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Robust Co-Prime Sensing with Underwater Inflatable Passive Sonar Arrays

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Robust Co-Prime Sensing with Underwater Inflatable Passive Sonar Arrays

1. Executive Summary

“Robust Co-Prime Sensing with Underwater Inflatable Passive Sonar Arrays” is a collaborative project between Temple University and Florida Atlantic University (FAU). The research objective of the overall project is to develop an underwater inflatable co-prime sonar array (UICSA) prototype with associated signal processing capabilities to demonstrate the feasibility of an underwater deployable sensor network (UDSN) with sensing nodes comprising vertical co-prime arrays using size-A sonobuoys. To this end, the aim of the research team at FAU is the development of a UICSA node with compact stowed dimensions within the significant size, weight and power (SWaP) constraints of the size-A sonobuoy, whereas the goal of the research team at Temple University is the development of associated signal processing algorithms optimized for the UICSA node.

This report presents the results of the research performed at Temple University under Office of Naval Research grant N00014-18-1-2460 over the period of May 1, 2018 to October 15, 2021. Prof. Fauzia Ahmad is the principal investigator (PI) on this grant, which also supported three graduate students, Tongdi Zhou, Kenneth Mills, and Christopher Campbell, all at Temple University. In addition to FAU, we have also collaborated with Prof. Panos Markopolous (Rochester Institute of Technology), Prof. Elias Aboutanios (University of New South Wales, Australia), and Prof. Piya Pal (University of California at San Diego).

The focus of our studies has been in the following areas: (i) Generalized design of optimal autocorrelation combining in the mean-square sense for direction finding based on co-prime processing; (ii) Design of iterative Fourier methods for exploitation of full degrees-of-freedom (DOFs) in direction-of-arrival (DOA) estimation with co-prime arrays; (iii) DOA estimation in the presence of unknown correlated ambient noise; and (iv) Narrowband and multi-frequency co-prime processing for validating the DOA estimation performance of the devised UICSA prototype. These efforts have resulted in three journal articles (published), one journal article under review and seven conference papers.

Below is a summary of the research accomplishments in each of the aforementioned individual areas. A list of the publications generated under the support of this project is provided in Section 2. The full text of the journal publications/manuscripts is included in Section 3.

1.1. Generalized Design of Optimal Autocorrelation Combining

Co-prime arrays enable DOA estimation of an increased number of sources. To that end, the receiver estimates the autocorrelation matrix of a larger virtual uniform linear array (difference coarray), by applying selection or averaging to the physical array’s autocorrelation estimates, followed by spatial-smoothing. Both selection and averaging have been designed under no optimality criterion and attain arbitrary (suboptimal) Mean-Squared-Error (MSE) estimation

performance. We have designed a novel generalized co-prime array receiver that estimates the coarray autocorrelation with Minimum-MSE (MMSE), for any probability distribution of the source DOAs [1]. We have demonstrated via extensive numerical evaluations that the proposed MMSE approach returns superior autocorrelation estimates which, in turn, enable higher DOA estimation performance compared to the standard counterparts.

1.2. Fourier Methods for Exploitation of Full DOFs in DOA Estimation

The co-prime array offers significantly higher DOFs over its uniformly-spaced counterpart by leveraging the difference coarray. However, the coarray of a co-prime array is non-uniform, with a contiguous central portion and the missing elements or holes symmetrically placed outside of this uniformly-spaced part. Because of this non-uniformity, DOA estimation methods devised for use with uniform configurations can only be applied to the contiguous portion of the coarray, and thus cannot exploit the full DOFs afforded by co-prime arrays. We have devised two low-complexity Fourier-based iterative methods to address the non-uniformity of the corresponding difference coarray, thereby permitting exploitation of all offered DOFs [2]. The first method specifically accounts for the non-uniformity of the coarray in the iterative algorithm. The second approach estimates the contributions of the missing elements, thus emulating a uniform coarray. We have also shown that both methods achieve asymptotically unbiased DOA estimates. The superiority of the proposed approaches in terms of DOA estimation accuracy and the number of resolvable targets were clearly demonstrated via extensive simulations.

1.3. DOA Estimation in Spatially Correlated Noise

Ocean ambient noise exhibits spatial correlations, making the typical assumption of spatially white noise invalid in co-prime array processing. To address the challenges brought on by the presence of unknown correlated noise, we have presented two separate approaches for DOA estimation using co-prime arrays in underwater environments [3, 4]. The first approach is the sparsity-based “correlation-aware” LASSO (Co-LASSO) method, which does not require strong assumptions on the distribution or correlation of the noise, and its theoretical performance guarantees can be established for any (bounded) noise pattern. We applied Co-LASSO to a finite number of signal snapshots received with a co-prime array under spatially correlated noise, modeled as a first-order autoregressive process, and compared its performance with coarray-based MUSIC algorithm for low and high noise spatial correlations [3]. We clearly demonstrated that the Co-LASSO provides superior DOA estimates as compared to coarray-based MUSIC, provided the number of sources does not approach the maximum level of recoverable sparsity. The second proposed approach caters not only to correlated noise, but also accounts for local scattering at the source leading to a spatially-spread source [4]. The angular spread of the source is described in terms of a Gaussian probability density function centered around its nominal DOA and the noise is modeled as a first-order autoregressive process. The source DOA and angular spread were estimated by solving a non-convex optimization problem using Covariance Matrix Adaptation Evolution Strategy (CMA-ES). Further, to address the susceptibility of the co-prime array to structural deflections, we extended this approach to accurately perform source parameter estimation in the presence of sensor perturbations and unknown correlated noise. More specifically, using Jacobi-Anger approximation of manifold separation, the array steering vector was decomposed into a matrix-vector product, with the matrix and the vector being functions of the array and source parameters, respectively.

The sensor perturbations along with the other unknown source and noise parameters were iteratively estimated using CMA-ES. Simulated data measurements with a co-prime array from a spatially-spread source were used for performance evaluation of the proposed approach.

1.4. Performance Validation of the UICSA Prototype

We provided array design specifications and signal processing support at various stages of the UICSA prototype development and testing. The UICSA will be a crucial component of a UDSN that can be rapidly deployed using compact autonomous underwater vehicles (AUVs). The UICSA initially is packed in a compact container to fit the payload space of an AUV. After deployment, the UICSA expands to its predetermined full length to acquire sensing data for source localization. More specifically, the mechanical compression of the UICSA is achieved through a non-rigid array support structure, which consists of flexible inflatable segments between adjoining hydrophones that are folded in order to package the UICSA for deployment. The system exploits compression in hydrophone layouts by utilizing a sparse array configuration, namely the co-prime array since it requires fewer hydrophones than a uniform linear array of the same length to estimate a given number of sources. With two-way compression, the storage, handling, and transportation of the compactly designed UICSA is convenient, particularly for the AUVs with limited payload space. Laboratory studies of the UICSA prototype were conducted and the measured data were processed to demonstrate the source DOA estimation performance [5]. The UICSA prototype was also deployed in FAU's Harbor Branch channel and field tests were conducted to validate the source localization performance [6]. Both narrowband and multi-frequency operation/processing were considered and demonstrated for laboratory and field tests.

1.5. References

- [1] S. Qin, D. G. Chachlakis, T. Zhou, F. Ahmad, and P. Markopoulos, "Minimum mean-squared-error autocorrelation processing in coprime arrays," *Digital Signal Processing*, vol. 114, Article ID 103034, July 2021.
- [2] K. R. Mills, F. Ahmad, and E. Aboutanios, "Coarray-domain iterative direction-of-arrival estimation with coprime arrays," *Digital Signal Processing*, vol. 122, Article ID 103332, April 2022.
- [3] F. Ahmad, P. Pal, and T. Zhou, "Reliable DOA estimation in spatially correlated noise with nonuniform arrays," *Proc. Asilomar Conference on Signals, Systems, and Computers*, Pacific Grove, CA, Oct./Nov. 2021.
- [4] T. Zhou and F. Ahmad, "DOA estimation of a spatially-spread source with a perturbed vertical co-prime array in correlated noise," *IEEE Journal of Oceanic Engineering*, Under review.
- [5] T. Zhou, F. Ahmad, B. Ouyang, and Y. Li, "Near-field source localization with an underwater inflatable sonar co-prime array," *Proc. SPIE Big Data: Learning, Analytics, and Applications*, Baltimore, MD, vol. 10989, Apr. 2019, paper ID 1098905.
- [6] Y. Li, B. Ouyang, T. Zhou, J. Thomas, S. Bavar, E. Weber, L. Michieletto, T.-C. Su, and F. Ahmad, "Laboratory and field experimental study of underwater inflatable co-prime sonar array (UICSA)," *Journal of Civil Engineering and Construction*, In press.

2. List of Publications

Note that the full manuscripts of the journal papers listed below are included in Chapter 3.

Journal Articles

1. D. G. Chachlakis, T. Zhou, F. Ahmad, and P. Markopoulos, “Minimum mean-squared-error autocorrelation processing in coprime arrays,” *Digital Signal Processing*, vol. 114, Article ID 103034, July 2021.
2. K. R. Mills, F. Ahmad, and E. Aboutanios, “Coarray-domain iterative direction-of-arrival estimation with coprime arrays,” *Digital Signal Processing*, vol. 122, Article ID 103332, April 2022.
3. Y. Li, B. Ouyang, T. Zhou, J. Thomas, S. Bavar, E. Weber, L. Michieletto, T.-C. Su, and F. Ahmad, “Laboratory and field experimental study of underwater inflatable co-prime sonar array (UICSA),” *Journal of Civil Engineering and Construction*, In press.
4. T. Zhou and F. Ahmad, “DOA estimation of a spatially-spread source with a perturbed vertical co-prime array in correlated noise,” *IEEE Journal of Oceanic Engineering*, Under review.

Conference Papers

1. T. Zhou, F. Ahmad, B. Ouyang, and Y. Li, “Near-field source localization with an underwater inflatable sonar co-prime array,” *Proc. SPIE Big Data: Learning, Analytics, and Applications*, Baltimore, MD, Apr. 2019.
2. Y. Li, J. Thomas, B. Ouyang, T.-C. Su, T. Zhou, and F. Ahmad, “Design and experimental study of underwater inflatable co-prime sonar array (UICSA),” *Proc. MTS/IEEE OCEANS Conference & Exposition*, Seattle, WA, Oct. 2019.
3. T. Zhou and F. Ahmad, “DOA estimation in the presence of local scattering and correlated noise using co-prime arrays,” *Proc. IEEE Underwater Acoustic Signal Processing Workshop*, West Greenwich, RI, Oct. 2019.
4. K. Mills, E. Aboutanios, and F. Ahmad, “Iterative interpolated beamforming for exploiting full degrees-of-freedom in DOA estimation with co-prime arrays,” *Proc. IEEE International Workshop on Computational Advances in Multi-sensor Adaptive Processing*, Guadeloupe, West Indies, Dec. 2019.
5. Y. Li, J. Thomas, B. Ouyang, T.-C. Su, and F. Ahmad, “Numerical study of underwater inflatable co-prime sonar array (UICSA),” *Proc. 39th ASME International Conference on Ocean, Offshore, and Arctic Engineering*, Fort Lauderdale, FL, Jun./Jul. 2020.
6. C. A. Schaff, Y. Li, B. Ouyang, C. D. Ouden, T. Zhou, and F. Ahmad, “Development of a low-cost subsea datalogger for passive sonar arrays,” *Proc. SPIE Big Data III: Learning, Analytics, and Applications*, Online, April 2021.
7. F. Ahmad, P. Pal, and T. Zhou, “Reliable DOA estimation in spatially correlated noise with nonuniform arrays,” *Proc. Asilomar Conference on Signals, Systems, and Computers*, Pacific Grove, CA, Oct./Nov. 2021.

3. Selected Publications

1. Minimum mean-squared-error autocorrelation processing in coprime arrays

Publish in *Digital Signal Processing*, vol. 114, Article ID 103034, July 2021

2. Coarray-domain iterative direction-of-arrival estimation with coprime arrays

Published in *Digital Signal Processing*, vol. 122, Article ID 103332, April 2022

3. DOA estimation exploiting a uniform linear array with multiple co-prime frequencies

Under review in *IEEE Journal on Oceanic Engineering*

4. Laboratory and field experimental study of underwater inflatable co-prime sonar array (UICSA)

Accepted for publication in *Journal of Civil Engineering and Construction*

3.1 Minimum Mean-Squared-Error Autocorrelation Processing in Coprime Arrays

Abstract

Coprime arrays enable Direction-of-Arrival (DoA) estimation of an increased number of sources. To that end, the receiver estimates the autocorrelation matrix of a larger virtual uniform linear array (coarray), by applying selection or averaging to the physical array's autocorrelation estimates, followed by spatial-smoothing. Both selection and averaging have been designed under no optimality criterion and attain arbitrary (suboptimal) Mean-Squared-Error (MSE) estimation performance. In this work, we design a novel coprime array receiver that estimates the coarray autocorrelation with Minimum-MSE (MMSE), for any probability distribution of the source DoAs. Our extensive numerical evaluation illustrates that the proposed MMSE approach returns superior autocorrelation estimates which, in turn, enable higher DoA estimation performance compared to standard counterparts.

1. Introduction

Coprime Arrays (CAs) are non-uniform linear arrays with element locations determined by a pair of distinct coprime numbers. CAs are a special class of sparse arrays [1, 2] which are often preferred due to their desirable bearing properties –i.e., enhanced Degrees-of-Freedom (DoF) and closed-form expressions for element-locations. CAs have attracted significant research interest over the past years and have been successfully employed in applications such as Direction-of-Arrival (DoA) estimation [3–15], beamforming [16–18], interference localization and mitigation in satellite systems [19], and space-time adaptive processing [20, 21], to name a few. Interpolation methods for CAs which further increase DoF [22–26] have also been extensively studied. Furthermore, scholars have considered CAs for underwater localization [27, 28], channel estimation in MIMO communications via tensor decomposition [29], receivers capable of two-dimensional DoA estimation [30], and receivers on moving platforms which also promote increased number of DoF [31–33].

In standard DoA estimation with CAs [1], the receiver conducts a series of intelligent processing steps and assembles an autocorrelation matrix which corresponds to a larger virtual Uniform Linear Array (ULA), known as *coarray*. Accordingly, CAs enable the identification of more sources than physical sensors compared to equal-length ULAs. Processing at a coprime array receiver commences with the estimation of the nominal (true) physical-array autocorrelations based on a collection of received-signal snapshots. The receiver processes the estimated autocorrelations so that each coarray element is represented by one autocorrelation estimate. Next, the processed autocorrelations undergo spatial smoothing, forming an autocorrelation matrix estimate which corresponds to the coarray. Finally, a DoA estimation approach such as the MUltiple SIgnal Classification (MUSIC) algorithm can be applied to the resulting autocorrelation matrix estimate for identifying the source directions.

At the autocorrelation processing step, the estimated autocorrelations are commonly processed by *selection combining* [1], retaining only one autocorrelation sample for each coarray element. Alternatively, an autocorrelation estimate for each coarray element is obtained by *averaging combining* [11] all available sample-estimates corresponding to a particular coarray element. The two methods coincide in Mean-Squared-Error (MSE) estimation performance when applied to the nominal physical-array autocorrelations –which the receiver could only estimate with asymptotically large number of received-signal snapshots. In practice, due to a finite number of received-signal snapshots available at the receiver and the fact that these methods have been designed under no optimality criterion, the estimated autocorrelations diverge from the nominal ones and attain arbitrary MSE performance. In this case, the two methods no longer coincide in MSE estimation performance. In fact, it was recently shown [3] that averaging combining attains superior estimation performance compared to selection combining with respect to the MSE metric.

Motivated by prior works which treat source angles as statistical random variables [34, 35], in this work, we make the mild assumption that the DoAs are independently and identically distributed random variables and design a novel coprime array receiver equipped with a linear autocorrelation combiner, which is designed under the Minimum-MSE (MMSE) optimality criterion. The pro-

posed MMSE combiner minimizes, in the mean –i.e., for any configuration of DoAs– the error in estimating the physical-array autocorrelations with respect to the MSE metric. We conduct extensive numerical studies and compare the performance of the proposed MMSE combiner to existing counterparts, with respect to autocorrelation estimation error and DoA estimation.

The contributions of this work are three-fold: (i) We introduce the MMSE combining solution for any DoA probability distribution.¹ (ii) We offer formal derivation of the proposed MMSE combiner and present a complexity analysis. (iii) We offer formal proofs for the closed-form MSE expressions of selection and averaging that were originally presented in the preliminary work [3].

The rest of this paper is organized as follows. In Section 2, we present the signal model and state the problem of interest. In Section 3, we review existing selection and averaging autocorrelation combining methods for coprime arrays, providing their closed-form MSE expressions [3], and offering formal mathematical proofs for these expressions. We present the proposed MMSE autocorrelation combining approach in Section 4. Next, in Section 5, we conduct extensive numerical performance evaluations of the proposed combining approach and compare against existing counterparts. Conclusions are drawn in Section 6.

2. Signal Model

Consider coprime naturals (M, N) such that $M < N$. A coprime array equipped with $L = 2M + N - 1$ elements is formed by overlapping a ULA with N antenna elements at positions $p_{M,i} = (i - 1)M\Delta$, $i = 1, 2, \dots, N$, and a ULA equipped with $2M - 1$ antenna elements at positions $p_{N,i} = iN\Delta$, $i = 1, 2, \dots, 2M - 1$. The reference unit-spacing Δ is typically set to one-half wavelength at the operating frequency. The positions of the L elements are the entries of the element-location vector $\mathbf{p} \stackrel{\text{def}}{=} \text{sort}([p_{M,1}, \dots, p_{M,N}, p_{N,1}, \dots, p_{N,2M-1}]^T)$, where $\text{sort}(\cdot)$ sorts the entries of its vector argument in ascending order and the superscript ‘T’ denotes matrix transpose. We assume that narrowband signals impinge on the array from $K < MN + M$ sources with propagation speed c and carrier frequency f_c . Assuming far-field conditions, a signal from source $k \in \{1, 2, \dots, K\}$ im-

¹The preliminary work [4] considered the special case of uniformly distributed DOAs over $(-\frac{\pi}{2}, \frac{\pi}{2}]$.

pinges on the array from direction $\theta_k \in (-\frac{\pi}{2}, \frac{\pi}{2}]$ with respect to the broadside. The array response vector for source k is $\mathbf{s}(\theta_k) \stackrel{\text{def}}{=} [v(\theta_k)^{[\mathbf{p}]_1}, \dots, v(\theta_k)^{[\mathbf{p}]_L}]^\top \in \mathbb{C}^{L \times 1}$, with $v(\theta) \stackrel{\text{def}}{=} \exp\left(\frac{-j2\pi f_c}{c} \sin(\theta)\right)$ for every $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2}]$. Accordingly, the q th collected received-signal snapshot is of the form

$$\mathbf{y}_q = \sum_{k=1}^K \mathbf{s}(\theta_k) \xi_{q,k} + \mathbf{n}_q \in \mathbb{C}^{L \times 1}, \quad (1)$$

where $\xi_{q,k} \sim \mathcal{CN}(0, d_k)$ is the q th symbol for source k (power-scaled and flat-fading-channel processed) and $\mathbf{n}_q \sim \mathcal{CN}(\mathbf{0}_L, \sigma^2 \mathbf{I}_L)$ models Additive White Gaussian Noise (AWGN). We make the common assumptions that the random variables are statistically independent across different snapshots and that symbols from different sources are independent of each other and of every entry of $\mathbf{n}_q$. The received-signal autocorrelation matrix is given by

$$\mathbf{R}_y \stackrel{\text{def}}{=} \mathbb{E}\{\mathbf{y}_q \mathbf{y}_q^H\} = \mathbf{S} \text{diag}(\mathbf{d}) \mathbf{S}^H + \sigma^2 \mathbf{I}_L, \quad (2)$$

where $\mathbf{d} \stackrel{\text{def}}{=} [d_1, d_2, \dots, d_K]^\top \in \mathbb{R}_+^{K \times 1}$ is the source-power vector and $\mathbf{S} \stackrel{\text{def}}{=} [\mathbf{s}(\theta_1), \mathbf{s}(\theta_2), \dots, \mathbf{s}(\theta_K)] \in \mathbb{C}^{L \times K}$ is the array-response matrix. We define

$$\mathbf{r} \stackrel{\text{def}}{=} \text{vec}(\mathbf{R}_y) = \sum_{i=1}^K \mathbf{a}(\theta_i) d_i + \sigma^2 \mathbf{i}_L \in \mathbb{C}^{L^2 \times 1}, \quad (3)$$

where $\text{vec}(\cdot)$ returns the column-wise vectorization of its matrix argument, $\mathbf{a}(\theta_i) \stackrel{\text{def}}{=} \mathbf{s}(\theta_i)^* \otimes \mathbf{s}(\theta_i)$, $\mathbf{i}_L \stackrel{\text{def}}{=} \text{vec}(\mathbf{I}_L)$, the superscript $*$ denotes complex conjugate, and $\otimes$ is the Kronecker product of matrices [36]. By coprime number theory [1], for every $n \in \{-L' + 1, -L' + 2, \dots, L' - 1\}$ with $L' \stackrel{\text{def}}{=} MN + M$, there exists a well-defined set of indices $\mathcal{J}_n \subset \{1, 2, \dots, L^2\}$, such that

$$[\mathbf{a}(\theta)]_j = v(\theta)^n \quad \forall j \in \mathcal{J}_n, \quad (4)$$

for every $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2}]$. We henceforth consider that $\mathcal{J}_n$ contains all j indices that satisfy (4). In view of (4), a coprime array receiver assembles a linear combining matrix $\mathbf{E} \in \mathbb{R}^{L^2 \times (2L' - 1)}$ and

forms a length- $(2L' - 1)$ autocorrelation-vector $\mathbf{r}_{\text{co}}$, each element of which corresponds to a single set $\mathcal{J}_n$, for every $n \in \{1 - L', 2 - L', \dots, L' - 1\}$, by conducting linear *processing*² (e.g., $\mathbf{E}^\top \mathbf{r}$) to the autocorrelations in $\mathbf{r}$. That is, there exists linear combiner $\mathbf{E}$ such that

$$\mathbf{r}_{\text{co}} = \mathbf{E}^\top \mathbf{r} = \sum_{k=1}^K \mathbf{a}_{\text{co}}(\theta_k) d_k + \sigma^2 \mathbf{e}_{L', 2L'-1}, \quad (5)$$

where $\mathbf{a}_{\text{co}} \stackrel{\text{def}}{=} [v(\theta)^{1-L'}, v(\theta)^{2-L'}, \dots, v(\theta)^{L'-1}]$ for any $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2}]$, and, for any $p \leq P \in \mathbb{N}_+$, $\mathbf{e}_{p,P}$ is the p th column of $\mathbf{I}_P$. Thereafter, the receiver applies spatial-smoothing to organize the sampled autocorrelations as the matrix

$$\mathbf{Z} \stackrel{\text{def}}{=} \Phi(\mathbf{r}_{\text{co}}) \stackrel{\text{def}}{=} \mathbf{F}(\mathbf{I}_{L'} \otimes \mathbf{r}_{\text{co}}) \in \mathbb{C}^{L' \times L'}, \quad (6)$$

where $\mathbf{F} \stackrel{\text{def}}{=} [\mathbf{F}_1, \mathbf{F}_2, \dots, \mathbf{F}_{L'}]$ and $\mathbf{F}_m \stackrel{\text{def}}{=} [\mathbf{0}_{L' \times (L'-m)}, \mathbf{I}_{L'}, \mathbf{0}_{L' \times (m-1)}] \forall m \in \{1, 2, \dots, L'\}$. Importantly, under nominal statistics $\mathbf{Z}$ becomes the autocorrelation matrix of a length- L' ULA with antenna elements at locations $\{0, 1, \dots, L' - 1\}\Delta$. That is,

$$\mathbf{Z} = \mathbf{S}_{\text{co}} \text{diag}(\mathbf{d}) \mathbf{S}_{\text{co}}^H + \sigma^2 \mathbf{I}_{L'}, \quad (7)$$

where it holds that $[\mathbf{S}_{\text{co}}]_{m,k} = v(\theta_k)^{m-1}$ for every $m \in \{1, 2, \dots, L'\}$ and $k \in \{1, 2, \dots, K\}$. Standard MUSIC DoA estimation can be applied to $\mathbf{Z}$. Let the columns of $\mathbf{U} \in \mathbb{C}^{L' \times K}$ be the dominant left-hand singular vectors of $\mathbf{Z}$, corresponding to its K highest singular values, acquired by means of Singular-Value-Decomposition (SVD). Defining $\mathbf{v}(\theta) = [1, v(\theta), \dots, v(\theta)^{L'-1}]^\top$, we can accurately decide that $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2}]$ belongs in $\Theta \stackrel{\text{def}}{=} \{\theta_1, \theta_2, \dots, \theta_K\}$ if $(\mathbf{I}_{L'} - \mathbf{U}\mathbf{U}^H)\mathbf{v}(\theta) = \mathbf{0}_{L'}$ is satisfied for some θ . Equivalently, we can resolve the angles in Θ by the K (smallest) local minima of the standard MUSIC spectrum [37]

$$P_{\text{MUSIC}}(\theta) = \left\| (\mathbf{I}_{L'} - \mathbf{U}\mathbf{U}^H) \mathbf{v}(\theta) \right\|_2^2. \quad (8)$$

²Existing coprime array autocorrelation processing methods in the literature are reviewed in Section 3.

