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Revisiting the Booker Quartic Dispersion Relation

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The Booker quartic is a dispersion relation that is important for propagation / reflection near a spitze. Although it was first developed in the 1930's, with recent interest in computational ray tracing, the quartic has seen a resurgence of use and importance. In *The propagation of radio waves*, K.G. Budden gives an equation for the Booker quartic in general coordinates that is different than what is given in his previous book, *Radio Waves in the Ionosphere*. We rederive this formulation following a slightly different approach and note an error in Budden's most recent representation of the formula that manifests in the absence of an extra term. Qualitatively, we show that this term does make a difference in general coordinates but not for the usual convention of propagation in the x – z plane. Our approach involves detailed derivations of his (3.55) and (6.17) which are not included in the text in which we also correct a typographical error in the former. We then proceed to discus the quantitative repercussions of the more serious error.

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CONTENTS

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REVISITING THE BOOKER QUARTIC DISPERSION RELATION

1. BACKGROUND AND USEFUL NOTATION

The formula known today as the Booker quartic that describes the propagation of a radio wave incident obliquely upon a slowly varying plane-stratified ionosphere in terms of the vertical component of the phase-propagation vector was originally developed by H.G. Booker in his seminal papers [1–3]. Specifically, although the notation is slightly different, the form of the quartic to be used in a general coordinate system (assuming z is still vertical but not constricting the propagation to the $x-z$ plane) is given by (3) and (17)-(21) in [3]. This form can be derived from the well-known Appleton-Hartree equation, as seen in [4]. In his publications, K.G. Budden uses a different notation, and because this paper's concern is with the equation given by Budden, we will use his notation, specifically that of his landmark book *The propagation of radio waves* [5].

The full Booker Quartic is an electromagnetic propagation dispersion relation that accounts for the collisional plasma effects on the electromagnetic wave in the presence of a magnetic field in the complex domain. The Booker Quartic reduces to the classical Appleton-Hartree dispersion relation in the real domain. It specifically handles magneto-ionic conditions when the wave propagation vector is perpendicular to the magnetic field that the Appleton-Hartree cannot in the real domain. This condition frequently occurs at high latitude for over the horizon high-frequency radio wave propagation where the earth's magnetic field is generally nearly vertical, but it also occurs at low and mid-latitude for near-vertical incidence propagation. Mathematically this condition is a cusp with a discontinuity in the spatial derivative, but in the ray tracing literature, it is called a spitze, from the German word meaning peak or point. In a near-realistic ionosphere with very slowly varying gradients this condition is a singularity. In the presence of field aligned ionospheric irregularities at scales comparable to the electromagnetic wave, this condition is met for a finite cone of incident electromagnetic waves where complete backscatter can occur. In order to be able to accurately model ionospheric ray behavior, especially in the HF (high frequency) band, it is important that a ray tracing code contain the Booker Quartic in a generalized coordinate system and that the mathematics representing the quartic is correct. It is with this topic and its application to over-the-horizon radar system performance prediction in mind that this report is written.

Let us approximate the ionosphere by a number of thin, discrete, horizontal strata, in each of which the medium is homogeneous. At each boundary between layers, an incident wave is partially reflected and partially transmitted. Similarly, each of those waves are partially reflected and partially transmitted when they reach the next boundary. Also, the ionosphere is birefringent, creating ordinary and extraordinary waves with each new transmission. Accounting for all of the waves via superposition, in each stratum there are two obliquely upgoing and two obliquely downgoing waves. Let us consider a plane wave incident on the ionosphere from below. Assuming the standard Cartesian coordinate system convention where z is aligned with the vertical axis, let θ be the angle that this plane wave makes with the vertical. We break the wave

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normal into its directional components, introducing the notation convention $C = \cos\theta$ and $S_1^2 + S_2^2 = \sin^2\theta$. Therefore each field component describing the wave before it reaches the ionosphere contains the factor

$$e^{-ik(S_1 x + S_2 y + Cz)}. \tag{1}$$

In the arbitrarily chosen r^{th} stratum, let us look at one of the four waves. If the refractive index for this wave in this particular stratum is denoted as n_r, each field component of the wave in this stratum will contain the factor

$$e^{-ik(S_1 x + S_2 y + n_r(\cos\theta_r) z)}. \tag{2}$$

Note that the choice of coordinates allows for boundary conditions such that only the z-components of the fields differ between the strata. Due to Snell's Law, $n_r \sin\theta_r = \sqrt{S_1^2 + S_2^2} = \sin\theta$, but the z-component of (2) cannot be determined as easily. n_r and θ_r are dependent on each other, but they are both unknown. To simplify things, denote $q = n_r \cos\theta_r$. Therefore,

$$n_r^2 = S_1^2 + S_2^2 + q^2, \tag{3}$$

and (2) becomes $e^{-ik(S_1 x + S_2 y + qz)}$. The four waves in each stratum all take this form, and thus there are four different possible values for q. These values are the roots of the Booker quartic.

The Booker quartic can be written in the form

$$F(q) \equiv \alpha q^4 + \beta q^3 + \gamma q^2 + \delta q + \epsilon = 0. \tag{4}$$

To understand where this formula comes from and what the coefficients are, we must first revisit more of the notation used by Budden. The capital letters X, Y, and Z are all used, but they have nothing to do with the x, y, and z coordinates. Because the ionosphere can be approximated as a cold plasma, an important quantity is the Langmuir frequency, also called the angular plasma frequency, ω_N. Knowing that the effect of the ions on radio waves is negligible, we can define the Langmuir frequency in SI units as

$$\omega_N = \sqrt{\frac{N_e e^2}{\varepsilon_0 m_e}}, \tag{5}$$

where N_e is the electron number density, e is the charge of an electron, m_e is the mass of an electron, and ε_0 is the permittivity of free space. It commonly appears in the equations in a very specific way, and we denote that as the quantity X, which is defined as

$$X \equiv \frac{\omega_N^2}{\omega^2} = \frac{N_e e^2}{\varepsilon_0 m_e \omega^2}, \tag{6}$$

where ω is the angular frequency of the wave. When a damping force is included, the equation of motion for an electron with displacement (from its position when no field is present)

$$\boldsymbol{r} = \Re\left(\boldsymbol{R} e^{i\omega t}\right), \tag{7}$$

where $\boldsymbol{R}$ is some complex vector, is

$$\boldsymbol{E}e = m_e\frac{\partial^2\boldsymbol{r}}{\partial t^2} + m_e\nu\frac{\partial\boldsymbol{r}}{\partial t}, \tag{8}$$

where ν is the electron-neutral and/or the electron-ion collision frequency. Replacing ν by the effective collision frequency ν_{eff}, (8) can be multiplied by $N_e e$ to give the relation

$$\boldsymbol{E}N_e e^2 = -\omega^2 m_e\left(1 - i\frac{\nu_{\text{eff}}}{\omega}\right)\boldsymbol{P}. \tag{9}$$

Here, $\boldsymbol{P}$ is the electric polarization vector. For more information on this switch to the effective collision frequency, see [6]. Using (6), we can solve for $\boldsymbol{P}$:

$$\boldsymbol{P} = -\varepsilon_0\frac{X}{1 - iZ}\boldsymbol{E}, \tag{10}$$

where

$$Z \equiv \frac{\nu_{\text{eff}}}{\omega}. \tag{11}$$

However, the term in the denominator of (10) is another quantity that appears frequently, so we will relabel it as

$$U \equiv 1 - iZ. \tag{12}$$

It is important to note that (8) does not take the Earth's magnetic field into account. Therefore, the correct relation is

$$\boldsymbol{E}e + e\frac{\partial\boldsymbol{r}}{\partial t}\times\boldsymbol{B} = m\frac{\partial^2\boldsymbol{r}}{\partial t^2} + m_e\nu\frac{\partial\boldsymbol{r}}{\partial t}, \tag{13}$$

where $\boldsymbol{B}$ is the constant magnetic induction of the Earth's field. Similarly to the derivation of (9), this equation gives us the relation

$$\frac{N_e e^2}{m_e\omega^2}\boldsymbol{E} + \frac{ie}{m_e\omega}\boldsymbol{P}\times\boldsymbol{B} = -\boldsymbol{P}\left(1 - iZ\right). \tag{14}$$

Using the same lowercase convention as in (7), the electromagnetic field in a plasma is governed by the four Maxwell equations:

$$\nabla\cdot\boldsymbol{d} = 0, \tag{15}$$

$$\nabla\cdot\boldsymbol{b} = 0, \tag{16}$$

$$\nabla\times\boldsymbol{e} = -\frac{\partial\boldsymbol{b}}{\partial t} = -\mu_0\frac{\partial\boldsymbol{h}}{\partial t}, \tag{17}$$

and

$$\nabla \times \boldsymbol{h} = \frac{\partial \boldsymbol{d}}{\partial t}. \tag{18}$$

Due to the fact that all the complex fields contain the same factor of $e^{i\omega t}$, we follow the standard practice (as does Budden) of using the uppercase letters with the implication that the $e^{i\omega t}$ is always present but the real part of the term is what is being used. That being said, we combine this with the convention

$$\mathscr{H} = Z_0 \boldsymbol{H}, \tag{19}$$

where

$$Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \tag{20}$$

is known as the characteristic impedance of free space, to write the third and fourth Maxwell equations, the Faraday law of induction and Ampère's law, respectively, as

$$\nabla \times \boldsymbol{E} = -ik\mathscr{H} \tag{21}$$

and

$$\nabla \times \mathscr{H} = i\frac{k}{\varepsilon_0}\boldsymbol{D}. \tag{22}$$

The μ_0 in (20) denotes the permeability of free space.

In (14) we see one more frequently occurring quantity that can be given new notation:

$$\boldsymbol{Y} = \frac{e\boldsymbol{B}}{m_e\omega}. \tag{23}$$

Thus, (14) becomes

$$-\varepsilon_0 X\boldsymbol{E} = U\boldsymbol{P} + i\boldsymbol{P} \times \boldsymbol{Y}. \tag{24}$$

Although, unlike X, Z, and U, $\boldsymbol{Y}$ is a vector, but its magnitude is not without physical significance:

$$Y = |\boldsymbol{Y}| = \frac{\omega_H}{\omega}, \tag{25}$$

where $\omega_H = \left|\frac{e\boldsymbol{B}}{m_e}\right|$ is the angular electron gyro-frequency, also known as the cyclotron frequency. Finally, the directional cosines of $\boldsymbol{Y}$ are denoted as l_x, l_y, and l_z so that, if Θ is the angle between the wave normal and $\boldsymbol{Y}$,

$$n_r \cos\Theta = l_x S_1 + l_y S_2 + l_z q. \tag{26}$$

2. DERIVATIONS OF THE BOOKER QUARTIC

Now that all of the useful notation is introduced, we return our attention to the Booker quartic as given in (4). In Section 6.3 of [5], Budden presents two different methods to determine the coefficients of the quartic. We will take a closer look at each of those methods and show that the formula Budden gives for one of the coefficients, namely γ, is wrong. It should be noted that in what can be considered the precursor to this book, namely [7], the coefficient is given correctly in (13.14). However, since [5] is much more widely used, it seemed the issue is worthy of attention. In fact, the importance of this book to researchers in the field of radio wave propagation over the past generation cannot be overstated. That makes it all the more imperative that any errors be addressed so that they are not promulgated throughout the citing literature such as [8], where the incorrect value of γ surfaces in (4.9). K.G. Budden is no longer with us, but we are confident he would have wanted to ensure that what he wrote and presented as fact was indeed true. It is our hope that this paper will remove a small blemish on his legacy and ensure his work continues to be the valuable resource it is today.

2.1 Method 1: The Dispersion Relation

2.1.1 The Permittivity Tensor

In a homogeneous electron plasma such as how the ionosphere is approximated in this paper, it is crucial to know the refractive indices of the medium. Because each refractive index is the square of a corresponding relative permittivity ε_r, we can use the constitutive relation

$$\boldsymbol{D} = \varepsilon_0 \boldsymbol{\varepsilon}_r \boldsymbol{E}, \tag{27}$$

where the displacement field $\boldsymbol{D}$ is defined by

$$\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}. \tag{28}$$

The tensor $\boldsymbol{\varepsilon}_r$ contains all of the relative permittivities ε_r. It is common in the literature to drop the r, and that convention will be followed here, as well. In addition, it should be noted that any mention of the permittivity tensor is really discussing the *relative* permittivity tensor.

Recall that (24) is a vector equation and can therefore be written as

$$-\varepsilon_0 X \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = \begin{pmatrix} U & iYl_z & -iYl_y \\ -iYl_z & U & iYl_x \\ iYl_y & -iYl_x & U \end{pmatrix} \begin{pmatrix} P_x \\ P_y \\ P_z \end{pmatrix}. \tag{29}$$

In Section 3.10 of [5], Budden simplifies things by choosing the z axis to be parallel to $\boldsymbol{Y}$. In (29), this is equivalent to making $l_x = l_y = 0$ and $l_z = 1$. He proceeds to find the permittivity tensor for this special case and concludes the section by giving it for two more general cases while not providing the derivation: one where $\boldsymbol{Y}$ is in the $x - z$ plane at an angle Θ to the z axis (where $\tan\Theta > 0$), and the most general case, where all that can be said about $\boldsymbol{Y}$ is that it has direction cosines l_x, l_y, and l_z. It is the latter case that we will derive here, showing that the method Budden used for the $z \parallel \boldsymbol{Y}$ case also works for the general one. In what follows, equation numbers in the form (#.#) refer to those in [5].

