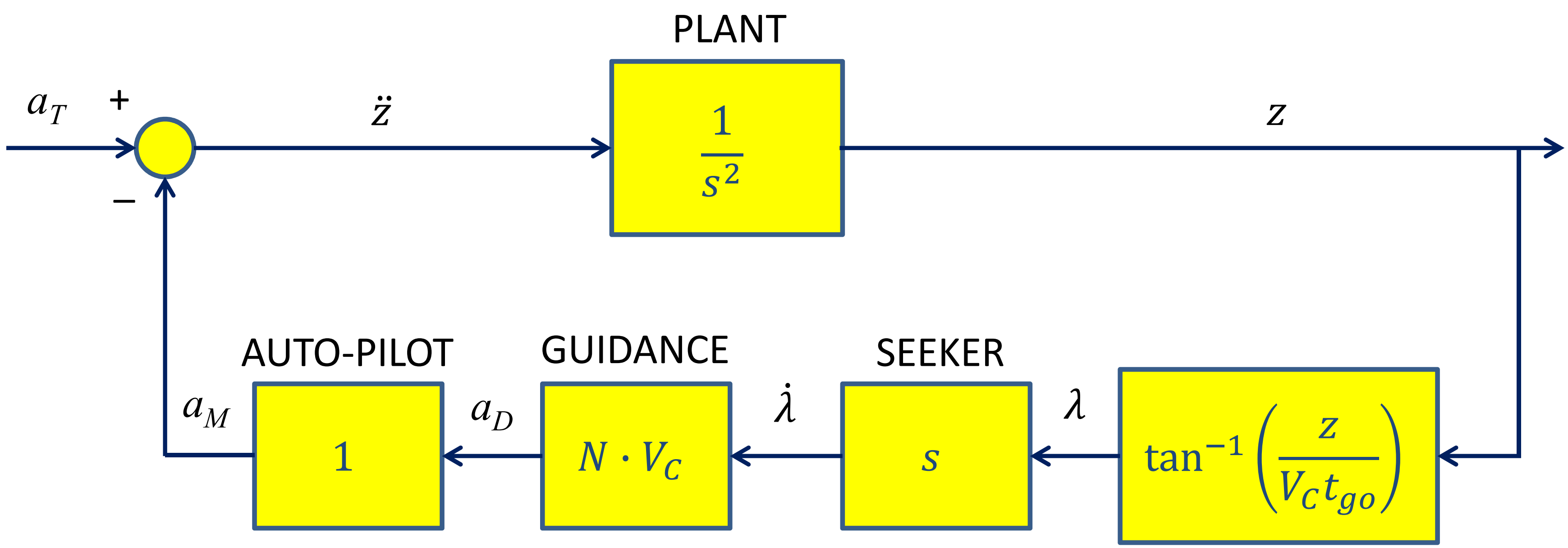
$$a_{m,z} = N \cdot ZEM / t_{go}{}^2$$

where

Z.E.M. is the zero effort miss. The miss distance if we continue on the present course, with no control effort.

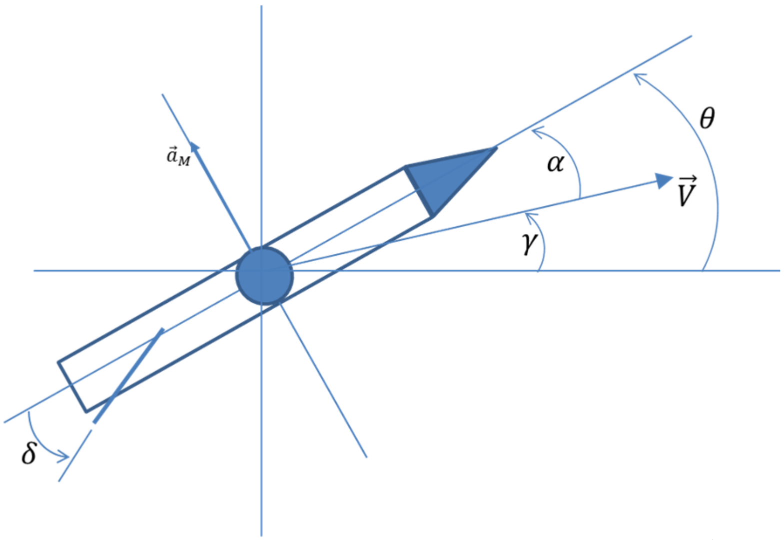
t_{go} is the time to go for intercept.

TERMINAL GUIDANCE



a_M	Missile acceleration
a_T	Target acceleration
a_D	Desired acceleration

TERMINAL GUIDANCE



PITCH PLANE DYNAMICS

$$a_M = K_1\alpha + K_2\alpha^3 + K_3\delta + K_4\alpha^2\delta$$
$$\dot{\theta} = q$$
$$\dot{q} = C_1\alpha + C_2\alpha^3 + C_3\delta + C_4\alpha^2\delta$$
$$\dot{\delta} = T\delta + T\delta_C$$

where, $\alpha = \theta - \frac{(\int a_M)}{V} = \theta - v/V$

K_i's > 0
C_i's < 0

Input: δ_C
Output: θ

We design a pitch plane control system $\delta_C = G(\theta_C, states)$, e.g., using feedback linearization; obtaining $\theta \rightarrow \theta_C$.

We make the assumption that $a_{LAT} \approx a_M$. With a desired lateral acceleration $a_{LAT,D} = nV_C\dot{\lambda}$, we want to determine a desired pitch angle $\theta_D = T(a_{LAT,D})$. We linearize the dynamics for use with this part of the control system.

$$a_M = K_1(\theta - v/V) + K_3\delta = K_1\alpha + K_3\delta$$

Then,

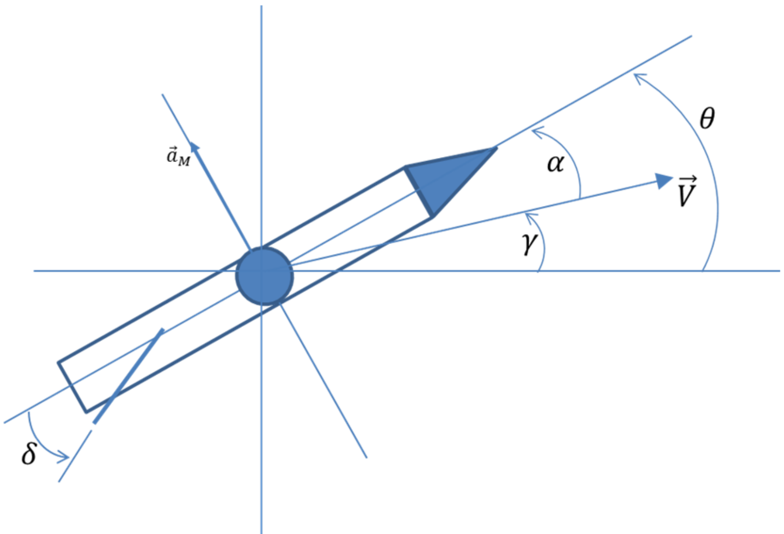
$$\alpha = (a_{LAT,D} - K_3\delta)/K_1$$

Finally,

$$\theta_D = \alpha + \gamma$$

This θ_D is used as input in the pitch plane control system. $\theta_D = T(a_{LAT,D})$ is a <u>transducer</u>: I/O of different units.

TERMINAL GUIDANCE



PITCH PLANE DYNAMICS

$$
\begin{aligned}
a_M &= K_1(\theta - v/V) + K_2(\theta - v/V)^3 + K_3\delta + K_4(\theta - v/V)^2\delta \\
\dot{\theta} &= q \\
\dot{q} &= C_1(\theta - v/V) + C_2(\theta - v/V)^3 + C_3\delta + C_4(\theta - v/V)^2\delta \\
\dot{\delta} &= T\delta + T\delta_C
\end{aligned}
$$

where, $\alpha = \theta - \frac{(\int a_M)}{V} = \theta - v/V$

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Then,

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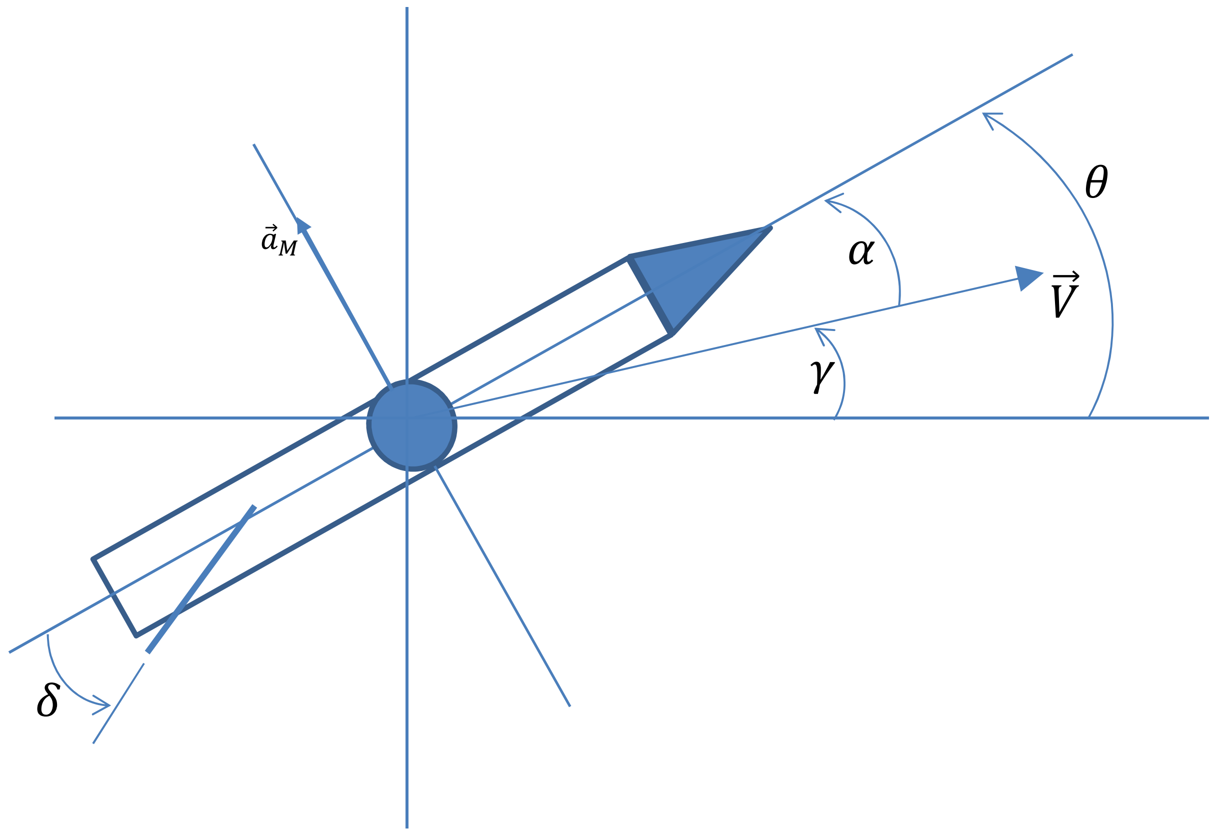