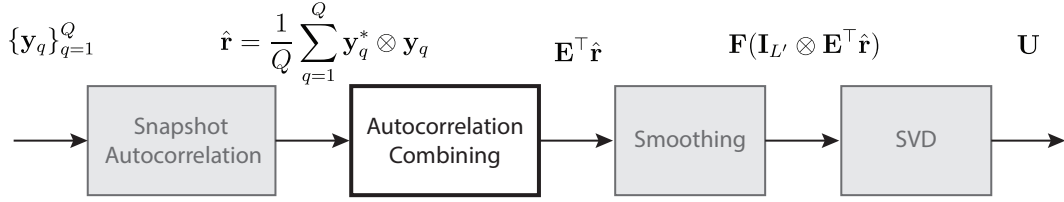


Figure 1: Coprime processing steps: from a collection of samples $\{\mathbf{y}_q\}_{q=1}^Q$ to the estimated coarray signal-subspace basis $\mathbf{U}$.

In practice, $\mathbf{R}_y$ in (2) is unknown to the receiver and sample-average estimated by a collection of Q received-signal snapshots in $\mathbf{Y} = [\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_Q]$ by

$$\hat{\mathbf{R}}_y = \frac{1}{Q} \sum_{q=1}^Q \mathbf{y}_q \mathbf{y}_q^H. \quad (9)$$

Accordingly, the physical-array autocorrelation-vector $\mathbf{r}$ in (3) is estimated by

$$\hat{\mathbf{r}} \stackrel{\text{def}}{=} \text{vec}(\hat{\mathbf{R}}_y) = \frac{1}{Q} \sum_{q=1}^Q \mathbf{y}_q^* \otimes \mathbf{y}_q. \quad (10)$$

The receiver then conducts linear combining to the estimated autocorrelation vector $\hat{\mathbf{r}}$ to obtain an estimate of $\mathbf{r}_{\text{co}}$, $\hat{\mathbf{r}}_{\text{co}} = \mathbf{E}^\top \hat{\mathbf{r}}$. The estimation error $\|\mathbf{r}_{\text{co}} - \hat{\mathbf{r}}_{\text{co}}\|_2$ depends on how well the linear combiner $\mathbf{E}$ estimates the nominal physical-array autocorrelations. Accordingly, $\mathbf{Z}$ in (7) is estimated by $\hat{\mathbf{Z}} \stackrel{\text{def}}{=} \Phi(\hat{\mathbf{r}}_{\text{co}})$.

Finally, MUSIC DoA estimation can be applied using the K dominant left-hand singular vectors of $\hat{\mathbf{Z}}$ instead of those of $\mathbf{Z}$. Of course, in practice, there is an inherent DoA estimation error due to the mismatch between $\mathbf{Z}$ and $\hat{\mathbf{Z}}$. A schematic illustration of the coprime array processing steps presented above is offered in Fig. 1. In the sequel, we review the most commonly considered autocorrelation combining approaches in the coprime array literature and conduct a formal MSE analysis.

3. Technical Background on Autocorrelation Combining

3.1. Selection Combining [1]

The most commonly considered autocorrelation combining method is *selection combining* based on which the receiver selects any single index $j_n \in \mathcal{J}_n$, for $n \in \{-L' + 1, \dots, L' - 1\}$, and forms the *selection combining* matrix

$$\mathbf{E}_{\text{sel}} \stackrel{\text{def}}{=} [\mathbf{e}_{j_{1-L'}, L^2}, \mathbf{e}_{j_{2-L'}, L^2}, \dots, \mathbf{e}_{j_{L'-1}, L^2}] \in \mathbb{R}^{L^2 \times (2L'-1)}, \quad (11)$$

by which it processes the autocorrelations in $\mathbf{r}$, discarding by selection all duplicates –i.e., every entry with index in $\mathcal{J}_n \setminus j_n$, for every n – to form autocorrelation vector

$$\mathbf{r}_{\text{sel}} \stackrel{\text{def}}{=} \mathbf{E}_{\text{sel}}^\top \mathbf{r} \in \mathbb{R}^{2L'-1}. \quad (12)$$

Importantly, when the nominal entries of $\mathbf{r}$ are known to the receiver, $\mathbf{r}_{\text{sel}}$ coincides with $\mathbf{r}_{\text{co}}$ in (5), thus, applying spatial smoothing on $\mathbf{r}_{\text{sel}}$ yields the exact coarray autocorrelation matrix $\mathbf{Z} = \Phi(\mathbf{r}_{\text{sel}})$. In contrast, when $\mathbf{r}$ is unknown to the receiver and estimated by $\hat{\mathbf{r}}$ in (10), $\mathbf{r}_{\text{sel}}$ in (12) is estimated by

$$\hat{\mathbf{r}}_{\text{sel}} = \mathbf{E}_{\text{sel}}^\top \hat{\mathbf{r}} \in \mathbb{R}^{2L'-1}. \quad (13)$$

Accordingly, the coarray autocorrelation matrix is estimated by $\hat{\mathbf{Z}}_{\text{sel}} \stackrel{\text{def}}{=} \Phi(\hat{\mathbf{r}}_{\text{sel}})$.

3.2. Averaging Combining [11]

Instead of selecting a single index in $\mathcal{J}_n$ by discarding duplicates, averaging combining conducts averaging on all autocorrelation estimates in $\mathcal{J}_n$, for every $n \in \{1 - L', 2 - L', \dots, L' - 1\}$. That is, the receiver forms the *averaging combining* matrix $\mathbf{E}_{\text{avg}}$, where, for every $i \in \{1, \dots, 2L' - 1\}$,

$$[\mathbf{E}_{\text{avg}}]_{:,i} \stackrel{\text{def}}{=} \frac{1}{|\mathcal{J}_{i-L'}|} \sum_{j \in \mathcal{J}_{i-L'}} \mathbf{e}_{j, L^2}. \quad (14)$$

$|\cdot|$ denotes the cardinality (number of elements in a set) of its argument. Then, it processes the autocorrelation vector $\mathbf{r}$ to obtain

$$\mathbf{r}_{\text{avg}} \stackrel{\text{def}}{=} \mathbf{E}_{\text{avg}}^{\top} \mathbf{r} \in \mathbb{R}^{2L'-1}. \quad (15)$$

By (4) and the fact that $[\mathbf{i}_L]_j$ equals 1, if $j \in \mathcal{J}_0$ and 0 otherwise, it holds that, for any $n \in \{-L' + 1, \dots, L' - 1\}$, $[\mathbf{r}]_j = \mathbf{e}_{j,L^2}^{\top} \mathbf{r}$ takes a constant value for every $j \in \mathcal{J}_n$. Thus, $\mathbf{r}_{\text{avg}}$ coincides with $\mathbf{r}_{\text{sel}}$ and $\mathbf{r}_{\text{co}}$. Therefore, similar to the selection combining approach, when $\mathbf{r}$ is known to the receiver, applying spatial smoothing on $\mathbf{r}_{\text{avg}}$ yields $\mathbf{Z} = \Phi(\mathbf{r}_{\text{avg}})$. In practice, when $\mathbf{R}_y$ in (2) is estimated by $\hat{\mathbf{R}}_y$ in (9), $\mathbf{r}_{\text{avg}}$ is estimated by

$$\hat{\mathbf{r}}_{\text{avg}} \stackrel{\text{def}}{=} \mathbf{E}_{\text{avg}}^{\top} \hat{\mathbf{r}} \in \mathbb{R}^{2L'-1}. \quad (16)$$

Accordingly, $\mathbf{Z}$ is estimated by $\hat{\mathbf{Z}}_{\text{avg}} \stackrel{\text{def}}{=} \Phi(\hat{\mathbf{r}}_{\text{avg}})$.

3.3. Closed-form MSE Expressions for Selection and Averaging Combining [3]

In general, estimates $\hat{\mathbf{r}}_{\text{sel}}$ and $\hat{\mathbf{r}}_{\text{avg}}$ diverge from $\mathbf{r}_{\text{co}}$ and attain $\text{MSE err}_r(\hat{\mathbf{r}}_{\text{sel}}) \stackrel{\text{def}}{=} \mathbb{E}\{\|\mathbf{r}_{\text{co}} - \hat{\mathbf{r}}_{\text{sel}}\|_2^2\}$ and $\text{err}_r(\hat{\mathbf{r}}_{\text{avg}})$ respectively. $\hat{\mathbf{Z}}_{\text{sel}}$ and $\hat{\mathbf{Z}}_{\text{avg}}$ diverge from the true $\mathbf{Z}$ and attain $\text{MSE err}_Z(\hat{\mathbf{Z}}_{\text{sel}}) \stackrel{\text{def}}{=} \mathbb{E}\{\|\mathbf{Z} - \hat{\mathbf{Z}}_{\text{sel}}\|_F^2\}$ and $\text{err}_Z(\hat{\mathbf{Z}}_{\text{avg}})$ respectively. Closed-form MSE expressions for the errors above were first presented in the form of Lemmas and Propositions (proofs omitted) in [3]. For completeness purposes, we present again the MSE expressions in the form of Lemmas and Propositions, this time, accompanied by formal mathematical proofs.

For any sample support Q , the following Lemma 1 and Lemma 2 express in closed-form the MSE attained by $\hat{\mathbf{r}}_{\text{sel}}$.

Lemma 1. *For any $n \in \{-L' + 1, -L' + 2, \dots, L' - 1\}$ and $j \in \mathcal{J}_n$, it holds $e = \mathbb{E}\{|\mathbf{r}]_j - [\hat{\mathbf{r}}]_j|^2\} = \frac{(\mathbf{1}_K^{\top} \mathbf{d} + \sigma^2)^2}{Q}$.* □

Lemma 2. *$\hat{\mathbf{r}}_{\text{sel}}$ attains $\text{MSE err}_r(\hat{\mathbf{r}}_{\text{sel}}) = (2L' - 1)e$.* □

In view of Lemma 2, the following Proposition naturally follows.

Proposition 1. $\hat{\mathbf{Z}}_{\text{sel}}$ attains $\text{MSE err}_Z(\hat{\mathbf{Z}}_{\text{sel}}) = L'^2 e$. $\square$

Expectedly, as the sample-support Q grows asymptotically, e , $\text{err}_r(\hat{\mathbf{r}}_{\text{sel}})$, and $\text{err}_Z(\hat{\mathbf{Z}}_{\text{sel}})$ converge to zero.

For any sample support Q , the following Lemma 3 and Lemma 4 express in closed-form the MSE attained by $\hat{\mathbf{r}}_{\text{avg}}$, where $\hat{\mathbf{p}} \stackrel{\text{def}}{=} \mathbf{p} \otimes \mathbf{1}_L$, $\hat{\omega}_{i,j} \stackrel{\text{def}}{=} [\hat{\mathbf{p}}]_i - [\hat{\mathbf{p}}]_j$, and $\mathbf{z}_{i,j} \stackrel{\text{def}}{=} [v(\theta_1)^{\hat{\omega}_{i,j}} v(\theta_2)^{\hat{\omega}_{i,j}} \dots v(\theta_K)^{\hat{\omega}_{i,j}}]^H$.

Lemma 3. For any $n \in \{-L' + 1, \dots, L' - 1\}$ and $j_n \in \mathcal{J}_n$, it holds that

$$e_n = \mathbb{E} \left\{ \left| [\mathbf{r}]_{j_n} - \frac{1}{|\mathcal{J}_n|} \sum_{j \in \mathcal{J}_n} [\hat{\mathbf{r}}]_j \right|^2 \right\} = \frac{1}{Q} \left(\frac{2\sigma^2 \mathbf{1}_K^T \mathbf{d} + \sigma^4}{|\mathcal{J}_n|} + \sum_{i \in \mathcal{J}_n} \sum_{j \in \mathcal{J}_n} \frac{|\mathbf{z}_{i,j}^H \mathbf{d}|^2}{|\mathcal{J}_n|^2} \right). \quad (17)$$

Lemma 4. $\hat{\mathbf{r}}_{\text{avg}}$ attains $\text{MSE err}_r(\hat{\mathbf{r}}_{\text{avg}}) = \sum_{n=1-L'}^{L'-1} e_n$. $\square$

By Lemma 4, the following Proposition naturally follows.

Proposition 2. $\hat{\mathbf{Z}}_{\text{avg}}$ attains $\text{MSE err}_Z(\hat{\mathbf{Z}}_{\text{avg}}) = \sum_{m=1}^{L'} \sum_{n=1-m}^{L'-m} e_n$. $\square$

Similar to selection combining, as the sample-support Q grows asymptotically, e_n , $\text{err}_r(\hat{\mathbf{r}}_{\text{avg}})$, and $\text{err}_Z(\hat{\mathbf{Z}}_{\text{avg}})$ converge to zero. Complete proofs for the above Lemmas and Propositions are offered for the first time in the Appendix. Further remarks, in view of the above closed-form MSE expressions, are offered in [3].

4. Proposed Minimum Mean-Squared-Error Autocorrelation Combining

We focus on the autocorrelation combining step of coprime-array processing (see Fig. 1) where the receiver applies linear combining matrix $\mathbf{E}$ to the estimated autocorrelations of the physical array. Arguably, a preferred receiver will attain consistently –i.e., for any possible configuration of DoAs, Θ – low squared-estimation error $\|\mathbf{r}_{\text{co}} - \mathbf{E}^T \hat{\mathbf{r}}\|_2^2$. If this error is exactly equal to zero, then at the fourth step of coprime-array processing, MUSIC will identify the exact DoAs in Θ .

In this work, we treat the DoAs in Θ as independently and identically distributed (i.i.d.) random variables and design a coprime array receiver equipped with linear combiner $\mathbf{E}$ such that $\|\mathbf{r}_{\text{co}} -$

$\mathbf{E}^\top \hat{\mathbf{r}}\|_2^2$ is minimized in the mean. We assume that, $\forall k$, $\theta_k \in \Theta$ is a random variable with probability distribution $\mathcal{D}(a, b)$ –e.g., uniform, truncated normal, or, other– where a and b denote the limits of the support of $\mathcal{D}$. Under this assumption, we seek the Minimum-MSE combining matrix $\mathbf{E}$ that minimizes $\mathbb{E}\{\|\mathbf{r}_{\text{co}} - \mathbf{E}^\top \hat{\mathbf{r}}\|_2^2\}$.

Probability distributions of angular variables is not a new concept. In fact, angular variables have been modeled by the *von Mises* Probability Density Function (PDF) which can include (or, nearly approximate) standard distributions such as uniform, Gaussian, and wrapped Gaussian, to name a few, by tuning a parameter in the PDF expression [34, 35]. Angular distributions have also been considered for Bayesian-based beamforming [38, 39].

In the most general case and in lieu of any pertinent prior information at the receiver, the DoAs in Θ can, for instance, be assumed to be the uniformly distributed in $(-\frac{\pi}{2}, \frac{\pi}{2}]$ –i.e., $\mathcal{D}(a, b) \equiv \mathcal{U}(-\frac{\pi}{2}, \frac{\pi}{2})$. In the sequel, we derive the Minimum-MSE combining matrix *for any* continuous probability distribution $\mathcal{D}(a, b)$, $-\frac{\pi}{2} < a < b \leq \frac{\pi}{2}$.

First, we introduce new notation to the problem statement and formulate the MSE-Minimization problem. We let

$$\mathbf{A} \stackrel{\text{def}}{=} \left[\mathbf{S} \text{diag}([\sqrt{d_1}, \sqrt{d_2}, \dots, \sqrt{d_K}], \sigma \mathbf{I}_L) \right] \in \mathbb{C}^{L \times (K+L)}. \quad (18)$$

The autocorrelation matrix $\mathbf{R}_y$ in (2) is factorized as $\mathbf{R}_y = \mathbf{A} \mathbf{A}^H$. Then, we define

$$\mathbf{V} \stackrel{\text{def}}{=} \mathbf{A}^* \otimes \mathbf{A} \in \mathbb{C}^{L^2 \times (K+L)^2} \quad (19)$$

and $\mathbf{i} \stackrel{\text{def}}{=} \text{vec}(\mathbf{I}_{K+L}) \in \mathbb{R}^{(K+L)^2}$, where $\mathbf{I}_{K+L}$ is the $(K+L)$ -size identity matrix. It follows that $\mathbf{r}$ takes the form

$$\mathbf{r} = \mathbf{V} \mathbf{i}. \quad (20)$$

In view of the above, $\mathbf{r}_{\text{co}}$ (or, $\mathbf{r}_{\text{sel}}$ and $\mathbf{r}_{\text{avg}}$ in (12) and (15), respectively) can be expressed as $\mathbf{E}_{\text{sel}}^\top \mathbf{V} \mathbf{i} =$

$\mathbf{E}_{\text{avg}}^\top \mathbf{V} \mathbf{i}$. Next, we observe that $\forall q \in \{1, 2, \dots, Q\}$, there exists vector $\mathbf{x}_q \sim \mathcal{CN}(\mathbf{0}_{K+L}, \mathbf{I}_{K+L})$, pertinent to $\mathbf{y}_q$, such that $\mathbf{y}_q = \mathbf{A} \mathbf{x}_q$. By the signal model, $\mathbf{x}_q$ is statistically independent from $\mathbf{x}_p$ for any $(p, q) \in \{1, 2, \dots, Q\}^2$ such that $p \neq q$. The physical-array autocorrelation matrix estimate $\hat{\mathbf{R}}_y$ in (9) is expressed as $\hat{\mathbf{R}}_y = \mathbf{A} \mathbf{W} \mathbf{A}^H$, where $\mathbf{W} \stackrel{\text{def}}{=} \frac{1}{Q} \sum_{q=1}^Q \mathbf{x}_q \mathbf{x}_q^H$. Moreover, by defining $\mathbf{w} \stackrel{\text{def}}{=} \text{vec}(\mathbf{W}) = \frac{1}{Q} \sum_{q=1}^Q \mathbf{x}_q^* \otimes \mathbf{x}_q$, the estimate $\hat{\mathbf{r}}$ in (10) takes the form

$$\hat{\mathbf{r}} = \mathbf{V} \mathbf{w}. \quad (21)$$

By (20) and (21), we propose to design linear combiner $\mathbf{E}_{\text{MMSE}}$ by formulating and solving the MSE-Minimization

$$\arg \min_{\mathbf{E} \in \mathbb{C}^{L^2 \times (2L'-1)}} \mathbb{E}_{\Theta, \mathbf{w}} \left\{ \left\| \mathbf{E}^H \mathbf{V} \mathbf{w} - \mathbf{E}_{\text{sel}}^\top \mathbf{V} \mathbf{i} \right\|_2^2 \right\}. \quad (22)$$

Of course, if we replace $\mathbf{E}_{\text{sel}}$ by $\mathbf{E}_{\text{avg}}$ in (22), the resulting problem will be equivalent. In the sequel, we show that a closed-form solution to (22) exists for any finite value of sample support Q and present it step-by-step.

We commence our solution by defining $\mathbf{G} \stackrel{\text{def}}{=} \mathbf{V} \mathbf{w} \mathbf{w}^H \mathbf{V}^H \in \mathbb{C}^{L^2 \times L^2}$ and $\mathbf{H} \stackrel{\text{def}}{=} \mathbf{V} \mathbf{w} \mathbf{i}^H \mathbf{V}^H \in \mathbb{C}^{L^2 \times L^2}$. Then, the problem in (22) simplifies to

$$\arg \min_{\mathbf{E} \in \mathbb{C}^{L^2 \times (2L'-1)}} \mathbb{E}_{\Theta, \mathbf{w}} \left\{ \text{Tr}(\mathbf{E}^H \mathbf{G} \mathbf{E}) - 2\Re \left\{ \text{Tr}(\mathbf{E}^H \mathbf{H} \mathbf{E}_{\text{sel}}) \right\} \right\}, \quad (23)$$

where $\Re\{\cdot\}$ extracts the real part of its argument and $\text{Tr}(\cdot)$ returns the sum of the diagonal entries of its argument. Furthermore, we define $\mathbf{G}_{\mathbb{E}} \stackrel{\text{def}}{=} \mathbb{E}_{\Theta, \mathbf{w}}\{\mathbf{G}\}$ and $\mathbf{H}_{\mathbb{E}} \stackrel{\text{def}}{=} \mathbb{E}_{\Theta, \mathbf{w}}\{\mathbf{H}\}$. Then, (23) takes the equivalent form

$$\arg \min_{\mathbf{E} \in \mathbb{C}^{L^2 \times (2L'-1)}} \left\{ \text{Tr}(\mathbf{E}^H \mathbf{G}_{\mathbb{E}} \mathbf{E}) - 2\Re \left\{ \text{Tr}(\mathbf{E}^H \mathbf{H}_{\mathbb{E}} \mathbf{E}_{\text{sel}}) \right\} \right\}. \quad (24)$$

Next, we focus on deriving closed-form expressions for $\mathbf{G}_{\mathbb{E}}$ and $\mathbf{H}_{\mathbb{E}}$ that will allow us to solve (24) and

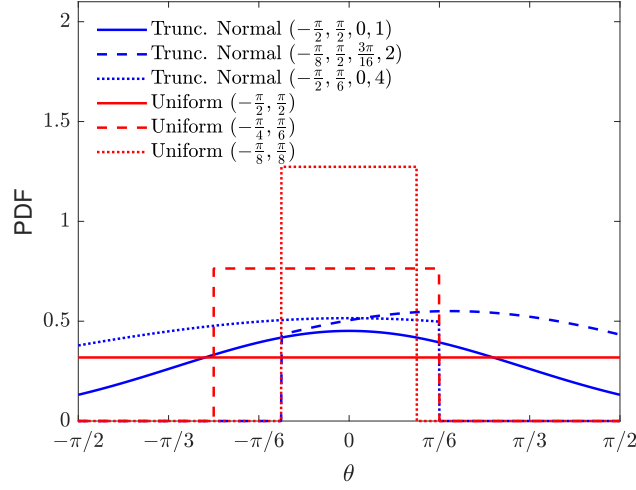


Figure 2: Probability density function $f(\theta)$ for different distributions and support sets.

Table 1

Entry-wise closed-form expression for matrix $\mathbf{V}$ defined in (19).

$[\mathbf{V}]_{i,j}$	Condition on (i, j)
$\sqrt{[\mathbf{d}]_{[\ddot{\mathbf{u}}]_j} [\mathbf{d}]_{[\dot{\mathbf{u}}]_j} \nu(\theta_{[\ddot{\mathbf{u}}]_j})^{[\ddot{\mathbf{p}}]_i} \nu(\theta_{[\dot{\mathbf{u}}]_j})^{-[\dot{\mathbf{p}}]_i}}$	$[\ddot{\mathbf{u}}]_j, [\dot{\mathbf{u}}]_j \leq K$
$\sigma \sqrt{[\mathbf{d}]_{[\ddot{\mathbf{u}}]_j} \nu(\theta_{[\ddot{\mathbf{u}}]_j})^{-[\dot{\mathbf{p}}]_i}}$	$[\dot{\mathbf{u}}]_j \leq K$ and $[\ddot{\mathbf{u}}]_j = [\dot{\mathbf{v}}]_i + K$
$\sigma \sqrt{[\mathbf{d}]_{[\ddot{\mathbf{u}}]_j} \nu(\theta_{[\ddot{\mathbf{u}}]_j})^{[\ddot{\mathbf{p}}]_i}}$	$[\ddot{\mathbf{u}}]_j \leq K$ and $[\dot{\mathbf{u}}]_j = [\dot{\mathbf{v}}]_i + K$
σ^2	$[\dot{\mathbf{u}}]_j = [\dot{\mathbf{v}}]_i + K$ and $[\ddot{\mathbf{u}}]_j = [\ddot{\mathbf{v}}]_i + K$
0	otherwise

Auxiliary variables used in the above conditions/expressions:
 $\mathbf{s}_x = [1, 2, \dots, x]^\top$, $\dot{\mathbf{u}} = \mathbf{1}_{K+L} \otimes \mathbf{s}_{K+L}$, $\ddot{\mathbf{u}} = \mathbf{s}_{K+L} \otimes \mathbf{1}_{K+L}$, $\dot{\mathbf{v}} = \mathbf{1}_L \otimes \mathbf{s}_L$, $\ddot{\mathbf{v}} = \mathbf{s}_L \otimes \mathbf{1}_L$.

obtain the Minimum-MSE linear combiner $\mathbf{E}_{\text{MMSE}}$. At the core of our developments lies the observation that, $\forall \theta \sim \mathcal{D}(a, b)$ with PDF $f(\theta)$ and scalar $x \in \mathbb{R}$, it holds

$$\mathcal{I}(x) \stackrel{\text{def}}{=} \mathbb{E}_{\theta} \{ \nu(\theta)^x \} = \int_a^b f(\theta) \exp \left(-jx \frac{2\pi f_c}{c} \sin \theta \right) d\theta. \quad (25)$$

The integral $\mathcal{I}(x)$ can be approximated within some numerical error tolerance with numerically efficient vectorized methods [40–42]. In Fig. 2, we offer visual illustration examples of $f(\theta)$ when θ follows a uniform distribution in (a, b) –i.e., $\theta \sim \mathcal{U}(a, b)$ – or, a truncated normal distribution in (a, b) with mean μ and variance σ^2 –i.e., $\theta \sim \mathcal{TN}(a, b, \mu, \sigma^2)$.