We start with the matrix given in (29). The characteristic polynomial for the matrix is

$$0 = (U - \lambda)^3 - \left(l_x^2 + l_y^2 - l_z^2\right) Y^2 (U - \lambda), \tag{30}$$

the roots of which provide the eigenvalues

$$\lambda_1 = U + \sqrt{l_x^2 + l_y^2 + l_z^2} Y, \quad \lambda_2 = U - \sqrt{l_x^2 + l_y^2 + l_z^2} Y, \quad \text{and } \lambda_3 = U. \tag{31}$$

This polynomial is found the standard way, setting

$$\det \begin{pmatrix} \lambda - U & iYl_z & -iYl_y \\ -iYl_z & \lambda - U & iYl_x \\ iYl_y & -iYl_x & \lambda - U \end{pmatrix} = 0. \tag{32}$$

Due to their roles as direction cosines, the variables l_x, l_y, and l_z are subject to the important property

$$l_x^2 + l_y^2 + l_z^2 = 1. \tag{33}$$

This relationship is used extensively in the derivation of (3.55). In fact, it shows that the eigenvalues (31) are equivalent to $U + Y$, $U - Y$, and U, which are the eigenvalues for the simpler case that Budden covers. However, the same cannot be said about the eigenvectors. When λ_1 is substitued into the matrix in (32) it has the reduced row echelon form of

$$\begin{pmatrix} 1 & 0 & \frac{l_x l_z + i l_y}{1 - l_z^2} \\ 0 & 1 & \frac{l_y l_z - i l_x}{1 - l_z^2} \\ 0 & 0 & 0 \end{pmatrix},$$

and arbitrarily setting the third component to zero, we have the first eigenvector:

$$\boldsymbol{v}_1 = \begin{pmatrix} \frac{-l_x l_z - i l_y}{1 - l_z^2} \\ \frac{-l_y l_z + i l_x}{1 - l_z^2} \\ 1 \end{pmatrix}. \tag{34a}$$

Similarly, substituting the second eigenvalue in the matrix, we find it has the reduced row echelon form of

$$\begin{pmatrix} 1 & 0 & \frac{l_x l_z - i l_y}{1 - l_z^2} \\ 0 & 1 & \frac{l_y l_z + i l_x}{1 - l_z^2} \\ 0 & 0 & 0 \end{pmatrix}.$$

Again, setting the third component to zero, we have the second eigenvector:

$$\boldsymbol{v}_2 = \begin{pmatrix} \frac{-l_z l_z - i l_y}{1 - l_z^2} \\ \frac{-l_y l_z - i l_x}{1 - l_z^2} \\ 1 \end{pmatrix}. \tag{34b}$$

The reduced row echelon form of the matrix that results from substituting the third eigenvalue in the matrix is not needed; assuming a nonzero l_z, it is easy to see that the eigenvector is

$$\boldsymbol{v}_3 = \begin{pmatrix} \frac{l_x}{l_z} \\ \frac{l_y}{l_z} \\ 1 \end{pmatrix}. \tag{34c}$$

The simplicity makes it irrelevant in obtaining Budden's (3.40), but for our purposes, it is important to remember that for complex vectors, the length of the vector is $||\boldsymbol{v}|| = \sqrt{\boldsymbol{v}^*\boldsymbol{v}}$, where $\boldsymbol{v}^*$ is the complex conjugate of $\boldsymbol{v}$. Keeping this and (33) in mind, we find that both $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ have lengths $\sqrt{\frac{2}{1-l_z^2}}$, and $\boldsymbol{v}_3$ has length $|l_z|^{-1}$ and is unity if l_z happens to be equal to zero. The sign of l_z is unknown, but we can choose it to be positive without loss of generality; otherwise the matrices that are analogous to (3.43) will be slightly different than what is presented here, but the outcome will remain the same.

Knowing the lengths of vectors (34), we conclude the normalized eigenvectors are

$$\boldsymbol{u}_1 = \frac{1}{\sqrt{2\left(1 - l_z^2\right)}} \begin{pmatrix} -l_x l_z - i l_y \\ -l_y l_z + i l_x \\ 1 - l_z^2 \end{pmatrix}, \tag{35a}$$

$$\boldsymbol{u}_2 = \frac{1}{\sqrt{2\left(1 - l_z^2\right)}} \begin{pmatrix} -l_x l_z + i l_y \\ -l_y l_z - i l_x \\ 1 - l_z^2 \end{pmatrix}, \tag{35b}$$

and

$$\boldsymbol{u}_3 = \begin{pmatrix} l_x \\ l_y \\ l_z \end{pmatrix}. \tag{35c}$$

Continuing Budden's argument, we want to apply a matrix transformation on the matrix in (24) so that the the matrix is diagonal. Just like for (3.41), the contravariant components E_1, E_2, E_3 of the vector $\boldsymbol{E}$ in the new axis system are a linear combination of E_x, E_y, E_z and are found by multiplication by a transforming matrix $\mathscr{U}$ thus

$$\begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} = \mathscr{U} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}, \tag{36}$$

and the same applies for any other vector, including $\boldsymbol{P}$. Now (29) may be multiplied on the left by $\mathscr{U}$ to give

$$-\varepsilon_0 X \mathscr{U} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = \mathscr{U} \begin{pmatrix} U & iYl_z & -iYl_y \\ -iYl_z & U & iYl_x \\ iYl_y & -iYl_x & U \end{pmatrix} \mathscr{U}^{-1} \mathscr{U} \begin{pmatrix} P_x \\ P_y \\ P_z \end{pmatrix}. \tag{37}$$

The matrix $\mathscr{U}^{-1}$ is comprised of the normalized eigenvectors so that

$$\mathscr{U}^{-1} = \frac{1}{\sqrt{2\left(1-l_z^2\right)}} \begin{pmatrix} -l_x l_z - i l_y & -l_x l_z + i l_y & l_x \sqrt{2\left(1-l_z^2\right)} \\ -l_y l_z + i l_x & -l_y l_z - i l_x & l_y \sqrt{2\left(1-l_z^2\right)} \\ 1-l_z^2 & 1-l_z^2 & l_z \sqrt{2\left(1-l_z^2\right)} \end{pmatrix}, \tag{38}$$

and therefore the transforming matrix $\mathscr{U}$ is given by

$$\mathscr{U} = \sqrt{2\left(1-l_z^2\right)} \begin{pmatrix} \frac{-l_x l_z + i l_y}{2\left(1-l_z^2\right)} & \frac{l_x l_y - i l_z}{2\left(l_x l_z + i l_y\right)} & \frac{1}{2} \\ \frac{-l_x l_z - i l_y}{2\left(1-l_z^2\right)} & \frac{-l_y l_z + i l_x}{2\left(1-l_z^2\right)} & \frac{1}{2} \\ \frac{l_x}{\sqrt{2\left(1-l_z^2\right)}} & \frac{l_y}{\sqrt{2\left(1-l_z^2\right)}} & \frac{l_z}{\sqrt{2\left(1-l_z^2\right)}} \end{pmatrix}. \tag{39}$$

Although it is not as easy to see as in Budden's simpler case, our $\mathscr{U}$ is also a unitary matrix.

Budden's (3.46)-(3.52) remain valid for the general case we are studying. The calculation of the eigenmatrix, seen in his (3.46), heavily involves property (33) and its corollary

$$1 - l_z^2 = l_x^2 + l_y^2, \tag{40}$$

but the diagonalization does work. As with the simpler case, (37) can be used to determine the elements of the diagonalized susceptibility tensor, $\frac{-X}{U+Y}$, $\frac{-X}{U-Y}$, and $\frac{-X}{U}$. Combining this with (27) and (28), we arrive at the principal axis values of the elements of the diagonalized electric permittivity tensor:

$$\varepsilon_1 = 1 - \frac{X}{U+Y}, \quad \varepsilon_2 = 1 - \frac{X}{U-Y}, \quad \text{and } \varepsilon_3 = 1 - \frac{X}{U}. \tag{41}$$

We shall delay further analysis using these values until Section 2.1.3. Returning to our present discussion, the equations may now be transformed back to give us the permittivity matrix $\boldsymbol{\varepsilon}$ in the general cartesian coordinate system using

$$\begin{aligned} \boldsymbol{\varepsilon} =& \mathscr{U}^{-1} \begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix} \mathscr{U} \\ =& \begin{pmatrix} -l_x l_z - i l_y & -l_x l_z + i l_y & l_x \sqrt{2\left(1-l_z^2\right)} \\ -l_y l_z + i l_x & -l_y l_z - i l_x & l_y \sqrt{2\left(1-l_z^2\right)} \\ 1-l_z^2 & 1-l_z^2 & l_z \sqrt{2\left(1-l_z^2\right)} \end{pmatrix} \cdot \begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix} \\ & \cdot \begin{pmatrix} \frac{-l_x l_z + i l_y}{2\left(1-l_z^2\right)} & \frac{l_x l_y - i l_z}{2\left(l_x l_z + i l_y\right)} & \frac{1}{2} \\ \frac{-l_x l_z - i l_y}{2\left(1-l_z^2\right)} & \frac{-l_y l_z + i l_x}{2\left(1-l_z^2\right)} & \frac{1}{2} \\ \frac{l_x}{\sqrt{2\left(1-l_z^2\right)}} & \frac{l_y}{\sqrt{2\left(1-l_z^2\right)}} & \frac{l_z}{\sqrt{2\left(1-l_z^2\right)}} \end{pmatrix}. \end{aligned} \tag{42a}$$

When multiplied out, the result becomes

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \frac{1}{2}(\varepsilon_1+\varepsilon_2) - l_x^2 G & -l_x l_y G + \frac{1}{2} i l_z (\varepsilon_1-\varepsilon_2) & -l_x l_z G - \frac{1}{2} i l_y (\varepsilon_1-\varepsilon_2) \\ -l_x l_y G - \frac{1}{2} i l_z (\varepsilon_1-\varepsilon_2) & \frac{1}{2}(\varepsilon_1+\varepsilon_2) - l_y^2 G & -l_y l_z G + \frac{1}{2} i l_x (\varepsilon_1-\varepsilon_2) \\ -l_x l_z G + \frac{1}{2} i l_y (\varepsilon_1-\varepsilon_2) & -l_y l_z G - \frac{1}{2} i l_x (\varepsilon_1-\varepsilon_2) & \frac{1}{2}(\varepsilon_1+\varepsilon_2) - l_z^2 G \end{pmatrix}, \tag{42b}$$

where we have followed Budden's notation simplification of

$$G = \frac{1}{2}(\varepsilon_1+\varepsilon_2) - \varepsilon_3. \tag{43}$$

One can see that (42b) agrees with (3.55) except for a typographical error in the $(3,2)$ term where the i has been left out. With (3.55) verified we may now return to the main focus of this paper.

2.1.2 Using the Permittivity to Derive the Booker Quartic

Budden uses (3.54), the case where $\boldsymbol{Y}$ is in the $x-z$ plane, to proceed further, but because (3.55) is more robust, it is what we will continue to use. It is implied in (2) that the refractive index n_r of one of the waves discussed in Section 1 is actually a vector. While that is true, for our purposes the values of the components of that vector are irrelevant; we only care about its length. Therefore, we may temporarily assume that the wave normal is parallel to the z axis. Under this assumption, the Maxwell equations can be used to show that in a homogeneous cold magnetoplasma one can determine three important relations:

$$D_x = \varepsilon_0 n^2 E_x, \quad D_y = \varepsilon_0 n^2 E_y, \quad \text{and } D_z = 0, \tag{44}$$

see Section 4.1 in [5]. Substituting these into (28), we arrive at (3.60), namely

$$\left.\begin{aligned} \left(\varepsilon_{xx} - n^2\right) E_x + \varepsilon_{xy} E_y + \varepsilon_{xz} Ez &= 0 \\ \varepsilon_{yx} E_x + \left(\varepsilon_{yy} - n^2\right) E_y + \varepsilon_{yz} E_z &= 0 \\ \varepsilon_{zx} E_x + \varepsilon_{zy} E_y + \varepsilon_{zz} E_z &= 0 \end{aligned}\right\}. \tag{45}$$

For there to be a solution to this set of equations, it is necessary that the determinant of the coefficients of E_x, E_y, and E_z be zero. While Budden substitutes the values of $\boldsymbol{\varepsilon}$ in the the form of (3.54), we shall use (42b), giving

$$\begin{vmatrix} \frac{1}{2}(\varepsilon_1+\varepsilon_2) - l_x^2 G - n^2 & -l_x l_y G + \frac{1}{2} i l_z (\varepsilon_1-\varepsilon_2) & -l_x l_z G - \frac{1}{2} i l_y (\varepsilon_1-\varepsilon_2) \\ -l_x l_y G - \frac{1}{2} i l_z (\varepsilon_1-\varepsilon_2) & \frac{1}{2}(\varepsilon_1+\varepsilon_2) - l_y^2 G - n^2 & -l_y l_z G + \frac{1}{2} i l_x (\varepsilon_1-\varepsilon_2) \\ -l_x l_z G + \frac{1}{2} i l_y (\varepsilon_1-\varepsilon_2) & -l_y l_z G - \frac{1}{2} i l_x (\varepsilon_1-\varepsilon_2) & \frac{1}{2}(\varepsilon_1+\varepsilon_2) - l_z^2 G \end{vmatrix} = 0. \tag{46}$$

Multiplying this out gives a quadratic equation for n^2 of the form

$$f(n^2) \equiv a n^4 - 2 b n^2 + c = 0, \tag{47a}$$

where

$$a = \frac{1}{2}\left(\varepsilon_1 + \varepsilon_2\right) - l_z^2 G, \tag{47b}$$

$$b = \frac{1}{4}\left(\varepsilon_1 + \varepsilon_2\right)^2 - \frac{1}{4}\left(1 + l_z^2\right)\left(\varepsilon_1 + \varepsilon_2\right) G - \frac{1}{8}\left(l_x^2 + l_y^2\right)\left(\varepsilon_1 - \varepsilon_2\right)^2, \tag{47c}$$

and

$$c = \frac{1}{8}\left(\varepsilon_1 + \varepsilon_2\right)^2 - \varepsilon_1\varepsilon_2 G - \frac{1}{8}\left(\varepsilon_1^2 - \varepsilon_2^2\right)\left(\varepsilon_1 - \varepsilon_2\right). \tag{47d}$$

