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New Computational Framework for Modeling Porous Materials

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14. ABSTRACT
The U.S. Air Force makes use of a large array of materials, many of which are quite ductile and may exhibit anisotropy and initial defects. While many efforts have been devoted to modeling such systems, most of the ductile models are empirical. For most advanced models involving 6-D yield surfaces, form analysis has not been done. In this effort, we deduced new mathematical results that enable to recognize that all the widely used isotropic models are in fact homogeneous polynomials of J2 and J3, the invariants of the stress deviator. This had very important implications, one of them being the possibility to obtain explicit expressions in terms of the components of the stress tensor of anisotropic extensions of these models, which are part of the material libraries of commercial and government codes. It was shown that these models are in fact of the same form with anisotropy coefficients that are not independent and impose un-physical couplings between shear and normal stress properties. Most importantly, for the first-time analytical expressions of the anisotropy coefficients of these criteria were derived. The new mathematical results lead to better understanding and analysis of anisotropic materials data sets, the correlations of anisotropy coefficients to physical properties/microstructures, reliability of parameter identification, and ultimately the fidelity of numerical simulations.

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NEW COMPUTATIONAL FRAMEWORK FOR MODELING POROUS MATERIALS

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Abstract

The U.S. Air Force makes use of a large array of materials, many of which are quite ductile and may exhibit anisotropy and initial defects. While many efforts have been devoted to modeling such systems, most of the ductile models are empirical. For most advanced models involving 6-D yield surfaces, form analysis has not been done. In this effort, we deduced new mathematical results that enable to recognize that all the widely used isotropic models are in fact homogeneous polynomials of J_2 and J_3 , the invariants of the stress deviator. This had very important implications, one of them being the possibility to obtain explicit expressions in terms of the components of the stress tensor of anisotropic extensions of these models, which are part of the material libraries of commercial and government codes. It was shown that these models are in fact of the same form with anisotropy coefficients that are not independent and impose un-physical couplings between shear and normal stress properties. Most importantly, for the first-time analytical expressions of the anisotropy coefficients of these criteria were derived. The new mathematical results lead to better understanding and analysis of anisotropic materials data sets, the correlations of anisotropy coefficients to physical properties/microstructures, reliability of parameter identification, and ultimately the fidelity of numerical simulations. These results also serve to eliminate redundancies of existing FE codes. Another key accomplishment is related to modeling localization of deformations in uniaxial tension. It was demonstrated that this phenomenon is deterministic, the number and inclination of the bands being correlated to intrinsic plastic anisotropy features.

1. Introduction

For predictions of ductile deformation and failure constitutive descriptions should be well-defined for any loadings in the 6-D stress space. Key in these formulations is the description of the onset of plastic deformation. The most widely used non-quadratic yielding descriptions are expressed in terms of principal values of the stress deviator and respectively its mapping through 4-th order anisotropic tensors. Parameterizations of these models are very difficult and often unreliable. Additionally, the quality of F.E. predictions using these models are strongly dependent on the quality of the parameterization, which in some cases prevent these models from being used.

One of the reasons for these difficulties is that parameterizations are not unique. The parametrizations depend on the type and extent of data used as input. Since for most advanced anisotropic models, expressions of the parameters in terms of mechanical properties are not known, the choice of the parameterization for a specific application is strongly dependent on the experience of the analyst. Moreover, mechanical data is readily available only for materials in sheet form. Therefore, it is impossible to determine the parameters associated to the out-of-the sheet plane properties, and when conducting 3-D simulations additional choices related to the values of these parameters need to be made. Consequently, the analysis of the capabilities of these advanced models to describe the anisotropy in stresses and strain ratios is generally done for a single symmetry plane, and very little is known on the quality of the predictions for multi-axial loadings.

As part of this grant, new mathematical results that provide answers to open questions concerning the form of yielding formulations for both isotropic and anisotropic yield functions have been obtained and issues related to the reliability of these formulations and their parametrizations have

been addressed. The research results have been published in a book published by Elsevier in 2021 and a number of peer-reviewed journals. Only key results are presented here.

After a brief introduction in which we present the background and motivation for our research, the proof that the most widely used non-quadratic isotropic yield functions expressed in terms of principal values have the same mathematical form is given in Section 2. Making use of these mathematical results, it is demonstrated that for the same value of the homogeneity exponent m , Barlat et al. (1991) and Karafillis and Boyce (1993) orthotropic criteria are identical in form. It is also proved that all the in-plane anisotropy coefficients involved in these formulation can be expressed solely in terms of flow stresses in the orthotropy directions (for a rolled plate these directions are the rolling direction (RD), transverse direction (TD), and normal direction (ND)) and at 45° to RD. The respective systems of algebraic equations that need to be solved to evaluate the anisotropy coefficients are provided for BCC and FCC materials, respectively. Moreover, it is shown that the same anisotropy coefficients can be expressed solely in terms of Lankford coefficients for the same orientations. The respective systems of equations that need to be solved are also given for both BCC and FCC materials (see Section 3). Illustrations of the new results and their implications in terms of predictions are given for a textured high-strength steel SAPH440 and an aluminum alloy AA 6451-T4 sheet and the implications of the findings are summarized in Section 4. New results concerning localization of deformation are summarized in Section 5, conclusions and final remarks are given in Section 6.

2. Form analysis and new expressions in terms of stresses of most widely used non-quadratic isotropic yield criteria

2.1. Background and Motivation

Using classic results of the theory of representation of scalar isotropic functions of a symmetric second-order tensor (see for example, Wang (1970), Boehler (1978)) it has been shown that the yield function of any isotropic material for which plastic deformation is not influenced by the mean stress is expressible only in terms of the invariants J_2 and J_3 of the stress deviator s , i.e.

$$\Phi(\boldsymbol{\sigma}) = \Phi(J_2, J_3), \quad (1)$$

where

$$J_2 = \frac{1}{2}(s_1^2 + s_2^2 + s_3^2) \quad \text{and} \quad J_3 = s_1 s_2 s_3. \quad (2)$$

In Eq.(2), s_k , $k = 1 \dots 3$ denote the eigenvalues of $s = \boldsymbol{\sigma} - (tr(\boldsymbol{\sigma})/3)\mathbf{I}$ with $\boldsymbol{\sigma}$ the Cauchy stress tensor, \mathbf{I} is the second-order unit tensor, and tr standing for the trace operator.

Drucker (1949) proposed the following isotropic yield criterion:

$$J_2^3 - cJ_3^2 = \tau_Y^6, \quad (3)$$

where τ_Y is the yield stress in pure shear, and c is a constant. The effective stress associated with this criterion is given by:

$$\bar{\sigma} = \sigma_Y = \left(\frac{729}{27 - 4c} \right)^{1/6} (J_2^3 - cJ_3^2)^{1/6}, \quad (4)$$

where σ_Y denotes the tensile uniaxial yield stress. The constant c is expressible as the ratio between the yield stress in uniaxial tension, σ_Y , and τ_Y (see for example, Cazacu, Revil-Baudard and Chandola, 2019). For $c = 0$, the Drucker (1949) yield surface reduces to the von Mises's one while for $c > 0$, it lies between those of von Mises (1913) and Tresca's. Hershey (1954) and Hosford (1972) proposed a yield function in terms of the eigenvalues of the stress deviator:

$$f_1(s_1, s_2, s_3) = |s_1 - s_2|^m + |s_2 - s_3|^m + |s_3 - s_1|^m = 2\sigma_Y^m, \quad (5)$$

where m is a constant.

It was shown that for $m = 6$ this criterion approximates well yielding of isotropic body-centered cubic (BCC) materials while for $m = 8$ there is an excellent agreement with the yield points obtained by crystal plasticity for isotropic face-centered cubic (FCC) materials. Note that for $m = 2$, the Hershey-Hosford criterion reduces to the von Mises yield criterion, while for $m \rightarrow \infty$ Tresca's yield criterion is recovered. Another isotropic criterion that is also widely used to describe yielding of isotropic metallic materials is:

$$f_2(s_1, s_2, s_3) = s_1^m + s_2^m + s_3^m = \left(\frac{2^m + 2}{3^m} \right) \sigma_Y^m, \quad (6)$$

where m is an even integer (see Karafillis and Boyce, 1993).

A generic isotropic yield function which is obtained by interpolation of Hershey-Hosford yield function and $f_2(s_1, s_2, s_3)$ (see Eq. (5)-(6)) was proposed by Karafillis and Boyce (1993). Its expression is:

$$\chi \frac{3^m}{2^{m-1} + 1} (s_1^m + s_2^m + s_3^m) + (1 - \chi) \left((s_1 - s_2)^m + (s_2 - s_3)^m + (s_3 - s_1)^m \right) = 2\sigma_Y^m, \quad 0 \leq \chi \leq 1 \quad (7)$$

It is important to note that these authors imposed that in Eq. (7) the exponent m is an even integer with values ranging from 2 to ∞ . In this way, it was ensured that the predicted yielding response is the same for tensile and compressive stress states (see Karafillis and Boyce (1993)).

To extend the isotropic yield function given by Eq.(7) such as to describe any type of plastic anisotropy (e.g. cubic symmetry, orthotropy, or transverse isotropy) the same authors proposed a very general and rigorous methodology. Specifically, the anisotropic yield function was obtained by replacing in Eq. (7) the stress tensor $\boldsymbol{\sigma}$ with a transformed stress tensor $\tilde{\mathbf{S}}$ defined as:

$$\tilde{\mathbf{S}} = \mathbf{C}\mathbf{s} = \mathbf{C}\mathbf{K}\boldsymbol{\sigma} = \mathbf{L}\boldsymbol{\sigma}, \quad (8)$$

where \mathbf{C} is a fourth-order tensor accounting for the specific symmetries of the given material and \mathbf{K} is the fourth-order symmetric deviatoric unit-tensor that transforms the Cauchy stress tensor $\boldsymbol{\sigma}$ to its deviator \mathbf{s} . In the same seminal paper, the authors showed that the orthotropic yield criterion developed by Barlat et al. (1991), denoted Yld91, can be obtained by applying this methodology in conjunction with the Hershey-Hosford isotropic yield function given by Eq. (5) (see also the detailed discussion and proofs given in Barlat et al. (2007)).

Another general and rigorous approach for extending any isotropic yield criterion such as to incorporate any type of material symmetry was proposed by Cazacu and Barlat (2001; 2003). Specifically, using theorems of representation of anisotropic tensor functions (Liu, 1982) these authors developed generalizations of the isotropic invariants J_2 and J_3 to various types of symmetries. Therefore, the anisotropic extension of any isotropic yield function can be obtained simply by replacing in Eq. (1) the isotropic invariants J_2 and J_3 with their respective anisotropic generalizations.

Since in this report we discuss modeling of materials with orthotropic symmetry, in the following we present only the expressions of the orthotropic generalizations of J_2 and J_3 , respectively. For the expressions of the generalized invariants corresponding to any other material symmetries, the reader is referred to Cazacu et al. (2018) and the monograph of Cazacu, Revil-Baudard and Chandola (2019).

Orthotropic materials are characterized by having through each material point three mutually orthogonal planes of symmetries. Let us consider the Cartesian coordinate system $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ associated to the orthotropy axes (i.e. the normals to the orthotropy planes). Relative to $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ the orthotropic generalizations of J_2 and J_3 , which are denoted J_2^o and J_3^o , are given by:

$$\begin{aligned}
J_2^o &= \frac{a_1}{6}(\sigma_{xx} - \sigma_{yy})^2 + \frac{a_2}{6}(\sigma_{yy} - \sigma_{zz})^2 + \frac{a_3}{6}(\sigma_{xx} - \sigma_{zz})^2 + a_4\sigma_{xy}^2 + a_5\sigma_{xz}^2 + a_6\sigma_{yz}^2, \\
J_3^o &= \frac{1}{27}(b_1 + b_2)\sigma_{xx}^3 + \frac{1}{27}(b_3 + b_4)\sigma_{yy}^3 + \frac{1}{27}[2(b_1 + b_4) - b_2 - b_3]\sigma_{zz}^3 \\
&\quad - \frac{1}{9}(b_1\sigma_{yy} + b_2\sigma_{zz})\sigma_{xx}^2 - \frac{1}{9}(b_3\sigma_{zz} + b_4\sigma_{xx})\sigma_{yy}^2 \\
&\quad - \frac{1}{9}[(b_1 - b_2 + b_4)\sigma_{xx} + (b_1 - b_3 + b_4)\sigma_{yy}]\sigma_{zz}^2 \\
&\quad + \frac{2}{9}(b_1 + b_4)\sigma_{xx}\sigma_{yy}\sigma_{zz} - \frac{\sigma_{xz}^2}{3}[2b_9\sigma_{yy} - b_8\sigma_{zz} - (2b_9 - b_8)\sigma_{xx}] \\
&\quad - \frac{\sigma_{xy}^2}{3}[2b_{10}\sigma_{zz} - b_5\sigma_{yy} - (2b_{10} - b_5)\sigma_{xx}] - \frac{\sigma_{yz}^2}{3}[2b_7\sigma_{xx} - b_6\sigma_{yy} - (2b_7 - b_6)\sigma_{zz}] \\
&\quad + 2b_{11}\sigma_{xy}\sigma_{xz}\sigma_{yz}
\end{aligned} \tag{9}$$

In the above expressions a_i ($i=1\dots6$) and b_k ($k=1\dots11$) are constants. Note that J_2^o is a second-order polynomial in stresses, insensitive to hydrostatic pressure ($J_2^o(\boldsymbol{\sigma} + p\mathbf{I}) = J_2^o(\boldsymbol{\sigma})$ for any p and $\boldsymbol{\sigma}$) and form-invariant under any orthogonal transformation belonging to the orthotropy group. If in the expression of J_2^o all the coefficients $a_i = 1$, we have $J_2^o = J_2$. Note that J_3^o is a

third-degree homogeneous polynomial in stresses that reduces to J_3 when all the anisotropy coefficients b_k are set equal to unity. The orthotropic invariant J_3^o is insensitive to hydrostatic pressure and respects the orthotropic symmetries.

The orthotropic extension of any isotropic yield function, say $\Phi(J_2, J_3)$ (see Eq.(1)) can be then easily obtained simply by replacing J_2 with J_2^o , and J_3 with J_3^o , respectively. This general approach was used by Cazacu and Barlat (2001) for formulating the orthotropic extension of Drucker's (1949) yield function (see Eq. (3)). Specifically, the expression of Cazacu and Barlat (2001) orthotropic yield criterion is:

$$H(\boldsymbol{\sigma}) = (J_2^o)^3 - c(J_3^o)^2, \quad (10)$$

where c is a parameter.

Another example of an orthotropic yield criterion expressed in terms of the orthotropic invariants J_2^o and J_3^o is the orthotropic criterion, developed as part of this research effort:

$$G(\boldsymbol{\sigma}) = (J_2^o)^4 - \alpha(J_2^o)(J_3^o)^2, \quad (11)$$

with α being a parameter. For full three-dimensional (3-D) loadings, this orthotropic yield criterion involves 17 independent anisotropy coefficients (i.e. the anisotropy coefficients involved in the expressions of the orthotropic invariants J_2^o and J_3^o given by Eq.(9)). Due to page limitations, the capabilities of this new criterion are not presented here. The reader is referred to the paper of Cazacu (2018) that was issued based on the research conducted with support of this grant.

In the following we briefly present the new results concerning the

2.2. New invariants-based expressions of widely used isotropic yield criteria

As mentioned, the most widely used non-quadratic isotropic criteria are expressed in terms of principal values of the stress deviator (see Eq. (5)-(6)). It is to be noted that in order to use these criteria for general 3-D loadings, the eigenvalues s_k , $k = 1 \dots 3$ need to be calculated. This entails solving the third-order characteristic equation:

$$\lambda^3 - (J_2)\lambda - J_3 = 0, \quad (12)$$

where J_2 and J_3 are the invariants of the stress deviator \mathbf{s} . While closed-form solutions of the cubic Eq. (12) do exist (see Malvern, 1969), the respective expressions involve trigonometric functions having as arguments specific combinations between J_2 and J_3 . Their use for the expression of Hershey-Hosford yield surface (see Eq.(5)) or for the Karafillis and Boyce (1993) isotropic yield criteria has led to rather complicated expressions.

For example, if $s_1 \geq s_2 \geq s_3$, Hershey-Hosford yield surface is given by:

$$\left(\sqrt{J_2}\right)^m \left[\left(\sqrt{3} \cos(\alpha_1) - \sin(\alpha_1)\right)^m + \left(\sqrt{3} \cos(\alpha_1) + \sin(\alpha_1)\right)^m + (2 \sin \alpha_1)^m \right] = 2\sigma_Y^m. \quad (13)$$

In Eq.(13) α_1 denotes the angle satisfying $0 \leq \alpha_1 \leq \pi / 3$ whose cosine is given by:

$$\cos(3\alpha_1) = \frac{J_3}{2} \left(\frac{3}{J_2} \right)^{\frac{3}{2}} \quad (14)$$

Moreover, the form given by Eq. (13) has led to perceived singularities of its derivatives (see derivatives used in the family of Barlat criteria, e.g. Barlat et al. (1991)).

In this research, for the first time new equivalent expressions of these isotropic formulations have been deduced.

Theorem 1: For any even integer m , the Hershey-Hosford yield surface is a homogeneous polynomial in J_2 and J_3^2 , where J_2 and J_3 are the invariants of the stress deviator \mathbf{s} .