That is,

$$f(\theta) = \begin{cases} \frac{\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left(\frac{\theta-\mu}{\sigma}\right)^2\right)}{\frac{1}{2}\left(\operatorname{erf}\left(\frac{b-\mu}{\sigma\sqrt{2}}\right)-\operatorname{erf}\left(\frac{a-\mu}{\sigma\sqrt{2}}\right)\right)}, & \theta \sim \mathcal{TN}(a, b, \mu, \sigma^2), \\ \frac{1}{b-a}, & \theta \sim \mathcal{U}(a, b), \end{cases} \quad (26)$$

where $\operatorname{erf}(\cdot)$ denotes the Error Function [43]. In the special case that $D(a, b) \equiv \mathcal{U}(-\frac{\pi}{2}, \frac{\pi}{2})$, $\mathcal{I}(x)$ coincides with $J_0(x\frac{2\pi f_c}{c})$: the 0th order Bessel function of the first kind [44] for which there exist look-up tables. Next, we define the indicator function $\delta(x)$ which equals 1 if $x = 0$ and assumes a value of zero otherwise. We provide in the following Lemma the statistics of the random variable $\mathbf{w}$ which appear in the closed-form expressions of $\mathbf{G}_E$ and $\mathbf{H}_E$.

Lemma 5. *The first-order and second-order statistics of the random variable $\mathbf{w}$ are given by $\mathbb{E}_{\mathbf{w}}\{\mathbf{w}\} = \mathbf{i} \in \mathbb{R}^{(K+L)^2}$ and $\mathbb{E}_{\mathbf{w}}\{\mathbf{w}\mathbf{w}^H\} = \mathbf{i}\mathbf{i}^T + \frac{1}{Q}\mathbf{I}_{(K+L)^2} \in \mathbb{R}^{(K+L)^2 \times (K+L)^2}$, respectively.* $\square$

A proof for Lemma 5 is offered in the Appendix. Next, we define $\ddot{\mathbf{p}} \stackrel{\text{def}}{=} \mathbf{1}_L \otimes \mathbf{p}$ and $\omega_i \stackrel{\text{def}}{=} [\dot{\mathbf{p}}]_i - [\ddot{\mathbf{p}}]_i$. In view of Lemma 5, we present an entry-wise closed-form expression for $\mathbf{H}_E$ in the following Lemma.

Lemma 6. *For any $(i, m) \in \{1, 2, \dots, L^2\}^2$,*

$$[\mathbf{H}_E]_{i,m} = \|\mathbf{d}\|_2^2 \mathcal{I}(\omega_i - \omega_m) + \sigma^4 \delta(\omega_i) \delta(\omega_m) + \mathcal{I}(\omega_i) \mathcal{I}(-\omega_m) \left((\mathbf{1}_K^T \mathbf{d})^2 - \|\mathbf{d}\|_2^2 \right) \quad (27)$$

$$+ \sigma^2 \left(\mathbf{1}_K^T \mathbf{d} \right) \left(\delta(\omega_i) \mathcal{I}(-\omega_m) + \mathcal{I}(\omega_i) \delta(-\omega_m) \right). \quad (28)$$

$\square$

A complete proof for Lemma 6 is also provided in the Appendix. Hereafter, we focus on deriving a closed-form expression for $\mathbf{G}_E$. First, we define $\tilde{\mathbf{V}} \stackrel{\text{def}}{=} \mathbf{V}\mathbf{V}^H$ the expectation of which, $\tilde{\mathbf{V}}_E \stackrel{\text{def}}{=} \mathbb{E}_{\Theta}\{\tilde{\mathbf{V}}\}$, appears in $\mathbf{G}_E$. We observe that each entry of $\tilde{\mathbf{V}}$ can be expressed as a linear combination

Table 2

Closed-form expression for $\gamma_j^{(i,m)}$ defined in (30) .

$\gamma_j^{(i,m)}$	Condition on (i, m, j)
$[\mathbf{d}]_{[\mathbf{u}]_j} [\mathbf{d}]_{[\mathbf{u}]_j} v(\theta_{[\mathbf{u}]_j})^{[\mathbf{p}]_i - [\mathbf{p}]_m} v(\theta_{[\mathbf{u}]_j})^{[\mathbf{p}]_m - [\mathbf{p}]_i}$	$[\mathbf{u}]_j = [\mathbf{u}]_j \leq K,$
$[\mathbf{d}]_{[\mathbf{u}]_j} [\mathbf{d}]_{[\mathbf{u}]_j} v(\theta_{[\mathbf{u}]_j})^{[\mathbf{p}]_i - [\mathbf{p}]_m} v(\theta_{[\mathbf{u}]_j})^{[\mathbf{p}]_m - [\mathbf{p}]_i}$	$[\mathbf{u}]_j, [\mathbf{u}]_j \leq K; [\mathbf{u}]_j \neq [\mathbf{u}]_j$
$\sigma^2 [\mathbf{d}]_{[\mathbf{u}]_j} v(\theta_{[\mathbf{u}]_j})^{[\mathbf{p}]_m - [\mathbf{p}]_i}$	$[\mathbf{u}]_j \leq K; [\mathbf{u}]_j - K = [\mathbf{v}]_i = [\mathbf{v}]_m$
$\sigma^2 [\mathbf{d}]_{[\mathbf{u}]_j} v(\theta_{[\mathbf{u}]_j})^{[\mathbf{p}]_i - [\mathbf{p}]_m}$	$[\mathbf{u}]_j \leq K; [\mathbf{u}]_j - K = [\mathbf{v}]_i = [\mathbf{v}]_m$
σ^4	$[\mathbf{u}]_j - K = [\mathbf{v}]_i = [\mathbf{v}]_m; [\mathbf{u}]_j - K = [\mathbf{v}]_i = [\mathbf{v}]_m$
0	otherwise

Auxiliary variables used in the above conditions/expressions:
 $\mathbf{p} = \mathbf{p} \otimes \mathbf{1}_L, \mathbf{p} = \mathbf{1}_L \otimes \mathbf{p}, \mathbf{s}_x = [1, 2, \dots, x]^\top, \mathbf{u} = \mathbf{1}_{K+L} \otimes \mathbf{s}_{K+L}, \mathbf{u} = \mathbf{s}_{K+L} \otimes \mathbf{1}_{K+L}, \mathbf{v} = \mathbf{1}_L \otimes \mathbf{s}_L, \mathbf{v} = \mathbf{s}_L \otimes \mathbf{1}_L.$

of the entries of $\mathbf{V}$. That is, for any $(i, m) \in \{1, 2, \dots, L^2\}^2$ and $j \in \{1, 2, \dots, (K + L)^2\}$, it holds

$$[\tilde{\mathbf{V}}]_{i,m} = \sum_{j=1}^{(K+L)^2} [\mathbf{V}]_{i,j} [\mathbf{V}^*]_{m,j}. \quad (29)$$

An entry-wise closed-form expression for $\mathbf{V}$, in terms of Θ , is offered in Table 1. Then, for any triplet (i, m, j) such that $(i, m) \in \{1, 2, \dots, L^2\}^2$ and $j \in \{1, 2, \dots, (K + L)^2\}$, we derive a closed-form expression for

$$\gamma_j^{(i,m)} \stackrel{\text{def}}{=} [\mathbf{V}]_{i,j} [\mathbf{V}^*]_{m,j}, \quad (30)$$

in Table 2. Accordingly, $\mathbb{E}_\Theta \{\gamma_j^{(i,m)}\}$ is offered in Table 3, based on which, the (i, m) th entry of $\tilde{\mathbf{V}}_\mathbb{E}$ is computed as

$$[\tilde{\mathbf{V}}_\mathbb{E}]_{i,m} = \sum_{j=1}^{(K+L)^2} \mathbb{E}_\Theta \{\gamma_j^{(i,m)}\}. \quad (31)$$

In view of the above, we offer a closed-form expression for matrix $\mathbf{G}_\mathbb{E}$ in the following Lemma.

Lemma 7. *Matrix $\mathbf{G}_\mathbb{E}$ is given by $\mathbf{G}_\mathbb{E} = \mathbf{H}_\mathbb{E} + \frac{1}{Q} \tilde{\mathbf{V}}_\mathbb{E}$.* □

A proof for Lemma 7 is offered in the Appendix. Next, we differentiate (24) with respect to $\mathbf{E}$,

Table 3

Closed-form expression for $\mathbb{E}_\Theta\{\gamma_j^{(i,m)}\}$.

$\mathbb{E}_\Theta\{\gamma_j^{(i,m)}\}$	Condition on (i, m, j)
$[\mathbf{d}]_{[\ddot{\mathbf{u}}]_j} [\mathbf{d}]_{[\dot{\mathbf{u}}]_j} \mathcal{I}(\ddot{\omega}_{m,i} + \dot{\omega}_{i,m})$	$[\dot{\mathbf{u}}]_j = [\ddot{\mathbf{u}}]_j \leq K,$
$[\mathbf{d}]_{[\ddot{\mathbf{u}}]_j} [\mathbf{d}]_{[\dot{\mathbf{u}}]_j} \mathcal{I}(\ddot{\omega}_{m,i}) \mathcal{I}(\dot{\omega}_{i,m})$	$[\dot{\mathbf{u}}]_j, [\ddot{\mathbf{u}}]_j \leq K; [\ddot{\mathbf{u}}]_j \neq [\dot{\mathbf{u}}]_j$
$\sigma^2 [\mathbf{d}]_{[\dot{\mathbf{u}}]_j} \mathcal{I}(\ddot{\omega}_{m,i})$	$[\dot{\mathbf{u}}]_j \leq K; [\ddot{\mathbf{u}}]_j - K = [\ddot{\mathbf{v}}]_i = [\ddot{\mathbf{v}}]_m$
$\sigma^2 [\mathbf{d}]_{[\ddot{\mathbf{u}}]_j} \mathcal{I}(\dot{\omega}_{i,m})$	$[\ddot{\mathbf{u}}]_j \leq K; [\dot{\mathbf{u}}]_j - K = [\ddot{\mathbf{v}}]_i = [\ddot{\mathbf{v}}]_m$
σ^4	$[\dot{\mathbf{u}}]_j - K = [\ddot{\mathbf{v}}]_i = [\ddot{\mathbf{v}}]_m; [\ddot{\mathbf{u}}]_j - K = [\ddot{\mathbf{v}}]_i = [\ddot{\mathbf{v}}]_m$
0	otherwise

Auxiliary variables used in the above conditions/expressions:
 $\dot{\mathbf{p}} = \mathbf{p} \otimes \mathbf{1}_L, \ddot{\mathbf{p}} = \mathbf{1}_L \otimes \mathbf{p}, \dot{\omega}_{i,j} = [\dot{\mathbf{p}}]_i - [\dot{\mathbf{p}}]_j, \ddot{\omega}_{i,j} = [\ddot{\mathbf{p}}]_i - [\ddot{\mathbf{p}}]_j, \mathbf{s}_x = [1, 2, \dots, x]^\top, \dot{\mathbf{u}} = \mathbf{1}_{K+L} \otimes \mathbf{s}_{K+L}, \ddot{\mathbf{u}} = \mathbf{s}_{K+L} \otimes \mathbf{1}_{K+L},$
 $\dot{\mathbf{v}} = \mathbf{1}_L \otimes \mathbf{s}_L, \ddot{\mathbf{v}} = \mathbf{s}_L \otimes \mathbf{1}_L.$

set its derivative to zero, and obtain

$$\left(\mathbf{H}_\mathbb{E} + \frac{1}{Q} \tilde{\mathbf{V}}_\mathbb{E} \right) \mathbf{E}_{\text{MMSE}} = \mathbf{H}_\mathbb{E} \mathbf{E}_{\text{sel}}. \quad (32)$$

We observe that (32) is, in practice, a collection of $(2L' - 1)$ systems of linear equations. Let $\mathbf{E}_{\text{MMSE}} = [\mathbf{e}_1, \dots, \mathbf{e}_{2L'-1}]$ and $\mathbf{c}_i = [\mathbf{H}_\mathbb{E} \mathbf{E}_{\text{sel}}]_{:,i} \forall i \in \{1, 2, \dots, 2L' - 1\}$. Solving (32) is equivalent to solving, for every i ,

$$\mathbf{G}_\mathbb{E} \mathbf{e}_i = \mathbf{c}_i. \quad (33)$$

For any i such that $\mathbf{c}_i \in \text{span}(\mathbf{G}_\mathbb{E})$, (33) has at least one exact solution $\mathbf{e}_i = \mathbf{V} \boldsymbol{\Sigma}^{-1} \mathbf{U}^H \mathbf{c}_i + \mathbf{b}_i$, where $\mathbf{G}_\mathbb{E}$ admits SVD $\mathbf{U}_{L^2 \times \rho} \boldsymbol{\Sigma}_{\rho \times \rho} \mathbf{V}_{\rho \times L^2}^H$, $\rho = \text{rank}(\mathbf{G}_\mathbb{E})$, and $\mathbf{b}_i$ is an arbitrary vector in the nullspace of $\mathbf{G}_\mathbb{E}$ which is denoted by $\mathcal{N}(\mathbf{G}_\mathbb{E})$. In the special case that $\rho = L^2$, that is, $\mathbf{G}_\mathbb{E}$ has full-rank, then $\mathcal{N}(\mathbf{G}_\mathbb{E}) = \mathbf{0}_{L^2}$ and there exists a unique solution $\mathbf{e}_i = \mathbf{V} \boldsymbol{\Sigma}^{-1} \mathbf{U}^H \mathbf{c}_i$. If, on the other hand, $\exists i$ such that $\mathbf{c}_i \notin \text{span}(\mathbf{G}_\mathbb{E})$, then (33) does not have an exact solution and a Least Squares (LS) approach can be followed by solving $\min_{\mathbf{e}_i} \|\mathbf{G}_\mathbb{E} \mathbf{e}_i - \mathbf{c}_i\|_2^2$. Interestingly, it is easy to show that the LS solution is the same as before –i.e., $\mathbf{e}_i = \mathbf{V} \boldsymbol{\Sigma}^{-1} \mathbf{U}^H \mathbf{c}_i + \mathbf{b}_i$, where $\mathbf{b}_i \in \mathcal{N}(\mathbf{G}_\mathbb{E})$. In every case, each column of

$\mathbf{E}_{\text{MMSE}}$ can be computed in closed-form as

$$\mathbf{e}_i = \mathbf{V}\mathbf{\Sigma}^{-1}\mathbf{U}^H\mathbf{c}_i + \mathbf{b}_i, \quad \mathbf{b}_i \in \mathcal{N}(\mathbf{G}_{\mathbb{E}}). \quad (34)$$

In view of the above, we propose to process the autocorrelations in $\hat{\mathbf{r}}$ by the linear combiner $\mathbf{E}_{\text{MMSE}}$ to obtain the MMSE estimate of $\mathbf{r}_{\text{co}}$,

$$\hat{\mathbf{r}}_{\text{MMSE}} \stackrel{\text{def}}{=} \mathbf{E}_{\text{MMSE}}^{\top} \hat{\mathbf{r}}. \quad (35)$$

In turn, we propose to Minimum-MSE estimate $\mathbf{Z}$ in (6) by $\hat{\mathbf{Z}}_{\text{MMSE}} = \Phi(\hat{\mathbf{r}}_{\text{MMSE}})$.

Next, we discuss the computational complexity of forming $\mathbf{E}_{\text{MMSE}}$. First, we consider the cost of numerically approximating the integral $\mathcal{I}(x)$ in (25) for any DoA probability distribution $f(\theta)$, support set (a, b) , and scalar x . There is rich literature on theory, methods, and complexity/accuracy trade-offs in numerical integration—a topic that extends well beyond the scope of this article. For the purpose of our complexity analysis, we consider that $\mathcal{I}(x)$ can be approximated within numerical error tolerance $\epsilon > 0$ by, e.g., trapezoidal, midpoint, and Simpson’s interpolatory quadrature rules [40–42], with cost $\mathcal{O}(C)$, where $C = \frac{1}{\sqrt{\epsilon}}$. Next, we note that evaluating $\mathcal{I}(x)$ for every $x \in \mathcal{V} \stackrel{\text{def}}{=} \{ \{\omega_i\}_{i=1}^{L^2} \cup \{\omega_i - \omega_m\}_{i,m=1}^{L^2} \cup \{\dot{\omega}_i\}_{i=1}^{L^2} \cup \{\ddot{\omega}_i\}_{i=1}^{L^2} \cup \{\ddot{\omega}_{m,i} + \dot{\omega}_{i,m}\}_{i,m=1}^{L^2} \}$ costs at most $\mathcal{O}(MNC)$. Having computed $\mathcal{I}(x)$ for every $x \in \mathcal{V}$, we can form matrices $\mathbf{H}_{\mathbb{E}}$ and $\tilde{\mathbf{V}}_{\mathbb{E}}$ with costs $\mathcal{O}(L^4)$ and $\mathcal{O}(L^6 + KL^5)$, respectively. Thereafter, the SVD of $\mathbf{G}_{\mathbb{E}} = \mathbf{H}_{\mathbb{E}} + \frac{1}{Q}\tilde{\mathbf{V}}_{\mathbb{E}}$ for solving (33) requires $\mathcal{O}(L^6)$ computations. Observing that $L = 2M + N - 1$ and $K \leq MN + M$ and keeping only the dominant terms, the computational complexity of forming $\mathbf{E}_{\text{MMSE}}$ is $\mathcal{O}(L^6 + MNC)$. In addition, we note that the computation of $\mathcal{I}(x)$ for different values of x can be performed in parallel, tremendously reducing the computation time. In this work, we conduct numerical integration by means of the vectorized adaptive quadrature approach of [40] with absolute error tolerance $\epsilon = 1e - 10$. As a simple numerical example, for $(M, N) = (2, 3)$, $K = 7$, $\epsilon = 1e - 10$, and serial computation of the numerical integrals, in MATLAB 2019a (generic software) running on an Intel(R) core(TM)

i7-8700 processor at 3.2 GHz and 32GB RAM (generic hardware), we computed $\mathbf{E}_{\text{MMSE}}$ in a very small fraction of a second.

Finally, a remark is in order on the source powers and noise variance. The proposed linear combiner $\mathbf{E}_{\text{MMSE}}$ depends on $\mathbf{H}_{\mathbb{E}}$ and $\tilde{\mathbf{V}}_{\mathbb{E}}$ which, in turn, depend on the powers $d_1, d_2, \dots, d_K, \sigma^2$ associated to the source DoAs $\theta_1, \theta_2, \dots, \theta_K$ and noise, respectively. In this paper, we assume the signal-source powers and noise variance to be known. Joint power/DoA estimation is beyond the scope of this paper. For the problem of the power estimation, we refer the interested reader to, e.g., the works in [45–50].

5. Numerical Studies

We consider coprime naturals (M, N) with $M < N$ and form a length- $(L = 2M + N - 1)$ coprime array which corresponds to a length- $(L' = MN + M)$ virtual ULA –i.e., the coarray. Signals from K sources impinge on the array with equal transmit power $d_1 = d_2 = \dots = d_K = \alpha^2 = 10$ dB. The noise variance is set to $\sigma^2 = 0$ dB. Accordingly, the Signal-to-Noise-Ratio (SNR) is set to 10dB for every DoA signal source.

We commence our studies by computing the empirical Cumulative Distribution Function (CDF) of the Normalized-MSE (NMSE) in estimating $\mathbf{Z}$ for a given DoA collection $\Theta = \{\theta_1, \dots, \theta_K\}$ such that the DoAs in Θ are i.i.d. –i.e., $\theta_k \sim \mathcal{D}(a, b) \forall k$. More specifically, we consider fixed sample-support $Q = 10$ and for each estimate $\hat{\mathbf{Z}} \in \{\hat{\mathbf{Z}}_{\text{sel}}, \hat{\mathbf{Z}}_{\text{avg}}, \hat{\mathbf{Z}}_{\text{MMSE}}\}$, we compute the estimation error

$$\text{NMSE} = \left\| \mathbf{Z} - \hat{\mathbf{Z}} \right\|_F^2 \left\| \mathbf{Z} \right\|_F^{-2}. \quad (36)$$

We repeat this process over 4000 statistically independent realizations of Θ and AWGN. We collect 4000 NMSE measurements based on which we plot, in Fig. 3, the empirical CDF of NMSE in estimating $\mathbf{Z}$ for fixed $(M, N) = (2, 3)$, $K \in \{5, 7\}$, and $\mathcal{D}(a, b) \in \{\mathcal{U}(-\frac{\pi}{2}, \frac{\pi}{2}), \mathcal{U}(-\frac{\pi}{4}, \frac{\pi}{6}), \mathcal{T}N(-\frac{\pi}{8}, \frac{\pi}{8}, 0, 1)\}$. We observe that the proposed MMSE combining approach attains superior MSE in estimating $\mathbf{Z}$ for any distribution and support set considered for DoAs in Θ . Moreover, we notice that the averag-

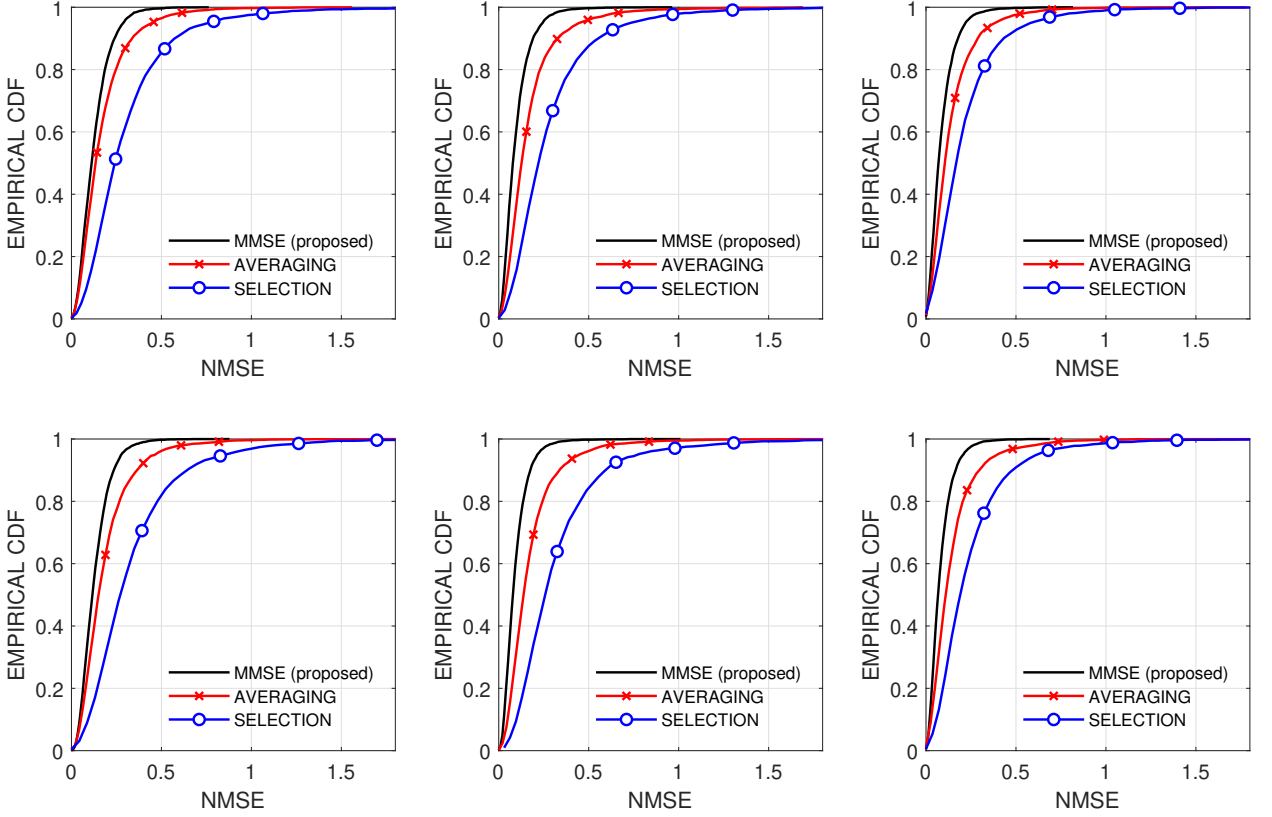


Figure 3: Empirical CDF of the MSE in estimating $\mathbf{Z}$ for $(M, N) = (2, 3)$, $\text{SNR} = 10\text{dB}$, $Q = 10$, $K = 5$ (top), and $K = 7$ (bottom). $\forall k, \theta_k \sim \mathcal{U}(-\frac{\pi}{2}, \frac{\pi}{2})$ (left), $\mathcal{U}(-\frac{\pi}{4}, \frac{\pi}{6})$ (center), $\mathcal{TN}(-\frac{\pi}{8}, \frac{\pi}{8}, 0, 1)$ (right).

ing combining approach consistently outperforms the selection combining approach in accordance with our theoretical finding that averaging combining attains superior performance compared to selection combining. Furthermore, we observe that the performance advantage of the proposed MMSE combining approach over its standard counterparts is more emphatic when $\theta_k \sim \mathcal{U}(-\frac{\pi}{4}, \frac{\pi}{6})$. Last, we notice that for $\theta_k \sim \mathcal{U}(-\frac{\pi}{4}, \frac{\pi}{6})$, the performance gap between the proposed MMSE combining and the standard averaging approach is greater for $K = 7$. We repeat the last study for $(M, N) = (2, 5)$, $K \in \{7, 9\}$, and $D(a, b) \in \{\mathcal{U}(-\frac{\pi}{2}, \frac{\pi}{2}), \mathcal{U}(-\frac{\pi}{4}, \frac{\pi}{6}), \mathcal{TN}(-\frac{\pi}{8}, \frac{\pi}{8}, 0, 1)\}$. We illustrate the new CDFs in Fig. 4. Similar observations as in Fig. 3 are made. The proposed MMSE combining approach clearly outperforms its standard counterparts for any distribution assumption and support set for $\theta_k \forall k$.