Our goal is to substitute n_r into (47) and use (3) to get a fourth degree polynomial $F(q)$ with respect to q, the Booker quartic. However, a quick glance will reveal that the dependence of q in n_r is only in its square. Thus, just substituting (3) into (47) will result in a quadratic equation for q^2. We know the Booker quartic has nonzero first and third order terms, so from where do they come? To answer that, we note that both (47b) and (47c) have terms that are dependent on the directional cosines of $\boldsymbol{Y}$. We also recall property (40) and can factor out the l_z^2 terms, rewriting (47) as

$$\begin{aligned} f(n^2) \equiv& \frac{1}{2}\left(\varepsilon_1 + \varepsilon_2\right) n^4 \\ &+ \left(-Gn^2 + \frac{1}{2}\left(\varepsilon_1 + \varepsilon_2\right) G - \frac{1}{4}\left(\varepsilon_1 - \varepsilon_2\right)^2\right) n^2 l_z^2 \\ &- \left(\frac{1}{2}\left(\varepsilon_1 + \varepsilon_2\right)^2 - \frac{1}{2}\left(\varepsilon_1 + \varepsilon_2\right) G - \frac{1}{4}\left(\varepsilon_1 - \varepsilon_2\right)^2\right) n^2 \\ &+ \frac{1}{8}\left(\varepsilon_1 + \varepsilon_2\right)^2 - \varepsilon_1\varepsilon_2 G - \frac{1}{8}\left(\varepsilon_1^2 - \varepsilon_2^2\right)\left(\varepsilon_1 - \varepsilon_2\right) \\ =& 0. \end{aligned} \tag{48}$$

We would like to find $f(n_r^2)$, and in the form of (48), we can see that there are terms of $f(n_r^2)$ that share the common factor $n_r^2 l_z^2$. In Section 2.1.1 we discussed how Budden solved for the permittivity tensor in the case where $\boldsymbol{Y}$ is in the $x - z$ plane at an angle of Θ with the z-axis. This allows one to make the simplifications $l_x = \sin\Theta$, $l_y = 0$, and $l_z = \cos\Theta$, and the formulas can all be written in terms of those trigonometry functions. We do not impose any assumptions on the direction of $\boldsymbol{Y}$, but we can still use a similar convention. If Θ is the angle between the z-axis and $\boldsymbol{Y}$, the relationship $l_z = \cos\Theta$ remains true. However, because $\boldsymbol{Y}$ makes an arbitrary angle with the $x - z$ plane, its projection onto the $x - y$ plane potentially has both x and y components. In order to satisfy (33), we must use the relationship

$$l_x^2 + l_y^2 = \sin^2\Theta. \tag{49}$$

Unlike the simpler case, we cannot determine $\sin\Theta$, but we do know its square. The important thing, though, is that we can use

$$l_z = \cos\Theta. \tag{50}$$

Thus, the factor $n_r^2 l_z^2$ can be written as $n_r^2 \cos^2\Theta$, the square of a quantity we have seen before in (26). The dependence of $n_r \cos\Theta$ on q is linear, and the square of it retains a linear term, as well. This is where the first and third order terms of the Booker quartic originate.

Before we proceed, there are a few ways to clean (48) up a bit. First, consider the zeroth order terms of the polynomial with respect to n^2 that is found inside the parentheses of the second line. Substituting (43) into that polynomial and multiplying everything out, we can rewrite it as

$$\frac{1}{2}\left(\varepsilon_1+\varepsilon_2\right)G-\frac{1}{4}\left(\varepsilon_1-\varepsilon_2\right)^2=\varepsilon_1\varepsilon_2-\frac{1}{2}\varepsilon_3\left(\varepsilon_1+\varepsilon_2\right)\equiv -J, \tag{51}$$

where we introduce the other variable defined in (3.51):

$$J=\frac{1}{2}\varepsilon_3\left(\varepsilon_1+\varepsilon_2\right)-\varepsilon_1\varepsilon_2. \tag{52}$$

Second, the same procedure can be used to simplify the term inside the parentheses in the third line:

$$\frac{1}{2}\left(\varepsilon_1+\varepsilon_2\right)^2-\frac{1}{2}\left(\varepsilon_1+\varepsilon_2\right)G-\frac{1}{4}\left(\varepsilon_1-\varepsilon_2\right)^2=\frac{1}{2}\varepsilon_3\left(\varepsilon_1+\varepsilon_2\right)+\varepsilon_1\varepsilon_2. \tag{53}$$

Similarly, the left hand side of the fourth line simplifies as well:

$$\frac{1}{8}\left(\varepsilon_1+\varepsilon_2\right)^3-\varepsilon_1\varepsilon_2 G-\frac{1}{8}\left(\varepsilon_1^2-\varepsilon_2^2\right)\left(\varepsilon_1-\varepsilon_2\right)=\varepsilon_1\varepsilon_2\varepsilon_3. \tag{54}$$

With these findings, we rewrite (48) as

$$f(n^2)\equiv\frac{1}{2}\left(\varepsilon_1+\varepsilon_2\right)n^4-\left(Gn^2+J\right)n^2 l_z^2-\left(\frac{1}{2}\varepsilon_3\left(\varepsilon_1+\varepsilon_2\right)+\varepsilon_1\varepsilon_2\right)n^2+\varepsilon_1\varepsilon_2\varepsilon_3=0. \tag{55}$$

Using (26), we are now ready to evaluate $f(n_r^2)$ in two steps. First, we write

$$\begin{aligned} f_c(n_r^2)\equiv&\frac{1}{2}\left(\varepsilon_1+\varepsilon_2\right)n_r^4-\left(Gn^2+J\right)\left(l_x S_1+l_y S_2+l_z q\right)\\ &-\left(\frac{1}{2}\varepsilon_3\left(\varepsilon_1+\varepsilon_2\right)+\varepsilon_1\varepsilon_2\right)n_r^2+\varepsilon_1\varepsilon_2\varepsilon_3=0, \end{aligned} \tag{56}$$

where the subscript c denotes that we have made the substitution of the cosine term, and second, we substitute (3) into $f_c(n_r^2)$ to achieve a new polynomial $F(q)$:

$$\begin{aligned} F(q)\equiv&\left(q^2+S_1^2+S_2^2\right)^2\left(\frac{1}{2}\left(\varepsilon_1+\varepsilon_2\right)\right)\\ &-\left(\left(S_1^2+S_2^2+q^2\right)G+J\right)\left(l_x S_1+l_y S_2+l_z q\right)\\ &-\left(q^2+S_1^2+S_2^2\right)\left(\frac{1}{2}\varepsilon_3\left(\varepsilon_1+\varepsilon_2\right)+\varepsilon_1\varepsilon_2\right)\\ &+\varepsilon_1\varepsilon_2\varepsilon_3=0. \end{aligned} \tag{57}$$

Written in the form of (4), and letting $S^2 = S_1^2 + S_2^2$, this becomes $F(q) \equiv \alpha q^4 + \beta q^3 + \gamma q^2 + \delta q + \epsilon = 0$, where

$$\left.\begin{aligned}
\alpha &= \frac{1}{2}(\varepsilon_1 + \varepsilon_2) - l_z^2 G, \\
\beta &= -2\left(l_x S_1 + l_y S_2\right) l_z G, \\
\gamma &= -\frac{1}{2}\varepsilon_3(\varepsilon_1 + \varepsilon_2) - \varepsilon_1\varepsilon_2 + S^2\left(\varepsilon_1 + \varepsilon_2 - l_z^2 G\right) - l_z^2 J - \left(l_x S_1 + l_y S_2\right)^2 G, \\
\delta &= -2\left(l_x S_1 + l_y S_2\right) l_z \left(J + S^2 G\right), \\
\epsilon &= \left(\varepsilon_1\varepsilon_2 - \frac{1}{2}(\varepsilon_1 + \varepsilon_2) S^2\right)\left(\varepsilon_3 - S^2\right) - \left(l_x S_1 + l_y S_2\right)^2\left(J + S^2 G\right).
\end{aligned}\right\} \tag{58}$$

This result matches (6.17) perfectly.

2.1.3 Where the Issue Arises

We have successfully derived the Booker quartic as given in (6.17) of [5]. This is a more general form of the quartic than the one Budden derives as it can allow for ions or velocity dependent electron collision frequencies. In order to see the problem with (6.16) we must use the specific values for the three permittivities given by (41). The values of certain repeating combinations of the permittivities are obtained from these:

$$\varepsilon_1 + \varepsilon_2 = 2 + \frac{2XU}{U^2 - Y^2} \quad \text{and} \quad \varepsilon_1\varepsilon_2 = \frac{U^2 - Y^2 - 2XU + X^2}{U^2 - Y^2}. \tag{59}$$

Also of note are

$$G = \frac{1}{2}(\varepsilon_1 + \varepsilon_2) - \varepsilon_3 = -\frac{XU}{U^2 - Y^2} + \frac{X}{U} = -\frac{XY^2}{U\left(U^2 - Y^2\right)} \tag{60}$$

and

$$\begin{aligned}
J =& \frac{1}{2}(\varepsilon_1 + \varepsilon_2)\varepsilon_3 - \varepsilon_1\varepsilon_2 \\
=& 1 - \frac{X}{U} - \frac{XU}{U^2 - Y^2} + \frac{X^2 U}{U^2 - Y^2} - \frac{U^2 - Y^2 - 2XU + X^2}{U^2 - Y^2} \\
=& \frac{XY^2}{U\left(U^2 - Y^2\right)},
\end{aligned} \tag{61}$$

which means for these permittivitites, $J = -G$. One can also see that

$$J + S^2 G = \left(1 - S_1^2 - S_2^2\right)\frac{XY^2}{U^2\left(U^2 - Y^2\right)}, \tag{62}$$

and, as seen below in (65), Budden uses the notation C^2 for what is contained within the parentheses. Out of the five coefficients of (58), the easiest to compute is β. Using (60), we get $\beta = 2\left(l_x S_1 + l_y S_2\right) l_z \frac{XY^2}{U\left(U^2 - Y^2\right)}$. This differs from (6.16β) only by the denominator, which illustrates the importance of remembering that

that Booker quartic is an equation and not an expression. Recall that the quartic comes from (46), where we set the determinant of the coefficient matrix from (45) to zero in order to find the nontrivial solution to the system of equations. The Booker quartic produces such a solution, albeit in an indirect way due to a change of variables. Because the determinant is set to zero, the quartic itself is equal to zero, and that is still the case for any constant multiple of the set of coefficients $\{\alpha, \beta, \gamma, \delta, \epsilon\}$. From our computed value of β, a logical multiple is $U\left(U^2 - Y^2\right)$. Including this multiple, the α, δ, and ϵ of (58) also become the α, δ, and ϵ of (6.16) where we leave it to the reader to verify. For the purposes of this paper, we *will* work out the coefficient γ.

In addition to those evaluated in (59), γ in (58) contains an expression that was previously computed within (61) to get J:

$$\frac{1}{2}\varepsilon_3\left(\varepsilon_1 + \varepsilon_2\right) = 1 - \frac{X}{U} - \frac{XU}{U^2 - Y^2} + \frac{X^2U}{U^2 - Y^2} = \frac{(U - X)\left(U^2 - Y^2 - XU\right)}{U\left(U^2 - Y^2\right)}. \tag{63}$$

Combining (59), (60), (61), and (63), we find

$$\begin{aligned}
\gamma =& -\frac{(U - X)\left(U^2 - Y^2 - XU\right)}{U\left(U^2 - Y^2\right)} - \frac{U^2 - Y^2 - 2XU + X^2}{U^2 - Y^2}S^2 + \left(2 - \frac{2XU}{U^2 - Y^2}\right)S^2 \\
&+ \left(\left(S^2 - 1\right)l_z^2 + \left(l_xS_1 + l_yS_2\right)^2\right)\frac{XY^2}{U\left(U^2 - Y^2\right)} \\
=&\frac{-2U^3 + 2UY^2 + 4XU^2 - XY^2 - 2X^2U}{U\left(U^2 - Y^2\right)} + \left(\frac{2U^3 - 2UY^2 - 2XU^2}{U\left(U^2 - Y^2\right)}\right)S^2 \\
&+ \left(\left(S^2 - 1\right)l_z^2 + \left(l_xS_1 + l_yS_2\right)^2\right)\frac{XY^2}{U\left(U^2 - Y^2\right)} \\
=&\frac{-2U^3 + 2UY^2 + 4XU^2 - 2XY^2 - 2X^2U}{U\left(U^2 - Y^2\right)} + \left(\frac{2U^3 - 2UY^2 - 2XU^2}{U\left(U^2 - Y^2\right)}\right)S^2 \\
&+ \left(1 + \left(S^2 - 1\right)l_z^2 + \left(l_xS_1 + l_yS_2\right)^2\right)\frac{XY^2}{U\left(U^2 - Y^2\right)}.
\end{aligned} \tag{64}$$

This can be simplified using Budden's notation

$$C^2 = 1 - S_1^2 - S_2^2 = 1 - S^2, \tag{65}$$

and, multiplying by $U\left(U^2 - Y^2\right)$ while referring to the product also as γ,

$$\begin{aligned}
\gamma =& \left(-2U^3 + 2UY^2 + 2XU^2\right)C^2 + 2XU^2 - 2XY^2 - 2X^2U \\
&+ XY^2\left(1 - l_z^2C^2 + \left(l_xS_1 + l_yS_2\right)^2\right) \\
=& -2U(U - X)\left(C^2U - X\right) + 2Y^2\left(C^2U - X\right) \\
&+ XY^2\left(1 - l_z^2C^2 + \left(l_xS_1 + l_yS_2\right)^2\right).
\end{aligned} \tag{66}$$