Proof: Using the method of induction we will show that:

$$(s_1 - s_2)^{2n} + (s_2 - s_3)^{2n} + (s_1 - s_3)^{2n} = \sum_{\substack{2p+6q=2n, \\ p,q \geq 0}} a_{pq} (J_2)^p (J_3^2)^q \quad (15)$$

where a_{pq} are constants. For any integer n , let us define:

$$\Lambda_{2n} = (s_1 - s_2)^{2n} + (s_2 - s_3)^{2n} + (s_1 - s_3)^{2n}. \quad \text{Note that } \Lambda_0 = 3, \quad \Lambda_2 = 6J_2, \quad \text{and } \Lambda_4 = 18J_2^2.$$

Therefore, the representation given by Eq. (15) is true for $n = 0$, $n = 1$, and $n = 2$. Next, assuming that the representation is true for $n = N-1$, $n = N$, and $n = N+1$ by using the definitions of the invariants J_2 and J_3 , we obtain that the relation given by Eq. (15) is true for $n = N + 2$.

Remark: Using this general mathematical result, one can easily obtain for any value of the exponent $m = 2n$, the expression of Hershey-Hosford yield surface in terms of J_2 and J_3^2 .

In particular,

- Hershey-Hosford yield surface for BCC materials ($m = 6$) is expressed as:

$$(s_1 - s_2)^6 + (s_2 - s_3)^6 + (s_1 - s_3)^6 = 66J_2^3 - 81J_3^2, \quad (16)$$

- Hershey-Hosford yield surface for FCC materials ($m = 8$) is expressed as:

$$(s_1 - s_2)^8 + (s_2 - s_3)^8 + (s_1 - s_3)^8 = 258J_2^4 - 648J_2J_3^2, \quad (17)$$

- Hershey-Hosford yield surface for $m = 12$ is expressed as

$$(s_1 - s_2)^{12} + (s_2 - s_3)^{12} + (s_1 - s_3)^{12} = 3(366J_2^6 - 6048J_2^3J_3^2 + 729J_3^4) \quad (18)$$

In the following, we will prove that the isotropic form of the Karafillis and Boyce (1993) criterion (see Eq. (6)) is a homogeneous polynomial in J_2 and J_3^2 .

Theorem 2:

If s_k , $k = 1 \dots 3$ denote the principal values of the stress deviator s , then:

$$s_1^{2n} + s_2^{2n} + s_3^{2n} = \sum_{\substack{2p+6q=2n, \\ p,q \geq 0}} b_{pq} (J_2)^p (J_3^2)^q \quad \text{for any integer } n \geq 1, \quad (19)$$

where b_{pq} are constants.

Proof: We shall use the method of induction. Let us note that by definition, $s_1^2 + s_2^2 + s_3^2 = 2J_2$, and that we have:

$$s_1^4 + s_2^4 + s_3^4 = 2J_2^2 \text{ and } s_1^6 + s_2^6 + s_3^6 = 3J_2^3 + 2J_3^2.$$

Therefore, the representation given by Eq. (19) is true for $n = 1$, $n = 2$, and $n = 3$. Next, assuming that the representation is true for $n = N-1$, $n = N$, and $n = N+1$ by using the definitions of the invariants J_2 and J_3 , we obtain that the relation given by Eq. (19) is true for $n = N + 2$.

Remark: Using this general mathematical result, one can easily obtain for any value of the exponent $m = 2n$, with $n \geq 1$ integer, the expression of $f_2(s_1, s_2, s_3)$ in terms of J_2 and J_3^2 .

We will write down explicitly the expressions of $f_2(s_1, s_2, s_3)$ for $m = 6$, $m = 8$, and $m = 12$, respectively. Indeed, using Theorem 2, we obtain:

$$f_2(s_1, s_2, s_3)_{m=6} = s_1^6 + s_2^6 + s_3^6 = 2J_2^3 + 3J_3^2, \quad (20)$$

$$f_2(s_1, s_2, s_3)_{m=8} = s_1^8 + s_2^8 + s_3^8 = 2J_2^4 + 8J_2J_3^2, \quad (21)$$

$$f_2(s_1, s_2, s_3)_{m=12} = s_1^{12} + s_2^{12} + s_3^{12} = 2J_2^6 + 24J_2^3J_3^2 + 3J_3^4. \quad (22)$$

On the basis of crystal plasticity calculations, Karrafillis and Boyce (1993) recommended for isotropic FCC materials to take $m = 12$ and $\chi = 0.3$ or $m = 30$ and $\chi = 0.835$ while for isotropic BCC materials to set $m = 6$ and $\chi = 0.17$ or take $m = 30$ and $\chi = 0.948$.

As shown in Theorem 1, the Hershey-Hosford yield function $f_1(s_1, s_2, s_3)$ (see Eq.(5)) is a homogeneous polynomial of J_2 and J_3^2 . Given that for isotropic materials the Karafillis and Boyce (1993) yield surface is obtained by interpolation of $f_1(s_1, s_2, s_3)$ and $f_2(s_1, s_2, s_3)$, we obtain the following general result:

Theorem 3: The Hershey-Hosford yield function and the isotropic Karafillis and Boyce (1993) yield function have the same mathematical form.

Remark: It is also worth noting that if in the Drucker (1949) yield function (see Eq. (3)) we set $c = 81/66$, we recover the Hershey-Hosford yield criterion for BCC materials, i.e.

$$\bar{\sigma} = \sigma_Y = \left(\frac{729}{27 - 324/66} \right)^{1/6} \left(J_2^3 - \frac{81}{66} J_3^2 \right)^{1/6}. \quad (23)$$

Moreover, it can be easily seen that the isotropic Karafillis and Boyce (1993) criterion for BCC materials ($m = 6$ and $\chi = 0.17$) can be written as:

$$\bar{\sigma} = \sigma_Y = \left(\frac{729}{27 - 2052/571} \right)^{1/6} \left(J_2^3 - \frac{513}{571} J_3^2 \right)^{1/6}, \quad (24)$$

In the next section, making use of the expressions in terms of the invariants J_2 and J_3^2 for $f_1(s_1, s_2, s_3)$ and $f_2(s_1, s_2, s_3)$, we develop new expressions in terms of all 6 stress components for the orthotropic yield criterion of Barlat et al (1991) and respectively the orthotropic Karafillis and Boyce (1993) for BCC materials ($m = 6$) and FCC materials ($m = 12$).

3. Expressions in terms of components of the stress tensor of widely used orthotropic yield criteria

3.1. New expression of the orthotropic yield criterion of Barlat et al (1991) for FCC and BCC materials

Barlat et al. (1991) proposed the following yield surface:

$$\text{Yld91} = \phi(\tilde{S}_1, \tilde{S}_2, \tilde{S}_3) = |\tilde{S}_1 - \tilde{S}_2|^m + |\tilde{S}_2 - \tilde{S}_3|^m + |\tilde{S}_3 - \tilde{S}_1|^m = 2\bar{\sigma}^m, \quad (25)$$

with \tilde{S}_k being the eigenvalues of the transformed stress tensor $\tilde{\mathbf{S}}$ given by Eq.(8) and the fourth-order tensor \mathbf{C} being orthotropic, symmetric (i.e. $C_{ijkl} = C_{klij} = C_{jikl}$ for $i, j, k, l = 1 \dots 3$), and deviatoric. If in Voigt notation the Cauchy stress tensor is represented by the 6-dimensional vector $\boldsymbol{\sigma} = (\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{yz}, \sigma_{xz}, \sigma_{xy})$ relative to the material frame $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ the orthotropic tensor \mathbf{C} is represented by a 6x6 matrix with six non-zero components, a, b, \bar{c}, f, g, h , i.e. :

$$\mathbf{C} = \frac{1}{3} \begin{bmatrix} b + \bar{c} & -\bar{c} & -b & 0 & 0 & 0 \\ -\bar{c} & \bar{c} + a & -a & 0 & 0 & 0 \\ -b & -a & a + b & 0 & 0 & 0 \\ 0 & 0 & 0 & 3f & 0 & 0 \\ 0 & 0 & 0 & 0 & 3g & 0 \\ 0 & 0 & 0 & 0 & 0 & 3h \end{bmatrix} \quad (26)$$

(see also Barlat et al. (2007)).

Theorem 4:

- For BCC orthotropic materials

$$\text{Yld91}_{\text{BCC}} = (\tilde{S}_1 - \tilde{S}_2)^6 + (\tilde{S}_2 - \tilde{S}_3)^6 + (\tilde{S}_1 - \tilde{S}_3)^6 = 66\tilde{J}_2^3 - 81\tilde{J}_3^2, \quad (27)$$

the associated equivalent stress being given by:

$$\bar{\sigma}_{\text{Yld91}_{\text{BCC}}} = M \left(\tilde{J}_2^3 - \frac{81}{66} \tilde{J}_3^2 \right)^{1/6}. \quad (28)$$

- For FCC orthotropic materials:

$$\text{Yld91}_{\text{FCC}} = (\tilde{S}_1 - \tilde{S}_2)^8 + (\tilde{S}_2 - \tilde{S}_3)^8 + (\tilde{S}_1 - \tilde{S}_3)^8 = 258\tilde{J}_2^4 - 648\tilde{J}_2\tilde{J}_3^2, \quad (29)$$

the associated equivalent stress being expressed as:

$$\bar{\sigma}_{\text{Yld91}_{\text{FCC}}} = N \left(\tilde{J}_2^4 - \frac{648}{258} \tilde{J}_2\tilde{J}_3^2 \right)^{1/8}, \quad (30)$$

with M and N being constants defined such that the effective stress reduces to the uniaxial yield stress along \mathbf{x} -orthotropy direction (or rolling direction (RD)); in the above equations,

$\tilde{J}_2 = tr(\tilde{\mathbf{S}}^2)/2$ and $\tilde{J}_3 = tr(\tilde{\mathbf{S}}^3)/3$ denote the invariants of the transformed tensor $\tilde{\mathbf{S}}$ their expressions in terms of the stress components in the orthotropy axes $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ are:

$$\begin{aligned} \tilde{J}_2 = & f^2 \sigma_{yz}^2 + g^2 \sigma_{xz}^2 + h^2 \sigma_{xy}^2 + [(b + \bar{c}) \sigma_{xx} - \bar{c} \sigma_{yy} - b \sigma_{zz}]^2 / 18 + \\ & [-\bar{c} \sigma_{xx} + (\bar{c} + a) \sigma_{yy} - a \sigma_{zz}]^2 / 18 + [-b \sigma_{xx} - a \sigma_{yy} + (a + b) \sigma_{zz}]^2 / 18 \end{aligned} \quad (31)$$

$$\begin{aligned} \tilde{J}_3 = & 2(fgh) \sigma_{xy} \sigma_{xz} \sigma_{yz} + \frac{1}{27} \left[\left[(b + \bar{c}) \sigma_{xx} - \bar{c} \sigma_{yy} - b \sigma_{zz} \right] \cdot \left[-\bar{c} \sigma_{xx} + (\bar{c} + a) \sigma_{yy} - a \sigma_{zz} \right] \cdot \right. \\ & \left. \left[-b \sigma_{xx} - a \sigma_{yy} + (a + b) \sigma_{zz} \right] \right] \\ & - \frac{f^2 \sigma_{yz}^2}{3} \left[(b + \bar{c}) \sigma_{xx} - \bar{c} \sigma_{yy} - b \sigma_{zz} \right] - \frac{g^2 \sigma_{xz}^2}{3} \left[-\bar{c} \sigma_{xx} + (\bar{c} + a) \sigma_{yy} - a \sigma_{zz} \right] \\ & - \frac{h^2 \sigma_{xy}^2}{3} \left[-b \sigma_{xx} - a \sigma_{yy} + (a + b) \sigma_{zz} \right] \end{aligned} \quad (32)$$

Proof: To obtain the analytical expressions in terms of stress components of the Yld91 yield function for BCC and FCC orthotropic materials given by Eq.(29) and Eq.(30) one needs to replace in Eq.(16) and Eq.(17) the invariants J_2 and J_3 with $\tilde{J}_2 = tr(\tilde{\mathbf{S}}^2)/2$ and $\tilde{J}_3 = tr(\tilde{\mathbf{S}}^3)/3$, respectively and further use the definition of $\tilde{\mathbf{S}}$ and the matrix representation of the fourth-order tensor \mathbf{C} given by Eq.(26).

Proposition 1: Of the six anisotropy coefficients a, b, \bar{c}, f, g, h involved in the Yld91 yield surface, only five are independent.

Proof: This is a direct consequence of the Yld91 yield function for orthotropic BCC and FCC materials being a sixth-order, and respectively eight-order polynomial in stresses.

Remark: This statement concerning the number of independent anisotropy coefficients in Yld91 yield surface holds true for any orthotropic yield surface that is obtained by replacing the stress tensor with a transformed stress tensor.

Theorem 4: For BCC orthotropic materials, the Yld91 yield function is a particular case of the orthotropic yield function of Cazacu and Barlat (2001) (Eq.(10)) with fewer anisotropy coefficients and $c = 81/66$. For FCC orthotropic materials, the Yld91 yield function is a particular case of the orthotropic yield function of Cazacu (2018) (Eq.(11)) corresponding to $\alpha = 648/258$ and including fewer anisotropy coefficients.

Let us denote by σ_θ the flow stress in simple tension along a direction at an angle θ to the \mathbf{x} -axis (RD) in the (\mathbf{x}, \mathbf{y}) plane, and by r_θ , the ratio between the in-plane transverse and through-thickness plastic strain increments measured in the same test.

3.1.1. New analytical identification procedures for Yld91 surface for orthotropic BCC materials ($m = 6$)

Identification of the anisotropy coefficients in Yld91 for BCC materials based only on yield stresses

For BCC materials, the anisotropy coefficients a, b, \bar{c} involved in Yld91 surface depend only on the uniaxial flow stresses in the directions of orthotropy, \mathbf{x}, \mathbf{y} , and \mathbf{z} , and are calculated by solving the following system of three non-linear algebraic equations:

$$\begin{cases} (b^2 + b\bar{c} + \bar{c}^2)^3 - (81/66)(b\bar{c})^2 (b + \bar{c})^2 = 1, \\ (a^2 + a\bar{c} + \bar{c}^2)^3 - (81/66)(a\bar{c})^2 (a + \bar{c})^2 = (\sigma_0 / \sigma_{90})^6, \\ (a^2 + ab + b^2)^3 - (81/66)(ab)^2 (a + b)^2 = (\sigma_0 / \sigma_{zz})^6 = (\sigma_0 / \sigma_b)^6, \end{cases} \quad (33)$$

where σ_0, σ_{90} , and σ_z denote the tensile uniaxial flow stresses in the \mathbf{x} -direction, \mathbf{y} -direction, and \mathbf{z} -direction, respectively while σ_b is the flow stress in equibiaxial tension in the (\mathbf{x}, \mathbf{y}) plane. The anisotropy coefficient h is the root of the following algebraic equation:

$$(a^2 + ab + b^2 + 9h^2)^3 - (81/66)(9h^2 - ab)^2 (a + b)^2 = (2\sigma_0 / \sigma_{45})^6, \quad (34)$$

where σ_{45} is the uniaxial flow stress in simple tension along a direction at 45° to the \mathbf{x} -axis in the (\mathbf{x}, \mathbf{y}) plane (for proof, see the paper issued from this research, Cazacu (2019)).

Identification of the anisotropy coefficients in Yld91 for BCC materials based on Lankford coefficients

The anisotropy coefficients a , b , \bar{c} involved in Yld91 for BCC materials depend only on the Lankford coefficients r_0 and r_{90} in the \mathbf{x} -direction and \mathbf{y} -direction of orthotropy, respectively; the anisotropy coefficients a , b , \bar{c} are the solutions of the following system of algebraic equations:

$$\left\{ \begin{array}{l} (b^2 + b\bar{c} + \bar{c}^2)^3 - (81/66)(b\bar{c})^2 (b + \bar{c})^2 = 1 \\ r_0 = \frac{(3/2)(b^2 + b\bar{c} + \bar{c}^2)^2 [\bar{c}(2\bar{c} + a + b) - ab] - (81/66)(b\bar{c})(b + \bar{c}) [a(b^2 - \bar{c}^2) + b\bar{c}(b + 2\bar{c})]}{(3/2)(b^2 + b\bar{c} + \bar{c}^2)^2 [b(2b + a + \bar{c}) - a\bar{c}] - (81/66)(b\bar{c})(b + \bar{c}) [a(\bar{c}^2 - b^2) + b\bar{c}(\bar{c} + 2b)]} \\ r_{90} = \frac{(3/2)(a^2 + a\bar{c} + \bar{c}^2)^2 [\bar{c}(2\bar{c} + a + b) - ab] - (81/66)(a\bar{c})(a + \bar{c}) [b(a^2 - \bar{c}^2) + a\bar{c}(a + 2\bar{c})]}{(3/2)(a^2 + a\bar{c} + \bar{c}^2)^2 [a(2a + b + \bar{c}) - b\bar{c}] - (81/66)(a\bar{c})(a + \bar{c}) [b(\bar{c}^2 - a^2) + a\bar{c}(\bar{c} + 2a)]} \end{array} \right. \quad (35)$$

The coefficient h depends only on a, b, \bar{c} and the Lankford coefficient for $\theta = 45^\circ$, and can be determined by solving the following non-linear equation:

$$r_{45} = \frac{(a^2 + ab + b^2 + 9h^2)^2 (9h^2 - a^2 - b^2 - ab) + (81/66)(a + b)^2 (ab - 9h^2)(ab + 3h^2)}{2(a^2 + ab + b^2)(a^2 + ab + b^2 + 9h^2)^2 - (81/33)(a + b)^2 (ab - 9h^2)(ab - 3h^2)} \quad (36)$$

Remark: With the original formulation of Yld91 given by Eq.(25), when calculating the derivatives, singularities arise for loadings when two of the principal stresses are equal (see also Barlat et al. (2007)). One of the benefits brought by the new formulation of Yld91 is that now it becomes possible to correctly define and calculate the derivatives of a function which is of class C^1 for any loadings.