For fixed $K = 7$ and every other parameter same as above, we plot the NMSE (averaged over

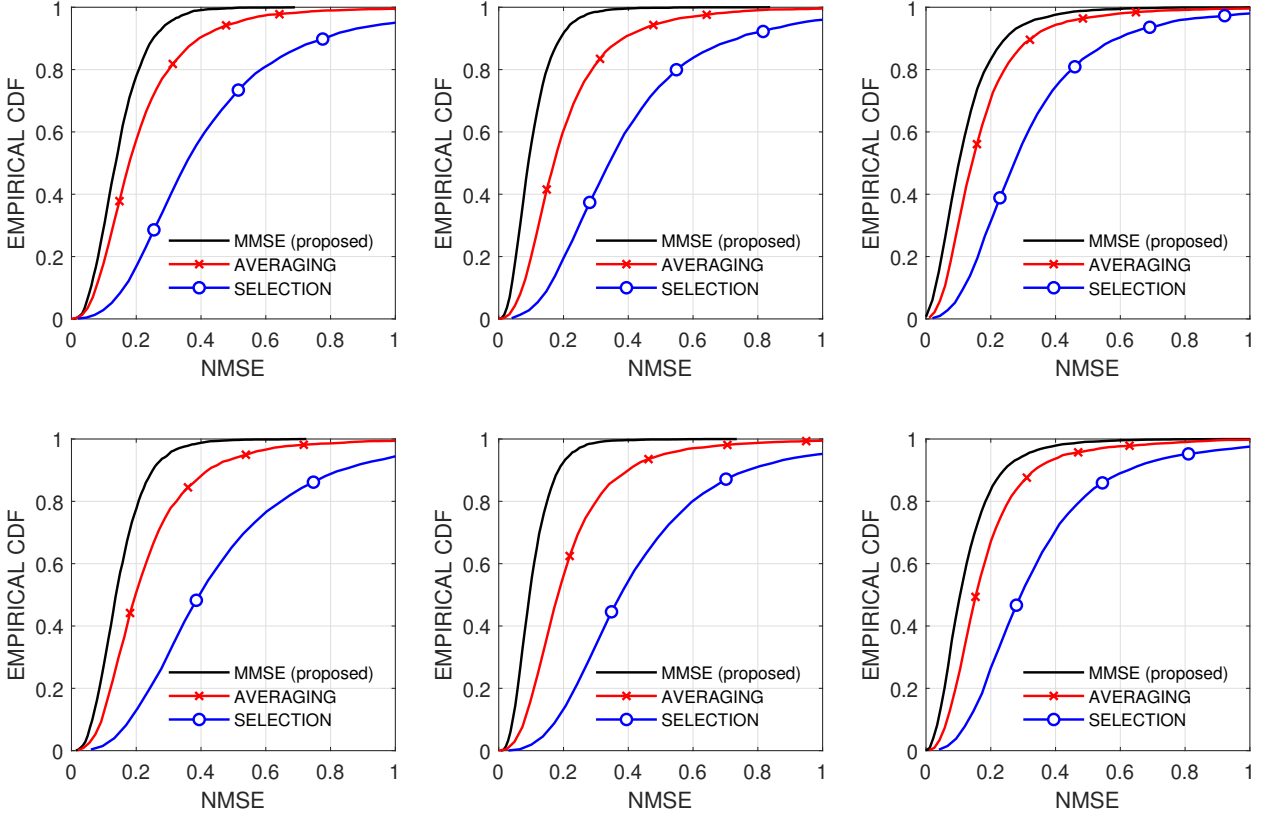


Figure 4: Empirical CDF of the NMSE in estimating $\mathbf{Z}$ for $(M, N) = (2, 5)$, $\text{SNR} = 10\text{dB}$, $Q = 10$, $K = 7$ (top), and $K = 9$ (bottom). $\forall k, \theta_k \sim \mathcal{U}(-\frac{\pi}{2}, \frac{\pi}{2})$ (left), $\mathcal{U}(-\frac{\pi}{4}, \frac{\pi}{6})$ (center), and $\mathcal{TN}(-\frac{\pi}{8}, \frac{\pi}{8}, 0, 1)$ (right).

4000 realizations) versus sample support $Q \in \{1, 10, 100, 1000, 10000\}$, in Fig. 5. Consistent with the observations above, we notice that selection attains the highest NMSE while the proposed MMSE combiner attains, expectedly, the lowest NMSE in estimating $\mathbf{Z}$ across the board. The performance gap between the proposed MMSE combining estimate and the estimates based on existing combining approaches decreases as the sample support Q increases. Nonetheless, it remains superior, in many cases, even for high values of Q —e.g., $Q = 10^4$. Moreover, in the first two subplots (uniform DoA distribution), we notice that the performance gap between the MMSE combining approach and the averaging approach is wider when the range of the support set (a, b) is narrower.

Next, we evaluate the performance of the proposed MMSE combining approach and its counterparts by measuring the Root-MSE (RMSE) in estimating the DoAs in Θ versus sample support Q .

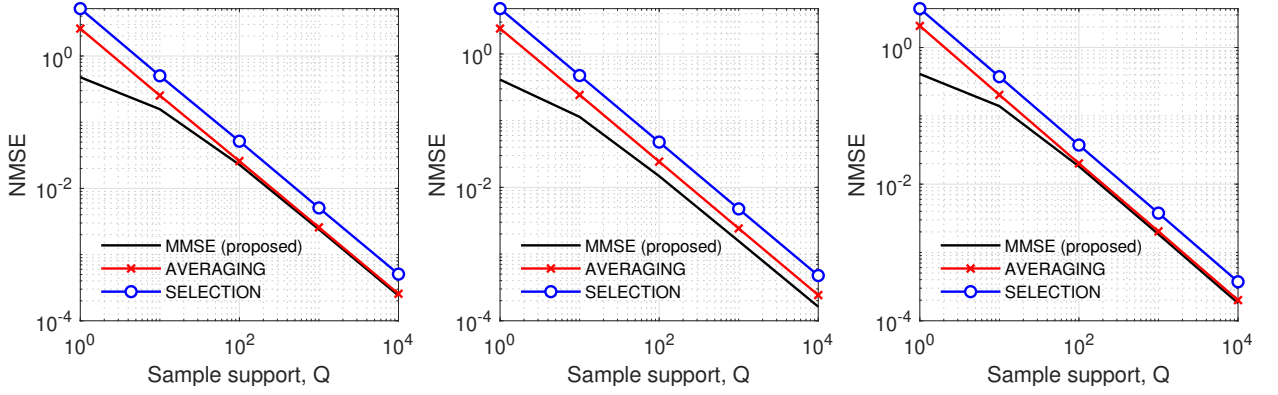


Figure 5: NMSE in estimating $\mathbf{Z}$, versus sample support, Q . $(M, N) = (2, 5)$, SNR= 10dB, and $K = 7$. $\forall k, \theta_k \sim \mathcal{U}(-\frac{\pi}{2}, \frac{\pi}{2})$ (left), $\mathcal{U}(-\frac{\pi}{4}, \frac{\pi}{6})$ (center), $\mathcal{TN}(-\frac{\pi}{8}, \frac{\pi}{8}, 0, 1)$ (right).

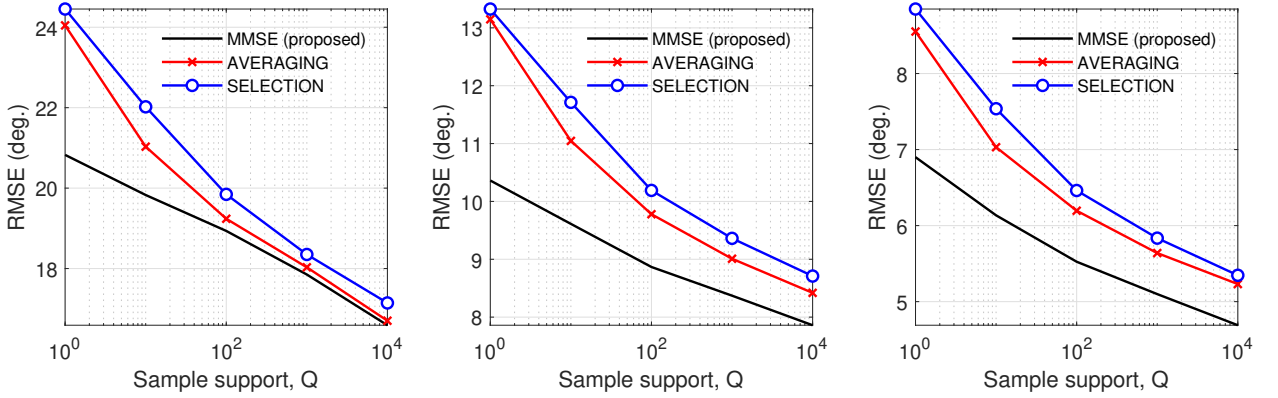


Figure 6: RMSE (degrees) in estimating the DoA set Θ , versus sample support, Q . $(M, N) = (2, 5)$, SNR= 10dB, and $K = 7$. $\forall k, \theta_k \sim \mathcal{U}(-\frac{\pi}{2}, \frac{\pi}{2})$ (left), $\mathcal{U}(-\frac{\pi}{4}, \frac{\pi}{6})$ (center), $\mathcal{TN}(-\frac{\pi}{8}, \frac{\pi}{8}, 0, 1)$ (right).

First, we evaluate the MUSIC spectrum for each estimate $\hat{\mathbf{Z}} \in \{\hat{\mathbf{Z}}_{\text{sel}}, \hat{\mathbf{Z}}_{\text{avg}}, \hat{\mathbf{Z}}_{\text{MMSE}}\}$ over the probability distribution support set (a, b) and obtain DoA estimates in $\hat{\Theta} = \{\hat{\theta}_1, \dots, \hat{\theta}_K\}$. Then, we compute the estimation error

$$\text{MSE}(\hat{\Theta}) = \frac{1}{K} \sum_{k=1}^K (\theta_k - \hat{\theta}_k)^2. \quad (37)$$

We compute the RMSE by taking the square root of the $\text{MSE}(\hat{\Theta})$ computed over 4000 statistically independent realizations of Θ and AWGN. The resulting RMSE curves are depicted in Fig. 6. For every DoA distribution (even the most general case of uniform distribution in $(-\frac{\pi}{2}, \frac{\pi}{2})$) and every sample support (even as high as 10^4), the proposed method attains the lowest RMSE.

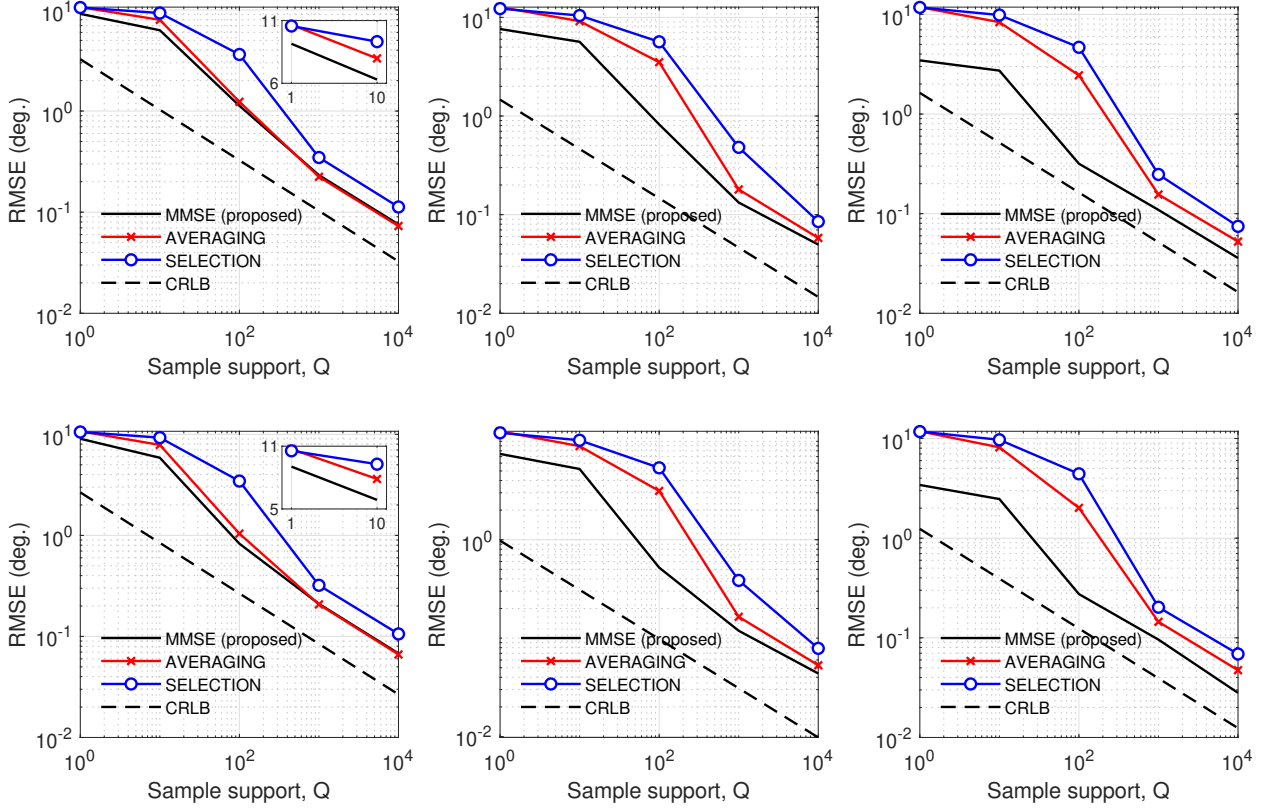


Figure 7: RMSE (degrees) in estimating the DoA set Θ , versus sample support, Q . $(M, N) = (3, 5)$ and $K = 11$. $(\Theta, \text{SNR}, \mathcal{D}(a, b)) = (\Theta_1, 0\text{dB}, \mathcal{U}(-\frac{\pi}{2}, \frac{\pi}{2}))$ –top left, $(\Theta_1, 8\text{dB}, \mathcal{U}(-\frac{\pi}{2}, \frac{\pi}{2}))$ –bottom left, $(\Theta_2, 0\text{dB}, \mathcal{U}(-\frac{\pi}{4}, \frac{\pi}{4}))$ –top center, $(\Theta_2, 8\text{dB}, \mathcal{U}(-\frac{\pi}{4}, \frac{\pi}{4}))$ –bottom center, $(\Theta_3, 0\text{dB}, \mathcal{U}(-\frac{\pi}{4}, \frac{\pi}{6}))$ –top right, $(\Theta_3, 8\text{dB}, \mathcal{U}(-\frac{\pi}{4}, \frac{\pi}{6}))$ –bottom right.

We conclude our studies by evaluating the DoA estimation performance of the proposed MMSE combiner and its counterparts for fixed DoAs. This time, we consider coprime naturals $(M, N) = (3, 5)$ forming an array with $L = 10$ sensors, fixed number of sources $K = 11$, and three distinct realizations of DoA set $\Theta = \{\theta_1, \dots, \theta_K\}$:

$$\Theta_1 = \{-79, -55, -43, -25, -15, -6, 7, 21, 34, 54, 63\}^\circ, \quad (38)$$

$$\Theta_2 = \{-44, -38, -27, -12, -3, 6, 17, 24, 31, 37, 44\}^\circ, \quad (39)$$

$$\Theta_3 = \{-44, -37, -31, -23, -15, -8, -2, 4, 17, 22, 29\}^\circ. \quad (40)$$

For every fixed Θ_i , $i \in \{1, 2, 3\}$, we proceed as follows in order to compute the RMSE curves.

First, we consider 4000 statistically independent realizations of AWGN for $\text{SNR} \in \{0, 8\}$ dB. For every realization, we compute DoA estimate $\hat{\Theta}_i$ as previously described and evaluate the MUSIC spectrum over a probability distribution support set. We then measure $\text{MSE}(\hat{\Theta}_i)$ as described above and compute the RMSE by taking the square root of the sample-average of the 4000 $\text{MSE}(\hat{\Theta}_i)$ measurements. As a benchmark, we also compute the Cramér Rao Lower Bound (CRLB) [14] curves.

In Fig. 7 (left), we plot the RMSE and CRLB curves versus sample support for $\Theta = \Theta_1$, and $\text{SNR} = 0$ dB (top-left) and $\text{SNR} = 8$ dB (bottom-left), respectively. We form the MMSE combiner by considering $D(a, b) = \mathcal{U}(-\frac{\pi}{2}, \frac{\pi}{2})$. We observe that for $\text{SNR} \in \{0, 8\}$ dB and low sample support, $Q \leq 100$, the proposed MMSE combiner clearly outperforms its counterparts in DoA estimation accuracy. For $Q \geq 1000$, the proposed MMSE and the averaging combiners attain almost identical performance, superior to that of the selection combiner.

In Fig. 7 (center), we plot the RMSE and CRLB curves versus sample support for $\Theta = \Theta_2$, and $\text{SNR} = 0$ dB (top-center) and $\text{SNR} = 8$ dB (bottom-center), respectively. We consider $D(a, b) = \mathcal{U}(-\frac{\pi}{4}, \frac{\pi}{4})$. In this case, we observe that the proposed MMSE combiner outperforms its counterparts across the board with respect to both sample support and SNR.

Finally, we consider $D(a, b) = \mathcal{U}(-\frac{\pi}{4}, \frac{\pi}{6})$ and in Fig. 7 (right), we plot the RMSE and CRLB curves versus sample support for $\Theta = \Theta_3$, and $\text{SNR} = 0$ dB (top-right) and $\text{SNR} = 8$ dB (bottom-right), respectively. In accordance with earlier observations, now that the probability distribution support set is even narrower, the proposed MMSE combiner attains an even more superior DoA estimation performance compared to its counterparts.

6. Conclusion

We proposed a novel coprime array receiver that attains minimum MSE in coarray autocorrelation estimation, for any probability distribution of the source DoAs. Moreover, we offered formal mathematical proofs for the closed-form MSE expressions of selection and averaging, which were first presented in [3]. Extensive numerical studies on various DoA distributions demonstrate that

the proposed MMSE combining method consistently outperforms its existing counterparts in auto-correlation estimation performance with respect to the MSE metric. In turn, the proposed MMSE combiner enables lower RMSE in DoA estimation.

7. Appendix

7.1. Proof of Lemma 1

Proof. Let $\mathbf{b}_q \stackrel{\text{def}}{=} \mathbf{y}_q^* \otimes \mathbf{y}_q \forall q$ and define auxiliary variables $\dot{\mathbf{v}} = \mathbf{1}_L \otimes \mathbf{s}_L$, $\dot{\mathbf{p}} = \mathbf{p} \otimes \mathbf{1}_L$, $\ddot{\mathbf{v}} = \mathbf{s}_L \otimes \mathbf{1}_L$, and $\ddot{\mathbf{p}} = \mathbf{1}_L \otimes \mathbf{p}$.³ For any $n \in \{1-L', 2-L', \dots, L'-1\}$ and $j_n \in \mathcal{J}_n$, it holds that $[\mathbf{b}_q]_{j_n} = [\mathbf{y}_q^*]_{[\dot{\mathbf{v}}]_{j_n}} [\mathbf{y}_q]_{[\ddot{\mathbf{v}}]_{j_n}}$. Further, it holds

$$[\mathbf{y}_q]_{[\ddot{\mathbf{v}}]_{j_n}} = \sum_{k=1}^K v(\theta_k)^{[\dot{\mathbf{p}}]_{j_n}} \xi_{q,k} + [\mathbf{n}_q]_{[\ddot{\mathbf{v}}]_{j_n}}, \quad (41)$$

$$[\mathbf{y}_q^*]_{[\dot{\mathbf{v}}]_{j_n}} = \sum_{k=1}^K v(\theta_k)^{[\dot{\mathbf{p}}]_{j_n}} \xi_{q,k}^* + [\mathbf{n}_q^*]_{[\dot{\mathbf{v}}]_{j_n}}. \quad (42)$$

Next, we compute $\mathbb{E}\{[\mathbf{b}_q]_{j_n}\} = \sum_{k=1}^K v(\theta_k)^{\omega_{j_n}} d_k + \sigma^2 [\mathbf{i}_L]_{j_n} = [\mathbf{r}]_{j_n}$,⁴ which implies that $\mathbb{E}\{\hat{\mathbf{r}}\} = \mathbf{r}$. Then, for every $n \in \{1-L', 2-L', \dots, L'-1\}$ and $(i, j) \in \mathcal{J}_n$, we define $b_{p,q,i,j}^{(n)} \stackrel{\text{def}}{=} [\mathbf{b}_q^*]_i [\mathbf{b}_p]_j$. After simple algebraic manipulations we find that

$$\mathbb{E}\left\{b_{p,q,i,j}^{(n)}\right\} = \left|\mathbf{g}_n^H \mathbf{d} + \delta(n)\sigma^2\right|^2 + \delta(q-p) \left|\mathbf{z}_{i,j}^H \mathbf{d} + \delta(i-j)\sigma^2\right|^2, \quad (43)$$

where $\mathbf{g}_n \stackrel{\text{def}}{=} [v(\theta_1)^n, v(\theta_2)^n, \dots, v(\theta_K)^n]^\top$. Next, we proceed as follows.

$$e = \mathbb{E}\left\{\left|[\mathbf{r}]_{j_n} - [\hat{\mathbf{r}}]_{j_n}\right|^2\right\} \quad (44)$$

$$= \left|[\mathbf{r}]_{j_n}\right|^2 + \mathbb{E}\left\{[\hat{\mathbf{r}}]_{j_n} [\hat{\mathbf{r}}]_{j_n}^*\right\} - 2\mathbb{E}\left\{\Re\left\{[\mathbf{r}]_{j_n} [\hat{\mathbf{r}}]_{j_n}^*\right\}\right\} \quad (45)$$

$$= \mathbb{E}\left\{[\hat{\mathbf{r}}]_{j_n} [\hat{\mathbf{r}}]_{j_n}^*\right\} - \left|[\mathbf{r}]_{j_n}\right|^2 \quad (46)$$

³Recall that for any $x \in \mathbb{N}_+$, $\mathbf{s}_x = [1, 2, \dots, x]^\top$.