Here is where Budden makes the crucial mistake: what is printed as (6.16γ) is equivalent to erroneously expanding the $\left(l_x S_1 + l_y S_2\right)^2$ in the final term of (66) as $l_x^2 S_1^2 + l_y^2 S_2^2$. Curiously, in (6.16$\epsilon$) the same expression is left unexpanded, preventing the mistake in γ from being repeated there as well. Nevertheless, with the correct expansion in mind, it becomes apparent that (6.16) should really state

$$\left.\begin{aligned}
\alpha =& U\left(U^2 - Y^2\right) + X\left(Y^2 l_z^2 - U^2\right),\\
\beta =& 2\left(l_x S_1 + l_y S_2\right) l_z X Y^2,\\
\gamma =& -2U\left(U - X\right)\left(C^2 U - X\right) + 2Y^2\left(C^2 U - X\right)\\
& + XY^2\left(1 - l_z^2 C^2 + l_x^2 S_1^2 + l_y^2 S_2^2 + 2 l_x l_y S_1 S_2\right),\\
\delta =& -2C^2\left(l_x S_1 + l_y S_2\right) l_z X Y^2,\\
\epsilon =& \left(U - X\right)\left(C^2 U - X\right)^2 - C^2 Y^2\left(C^2 U - X\right) - \left(l_x S_1 + l_y S_2\right)^2 C^2 X Y^2.
\end{aligned}\right\} \tag{67}$$

2.1.4 Comparison to Budden's First Method

The procedure used in Sections 2.1.1 and 2.1.2 is a modification of the way that [5] derives the version of the Booker quartic given in (6.16) and corrected in our (67). We kept things as general as possible, deriving the second form of the quartic given in (6.17) – our (58), before using the specific values of the permittivities that describe the physical system being modeled to arrive at (67). This is somewhat the opposite approach Budden takes as he starts making geometrical modifications right from the very beginning. He simplifies (29) so that the z-axis is parallel to $\boldsymbol{Y}$ to arrive at a block diagonal permittivity matrix of

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \frac{1}{2}\left(\varepsilon_1 + \varepsilon_2\right) & \frac{1}{2} i\left(\varepsilon_1 - \varepsilon_2\right) & 0\\ -\frac{1}{2} i\left(\varepsilon_1 - \varepsilon_2\right) & \frac{1}{2}\left(\varepsilon_1 + \varepsilon_2\right) & 0\\ 0 & 0 & \varepsilon_3 \end{pmatrix}$$

and then mentions that a similar procedure could be done for the geometry of the physical problem at hand, where $\boldsymbol{Y}$ is in the $x - z$ plane at an angle of Θ to the z-axis. The result he obtains comes from making that assumption after (29) and then simplifying, although he chooses the aforementioned nicer looking case for demonstration purposes. With the geometrical framework in place he continues with his equivalent of (44)—(46). It is at this point where he inserts the permittivities given by (41) instead of saving them for the end as with our method. This allows for a direct computation of the Booker quartic with the coefficients (67) via the equivalent of (57):

$$\begin{aligned}
&(q^2 + S_1^2 + S_2^2)^2\left(U^2\left(U - X\right) - UY^2\right)\\
&\quad + \left(q^2 - C^2\right)\left(l_x S_1 + l_y S_2 + l_z q\right)^2 XY^2\\
&\quad - \left(q^2 + S_1^2 + S_2^2\right)\left(2U\left(U - X\right)^2 - 2Y^2\left(U - X\right) - XY^2\right)\\
&\quad + \left(U - X\right)^3 - Y^2\left(U - X\right) = 0.
\end{aligned} \tag{68}$$

Personal verification of this computation is what initially alerted us to the error in Budden's work. One can see the term in question arise in the second line of (68) from the multiplication of q^2 with the $\left(l_x S_1 + l_y S_2\right)^2$ that is part of the expansion of $\left(l_x S_1 + l_y S_2 + l_z q\right)^2$.

2.2 Method 2: The Direct Approach

While it is clear that (67) is what is really obtained from (68), having just been introduced to the concept of the Booker quartic, our initial assumption was that perhaps (6.16) is right and that the error lies in (68). While we have now shown that not to be the case, at the time it seemed prudent to look at the quartic from a different approach, and Budden provides such a method immediately following (6.17). In what follows, we reproduce this alternative method and fill in its details to demonstrate that (67) is correct.

Recall from (2) that each field component of a wave in the rth stratum of the inonosphere using the geometry we established will contain the factor

$$e^{-ik(S_1 x + S_2 y + qz)},$$

which is the spatial dependence of the waves. Selecting one such wave to examine, we see that

$$\frac{\partial}{\partial x} \equiv -ikS_1, \quad \frac{\partial}{\partial y} \equiv -ikS_2, \quad \text{and} \quad \frac{\partial}{\partial z} \equiv -ikq \tag{69}$$

for all field variables. Breaking (21) and (22) into their individual components produces

$$\begin{aligned} \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} &= -ik\mathscr{H}_x, & \frac{\partial \mathscr{H}_z}{\partial y} - \frac{\partial \mathscr{H}_y}{\partial z} &= i\frac{k}{\varepsilon_0}D_x, \\ \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} &= -ik\mathscr{H}_y, \text{ and } & \frac{\partial \mathscr{H}_x}{\partial z} - \frac{\partial \mathscr{H}_z}{\partial x} &= i\frac{k}{\varepsilon_0}D_y, \\ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} &= -ik\mathscr{H}_z & \frac{\partial \mathscr{H}_y}{\partial x} - \frac{\partial \mathscr{H}_x}{\partial y} &= i\frac{k}{\varepsilon_0}D_z. \end{aligned} \tag{70}$$

Using (2) and (69), formulae (70) can be written using matrix notation as

$$\mathbf{\Gamma}\boldsymbol{E} = \boldsymbol{\mathscr{H}} \quad \text{and} \quad \mathbf{\Gamma}\boldsymbol{\mathscr{H}} = -\left(\boldsymbol{I} + \boldsymbol{M}\right)\boldsymbol{E} \tag{71}$$

where

$$\mathbf{\Gamma} = \begin{pmatrix} 0 & -q & S_2 \\ q & 0 & -S_1 \\ -S_2 & S_1 & 0 \end{pmatrix}, \tag{72}$$

and $\boldsymbol{M}$ is the susceptibility matrix given by

$$\boldsymbol{M} = -\frac{X}{U\left(U^2 - Y^2\right)} \begin{pmatrix} U^2 - l_x^2 Y^2 & -il_z YU - l_x l_y Y^2 & il_y YU - l_x l_z Y^2 \\ il_z YU - l_x l_y Y^2 & U^2 - l_y^2 Y^2 & -il_x YU - l_y l_z Y^2 \\ -il_y YU - l_x l_z Y^2 & il_x YU - l_y l_z Y^2 & U^2 - l_z^2 Y^2 \end{pmatrix}. \tag{73}$$

The common denominator for each entry of $\boldsymbol{M}$ should look familiar to the reader from the work in the previous sections. In fact, $\boldsymbol{M}$ is the matrix that appears in the solution of $\frac{1}{\varepsilon_0}\boldsymbol{P}$ in (29) multiplied by X. That is to say,

$$\frac{1}{\varepsilon_0}\boldsymbol{P} = -\frac{X}{U\left(U^2 - Y^2\right)}\boldsymbol{M}\boldsymbol{E}, \tag{74}$$

or, to put it another way,

$$\boldsymbol{M} = -X\begin{pmatrix} U & iYl_z & -iYl_y \\ -iYl_z & U & iYl_x \\ iYl_y & -iYl_x & U \end{pmatrix}^{-1} \tag{75}$$

when property (33) and its corollaries are kept in mind. Moving on, we can reduce (71) to one equation by eliminating $\mathscr{H}$:

$$\left(\boldsymbol{\Gamma}^2 + \boldsymbol{M} + \boldsymbol{I}\right)\boldsymbol{E} = 0. \tag{76}$$

In order for (76) to have a nontrivial solution, which we can assume it does, it must be true that $\det\left(\boldsymbol{\Gamma}^2 + \boldsymbol{M} + \boldsymbol{I}\right) = 0$. From (72) we determine that

$$\boldsymbol{\Gamma}^2 = \begin{pmatrix} -q^2 - S_2^2 & S_1S_2 & qS_1 \\ S_1S_2 & -q^2 - S_1^2 & qS_2 \\ qS_1 & qS_2 & -S_1^2 - S_2^2 \end{pmatrix}, \tag{77}$$

and combining this knowledge with (73) we see that finding a nontrivial solution to (76) is equivalent to solving

$$0 = \begin{vmatrix} -q^2 - S_2^2 - \frac{(U^2 - l_x^2Y^2)X}{U(U^2-Y^2)} + 1 & S_1S_2 + \frac{(il_zUY + l_xl_yY^2)X}{U(U^2-Y^2)} & qS_1 - \frac{(il_yUY - l_xl_zY^2)X}{U(U^2-Y^2)} \\ S_1S_2 - \frac{(il_zUY - l_xl_yY^2)X}{U(U^2-Y^2)} & -q^2 - S_1^2 - \frac{\left(U^2 - l_y^2Y^2\right)X}{U(U^2-Y^2)} + 1 & qS_2 + \frac{(il_xUY + l_yl_zY^2)X}{U(U^2-Y^2)} \\ qS_1 + \frac{(il_yUY + l_xl_zY^2)X}{U(U^2-Y^2)} & qS_2 - \frac{(il_xUY - l_yl_zY^2)X}{U(U^2-Y^2)} & -S_1 - S_2^2 - \frac{(U^2 - l_z^2Y^2)X}{U(U^2-Y^2)} + 1 \end{vmatrix}. \tag{78}$$

But for what are we solving? We know that the Booker quartic is a polynomial in q that we set equal to zero in order to find the distinct vertical components of the fields of the four waves within a particular layer of the ionosphere. In (78), the highest degree of q is four, which comes from the product of the terms on the main diagonal subtracted by the product of the terms along the other diagonal: the product of the $(1,3)$, $(2,2)$, and $(3,1)$ terms. The coefficient of this fourth order term is

$$\begin{aligned} &-S_1^2 - S_2^2 - \frac{\left(U^2 - l_z^2Y^2\right)X}{U\left(U^2 - Y^2\right)} + 1 + S_1^2 + S_2^2 \\ =&C^2 + S_1^2 + S_2^2 - \frac{\left(U^2 - l_z^2Y^2\right)X}{U\left(U^2 - Y^2\right)} \\ =&1 - \frac{\left(U^2 - l_z^2Y^2\right)X}{U\left(U^2 - Y^2\right)} \\ =&\frac{U\left(U^2 - Y^2\right) + X\left(Y^2l_z^2 - U^2\right)}{U\left(U^2 - Y^2\right)}. \end{aligned} \tag{79}$$

If we multiply this result by its denominator, we get α from (67)! Indeed, rearranging (78) will produce the Booker quartic. Of course, our focus is what the coefficient for the second order term ends up being. That computation is not quite as easy as the fourth order term due to the fact that there are many more terms at which to look, and the same is true for the zeroth, first, and third order terms. While the determinant as a whole can be computed on a computer algebra system, for the readers who like to work out the algebra by hand, we provide the major steps in the Appendix so one can see the math involved in determining the remaining four coefficients. The results from (79), (A5), (A18), (A22), and (A43) do match up with the coefficients in (67) when the former are multiplied by $U\left(U^2 - Y^2\right)$. Most importantly, (A18) is another verification that Budden's (6.16γ) is incorrect.

3. DOES IT MATTER?

In Section 2 it was determined through three separate methods (two related and one independent) that the second order coefficient of the first statement of the Booker quartic in [5] is missing a term:

$$2S_1 S_2 l_x l_y X Y^2. \tag{80}$$

That means that the formula given in the book is qualitatively wrong, but we would also like to know if it makes a quantitative difference. If the term's contribution is negligible, one might make the argument that it is okay to continue neglecting it and that anyone who has used this formula as a reference source does not have to worry about the validity of his or her work. On the other hand, if the term contributes substantially to the value of the coefficient, it could have serious implications for computation based on this representation of the quartic.

The geometrical configuration that is used most often is when propagation is in the $x - z$ plane. In this special case $S_2 = 0$. A quick glance at (80) will reveal that when this is true, the extra term vanishes. This fact may be why the error has gone unnoticed for so long. In fact, Budden correctly gives the Booker quartic coefficients for this case in (6.23). That is the reference formula of choice for the standard geometrical convention, and those who use it without question would never be aware of the error in the more general formula from which it is derived. It is worth noting that the extra term would also vanish with propagation exclusively in the $y - z$ plane because this would imply $S_1 = 0$. Furthermore, as seen in the paragraph above (49), $\boldsymbol{Y}$ cannot be confined to the $x - z$ plane ($l_y = 0$) nor, analogously, the $y - z$ plane ($l_x = 0$).