$\vec{a}_M$
θ
α
$\vec{V}$
γ
δ

TERMINAL GUIDANCE

PLANT DYNAMICS

$$\frac{d}{dt}\begin{bmatrix} z \\ z' \\ a_T \\ a_T' \\ a_M \end{bmatrix} = \underbrace{\begin{bmatrix} 0100 & 0 \\ 0010 & -1 \\ 0001 & 0 \\ 0000 & 0 \\ 0000 & -1/T \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} z \\ z' \\ a_T \\ a_T' \\ a_M \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1/T \end{bmatrix}}_{B} a_C$$

COST

$$J = \int_{t_0}^{t_f} \frac{1}{2} a_C{}^2 dt$$

TERMINAL CONSTRAINT

$$z(t_f) = 0$$

ADJOINT DYNAMICS

$$\frac{dp}{dt} = -A'p$$

HAMILTONIAN

$$H = \frac{1}{2} a_C{}^2 + \langle p, Ax + Ba_C \rangle$$

- The solution to this optimal control problem is, after a few simplifications, proportional navigation.

$$a_C = \left(\frac{N(x)}{\left(t_{go}\right)^2}\right)\left(z(t_0) + t_{go}\dot{z}(t_0) + \frac{\left(t_{go}\right)^2}{2} a_T(t_0) + \frac{\left(t_{go}\right)^3}{3} \dot{a}_T(t_0) + T^2(1 - x - e^{-x}) a_M(t_0)\right)$$

with $x = {}^{t_{go}}/_{T}$, where $t_{go} = t - t_f$

$$N(x) = \left(\frac{6x^2(x + e^{-x} - 1)}{6x(1 - 2e^{-x}) + 2x^3 + 3(1 - e^{-2x}) - 6x^2}\right)$$

TERMINAL GUIDANCE

- The Zero Effort Miss (ZEM) is determined by setting a_C=0 and calculating $z(t=t_f)$
- Finding the state transition matrix facilitates this determination.

$$z(t_f)|_{a_M=0} = ZEM = z(t_0) + t_{go}\dot{z}(t_0) + \frac{(t_{go})^2}{2} a_T(t_0) + \frac{(t_{go})^3}{3} \dot{a}_T(t_0) + T^2(1 - x - e^{-x}) a_M(t_0)$$

- This term appeared above. Substituting,

$$a_C = \left(\frac{N(x)}{(t_{go})^2} \right) (ZEM)$$

- An interesting case is when $T \rightarrow 0$.

$$\lim_{T \rightarrow 0} N\,(x) = 3$$

TERMINAL GUIDANCE

The variation of parameters solution to the time invariant differential equation,

$$\dot{X} = AX + Bu, \qquad Y = CX \qquad \text{is} \qquad Y(t) = Ce^{A(t)}X(t_0) + C\int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$

Note that the solution has a part due to the initial condition $X(t_0)$ and a forced part due to the input u. In our case we have, with A and B as before,

$$X = \begin{bmatrix} z \\ z' \\ a_T \\ a_T{}' \\ a_M \end{bmatrix}; C = [1 \;\; 0 \;\; 0 \;\; 0 \;\; 0]$$

$$z(t_f) = z(t_0) + (t_f - t_0)\dot{z}(t_0) + \frac{(t_f - t_0)^2}{2}a_T(t_0) + \frac{(t_f - t_0)^3}{3}\dot{a}_T(t_0)$$

$$+ T^2\left(1 - \frac{(t_f - t_0)}{T} - e^{-(t_f - t_0)/T}\right)a_M(t_0) + \underbrace{\int_{t_0}^{t_f}\left(T - (t_f - \tau) - Te^{-(t_f-\tau)/T}\right)u(\tau)d\tau}_{\text{forced part}}$$

Note: $t_{go} = t_f - t_0$

TERMINAL GUIDANCE

The previous, setting the forcing term to zero is: ($u = a_C = 0$),

$$Z.E.M. = z(t_f) = z(t_0) + t_{go}\dot{z}(t_0) + \frac{(t_{go})^2}{2} a_T(t_0) + \frac{(t_{go})^3}{3} \dot{a}_T(t_0) \; + T^2\left(1 - \frac{t_{go}}{T} - e^{-t_{go}/T}\right) a_M(t_0)$$

TERMINAL GUIDANCE

- Consider the line of sight angle $\lambda = z/R_{TM} = z\,/\,V_C \cdot t_{go}$. Then,

$$\dot{\lambda} = \frac{1}{V_C}\left(\frac{t_{go}\dot{z} - (-1)z}{\left(t_{go}\right)^2}\right) = \frac{z + t_{go}\dot{z}}{V_C\left(t_{go}\right)^2} = \frac{ZEM}{V_C\left(t_{go}\right)^2}$$