3.1.2. New analytical identification procedures for Yld91 surface for orthotropic FCC materials ($m = 8$)

The new expression of the Yld91 for FCC materials (see Eq.(30)) makes possible the derivation of analytical formulas for the variation of the uniaxial flow stresses σ_θ and strain-ratios r_θ , respectively with the tensile loading orientation θ . For example, according to Yld91 for FCC orthotropic materials ($m = 8$):

$$\sigma_\theta / \sigma_0 = \left\{ \left[\lambda_1 \cos^4 \theta + \lambda_2 \cos^2 \theta \sin^2 \theta + \lambda_3 \sin^4 \theta \right]^4 - (648/258) \left[\lambda_4 \cos^6 \theta + \lambda_5 \sin^6 \theta - [\lambda_6 \cos^2 \theta + \lambda_7 \sin^2 \theta] \sin^2 \theta \cos^2 \theta \right]^2 \times \left[\begin{array}{l} \lambda_1 \cos^4 \theta \\ + \lambda_2 \cos^2 \theta \sin^2 \theta + \lambda_3 \sin^4 \theta \end{array} \right] \right\}^{-1/8} \quad (37)$$

with $B = [\lambda_1^4 - (648/258)\lambda_1\lambda_4^2]^{1/8}$ and $\lambda_k, k = 1...7$ given by Eq. Error! Reference source not found..

Identification of the anisotropy coefficients in Yld91 for FCC materials based solely on experimental yield stresses

The anisotropy coefficients a, b, \bar{c} involved in the expression of Yld91 for FCC materials depend only on the uniaxial tensile flow stresses in the directions of orthotropy, \mathbf{x}, \mathbf{y} , and \mathbf{z} , and can be evaluated by solving the following system of algebraic equations:

$$\begin{aligned} (b^2 + b\bar{c} + \bar{c}^2)^4 - (648/258)(b\bar{c})^2 (b^2 + b\bar{c} + \bar{c}^2)(b + \bar{c})^2 &= 1 \\ (a^2 + a\bar{c} + \bar{c}^2)^4 - (648/258)(a\bar{c})^2 (a^2 + a\bar{c} + \bar{c}^2)(a + \bar{c})^2 &= (\sigma_0 / \sigma_{90})^8 \\ (a^2 + ab + b^2)^4 - (648/258)(ab)^2 (a^2 + ab + b^2)(a + b)^2 &= (\sigma_0 / \sigma_z)^8 = (\sigma_0 / \sigma_b)^8 \end{aligned} \quad (38)$$

The anisotropy coefficient h depends only on σ_{45} and the coefficients a, b, \bar{c} :

$$(a^2 + ab + b^2 + 9h^2)^4 - (648/258)(9h^2 - ab)^2 (9h^2 + ab + a^2 + b^2)(a + b)^2 = (2\sigma_0 / \sigma_{45})^8 \quad (39)$$

For Yld91 for FCC materials the Lankford coefficients r_θ are calculated based on the formula:

$$r_\theta = - \frac{(\sin^2 \theta) \frac{\partial \bar{\sigma}_{Yld91_FCC}}{\partial \sigma_{xx}} - (\sin 2\theta) \frac{\partial \bar{\sigma}_{Yld91_FCC}}{\partial \sigma_{xy}} + (\cos^2 \theta) \frac{\partial \bar{\sigma}_{Yld91_FCC}}{\partial \sigma_{yy}}}{\frac{\partial \bar{\sigma}_{Yld91_FCC}}{\partial \sigma_{xx}} + \frac{\partial \bar{\sigma}_{Yld91_FCC}}{\partial \sigma_{yy}}} \quad (40)$$

$$\text{with } \frac{\partial \bar{\sigma}_{Yld91_FCC}}{\partial \sigma_{ij}} = \left[4(\tilde{J}_2)^3 - \frac{648}{258}(\tilde{J}_3)^2 \right] \frac{\partial \tilde{J}_2}{\partial \sigma_{ij}} - \frac{1296}{258}(\tilde{J}_3)(\tilde{J}_2) \frac{\partial \tilde{J}_3}{\partial \sigma_{ij}},$$

where $i, j = 1 \dots 3$.

Identification of the anisotropy coefficients in Yld91 for FCC materials using solely experimental Lankford coefficients

The anisotropy coefficients a, b, \bar{c} involved in the expression of Yld91 for orthotropic FCC materials depend only on the Lankford coefficients r_0 and r_{90} , and can be determined by solving the following system of algebraic equations:

$$(b^2 + b\bar{c} + \bar{c}^2)^4 - (648/258)(b\bar{c})^2 (b^2 + b\bar{c} + \bar{c}^2)(b + \bar{c})^2 = 1$$

$$r_0 = \frac{H_1(a, b, \bar{c}) + (648/258)F_1(a, b, \bar{c})}{H_2(a, b, \bar{c}) + (648/258)F_2(a, b, \bar{c})} \quad (41)$$

$$r_{90} = \frac{T_1(a, b, \bar{c}) + (648/258)G_1(a, b, \bar{c})}{T_2(a, b, \bar{c}) + (648/258)G_2(a, b, \bar{c})}$$

The coefficient h depends only on r_{45} and a, b, \bar{c} and can be easily determined once a, b, \bar{c} are known in conjunction with the following equation

$$r_{45} = \frac{K_1(a, b, h) - (648/258)(a+b)^2(9h^2 - ab)P_1(a, b, h)}{K_2(a, b, h) + (648/258)(a+b)^2(9h^2 - ab)P_2(a, b, h)} \quad (42)$$

In the expression of r_0 (Eq.(41)), we have:

$$H_1(a,b,\bar{c}) = -4(b^2 + b\bar{c} + \bar{c}^2)^3(ab - \bar{c}(a + b + 2\bar{c}))$$

$$F_1(a,b,\bar{c}) = -b\bar{c}(b + \bar{c})\left(2b^4\bar{c} + 2ab^4 + 7b^3\bar{c}^2 + ab^3\bar{c} + 9b^2\bar{c}^3 + 6b\bar{c}^4 - ab\bar{c}^3 - 2a\bar{c}^4\right) \quad (43)$$

while $H_2(a,b,\bar{c})$ and $F_2(a,b,\bar{c})$ are obtained by replacing \bar{c} with b and b with \bar{c} in the expressions of $H_1(a,b,\bar{c})$ and $F_1(a,b,\bar{c})$, respectively.

Similarly, in the expression of r_{90} (see Eq.(41)₂), $T_1(a,b,\bar{c})$ and $G_1(a,b,\bar{c})$ are obtained by interchanging a and b in the expressions of $H_1(a,b,\bar{c})$ and $F_1(a,b,\bar{c})$, respectively; on the other hand, $T_2(a,b,\bar{c})$ and $G_2(a,b,\bar{c})$ are obtained by interchanging a and b in the expressions of $H_2(a,b,\bar{c})$ and $F_2(a,b,\bar{c})$, respectively. Finally, in the expression of r_{45} we have:

$$K_1(a,b,\bar{c}) = -2(a^2 + ab + b^2 - 9h^2)(a^2 + ab + b^2 + 9h^2)^3 \quad (44)$$

$$K_2(a,b,\bar{c}) = 4(a^2 + ab + b^2)(a^2 + ab + b^2 + 9h^2)^3 \quad (45)$$

$$P_1(a,b,h) = 2a^3b + 2a^2b^2 + 2ab^3 + 9abh^2 + 81h^4 \quad (46)$$

$$P_2(a,b,h) = 4a^3b + 4a^2b^2 + 4ab^3 + 9abh^2 - 81h^4 - 18a^2h^2 - 18b^2h^2$$

To summarize:

For any orthotropic material the anisotropy coefficients a, b, \bar{c}, h involved in the Yld91 surface can be obtained by:

- solving a system of four equations involving only the experimental tensile uniaxial flow stresses in the $0^\circ, 45^\circ, 90^\circ$ orientations and the equibiaxial flow stress, σ_b

or

- solving a system of equations involving only the experimental Lankford coefficients for the same orientations.

Remark: Since the coefficients a, b, \bar{c} are expressible solely in terms of σ_0, σ_{90} or solely in terms of r_0, r_{90} , it follows that Yld91 imposes a very specific non-linear relation between these material properties.

Either the stress-based or r-values based method for evaluation of the anisotropy coefficients provides a parametrization of Yld91 that leads to a reliable representation of the yield surface of a given orthotropic material. Illustrative examples will be presented in later sections.

3.2. New expressions of the orthotropic yield criterion of Karafillis and Boyce (1993) for FCC and BCC materials

As previously mentioned, Karafillis and Boyce (1993) developed the orthotropic generalization of the isotropic yield criterion given by Eq. (7) by means of the transformed stress tensor $\tilde{\mathbf{S}}$ (see Eq.(8)). The expression of the orthotropic Karafillis and Boyce (1993) yield criterion is:

$$\chi \frac{3^m}{2^{m-1} + 1} (\tilde{S}_1^m + \tilde{S}_2^m + \tilde{S}_3^m) + (1 - \chi) \left((\tilde{S}_1 - \tilde{S}_2)^m + (\tilde{S}_2 - \tilde{S}_3)^m + (\tilde{S}_3 - \tilde{S}_1)^m \right) = 2\sigma_Y^m, \quad (47)$$

with \tilde{S}_k being the eigenvalues of the transformed stress tensor $\tilde{\mathbf{S}}$.

Theorem 5: For orthotropic materials, the effective stress associated with the KB93 criterion are given by:

- For BCC orthotropic materials: $\bar{\sigma}_{KB93_BCC} = A \left(\tilde{J}_2^3 - \frac{513}{571} \tilde{J}_3^2 \right)^{1/6}, \quad (48)$

- For FCC orthotropic materials: $\bar{\sigma}_{KB93_FCC} = B \left(\tilde{J}_2^6 + c_1 \tilde{J}_2^3 \tilde{J}_3^2 + c_2 \tilde{J}_3^4 \right)^{1/12}, \quad (49)$

$$c_1 = -3700/1033 \text{ and } c_2 = 1205/2066,$$

The expressions of $\tilde{J}_2 = tr(\tilde{\mathbf{S}}^2)/2$ and $\tilde{J}_3 = tr(\tilde{\mathbf{S}}^3)/3$ are given by Eq.(31)-(32); the constants A and B in the above equations are defined such that the respective effective stress reduces to the uniaxial yield stress along the \mathbf{x} -axis (or RD).

Indeed, the new expressions of KB93 in terms of stress components enable to easily correlate the anisotropy coefficients a, b, \bar{c}, f, g, h of the criterion to mechanical properties. First, let us note that the KB93 criterion involves only five independent anisotropy coefficients. This is a direct consequence of the KB93 yield function for BCC and FCC orthotropic materials being a sixth-order, and respectively a twelve-order polynomial in stresses (see Eq. (48)-(49) with \tilde{J}_2 and \tilde{J}_3 given by Eq. (31-32). As mentioned, this statement about the number of independent anisotropy coefficients in KB93 yield function holds true for any orthotropic yield function that is obtained by replacing in the expression of a homogeneous isotropic function the stress tensor with a transformed stress tensor. In particular, this statement is valid for Hill (1948) and Yld91 (see Section 3.1.). Indeed, Hill (1948) is in fact of the same mathematical form as \tilde{J}_2 . Therefore, we have the following general result:

Theorem 6: For BCC orthotropic materials, the Yld91 and the KB93 yield functions have the same mathematical form and involve the same number of anisotropy coefficients. Moreover, Yld 91 is a particular case of the orthotropic yield criterion of Cazacu and Barlat (2001) corresponding to $c = 81/66$ while KB93 is a particular case of the same criterion corresponding to $c = 513/571$, and involves fewer anisotropy coefficients.

These results have a direct bearing on the evaluation of the anisotropy coefficients involved in KB93.

3.2.1. Analytical expressions for the anisotropy coefficients involved in the KB93 yield surface for orthotropic BCC materials

Identification of the anisotropy coefficients in KB93 for BCC materials based only on flow stresses

For BCC materials, the anisotropy coefficients a , b , \bar{c} involved in KB93 depend only on the uniaxial flow stresses in the directions of orthotropy, \mathbf{x} , \mathbf{y} , and \mathbf{z} , and are calculated by solving the following system of three non-linear algebraic equations:

$$\begin{cases} (b^2 + b\bar{c} + \bar{c}^2)^3 - (513/571)(b\bar{c})^2 (b + \bar{c})^2 = 1, \\ (a^2 + a\bar{c} + \bar{c}^2)^3 - (513/571)(a\bar{c})^2 (a + \bar{c})^2 = (\sigma_0 / \sigma_{90})^6, \\ (a^2 + ab + b^2)^3 - (513/571)(ab)^2 (a + b)^2 = (\sigma_0 / \sigma_z)^6 = (\sigma_0 / \sigma_b)^6, \end{cases} \quad (50)$$

where σ_0 , σ_{90} , and σ_z denote the tensile uniaxial flow stresses in the \mathbf{x} -direction, \mathbf{y} -direction, and \mathbf{z} -direction, respectively while σ_b is the flow stress in equibiaxial tension in the (\mathbf{x}, \mathbf{y}) plane. The anisotropy coefficient h is the root of the following algebraic equation:

$$(a^2 + ab + b^2 + 9h^2)^3 - (81/66)(9h^2 - ab)^2 (a + b)^2 = (2\sigma_0 / \sigma_{45})^6, \quad (51)$$

where σ_{45} is the uniaxial flow stress in simple tension along a direction at 45° to the \mathbf{x} -axis in the (\mathbf{x}, \mathbf{y}) plane.

Alternatively, since for orthotropic BCC materials, KB93 has the same mathematical form as Yld91, the above expressions of the anisotropy coefficients could have been obtained by replacing $c = 81/66$ with $c = 513/571$ in the corresponding system of equations for Yld91.

Remark: Note that according to either KB93 or Yld91 criterion the flow stress in pure shear in the (\mathbf{x}, \mathbf{y}) plane depends only of the anisotropy coefficient h . Therefore, this coefficient can be evaluated using this experimental value. Similarly, the anisotropy coefficients f and g can also be easily determined based on the experimental yield stresses in pure shear in the planes (\mathbf{y}, \mathbf{z}) and (\mathbf{x}, \mathbf{z}) , respectively. Recall that we demonstrated that the anisotropy coefficients a, b, \bar{c} involved in the Yld91 yield function depend only on r_0 and r_{90} , the strain ratios along the \mathbf{x} and \mathbf{y} directions of orthotropy, and the coefficient h depends also on r_{45} , the strain ratio for the 45° direction (see Section 3.1). Therefore, the anisotropy coefficients a, b, \bar{c} and h involved in KB93 for BCC materials can also be correlated to Lankford coefficients in these three orientations, the respective system of algebraic equations to be solved being obtained simply by replacing $c = 81/66$ with $c = 513/571$ in the corresponding system of equations for Yld91 for BCC materials.

Identification of the anisotropy coefficients in KB93 for BCC materials using solely experimental Lankford coefficients

The anisotropy coefficients a, b, \bar{c} involved in KB93 for BCC materials depend only on the Lankford coefficients r_0 and r_{90} , in the \mathbf{x} -direction and \mathbf{y} -direction of orthotropy, respectively; these anisotropy coefficients are calculated using the following system of three non-linear algebraic equations:

$$\begin{aligned}
& (b^2 + b\bar{c} + \bar{c}^2)^3 - (513/571)(b\bar{c})^2 (b + \bar{c})^2 = 1 \\
r_0 &= \frac{(3/2)(b^2 + b\bar{c} + \bar{c}^2)^2 [\bar{c}(2\bar{c} + a + b) - ab] - (513/571)(b\bar{c})(b + \bar{c}) [a(b^2 - \bar{c}^2) + b\bar{c}(b + 2\bar{c})]}{(3/2)(b^2 + b\bar{c} + \bar{c}^2)^2 [b(2b + a + \bar{c}) - a\bar{c}] - (513/571)(b\bar{c})(b + \bar{c}) [a(\bar{c}^2 - b^2) + b\bar{c}(\bar{c} + 2b)]} \\
r_{90} &= \frac{(3/2)(a^2 + a\bar{c} + \bar{c}^2)^2 [\bar{c}(2\bar{c} + a + b) - ab] - (513/571)(a\bar{c})(a + \bar{c}) [b(a^2 - \bar{c}^2) + a\bar{c}(a + 2\bar{c})]}{(3/2)(a^2 + a\bar{c} + \bar{c}^2)^2 [a(2a + b + \bar{c}) - b\bar{c}] - (513/571)(a\bar{c})(a + \bar{c}) [b(\bar{c}^2 - a^2) + a\bar{c}(\bar{c} + 2a)]}
\end{aligned} \tag{52}$$

The coefficient h depends only on a, b, \bar{c} and the Lankford coefficient for $\theta = 45^\circ$, and can be determined by solving the following non-linear equation:

$$r_{45} = \frac{(a^2 + ab + b^2 + 9h^2)^2 (9h^2 - a^2 - b^2 - ab) + (513/571)(a + b)^2 (ab - 9h^2)(ab + 3h^2)}{2(a^2 + ab + b^2)(a^2 + ab + b^2 + 9h^2)^2 - (1026/571)(a + b)^2 (ab - 9h^2)(ab - 3h^2)} \tag{53}$$

3.2.2. Analytical evaluation of the anisotropy coefficients involved in KB93 orthotropic for FCC materials

The new expression of the KB93 criterion for FCC materials (see Eq.(49)) makes possible to obtain analytical formulas for the variation of the uniaxial flow stresses σ_θ and strain-ratios r_θ , respectively with the tensile loading orientation θ and further analytically identify the anisotropy coefficients of this criterion.