⁴Recall that for any $i \in \{1, 2, \dots, L^2\}$, $\omega_i = [\dot{\mathbf{p}}]_i - [\ddot{\mathbf{p}}]_i$.

$$= \mathbb{E} \left\{ \frac{1}{Q^2} \sum_{q=1}^Q \sum_{p=1}^Q [\mathbf{b}_q]_{j_n} [\mathbf{b}_p]_{j_n}^* \right\} - |[\mathbf{r}]_{j_n}|^2 \quad (47)$$

$$= \frac{1}{Q^2} \sum_{q=1}^Q \sum_{p=1}^Q \mathbb{E} \left\{ b_{q,p,j_n,j_n}^{(n)} \right\} - |[\mathbf{r}]_{j_n}|^2 \quad (48)$$

$$\stackrel{(43)}{=} \frac{(\mathbf{1}_K^\top \mathbf{d} + \sigma^2)^2}{Q}. \quad (49)$$

□

7.2. Proof of Lemma 2

Proof. By Lemma 1, $\text{err}_r(\hat{\mathbf{r}}_{\text{sel}}) = \mathbb{E}\{\|\mathbf{r}_{\text{co}} - \hat{\mathbf{r}}_{\text{sel}}\|_2^2\} = \sum_{n=1}^{L'-1} \mathbb{E}\{|[\mathbf{r}]_{j_n} - [\hat{\mathbf{r}}]_{j_n}|^2\} = (2L' - 1)e$. □

7.3. Proof of Proposition 1

Proof. Notice that $\mathbf{Z} = \mathbf{F}(\mathbf{I}_{L'} \otimes \mathbf{r}_{\text{co}}) = [\mathbf{F}_1 \mathbf{r}_{\text{co}}, \mathbf{F}_2 \mathbf{r}_{\text{co}}, \dots, \mathbf{F}_{L'} \mathbf{r}_{\text{co}}]$. By definition, $\mathbf{F}_m$ is a selection matrix that selects the $\{L' - (m-1), L' - (m-2), \dots, 2L' - m\}$ th entries of the length- $(2L' - 1)$ vector it multiplies, for every $m \in \{1, 2, \dots, L'\}$. That is, $\mathbf{F}_m \mathbf{r}_{\text{co}} = [\mathbf{r}_{\text{co}}]_{L'-(m-1):2L'-m}$. Similarly, $\hat{\mathbf{Z}}_{\text{sel}} = [\mathbf{F}_1 \hat{\mathbf{r}}_{\text{sel}}, \mathbf{F}_2 \hat{\mathbf{r}}_{\text{sel}}, \dots, \mathbf{F}_{L'} \hat{\mathbf{r}}_{\text{sel}}]$ with $\mathbf{F}_m \hat{\mathbf{r}}_{\text{sel}} = [\hat{\mathbf{r}}_{\text{sel}}]_{L'-(m-1):2L'-m}$. In view of the above,

$$\text{err}_Z(\hat{\mathbf{Z}}_{\text{sel}}) = \mathbb{E} \left\{ \left\| \mathbf{Z} - \hat{\mathbf{Z}}_{\text{sel}} \right\|_F^2 \right\} \quad (50)$$

$$= \mathbb{E} \left\{ \sum_{m=1}^{L'} \left\| \mathbf{F}_m \mathbf{r}_{\text{co}} - \mathbf{F}_m \hat{\mathbf{r}}_{\text{sel}} \right\|_2^2 \right\} \quad (51)$$

$$= \sum_{m=1}^{L'} \mathbb{E} \left\{ \left\| [\mathbf{r}_{\text{co}}]_{L'-(m-1):2L'-m} - [\hat{\mathbf{r}}_{\text{sel}}]_{L'-(m-1):2L'-m} \right\|_2^2 \right\} \quad (52)$$

$$= \sum_{m=1}^{L'} \sum_{n=1}^{L'-m} \mathbb{E} \left\{ |[\mathbf{r}_{\text{co}}]_{L'+n} - [\hat{\mathbf{r}}_{\text{sel}}]_{L'+n}|^2 \right\} \quad (53)$$

$$= \sum_{m=1}^{L'} \sum_{n=1}^{L'-m} e \quad (54)$$

$$= L'e. \quad (55)$$

□

7.4. Proof of Lemma 3

Proof.

$$e_n = \mathbb{E} \left\{ \left| [\mathbf{r}]_{j_n} - \frac{1}{|\mathcal{J}_n|} \sum_{j \in \mathcal{J}_n} [\hat{\mathbf{r}}]_j \right|^2 \right\} \quad (56)$$

$$= |\mathbf{r}|_{j_n}^2 + \frac{1}{|\mathcal{J}_n|^2} \mathbb{E} \left\{ \left| \sum_{j \in \mathcal{J}_n} [\hat{\mathbf{r}}]_j \right|^2 \right\} - 2 \mathbb{E} \left\{ \Re \left\{ [\mathbf{r}]_{j_n}^* \left(\frac{1}{|\mathcal{J}_n|} \sum_{j \in \mathcal{J}_n} [\hat{\mathbf{r}}]_j \right) \right\} \right\} \quad (57)$$

$$= |\mathbf{r}|_{j_n}^2 + \frac{1}{|\mathcal{J}_n|^2} \sum_{j \in \mathcal{J}_n} \sum_{i \in \mathcal{J}_n} \frac{1}{Q^2} \sum_{q=1}^Q \sum_{p=1}^Q \mathbb{E} \left\{ b_{q,p,i,j}^{(n)} \right\} - 2 \Re \left\{ [\mathbf{r}]_{j_n}^* \left(\frac{1}{|\mathcal{J}_n|} \sum_{j \in \mathcal{J}_n} [\mathbf{r}]_j \right) \right\} \quad (58)$$

$$\stackrel{(43)}{=} \frac{1}{Q} \left(\frac{2\sigma^2 \mathbf{1}_K^\top \mathbf{d} + \sigma^4}{|\mathcal{J}_n|} + \sum_{i \in \mathcal{J}_n} \sum_{j \in \mathcal{J}_n} \frac{|\mathbf{z}_{i,j}^H \mathbf{d}|^2}{|\mathcal{J}_n|^2} \right). \quad (59)$$

□

7.5. Proof of Lemma 4

Proof. By Lemma 3, $\text{err}_r(\hat{\mathbf{r}}_{\text{avg}}) = \mathbb{E} \{ \|\mathbf{r}_{\text{co}} - \hat{\mathbf{r}}_{\text{avg}}\|_2^2 \} = \sum_{n=1-L'}^{L'-1} \mathbb{E} \{ |[\mathbf{r}]_{j_n} - \frac{1}{|\mathcal{J}_n|} \sum_{j \in \mathcal{J}_n} [\hat{\mathbf{r}}]_j|^2 \} = \sum_{n=1-L'}^{L'-1} e_n$. □

7.6. Proof of Proposition 2

Proof. We know that $\mathbf{Z} = [\mathbf{F}_1 \mathbf{r}_{\text{co}}, \mathbf{F}_2 \mathbf{r}_{\text{co}}, \dots, \mathbf{F}_{L'} \mathbf{r}_{\text{co}}]$. Similarly, $\hat{\mathbf{Z}}_{\text{avg}} = [\mathbf{F}_1 \hat{\mathbf{r}}_{\text{avg}}, \mathbf{F}_2 \hat{\mathbf{r}}_{\text{avg}}, \dots, \mathbf{F}_{L'} \hat{\mathbf{r}}_{\text{avg}}]$.

By the definition of $\mathbf{F}_m$, for every $m \in \{1, 2, \dots, L'\}$ it holds that $\mathbf{F}_m \mathbf{r}_{\text{co}} = [\mathbf{r}_{\text{co}}]_{L'-(m-1):2L'-m}$

and $\mathbf{F}_m \hat{\mathbf{r}}_{\text{avg}} = [\hat{\mathbf{r}}_{\text{avg}}]_{L'-(m-1):2L'-m}$. In view of the above,

$$\text{err}_Z(\hat{\mathbf{Z}}_{\text{avg}}) = \mathbb{E} \left\{ \left\| \mathbf{Z} - \hat{\mathbf{Z}}_{\text{avg}} \right\|_F^2 \right\} \quad (60)$$

$$= \mathbb{E} \left\{ \sum_{m=1}^{L'} \left\| \mathbf{F}_m \mathbf{r}_{\text{co}} - \mathbf{F}_m \hat{\mathbf{r}}_{\text{avg}} \right\|_2^2 \right\} \quad (61)$$

$$= \sum_{m=1}^{L'} \mathbb{E} \left\{ \left\| [\mathbf{r}_{\text{co}}]_{L'-(m-1):2L'-m} - [\hat{\mathbf{r}}_{\text{avg}}]_{L'-(m-1):2L'-m} \right\|_2^2 \right\} \quad (62)$$

$$= \sum_{m=1}^{L'} \sum_{n=1-m}^{L'-m} \mathbb{E} \left\{ \left| [\mathbf{r}_{\text{co}}]_{L'+n} - [\hat{\mathbf{r}}_{\text{avg}}]_{L'+n} \right|^2 \right\} \quad (63)$$

$$= \sum_{m=1}^{L'} \sum_{n=1-m}^{L'-m} e_n. \quad (64)$$

□

7.7. Proof of Lemma 5

We recall that $\mathbf{w} = \frac{1}{Q} \sum_{q=1}^Q \mathbf{x}_q^* \otimes \mathbf{x}_q$. Next, we notice that by utilizing the auxiliary variables⁵ $\dot{\mathbf{u}} = \mathbf{1}_{K+L} \otimes \mathbf{s}_{K+L}$ and $\ddot{\mathbf{u}} = \mathbf{s}_{K+L} \otimes \mathbf{1}_{K+L}$, we obtain $[\mathbf{w}]_i = \frac{1}{Q} \sum_{q=1}^Q [\mathbf{x}_q^*]_{[\ddot{\mathbf{u}}]_i} [\mathbf{x}_q]_{[\dot{\mathbf{u}}]_i}$. Then, we define $\mathcal{I} \stackrel{\text{def}}{=} \{i \in \{1, 2, \dots, (K+L)^2\} : [\mathbf{i}]_i = 1\}$. We observe that $\mathbb{E}\{[\mathbf{x}_q^*]_{[\ddot{\mathbf{u}}]_i} [\mathbf{x}_q]_{[\dot{\mathbf{u}}]_i}\} = \delta([\dot{\mathbf{u}}]_i - [\ddot{\mathbf{u}}]_i) = 1$ if $i \in \mathcal{I}$ and 0 if $i \notin \mathcal{I}$. The latter implies

$$\mathbb{E}\{\mathbf{w}\} = \mathbf{i}. \quad (65)$$

Next, for $(i, m) \in \{1, 2, \dots, (K+L)^2\}^2$, we define $\eta_{i,m} \stackrel{\text{def}}{=} [\mathbf{x}_q^*]_{[\ddot{\mathbf{u}}]_i} [\mathbf{x}_q]_{[\dot{\mathbf{u}}]_i} [\mathbf{x}_p^*]_{[\ddot{\mathbf{u}}]_m} [\mathbf{x}_p]_{[\dot{\mathbf{u}}]_m}$. It holds $[\mathbf{w}]_i [\mathbf{w}^*]_m = \frac{1}{Q^2} \sum_{q=1}^Q \sum_{p=1}^Q \eta_{i,m}$. By the 2nd and 4th order moments of zero-mean independent normal variables, we find that $\mathbb{E}\{\eta_{i,m}\}$ is equal to $1 + \delta(p-q)\delta(i-m)$ if $(i, m) \in \mathcal{I}$ and 0 otherwise. The latter implies that $\mathbb{E}\{[\mathbf{w}]_i [\mathbf{w}^*]_m\} = \frac{1}{Q^2} \sum_{q=1}^Q \sum_{p=1}^Q \mathbb{E}\{\eta_{i,m}\}$ is equal to $1 + \frac{1}{Q} \delta(i, m)$, if $(i, m) \in \mathcal{I}$ and 0 otherwise. Altogether, we have

$$\mathbb{E}_{\mathbf{w}}\{\mathbf{w}\mathbf{w}^H\} = \mathbf{i}\mathbf{i}^\top + \frac{1}{Q} \mathbf{I}_{K+L}. \quad (66)$$

7.8. Proof of Lemma 6.

Proof. For $(i, m) \in \{1, 2, \dots, L^2\}^2$, $[\mathbf{H}]_{i,m} = [\mathbf{V}\mathbf{w}]_i [(\mathbf{V}\mathbf{i})^*]_m = \sum_{j=1, l=1}^{(K+L)^2} [\mathbf{V}]_{i,j} [\mathbf{w}]_j [\mathbf{V}^*]_{m,l} [\mathbf{i}]_l$. Accordingly, $[\mathbf{H}_{\mathbb{E}}]_{i,m} = \sum_{j=1, l=1}^{(K+L)^2} \mathbb{E}_{\Theta} \mathbb{E}_{\mathbf{w}}\{[\mathbf{V}]_{i,j} [\mathbf{w}]_j [\mathbf{V}^*]_{m,l} [\mathbf{i}]_l\}$. Considering that the random variables $\mathbf{w}$ and $\mathbf{V}$ are statistically independent from each other and that $\mathbb{E}_{\mathbf{w}}\{\mathbf{w}\} = \mathbf{i}$ (see Lemma 5), we obtain $[\mathbf{H}_{\mathbb{E}}]_{i,m} = \mathbb{E}_{\Theta}\{[\mathbf{V}\mathbf{i}]_i [(\mathbf{V}\mathbf{i})^*]_m\} = \mathbb{E}_{\Theta}\{[\mathbf{r}]_i [\mathbf{r}^*]_m\}$. Then, we substitute⁶ $[\mathbf{r}]_i = \sum_{k=1}^K v(\theta_k)^{\omega_i} d_k + \sigma^2 [\mathbf{i}_L]_i$ in $\mathbb{E}_{\Theta}\{[\mathbf{r}]_i [\mathbf{r}^*]_m\}$.

⁵Recall that for any $x \in \mathbb{N}_+$, $\mathbf{s}_x = [1, 2, \dots, x]^\top$.

⁶Recall that for any $i \in \{1, 2, \dots, L^2\}$, $\omega_i = [\mathbf{p}]_i - [\mathbf{\ddot{p}}]_i$.

After plain algebraic operations, we obtain

$$[\mathbf{H}_{\mathbb{E}}]_{i,m} = \|\mathbf{d}\|_2^2 \mathcal{I}(\omega_i - \omega_m) + \sigma^4 \delta(\omega_i) \delta(\omega_m) + \sigma^2 (\mathbf{1}_K^\top \mathbf{d}) (\delta(\omega_i) \mathcal{I}(-\omega_m) + \mathcal{I}(\omega_i) \delta(\omega_m)) \quad (67)$$

$$+ \mathcal{I}(\omega_i) \mathcal{I}(-\omega_m) ((\mathbf{1}_K^\top \mathbf{d})^2 - \|\mathbf{d}\|_2^2). \quad (68)$$

□

7.9. Proof of Lemma 7.

Proof. For $(i, m) \in \{1, 2, \dots, L^2\}^2$ it holds $[\mathbf{G}]_{i,m} = [\mathbf{V}\mathbf{w}]_i [(\mathbf{V}\mathbf{w})^*]_m = \sum_{j=1, l=1}^{(K+L)^2} [\mathbf{V}]_{i,j} [\mathbf{w}]_j [\mathbf{V}^*]_{m,l} [\mathbf{w}^*]_l$.

Accordingly, $[\mathbf{G}_{\mathbb{E}}]_{i,m} = \sum_{j=1, l=1}^{(K+L)^2} \mathbb{E}_{\Theta} \mathbb{E}_{\mathbf{w}} \{[\mathbf{V}]_{i,j} [\mathbf{w}]_j [\mathbf{V}^*]_{m,l} [\mathbf{w}^*]_l\}$. Next, we recall that the random variables Θ and $\mathbf{w}$ are statistically independent from each other. Thus, $[\mathbf{G}_{\mathbb{E}}]_{i,m} = \sum_{j=1, l=1}^{(K+L)^2} \mathbb{E}_{\Theta} \{[\mathbf{V}]_{i,j} \mathbb{E}_{\mathbf{w}} \{[\mathbf{w}]_j [\mathbf{w}^*]_l\}\}$.

The latter is equivalent to $\mathbf{G}_{\mathbb{E}} = \mathbb{E}_{\Theta} \{\mathbf{V} \mathbb{E}_{\mathbf{w}} \{\mathbf{w} \mathbf{w}^H\} \mathbf{V}^H\}$. By Lemma 5, we obtain $\mathbf{G}_{\mathbb{E}} = \mathbb{E}_{\Theta} \{\mathbf{V} \mathbf{i} \mathbf{i}^\top \mathbf{V}^H + \frac{1}{Q} \mathbf{V} \mathbf{V}^H\}$. By Lemma 6, we find that $\mathbf{G}_{\mathbb{E}} = \mathbf{H}_{\mathbb{E}} + \frac{1}{Q} \tilde{\mathbf{V}}_{\mathbb{E}}$. □

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3.2 Coarray-Domain Iterative Direction-of-Arrival Estimation With Coprime Arrays

Abstract

We consider source direction-of-arrival (DOA) estimation with coprime arrays via coarray domain processing. We present two Fourier-based iterative methods which address the non-uniformity of the corresponding difference coarray, thereby permitting exploitation of all offered degrees-of-freedom. The proposed approaches are based on the fast iterative interpolated beamforming (FIIB), which is a low-complexity algorithm for DOA estimation using uniform linear arrays. The first method specifically accounts for the non-uniformity of the coarray in the FIIB algorithm. The second approach modifies the FIIB to estimate the contributions of the missing elements, thus emulating a uniform coarray. We show that both methods achieve asymptotically unbiased DOA estimates. Extensive simulation results are provided to demonstrate the effectiveness of the proposed approaches.

1. Introduction

Sparse array configurations, such as minimum redundancy arrays, nested arrays, and coprime arrays, have garnered considerable research interest due to their ability to estimate more sources than physical sensors [1–11]. The significantly higher degrees-of-freedom (DOFs), offered by a non-uniform linear array over its uniformly-spaced counterpart, are achieved by leveraging the difference coarray (set of achievable spatial lags) [12, 13]. However, unlike the minimum redundancy or nested arrays that have uniformly-spaced coarrays, the coprime array yields a non-uniform coarray. More specifically, the coarray has a contiguous central portion and the missing elements or holes are symmetrically placed outside of this uniformly-spaced part. Because of this non-uniformity, DOA estimation methods devised for use with uniform configurations can only be applied to the contiguous portion of the coarray [14–16]. As such, these algorithms cannot exploit the full DOFs afforded by the coprime arrays.

Various approaches have been proposed to handle the non-uniformity of the coarray corre-

sponding to a coprime array, thereby utilizing all available DOFs. The first category of algorithms assumes the sources to be sparse and recovers the source DOAs via sparse reconstruction methods [17–20]. The second category addresses the non-uniformity by interpolating the holes to obtain a uniform coarray where traditional algorithms can be applied to estimate the DOAs [21–27]. Another class of approaches considers array motion to synthetically fill the holes in the coarray [28, 29]. Instead of physically moving the sensors, the electrical distance between the sensors can be changed using a multi-frequency approach to generate a coarray with uniformly-spaced elements [6, 30]. New coprime array configurations have also been recently proposed in which more sensors are added to the physical array in strategic locations to extend the contiguous portion of the coarray [31–33]. However, the aforementioned techniques for allowing the exploitation of the entire coarray for DOA estimation suffer from high computational and/or hardware complexity, require fine tuning of parameters, or impose additional restrictions on source models.

Owing to their lower computational complexity, Fourier-based methods have also been used with coprime arrays for DOA estimation. Such methods include the product processor [4], the min processor [8], and coarray beamforming [3]. The first two processors apply conventional beamforming to each subarray individually, followed by multiplication of the output of one subarray with the complex conjugate of the output of the other subarray in the case of the product processor and selecting the minimum of the two subarray outputs in the case of the min processor. Both methods provide comparable performance for a coprime array which has not been extended to include additional periods of the spatial sampling pattern [34]. The third technique operates in the coarray domain and performs beamforming with respect to signal powers instead of signal amplitudes. An algorithm based on the inverse Discrete Fourier transform was proposed in [35] for DOA estimation with coprime arrays, but it is applicable only to the contiguous portion of the coarray. Initial spatial frequency estimates are determined using the Discrete Fourier transform (DFT) in [36], which are further refined through phase rotation. However, as this algorithm was developed for a uniform linear array (ULA), it can also be applied only to the contiguous portion of the coarray.

A low-complexity Fourier-based DOA estimation method, called fast iterative interpolated

beamforming (FIIB), was proposed for use with ULAs [15]. The FIIB builds upon a conventional beamformer and employs nested iterations to successively estimate source DOAs in conjunction with an interpolation strategy to remove spectral leakage from adjacent sources. In so doing, it avoids the known issue of biased DOA estimation with a conventional beamformer. The FIIB in its original form can be applied to minimum redundancy and nested arrays, which have uniform coarrays [37]. It was also shown to accurately estimate source DOAs with coprime arrays in [38, 39], but only operating on the contiguous portion of the coarray. Although FIIB is based on the conventional beamformer, it was shown to provide high-resolution performance rivaling that of Root-MUSIC [38].

We focus on Fourier-based methods in this paper and propose two modifications to the FIIB algorithm to enable use of all available coarray measurements for DOA estimation with coprime arrays. In the first approach, we retain the same functional form of the interpolation function as in the original FIIB for fine-tuning the coarse DOA estimates obtained with the conventional beamformer and devise a simple strategy to correct for the bias arising due to the presence of the holes. The second method, inspired by [40], estimates the observations corresponding to the holes and appends these to the actual measurements to produce the effect of a fully populated difference coarray with no missing elements. We show that both proposed methods provide asymptotically unbiased DOA estimates. Extensive simulation results are provided, which demonstrate the effectiveness of the proposed adjustments to the FIIB.

The remainder of this paper is organized as follows. In Section 2, we present the signal model. We review the original FIIB and introduce the proposed modifications in Section 3. A theoretical bias analysis of the modified FIIB techniques is provided in Section 4. Next, in Section 5, we perform extensive evaluations of the proposed methods in terms of the DOA estimation accuracy and compare against existing Fourier-based schemes. Finally, Section 6 presents the concluding remarks.

2. Signal model

Consider a coprime array with $M_a = \dot{M} + \dot{N} - 1$ elements, formed by combining an $\dot{N}$ -element ULA with element positions $P_{\dot{N}} = \{(i-1)\dot{M}d, i = 1, 2, \dots, \dot{N}\}$ and a ULA with $\dot{M} - 1$ elements at positions $P_{\dot{M}} = \{i\dot{N}d, i = 1, 2, \dots, \dot{M} - 1\}$. The unit spacing d equals $\lambda/2$, with λ being the operating wavelength. We denote the element positions of the coprime array by $\mathbf{p} = [p_1d, p_2d, \dots, p_{M_a}d]^T$, where $p_{m_a} \in (P_{\dot{M}} \cup P_{\dot{N}})$, $m_a = 1, 2, \dots, M_a$ are sorted in ascending order with $p_1 = 0$, and $(\cdot)^T$ denotes matrix transpose. The corresponding coarray has an extent of $(N-1)d$, where $N = (2(\dot{N}-1)\dot{M}) + 1$. However, not all uniformly-spaced available locations, $\{nd, n = 0, 1, \dots, N-1\}$, are populated. More specifically, the coarray has $K = \dot{M}\dot{N} + \dot{M} + \dot{N} - 2$ elements, which occupy a subset $\mathcal{S}$ of the available locations, and $M = N - K = (\dot{M}-1)(\dot{N}-3)$ holes exist at the remaining locations (the complement set $\bar{\mathcal{S}}$). Further, the holes appear in pairs at symmetrical locations about $(N-1)d/2$. Also, a symmetrical portion of the coarray, centered at $(N-1)d/2$ and of extent $2(\dot{N} + \dot{M} - 1)$, has contiguous elements and the holes occur outside of this uniformly-spaced part.

Assume that Q uncorrelated narrowband signals are impinging on the array from directions $\theta_1, \theta_2, \dots, \theta_Q$, with angles measured relative to broadside. Under far-field conditions, the received signal at time t can be expressed as

$$\mathbf{x}(t) = \sum_{q=1}^Q \dot{s}_q(t) \mathbf{v}_q + \dot{\mathbf{w}}(t), \quad (1)$$

where $\dot{s}_q(t)$ is the q th source signal, $\dot{\mathbf{w}}(t)$ is the additive white Gaussian noise with covariance $\sigma^2 \mathbf{I}_{M_a}$, and $\mathbf{I}_{M_a}$ is the $M_a \times M_a$ identity matrix. The steering vector, $\dot{\mathbf{v}}_q$ is defined as

$$\dot{\mathbf{v}}_q = [e^{j2\pi p_1 v_q}, \dots, e^{j2\pi p_{M_a} v_q}]^T, \quad (2)$$

where $v_q = \frac{d}{\lambda} \sin \theta_q \in (-0.5, 0.5]$ is the spatial frequency associated with direction θ_q .

Defining the $M_a \times Q$ steering matrix as $\dot{\mathbf{V}} = [\dot{\mathbf{v}}_1, \dot{\mathbf{v}}_2, \dots, \dot{\mathbf{v}}_Q]$ and assuming the noise to be

statistically independent of the sources, the signal autocorrelation matrix can be expressed as

$$\begin{aligned}\mathbf{R}_x &= \mathbb{E}\{\mathbf{x}(t)\mathbf{x}^H(t)\} \\ &= \hat{\mathbf{V}}\text{diag}([\hat{\beta}_1, \dots, \hat{\beta}_Q])\hat{\mathbf{V}}^H + \sigma^2\mathbf{I}_{M_a},\end{aligned}\quad (3)$$

where $\text{diag}(\cdot)$ returns a diagonal matrix with the elements of its vector argument on the main diagonal, $\hat{\beta}_q$ is the q th source power, and $(\cdot)^H$ denotes the Hermitian operator. Vectorizing $\mathbf{R}_x$, we obtain the $M_a^2 \times 1$ vector

$$\mathbf{r} = \text{vec}(\mathbf{R}_x) = \sum_{q=1}^Q \hat{\beta}_q \mathbf{u}_q + \sigma^2 \mathbf{i}_{M_a}, \quad (4)$$

where $\text{vec}(\cdot)$ returns the column-wise vectorization of its matrix argument, $\mathbf{u}_q = \hat{\mathbf{v}}_q^* \otimes \hat{\mathbf{v}}_q$, $\mathbf{i}_{M_a} = \text{vec}(\mathbf{I}_{M_a})$, $(\cdot)^*$ denotes complex conjugation, and ' $\otimes$ ' is the Kronecker product operator.

The receiver assembles a linear combining matrix $\mathbf{E} \in \mathbb{R}^{M_a^2 \times K}$ and forms a length- K coarray signal vector $\mathbf{r}_{co}$ as

$$\mathbf{r}_{co} = \mathbf{E}^T \mathbf{r} = \sum_{q=1}^Q \beta_q \mathbf{u}_{co,q} + \mathbf{w}, \quad (5)$$

where $\mathbf{u}_{co,q}$ is the coarray response vector for the q th source (with coarray locations in $\mathcal{S}$), $\beta_q = \hat{\beta}_q e^{-j2\pi \frac{(N-1)}{2} \nu_q}$ is the complex amplitude corresponding to the q th source, and $\mathbf{w} = \sigma^2 \mathbf{e}_{\frac{K+1}{2}, K}$, with $\mathbf{e}_{\frac{K+1}{2}, K}$ being the $((K+1)/2)$ th column of $\mathbf{I}_K$. For more details on the design of the matrix $\mathbf{E}$, see [41]. The receiver can estimate the sources by processing the coarray measurements in $\mathbf{r}_{co}$ via a suitable DOA estimation method. In practice, $\mathbf{R}_x$ in (3) is estimated as an average over T snapshots,

$$\hat{\mathbf{R}}_x = \frac{1}{T} \sum_{t=0}^{T-1} \mathbf{x}(t)\mathbf{x}(t)^H, \quad (6)$$

and the receiver forms $\mathbf{r}_{co}$ using $\hat{\mathbf{R}}_x$.

3. DOA Estimation with Coprime Arrays Using FIIB

We first review the original FIIB, which is applicable to ULAs [15]. Then, we present the proposed modifications to the FIIB for use with the non-uniform coarray measurement vector $\mathbf{r}_{co}$. In this and subsequent sections, we assume the number of sources, Q , to be known. We note, however, that the number of sources can be estimated by incorporating information theoretic criteria directly in the FIIB [42].

Algorithm 1: Original FIIB [15].