And

$$N \cdot V_C \cdot \dot{\lambda} = N \cdot \frac{ZEM}{\left(t_{go}\right)^2}$$

PLOTS

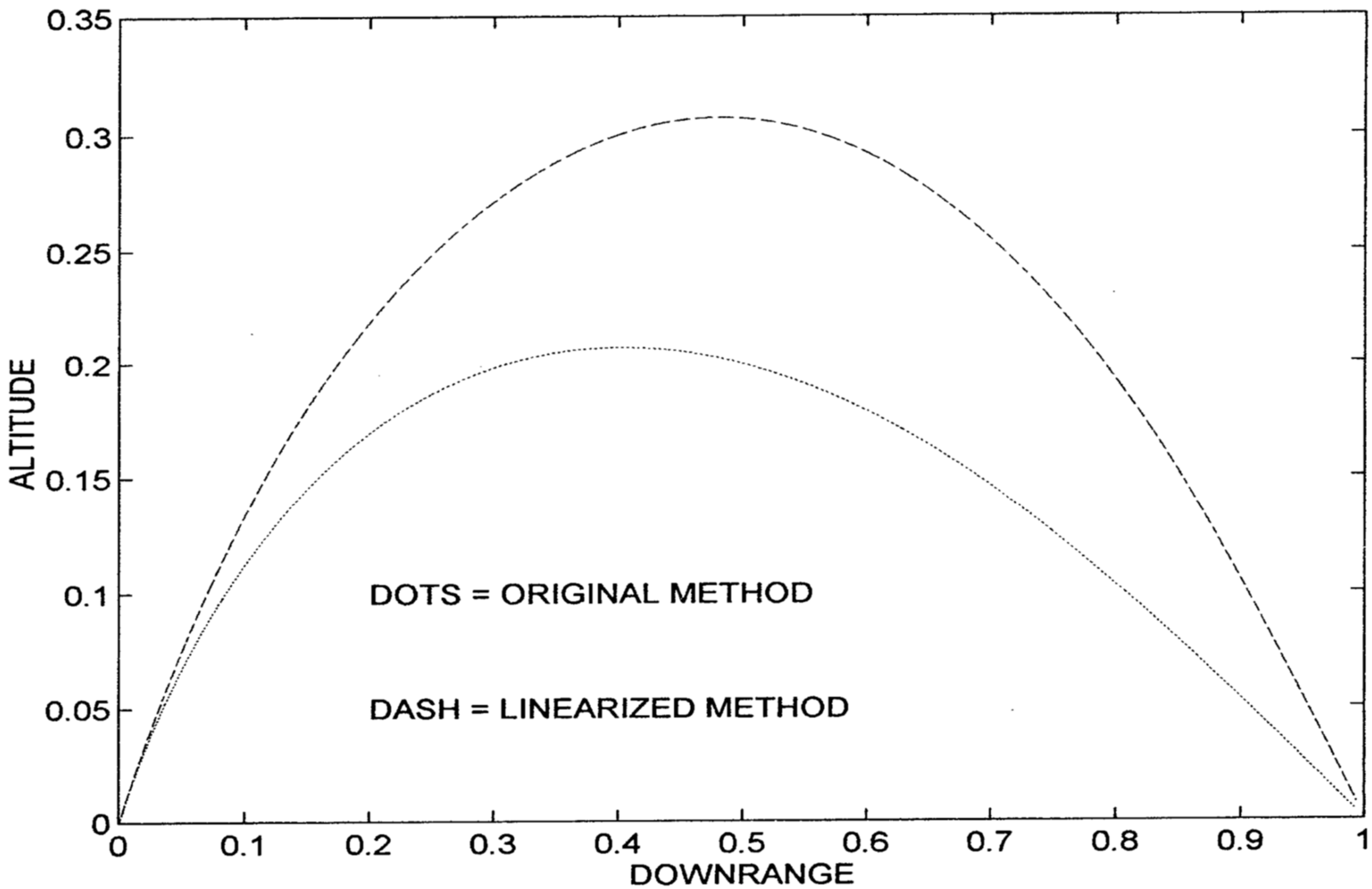


FIGURE 6-1. $\gamma_0 = \delta_0 = \pi/3, \gamma_f = \text{FREE}$

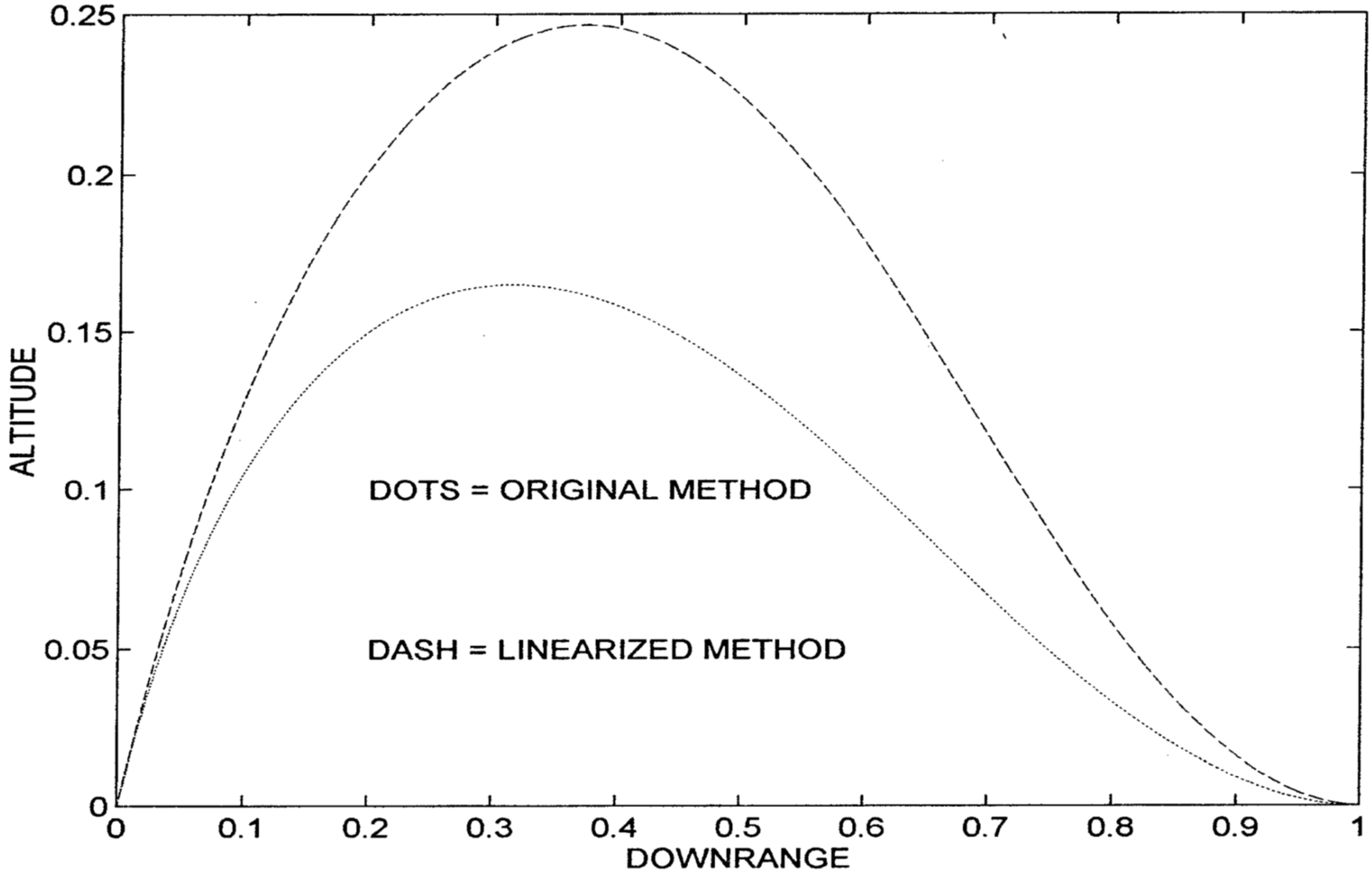