Proposition: The anisotropy coefficients a, b, \bar{c} involved in the expression of KB93 for FCC materials depend only on the uniaxial tensile flow stresses in the directions of orthotropy, \mathbf{x}, \mathbf{y} , and \mathbf{z} , and can be evaluated by solving the following system of algebraic equations:

$$(b^2 + b\bar{c} + \bar{c}^2)^6 + (c_1)(b\bar{c})^2 (b + \bar{c})^2 (b^2 + b\bar{c} + \bar{c}^2)^3 + (c_2)(b\bar{c})^4 (b + \bar{c})^4 = 1,$$

$$\begin{aligned}
& (a^2 + a\bar{c} + \bar{c}^2)^6 + c_1 (a\bar{c})^2 (a + \bar{c})^2 (a^2 + a\bar{c} + \bar{c}^2)^3 \\
& + c_2 (a\bar{c})^4 (a + \bar{c})^4 = \left(\sigma_0 / \sigma_{90} \right)^{12}, \\
& (a^2 + ab + b^2)^6 + c_1 (ab)^2 (a + b)^2 (a^2 + ab + b^2)^3 \\
& + c_2 (ab)^4 (a + b)^4 = \left(\sigma_0 / \sigma_z \right)^{12} = \left(\sigma_0 / \sigma_b \right)^{12}.
\end{aligned} \tag{54}$$

The anisotropy coefficient h depends only on σ_{45} and the coefficients a, b, \bar{c} :

$$\begin{aligned}
& (a^2 + ab + b^2 + 9h^2)^6 + c_1 (a + b)^2 (9h^2 - ab)^2 (9h^2 + a^2 + b^2 + ab)^3 \\
& + c_2 (a + b)^4 (9h^2 - ab)^4 = \left(2\sigma_0 / \sigma_{45} \right)^{12},
\end{aligned} \tag{55}$$

with $c_1 = -3700/383$ and $c_2 = 1205/2066$.

For KB93 for FCC materials the Lankford coefficients r_θ are calculated based on the formula:

$$r_\theta = - \frac{(\sin^2 \theta) \frac{\partial \bar{\sigma}_{KB93_FCC}}{\partial \sigma_{xx}} - (\sin 2\theta) \frac{\partial \bar{\sigma}_{KB93_FCC}}{\partial \sigma_{xy}} + (\cos^2 \theta) \frac{\partial \bar{\sigma}_{KB93_FCC}}{\partial \sigma_{yy}}}{\frac{\partial \bar{\sigma}_{KB93_FCC}}{\partial \sigma_{xx}} + \frac{\partial \bar{\sigma}_{KB93_FCC}}{\partial \sigma_{yy}}} \tag{56}$$

where $\frac{\partial \bar{\sigma}_{KB93_FCC}}{\partial \sigma_{ij}} = \left[6\tilde{J}_2^5 + 3c_1 \tilde{J}_3^2 \tilde{J}_2^2 \right] \frac{\partial \tilde{J}_2}{\partial \sigma_{ij}} + \left[2c_1 \tilde{J}_3 \tilde{J}_2^3 + 4c_2 \tilde{J}_3^3 \right] \frac{\partial \tilde{J}_3}{\partial \sigma_{ij}}, i, j = 1 \dots 3$.

Using Eq.(56), it can be easily shown that anisotropy coefficients a, b, \bar{c}, h involved in the expression of KB93 for orthotropic FCC materials depend only on r_0, r_{45} , and r_{90} .

Proposition: The anisotropy coefficients a, b, \bar{c} involved in the expression of KB93 for orthotropic FCC materials depend only on the Lankford coefficients r_0 and r_{90} , and can be determined by solving the following system of algebraic equations:

$$(b^2 + b\bar{c} + \bar{c}^2)^6 + (c_1)(b\bar{c})^2 (b + \bar{c})^2 (b^2 + b\bar{c} + \bar{c}^2)^3 + (c_2)(b\bar{c})^4 (b + \bar{c})^4 = 1$$

$$r_0 = \frac{E_0(a, b, \bar{c}) + c_1 (b\bar{c})(b + \bar{c})(b^2 + b\bar{c} + \bar{c}^2)^2 E_1(a, b, \bar{c}) + c_2 E_2(a, b, \bar{c})}{M_0(a, b, \bar{c}) + c_1 (b\bar{c})(b + \bar{c})(b^2 + b\bar{c} + \bar{c}^2)^2 M_1(a, b, \bar{c}) + c_2 M_2(a, b, \bar{c})} \quad (57)$$

$$r_{90} = \frac{S_0(a, b, \bar{c}) + c_1 (a\bar{c})(a + \bar{c})(a^2 + a\bar{c} + \bar{c}^2)^2 S_1(a, b, \bar{c}) + c_2 S_2(a, b, \bar{c})}{P_0(a, b, \bar{c}) + c_1 P_1(a, b, \bar{c})(a\bar{c})(a + \bar{c})(a^2 + a\bar{c} + \bar{c}^2)^2 + c_2 P_2(a, b, \bar{c})} \quad (58)$$

The coefficient h depends only on r_{45} and a, b, \bar{c} and can be easily determined once a, b, \bar{c} are

known in conjunction with the following equation

$$r_{45} = \frac{-1}{2} + \left(\frac{3}{2}\right) \frac{H_0(a, b, \bar{c}) + c_1 (a + b)^2 (9h^2 - ab) H_1(a, b, \bar{c}) + c_2 H_2(a, b, \bar{c})}{K_0(a, b, \bar{c}) + c_1 (a + b)^2 (9h^2 - ab) K_1(a, b, \bar{c}) + c_2 K_2(a, b, \bar{c})} \quad (59)$$

where $c_1 = -3700/383$ and $c_2 = 1205/2066$.

In the expression of r_0 (Eq. (57)), we have:

$$E_0(a, b, \bar{c}) = 6(b^2 + b\bar{c} + \bar{c}^2)^5 (ab - \bar{c}(a + b + 2\bar{c})) \quad (60)$$

$$E_1(a, b, \bar{c}) = -(2b^4\bar{c} + 2ab^4 + 9b^3\bar{c}^2 - ab^3\bar{c} + 15b^2\bar{c}^3 + 10b\bar{c}^4 + ab\bar{c}^3 - 2a\bar{c}^4) \quad (61)$$

$$E_2(a, b, \bar{c}) = -4(b\bar{c})^3 (b + \bar{c})^3 (b^2\bar{c} + ab^2 + 2b\bar{c}^2 - a\bar{c}^2), \quad (62)$$

while $M_0(a, b, \bar{c})$, $M_1(a, b, \bar{c})$, and $M_2(a, b, \bar{c})$ are obtained by replacing \bar{c} with b and b with \bar{c} in the expressions of $E_0(a, b, \bar{c})$, $E_1(a, b, \bar{c})$, and $E_2(a, b, \bar{c})$, respectively (see Eq. (A.1)-(A.3) in Appendix 1 for the explicit expressions). Similarly, in the expression of r_{90} (see Eq. (58)),

$S_0(a,b,\bar{c})$, $S_1(a,b,\bar{c})$, and $S_2(a,b,\bar{c})$ are obtained by interchanging a and b in the expressions of $E_0(a,b,\bar{c})$, $E_1(a,b,\bar{c})$, and $E_2(a,b,\bar{c})$, respectively; on the other hand, $P_0(a,b,\bar{c})$, $P_1(a,b,\bar{c})$, and $P_2(a,b,\bar{c})$ are obtained by interchanging a and b in the expressions of $M_0(a,b,\bar{c})$, $M_1(a,b,\bar{c})$, and $M_2(a,b,\bar{c})$, respectively (see Eq. (A.4)-(A.9) in Appendix 1). Finally, in the expression of r_{45} (see Eq.(59)), we have:

$$H_0(a,b,\bar{c}) = 6h^2(a^2 + ab + b^2 + 9h^2)^5 \quad (63)$$

$$H_1(a,b,\bar{c}) = h^2(a^2 + ab + b^2 + 9h^2)^2(2a^2 - ab + 2b^2 + 45h^2) \quad (64)$$

$$H_2(a,b,\bar{c}) = 4h^2(9h^2 - ab)^3(a + b)^4 \quad (65)$$

$$K_0(a,b,\bar{c}) = 2(a^2 + ab + b^2)(a^2 + ab + b^2 + 9h^2)^5 \quad (66)$$

$$K_1(a,b,\bar{c}) = -(a^2 + ab + b^2 + 9h^2)^2(2a^3b + 2a^2b^2 - 12a^2h^2 + 2ab^3 - 3abh^2 - 12b^2h^2 - 27h^4)$$

$$K_2(a,b,\bar{c}) = -2(ab - 3h^2)(9h^2 - ab)^3(a + b)^4, \quad (67)$$

To summarize: for any orthotropic material the anisotropy coefficients a, b, \bar{c}, h involved in KB93 criterion can be evaluated by solving a non-linear system of equations involving only the experimental tensile uniaxial flow stresses in the $0^\circ, 45^\circ, 90^\circ$ orientations and the equibiaxial flow stress, σ_b

As in the case of the Hill (1948) criterion, alternatively the anisotropy coefficients a, b, \bar{c}, h can be determined by solving a non-linear system of equations involving only the experimental

Lankford coefficients for the same orientations. Since the coefficients a, b, \bar{c} are expressible solely in terms of σ_0, σ_{90} or solely in terms of r_0, r_{90} , it follows that KB93 imposes a specific non-linear relation between these material properties.

Given that these methods of identification of the anisotropy coefficients are analytical and tied to mechanical properties, the parametrizations obtained lead to a physical and reliable representation of the yield surface of a given orthotropic material. Illustrative examples are presented in the next section.

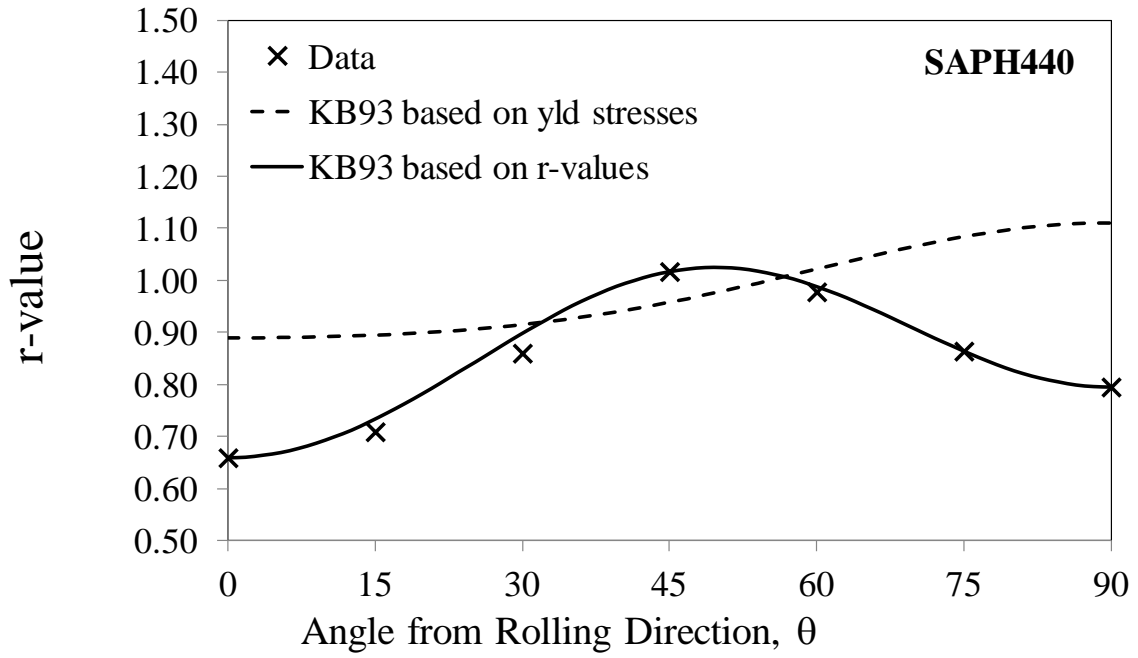
4. Applications to high-strength steel SAPH440 and 6451-T4 aluminum sheets

Using the new mathematical results derived in this research and the identification procedures based on analytical formulas involving either experimental flow stresses or experimental r-values for three orientations, it becomes possible to evaluate the anisotropy coefficients of non-quadratic orthotropic criteria with the same ease as in the case of the quadratic orthotropic criterion of Hill (1948). In the following, we illustrate this approach by applying the Yld91, KB93, and the Hill (1948) criterion to the description of the plastic anisotropy of sheets of high-strength steel SAPH440 (Japanese Industrial Standard designation of the alloy given in Numisheet 2018 Benchmark 2, see Hama et al., 2018) and AA 6451-T4 aluminum alloy, respectively (data from Numisheet 2016, Benchmark 2 reported by Allen et al., 2016).

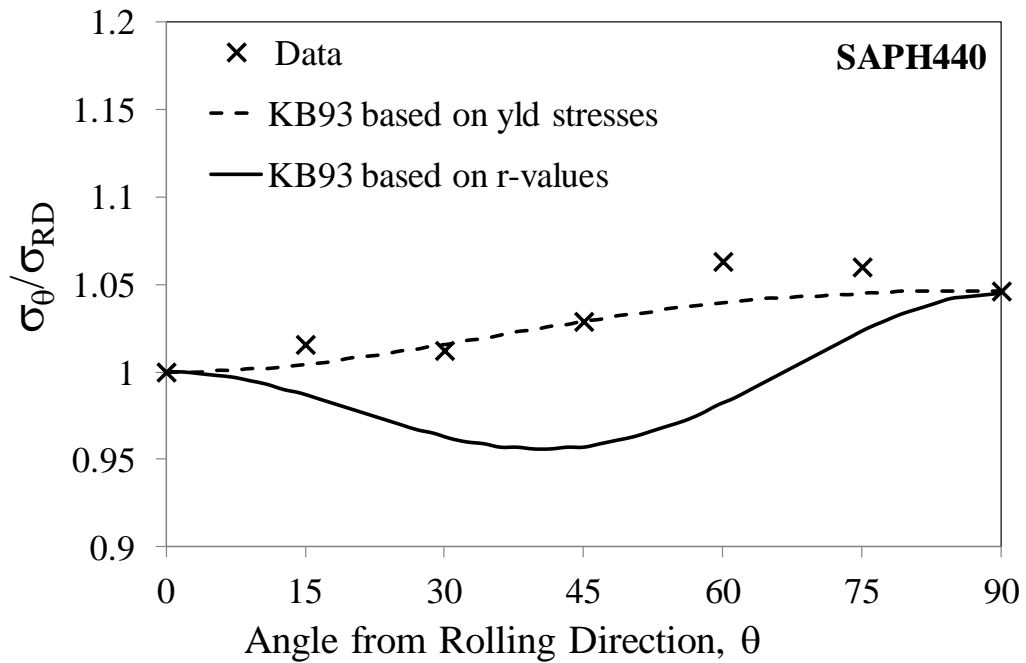
For the SAPH440, data consist of experimental uniaxial tensile flow stresses and r-values for seven orientations in the plane of the sheet and flow stresses measured in biaxial tension tests for seven linear stress paths, namely $\sigma_{xx} : \sigma_{yy} = 4:1, 2:1, 4:3, 1:1, 3:4, 1:2,$ and $1:4$.

Figure 1(a)-(b) show the comparison between the experimental and theoretical predictions of the anisotropy in uniaxial tensile flow stresses and r-values in the plane (RD-TD) according to KB93

criterion for BCC materials with anisotropy coefficients determined using the stress-based and r-values based strategy, respectively. The values of the anisotropy coefficients obtained by solving the algebraic system of equations (50) and the experimental flow stresses at 0° and 90° and in equibiaxial tension, denoted $\sigma_0^{\text{exp}}, \sigma_{90}^{\text{exp}}, \sigma_b^{\text{exp}}$, are: $a = 0.553$, $b = 0.606$, $\bar{c} = 0.577$. Next, these values are introduced in Eq. (51) and the experimental flow stresses at 45° , denoted σ_{45}^{exp} , is used to determine $h = 0.573$. Using the experimental r-values at 0° , 45° , and 90° , denoted $r_0^{\text{exp}}, r_{45}^{\text{exp}}, r_{90}^{\text{exp}}$, the solution of the algebraic system given by Eq.(52) leads to: $a = 0.591$, $b = 0.64$, $\bar{c} = 0.541$, $h = 0.618$. It is worth noting that the KB93 predictions of the r-values obtained with this latter set of values for the anisotropy coefficients are excellent although only $r_0^{\text{exp}}, r_{45}^{\text{exp}}, r_{90}^{\text{exp}}$ were used (see Fig. 1(a) solid line). Reasonable predictions of the uniaxial tensile flow stresses are obtained for all orientations, the most difference between experimental and predictions is of 5% (see Fig. 1(b) solid line). Most importantly, the shape of the theoretical KB93 yield locus in the biaxial plane (σ_{xx}, σ_{yy}) is very close to the experimental one, the most differences in curvature being for stress states in the vicinity of the equibiaxial tension state (see Fig. 1(c) solid line). If the anisotropy coefficients are evaluated using $\sigma_0^{\text{exp}}, \sigma_{90}^{\text{exp}}, \sigma_b^{\text{exp}}, \sigma_{45}^{\text{exp}}$ an excellent agreement is obtained between the theoretical predictions and all the experimental biaxial loadings points (see Fig. 1(c) interrupted line). No error bars were given for the experimental uniaxial flow stress data in the (RD-TD) plane. Nevertheless, the predicted anisotropy in flow stresses in this plane is probably within experimental error for all orientations. However, the r-values are overpredicted (with the exception of the 45° orientation) and the theoretical trend is different than the experimental one. Indeed, the KB93 criterion predicts that the r-values monotonically increase with the angle between the loading axis and RD (see Fig. 1(a) interrupted line).