Input : The received data vector $\mathbf{y}$

- 1 Put $\hat{v}_q = 0$ and $\hat{\beta}_q = 0$ for $q = 1, \dots, Q$
- 2 Let $\mathbf{Y} = \text{FFT}(\mathbf{y}, N)$.
- 3 Set $l = 0$
- 4 **for** L iterations or until convergence **do**
- 5 $l = l + 1$
- 6 **for** $q = 1 : Q$ **do**
- 7 **if** $l = 1$ **then**
- 8 $\tilde{Y}[n] = Y[n] - \sum_{\substack{i=1 \\ i \neq q}}^Q \hat{\beta}_i A_i[n], n = 0, 1, \dots, N - 1$
- 9 $\hat{v}_q = \frac{1}{N} \max_{0 \leq n \leq N-1} |\tilde{Y}[n]|^2$
- 10 **end**
- 11 $\tilde{Y}_{\pm p}(\hat{v}_q) = Y_{\pm p}(\hat{v}_q) - \sum_{\substack{i=1 \\ i \neq q}}^Q \hat{\beta}_i A_i(\hat{v}_q \pm p), p = 0.5$
- 12 where $\tilde{Y}_{\pm p}(\hat{v}_q) := \tilde{Y}(\hat{v}_q \pm p), Y_{\pm p}(\hat{v}_q) := Y(\hat{v}_q \pm p)$
- 13 $h = \frac{\tilde{Y}_{0.5} + \tilde{Y}_{-0.5}}{\tilde{Y}_{0.5} - \tilde{Y}_{-0.5}}$
- 14 $\hat{z} = \frac{1}{\cos\left(\frac{\pi}{N}\right) - jh \sin\left(\frac{\pi}{N}\right)}$
- 15 $\hat{\delta}_q = N \frac{\angle \hat{z}}{2\pi}$
- 16 $\hat{v}_q \leftarrow \hat{v}_q + \frac{\hat{\delta}_q}{N}$
- 17 $\hat{\beta}_q = \frac{1}{N} \left\{ \sum_{k=0}^{N-1} y[k] e^{-j \frac{2\pi}{N} k \hat{v}_q} - \sum_{\substack{i=1 \\ i \neq q}}^Q \hat{\beta}_i A_i(\hat{v}_q) \right\}$
- 18 **end**
- 19 **end**

Output: $[\hat{\beta}_q, \hat{v}_q], q = 1, 2, \dots, Q$

3.1. Original FIIB

In order to describe the FIIB algorithm, we first define the response vector of a fully populated coarray with N elements at positions $\{nd, n = 0, 1, \dots, N - 1\}$ as,

$$\mathbf{a} = [1, e^{j2\pi v}, \dots, e^{j2\pi(N-1)v}]^T, \quad (7)$$

and the corresponding coarray signal vector is modeled as,

$$\mathbf{y} = \sum_{q=1}^Q \beta_q \mathbf{a}_q + \mathbf{e}_{\frac{N+1}{2}, N}. \quad (8)$$

The original FIIB technique is provided in Algorithm 1. The DFT, $\mathbf{Y}$, of the length- N vector $\mathbf{y}$ is obtained by taking an N -point Fast Fourier Transform (FFT) in line 2. It is noted that the number of FFT points can be increased to $2N$ [15]. However, in this work, we focus on the N -point FFT only. The extension to the $2N$ -FFT is straightforward. The coarse spatial frequencies of the Q sources are sequentially estimated during the first iteration, starting with the strongest source. More specifically, the q th source is exposed by subtracting the $(q - 1)$ previously estimated sources from $\mathbf{Y}$ in line 8, where $A_i[n]$ represents the n th value of the DFT of the i th coarray response vector, $\mathbf{a}_i$, given by

$$A_i[n] = A_i(v)|_{v=\frac{n}{N}} = \frac{1 - e^{j2\pi N(\hat{v}_i - \frac{n}{N})}}{1 - e^{j2\pi(\hat{v}_i - \frac{n}{N})}}. \quad (9)$$

In line 9, the index corresponding to the highest peak is taken as the coarse estimate of the spatial frequency of the q th source which is refined using interpolation and leakage subtraction in lines 11 to 16. With $Y_{\pm p}(\hat{v}_q)$ denoting the interpolated DFT coefficients that are $\pm \frac{1}{2}$ bins away from $\hat{v}_q$ and $A_i(\hat{v}_q \pm p)$ representing the spectral leakage terms, given by

$$A_i(\hat{v}_q \pm p) = \frac{1 + e^{j2\pi N(\hat{v}_i - \hat{v}_q)}}{1 - e^{j2\pi(\hat{v}_i - \hat{v}_q \mp \frac{p}{N})}}, \quad p = 0.5, \quad (10)$$

the leakage-free interpolated DFT coefficients are computed in line 11 as

$$\tilde{Y}_{\pm p}(\hat{v}_q) = Y_{\pm p}(\hat{v}_q) - \sum_{\substack{i=1 \\ i \neq q}}^Q \hat{\beta}_i A_i(\hat{v}_q \pm p), \quad p = 0.5 \quad (11)$$

These coefficients are used to obtain the interpolation function h in line 13, which is then employed to estimate the frequency residual $\hat{\delta}_q$ in line 15. Line 17 estimates the amplitude of the q th signal using a maximum likelihood estimator [43]. The refinement process is executed for each source and repeated L times or until convergence is achieved. Finally, the DOA estimates are obtained from the estimated spatial frequencies as

$$\hat{\theta}_q = \sin^{-1} \left(\frac{\hat{v}_q \lambda}{d} \right). \quad (12)$$

In order to appreciate the proposed modifications, it is essential to understand the motivation behind the specific formulation of the interpolation function h . Let the spatial frequency of the q th source be $v_q = \frac{m_q + \delta_q}{N}$ with m_q and δ_q denoting the coarse and fine frequency components, respectively. Assume that the coarse frequency estimate of the q th source equals m_q , which is the case with high probability as shown in [44]. The noise-free interpolated DFT coefficients in each iteration for $p = 0.5$ assume the form

$$\begin{aligned} Y_{\pm p}(\hat{v}_q) &= \sum_{n=0}^{N-1} \beta_q e^{j2\pi n v_q} e^{-j2\pi n(\hat{v}_q \pm \frac{p}{N})} \\ &= \beta_q \frac{\rho}{1 - \alpha^{\mp 1} z} \end{aligned} \quad (13)$$

where $\rho = 1 + e^{j2\pi(\delta_q - \hat{\delta}_q)}$, $z = e^{j2\pi(\delta_q - \hat{\delta}_q)/N}$, and $\alpha = e^{j\pi/N}$. Using (13), the interpolation function can be expressed as

$$h = \frac{Y_{0.5} + Y_{-0.5}}{Y_{0.5} - Y_{-0.5}} = \frac{2 - (\alpha + \alpha^{-1})z}{(\alpha^{-1} - \alpha)z}, \quad (14)$$

where the arguments of $Y_{\pm p}$, $p = 0.5$, have been omitted for notational simplification. Solving (14)

for z leads to

$$z = (\cos(\pi/N) - jh \sin(\pi/N))^{-1}, \quad (15)$$

from which the frequency residual is obtained as $(\delta_q - \hat{\delta}_q) = \frac{N\angle z}{2\pi}$.

Algorithm 2: Modified FIIB I.

Input : The received data vector $\check{\mathbf{y}}$

- 1 Put $\hat{v}_q = 0$ and $\hat{\beta}_q = 0$ for $q = 1, \dots, Q$
- 2 Let $\check{\mathbf{Y}} = \text{FFT}(\check{\mathbf{y}}, N)$.
- 3 Set $l = 0$
- 4 **for** L iterations or until convergence **do**
- 5 $l = l + 1$
- 6 **for** $q = 1 : Q$ **do**
- 7 **if** $l = 1$ **then**
- 8 $\hat{\check{Y}}[n] = \check{Y}[n] - \sum_{\substack{i=1 \\ i \neq q}}^Q \hat{\beta}_i \check{A}_i[n], n = 0, 1, \dots, N - 1$
- 9 $\hat{v}_q = \frac{1}{N} \max_{0 \leq n \leq N-1} |\hat{\check{Y}}[n]|^2$
- 10 **end**
- 11 $\hat{\check{Y}}_{\pm p}(\hat{v}_q) = \check{Y}_{\pm p}(\hat{v}_q) - \sum_{\substack{i=1 \\ i \neq q}}^Q \hat{\beta}_i \check{A}_i(\hat{v}_q \pm p), p = 0.5$
- 12 where $\hat{\check{Y}}_{\pm p}(\hat{v}_q) := \hat{\check{Y}}(\hat{v}_q \pm p), \check{Y}_{\pm p}(\hat{v}_q) := \check{Y}(\hat{v}_q \pm p)$
- 13 $\check{h} = \frac{\hat{\check{Y}}_{0.5} + \hat{\check{Y}}_{-0.5}}{\hat{\check{Y}}_{0.5} - \hat{\check{Y}}_{-0.5}}$
- 14 $h = \check{h} - \check{b}$
- 15 $\hat{z} = \frac{1}{\cos\left(\frac{\pi}{N}\right) - jh \sin\left(\frac{\pi}{N}\right)}$
- 16 $\hat{\delta}_q = N \frac{\angle \hat{z}}{2\pi}$
- 17 $\hat{v}_q \leftarrow \hat{v}_q + \frac{\hat{\delta}_q}{N}$
- 18 $\hat{\beta}_q = \frac{1}{N} \left\{ \sum_{k=0}^{N-1} \check{y}[k] e^{-j \frac{2\pi}{N} k \hat{v}_q} - \sum_{\substack{i=1 \\ i \neq q}}^Q \hat{\beta}_i \check{A}_i(\hat{v}_q) \right\}$
- 19 **end**
- 20 **end**

Output: $[\hat{\beta}_q, \hat{v}_q], q = 1, 2, \dots, Q$

3.2. Proposed Modification I

We first define a length- N response vector, $\check{\mathbf{a}}$, for the non-uniform coarray by replacing with zero the elements of $\mathbf{a}$ in (7) corresponding to the holes. Likewise, a length- N coarray signal vector, $\check{\mathbf{y}}$, is constructed by augmenting $\mathbf{r}_{co}$ in (5) with zeros at indices corresponding to the hole locations. The modified procedure, provided in Algorithm 2, essentially follows the steps in Algorithm 1, but with some changes as detailed below.

The DFT, $\check{\mathbf{Y}}$, of the length- N vector $\check{\mathbf{y}}$ is obtained by taking an N -point FFT in line 2. During the first iteration, sequential estimation of the coarse spatial frequencies of the Q sources is performed, starting with the strongest source. That is, the $(q - 1)$ previously estimated sources are first subtracted from $\check{\mathbf{Y}}$ in line 8, where $\check{A}_i[n]$ represents the n th value of the DFT of the i th coarray response vector, $\check{\mathbf{a}}_i$, given by

$$\check{A}_i[n] = A_i[n] - \sum_{l \in \tilde{S}} e^{j2\pi l(\hat{v}_i - \frac{n}{N})} \quad (16)$$

with $A_i[n]$ as in (9). Note that since $A_i[n]$ corresponds to a fully-populated coarray, the contribution from the elements corresponding to hole locations is subtracted in (16) to account for the non-uniformity of the coarray. In step 9, the location of the highest peak of the resulting spectrum yields the coarse spatial frequency estimate of the q th source. Fine frequency estimation proceeds in steps 11 to 17. Specifically, the leakage DFT coefficients, $\check{A}_i(\hat{v}_q \pm p)$, in line 11 are given by

$$\check{A}_i(\hat{v}_q \pm p) = A_i(\hat{v}_q \pm p) - \sum_{l \in \tilde{S}} e^{j2\pi l(\hat{v}_i - \hat{v}_q \mp \frac{p}{N})} \quad (17)$$

where $A_i(\hat{v}_q \pm p)$ is defined in (10). Again, as (10) assumes a fully-populated coarray, the term being subtracted in (17) removes the contributions of the elements corresponding to the holes. The leakage-free interpolated DFT coefficients $\hat{Y}_{\pm p}(\hat{v}_q)$ are computed in line 11 for $p = 0.5$. We retain the same functional form of the interpolation function in line 13 as in the original FIIB algorithm. This, however, introduces a bias in the interpolation function due to the non-uniformity of the coarray. If not corrected, the biased interpolation function would produce erroneous results in the

subsequent computations. Below, we determine the bias term, $\breve{b}$, which is then subtracted from the interpolation function in line 14 to ensure accuracy of the various estimates in lines 15 through 18.

Assuming the coarse frequency estimate of the q th source to be m_q , the noise-free interpolated DFT coefficients $\breve{Y}_{\pm p}(\hat{v}_q)$ for $p = 0.5$ can be expressed as,

$$\begin{aligned}
\breve{Y}_{\pm p}(\hat{v}_q) &= \sum_{\substack{n=0 \\ n \notin \bar{S}}}^{N-1} \beta_q e^{j2\pi n v_q} e^{-j2\pi n(\hat{v}_q \pm \frac{p}{N})} \\
&= Y_{\pm p}(\hat{v}_q) - \sum_{n \in \bar{S}} \beta_q e^{j2\pi n \frac{\delta_q - \hat{\delta}_q \mp p}{N}} \\
&= Y_{\pm p}(\hat{v}_q) - \beta_q \sum_{n \in \bar{S}} z^n \alpha^{\mp n} \\
&:= Y_{\pm p}(\hat{v}_q) - \mathcal{E}_{\pm p}.
\end{aligned} \tag{18}$$

where $Y_{\pm p}(\hat{v}_q)$ is defined in (13). Using $\breve{Y}_{\pm p}(\hat{v}_q)$, the interpolation function can be expressed as

$$\breve{h} = \frac{\breve{Y}_{0.5} + \breve{Y}_{-0.5}}{\breve{Y}_{0.5} - \breve{Y}_{-0.5}}. \tag{19}$$

Substituting $\breve{Y}_{\pm 0.5}$ from (18) in (19) and using (13), $\breve{h}$ assumes the form

$$\breve{h} = \frac{Y_{0.5} + Y_{-0.5} - \Sigma}{Y_{0.5} - Y_{-0.5} - \Delta} = \frac{h - \Sigma \frac{(1-\alpha^{-1}z)(1-\alpha z)}{\rho(\alpha^{-1}-\alpha)z}}{1 - \Delta \frac{(1-\alpha^{-1}z)(1-\alpha z)}{\rho(\alpha^{-1}-\alpha)z}},$$

where

$$\begin{aligned}
\Sigma &= \frac{1}{\beta_q} (\mathcal{E}_{0.5} + \mathcal{E}_{-0.5}) = \sum_{n \in \bar{S}} z^n (\alpha^{-n} + \alpha^n), \\
\Delta &= \frac{1}{\beta_q} (\mathcal{E}_{0.5} - \mathcal{E}_{-0.5}) = \sum_{n \in \bar{S}} z^n (\alpha^{-n} - \alpha^n).
\end{aligned} \tag{20}$$

Cross-multiplying and rearranging, we have

$$\check{h} = h + \check{b}, \quad \check{b} = \frac{(1 - \alpha^{-1}z)(1 - \alpha z)}{\rho z(\alpha^{-1} - \alpha)} (\check{h}\Delta - \Sigma). \quad (21)$$

Therefore, subtracting $\check{b}$ from the interpolation function in line 14 restores the relation (14) between h and z , thereby rendering the subsequent steps valid.

3.3. Proposed Modification II

The second proposed modification to the FIIB, presented in Algorithm 3, calculates the contributions of the holes in each iteration using the signal parameters estimated in the previous iterations and adds these to the coarray measurement vector to emulate a filled coarray. This approach is inspired by the work in [40], wherein the authors present an estimate-and-add method for two-dimensional frequency estimation using non-uniform time samples.

This alternative algorithm differs from the original FIIB in that each iteration begins by adding the estimated frequency-domain contribution of the holes, $\hat{\mathcal{E}}$, to the DFT, $\check{\mathbf{Y}}$, of the received data vector (line 7). The vector $\hat{\mathcal{E}}$ is initialized to a zero vector and is updated at the end of each iteration of the inner loop, right after the spatial frequency and amplitude estimates of the q th source are obtained. As the signal $\check{\mathbf{Y}}$ in line 7 emulates measurements from a uniform coarray, the algorithm proceeds as the original FIIB until we evaluate the interpolation function in line 14. Similar to Algorithm 2, a bias term arises in the interpolation function in this case as shown below.

Considering the noise-free interpolated DFT coefficients, $\check{Y}_{\pm p}(\hat{\nu}_q)$ and assuming no error in the coarse frequency estimate of the q th source, we can express the interpolation function as

$$\begin{aligned} \check{h} &= \frac{\check{Y}_{0.5} + \check{Y}_{-0.5}}{\check{Y}_{0.5} - \check{Y}_{-0.5}} = \frac{Y_{0.5} - \mathcal{E}_{0.5} + \hat{\mathcal{E}}_{0.5} + Y_{-0.5} - \mathcal{E}_{-0.5} + \hat{\mathcal{E}}_{-0.5}}{Y_{0.5} - \mathcal{E}_{0.5} + \hat{\mathcal{E}}_{0.5} - Y_{-0.5} + \mathcal{E}_{-0.5} - \hat{\mathcal{E}}_{-0.5}} \\ &= \frac{h - \frac{(1 - \alpha^{-1}z)(1 - \alpha z)}{\rho z(\alpha^{-1} - \alpha)} \left(\Sigma - \frac{\hat{\beta}_q}{\beta_q} \chi \right)}{1 - \frac{(1 - \alpha^{-1}z)(1 - \alpha z)}{\rho z(\alpha^{-1} - \alpha)} \left(\Delta + \frac{\hat{\beta}_q}{\beta_q} \psi \right)} = h + \check{b}, \end{aligned} \quad (22)$$

where $\chi = \sum_{n \in \bar{S}} (\alpha^{-n} + \alpha^n)$, $\psi = \sum_{n \in \bar{S}} (\alpha^{-n} - \alpha^n)$, $\hat{\mathcal{E}}_{\pm 0.5} = \sum_{n \in \bar{S}} \hat{\beta}_q \alpha^{\mp n}$ and

$$\check{b} = \frac{(1 - \alpha^{-1}z)(1 - \alpha z)}{\rho z(\alpha^{-1} - \alpha)} \left((\check{h}\Delta - \Sigma) + \frac{\hat{\beta}_q}{\beta_q} (\check{h}\psi + \chi) \right). \quad (23)$$

We note that, unlike the bias term in Algorithm 2, the right side of (23) becomes zero when the amplitude and spatial frequency estimates of the q th source are equal to the true values.

We correct for this bias term in line 15 and use the bias-corrected interpolation function to update the spatial frequency estimate in lines 16 to 18. The amplitude of the q th source is estimated in line 19 using the “full” coarray measurement vector

$$\check{\mathbf{y}} = \check{\mathbf{y}} + \hat{\boldsymbol{\epsilon}}, \quad (24)$$

where the non-zero elements of the vector, $\hat{\boldsymbol{\epsilon}}$, contain the estimated coarray measurements corresponding to the holes. That is,

$$\hat{\epsilon}[n] = \begin{cases} \sum_{q=1}^Q \hat{\beta}_q e^{j2\pi \hat{\nu}_q n} & n \in \bar{S} \\ 0 & n \in S \end{cases}. \quad (25)$$

Finally, in line 21, the estimates of the coarray measurements corresponding to the holes are updated for use in the next iteration.

3.4. Computational Complexity

All FIIB algorithms require a single N -point FFT computation ($\mathcal{O}(N \log_2 N)$), followed by calculation of two interpolated DFT coefficients for each source per iteration which imposes an additional $\mathcal{O}(2LQN)$ operations. Thus, as noted in [15], the original FIIB has a complexity of $\mathcal{O}(N \log_2 N + L(2QN + \kappa))$, where κ accounts for additional overhead per iteration. In modification

I, we also compute the interpolated DFT coefficients of the hole contributions for each source per iteration, thereby adding an additional $\mathcal{O}(2LQM)$ operations. Thus, the complexity of modified FIIB I is $\mathcal{O}(N \log_2 N + L(2Q(N + M) + \kappa))$, where κ now includes the bias correction per iteration as well. Compared to the original FIIB, modification II only requires computation of the DFT of the holes response, adding $\mathcal{O}(LQM)$ operations to the algorithm. Thus, modified FIIB II has a complexity of $\mathcal{O}(N \log_2 N + L(2QN + QM + \kappa))$. For large N , the FFT is the highest complexity operation and, therefore, all FIIB algorithms have a computational complexity $\sim \mathcal{O}(N \log_2 N)$.

4. Bias Analysis of the Proposed DOA Estimators

In this section, we derive the bias in the spatial frequency estimates obtained with the proposed FIIB-based methods. We consider the case of $Q = 2$ sources with respective spatial frequencies $\nu_1 = (m_1 + \delta_1)/N$ and $\nu_2 = (m_2 + \delta_2)/N$ and note that the results can be readily generalized to the case of $Q > 2$. We assume the coarse frequency estimates to be unbiased, i.e., $\mathbb{E}[\hat{m}_i] = m_i, i = 1, 2$, with $\mathbb{E}[\cdot]$ denoting the expectation operator. The estimates of $\delta_i, i = 1, 2$, are denoted by $\hat{\delta}_i, i = 1, 2$, and let $M_{2,1} = m_2 - m_1$. Further, we assume the number of time snapshots T to be sufficiently large such that the noise term in (5) is approximated by $w[n] = \hat{\sigma}^2 \delta[n - \frac{N-1}{2}], n = 0, 1, \dots, N-1$, where $\hat{\sigma}^2$ is the noise sample variance. That is, the noise only impacts the measurement corresponding to the center element of the coarray.

4.1. Proposed Modification I

The “leakage-free” DFT coefficients of the first source with spatial frequency ν_1 can be expressed as

$$\hat{Y}_{1,\pm 0.5} = Y_{1,\pm 0.5} - \mathcal{E}_{1,\pm 0.5} + \check{V}_{\pm 0.5} + W_{1,\pm 0.5}, \quad (26)$$

which, using (13) and (18), can be rewritten as

$$\hat{Y}_{1,\pm 0.5} = \frac{\rho_1 \beta_1}{1 - \alpha^{\mp 1} z_1} - \beta_1 \sum_{n \in \bar{S}} \alpha^{\mp n} z_1^n + \check{V}_{\pm 0.5} + W_{1,\pm 0.5}, \quad (27)$$

with $\rho_1 = 1 + e^{j2\pi(\delta_1 - \hat{\delta}_1)}$ and $z_1 = e^{j\frac{2\pi}{N}(\delta_1 - \hat{\delta}_1)}$. The noise term, $W_{1,\pm 0.5}$, at $\hat{v}_1 = (m_1 + \hat{\delta}_1)/N$ is given by

$$\begin{aligned} W_{1,\pm 0.5} &= \sum_{n=0}^{N-1} w[n] e^{-j\frac{2\pi}{N}n(m_1 + \hat{\delta}_1 \pm 0.5)} - \sum_{n \in \tilde{S}} w[n] e^{-j\frac{2\pi}{N}n(m_1 + \hat{\delta}_1 \pm 0.5)} \\ &= \hat{\sigma}^2 \alpha^{-(N-1)(m_1 + \hat{\delta}_1 \pm 0.5)}. \end{aligned} \quad (28)$$

The residual leakage term $\check{V}_{\pm 0.5}$ takes the form

$$\begin{aligned} \check{V}_{\pm 0.5} &= \check{Y}_{2,\pm 0.5} - \hat{Y}_{2,\pm 0.5} = (Y_{2,\pm 0.5} - \mathcal{E}_{2,\pm 0.5}) - (\hat{Y}_{2,\pm 0.5} - \hat{\mathcal{E}}_{2,\pm 0.5}) \\ &= \beta_2 \left(\frac{1 + e^{j2\pi(\delta_2 - \hat{\delta}_1)}}{1 - \alpha^{\mp 1} e^{j\frac{2\pi}{N}(M_{2,1} + \delta_2 - \hat{\delta}_1)}} - \sum_{n \in \tilde{S}} \alpha^{\mp 1} e^{j\frac{2\pi}{N}n(M_{2,1} + \delta_2 - \hat{\delta}_1)} \right) \\ &\quad - \hat{\beta}_2 \left(\frac{1 + e^{j2\pi(\hat{\delta}_2 - \hat{\delta}_1)}}{1 - \alpha^{\mp 1} e^{j\frac{2\pi}{N}(M_{2,1} + \hat{\delta}_2 - \hat{\delta}_1)}} - \sum_{n \in \tilde{S}} \alpha^{\mp 1} e^{j\frac{2\pi}{N}n(M_{2,1} + \hat{\delta}_2 - \hat{\delta}_1)} \right) \end{aligned} \quad (29)$$

where $\hat{\beta}_2$ is the estimated amplitude of the second source and is given by

$$\begin{aligned} \hat{\beta}_2 &= \frac{1}{N} \sum_{n=0}^{N-1} (\check{y}_1[n] + \check{y}_2[n] + w[n] - \hat{y}_1[n]) e^{-j2\pi n \frac{m_2 + \hat{\delta}_2}{N}} \\ &= \frac{\beta_2}{N} \left(\frac{1 - e^{j2\pi(\delta_2 - \hat{\delta}_2)}}{1 - e^{j\frac{2\pi}{N}(\delta_2 - \hat{\delta}_2)}} - \sum_{n \in \tilde{S}} e^{j\frac{2\pi}{N}n(\delta_2 - \hat{\delta}_2)} \right) \\ &\quad + \frac{\beta_1}{N} \left(\frac{1 - e^{j2\pi(\delta_1 - \hat{\delta}_2)}}{1 - e^{j\frac{2\pi}{N}(\delta_1 - \hat{\delta}_2 - M_{2,1})}} - \sum_{n \in \tilde{S}} e^{j\frac{2\pi}{N}n(\delta_1 - \hat{\delta}_2 - M_{2,1})} \right) \\ &\quad - \frac{\hat{\beta}_1}{N} \left(\frac{1 - e^{j2\pi(\hat{\delta}_1 - \hat{\delta}_2)}}{1 - e^{j\frac{2\pi}{N}(\hat{\delta}_1 - \hat{\delta}_2 - M_{2,1})}} - \sum_{n \in \tilde{S}} e^{j\frac{2\pi}{N}n(\hat{\delta}_1 - \hat{\delta}_2 - M_{2,1})} \right) + \frac{W_2}{N}, \end{aligned} \quad (30)$$

with $W_2 = \hat{\sigma}^2 \alpha^{-(N-1)(m_2 + \hat{\delta}_2)}$ being the noise term at $\hat{v}_2 = (m_2 + \hat{\delta}_2)/N$. We have $M_{2,1} \sim \mathcal{O}(N)$ and $\frac{M_{2,1}}{N} \sim \mathcal{O}(1)$ under the assumption that $\min_{q, q'=1, \dots, Q; q \neq q'} |v_q - v_{q'}| = |\frac{m_q - m_{q'} + \delta_q - \delta_{q'}}{N}| \sim \mathcal{O}(1)$ [45]. As

a result, the second and the third terms in (30) are $\mathcal{O}(N^{-1})$, leading to

$$\hat{\beta}_2 = \frac{\beta_2}{N} \left(\frac{1 - e^{j2\pi(\delta_2 - \hat{\delta}_2)}}{1 - e^{j\frac{2\pi}{N}(\delta_2 - \hat{\delta}_2)}} - \sum_{n \in \bar{S}} e^{j\frac{2\pi}{N}n(\delta_2 - \hat{\delta}_2)} \right) + \frac{W_2}{N} + \mathcal{O}(N^{-1}). \quad (31)$$

Substituting (31) in (29) yields

$$\check{V}_{\pm 0.5} = \beta_2 \check{\Xi}_{\pm} - \check{\eta}_{\pm} W_2 + \mathcal{O}(N^{-1}), \quad (32)$$

where

$$\check{\Xi}_{\pm} = \frac{1 + e^{j2\pi(\delta_2 - \hat{\delta}_1)}}{1 - \alpha^{\mp 1} e^{j\frac{2\pi}{N}(M_{2,1} + \delta_2 - \hat{\delta}_1)}} - \sum_{n \in \bar{S}} \alpha^{\mp n} e^{j\frac{2\pi}{N}n(M_{2,1} + \delta_2 - \hat{\delta}_1)} - \check{\eta}_{\pm} \left(\frac{1 - e^{j2\pi(\delta_2 - \hat{\delta}_2)}}{1 - e^{j\frac{2\pi}{N}(\delta_2 - \hat{\delta}_2)}} - \sum_{n \in \bar{S}} e^{j\frac{2\pi}{N}n(\delta_2 - \hat{\delta}_2)} \right) \quad (33)$$

and

$$\check{\eta}_{\pm} = \frac{1}{N} \left(\frac{1 + e^{j2\pi(\hat{\delta}_2 - \hat{\delta}_1)}}{1 - \alpha^{\mp 1} e^{j\frac{2\pi}{N}(M_{2,1} + \hat{\delta}_2 - \hat{\delta}_1)}} - \sum_{n \in \bar{S}} \alpha^{\mp n} e^{j\frac{2\pi}{N}n(M_{2,1} + \hat{\delta}_2 - \hat{\delta}_1)} \right). \quad (34)$$

We are now in a position to evaluate the interpolation function corresponding to the first source.