FIGURE 6-3. $\gamma_0 = \delta_0 = \pi/3, \gamma_f = 0$

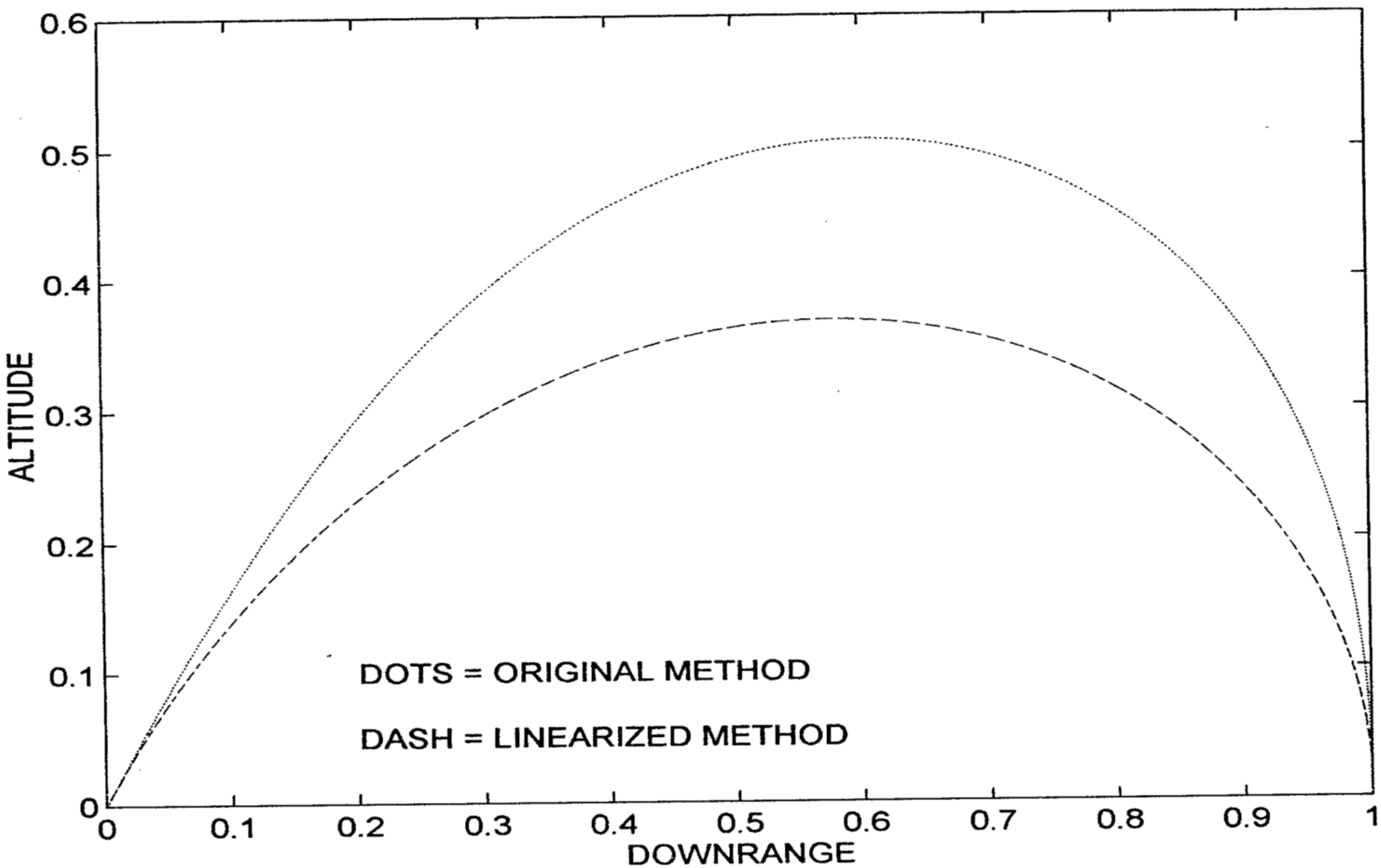


FIGURE 6-2. $\gamma_0 = \delta_0 = \pi/3, \gamma_f = -\pi/2$

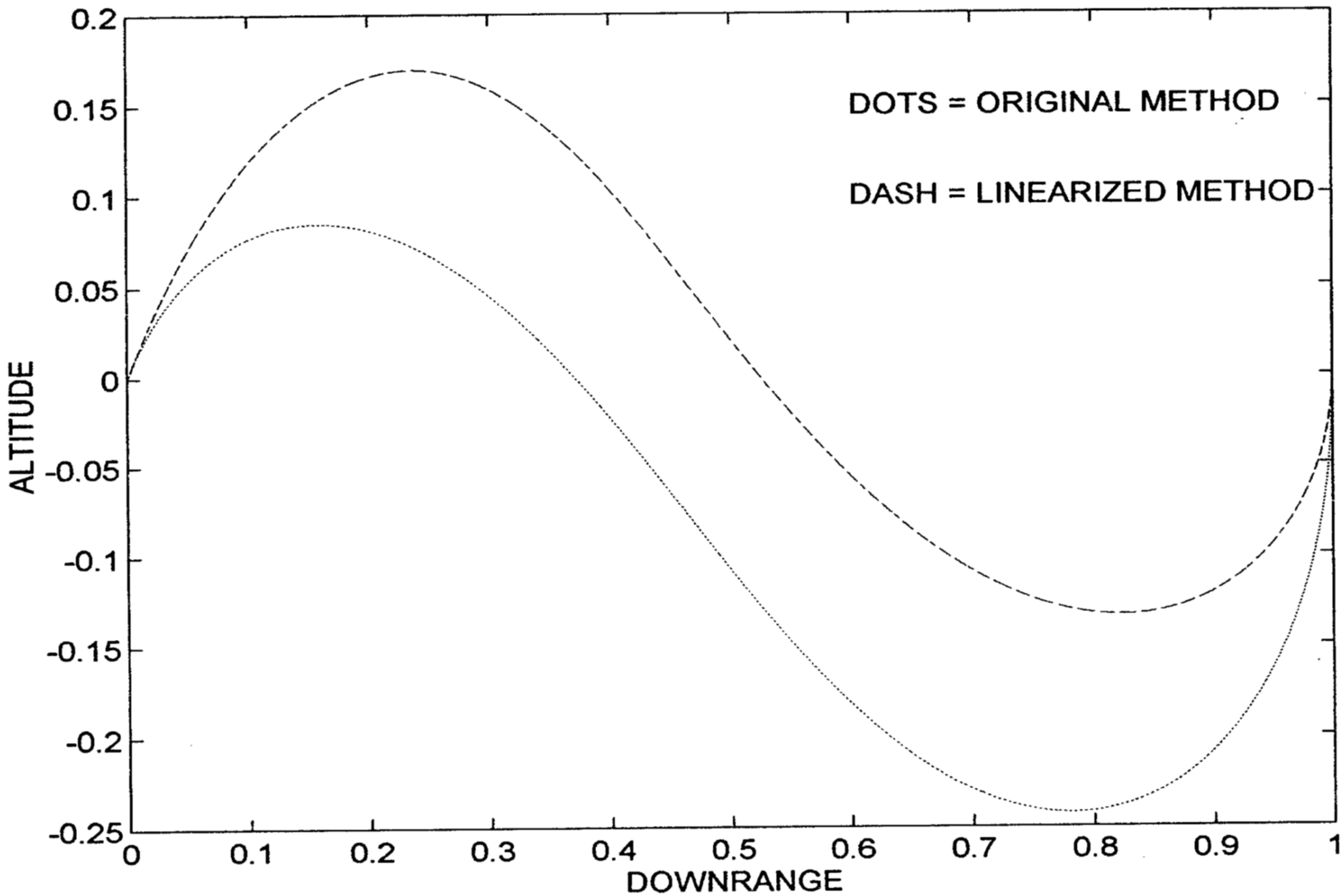
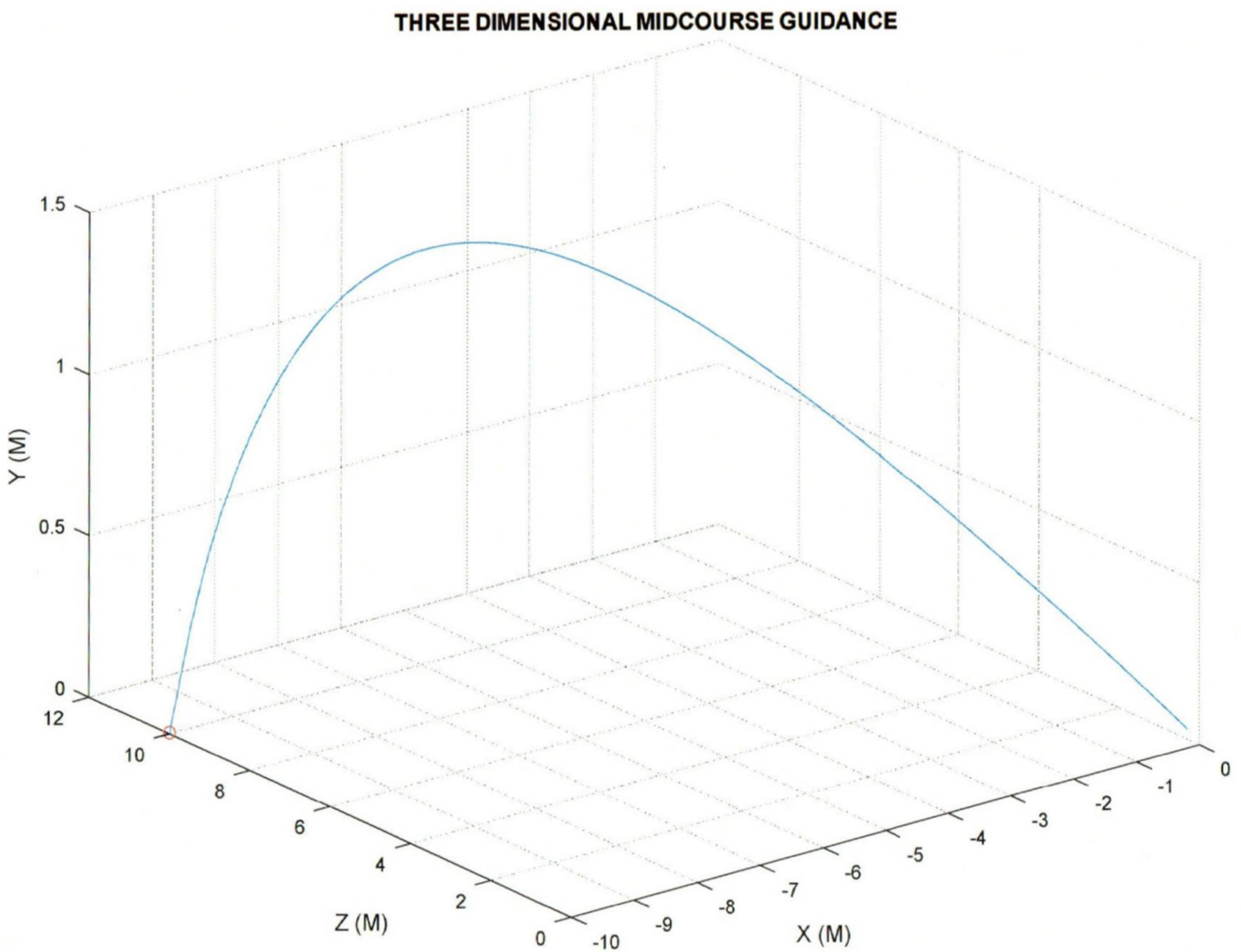
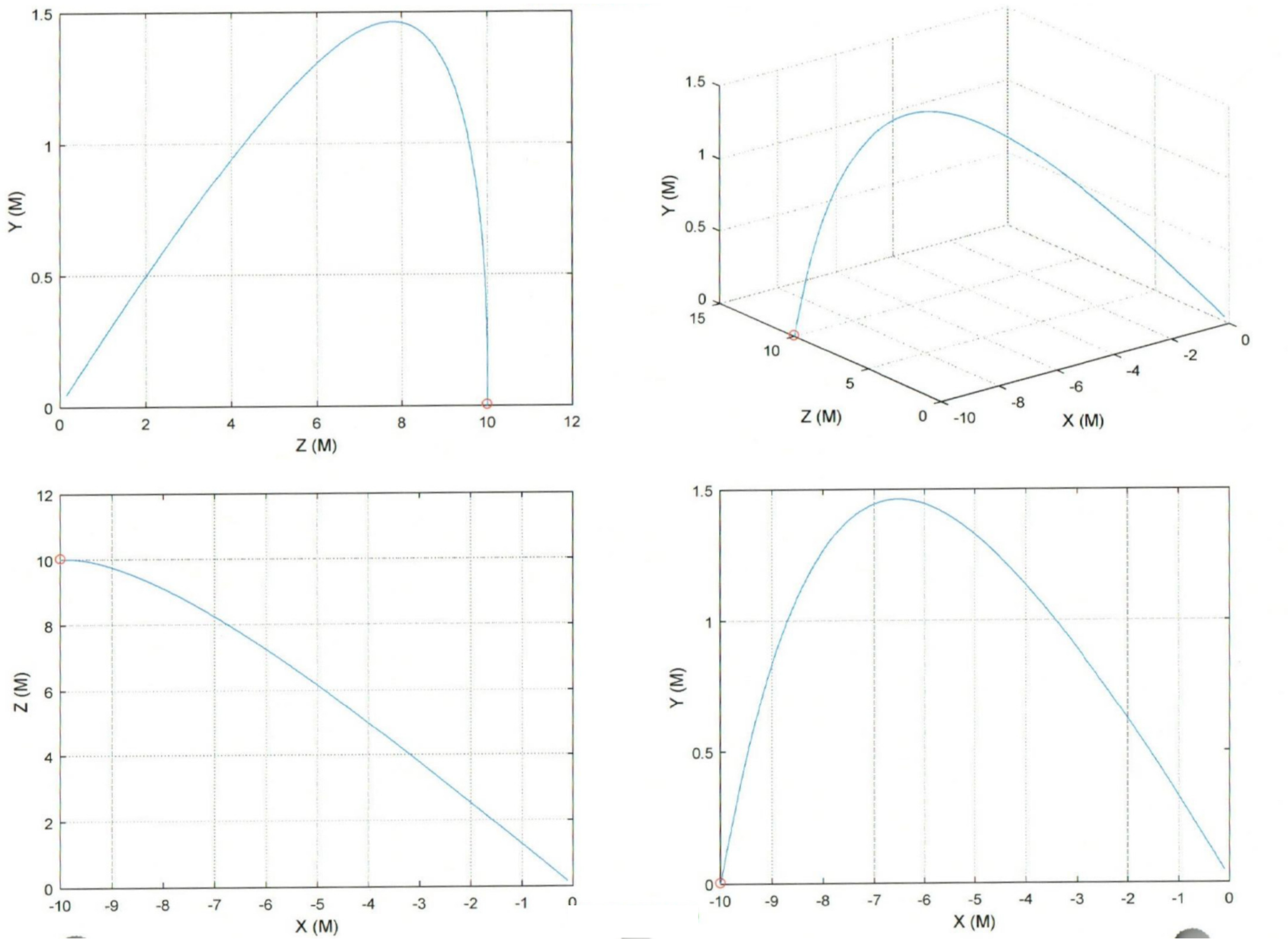