(a)



(b)

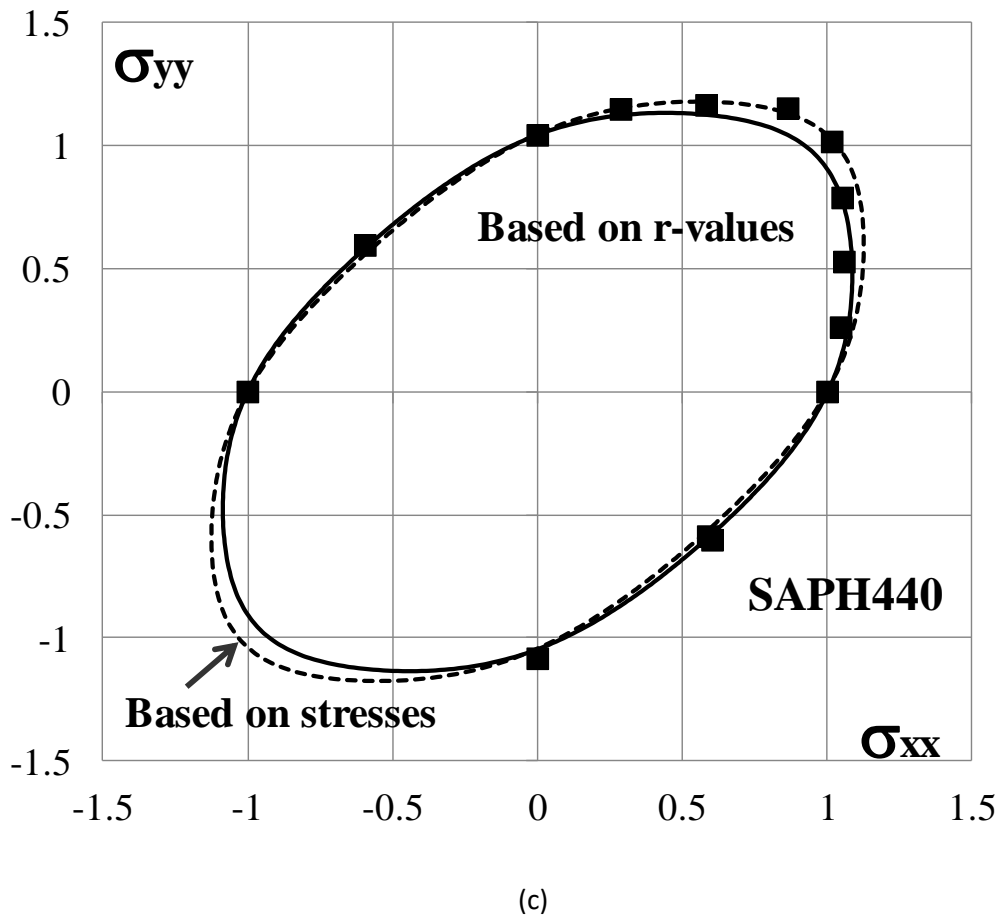


Figure 1: Predicted anisotropy for SAPH440 steel according to Karafillis and Boyce (1993) (KB93) criterion for orthotropic BCC materials ($m = 6$). Anisotropy coefficients determined using analytical expressions and either the experimental flow stresses σ_0^{exp} , σ_{45}^{exp} , σ_{90}^{exp} , σ_b^{exp} or the experimental r-values r_0^{exp} , r_{45}^{exp} , r_{90}^{exp} : (a) r-values; (b) uniaxial flow stresses; (c) surfaces in the sheet plane. Stresses are normalized by the flow stress in the x-direction (RD). Data after Numisheet 2018, Benchmark 2.

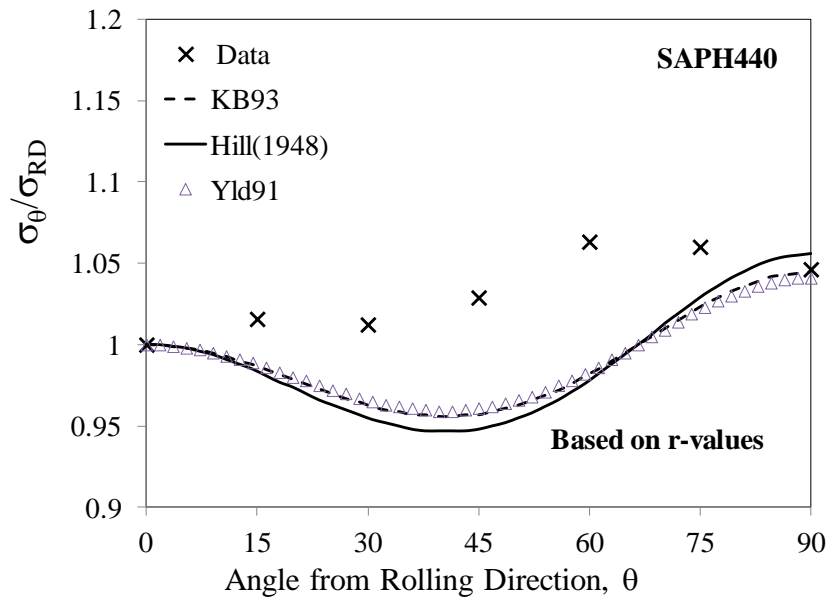
Comparison between the projections in the $(\sigma_{xx}, \sigma_{yy})$ of the KB93 yield surfaces corresponding to $\sigma_{xy} = 0$ for the two sets of values for the anisotropy parameters indicates that the stress-based surface is external to the other one, the most difference in flow stresses and curvature being in the vicinity of the equibiaxial states (see Fig. 1(c)). Specifically, if the r-values based parameters are used $\sigma_b^{th} = 0.961$ as compared to $\sigma_b^{exp} = 1.02$. For comparison purposes, Yld91 with $m = 6$ and Hill (1948) were applied to the same material (see Fig. 2-3). The values of the anisotropy coefficients involved in these criteria were also calculated using stress-based and r-based systems of equations involving either $\sigma_0^{exp}, \sigma_{90}^{exp}, \sigma_b^{exp}$ or $r_0^{exp}, r_{45}^{exp}, r_{90}^{exp}$ (see Table 1-2 for the values of the respective anisotropy coefficients). Note that irrespective of the strategy used for identification of the anisotropy coefficients for SAPH440 the predictions according to KB93 and Yld91 are practically the same (see Fig. 2-3). For the calculation of the coefficients F, G, H, N involved in Hill (1948) criterion we used analytical formulas in terms of $\sigma_0^{exp}, \sigma_{90}^{exp}, \sigma_b^{exp}$ and $r_0^{exp}, r_{45}^{exp}, r_{90}^{exp}$, respectively.

Table 1. Parameters for Yld91 criterion for high-strength steel SAPH 440.

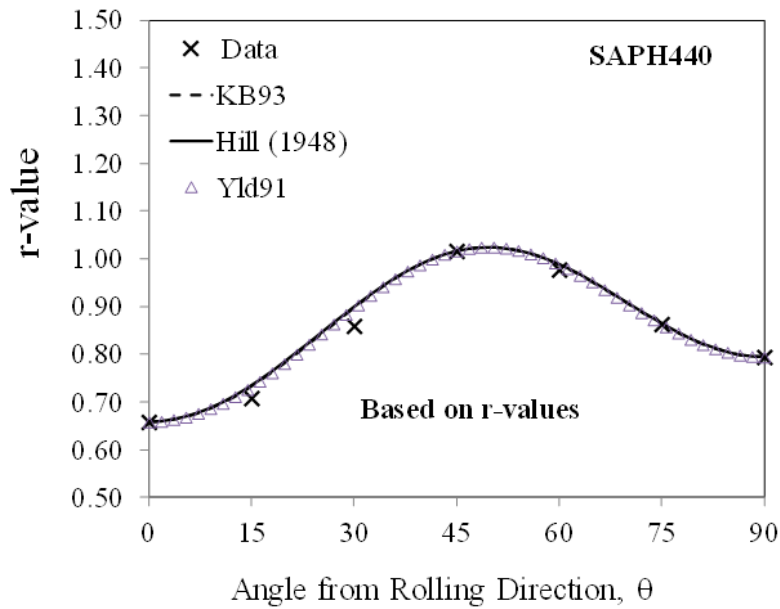
Yld 91 ($m=6$)	a	b	\bar{c}	h
Stress-based	0.559	0.611	0.582	0.579
r-values-based	0.597	0.642	0.55	0.622

Table 2. Parameters for Hill (1948) criterion for high-strength steel SAPH 440.

Hill (1948)	F	G	H	N
Stress-based	0.437	0.524	0.476	1.409
r-values-based	0.5	0.603	0.397	1.672



(a)



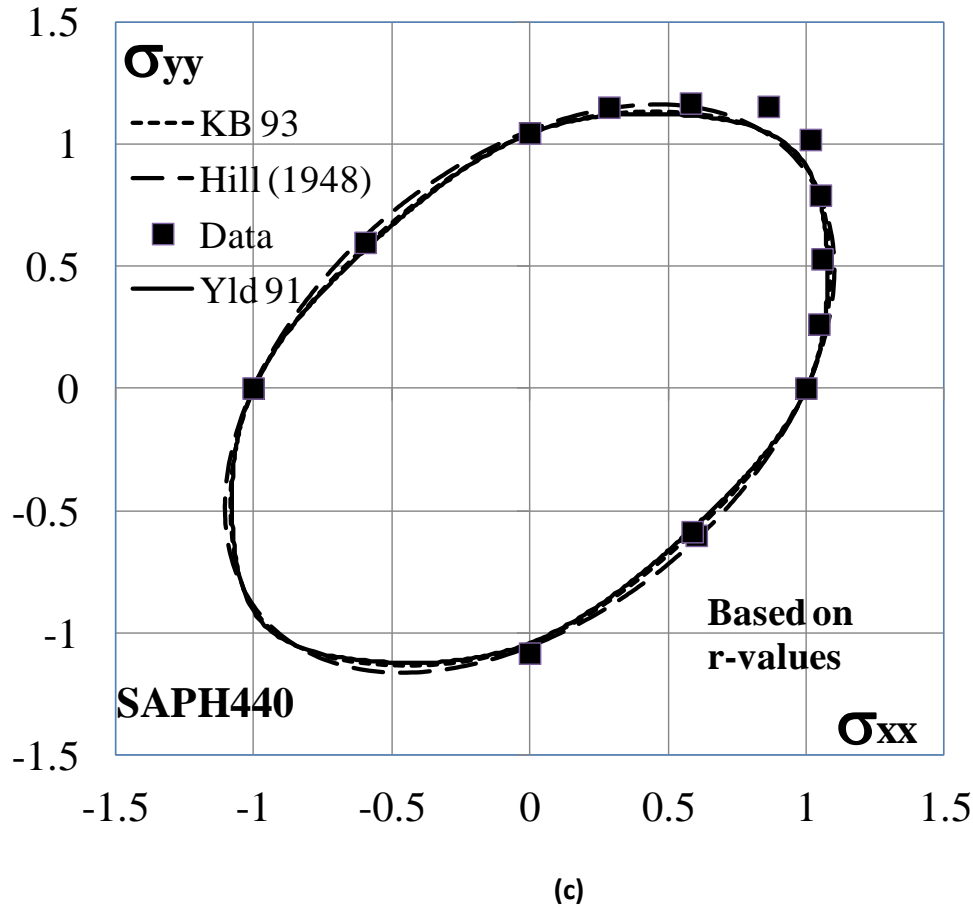
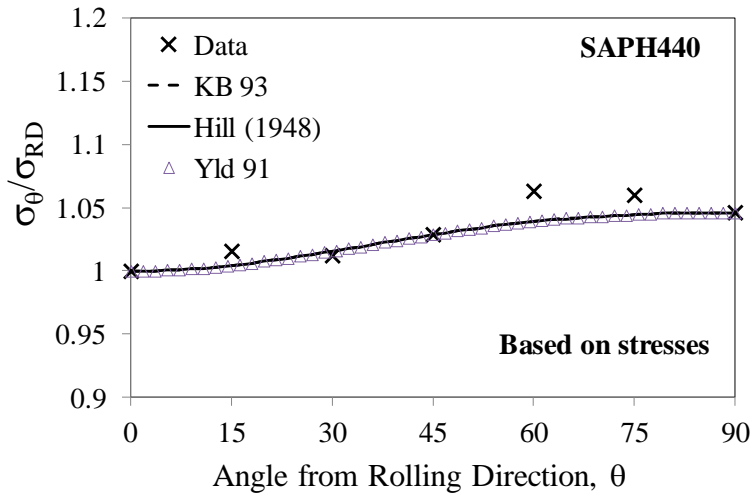
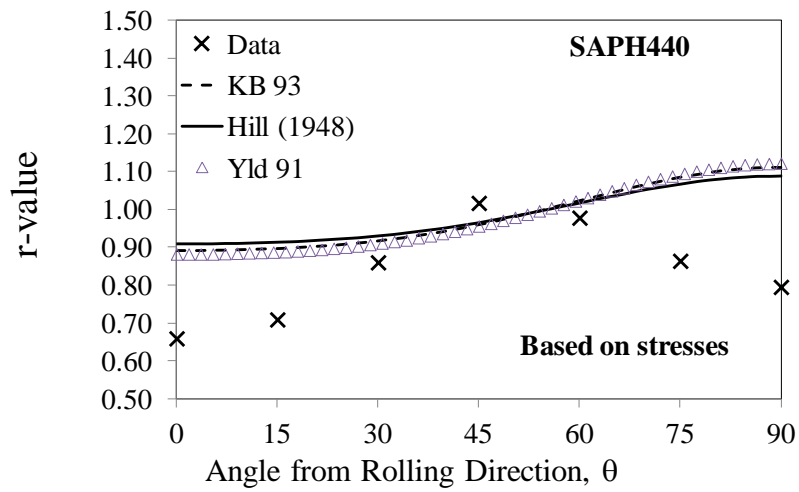


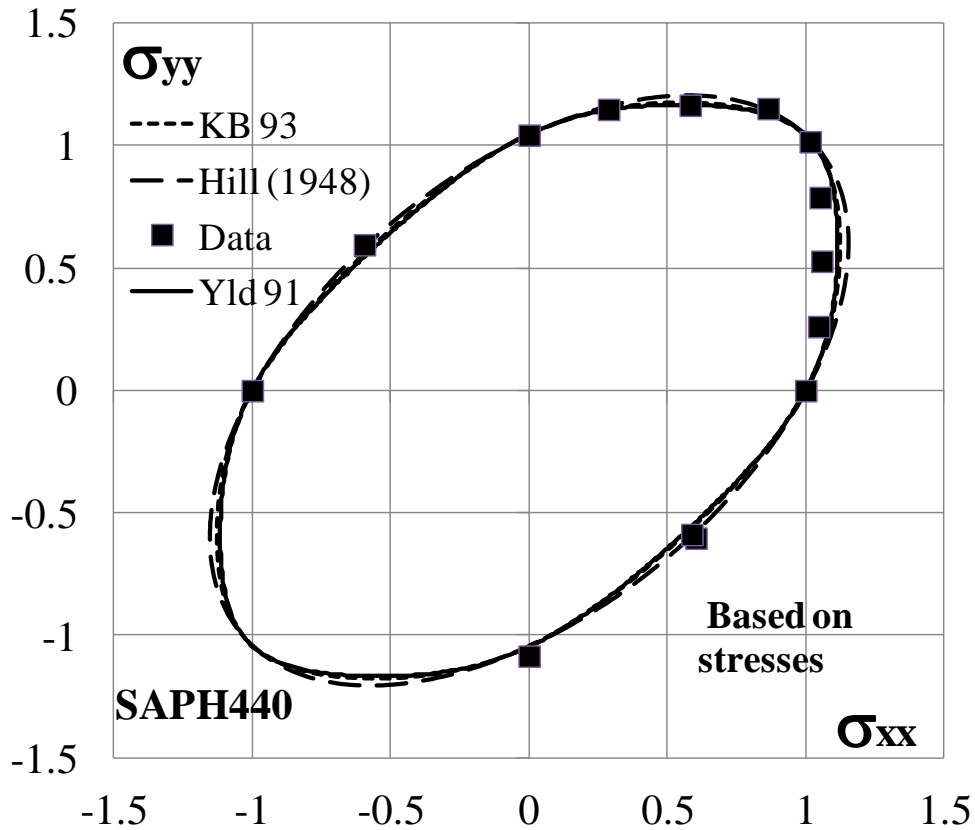
Figure 2. Predicted anisotropy for SAPH440 steel according to Karafillis and Boyce (1993) (KB93) criterion for orthotropic BCC materials ($m = 6$), Hill (1948), and Barlat et al. (1991) (Yld91) criterion ($m = 6$), respectively with anisotropy coefficients determined using analytical formulas and the experimental r-values r_0^{exp} , r_{45}^{exp} , r_{90}^{exp} : (a) uniaxial flow stresses; (b) r-values; (c) surfaces in the sheet plane. Stresses are normalized by the flow stress in the x-direction (RD). Data after Numisheet 2018, Benchmark 2.