When is the error the most damaging, then? While it could be determined intuitively, a more mathematical approach such as the method of Lagrange multipliers would be ideal to conclusively answer that question. Let

$$f(l_x, l_y, S_1, S_2) = 2 l_x l_y S_1 S_2, \tag{81}$$

and let

$$0 = g_1(l_x, l_y, S_1, S_2) = l_x^2 + l_y^2 - \left(1 - l_z^2\right), \tag{82a}$$

$$0 = g_2(l_x, l_y, S_1, S_2) = S_1^2 + S_2^2 - \left(n_r^2 - q^2\right), \tag{82b}$$

and

$$0 = g_3(l_x, l_y, S_1, S_2 = l_x S_1 + l_y S_2 - (n_r l_z - l_z q)\,, \tag{82c}$$

where (82a), (82b), and (82c) come from (33), (3), and (26) with (50) substituted in, respectively. For our purposes, the method of Lagrange multipliers reveals that there exist λ, μ, and η such that the equations given by

$$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2 + \eta \nabla g_3, \tag{83}$$

combined with (82) produce a system of equations where the solution(s) will give us the values of the variables $\{l_x, l_y, S_1, S_2\}$ that create the maximum possible value of $2l_x l_y S_1 S_2$ with the given constraints. Expanding the gradients in (83), we find

$$\frac{\partial f}{\partial l_x} = 2l_y S_1 S_2, \quad \frac{\partial f}{\partial l_y} = 2l_x S_1 S_2, \quad \frac{\partial f}{\partial S_1} = 2l_x l_y S_2, \quad \frac{\partial f}{\partial S_2} = 2l_x l_y S1, \tag{84a}$$

$$\frac{\partial g_1}{\partial l_x} = 2l_x, \quad \frac{\partial g_1}{\partial l_y} = 2l_y, \quad \frac{\partial g_1}{\partial S_1} = 0, \quad \frac{\partial g_1}{\partial S_2} = 0, \tag{84b}$$

$$\frac{\partial g_2}{\partial l_x} = 0, \quad \frac{\partial g_2}{\partial l_y} = 0, \quad \frac{\partial g_2}{\partial S_1} = 2S_1, \quad \frac{\partial g_2}{\partial S_2} = 2S_2, \tag{84c}$$

$$\frac{\partial g_3}{\partial l_x} = S_1, \quad \frac{\partial g_3}{\partial l_y} = S_2, \quad \frac{\partial g_3}{\partial S_1} = l_x, \quad \frac{\partial g_3}{\partial S_2} = l_y. \tag{84d}$$

We are now ready to write the system of seven equations with seven unknowns (l_z, n_r, and q are treated as constants):

$$\left.\begin{aligned} 2l_x\lambda + S_1\eta &= 2l_y S_1 S_2 \\ 2l_y\lambda + S_2\eta &= 2l_x S_1 S_2 \\ 2S_1\mu + l_x\eta &= 2l_x l_y S_2 \\ 2S_2\mu + l_y\eta &= 2l_x l_y S_1 \\ l_x^2 + l_y^2 &= 1 - l_z^2 \\ S_1^2 + S_2^2 &= n_r^2 - q^2 \\ l_x S_1 + l_y S_2 &= (n_r - q)\, l_z. \end{aligned}\right\} \tag{85}$$

System (85) is nonlinear; solving it by hand is rather complicated. Therefore, it is easiest to rely on a computer algebra system to find the solution for us. We chose to use the symbolic toolbox in MATLAB, which admits that it cannot solve the system symbolically but returns a numerical approximation instead.

Thankfully, its solution can easily be converted back to being symbolic, and we get:

$$\left.\begin{aligned}
l_x &= \frac{\sqrt{2\left(n_r-q\right)\left(n_r+q-R\right)}\left(1-l_z^2\right)\left(n_r+q+R\right)}{2l_z\left(n_r+q\right)\left(n_r-q\right)}\\
l_y &= \frac{\sqrt{2\left(n_r-q\right)\left(n_r+q+R\right)}\left(1-l_z^2\right)\left(n_r+q-R\right)}{2l_z\left(n_r+q\right)\left(n_r-q\right)}\\
S_1 &= \frac{\sqrt{2\left(n_r-q\right)\left(n_r+q-R\right)}}{2}\\
S_2 &= \frac{\sqrt{2\left(n_r-q\right)\left(n_r+q+R\right)}}{2}\\
\lambda &= 0\\
\mu &= 0\\
\eta &= \left(n_r-q\right)l_z,
\end{aligned}\right\} \tag{86a}$$

where

$$R \equiv \sqrt{\frac{\left(n_r+q\right)\left(n_r+q-2n_r l_z^2\right)}{1-l_z^2}}. \tag{86b}$$

As "nice" as this may look, there is a major issue with these results. This system of equations could have multiple solutions. The way the MATLAB algorithm works in this case is that it stops after finding one solution. Curiously, something peculiar seems to happen within the internal MATLAB code. In reality, we did not input (85) into MATLAB's solver as written but instead used linear algebra on the first four equations to solve for λ, μ, and η in terms of l_x, l_y, S_1, and S_2. Because that system is overdetermined, one of the equations will yield an expression that is equal to zero, which we count as the fourth equation. Namely, we have

$$\left.\begin{aligned}
\lambda &= \frac{l_y S_1^2\left(l_y S_2-l_x S_1\right)}{l_x\left(l_y S_1-l_x S_2\right)},\\
0 &= \frac{2\left(l_y S_1+l_x S_2\right)\left(l_x S_1-l_y S_2\right)}{l_x},\\
\mu &= \frac{l_x l_y\left(l_y S_2-l_x S_1\right)}{l_y S_1-l_x S_2},\\
\eta &= \frac{2l_x l_y\left(S_1^2-S_2^2\right)}{l_y S_1-l_x S_2}.
\end{aligned}\right\} \tag{87}$$

If (87), along with the final three equations in (85) are passed along to MATLAB, it produces (86) as the solution, with the aforementioned warning. However, if (85) is what is passed into MATLAB without any modification, it finds 32 possible solutions for the system, none of which are (86). On top of that, when it is asked to provide the conditions under which each possible solution arises, it determines that it cannot find an explicit solution. This is quite puzzling. We know we could reduce the number of solutions by imposing more assumptions such as the nonnegativity of l_x, l_y, l_z, S_1, and S_2, for instance, but that does not answer the question of why the solution we analyze above is not included in that set. The answer may lie in the fact that (87) requires two assumptions: $l_x \neq 0$ and $l_y S_1 \neq l_x S_2$, and thus the systems are not completely identical. The discussion at the bottom of page 17 establishes that the former assumption is valid, but there is still the matter of the second one. We also attempted to solve (85) using Mathematica without supplying any additional assumptions for the variables, but it could not determine the solution set, as well.

If we only consider (86), with a little algebra one can determine that we get a value for the missing term (neglecting the XY^2):

$$\begin{aligned}
2l_xS_1 &= 2\frac{\sqrt{2\left(n_r-q\right)\left(n_r+q-R\right)}\left(1-l_z^2\right)\left(n_r+q+R\right)}{2l_z\left(n_r+q\right)\left(n_r-q\right)}\frac{\sqrt{2\left(n_r-q\right)\left(n_r+q-R\right)}}{2} \\
&= \frac{\left(n_r-q\right)\left(n_r+q-R\right)\left(1-l_z^2\right)\left(n_r+q+R\right)}{l_z\left(n_r+q\right)\left(n_r-q\right)} \\
&= \frac{\left(1-l_z^2\right)\left(\left(n_r+q\right)^2-R^2\right)}{l_z\left(n_r+q\right)} \\
&= \frac{\left(1-l_z^2\right)\left(\left(n_r+q\right)^2-\frac{\left(n_r+q\right)\left(n_r+q-2n_rl_z^2\right)}{1-l_z^2}\right)}{l_z\left(n_r+q\right)} \\
&= \frac{\left(n_r+q\right)^2\left(1-l_z^2\right)-\left(n_r+q\right)\left(n_r+q-2n_rl_z^2\right)}{l_z\left(n_r+q\right)} \\
&= \frac{\left(n_r+q\right)\left(1-l_z^2\right)-\left(n_r+q\right)+2n_rl_z^2}{l_z} \\
&= -l_z\left(n_r+q\right)+2n_rl_z \\
&= \left(n_r-q\right)l_z,
\end{aligned} \tag{88}$$

and similarly

$$2l_yS_2 = \left(n_r-q\right)l_z, \tag{89}$$

and thus

$$2l_xl_yS_1S_2 = \frac{1}{2}\left(n_r-q\right)^2l_z^2. \tag{90}$$

This can be done with the other second order coefficients in (67), and one of the findings is that for this solution,

$$2l_xl_yS_1S_2 = \frac{1}{2}\left(n_r-q\right)^2l_z^2 = \frac{1}{4}\left(n_r-q\right)^2l_z^2+\frac{1}{4}\left(n_r-q\right)^2l_z^2 = \frac{1}{4}\left(2l_xS_1\right)^2+\frac{1}{4}\left(2l_yS_2\right)^2 = l_x^2S_1^2+l_y^2S_2^2. \tag{91}$$

That makes it clear that there exists at least one case in which the missing term is on the same order as the other factors of XY^2 in γ of (67), and therefore it is significant and cannot be neglected.

This is only a preliminary look at the possible ramifications of this error. A more detailed study is needed to fully determine precisely when and how it is the most damaging. We hope to be able to ascertain that through the full-physics ray tracing software package Modernized Jones-Stephenson, or MoJo [9], but initial attempts have shown that the question is far more difficult to answer than was initially thought. The full and most general form of the Booker Quartic using the coefficients in (67) has never been put into code as far as we can tell. The reason for this is the difficulty of the task. However, we discovered a way to do it. The effort and results of our findings will be documented in an upcoming paper. In this paper, we plan to compare ray paths at different scenarios using both the correct coefficients (67) and those given in [5] and present the results.

4. CONCLUSION

Our results indicate that equation (6.16) of [5] contains an error. Two of our methods (Sections 2.1.4 and 2.2) are taken directly from the book but fleshed out, especially the latter (see Appendix A). Our other, most prominent, method, covered in Sections 2.1.1, 2.1.2, and 2.1.3, neglects some geometrical simplifications to give a full derivation of the more robust version of the Booker quartic seen in (6.17) whereupon the correct substitutions and simplifications are made to reduce it to the form of (67). This approach revealed a typographical error in the third row, second column entry of (3.55). All three methods agree that what is printed in the book is erroneous. There is a missing second order term that will affect the four values of q, which characterizes the vertical components of the fields of an EM wave propagating through a stratum of a cold plasma such as the ionosphere. This book is widely used as a reference source, and anyone using the coefficients given by (6.16) will produce incorrect results.

Exactly *how* incorrect results may be is something we unfortunately cannot say at this time. Unless a correct set of solutions to (85) can be found or perhaps a different method to analyze that term and determine its impact is utilized, solving this problem will require a deeper investigation of why MATLAB seems to be contradicting itself. It could be something regarding the discussion in the middle of the main paragraph on page 20, but the issue may go beyond that. Furthermore, if there really are at least 33 solutions to that set of equations $(32 + 1)$, it is beyond the scope of this paper to analyze all of them. Both of these issues may be the subject of future research because the precise quantitative effect this error produces is something that would be helpful for those who might have relied on this equation in the past to know, and it would also satisfy general curiosity. As to how the study would proceed, a more numerical approach might be necessary to determine this term's maximum impact. At the time of writing, it seems most likely that additional validation regarding the importance of this term will come from ray tracing software, specifically MoJo. If the code of such software was modified to generate a side by side comparison of computed ray behavior using the Booker quartic with and without the term, one could get a much clearer picture of its effect than by just analyzing the extreme cases. There are plans to do this in the near future and the results will be presented in an upcoming paper.

For now, we have shown that what is in the *The propagation of radio waves* is incorrect, and the Booker quartic as printed should not be used as a reference source without the addition of the extra term, as seen in (67). We end by reiterating that if propagation is confined to the $x - z$ plane, there is no issue with what is given in the book; it is perfectly safe to use for that particular framework. One must be careful otherwise.

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Appendix A

COMPUTING THE DIRECT APPROACH

Computation of a 3×3 determinant is taught slightly different ways at the undergraduate level. We prefer to use the method of finding the difference between the sum of the forward diagonals and the sum of the backward diagonals. This breaks the determinant into six parts for easier consumption. We shall label these six parts a—f. Specifically, element wise, the six parts are as follows: a is $(1,1)\cdot(2,2)\cdot(3,3)$, b is $(1,2)\cdot(2,3)\cdot(3,1)$, c is $(1,3)\cdot(2,1)\cdot(3,2)$, d is $(1,3)\cdot(2,2)\cdot(3,1)$, e is $(1,2)\cdot(2,1)\cdot(3,3)$, and f is $(1,1)\cdot(2,3)\cdot(3,2)$. The value of the determinant is then $a+b+c-d-e-f$. Plugging in the values from (78), we have

$$\begin{aligned}&\left(-q^2 - S_2^2 - \frac{\left(U^2 - l_x^2 Y^2\right) X}{U\left(U^2 - Y^2\right)} + 1\right) \cdot \left(-q^2 - S_1^2 - \frac{\left(U^2 - l_y^2 Y^2\right) X}{U\left(U^2 - Y^2\right)} + 1\right)\\ &\cdot \left(-S_1^2 - S_2^2 - \frac{\left(U^2 - l_z^2 Y^2\right) X}{U\left(U^2 - Y^2\right)} + 1\right),\end{aligned} \tag{A1a}$$

$$\begin{aligned}&\left(S_1 S_2 + \frac{\left(i l_z U Y + l_x l_y Y^2\right) X}{U\left(U^2 - Y^2\right)}\right) \cdot \left(q S_2 + \frac{\left(i l_x U Y + l_y l_z Y^2\right) X}{U\left(U^2 - Y^2\right)}\right)\\ &\cdot \left(q S_1 + \frac{\left(i l_y U Y + l_x l_z Y^2\right) X}{U\left(U^2 - Y^2\right)}\right),\end{aligned} \tag{A1b}$$

$$\begin{aligned}&\left(q S_1 - \frac{\left(i l_y U Y - l_x l_z Y^2\right) X}{U\left(U^2 - Y^2\right)}\right) \cdot \left(S_1 S_2 - \frac{\left(i l_z U Y - l_x l_y Y^2\right) X}{U\left(U^2 - Y^2\right)}\right)\\ &\cdot \left(q S_2 - \frac{\left(i l_x U Y - l_y l_z Y^2\right) X}{U\left(U^2 - Y^2\right)}\right),\end{aligned} \tag{A1c}$$

$$\begin{aligned}&\left(q S_1 - \frac{\left(i l_y U Y - l_x l_z Y^2\right) X}{U\left(U^2 - Y^2\right)}\right) \cdot \left(-q^2 - S_1^2 - \frac{\left(U^2 - l_y^2 Y^2\right) X}{U\left(U^2 - Y^2\right)} + 1\right)\\ &\cdot \left(q S_1 + \frac{\left(i l_y U Y - l_x l_z Y^2\right) X}{U\left(U^2 - Y^2\right)}\right),\end{aligned} \tag{A1d}$$

$$\begin{aligned}&\left(S_1 S_2 + \frac{\left(i l_z U Y - l_x l_y Y^2\right) X}{U\left(U^2 - Y^2\right)}\right) \cdot \left(S_1 S_2 - \frac{\left(i l_z U Y - l_x l_y Y^2\right) X}{U\left(U^2 - Y^2\right)}\right)\\ &\cdot \left(-S_1^2 - S_2^2 - \frac{\left(U^2 - l_z^2 Y^2\right) X}{U\left(U^2 - Y^2\right)} + 1\right),\end{aligned} \tag{A1e}$$

$$\left(-q^2 - S_2^2 - \frac{\left(U^2 - l_x^2 Y^2\right) X}{U\left(U^2 - Y^2\right)} + 1\right) \cdot \left(qS_2 + \frac{\left(il_x UY + l_y l_z Y^2\right) X}{U\left(U^2 - Y^2\right)}\right) \cdot \left(qS_2 - \frac{\left(il_x UY - l_y l_z Y^2\right) X}{U\left(U^2 - Y^2\right)}\right). \tag{A1f}$$