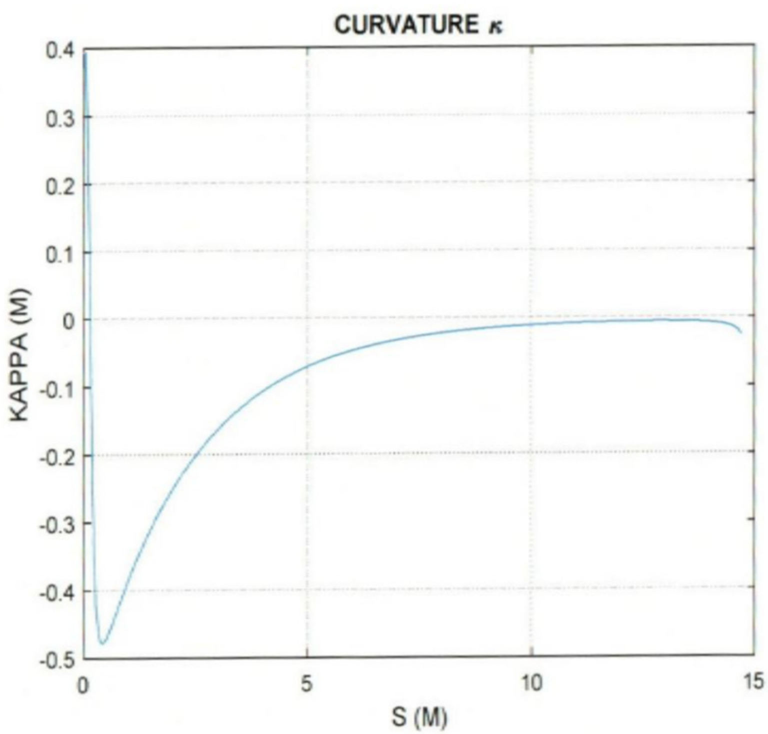
FIGURE 6-4. $\gamma_0 = \delta_0 = \pi/3, \gamma_f = \pi/2$

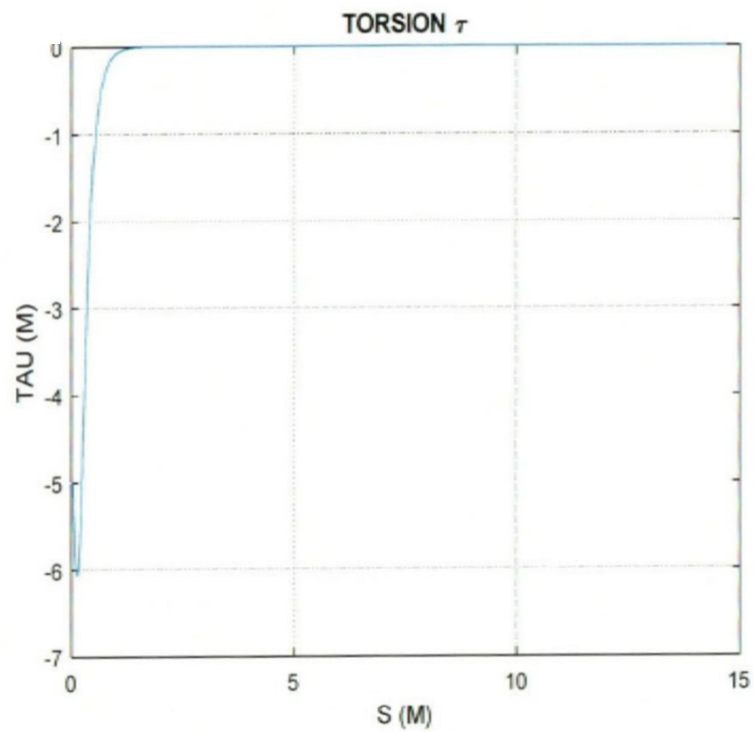


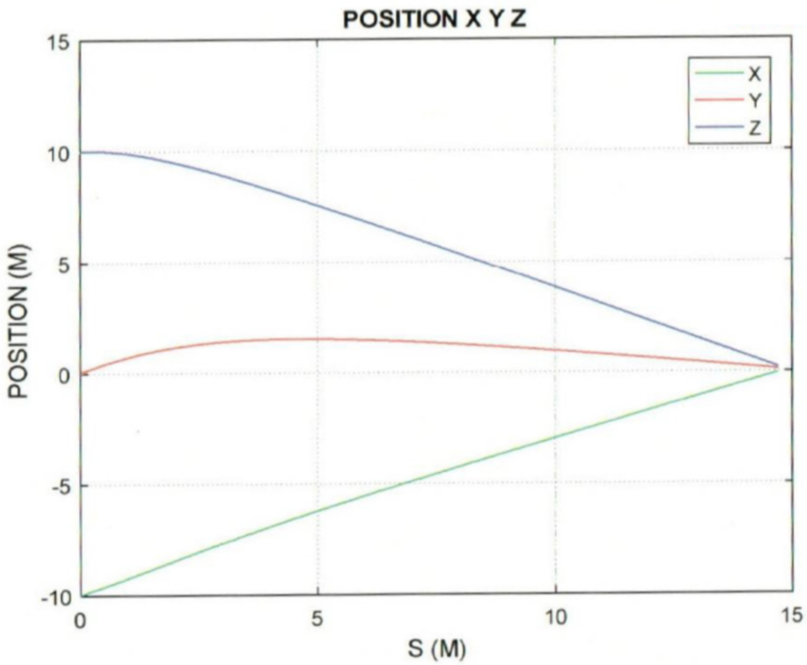
NOTE: This is a mirror image. Mirror in the XY plane.

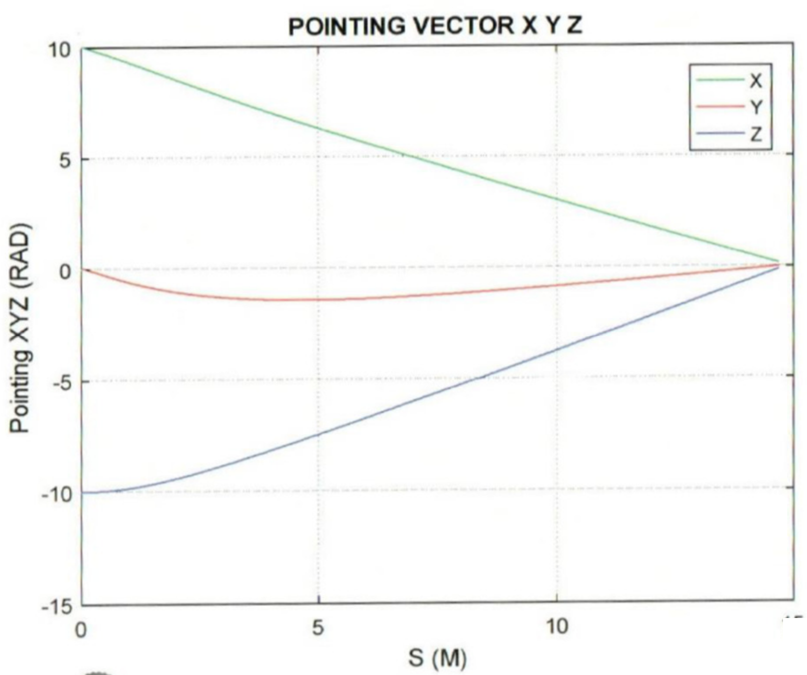


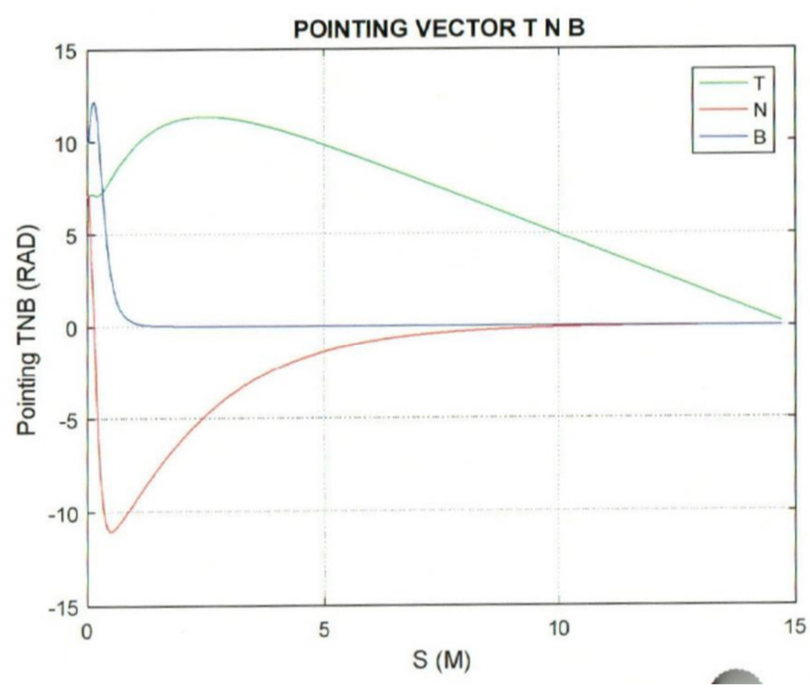
Y (M)
Z (M)
Y (M)
Z (M)
X (M)
Z (M)
X (M)
Y (M)
X (M)