(a)



(b)



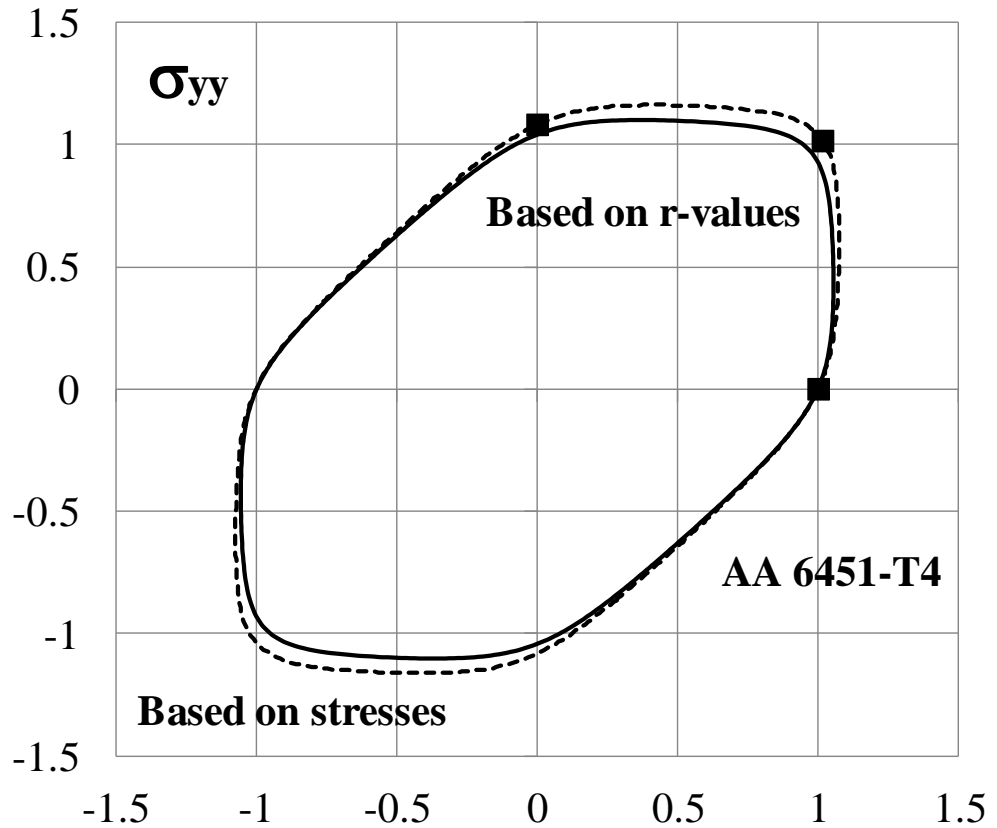
(c)

Figure 3. Predicted anisotropy for SAPH440 steel according to Karafillis and Boyce (1993) (KB93) criterion for orthotropic BCC materials ($m = 6$), Hill (1948), and Barlat et al. (1991) (Yld91) criterion ($m = 6$), respectively with anisotropy coefficients determined using analytical formulas and the experimental flow stresses σ_0^{exp} , σ_{45}^{exp} , σ_{90}^{exp} , σ_b^{exp} : (a) uniaxial flow stresses; (b) r-values; (c) surfaces in the sheet plane. Stresses are normalized by the flow stress in the x -direction (RD). Data after Numisheet 2018, Benchmark 2.

Note that irrespective of the strategy used for evaluating the anisotropy coefficients Hill (1948) predicts the same trends as KB93 and Yld91 (see Fig. 2-3). Specifically, if the anisotropy coefficients are determined using only the experimental r-values at 0° , 45° , 90° orientations, the theoretical r-values according to Hill (1948), KB93 and Yld91 are practically the same (to the order of 10^{-3}) and in excellent agreement with all experimental r-values (see Fig. 2(b)). For orientations $15^\circ < \theta < 60^\circ$, Hill (1948) predicts flow stresses slightly smaller than KB93_{BCC} and Yld91_{BCC} while for $60^\circ < \theta < 90^\circ$ the reverse holds true. For all the other orientations the theoretical uniaxial flow stresses according to Hill (1948), Yld91, and KB93 are practically the same. As concerns the predictions of the biaxial flow stresses, all criteria are in excellent agreement with the data points (see Figure 2(c)). Only the experimental equibiaxial tension flow stress $\sigma_b^{\text{exp}} / \sigma_0 = 1.02$ is underpredicted. Note that Yld91 and KB93 provide a slight improvement over Hill (1948). For example, KB93, Yld91, and Hill (1948) predictions of the flow stress in equibiaxial tension are: $\sigma_b^{\text{th}} / \sigma_0 = 0.961, 0.963, \text{ and } 0.952$, respectively against $\sigma_b^{\text{exp}} = 1.02$. On the other hand, if the anisotropy coefficients are determined using as input $\sigma_0^{\text{exp}}, \sigma_{45}^{\text{exp}}, \sigma_{90}^{\text{exp}}, \sigma_b^{\text{exp}}$ for SAPH 440 the predictions according to all three criteria are very close (see Fig. 3).

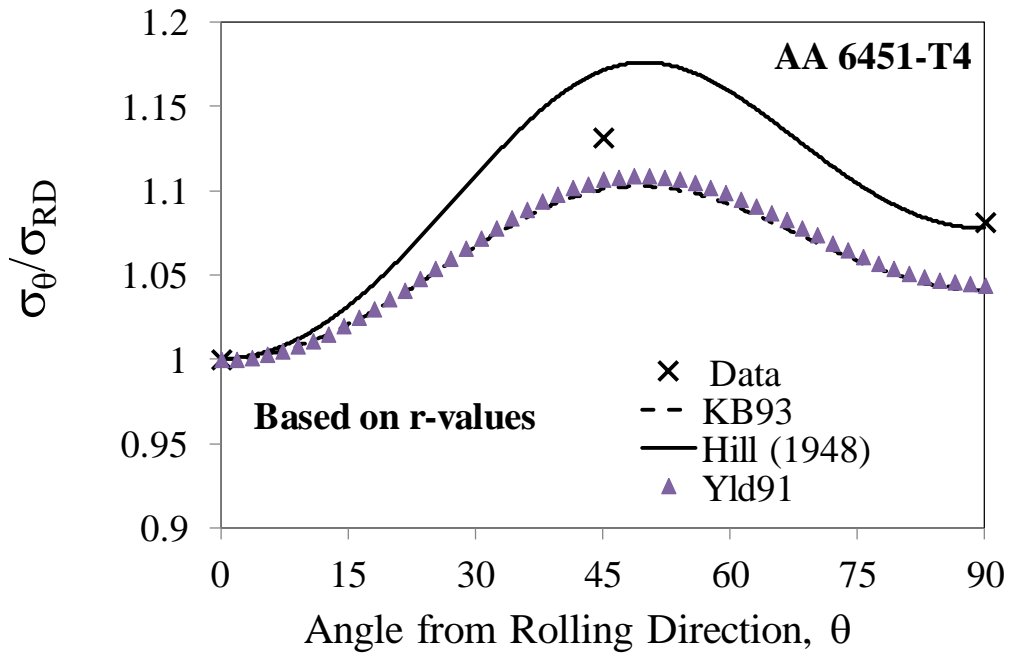
The next example concerns AA 6451-T4 aluminum sheet. For this material, only the uniaxial tensile flow stresses and r-values for the $0^\circ, 45^\circ, 90^\circ$ orientations and the equibiaxial tension flow stress were provided in the Benchmark 2 of Numisheet 2016 (see Allen et al., 2016). Figure 4(a)-(b) show the comparison between experimental data and the predicted yield stresses and r-values variation with the loading orientation according to KB93 for FCC materials (see Eq.(49)). The stress-based identification of the anisotropy coefficients was done by solving the algebraic system of equations given by Eq. (54) - (55), the corresponding values being: $a = 0.557, b = 0.648, \bar{c} = 0.577, h = -0.517$.

The values of these coefficients determined by solving the algebraic system given by Eq.(57)-(59) and using as input $r_0^{\text{exp}}, r_{45}^{\text{exp}}, r_{90}^{\text{exp}}$ are: $a = 0.606, b = 0.653, \bar{c} = 0.571, h = 0.527$. Note that if the anisotropy coefficients a, b, \bar{c}, h are determined using as input the experimental r-values, KB93 criterion predicts qualitatively the anisotropy in uniaxial tensile flow stresses in the plane (RD-TD), namely it predicts maximum flow stress at 45° to RD, with extrema also at 0° and 90° with the flow stress along RD being less than that along TD; the most difference between experimental and predicted flow stresses is along TD (see Fig. 4(b) solid line). On the other hand, if the anisotropy coefficients are determined using as input $\sigma_0^{\text{exp}}, \sigma_{45}^{\text{exp}}, \sigma_{90}^{\text{exp}}, \sigma_b^{\text{exp}}$, KB93 captures well the experimental trends in r-values anisotropy. It predicts a minimum at 45° and r-value at 0° less than that at 90° (see Figure 4(a) interrupted line). Figure 4(c) shows the comparison between the $(\sigma_{xx}, \sigma_{yy})$ projections of the yield surfaces according to KB93 for the two sets of values for the anisotropy parameters. Note that for stress states in the compression-compression and tension-tension quadrants respectively, the KB93 yield surface corresponding to the anisotropy coefficients obtained based on stress values is external to the other one (see Fig. 4(c)). The curvatures of the surfaces are similar, the most difference is for the equibiaxial stress points (for the case when the anisotropy coefficients are determined based on r-values $\sigma_b^{\text{th}} / \sigma_0 = 0.974$ vs. $\sigma_b^{\text{exp}} / \sigma_0 = 1.015$). For Yld91, the values of the anisotropy coefficients obtained analytically with the two respective types of data are given in Table 3. For Hill (1948) the values of the anisotropy coefficients are given in Table 4.

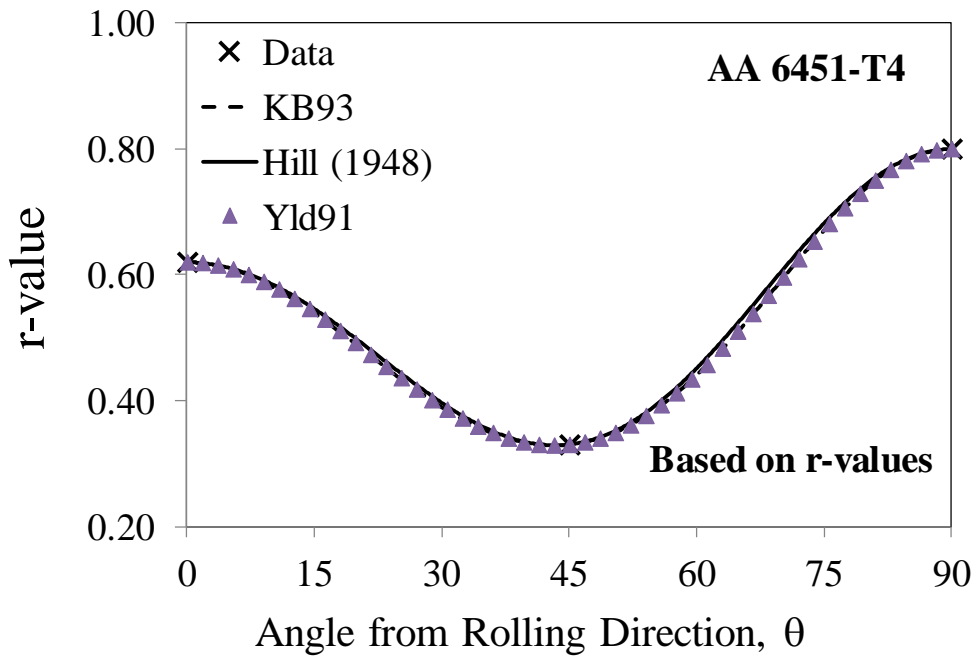


(c)

Figure 4. Predicted anisotropy for AA 6451-T4 aluminum according to Karafillis and Boyce (1993) (KB93) orthotropic criterion ($m = 12$) for FCC materials with anisotropy coefficients determined using analytical expressions and either the experimental flow stresses σ_0^{exp} , σ_{45}^{exp} , σ_{90}^{exp} or the experimental r-values r_0^{exp} , r_{45}^{exp} , r_{90}^{exp} : (a) r-values; (b) uniaxial flow stresses; (c) surfaces in the sheet plane. Stresses are normalized by the flow stress in the x-direction (RD). Data after Numisheet 2016, Benchmark 2.



(a)



(b)

In Figure 5-6 is shown a comparison between the predictions according to Yld91 ($m = 8$) and KB93 ($m = 12$) and Hill (1948) calibrated using the same data sets.

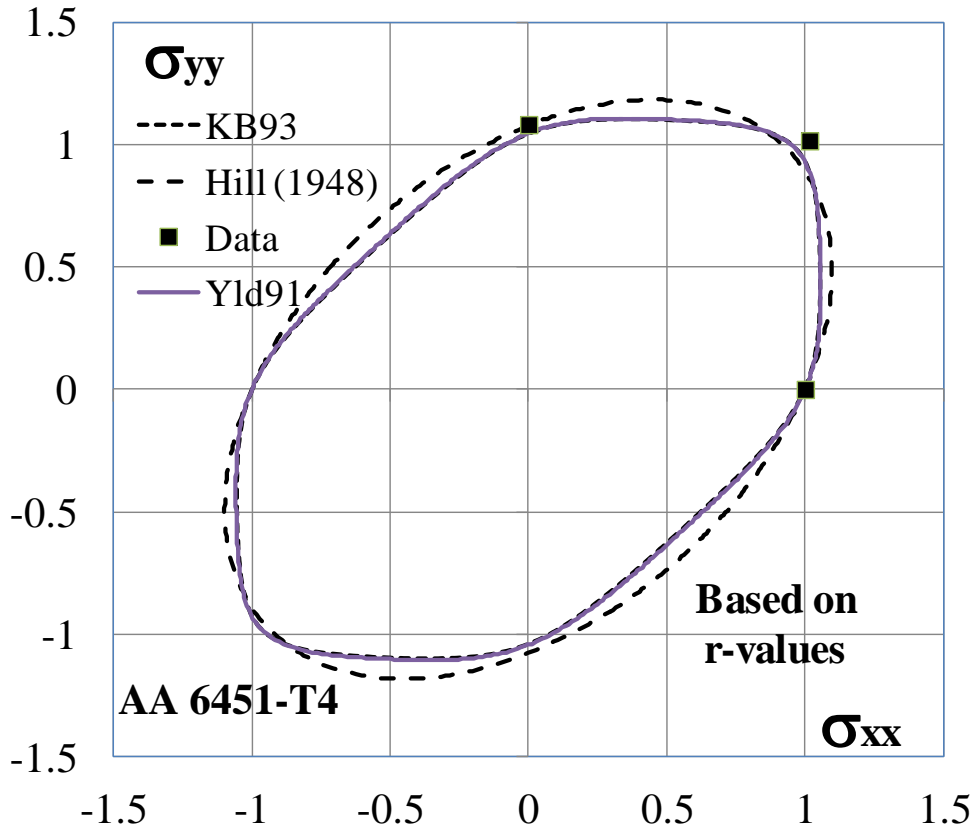


Figure 5. Predicted anisotropy for AA 6451-T4 aluminum according to Karafillis and Boyce (1993) (KB93) criterion for orthotropic FCC materials ($m = 12$), Hill (1948), and Barlat et al. (1991) (Yld91) criterion ($m = 6$), respectively with anisotropy coefficients determined using analytical formulas and the experimental r-values r_0^{exp} , r_{45}^{exp} , r_{90}^{exp} : (a) uniaxial flow stresses; (b) r-values; (c) surfaces in the sheet plane. Stresses are normalized by the flow stress in the x -direction (RD). Data after Numisheet 2016, Benchmark 2.

Yld 91 ($m = 8$)	a	b	\bar{c}	h
Stress-based	0.556	0.647	0.576	0.516
r-values-based	0.604	0.654	0.567	0.521

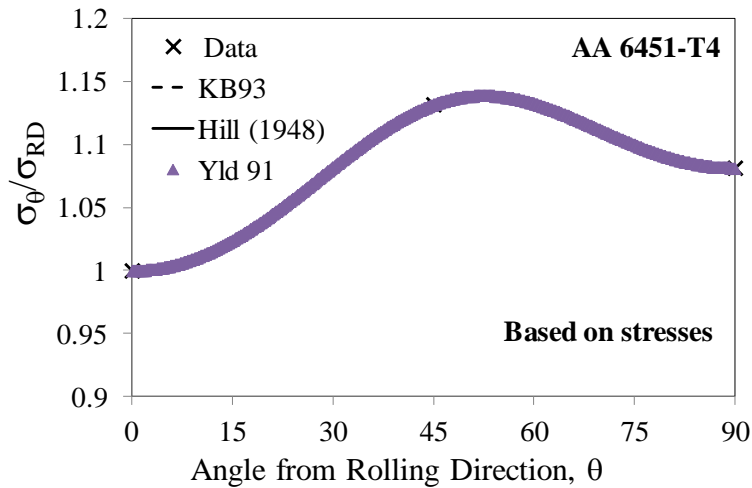
Table 3. Parameters values for Yld91 criterion ($m = 8$) for aluminum alloy AA 6451-T4.

Hill (1948)	F	G	H	N
Stress-based	0.413	0.577	0.443	1.077
r-values-based	0.478	0.617	0.383	0.909

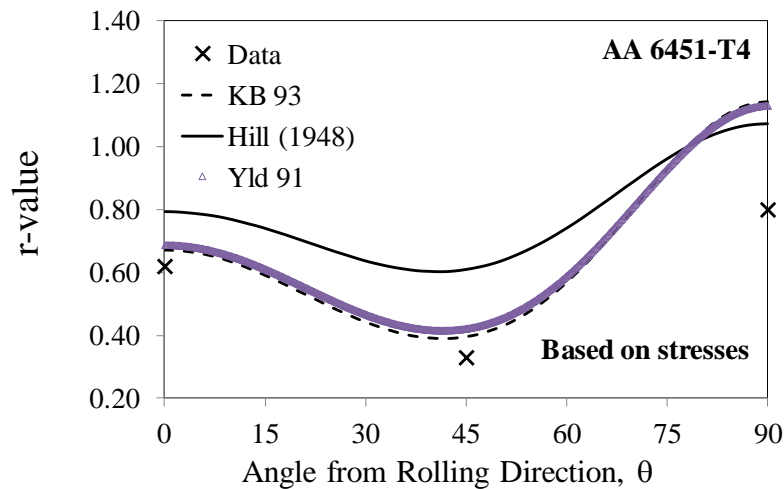
Table 4. Parameters values for Hill (1948) criterion for aluminum alloy AA 6451-T4.