For our purposes, we need to group everything so that each part is written as a polynomial in q. We can also use (65) to make things a little nicer. Therefore, we get that a becomes

$$\begin{aligned}
&\left(C^2 - \frac{\left(U^2 - l_z^2 Y^2\right) X}{U\left(U^2 - Y^2\right)}\right) q^4 \\
&+ \left(\frac{\left(U^2 - l_x^2 Y^2\right) X}{U\left(U^2 - Y^2\right)} + \frac{\left(U^2 - l_y^2 Y^2\right) X}{U\left(U^2 - Y^2\right)} - \left(C^2 + 1\right)\right) \left(C^2 - \frac{\left(U^2 - l_z^2 Y^2\right) X}{U\left(U^2 - Y^2\right)}\right) q^2 \\
&+ \left[C^2 + S_1^2 S_2^2 + \left(S_1^2 - 1\right) \frac{\left(U^2 - l_x^2 Y^2\right) X}{U\left(U^2 - Y^2\right)} + \left(S_2^2 - 1\right) \frac{\left(U^2 - l_y^2 Y^2\right) X}{U\left(U^2 - Y^2\right)} \right. \\
&\left. + \frac{\left(U^2 - l_x^2 Y^2\right)\left(U^2 - l_y^2 Y^2\right) X^2}{U^2\left(U^2 - Y^2\right)^2}\right] \left(C^2 - \frac{\left(U^2 - l_z^2 Y^2\right) X}{U\left(U^2 - Y^2\right)}\right),
\end{aligned} \tag{A2a}$$

b becomes

$$\begin{aligned}
&S_1 S_2 \left(S_1 S_2 + \frac{\left(il_z UY + l_x l_y Y^2\right) X}{U\left(U^2 - Y^2\right)}\right) q^2 \\
&+ \left[S_1 \left(S_1 S_2 + \frac{\left(il_z UY + l_x l_y Y^2\right) X}{U\left(U^2 - Y^2\right)}\right) \frac{\left(il_x UY + l_y l_z Y^2\right) X}{U\left(U^2 - Y^2\right)} \right. \\
&\left. + S_2 \left(S_1 S_2 + \frac{\left(il_z UY + l_x l_y Y^2\right) X}{U\left(U^2 - Y^2\right)}\right) \frac{\left(il_y UY + l_x l_z Y^2\right) X}{U\left(U^2 - Y^2\right)}\right] q \\
&+ \left(S_1 S_2 + \frac{\left(il_z UY + l_x l_y Y^2\right) X}{U\left(U^2 - Y^2\right)}\right) \frac{\left(il_x UY + l_y l_z Y^2\right) X}{U\left(U^2 - Y^2\right)} \frac{\left(il_y UY + l_x l_z Y^2\right) X}{U\left(U^2 - Y^2\right)},
\end{aligned} \tag{A2b}$$

c becomes

$$\begin{aligned}
&S_1 S_2 \left(S_1 S_2 - \frac{\left(il_z UY - l_x l_y Y^2\right) X}{U\left(U^2 - Y^2\right)}\right) q^2 \\
&- \left[S_1 \left(S_1 S_2 - \frac{\left(il_z UY - l_x l_y Y^2\right) X}{U\left(U^2 - Y^2\right)}\right) \frac{\left(il_x UY - l_y l_z Y^2\right) X}{U\left(U^2 - Y^2\right)} \right. \\
&\left. + S_2 \left(S_1 S_2 - \frac{\left(il_z UY - l_x l_y Y^2\right) X}{U\left(U^2 - Y^2\right)}\right) \frac{\left(il_y UY - l_x l_z Y^2\right) X}{U\left(U^2 - Y^2\right)}\right] q \\
&+ \left(S_1 S_2 - \frac{\left(il_z UY - l_x l_y Y^2\right) X}{U\left(U^2 - Y^2\right)}\right) \frac{\left(il_y UY - l_x l_z Y^2\right) X}{U\left(U^2 - Y^2\right)} \frac{\left(il_x UY - l_y l_z Y^2\right) X}{U\left(U^2 - Y^2\right)},
\end{aligned} \tag{A2c}$$

d becomes

$$
\begin{aligned}
&-S_1^2 q^4 \\
&-S_1\left(\frac{\left(il_yUY+l_xl_zY^2\right)X}{U\left(U^2-Y^2\right)}-\frac{\left(il_yUY-l_xl_zY^2\right)X}{U\left(U^2-Y^2\right)}\right)q^3 \\
&+\left(\frac{\left(il_yUY-l_xl_zY^2\right)X}{U\left(U^2-Y^2\right)}\frac{\left(il_yUY+l_xl_zY^2\right)X}{U\left(U^2-Y^2\right)}-S_1^2\left(S_1^2-1+\frac{\left(U^2-l_y^2Y^2\right)X}{U\left(U^2-Y^2\right)}\right)\right)q^2 \\
&+S_1\left[\left(S_1^2-1+\frac{\left(U^2-l_y^2Y^2\right)X}{U\left(U^2-Y^2\right)}\right)\frac{\left(il_yUY-l_xl_zY^2\right)X}{U\left(U^2-Y^2\right)}\right. \\
&\left.-\left(S_1^2-1+\frac{\left(U^2-l_y^2Y^2\right)X}{U\left(U^2-Y^2\right)}\right)\frac{\left(il_yUY+l_xl_zY^2\right)X}{U\left(U^2-Y^2\right)}\right]\quad q \\
&+\left(S_1^2-1+\frac{\left(U^2-l_y^2Y^2\right)X}{U\left(U^2-Y^2\right)}\right)\frac{\left(il_yUY-l_xl_zY^2\right)X}{U\left(U^2-Y^2\right)}\frac{\left(il_yUY+l_xl_zY^2\right)X}{U\left(U^2-Y^2\right)},
\end{aligned}
\tag{A2d}
$$

e, which only has zero order terms, remains

$$
\begin{aligned}
&\left(S_1S_2+\frac{\left(il_zUY-l_xl_yY^2\right)X}{U\left(U^2-Y^2\right)}\right)\cdot\left(S_1S_2-\frac{\left(il_zUY-l_xl_yY^2\right)X}{U\left(U^2-Y^2\right)}\right) \\
&\cdot\left(C^2-\frac{\left(U^2-l_z^2Y^2\right)X}{U\left(U^2-Y^2\right)}\right),
\end{aligned}
\tag{A2e}
$$

and f becomes

$$
\begin{aligned}
&-S_2 q^4 \\
&+S_2\left(\frac{\left(il_xUY-l_yl_zY^2\right)X}{U\left(U^2-Y^2\right)}-\frac{\left(il_xUY+l_yl_zY^2\right)X}{U\left(U^2-Y^2\right)}\right)q^3 \\
&+\left(\frac{\left(il_xUY+l_yl_zY^2\right)X}{U\left(U^2-Y^2\right)}\frac{\left(il_xUY-l_yl_zY^2\right)X}{U\left(U^2-Y^2\right)}-S_2^2\left(S_2^2-1+\frac{\left(U^2-l_x^2Y^2\right)X}{U\left(U^2-Y^2\right)}\right)\right)q^2 \\
&+S_2\left[\left(S_2^2-1+\frac{\left(U^2-l_x^2Y^2\right)X}{U\left(U^2-Y^2\right)}\right)\frac{\left(il_xUY-l_yl_zY^2\right)X}{U\left(U^2-Y^2\right)}\right. \\
&\left.-\left(S_2^2-1+\frac{\left(U^2-l_x^2Y^2\right)X}{U\left(U^2-Y^2\right)}\right)\frac{\left(il_xUY+l_yl_zY^2\right)X}{U\left(U^2-Y^2\right)}\right]\quad q \\
&+\left(S_2^2-1+\frac{\left(U^2-l_x^2Y^2\right)X}{U\left(U^2-Y^2\right)}\right)\frac{\left(il_xUY+l_yl_zY^2\right)X}{U\left(U^2-Y^2\right)}\frac{\left(il_xUY-l_yl_zY^2\right)X}{U\left(U^2-Y^2\right)}.
\end{aligned}
\tag{A2f}
$$

Due to the canceling out of some terms, d, e and f can be simplified, so we rewrite (A2) as

$$\begin{aligned}
&\left(C^2-\frac{\left(U^2-l_z^2Y^2\right)X}{U\left(U^2-Y^2\right)}\right)q^4\\
&+\left(\frac{\left(U^2-l_x^2Y^2\right)X}{U\left(U^2-Y^2\right)}+\frac{\left(U^2-l_y^2Y^2\right)X}{U\left(U^2-Y^2\right)}-\left(C^2+1\right)\right)\left(C^2-\frac{\left(U^2-l_z^2Y^2\right)X}{U\left(U^2-Y^2\right)}\right)q^2\\
&+\left[C^2+S_1^2S_2^2+\left(S_1^2-1\right)\frac{\left(U^2-l_x^2Y^2\right)X}{U\left(U^2-Y^2\right)}+\left(S_2^2-1\right)\frac{\left(U^2-l_y^2Y^2\right)X}{U\left(U^2-Y^2\right)}\right.\\
&\left.+\frac{\left(U^2-l_x^2Y^2\right)\left(U^2-l_y^2Y^2\right)X^2}{U^2\left(U^2-Y^2\right)^2}\right]\left(C^2-\frac{\left(U^2-l_z^2Y^2\right)X}{U\left(U^2-Y^2\right)}\right),
\end{aligned} \tag{A3a}$$

$$\begin{aligned}
&S_1S_2\left(S_1S_2+\frac{\left(il_zUY+l_xl_yY^2\right)X}{U\left(U^2-Y^2\right)}\right)q^2\\
&+\left[S_1\left(S_1S_2+\frac{\left(il_zUY+l_xl_yY^2\right)X}{U\left(U^2-Y^2\right)}\right)\frac{\left(il_xUY+l_yl_zY^2\right)X}{U\left(U^2-Y^2\right)}\right.\\
&\left.+S_2\left(S_1S_2+\frac{\left(il_zUY+l_xl_yY^2\right)X}{U\left(U^2-Y^2\right)}\right)\frac{\left(il_yUY+l_xl_zY^2\right)X}{U\left(U^2-Y^2\right)}\right]q\\
&+\left(S_1S_2+\frac{\left(il_zUY+l_xl_yY^2\right)X}{U\left(U^2-Y^2\right)}\right)\frac{\left(il_xUY+l_yl_zY^2\right)X}{U\left(U^2-Y^2\right)}\frac{\left(il_yUY+l_xl_zY^2\right)X}{U\left(U^2-Y^2\right)},
\end{aligned} \tag{A3b}$$

$$\begin{aligned}
&S_1S_2\left(S_1S_2-\frac{\left(il_zUY-l_xl_yY^2\right)X}{U\left(U^2-Y^2\right)}\right)q^2\\
&-\left[S_1\left(S_1S_2-\frac{\left(il_zUY-l_xl_yY^2\right)X}{U\left(U^2-Y^2\right)}\right)\frac{\left(il_xUY-l_yl_zY^2\right)X}{U\left(U^2-Y^2\right)}\right.\\
&\left.+S_2\left(S_1S_2-\frac{\left(il_zUY-l_xl_yY^2\right)X}{U\left(U^2-Y^2\right)}\right)\frac{\left(il_yUY-l_xl_zY^2\right)X}{U\left(U^2-Y^2\right)}\right]q\\
&+\left(S_1S_2-\frac{\left(il_zUY-l_xl_yY^2\right)X}{U\left(U^2-Y^2\right)}\right)\frac{\left(il_yUY-l_xl_zY^2\right)X}{U\left(U^2-Y^2\right)}\frac{\left(il_xUY-l_yl_zY^2\right)X}{U\left(U^2-Y^2\right)},
\end{aligned} \tag{A3c}$$