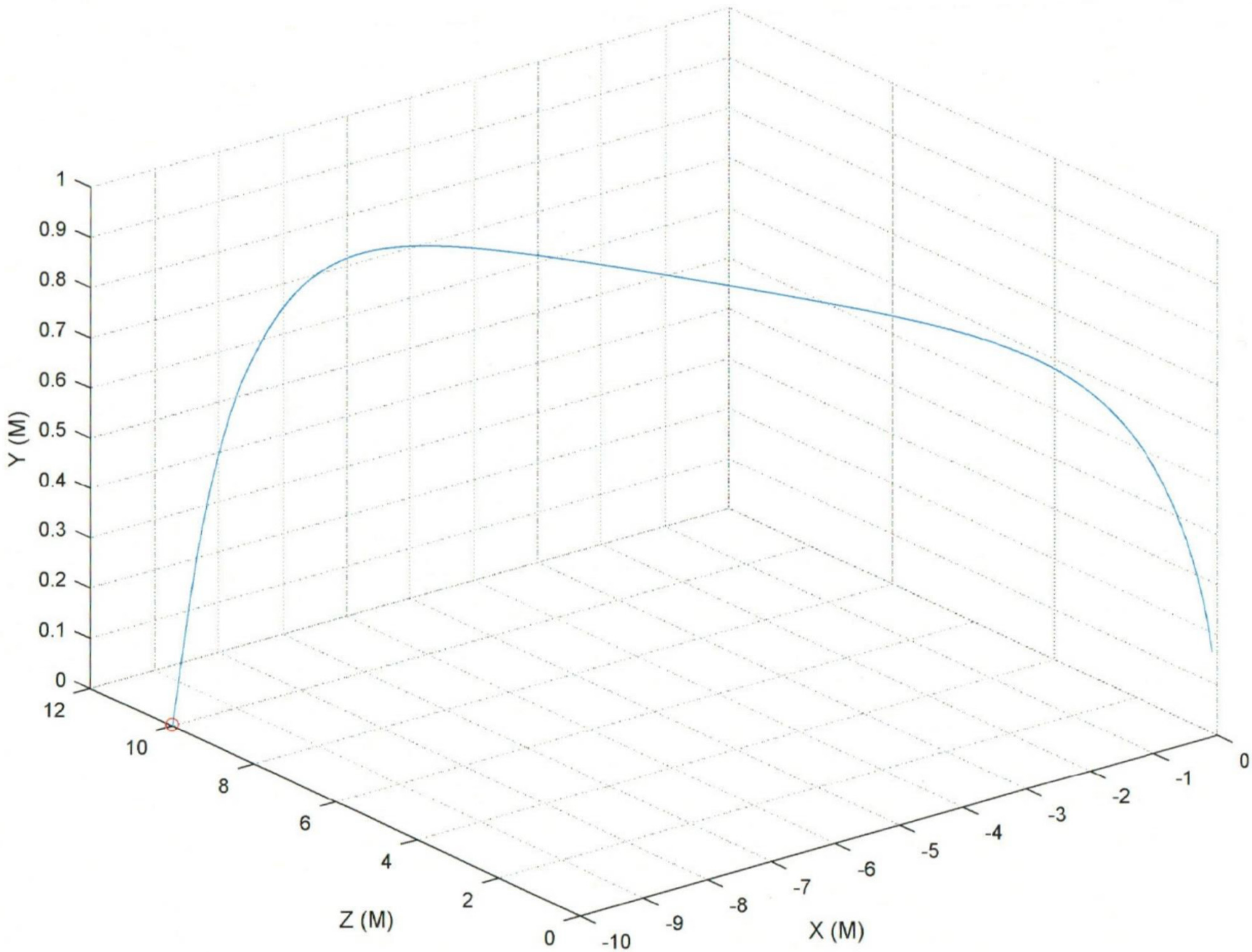

CURVATURE κ
KAPPA (M)
S (M)
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
0
5
10
15


TORSION τ
TAU (M)
S (M)
0
-1
-2
-3
-4
-5
-6
-7
0
5
10
15


POSITION X Y Z
X
Y
Z
POSITION (M)
S (M)


POINTING VECTOR X Y Z
X
Y
Z
Pointing XYZ (RAD)
S (M)


POINTING VECTOR T N B
T
N
B
Pointing TNB (RAD)
S (M)