It can be concluded that the r-based predictions of the plastic anisotropy in flow stresses and r-values in the plane of the sheet and the $(\sigma_{xx}, \sigma_{yy})$ projections of the yield surfaces ($\sigma_{xy} = 0$) according to Yld91 and KB93 are practically the same (see Fig.5) while the Hill (1948) surface is external to both. As concerns, the prediction of the yield stresses in the plane of the sheet, KB93 and Yld91 predictions are slightly lower than the experimental ones and those according to Hill (1948). On the other hand, Hill (1948) slightly overpredicts σ_{45}^{exp} and approximates well σ_{90}^{exp} . Nevertheless, the differences between the predictions of the three criteria are negligible. If the experimental stresses are used for calibration all criteria overestimate the experimental r-values. However, the overall experimental trends are qualitatively predicted.

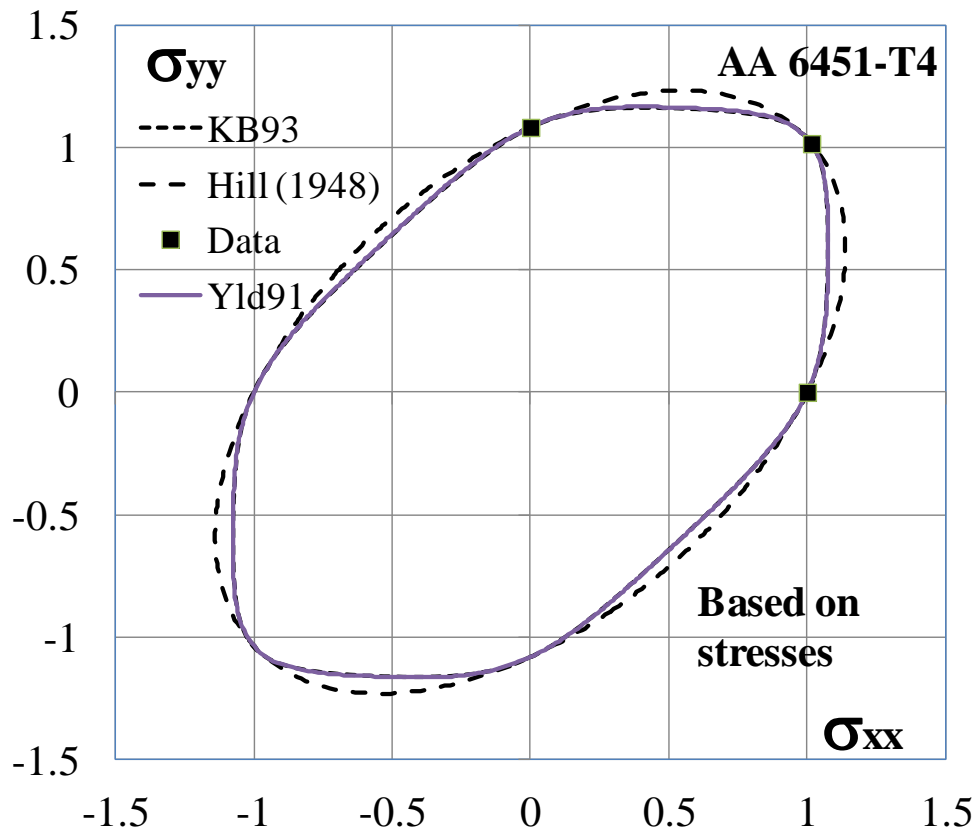
The r-values according to KB93 are slightly lower than those according to Yld91, but the differences are negligible. Hill (1948) predicts larger r-values than the non-quadratic criteria while the predictions of the yield stress anisotropy are the same. The yield surface according to Yld91 is a very slight upper-bound of KB93 while Hill (1948) is external to both (see Fig.6).



(a)



(b)



(c)

Figure 6. Predicted anisotropy for AA 6451-T4 aluminum according to Karafillis and Boyce (1993) (KB93) criterion for orthotropic FCC materials ($m = 12$), Hill (1948), and Barlat et al. (1991) (Yld91) criterion ($m = 6$), respectively with anisotropy coefficients determined using analytical formulas and the experimental flow stresses σ_0^{exp} , σ_{45}^{exp} , σ_{90}^{exp} : (a) uniaxial flow stresses; (b) r-values; (c) surfaces in the sheet plane. Stresses are normalized by the flow stress in the x -direction (RD). Data after Numisheet 2016, Benchmark 2.

4. Summary and concluding remarks on the implications of the new mathematical results for isotropic and anisotropic yielding formulations

In this research, new expressions of widely used yield formulations were deduced for isotropic and orthotropic BCC and FCC materials. It was demonstrated that for any positive and even integer m , the isotropic Hershey-Hosford criterion and the isotropic form of the Karafillis and Boyce (1993) criterion is a homogeneous polynomial in J_2 and J_3^2 . A direct consequence is that irrespective of the exponent m the respective orthotropic generalizations of these yield functions are homogeneous polynomials in terms of \tilde{J}_2 and \tilde{J}_3^2 , the second and third-invariant of a transformed stress tensor $\tilde{\mathbf{S}}$. Moreover, new expressions of the orthotropic yield functions of Barlat et al (1991) and Karafillis and Boyce (1993) yield function in terms of stress components were given for orthotropic BCC and FCC materials. Using these new representations, it was demonstrated that for BCC materials ($m = 6$) Karafillis and Boyce (1993) yield function and Barlat et al. (1991) yield function are identical in form. Moreover, Yld 91 is a particular case of the orthotropic yield criterion of Cazacu and Barlat (2001) involving fewer anisotropy coefficients and corresponding to fixed values of the coefficient c . Most importantly, it was shown that the anisotropy coefficients a, b, \bar{c} involved in the Karafillis and Boyce (1993) and respectively Barlat et al (1991) criterion are expressible only in terms of the yield stresses in the rolling, transverse, and through-thickness direction. The respective system of three algebraic equations in the unknown a, b, \bar{c} that need to be solved was provided. Once these coefficients are known, it was shown that the coefficient h can be easily evaluated by solving a non-linear equation involving only the experimental value of the uniaxial tensile flow stress at 45° to RD. Furthermore, it was shown that as in the case of Hill (1948) criterion, the coefficients a, b, \bar{c} can be evaluated using only the experimental r-values corresponding to RD and TD while the coefficient h is expressible in terms of a, b, \bar{c} and the r-

value at 45° to the RD. The benefits brought about by these new representations in terms of stress components of the Barlat et al (1991) and Karafillis and Boyce (1993) yield functions were illustrated for a textured steel SAPH440 and the aluminum alloy 6451-T4. Both the stress-based and the r-values based method of evaluation of the anisotropy coefficients led to a set of values for the anisotropy coefficients that give a reliable representation of the yield surface of the respective materials.

For mechanical characterization, generally uniaxial tensile tests are performed in three orientations: the rolling direction (0°), 45° to RD in the plane of the sheet, and transverse direction (90°), and additionally an equibiaxial tensile test or a bulge test may be conducted (e.g. see Allen et al., 2016). For this reason, in order to model the plastic anisotropy of the respective materials mainly Hill (1948) criterion is used. Using the new mathematical results derived as part of this research and the analytical expressions for anisotropy coefficients provided now it becomes possible to use advanced non-quadratic 3-D yield criteria with the same level of confidence in terms of reliability and ease of the identification procedure even when the mechanical characterization data is limited.

5. Influence of plastic anisotropy on the inception of localization bands under tensile loading: new analytical expressions for the band orientations

Localized deformation such as Luders bands (e.g. see Nádai and Wahl (1931)) or necking in flat specimens (e.g. see data of Körber and Hoff (1928)) have been the object of numerous investigations. Beginning with the pioneering papers of Hill (1948), Rudnicki and Rice (1975), Hill and Hutchinson (1975) and Rice (1976), the localization phenomenon is modeled as a

bifurcation problem for the velocity gradient (or incremental displacement field). However, most of the theoretical investigations concern isotropic materials. For such materials, analytical expressions for the bifurcations directions for either plane stress or plane strain conditions were derived (e.g. see Neilsen and Schreyer (1993)). In particular, it was demonstrated that under uniaxial tension, an isotropic pressure-insensitive metallic material develops two localization bands (from now on indistinctly referred to as necking bands) that are inclined at an angle of $\pm 54.76^\circ$ to the loading axis.

The study of Körber and Hoff (1928) is among the earliest experimental investigations on the influence of plastic anisotropy on localization. These authors carried out uniaxial tension tests on flat specimens taken from rolled sheets of Al, Cu, Brass, Iron, Ni. Both the yield stress and the orientation of the necking bands that developed in the specimens prior to failure were reported. Irrespective of the material, it was observed that the directions of the necking bands, β , that developed in the specimens were dependent on the sample orientation θ between the loading axis and the rolling direction of the sheet (see Fig.7). Furthermore, in contrast to isotropic materials, the orientation of the localization bands did not make an angle of $\sim 55^\circ$ with the loading axis. A qualitative discussion of these experimental results was done by Hill (Hill (1948)). Moreover, he demonstrated that two necking bands equally inclined with respect to the direction of loading develop in the RD and TD specimens. Specifically, for the RD specimen, the inclinations of the necking bands are:

$$\beta_{Hill}^{RD} = \pm \operatorname{atan}\left(\sqrt{(G+H)/H}\right) \quad (68)$$

while for the TD specimen the bands are at:

$$\beta_{Hill}^{TD} = \pm \operatorname{atan}\left(\sqrt{(F+H)/H}\right) \quad (69)$$

Moreover, Hill (1948) showed that if (N-G-2H) and (N-F-2H) have the same sign, two bands equally inclined to the loading direction will also develop in a specimen of an intermediate orientation θ^* , (i.e. $0 < \theta^* < 90$), given by :

$$\tan^2(\theta^*) = (N - G - 2H) / (N - F - 2H) \quad (70)$$

Remark: From Eq. (70), it follows that for an orthotropic material for which the plastic behavior is governed by Hill (1948)'s criterion, depending on the ordering of the anisotropy coefficients F , G , H , N , the band angles could be either smaller or larger than $\pm 54.76^\circ$. Moreover, for isotropy (i.e. $F=G=H=N/3$), $\beta_{Hill}^{RD} = \beta_{Hill}^{TD} = 54.76^\circ$.

Using Nixon et al. (2010) orthotropic yield criterion, we have conducted a F.E. study on the combined effects of anisotropy and tension-compression asymmetry on the inception of necking bands in tensile specimens subjected to dynamic uniaxial tensile loading (see N'souglo et al. (2019)). These F.E. results indicated that for the values of the material parameters used, there are only three specimen orientations, namely $\theta = 0^\circ$, $\theta = 90^\circ$, and $\theta = \theta^* \sim 45^\circ$ for which the localization bands are equally inclined to the loading direction. For specimens of orientation $\theta < \theta^*$, the two necking bands have different inclinations, and the band that grows faster makes an acute clockwise angle with the loading axis (see Fig. 7(a)); on the other hand, for specimens having the orientation $\theta > \theta^*$, the band that grows faster makes an acute counter-clockwise angle with the loading axis (see Fig. 7(b)). In addition, we used classical results of bifurcation theory to numerically determine the bands orientations for quasi-static loadings. It was shown that according to the Nixon et al. (2010) criterion the effects of plastic anisotropy on the localization of

deformation under uniaxial tensile loadings is the same irrespective of the loading rate (i.e. same trends for both quasi-static and dynamic uniaxial loadings).

We have also demonstrated that irrespective of the mathematical form of the plastic potential (i.e. of the effective stress $\bar{\sigma}$), under uniaxial tension two necking bands develop. Moreover, their orientations with respect to the loading axis, denoted β_1 and β_2 , are the roots of the following quadratic algebraic equation in $u = \tan \beta$, i.e.:

$$A \left(u^2 - \frac{1+r(\theta)}{r(\theta)} \right) + 2Bu = 0, \quad (71)$$

where

$$A = (\sin^2 \theta) \frac{\partial \bar{\sigma}}{\partial \sigma_{xx}} - \sin(2\theta) \frac{\partial \bar{\sigma}}{\partial \sigma_{xy}} + (\cos^2 \theta) \frac{\partial \bar{\sigma}}{\partial \sigma_{yy}} \quad (72)$$

$$B = \sin \theta \cos \theta \left(\frac{\partial \bar{\sigma}}{\partial \sigma_{yy}} - \frac{\partial \bar{\sigma}}{\partial \sigma_{xx}} \right) + \cos(2\theta) \left(\frac{\partial \bar{\sigma}}{\partial \sigma_{xy}} \right) \quad (73)$$

The results were reported in Cazacu and Rodríguez-Martínez (2019).

The proof is based on the fact that under uniaxial tension, a necking band corresponds to a direction of zero elongation. Let us denote by ξ the angle between the direction of zero elongation, Ox'' and RD (see Fig.7) i.e.

$$d''_{xx} = d_{xx} \cos^2 \xi + d_{yy} \sin^2 \xi + 2d_{xy} \sin \xi \cos \xi = 0 \quad (74)$$

Because for any material, the Lankford coefficient corresponding to $\theta = 0^\circ$ is always non-zero, it follows that $\xi \neq 90^\circ$. Thus, Eq. (74) can be rewritten as:

$$d_{yy} \tan^2 \xi + 2d_{xy} \tan \xi + d_{xx} = 0$$

Using the trigonometric identity,

$$\tan(\theta + \beta) = \frac{\tan \theta + \tan \beta}{1 - \tan \theta \tan \beta},$$

the assumption of associated flow rule and the definition of the Lankford coefficient $r(\theta)$, we arrive at the quadratic algebraic equation for $u = \tan \beta$ given by Eq.(71).

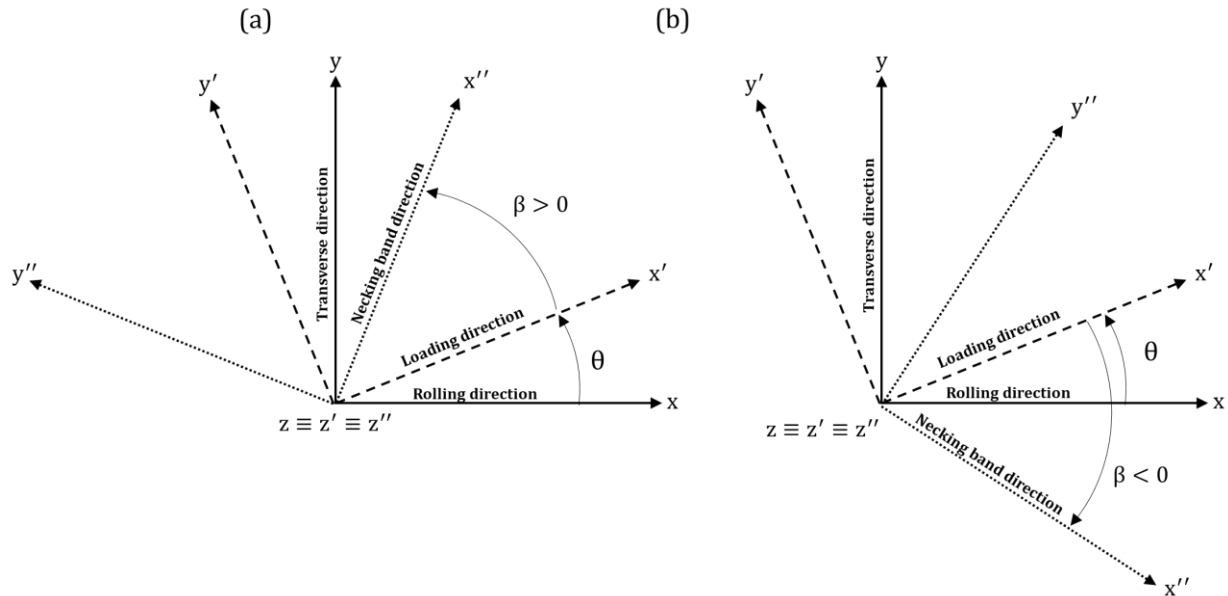


Figure 7: Definition of the two possible necking band angles β for a flat specimen loaded in uniaxial tension in a direction θ with respect to the rolling direction: (a) necking band forming an acute counter-clockwise angle with respect to the loading direction ($\beta > 0$); (b) necking band forming an acute clockwise angle with respect to the loading direction ($\beta < 0$).