$$
\begin{aligned}
&-S_1^2 q^4 \\
&-S_1 \frac{2l_x l_z Y^2 X}{U\left(U^2-Y^2\right)} q^3 \\
&-\left(\frac{\left(l_y^2 U^2 Y^2 + l_x^2 l_z^2 Y^4\right) X^2}{U^2\left(U^2-Y^2\right)^2} + S_1^2\left(S_1^2 - 1 + \frac{\left(U^2 - l_y^2 Y^2\right) X}{U\left(U^2-Y^2\right)}\right)\right) q^2 \\
&-S_1\left(S_1^2 - 1 + \frac{\left(U^2 - l_y^2 Y^2\right) X}{U\left(U^2-Y^2\right)}\right) \frac{2l_x l_z Y^2 X}{U\left(U^2-Y^2\right)} q \\
&-\left(S_1^2 - 1 + \frac{\left(U^2 - l_y^2 Y^2\right) X}{U\left(U^2-Y^2\right)}\right) \frac{\left(l_y^2 U^2 Y^2 + l_x^2 l_z^2 Y^4\right) X^2}{U^2\left(U^2-Y^2\right)^2},
\end{aligned}
\tag{A3d}
$$

$$
\left(S_1^2 S_2^2 + S_1 S_2 \frac{2l_x l_y Y^2 X}{U\left(U^2-Y^2\right)} + \frac{\left(l_z^2 U^2 Y^2 + l_x^2 l_y^2 Y^4\right) X^2}{U^2\left(U^2-Y^2\right)^2}\right)\left(C^2 - \frac{\left(U^2 - l_z^2 Y^2\right) X}{U\left(U^2-Y^2\right)}\right),
\tag{A3e}
$$

$$
\begin{aligned}
&-S_2^2 q^4 \\
&-S_2 \frac{2l_y l_z Y^2 X}{U\left(U^2-Y^2\right)} q^3 \\
&-\left(\frac{\left(l_x^2 U^2 Y^2 + l_y^2 l_z^2 Y^4\right) X^2}{U^2\left(U^2-Y^2\right)^2} + S_2^2\left(S_2^2 - 1 + \frac{\left(U^2 - l_x^2 Y^2\right) X}{U\left(U^2-Y^2\right)}\right)\right) q^2 \\
&-S_2\left(S_2^2 - 1 + \frac{\left(U^2 - l_x^2 Y^2\right) X}{U\left(U^2-Y^2\right)}\right) \frac{2l_y l_z Y^2 X}{U\left(U^2-Y^2\right)} q \\
&-\left(S_2^2 - 1 + \frac{\left(U^2 - l_x^2 Y^2\right) X}{U\left(U^2-Y^2\right)}\right) \frac{\left(l_x^2 U^2 Y^2 + l_y^2 l_z^2 Y^4\right) X^2}{U^2\left(U^2-Y^2\right)^2}.
\end{aligned}
\tag{A3f}
$$

Finding the determinant is equivalent to subtracting the sum of the last three parts of (A3) from the sum of the first three parts of (A3). We look at the coefficients for each order of magnitude of q separately.

For the $O(q^4)$ terms, we have from a, d and f:

$$
C^2 - \frac{\left(U^2 - l_z^2 Y^2\right) X}{U\left(U^2-Y^2\right)} + S_1^2 + S_2^2,
\tag{A4}
$$

and we know from (79) that is equivalent to $\frac{1}{U(U^2-Y^2)}\alpha$ from (67). The Greek letters present for the remainder of the appendix will also refer to the ones in that equation as opposed to (58). The $O(q^3)$ term comes from d and f, and, remembering that we are subtracting those terms, we find

$$S_1\frac{2l_xl_zY^2X}{U\left(U^2-Y^2\right)}+S_2\frac{2l_yl_zY^2X}{U\left(U^2-Y^2\right)}=\frac{2\left(l_xS_1+l_yS_2\right)l_zXY^2}{U\left(U^2-Y^2\right)}=\frac{1}{U\left(U^2-Y^2\right)}\beta. \tag{A5}$$

It is the $O(q^2)$ term with which we are the most concerned, and its parts are present in a, b, c, d, and f:

$$\begin{aligned}
&\left(\frac{\left(U^2-l_x^2Y^2\right)X}{U\left(U^2-Y^2\right)}+\frac{\left(U^2-l_y^2Y^2\right)X}{U\left(U^2-Y^2\right)}-\left(C^2+1\right)\right)\left(C^2-\frac{\left(U^2-l_z^2Y^2\right)X}{U\left(U^2-Y^2\right)}\right)\\
&+S_1S_2\left(S_1S_2+\frac{\left(il_zUY+l_xl_yY^2\right)X}{U\left(U^2-Y^2\right)}\right)+S_1S_2\left(S_1S_2-\frac{\left(il_zUY-l_xl_yY^2\right)X}{U\left(U^2-Y^2\right)}\right)\\
&+\left(\frac{\left(l_y^2U^2Y^2+l_x^2l_z^2Y^4\right)X^2}{U^2\left(U^2-Y^2\right)^2}+S_1^2\left(S_1^2-1+\frac{\left(U^2-l_y^2Y^2\right)X}{U\left(U^2-Y^2\right)}\right)\right)\\
&+\left(\frac{\left(l_x^2U^2Y^2+l_y^2l_z^2Y^4\right)X^2}{U^2\left(U^2-Y^2\right)^2}+S_2^2\left(S_2^2-1+\frac{\left(U^2-l_x^2Y^2\right)X}{U\left(U^2-Y^2\right)}\right)\right).
\end{aligned} \tag{A6}$$

It is helpful to break this up with regards to different powers of $\frac{1}{U(U^2-Y^2)}$. The $O\left(\left(\frac{1}{U(U^2-Y^2)}\right)^2\right)$ coefficients are

$$\begin{aligned}
&\Big(-\left(U^2-l_x^2Y^2\right)\left(U^2-l_z^2Y^2\right)-\left(U^2-l_z^2Y^2\right)^2\\
&\quad+l_y^2U^2Y^2+l_x^2l_z^2Y^4+l_x^2U^2Y^2+l_y^2l_z^2Y^4\Big)X^2,
\end{aligned} \tag{A7}$$

but many of the terms cancel out after everything is expanded, and (A7) simplifies to

$$\begin{aligned}
&\left(-2U^4+2l_z^2U^2Y^2+2l_x^2U^2Y^2+2l_y^2U^2Y^2\right)X^2\\
=&\left(-2U^4+2U^2Y^2\right)X^2\\
=&-2\left(U^2\left(U^2-Y^2\right)\right)X^2
\end{aligned} \tag{A8}$$

which gives one term of

$$-\frac{2UX^2}{U\left(U^2-Y^2\right)}. \tag{A9}$$

The $O\left(\frac{1}{U(U^2-Y^2)}\right)$ coefficients are

$$\begin{aligned}&\Big(C^2\left(U^2-l_x^2Y^2\right)+C^2\left(U^2-l_y^2Y^2\right)+\left(C^2+1\right)\left(U^2-l_z^2Y^2\right)\\&+S_1S_2\left(il_zUY+l_xl_yY^2-il_zUY+l_xl_yY^2\right)+S_1^2\left(U^2-l_y^2Y^2\right)+S_2^2\left(U^2-l_x^2Y^2\right)\Big)X,\end{aligned}\tag{A10}$$

which simplifies to

$$\begin{aligned}&\Big(C^2\left(3U^2-\left(l_x^2+l_y^2+l_z^2\right)Y^2\right)+U^2-l_z^2Y^2+S_1^2U^2+S_2^2U^2\\&-S_1^2l_y^2Y^2-S_2^2l_x^2Y^2+2l_xl_yY^2S_1S_2\Big)X.\end{aligned}\tag{A11}$$

Instead of using property (33) we use property (65) to get

$$\begin{aligned}&\Big(3C^2U^2-\left(1-S_1^2-S_2^2\right)\left(l_x^2+l_y^2+l_z^2\right)Y^2\\&+U^2-l_z^2Y^2+S_1^2U^2+S_2^2U^2-S_1^2l_y^2Y^2-S_2^2l_x^2Y^2+2l_xl_yY^2S_1S_2\Big)X.\end{aligned}\tag{A12}$$

Expanding the second term allows for the cancellation of any terms that are products of $S_2^2l_x^2$ or $S_1^2l_y^2$, and (A12) reduces to

$$\begin{aligned}&\Big(3C^2U^2-l_z^2Y^2+U^2-l_z^2Y^2+\left(S_1^2+S_2^2\right)U^2\\&-l_x^2Y^2+l_x^2S_1^2Y^2-l_y^2Y^2+l_y^2S_2^2Y^2+2l_xl_yY^2S_1S_2\Big)X.\end{aligned}\tag{A13}$$

We now can invoke (33) and use (65) once more to cancel out the $\left(S_1^2+S_2^2\right)U^2$, leaving us with

$$\left(2C^2U^2+2U^2-l_z^2Y^2-Y^2+l_x^2S_1^2Y^2+l_y^2S_2^2Y^2+2l_xl_yY^2S_1S_2\right)X.\tag{A14}$$

Thus, the contribution from this part is

$$\frac{\left(2C^2U^2+2U^2-l_z^2Y^2-Y^2+l_x^2S_1^2Y^2+l_y^2S_2^2Y^2+2l_xl_yY^2S_1S_2\right)X}{U\left(U^2-Y^2\right)}.\tag{A15}$$

The remaining terms of (A6) are $O(1)$:

$$-C^2\left(C^2+1\right)+\left(S_1^2+S_2^2\right)^2-S_1^2-S_2^2=-2+2S_1^2+2S_2^2=-2C^2.\tag{A16}$$

In order for it to have the same common denominator as the other two parts, we need to multiply (A16) by $\frac{U(U^2-Y^2)}{U(U^2-Y^2)}$, producing

$$\frac{-2C^2U\left(U^2-Y^2\right)}{U\left(U^2-Y^2\right)}.\tag{A17}$$

Finally we combine (A9), (A15), and (A17) to get the $O(q^2)$ coefficient:

$$\begin{aligned}
&\frac{-2UX^2+\left(2C^2U^2+2U^2-l_z^2Y^2-Y^2+l_x^2S_1^2Y^2+l_y^2S_2^2Y^2+2l_xl_yY^2S_1S_2\right)X}{U\left(U^2-Y^2\right)}\\
&-\frac{2UC^2\left(U^2-Y^2\right)}{U\left(U^2-Y^2\right)}\\
=&\frac{-2U\left(U-X\right)\left(C^2U-X\right)+2Y^2\left(C^2U-X\right)}{U\left(U^2-Y^2\right)}\\
&+\frac{XY^2\left(1-l_z^2C^2+l_x^2S_1^2+l_y^2S_2^2+2l_xl_yS_1S_2\right)}{U\left(U^2-Y^2\right)}\\
=&\frac{1}{U\left(U^2-Y^2\right)}\gamma.
\end{aligned} \tag{A18}$$

The $O(q)$ term comes next. These coefficients exist in b, c, d, and f (Recall it's $b+c-d-f$):

$$\begin{aligned}
&S_1\left(S_1S_2+\frac{\left(il_zUY+l_xl_yY^2\right)X}{U\left(U^2-Y^2\right)}\right)\frac{\left(il_xUY+l_yl_zY^2\right)X}{U\left(U^2-Y^2\right)}\\
&+S_2\left(S_1S_2+\frac{\left(il_zUY+l_xl_yY^2\right)X}{U\left(U^2-Y^2\right)}\right)\frac{\left(il_yUY+l_xl_zY^2\right)X}{U\left(U^2-Y^2\right)}\\
&-S_1\left(S_1S_2-\frac{\left(il_zUY-l_xl_yY^2\right)X}{U\left(U^2-Y^2\right)}\right)\frac{\left(il_xUY-l_yl_zY^2\right)X}{U\left(U^2-Y^2\right)}\\
&-S_2\left(S_1S_2-\frac{\left(il_zUY-l_xl_yY^2\right)X}{U\left(U^2-Y^2\right)}\right)\frac{\left(il_yUY-l_xl_zY^2\right)X}{U\left(U^2-Y^2\right)}\\
&+S_1\left(S_1^2-1+\frac{\left(U^2-l_y^2Y^2\right)X}{U\left(U^2-Y^2\right)}\right)\frac{2l_xl_zY^2X}{U\left(U^2-Y^2\right)}\\
&+S_2\left(S_2^2-1+\frac{\left(U^2-l_x^2Y^2\right)X}{U\left(U^2-Y^2\right)}\right)\frac{2l_yl_zY^2X}{U\left(U^2-Y^2\right)}.
\end{aligned} \tag{A19}$$

Due to the pattern to the left of the parentheses, it's convenient to break this up into the coefficients of S_1 from the first, third and fifth rows and the coefficients of S_2 from the second, fourth, and sixth rows. Multiplying things out, we get

$$\begin{aligned}
&S_1\left(\frac{2S_1S_2l_yl_zY^2X}{U\left(U^2-Y^2\right)}-\frac{2l_xl_zU^2Y^2X}{U^2\left(U^2-Y^2\right)^2}+\frac{2l_xl_y^2l_zY^4X}{U^2\left(U^2-Y^2\right)^2}\right.\\
&\left.+\frac{2l_xl_zU^2Y^2X-2l_xl_y^2l_zY^4X}{U^2\left(U^2-Y^2\right)^2}+\frac{2S_1^2l_xl_zY^2X}{U\left(U^2-Y^2\right)}-\frac{2l_xl_zY^2X}{U\left(U^2-Y^2\right)}\right)\\
&+S_2\left(\frac{2S_1S_2l_xl_zY^2X}{U\left(U^2-Y^2\right)}-\frac{2l_yl_zU^2Y^2X}{U^2\left(U^2-Y^2\right)^2}+\frac{2l_x^2l_yl_zY^4}{U^2\left(U^2-Y^2\right)^2}\right.\\
&\left.+\frac{2l_yl_zU^2Y^2X-2l_x^2l_yl_zY^4X}{U^2\left(U^2-Y^2\right)^2}+\frac{2S_2^2l_yl_zY^2X}{U\left(U^2-Y^2\right)}-\frac{2l_yl_zY^2X}{U\left(U^2-Y^2\right)}\right),
\end{aligned} \tag{A20}$$

and since all of the $O\left(\left(\frac{1}{U\left(U^2-Y^2\right)}\right)^2\right)$ terms cancel out, we are left with

$$\begin{aligned}
&2S_1\left(\frac{S_1S_2l_yl_zXY^2}{U\left(U^2-Y^2\right)}+\frac{S_1^2l_xl_zXY^2}{U\left(U^2-Y^2\right)}-\frac{l_xl_zXY^2}{U\left(U^2-Y^2\right)}\right)\\
&+2S_2\left(\frac{S_1S_2l_xl_zXY^2}{U\left(U^2-Y^2\right)}+\frac{S_2^2l_yl_zXY^2}{U\left(U^2-Y^2\right)}-\frac{l_yl_zXY^2}{U\left(U^2-Y^2\right)}\right).
\end{aligned} \tag{A21}$$