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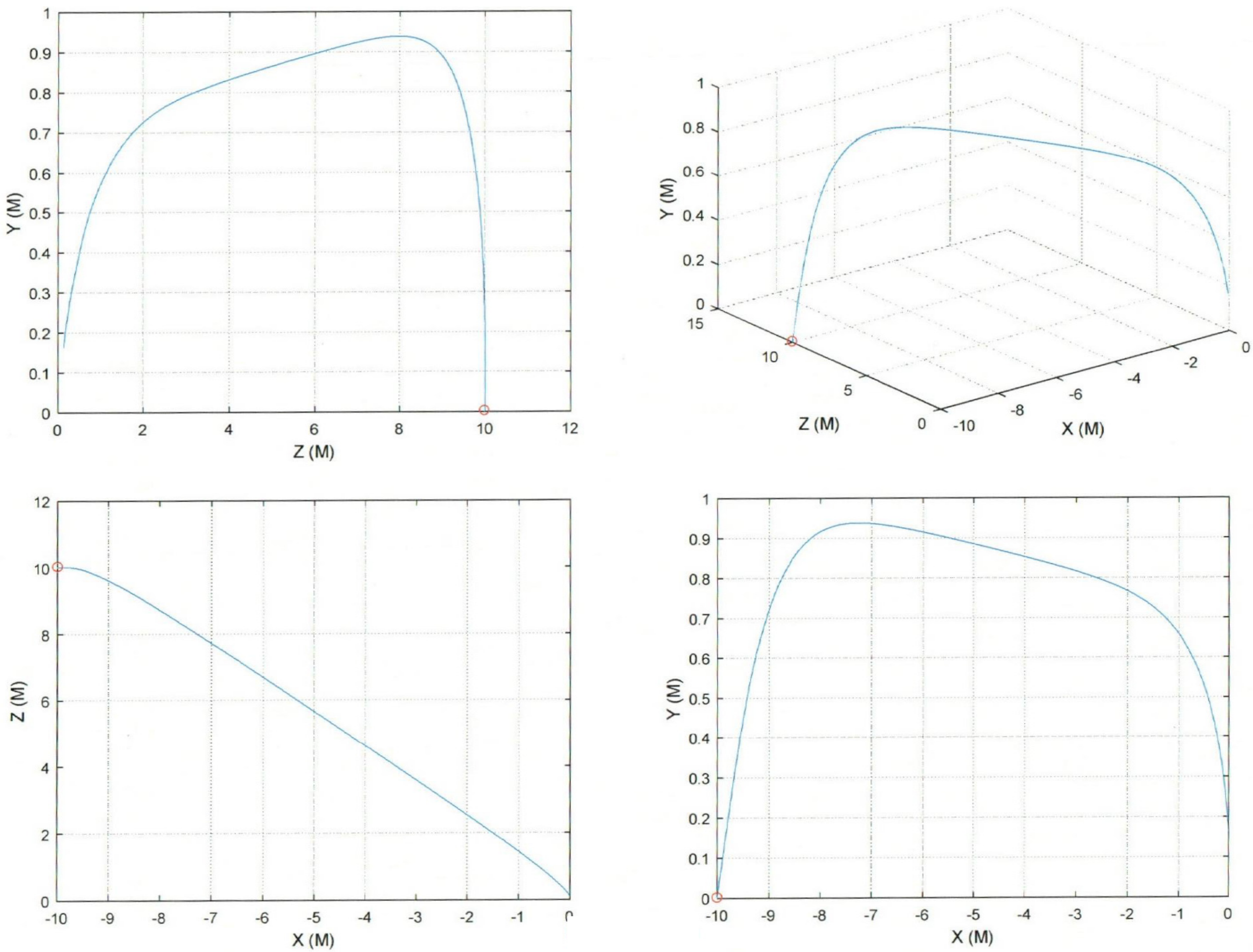
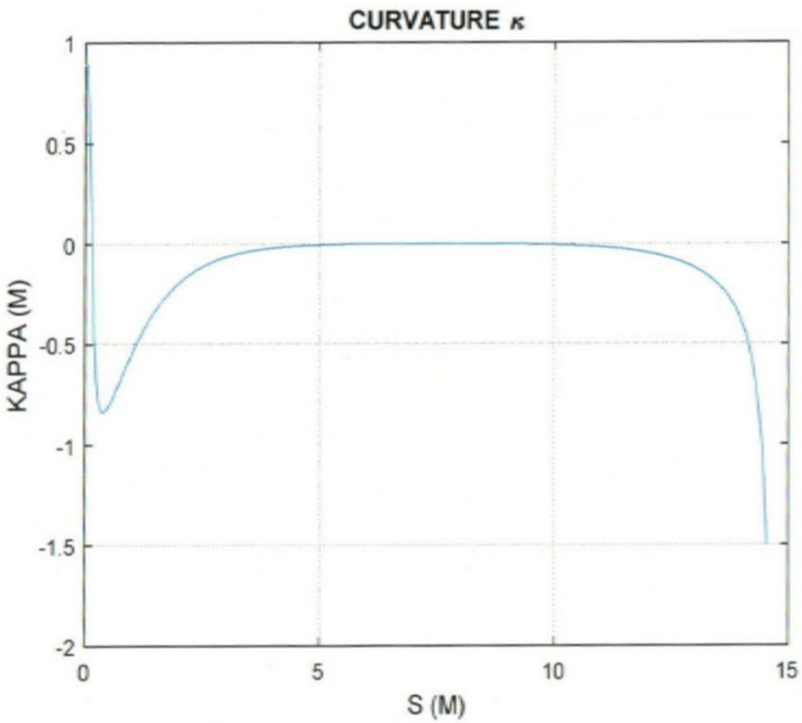
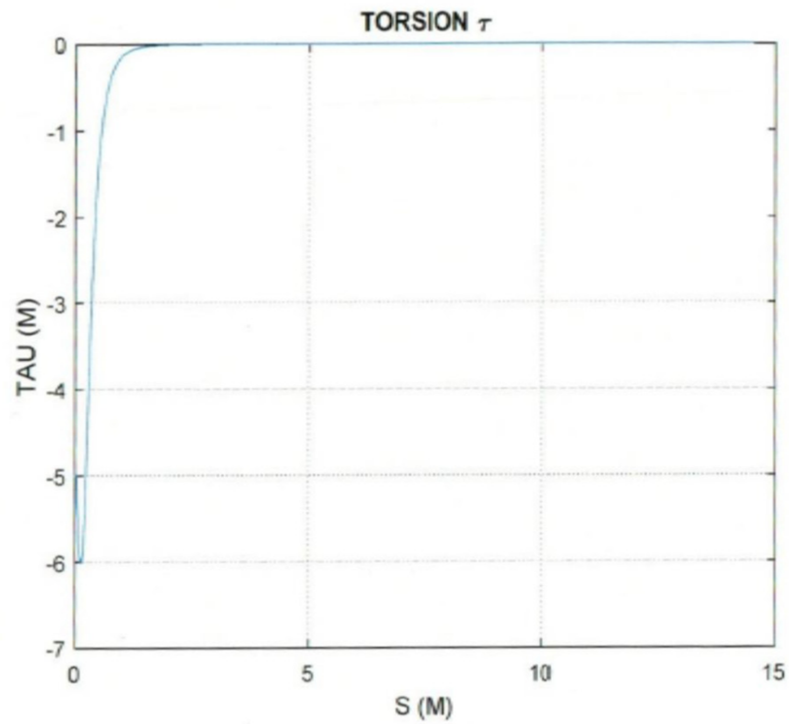
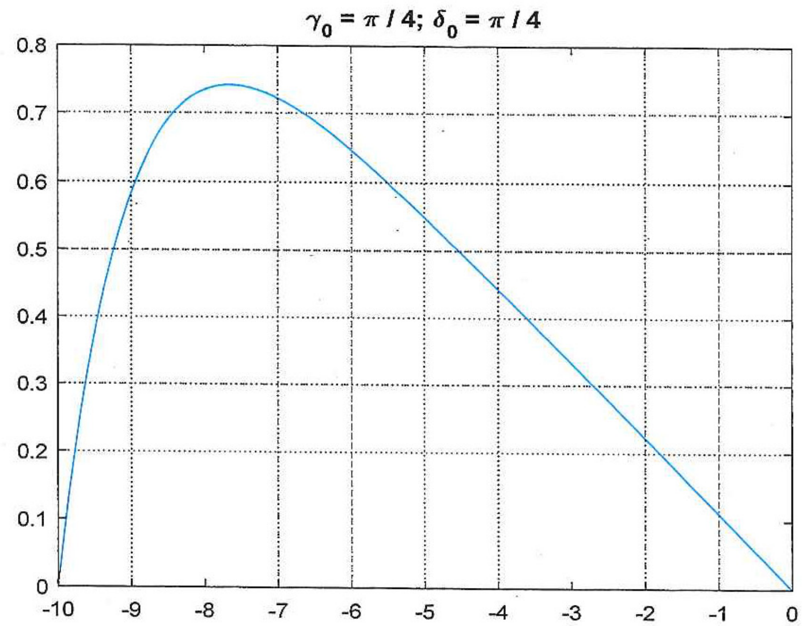
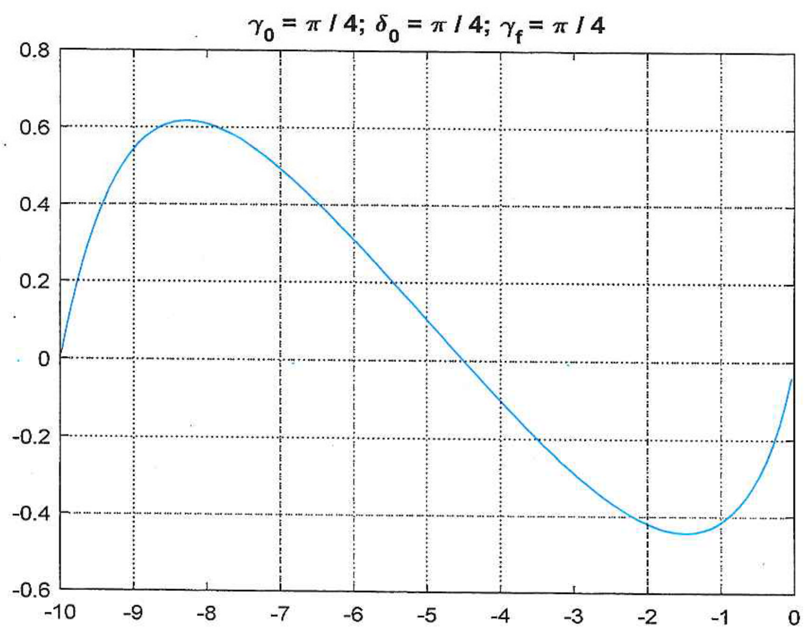
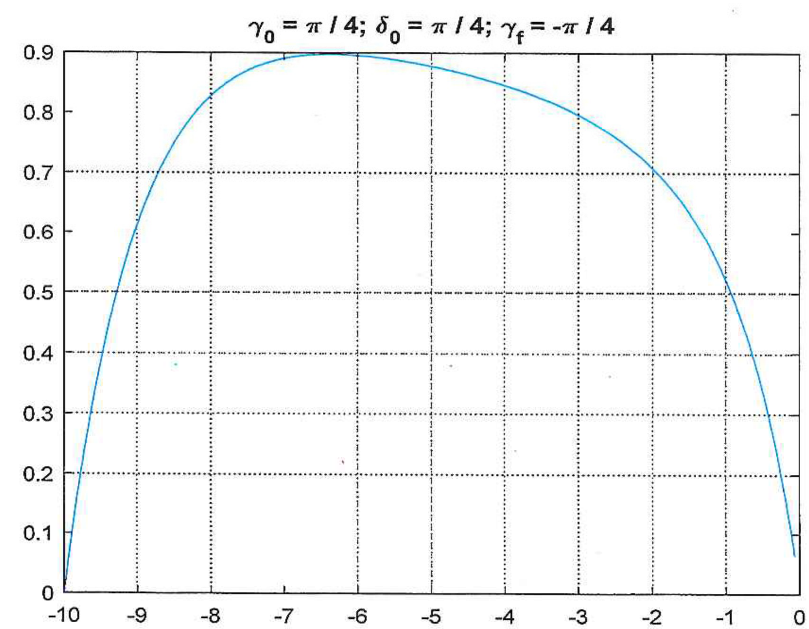

Y (M)
Z (M)
Y (M)
Z (M)
X (M)
Z (M)
X (M)
Y (M)
X (M)


CURVATURE κ
KAPPA (M)
S (M)
1
0.5
0
-0.5
-1
-1.5
-2
0
5
10
15


TORSION τ
TAU (M)
S (M)
0
-1
-2
-3
-4
-5
-6
-7
0
5
10
15


$\gamma_0 = \pi / 4;\ \delta_0 = \pi / 4$
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-10
-9
-8
-7
-6
-5
-4
-3
-2
-1
0


$\gamma_0 = \pi / 4;\ \delta_0 = \pi / 4;\ \gamma_f = \pi / 4$
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-10
-9
-8
-7
-6
-5
-4
-3
-2
-1
0