Proposition Two necking bands with the same inclination to the loading axis develop in the specimens of orientations θ^* corresponding to the minima or maxima of the uniaxial tensile flow stresses in the plane of the sheet. Moreover, the bands angles depend only on the Lankford coefficient $r(\theta^*)$, i.e.

$$\beta = \pm \text{atan}\left(\sqrt{(1+r(\theta^*))/r(\theta^*)}\right) \quad (75)$$

Proof: The orientations θ^* which correspond to the minima or maxima of the uniaxial tensile flow stresses σ_θ satisfy

$$\left. \frac{\partial \bar{\sigma}}{\partial \theta} \right|_{\theta=\theta^*} = 0$$

Using the chain rule,

$$\frac{\partial \bar{\sigma}}{\partial \theta} = \left(\frac{\partial \bar{\sigma}}{\partial \sigma_{xx}} \right) \left(\frac{\partial \sigma_{xx}}{\partial \theta} \right) + \left(\frac{\partial \bar{\sigma}}{\partial \sigma_{yy}} \right) \left(\frac{\partial \sigma_{yy}}{\partial \theta} \right) + 2 \left(\frac{\partial \bar{\sigma}}{\partial \sigma_{xy}} \right) \left(\frac{\partial \sigma_{xy}}{\partial \theta} \right) \quad (76)$$

and the transformation rules for the components of the stress tensor, we obtain that the extrema correspond to the orientations $\theta = \theta^*$ for which:

$$2 \sin \theta \cos \theta \left(\frac{\partial \bar{\sigma}}{\partial \sigma_{yy}} - \frac{\partial \bar{\sigma}}{\partial \sigma_{xx}} \right) + 2 \cos(2\theta) \left(\frac{\partial \bar{\sigma}}{\partial \sigma_{xy}} \right) = 0. \quad (77)$$

i.e. $B = 0$ in Eq.(71). Note that if $B = 0$, the two roots of Eq.(71) have the same absolute value and are opposite in sign, i.e. $u_{1,2} = \pm \sqrt{(1+r(\theta^*))/r(\theta^*)}$. Therefore, the two possible bands are symmetrical with respect to the specimen axis, and the bands angles are given by Eq. (75).

It is worth verifying that if the plastic anisotropy of the material is described by Hill (1948)'s orthotropic yield criterion, Eq.(75) leads to the expressions of the bands angles given by Hill (1954). Indeed, for the case when yielding is according to Hill (1948)'s criterion there is an analytical formula for the variation of the normalized uniaxial flow stress $\sigma(\theta)/\bar{\sigma} = \sigma(\theta)/\sigma(0)$ with the orientation θ of the specimen with RD in the (RD,TD) plane. Using this formula, it can be easily demonstrated that irrespective of the values of the coefficients F, G, H , and N , the extrema of $\sigma(\theta)$ vs. (θ) correspond to: $\theta = 0^\circ$ and $\theta = 90^\circ$.

Moreover, if $(N - G - 2H)(N - F - 2H) > 0$, there is an additional minimum or maximum which corresponds to the orientation θ^* given by :

$$\theta^* = \text{atan}\left(\sqrt{(N - G - 2H)/(N - F - 2H)}\right)$$

i.e. we recover Eq. (70). Since according to Hill (1948)'s criterion, $r_0 = H / G$ and $r_{90} = H / F$, further substitution in Eq. (75) leads to β_{Hill}^{RD} and β_{Hill}^{TD} given previously while for the specimen of orientation θ^* , the predicted bands angle is:

$$\beta^* = \pm \text{atan} \sqrt{\frac{4(F + H) \tan^4(\theta^*) + (3F + 3G + 2N) \tan^2(\theta^*) + 4(G - H)}{(2N - G - F) \tan^2(\theta^*) + 4H(\tan^4(\theta^*) - 1)}}$$

(78)

Remark: Another consequence of the above Propositions is that irrespective of the model used to describe the plastic anisotropy of the material, the specimens orientations θ for which the bands are equally inclined to the specimen axis are those for which the principal directions of stress and plastic strain rates coincide.

Proof: Indeed, for uniaxial tension along a direction θ with respect to the \mathbf{x} -direction (RD):

$$\begin{aligned}d'_{xx} &= d_{xx} \cos^2 \theta + d_{yy} \sin^2 \theta + \sin(2\theta) d_{xy} \\d'_{yy} &= d_{xx} \sin^2 \theta + d_{yy} \cos^2 \theta - \sin(2\theta) d_{xy} \\d'_{xy} &= \frac{1}{2}(d_{yy} - d_{xx}) \sin(2\theta) + d_{xy} \cos(2\theta)\end{aligned}\tag{79}$$

Therefore, the specimen orientation θ for which the principal axes of stress and plastic strain rates coincide correspond to:

$$(d_{yy} - d_{xx}) \sin(2\theta) + 2d_{xy} \cos(2\theta) = 0\tag{80}$$

Given that the flow rule is associated, Eq.(80) coincides with Eq. (77).

Proposition: For specimen of orientations other than $\theta = \theta^*$, there are two possible bands of different inclinations, β_1 and β_2 with respect to the loading axis.

- If $\sigma(\theta)$ vs. θ admits one absolute minimum $0^\circ < \theta_1^* < 90^\circ$: for specimen orientations $\theta < \theta_1^*$ the inclination of the band which makes an acute counter-clockwise angle with the loading direction is greater than the inclination of the band which makes an acute clockwise angle with the loading direction, i.e. $\beta_1 > |\beta_2|$ (see Fig.7). For specimen orientations $\theta > \theta_1^*$, the reverse occurs, i.e. $\beta_1 < |\beta_2|$.

- If $\sigma(\theta)$ vs. θ variation admits one absolute maximum $0^\circ < \theta_2^* < 90^\circ$: for $\theta < \theta_2^*$ we have that $\beta_1 < |\beta_2|$ and vice-versa for $\theta > \theta_2^*$.

Proof: Note that the product of the roots of Eq. (71) is equal to $-(1+r(\theta))/r(\theta)$ while the sum of the roots is equal to $(-B/A)$. Therefore, the band angles should be opposite in sign. Since A is always negative (see details in Cazacu and Rodríguez-Martínez (2019)), the sign of the sum of the roots is given by the sign of B . Note that the sign of B is the opposite of the sign of $\partial\sigma(\theta)/\partial\theta$ (see Eq. (73)). Therefore, if $0^\circ < \theta_1^* < 90^\circ$ is an absolute minimum for $\sigma(\theta)$ vs. θ , then for $\theta < \theta_1^*$ we have $B > 0$ while for $\theta > \theta_1^*$, $B < 0$. It means that $\beta_1 > |\beta_2|$ for $\theta < \theta_1^*$, and vice-versa for $\theta > \theta_1^*$.

Remarks : First, let us note that based only on the experimentally observed dependence of the yield stress on the orientation in the plane of the sheet, one can infer for which specimen orientations the necking bands are equally inclined with respect to the loading axis. Furthermore, the respective band angles can be easily calculated using only the experimental r -values in conjunction with Eq.(75). Moreover, if $r(\theta^*) < 1$, the absolute value of the bands angles are greater than 54.76° , and vice-versa if $r(\theta^*) > 1$. Obviously, for an isotropic material $r(\theta) = 1$ irrespective of the loading direction θ , therefore according to Eq. (75) the band angles are equal to: $\pm 54.76^\circ$. For all the other specimen orientations θ , there are two necking bands which correspond to the roots of Eq. (71), and the bands angles can be determined using any orthotropic plastic potential. Therefore, the predictions of the bands orientations will strongly depend on the choice of the yield

criterion, the type of data and the extent of the data set available for the identification of the parameters involved in the given yield criterion.

In the following, we present for AA 2090-T3 alloy an analysis based solely on data, and discuss the predictions of necking bands orientations obtained with Hill (1948)'s criterion and Cazacu (2018)'s criterion.

Based on the data reported in Barlat et al. (1991), it appears that in the interval $(0,90^\circ)$, the yield stresses $\sigma(\theta)$ may admit minima at $\theta_1^* = 45^\circ$ and $\theta_2^* = 60^\circ$, respectively. According to the above Proposition, it follows that for the specimens with orientations: 0° , 45° , 60° and 90° , the two necking bands will be equally inclined with respect to the loading axis. The respective band orientations, calculated using the experimental r-values are given in Table 5. Since for this material, both r_0 and r_{90} are less than 1, for the RD and TD specimens the bands are inclined at an angle greater than 54.736° . Moreover, since $r_0 = 0.212 < r_{90} = 0.692$, the inclination of the bands is greater for the RD specimen than for the TD specimen (see Eq. (75)). On the other hand, for the 45° and 60° specimens, for which the r-values are greater than one, the inclination of the bands is less than the isotropic value.

Table 5. Specimen orientations for which bands are equally inclined to the loading axis for the 2090-T3 Al alloy.

Specimen orientation to the rolling direction, θ (degrees)	Bands orientation, β (degrees)
0	± 67.3
45	± 51.96

60	± 54.486
90	± 57.4

Since the yield stresses $\sigma(\theta)$ may admit minima at $\theta_1^* = 45^\circ$ and $\theta_2^* = 60^\circ$, according to our theory for the specimens of orientation $\theta < \theta_1^*$, the angle of the band corresponding to $\beta_1 > 0$ is numerically larger than $|\beta_2|$ (the absolute value of the angle for the band that forms an acute clockwise angle with respect to the loading axis), and vice-versa for the specimens of orientation $\theta > \theta_2^*$ (see Figure 7). In view of the analysis of the necking for the AA 2090-T3 alloy, Hill (1948)'s criterion was identified using an analytical procedure based only on r-values. In this manner, it was ensured that the Hill's predictions for the band angles for the RD and TD specimens match exactly the estimates obtained using the experimental r-values (see Table 5). The values of Hill (1948)'s parameters are: $F=0.252$, $G=0.825$, $H=0.175$, $N=2.238$. As discussed, Hill (1948) identified on r-values cannot predict correctly the yield stresses. Specifically, since $N > F+2H$ and $N > G+2H$, there is an intermediate specimen orientation for which bands are equally inclined which corresponds to $\theta^* = 39.42^\circ$ (see Eq.(70)). Moreover, using the above Proposition, it follows that according to Hill (1948) for specimens of orientation $\theta > 39.42^\circ$, the predominant band corresponds to $\beta_2 < 0$ and $|\beta_2| > \beta_1$ (see Figure for the definition of the bands angles, and Figure 9 for the predictions for AA 2090-T3).

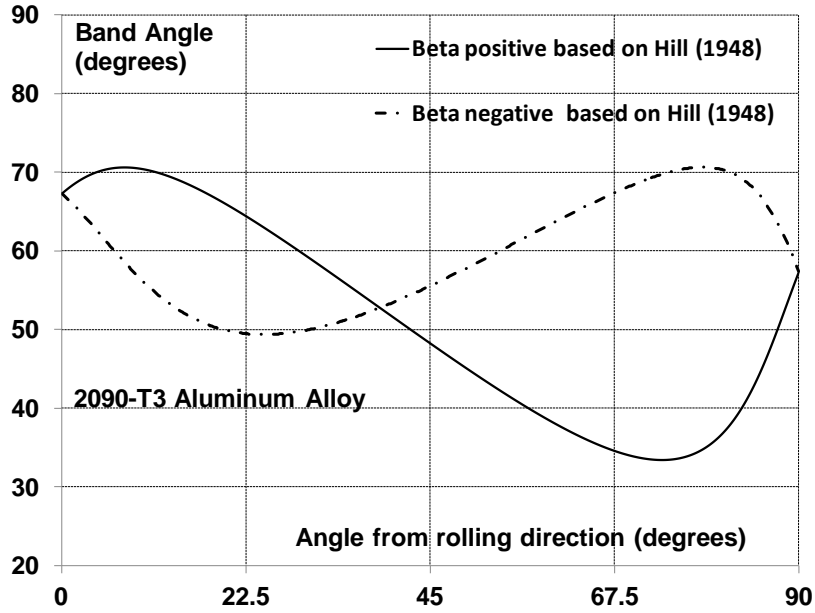


Figure 8: Predicted values of the bands angles according to Hill (1948) yield criterion as a function of the specimen orientation with respect to RD for the AA 2090-T3 alloy.

For AA 2090-T3, the parameters of the Cazacu (2018) criterion are: $a_1=1.76$, $a_2=1.715$; $a_3=1.16$, $a_4=0.91$, $b_1=-6.4$, $b_2=-0.006$, $b_3=2.61$, $b_4=4.88$, $b_5=6.6$, $b_{10}=0.913$, and $\alpha=1.2$. For these parameter values, Cazacu (2018) criterion predicts both the observed anisotropy in r-values and the $\sigma(\theta)$ vs. θ variation (for details, see Cazacu and Rodríguez-Martínez (2019)).

Moreover, a minimum for $\sigma(\theta)$ vs. θ is predicted for $\theta^*=60^\circ$. Therefore, according to this criterion, the intermediate specimen orientation for which the bands are equally inclined is at $\theta^*=60^\circ$; for specimens with orientation $0<\theta<60^\circ$, the predominant band (i.e. the band that grows faster) corresponds to $\beta_1>0$ and $\beta_1>|\beta_2|$. On the other hand, for specimens with $\theta>60^\circ$, the

reverse holds true (see Figure 7 and Fig. 9). Note that according to the Cazacu (2018) criterion the maximum of $(|\beta_2| - \beta_1)$ is significantly smaller, 18.26° versus 37° according to Hill (1948).

The same trends and general conclusions as for Hill (1948) can be drawn concerning the predictions of the orientation of the necking bands obtained with the Yld91, and KB93 criteria.

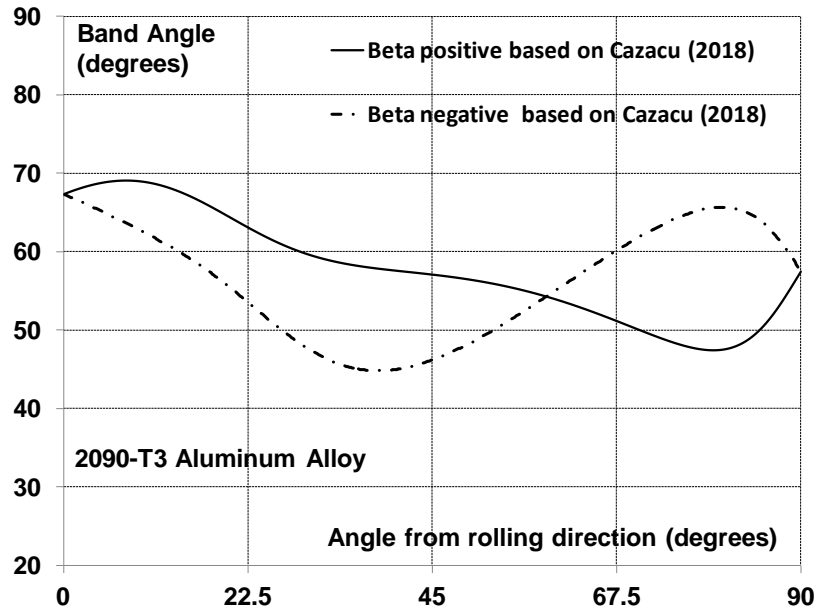


Fig. 9: Predicted values of the necking angles according to the Cazacu (2018) yield criterion as a function of the specimen orientation with respect to RD for the AA 2090-T3 alloy.

6. Conclusions

For predictions of ductile deformation and failure constitutive descriptions should be well-defined for any loadings in the 6-D stress space. Key in these formulations is the description of the onset of plastic deformation. The most widely used non-quadratic yielding descriptions are expressed in terms of principal values of the stress deviator and respectively its mapping through 4-th order anisotropic tensors. Parameterizations of these models are very difficult and often unreliable. Additionally, the quality of F.E. predictions using these models are strongly dependent on the quality of the parameterization, which in some cases prevent these models from being used. One of the reasons for these difficulties is that parameterizations are not unique. The parametrizations depend on the type and extent of data used as input. Since for most advanced anisotropic models, expressions of the parameters in terms of mechanical properties are not known, the choice of the parameterization for a specific application is strongly dependent on the experience of the analyst. Moreover, mechanical data is readily available only for materials in sheet form. Therefore, it is impossible to determine the parameters associated to the out-of-the sheet plane properties, and when conducting 3-D simulations additional choices related to the values of these parameters need to be made. Consequently, the analysis of the capabilities of these advanced models to describe the anisotropy in stresses and strain ratios is generally done for a single symmetry plane, and very little is known on the quality of the predictions for multi-axial loadings.

As part of this grant, the PI has deduced new mathematical results that have enabled to recognize that all the widely used isotropic models are in fact homogeneous polynomials of J_2 and J_3 , the invariants of the stress deviator. This had very important implications, one of them being the possibility to obtain explicit expressions in terms of the components of the stress tensor of orthotropic yield functions. The new explicit expressions in terms of stress components for

isotropic and orthotropic yield surfaces obtained lead to the understanding of what these models do for general loadings, their respective merits and limitations. Specifically, new explicit expressions were obtained for the most widely used non-quadratic anisotropic formulations, which are part of the material libraries of both commercial and government codes (e.g. LS-Dyna, Sierra). These new mathematical representations enabled to reveal that for Yld91 and KB93 models for both FCC and BCC materials the anisotropy coefficients are not independent. Most importantly, for the first-time analytical expressions of these coefficients were derived in terms of mechanical properties. This enabled to better understand these models. It was shown that the mathematical form of these yield surfaces imposes very specific relations between mechanical properties. Namely, according to either model the Lankford coefficients and yield stresses along any two orthotropy directions are not independent. It was also put into evidence that the mathematical form of these models leads to very specific couplings between shear and normal stresses at yielding. These new mathematical results obtained have also important implications in terms of determination of these anisotropy coefficients and the reliability of the obtained parameterizations. Specifically, the new representations in terms of stress components and generalized invariants greatly improve our understanding and analysis of anisotropic materials data sets, the correlations of these anisotropy coefficients to physical properties/microstructures, and ultimately the fidelity of numerical simulations. Interactions with DoD (AFRL/RW) and DOE (Sandia National Laboratory) researchers have been established in order to enable the transfer of these new findings. Moreover, as a first step the localization of deformation, which is a precursor to failure was investigated for uniaxial tensile loadings. It was demonstrated that the number of bands is deterministic being intrinsic to the type of symmetries of the material. Moreover, it was shown

that irrespective of the mathematical form of the plastic potential used for the description of plastic anisotropy, under uniaxial tension the bands angles can be determined analytically.

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