Pulling all of the common factors out, (A21) becomes

$$\begin{aligned}
&\frac{2S_2l_yl_zXY^2}{U\left(U^2-Y^2\right)}\left(S_1^2+S_2^2-1\right)+\frac{2S_1l_xl_zXY^2}{U\left(U^2-Y^2\right)}\left(S_1^2+S_2^2-1\right)\\
&=-\frac{2C^2\left(l_xS_1+l_yS_2\right)l_zXY^2}{U\left(U^2-Y^2\right)}\\
&=\frac{1}{U\left(U^2-Y^2\right)}\delta.
\end{aligned} \tag{A22}$$

The only terms left of (A3) are those of $O(1)$. They exist in all six parts:

$$\begin{aligned}
&\Bigg[C^2 + S_1^2S_2^2 + \left(S_1^2 - 1\right)\frac{\left(U^2 - l_x^2Y^2\right)X}{U\left(U^2 - Y^2\right)} + \left(S_2^2 - 1\right)\frac{\left(U^2 - l_y^2Y^2\right)X}{U\left(U^2 - Y^2\right)} \\
&\quad + \frac{\left(U^2 - l_x^2Y^2\right)\left(U^2 - l_y^2Y^2\right)X^2}{U^2\left(U^2 - Y^2\right)^2}\Bigg]\left(C^2 - \frac{\left(U^2 - l_z^2Y^2\right)X}{U\left(U^2 - Y^2\right)}\right) \\
&+ \left(S_1S_2 + \frac{\left(il_zUY + l_xl_yY^2\right)X}{U\left(U^2 - Y^2\right)}\right)\frac{\left(il_xUY + l_yl_zY^2\right)X}{U\left(U^2 - Y^2\right)}\frac{\left(il_yUY + l_xl_zY^2\right)X}{U\left(U^2 - Y^2\right)} \\
&+ \left(S_1S_2 - \frac{\left(il_zUY - l_xl_yY^2\right)X}{U\left(U^2 - Y^2\right)}\right)\frac{\left(il_yUY - l_xl_zY^2\right)X}{U\left(U^2 - Y^2\right)}\frac{\left(il_xUY - l_yl_zY^2\right)X}{U\left(U^2 - Y^2\right)} \\
&+ \left(S_1^2 - 1 + \frac{\left(U^2 - l_y^2Y^2\right)X}{U\left(U^2 - Y^2\right)}\right)\frac{\left(l_y^2U^2Y^2 + l_x^2l_z^2Y^4\right)X^2}{U^2\left(U^2 - Y^2\right)^2} \\
&- \left(S_1^2S_2^2 + S_1S_2\frac{2l_xl_yY^2X}{U\left(U^2 - Y^2\right)} + \frac{\left(l_z^2U^2Y^2 + l_x^2l_y^2Y^4\right)X^2}{U^2\left(U^2 - Y^2\right)^2}\right)\left(C^2 - \frac{\left(U^2 - l_z^2Y^2\right)X}{U\left(U^2 - Y^2\right)}\right) \\
&+ \left(S_2^2 - 1 + \frac{\left(U^2 - l_x^2Y^2\right)X}{U\left(U^2 - Y^2\right)}\right)\frac{\left(l_x^2U^2Y^2 + l_y^2l_z^2Y^4\right)X^2}{U^2\left(U^2 - Y^2\right)^2}.
\end{aligned} \tag{A23}$$

This is a rather daunting expression, but we can employ the same strategy we used for the $O\left(q^2\right)$ coefficients. First, let us look at the $O\left(\left(\frac{1}{U\left(U^2-Y^2\right)}\right)^3\right)$ terms. Their combined coefficients become

$$\begin{aligned}
&\Big[- \left(U^2 - l_x^2Y^2\right)\left(U^2 - l_y^2Y^2\right)\left(U^2 - l_z^2Y^2\right) \\
&\quad + \left(il_zUY + l_xl_yY^2\right)\left(il_xUY + l_yl_zY^2\right)\left(il_yUY + l_xl_zY^2\right) \\
&\quad - \left(il_zUY - l_xl_yY^2\right)\left(il_yUY - l_xl_zY^2\right)\left(il_xUY - l_yl_zY^2\right) \\
&\quad + \left(U^2 - l_y^2Y^2\right)\left(l_y^2U^2Y^2 + l_x^2l_z^2Y^4\right) \\
&\quad + \left(l_z^2U^2Y^2 + l_x^2l_y^2Y^4\right)\left(U^2 - l_z^2Y^2\right) \\
&\quad + \left(U^2 - l_x^2Y^2\right)\left(l_x^2U^2Y^2 + l_y^2l_z^2Y^4\right)\Big]X^3.
\end{aligned} \tag{A24}$$

When multiplied out, the first line inside the brackets becomes

$$-U^6 + \left(l_x^2 + l_y^2 + l_z^4\right)U^4Y^2 - \left(l_x^2l_y^2 + l_x^2l_z^2 + l_y^2l_z^2\right)U^2Y^4 + l_x^2l_y^2l_z^2Y^6, \tag{A25}$$

the second and third lines of (A24) inside the brackets reduce to

$$-2\left(l_x^2 l_y^2 + l_x^2 l_z^2 + l_y^2 l_z^2\right) U^2 Y^4 + 2 l_x^2 l_y^2 l_z^2 Y^6, \tag{A26}$$

and the final three lines inside the bracket are equivalent to

$$\left(l_x^2 + l_y^2 + l_z^2\right) U^4 Y^2 + \left(l_x^2 l_z^2 - l_y^4 - l_z^4 + l_x^2 l_y^2 + l_y^2 l_z^2 - l_x^4\right) U^2 Y^4 - 3 l_x^2 l_y^2 l_z^2 Y^6. \tag{A27}$$

Employing (33) and canceling where appropriate, (A25), (A26), and (A27) combine to give

$$-U^6 + 2U^4 Y^2 - U^2 Y^4 = -U^2 \left(U^2 - Y^2\right)^2, \tag{A28}$$

which means, the contribution of the $O\left(\left(\frac{1}{U\left(U^2-Y^2\right)}\right)^3\right)$ terms is

$$-\frac{X^3}{U\left(U^2 - Y^2\right)}. \tag{A29}$$

Returning to (A23), we extract the $O\left(\left(\frac{1}{U\left(U^2-Y^2\right)}\right)^2\right)$ terms:

$$\begin{aligned}
\Big[&-\left(S_1^2 - 1\right)\left(U^2 - l_x^2 Y^2\right)\left(U^2 - l_z^2 Y^2\right) - \left(S_2^2 - 1\right)\left(U^2 - l_y^2 Y^2\right)\left(U^2 - l_z^2 Y^2\right) \\
&+ C^2 \left(U^2 - l_x^2 Y^2\right)\left(U^2 - l_y^2 Y^2\right) \\
&+ S_1 S_2 \left(i l_x U Y + l_y l_z Y^2\right)\left(i l_y U Y + l_x l_z Y^2\right) \\
&+ S_1 S_2 \left(i l_y U Y - l_x l_z Y^2\right)\left(i l_x U Y - l_y l_z Y^2\right) \\
&+ \left(S_1^2 - 1\right)\left(l_y^2 U^2 Y^2 + l_x^2 l_z^2 Y^4\right) \\
&+ 2 S_1 S_2 l_x l_y \left(U^2 - l_z^2 Y^2\right) Y^2 - C^2 \left(l_z^2 U^2 Y^2 + l_x^2 l_y^2 Y^4\right) \\
&+ \left(S_2^2 - 1\right)\left(l_x^2 U^2 Y^2 + l_y^2 l_z^2 Y^4\right) \Big] X^2.
\end{aligned} \tag{A30}$$

Using both (33) and (65), the first two lines inside the brackets multiply out to

$$\begin{aligned}
&\left(2C^2 + 1\right) U^4 + \left(S_1^2 l_x^2 + S_2 l_y^2 - C^2 - 1\right) U^2 Y^2 \\
&+ \left(\left(1 - S_1^2\right) l_x^2 l_z^2 + \left(1 - S_2^2\right) l_y^2 l_z^2 + C^2 l_x^2 l_y^2\right) Y^4,
\end{aligned} \tag{A31}$$

the third and fourth lines inside the bracket become

$$-2 l_x l_y U^2 Y^2 + 2 l_x l_y l_z^2 Y^4, \tag{A32}$$

and the final three lines inside the bracket can be rearranged as

$$\begin{aligned}&\left(S_1^2\left(l_y^2+l_z^2\right)+2S_1S_2l_xl_y+S_2^2\left(l_x^2+l_z^2\right)-1\right)U^2Y^2\\+&\left(\left(S_1^2-1\right)l_x^2l_z^2-2S_1S_2l_xl_yl_z^2-C^2l_x^2l_y^2+\left(S_2^2-1\right)l_y^2l_z^2\right)Y^4.\end{aligned}\tag{A33}$$

Combing (A31), (A32), (A33), and canceling terms when appropriate yields

$$\left(2C^2+1\right)U^4+\left(S_1^2+S_2^2-C^2-2\right)U^2Y^2=\left(2C^2+1\right)U^2\left(U^2-Y^2\right).\tag{A34}$$

Thus, the contribution of the $O\left(\left(\frac{1}{U(U^2-Y^2)}\right)^2\right)$ terms is

$$\frac{\left(2C^2+1\right)UX^2}{U\left(U^2-Y^2\right)}.\tag{A35}$$

The next terms of (A23) we analyze are those that are $O\left(\frac{1}{U(U^2-Y^2)}\right)$:

$$\begin{aligned}&\left(-\left(C^2+S_1^2S_2^2\right)\left(U^2-l_z^2Y^2\right)+\left(S_1^2-1\right)C^2\left(U^2-l_x^2Y^2\right)+\left(S_2^2-1\right)\left(U^2-l_y^2Y^2\right)\right.\\&\left.-2S_1S_2C^2l_xl_yY^2+S_1^2S_2^2\left(U^2-l_z^2Y^2\right)\right)X.\end{aligned}\tag{A36}$$

When this is multiplied out and the terms that cancel out each other are removed, we get

$$\begin{aligned}&\Big(-3C^2U^2+\left(l_z^2+l_x^2+l_y^2\right)C^2Y^2+\left(S_1^2+S_2^2\right)C^2U^2\\&-\left(l_x^2S_1^2+l_y^2S_2^2\right)C^2Y^2-2S_1S_2C^2l_xl_yY^2\Big)X.\end{aligned}\tag{A37}$$

Applying (33) to (A37) gives us

$$\left(-3C^2U^2+C^2Y^2+\left(S_1^2+S_2^2\right)C^2U^2-\left(l_x^2S_1^2+l_y^2S_2^2\right)C^2Y^2-2S_1S_2C^2l_xl_yY^2\right)X,\tag{A38}$$

and for a reason that becomes apparent later, we use $S_1^2+S_2^2=1-C^2$ to write this as

$$\left(-2C^2U^2+C^2Y^2-C^4U^2-\left(l_x^2S_1^2+l_y^2S_2^2\right)C^2Y^2-2S_1S_2C^2l_xl_yY^2\right)X.\tag{A39}$$

From (A39) we obtain the contribution of the $O\left(\frac{1}{U(U^2-Y^2)}\right)$ terms of (A23):

$$\frac{\left(-2C^2U^2+C^2Y^2-C^4U^2-\left(l_x^2S_1^2+l_y^2S_2^2\right)C^2Y^2-2S_1S_2C^2l_xl_yY^2\right)X}{U\left(U^2-Y^2\right)},\tag{A40}$$

Finally we come to the terms that are $O(1)$ with respect to $\frac{1}{U(U^2-Y^2)}$:

$$\left(C^2 + S_1^2 S_2^2\right) C^2 - S_1^2 S_2^2 C^2 = C^4. \tag{A41}$$

Similar to the reasoning behind (A17), we find that the contribution of the $O(1)$ terms of (A23) is

$$\frac{C^4 U \left(U^2 - Y^2\right)}{U \left(U^2 - Y^2\right)}. \tag{A42}$$

We are now ready to combine (A29), (A35), (A40), and (A42) to determine the $O(1)$ part of (A3):

$$\begin{aligned}
&\frac{-X^3 + \left(2C^2 + 1\right) U X^2}{U \left(U^2 - Y^2\right)} \\
&+ \frac{\left(-2C^2U^2 + C^2Y^2 - C^4U^2 - \left(l_x^2 S_1^2 + l_y^2 S_2^2\right) C^2Y^2 - 2S_1S_2C^2 l_x l_y Y^2\right) X}{U \left(U^2 - Y^2\right)} \\
&+ \frac{C^4 U \left(U^2 - Y^2\right)}{U \left(U^2 - Y^2\right)} \\
=&\frac{(U - X) \left(C^2U - X\right)^2 - C^2Y^2 \left(C^2U - X\right) - \left(l_x S_1 + l_y S_2\right)^2 C^2 X Y^2}{U \left(U^2 - Y^2\right)} \\
=&\frac{1}{U \left(U^2 - Y^2\right)} \epsilon.
\end{aligned} \tag{A43}$$