$\gamma_0 = \pi / 4;\ \delta_0 = \pi / 4;\ \gamma_f = -\pi / 4$
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-10
-9
-8
-7
-6
-5
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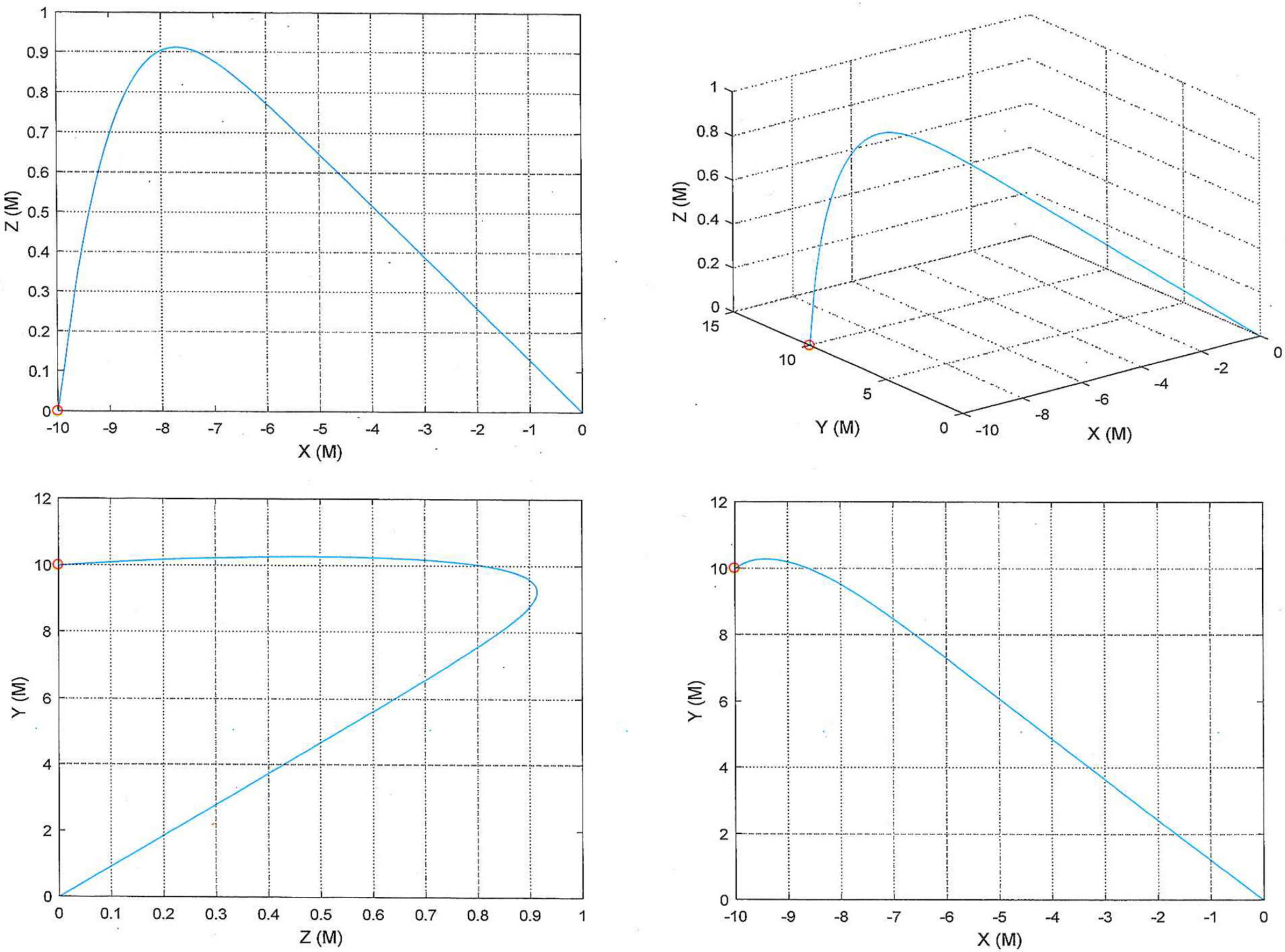
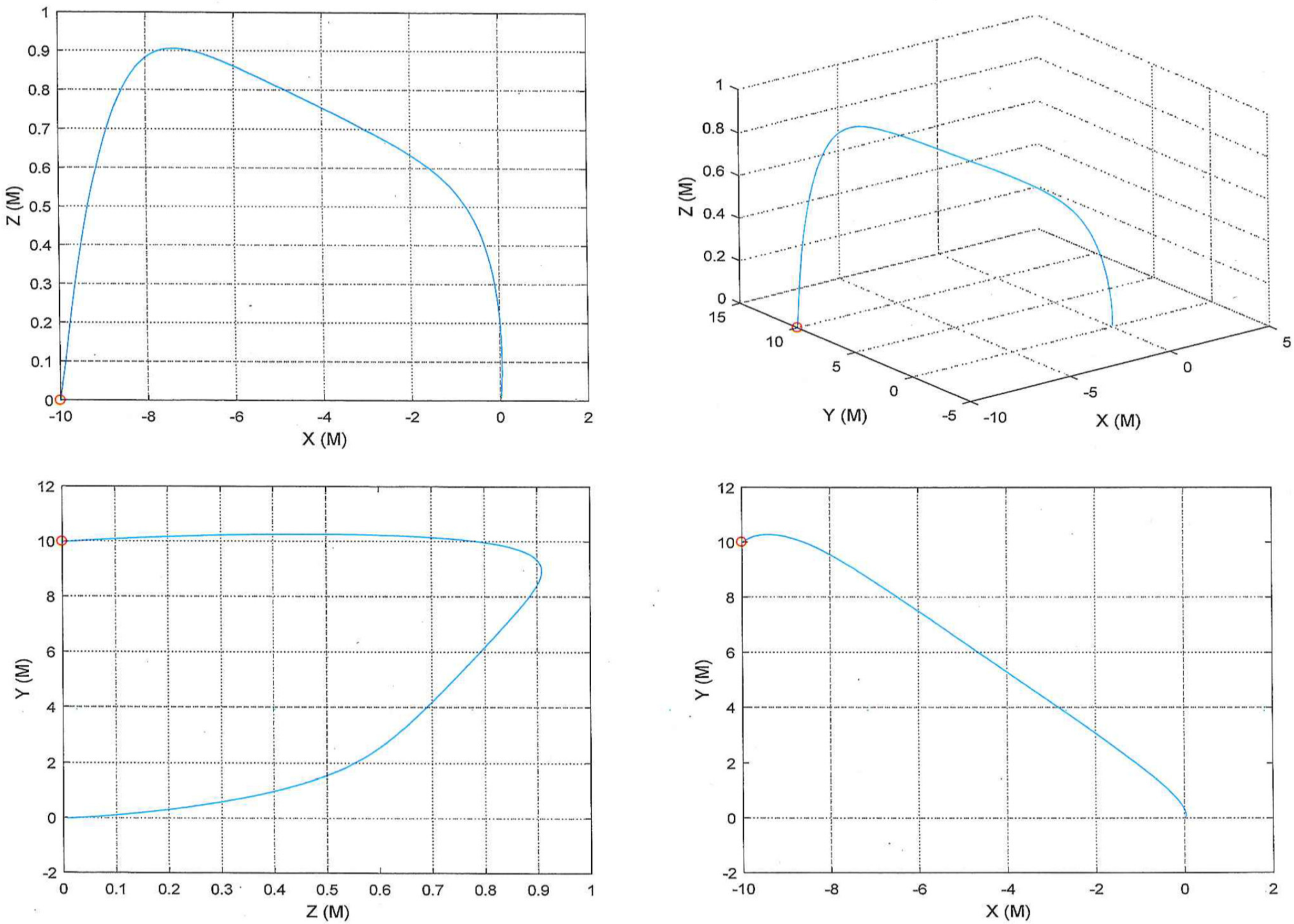
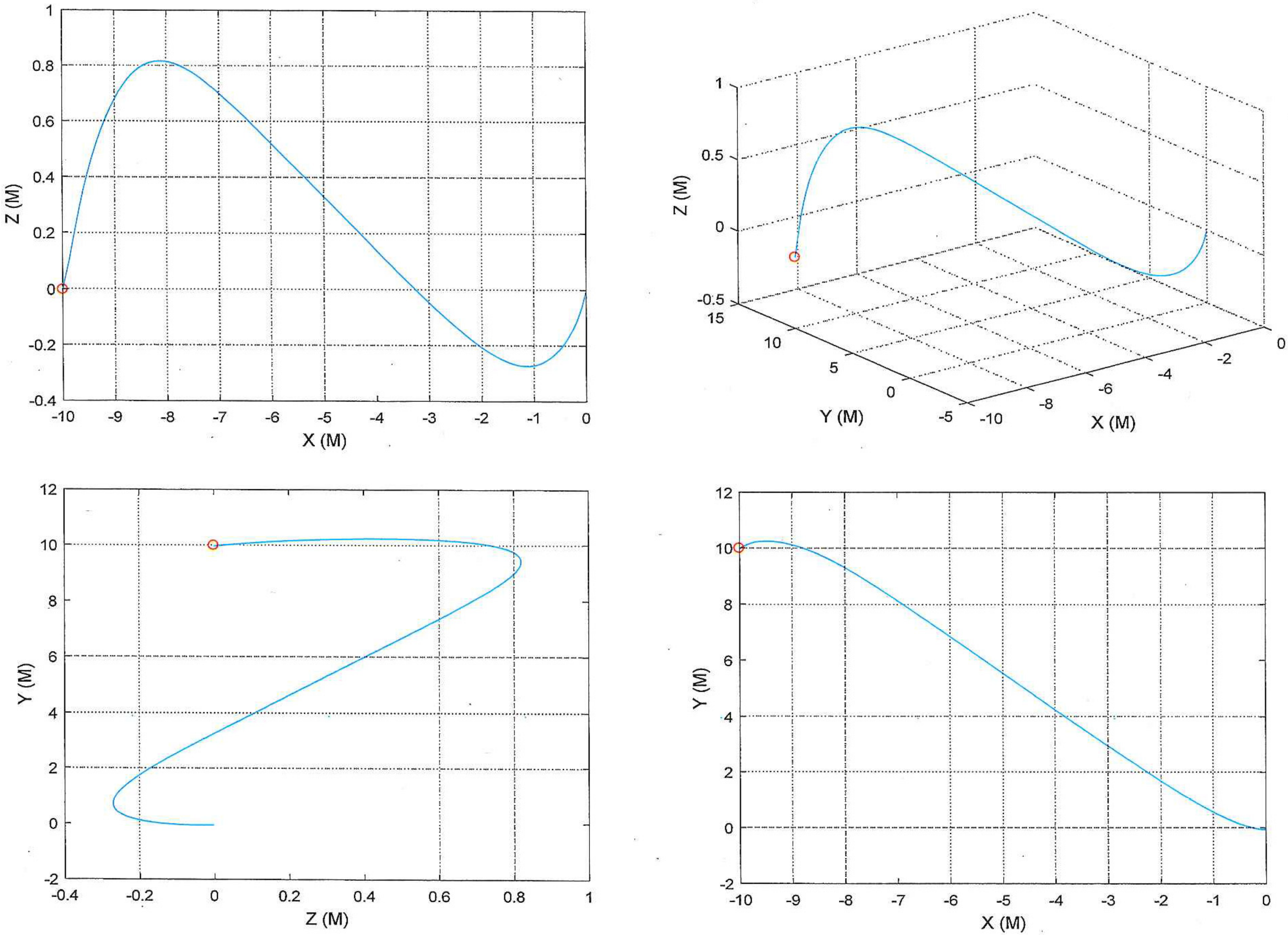
-3
-2
-1
0


Z (M)
X (M)
Z (M)
Y (M)
X (M)
Y (M)
Z (M)
Y (M)
X (M)


Z (M)
X (M)
Z (M)
Y (M)
X (M)
Y (M)
Z (M)
Y (M)
X (M)


Z (M)
X (M)
Z (M)
Y (M)
X (M)
Y (M)
Z (M)
Y (M)
X (M)

THE YELLOW AND BLUE

Sing to the colors that float in the light;
Hurrah for the Yellow and Blue!
Yellow the stars as they ride through the night
And reel in a rollicking crew;
Yellow the field where ripens the grain
And yellow the moon on the harvest wain;
-Hail!
Hail to the colors that float in the light
Hurrah for the Yellow and Blue!

THE END