To this end, we have

$$\begin{aligned} \hat{h}_1 &= \frac{\hat{Y}_{1,+0.5} + \hat{Y}_{1,-0.5}}{\hat{Y}_{1,+0.5} - \hat{Y}_{1,-0.5}} \\ &= \frac{h_1 + \frac{1}{Y_{1,+0.5} - Y_{1,-0.5}} (-\mathcal{E}_{1,+0.5} + \check{V}_{+0.5} + W_{1,+0.5} - \mathcal{E}_{1,-0.5} + \check{V}_{-0.5} + W_{1,-0.5})}{1 + \frac{1}{Y_{1,+0.5} - Y_{1,-0.5}} (-\mathcal{E}_{1,+0.5} + \check{V}_{+0.5} + W_{1,+0.5} + \mathcal{E}_{1,-0.5} - \check{V}_{-0.5} - W_{1,-0.5})}, \end{aligned} \quad (35)$$

where $h_1 = \frac{Y_{1,+0.5} + Y_{1,-0.5}}{Y_{1,+0.5} - Y_{1,-0.5}} = \frac{2 - (\alpha^{-1} + \alpha)z_1}{(\alpha^{-1} - \alpha)z_1} \sim \mathcal{O}(N^{-1})$. Cross-multiplying and simplifying, we obtain

$$\hat{h}_1 = h_1 + \check{b}_1 + \frac{(1 - \alpha^{-1}z_1)(1 - \alpha z_1)}{\rho_1 \beta_1 z_1 (\alpha^{-1} - \alpha)} U. \quad (36)$$

where $\check{b}_1$ is given by (21) for $q = 1$ and $U = (1 - \hat{h}_1)(\check{V}_{+0.5} + W_{1,+0.5}) + (1 + \hat{h}_1)(\check{V}_{-0.5} + W_{1,-0.5})$.

Following the bias correction step in line 14 of Algorithm 2, the estimate of h_1 is given by

$$\hat{h}_1 = h_1 + \frac{(1 - \alpha^{-1}z_1)(1 - \alpha z_1)}{\rho_1 \beta_1 z_1 (\alpha^{-1} - \alpha)} U. \quad (37)$$

Also, we have $W_{1,\pm 0.5} \sim \mathcal{O}(1)$ and $W_2 \sim \mathcal{O}(1)$. Since $\check{\Xi}_{\pm} \sim \mathcal{O}(1)$ and $\check{\eta}_{\pm} \sim \mathcal{O}(N^{-1})$, (32) yields $\check{V}_{\pm 0.5} \sim \mathcal{O}(1)$. Further, the term $(Y_{1,+0.5} - Y_{1,-0.5})^{-1} = \frac{(1 - \alpha^{-1}z_1)(1 - \alpha z_1)}{\rho_1 z_1 (\alpha^{-1} - \alpha)} \sim \mathcal{O}(N^{-1})$ and $\mathcal{E}_{1,\pm 0.5} \sim \mathcal{O}(1)$. This implies that $\hat{h}_1 \sim \mathcal{O}(N^{-1})$ from (35). As such, $U \sim \mathcal{O}(1)$.

Referring to (15), the estimation error of z_1^{-1} can be expressed as

$$\begin{aligned} \zeta_1 &= \hat{z}_1^{-1} - z_1^{-1} = -2j(\hat{h}_1 - h_1) \sin\left(\frac{\pi}{N}\right) \\ &= -2j \frac{(1 - \alpha^{-1}z_1)(1 - \alpha z_1)}{\rho_1 \beta_1 z_1 (\alpha^{-1} - \alpha)} U \sin\left(\frac{\pi}{N}\right) \\ &= \frac{(1 - \alpha^{-1}z_1)(1 - \alpha z_1)}{\rho_1 \beta_1 z_1} U. \end{aligned} \quad (38)$$

Expanding $\ln(\hat{z}_1^{-1})$ as a Taylor series around $\ln(z_1^{-1})$ yields

$$\ln(\hat{z}_1^{-1}) = -j \frac{2\pi}{N} \delta_{\prime 1} + z_1 \zeta_1 + \mathcal{O}(z_1^2 \zeta_1^2), \quad (39)$$

with $\delta_{\prime 1} = \delta_1 - \hat{\delta}_1$. Therefore, we have

$$\hat{\delta}_1 = -\frac{N}{2\pi} \Im[\ln(\hat{z}_1^{-1})] = \delta_{\prime 1} - \frac{N}{2\pi} \Im[z_1 \zeta_1 + \mathcal{O}(z_1^2 \zeta_1^2)], \quad (40)$$

where $\Im[\cdot]$ denotes the imaginary part of its complex argument. Then, using (38) and (40), the estimation error of $\delta_{\prime 1}$ can be computed as

$$\begin{aligned} \hat{\delta}_{\prime 1} - \delta_{\prime 1} &= -\frac{N}{2\pi} \Im[z_1 \zeta_1 + \mathcal{O}(z_1^2 \zeta_1^2)] \\ &= \frac{N}{2\pi} \Im \left[\frac{(1 - \alpha^{-1}z_1)(1 - \alpha z_1)}{\rho_1 \beta_1} U + \mathcal{O} \left(\left[\frac{(1 - \alpha^{-1}z_1)(1 - \alpha z_1)}{\rho_1 \beta_1} U \right]^2 \right) \right] \sim \mathcal{O}(N^{-1}), \end{aligned} \quad (41)$$

where we have used the fact that $\frac{(1-\alpha^{-1}z_1)(1-\alpha z_1)}{\rho_1\beta_1} \sim \mathcal{O}(N^{-2})$. Therefore, the fine frequency estimates are asymptotically unbiased. That is,

$$\mathbb{E}[\hat{\delta}_{r_1} - \delta_{r_1}] \sim \mathcal{O}(N^{-1}). \quad (42)$$

4.2. Proposed Modification II

In this case, the “leakage-free” DFT coefficients of the first source are represented by

$$\begin{aligned} \hat{Y}_{1,\pm 0.5} &= Y_{1,\pm 0.5} - \mathcal{E}_{1,\pm 0.5} + \hat{\mathcal{E}}_{1,\pm 0.5} + \check{V}_{\pm 0.5} + W_{1,\pm 0.5} \\ &= \frac{\rho_1\beta_1}{1 - \alpha^{\mp 1}z_1} - \beta_1 \sum_{n \in \bar{S}} \alpha^{\mp n} z_1^n + \hat{\beta}_1 \sum_{n \in \bar{S}} \alpha^{\mp n} + \check{V}_{\pm 0.5} + W_{1,\pm 0.5}. \end{aligned} \quad (43)$$

where $W_{1,\pm 0.5}$ and $\check{V}_{\pm 0.5}$ are given by (28) and (32), respectively. The interpolation function corresponding to the first source is given by

$$\begin{aligned} \hat{h}_1 &= \frac{\hat{Y}_{1,+0.5} + \hat{Y}_{1,-0.5}}{\hat{Y}_{1,+0.5} - \hat{Y}_{1,-0.5}} \\ &= \frac{h_1 + \frac{1}{Y_{1,+0.5} - Y_{1,-0.5}}(-\mathcal{E}_{1,+0.5} + \hat{\mathcal{E}}_{1,+0.5} + \check{V}_{+0.5} + W_{1,+0.5} - \mathcal{E}_{1,-0.5} + \hat{\mathcal{E}}_{1,-0.5} + \check{V}_{-0.5} + W_{1,-0.5})}{1 + \frac{1}{Y_{1,+0.5} - Y_{1,-0.5}}(-\mathcal{E}_{1,+0.5} + \hat{\mathcal{E}}_{1,+0.5} + \check{V}_{+0.5} + W_{1,+0.5} + \mathcal{E}_{1,-0.5} - \hat{\mathcal{E}}_{1,-0.5} - \check{V}_{-0.5} - W_{1,-0.5})}. \end{aligned} \quad (44)$$

Rearranging the terms in (44), we obtain

$$\hat{h}_1 = h_1 + \check{b}_1 + \frac{(1 - \alpha^{-1}z_1)(1 - \alpha z_1)}{\rho_1\beta_1 z_1(\alpha^{-1} - \alpha)} \check{U} \quad (45)$$

where $\check{b}_1$ is given by (23) for $q = 1$ and $\check{U} = (1 - \hat{h}_1)(\check{V}_{+0.5} + W_{1,+0.5}) + (1 + \hat{h}_1)(\check{V}_{-0.5} + W_{1,-0.5})$.

After correcting for the bias in line 15 of Algorithm 3, the estimate of h_1 becomes

$$\hat{h}_1 = h_1 + \frac{(1 - \alpha^{-1}z_1)(1 - \alpha z_1)}{\rho_1\beta_1 z_1(\alpha^{-1} - \alpha)} \check{U}. \quad (46)$$

Following the same arguments as in Section 4.1, we can show from (44) that $\hat{h}_1 \sim \mathcal{O}(N^{-1})$, implying that $\check{U} \sim \mathcal{O}(1)$. As such, for proposed method 2, the estimation error of δ_{i_1} can be computed as

$$\hat{\delta}_{i_1} - \delta_{i_1} = \frac{N}{2\pi} \Im \left[\frac{(1 - \alpha^{-1}z_1)(1 - \alpha z_1)}{\rho_1 \beta_1} \check{U} + \mathcal{O} \left(\left[\frac{(1 - \alpha^{-1}z_1)(1 - \alpha z_1)}{\rho_1 \beta_1} \check{U} \right]^2 \right) \right] \sim \mathcal{O}(N^{-1}), \quad (47)$$

which leads to

$$\mathbb{E}[\hat{\delta}_{i_1} - \delta_{i_1}] \sim \mathcal{O}(N^{-1}). \quad (48)$$

Therefore, the fine frequency estimates for proposed method 2 are also asymptotically unbiased.

5. Simulation Results

In this section, we conduct several experiments to perform a detailed evaluation of the proposed DOA estimation methods. We specifically consider underdetermined and overdetermined cases with both equal and disparate source powers. We determine the DOA estimation performance, in terms of the root-mean-square-error (RMSE), as a function of the signal-to-noise ratio (SNR), the number of snapshots, and source separation. We also evaluate the performance of the proposed methods in the presence of colored noise. For comparison, we include the results obtained with the coarray beamformer applied to the entire coarray and the original FIIB applied to the contiguous part of the coarray. We assume the number of sources to be known a priori. For each example, the coarray signal is synthesized with $T = 500$ snapshots, unless otherwise specified, and 1,000 trials are used to compute the RMSE.

5.1. Multiple Sources With Equal Power

We consider an $(\acute{M}, \acute{N}) = (5, 7)$ coprime array with $M_a = 11$ elements. The corresponding coarray has an extent of $(N - 1)d = 60d$ with $K = 45$ elements and $M = 16$ holes. The contiguous portion of the coarray comprises 23 elements uniformly spaced over an extent of $22d$. We consider

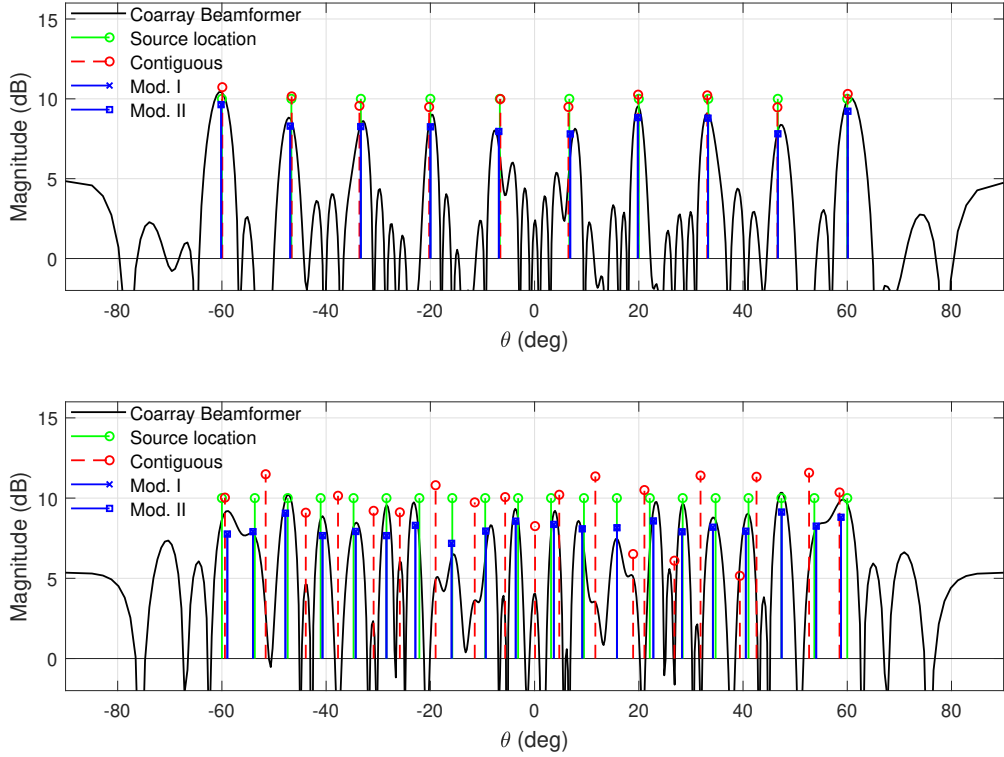


Figure 1: Beamformer outputs for a $(5, 7)$ array with $Q = 10$ (top) and $Q = 20$ (bottom) uniformly spaced sources over $[-60^\circ, 60^\circ]$.

$Q \in \{10, 20\}$ uniformly spaced uncorrelated sources between -60° to 60° , all at 10 dB SNR. The DOA estimates corresponding to proposed FIIB modifications and the original FIIB applied to the contiguous coarray portion are depicted in Fig. 1. The true DOAs and the coarray beamforming result are also included. We observe that both proposed methods are able to successfully estimate all sources. While the original FIIB works well for $Q = 10$ sources, it shows significant bias for some of the sources for $Q = 20$ case. The superior performance of the proposed methods is attributed to the exploitation of all offered DOFs for DOA estimation. Also, the coarray beamformer exhibits bias for some of the sources for both $Q = 10$ and 20; for the latter case, the spatial spectrum has higher sidelobe levels than source responses at some angles other than the true DOAs.

Next, we consider $Q \in \{10, 20\}$ uncorrelated sources impinging on the $(5, 7)$ coprime array with equal power. The SNR is varied from -20 dB to 10 dB in 1 dB increments. In Fig. 2, the RMSE of the DOA estimates from the different algorithms is plotted vs SNR for $Q \in \{10, 20\}$

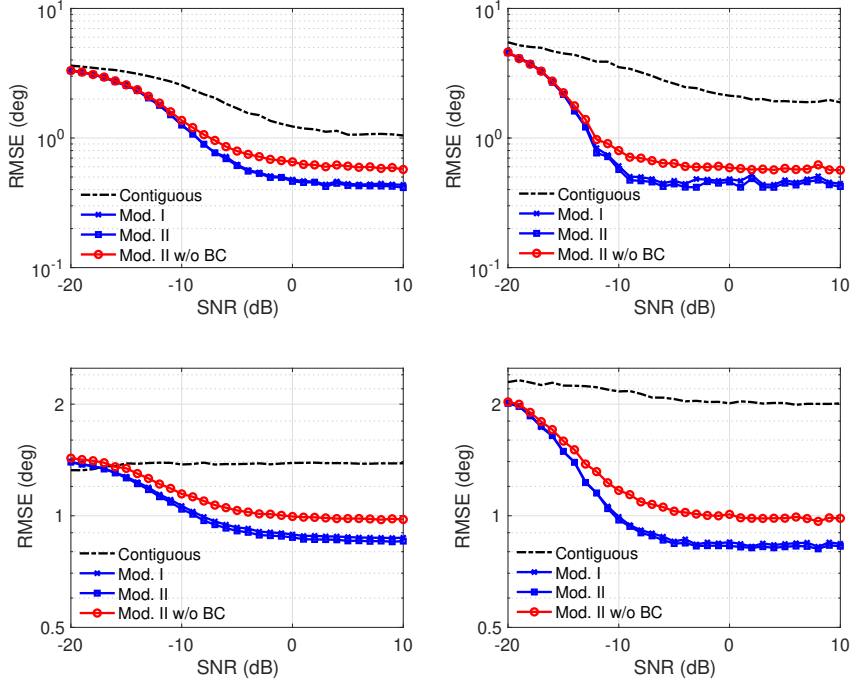


Figure 2: RMSE vs. SNR for $(5, 7)$ array with $Q = 10$ (top) and $Q = 20$ (bottom) uniformly-spaced (left) and randomly chosen (right) sources over $[-60^\circ, 60^\circ]$.

uniformly spaced and randomly chosen source DOAs between -60° to 60° . For random DOAs, no two sources are less than 2.5° apart. We observe that both proposed methods provide similar performance and clearly outperform the original FIIB applied to the contiguous part of the coarray. Further, the RMSE obtained with proposed method II without the bias correction (BC) step is also included in Fig. 2. We note that this version of method II yields a higher RMSE compared to its bias-corrected version, thereby illustrating the value of the bias subtraction step in Algorithm 3.

Next, we repeat the previous example with an $(\hat{M}, \hat{N}) = (10, 11)$ coprime array with 20 elements. The corresponding coarray has an extent of $(N - 1)d = 200d$ with $K = 129$ elements and $M = 72$ holes, while its contiguous part consists of 41 uniformly-spaced elements. The resulting RMSE plots are depicted in Fig. 3. In all plots, we see that proposed method I and both versions of proposed method II provide comparable performance, while the original FIIB applied to the contiguous part of the coarray produces a relatively higher RMSE. The lack of performance difference between FIIB modification II with and without bias subtraction step is attributed to the fact that the bias term of (23) is $\mathcal{O}(N^{-1})$, which is negligible for the $(10, 11)$ coprime array relative to the $(5, 7)$

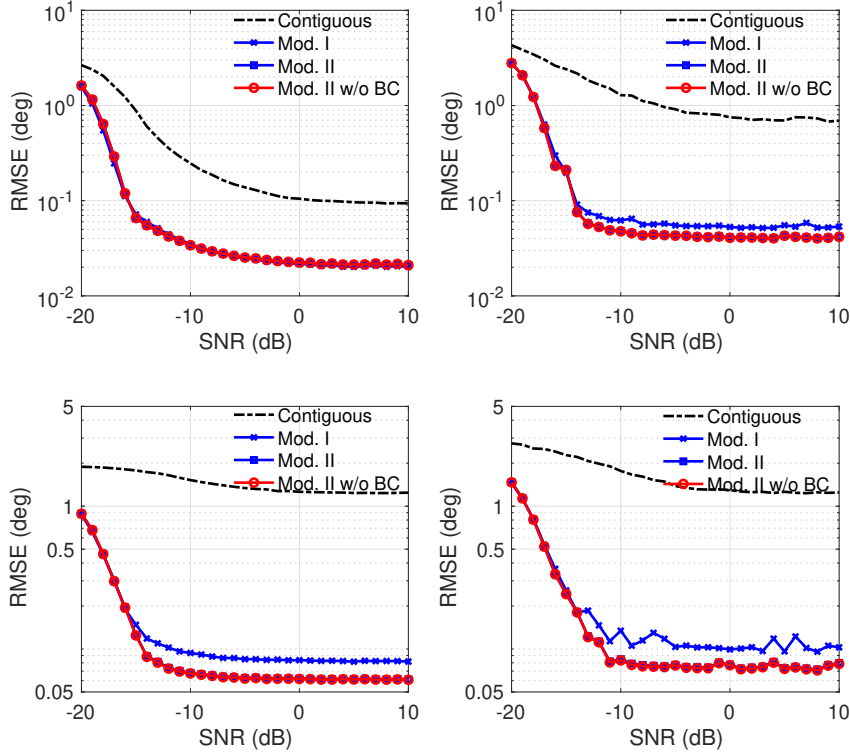


Figure 3: RMSE vs. SNR for (10, 11) array with $Q = 10$ (top) and $Q = 20$ (bottom) uniformly spaced (left) and randomly chosen (right) sources over $[-60^\circ, 60^\circ]$.

array.

5.2. Multiple Sources of Unequal Strength

We also investigate the performance of the proposed methods when sources have varying powers. We consider $Q = 10$ uncorrelated sources, five of these with 20 dB power each and the remaining sources with powers randomly chosen between 12 and 20 dB. The noise variance is set to 0 dB. The DOA estimates corresponding to the proposed FIIB modifications and the original FIIB applied to the contiguous coarray portion are depicted in Fig. 4 for uniformly spaced and randomly spaced sources between -60° to 60° . The true DOAs and the coarray beamforming result are also included therein. We observe that both proposed methods are able to successfully estimate all sources and outperform the other two methods. The original FIIB accurately estimates the uniformly spaced sources, but fails to accurately localize all sources in case of randomly spaced DOAs. The coarray beamformer, on the other hand, provides biased DOA estimates for some of the

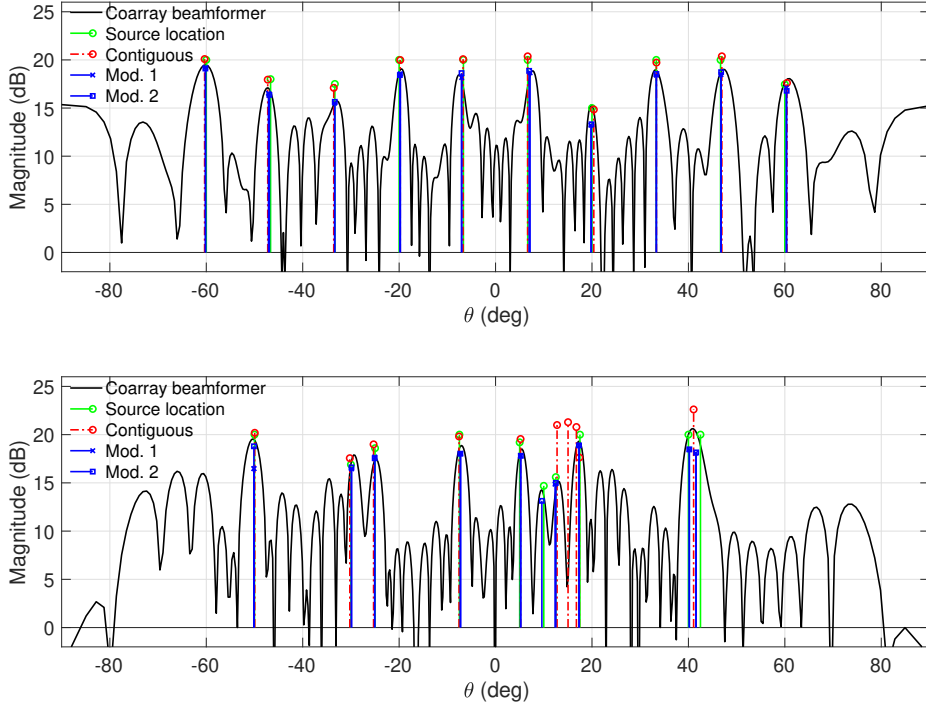


Figure 4: Beamformer outputs for a (5,7) array with $Q = 10$ uniformly spaced (top) and randomly spaced (bottom) sources of unequal powers over $[-60^\circ, 60^\circ]$.

sources under both uniformly and randomly spaced directions; for the latter case, the spatial spectrum also fails to resolve some sources and exhibits higher sidelobe levels than source responses at certain angles other than the true DOAs.

Next, we consider 20 uncorrelated sources of equal power at 20 dB initially. Then, we randomly choose a subset, consisting of a fraction of these sources, to have decreased power. The source powers associated with this subset are assumed to independently follow a uniform distribution on the [8, 20] dB interval. The noise power is kept at 0 dB. It is observed that the RMSE is independent of the fraction of sources with different powers. Table 1 shows the RMSE of the (5, 7) and (10, 11) coprime arrays for uniformly spaced and randomly spaced DOAs over $[-60^\circ, 60^\circ]$, with 2.5° minimum spacing for the latter case.

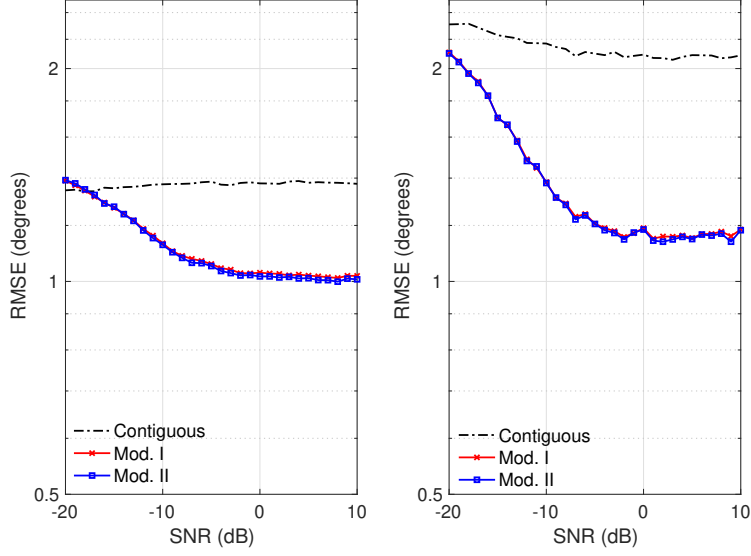


Figure 5: RMSE vs SNR for $Q = 20$ uniformly spaced (left) and randomly spaced (right) sources using 50 snapshots with a (5, 7) array.

5.3. Number of Snapshots

In this example, we show the RMSE vs SNR performance for a limited number of snapshots. More specifically, we consider the same simulation parameters as in Fig. 2, but with only $T = 50$ snapshots and $Q = 20$ sources. The corresponding results are plotted in Fig. 5 for both uniformly-spaced and randomly chosen source DOAs. Comparing with Fig. 2, we observe that the DOA estimation performance is better with a higher number of snapshots. This is because the accuracy of the estimated sample covariance matrix, and hence the coarray measurements, increases with increasing value of T . In particular, the off-diagonal terms in the sample covariance matrix may persist after averaging for small T and may no longer be negligible.

5.4. Two Sources With Varying Separation and Powers

We now focus on $Q = 2$ uncorrelated sources and first evaluate the RMSE as a function of source separation. In each trial, the DOA of source 1 is chosen randomly between -60° to 60° . For source 2, we set $\theta_2 = \theta_1 + \Delta_\theta$, where the source separation Δ_θ is varied from 2° to 40° in 2° increments. The SNR is set to 20 dB. The resulting RMSE vs Δ_θ is depicted in Fig. 6 for the (5, 7)

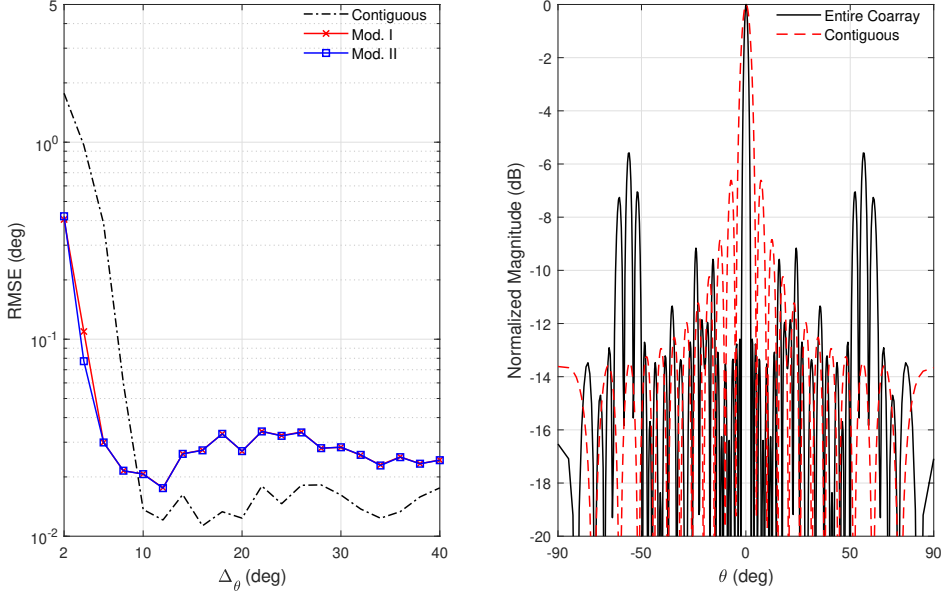


Figure 6: Two sources with equal power. RMSE vs source separation (left) and coarray beampattern (right) for the (5, 7) coprime array.

array. We observe that by exploiting all available DOFs, the proposed method achieves significantly higher estimation accuracy for $\Delta_\theta \leq 10^\circ$ than the original FIIB applied to the contiguous portion of the coarray. However, for $\Delta_\theta > 12^\circ$, the RMSE of the proposed methods is slightly higher than that of the original FIIB applied to the contiguous portion. This is attributed to the fact that the DFT of the response vector corresponding to the non-uniform coarray exhibits an irregular pattern with higher sidelobe levels outside $[-13^\circ, 13^\circ]$ compared to the response vector of its contiguous portion, as seen in Fig. 6, causing a higher degree of interference and hence a higher RMSE, for larger source separations.

Next, for $Q = 2$ sources, we fix $\Delta_\theta = 5^\circ$ and start with both sources having equal power of 20 dB, but then one source is incrementally decreased by 1 dB to 10 dB. The noise variance is kept at 0 dB. We plot the RMSE of the DOA estimates for each SNR difference in Fig. 7 for the (5, 7) array. We observe that the proposed methods provide improved performance over the original FIIB applied to the contiguous coarray portion at smaller power differences, but their performance degrades at larger differences. The degradation occurs at earlier power difference for modification I as compared to modification II. The high sidelobe levels in the beampattern of the non-uniform

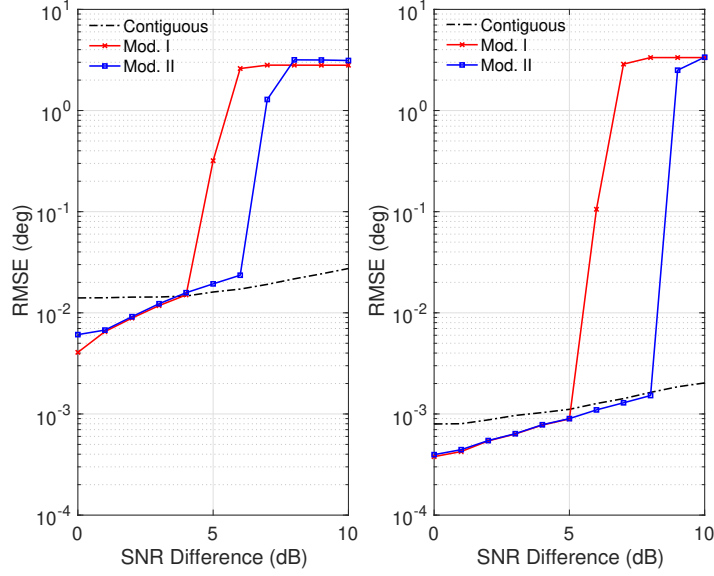


Figure 7: RMSE vs Difference in SNR for $Q = 2$ sources of unequal power with 5° separation using the (5, 7) coprime array (left) and (5, 7) coprime array extended to include 3 periods of the spatial sampling pattern (right).

coprime array are again responsible for this performance degradation, with modification II faring better owing to the "addition" of the hole contributions. We extend the coprime array by including three periods of the spatial sampling pattern to yield an array with 33 elements. Such an extension has been shown to reduce the peak sidelobe level [8]. The RMSE plot corresponding to the extended coprime array is also shown in Fig. 7. Clearly, the reduction of the sidelobe levels enables improved DOA estimation performance with the proposed methods over larger power differences. We note that the FIIB applied to the contiguous portion of the coarray does not encounter this issue over the considered range of power difference because of its uniform configuration.

5.5. Different Coprime Arrays

In this example, we evaluate the performance of the proposed methods for different coprime arrays listed in Table 2. We consider a scene with two uncorrelated sources at a separation of 5° , where the DOA of the first source is randomly selected from $[-60^\circ, 60^\circ]$, independently in each trial. Both sources have equal power with $\text{SNR} = 20$ dB. Fig. 8 compares the RMSE of the proposed methods to that of the FIIB applied to the contiguous portion of the coarray. Note that

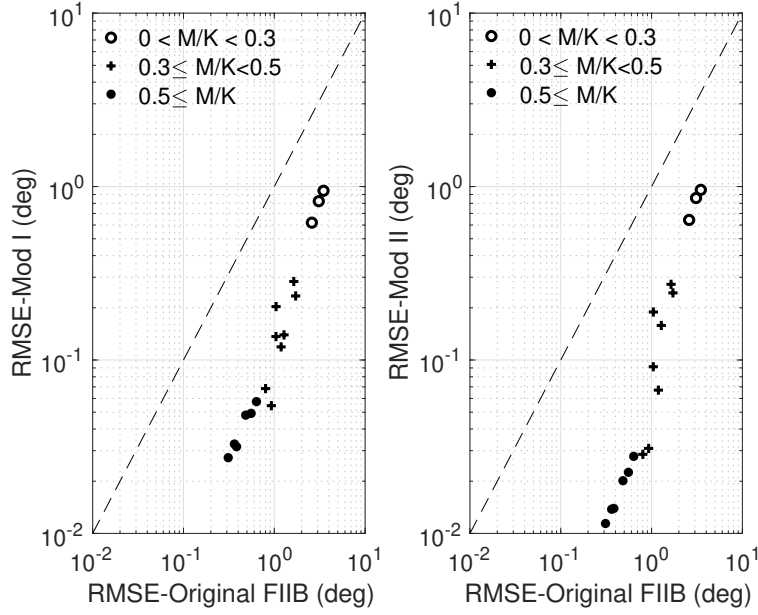


Figure 8: RMSE of Modification I vs RMSE of original FIIB (left); RMSE of Modification II vs RMSE of original FIIB (right). The original FIIB is applied to the contiguous portion of the coarray.

the dashed diagonal line in each plot corresponds to the case when the methods being compared have equal RMSE. All points that lie below (above) this line indicate arrays for which the proposed methods outperform (underperform) the FIIB applied to the contiguous portion. We observe that while the RMSE for all methods decreases with increasing M/K values, the proposed methods outperform the FIIB applied to the contiguous coarray portion for all considered coprime arrays. For array configurations whose coarrays have M/K values exceeding 0.37, proposed modification II provides a lower RMSE than proposed modification I. This behavior is attributed to the fact that the peak sidelobe level and total sidelobe area (normalized by the mainlobe area) of the coprime arrays increases with increasing M/K values, as shown in Fig. 9. Beyond a certain point, i.e., total sidelobe area exceeding 4 times that of the contiguous coarray and peak sidelobe level higher than that of the contiguous portion by 1.5 dB approximately, proposed modification II fairs better than modification I due to its hole filling characteristic.

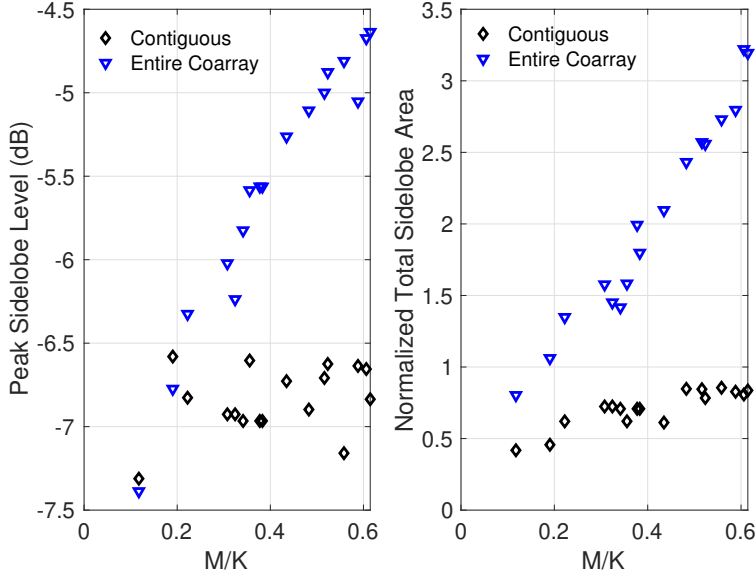


Figure 9: Peak sidelobe level (left) and normalized total sidelobe area (right), as a function of M/K .

5.6. Colored Noise

The original FIIB applied to the contiguous portion of the coarray has been shown to be effective against colored noise [39]. Although the presented model in Section 2 assumes signals corrupted by white noise, we examine in this example the performance of the proposed modifications under colored noise. We consider $Q = 2$ uncorrelated sources of equal power corrupted by pink Gaussian noise [46], with the DOA of the first source randomly chosen from $[-60^\circ, 60^\circ]$ and the other source at a 5° separation from the first source. The DOA estimates, obtained with the different FIIB approaches and the coarray beamformer, at 10 dB SNR using the (5, 7) array are provided in Fig. 10. The coarray beamformer provides biased estimates, whereas the FIIB-based methods accurately localize both targets. Next, we vary the SNR from -20 dB to 10 dB in 1 dB increments. The DOA of the first source is independently selected from $[-60^\circ, 60^\circ]$ in each trial and the source separation is fixed at 5° . Both the (5, 7) and (10, 11) arrays are employed for DOA estimation and the corresponding RMSE vs. SNR plots are shown in Fig. 11. We see that the proposed modifications are effective against colored noise. Moreover, the proposed methods have a lower breakdown threshold than the original FIIB applied to the contiguous coarray portion.

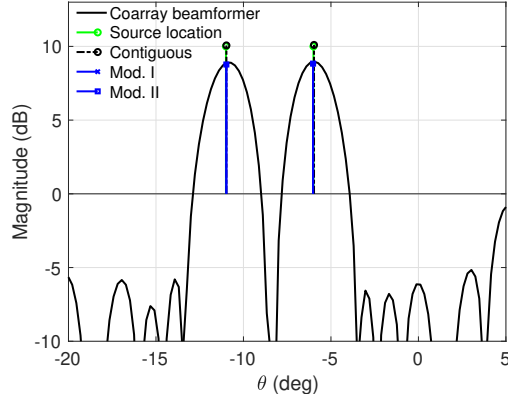


Figure 10: DOA estimates for a single realization with the (5, 7) array.

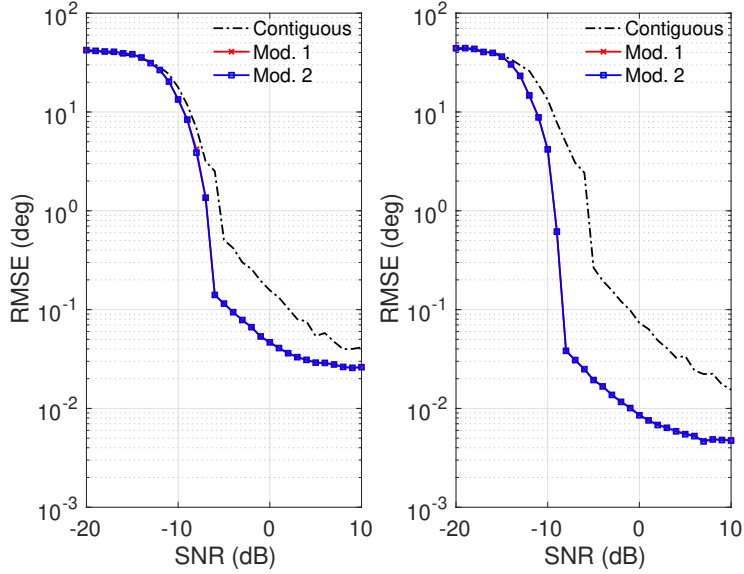


Figure 11: RMSE vs. SNR in pink Gaussian noise for (5, 7) array (left) and (10, 11) array (right).

6. Conclusion

In this paper, we extended the FIIB algorithm to estimate source directions with coprime arrays using all offered degrees of freedom. We presented two different modifications, which approach the non-uniformity of the difference coarray corresponding to a coprime array in a fundamentally different manner. The first method provides a simple strategy to remove the bias induced in the interpolation function by the presence of the holes during the refinement step of the FIIB. The second method estimates the contributions of the holes and adds these to the coarray measurements

to emulate a uniform coarray. Extensive numerical studies were conducted which demonstrated that the proposed methods consistently outperform existing Fourier-based techniques with respect to the RMSE metric, thereby permitting successful exploitation of all available degrees-of-freedom for DOA estimation using coprime arrays.

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Algorithm 3: Modified FIIB II.

Input : The received data vector $\check{\mathbf{y}}$

- 1 Put $\hat{v}_q = 0$ and $\hat{\beta}_q = 0$ for $q = 1, \dots, Q$, and $\hat{\mathcal{E}} = 0$
- 2 Let $\check{\mathbf{Y}} = \text{FFT}(\check{\mathbf{y}}, N)$.
- 3 Set $l = 0$
- 4 **for** L iterations or convergence **do**
- 5 $l = l + 1$
- 6 **for** $q = 1 : Q$ **do**
- 7 $\check{Y}[n] = \check{Y}[n] + \hat{\mathcal{E}}[n]$
- 8 **if** $l = 1$ **then**
- 9 $\hat{Y}[n] = \check{Y}[n] - \sum_{\substack{i=1 \\ i \neq q}}^Q \hat{\beta}_i A_i[n], n = 0, 1, \dots, N - 1$
- 10 $\hat{v}_q = \frac{1}{N} \max_{0 \leq n \leq N-1} |\hat{Y}[n]|^2$
- 11 **end**
- 12 $\hat{Y}_{\pm p}(\hat{v}_q) = \check{Y}_{\pm p}(\hat{v}_q) - \sum_{\substack{i=1 \\ i \neq q}}^Q \hat{\beta}_i A_i(\hat{v}_q \pm p), p = 0.5$
- 13 where $\hat{Y}_{\pm p}(\hat{v}_q) := \hat{Y}(\hat{v}_q \pm p), \check{Y}_{\pm p}(\hat{v}_q) := \check{Y}(\hat{v}_q \pm p)$
- 14 $\check{h} = \frac{\hat{Y}_{0.5} + \hat{Y}_{-0.5}}{\hat{Y}_{0.5} - \hat{Y}_{-0.5}}$
- 15 $h = \check{h} - \check{b}$
- 16 $\hat{z} = \frac{1}{\cos\left(\frac{\pi}{N}\right) - jh \sin\left(\frac{\pi}{N}\right)}$
- 17 $\hat{\delta}_q = N \frac{\angle \hat{z}}{2\pi}$
- 18 $\hat{v}_q \leftarrow \hat{v}_q + \frac{\hat{\delta}_q}{N}$
- 19 $\hat{\beta}_q = \frac{1}{N} \left\{ \sum_{k=0}^{N-1} \check{y}[k] e^{-j \frac{2\pi}{N} k \hat{v}_q} - \sum_{\substack{i=1 \\ i \neq q}}^Q \hat{\beta}_i A_i(\hat{v}_q) \right\}$
- 20 where $\check{y}[k] := \check{y}[k] + \hat{\mathcal{E}}[k], \hat{\mathcal{E}}[k] = \begin{cases} \sum_{q=1}^Q \hat{\beta}_q e^{j2\pi \hat{v}_q k} & k \in \bar{S} \\ 0 & k \in S \end{cases}$
- 21 $\hat{\mathcal{E}}[n] = \sum_{q=1}^Q \sum_{k \in \bar{S}} \hat{\beta}_q e^{j2\pi(\hat{v}_q - \frac{n}{N})k}$
- 22 **end**
- 23 **end**

Output: $[\hat{\beta}_q, \hat{v}_q], q = 1, 2, \dots, Q$

	Uniformly spaced		Randomly spaced	
	(5, 7) array	(10, 11) array	(5, 7) array	(10, 11) array
Contiguous	1.9464°	1.2406°	2.3040°	1.2450°
Mod. I	0.6443°	0.0612°	0.7072°	0.0732°
Mod. II	0.6508°	0.0616°	0.7219°	0.0740°

Table 1: RMSE with fraction of sources having unequal power.

coprime pair ($\acute{M}$, $\acute{N}$)	Array elements M_a	Coarray elements K	Holes M	Ratio M/K
(3, 4)	6	17	2	0.1176
(3, 5)	7	21	4	0.1905
(4, 5)	8	27	6	0.2222
(5, 6)	10	39	12	0.3077
(4, 7)	10	37	12	0.3243
(3, 10)	12	41	14	0.3415
(5, 7)	11	45	16	0.3556
(6, 7)	12	53	20	0.3774
(4, 9)	12	47	18	0.3830
(7, 8)	14	69	30	0.4348
(8, 9)	16	87	42	0.4828
(7, 11)	17	93	48	0.5161
(9, 10)	18	107	56	0.5234
(10, 11)	20	129	72	0.5581
(11, 12)	22	153	90	0.5882
(11, 13)	23	165	100	0.6061
(12, 13)	24	179	110	0.6145

Table 2: Arrays for various coprime pairs.

3.3 DOA Estimation of a Spatially-Spread Source With a Perturbed Vertical Co-prime Array in Correlated Noise

Abstract

In this paper, we consider the problem of estimating the direction-of-arrival (DOA) of a spatially-spread source in the presence of unknown correlated noise using a vertical co-prime array. The angular spread of the source is described in terms of a Gaussian probability density function centered around its nominal DOA. We propose to determine the source DOA and angular spread by solving a non-convex optimization problem using Covariance Matrix Adaptation Evolution Strategy. Further, to address the susceptibility of the vertical array to structural deflections, we extend the proposed method to accurately perform source parameter estimation in the presence of sensor perturbations and unknown correlated noise. Supporting simulation results are provided to demonstrate the effectiveness of the proposed method.

I. INTRODUCTION

Underwater sensor networks (UWSNs) are gaining impetus as effective means for ocean surveillance, monitoring, and exploration [1]–[3]. A UWSN comprises multiple sensor nodes which are deployed by marine platforms in a designated area for data collection. Owing to the advances in underwater vehicle technology over the last decade, sensor deployment can be accomplished in an efficient manner via remotely operated vehicles (ROVs) or autonomous underwater vehicles (AUVs). However, such platforms impose additional design constraints on a sensor node to not only prevent restrictions on the vehicle itself, but also avoid any hinderance to its maneuverability [4]. As such, design of compact deployable sensor nodes is critical to establishing a UWSN with autonomous vehicles. CA To this end, an underwater inflatable passive co-prime sonar array was recently proposed as a sensor node for a UWSN to detect acoustic sources within the water volume under surveillance [5], [6]. A co-prime array (CA) comprises two undersampled uniform linear subarrays with co-prime spatial sampling rates, which are interleaved to yield a nonuniform configuration [7], [8]. A CA offers the following desirable sensing characteristics: i) The extent of the aperture spanned by a CA exceeds that of a uniform linear array (ULA) with the same number of elements spaced one-half wavelength apart, resulting in a higher spatial resolution; ii) Unlike the ULA, a CA provides the ability to estimate more acoustic sources than sensors using subspace techniques, such as MUSIC [9] and ESPRIT [10]. The initial physical volume and weight of the inflatable CA is reduced via a compact design to make it convenient to store, handle, and transport, particularly for the autonomous vehicles with limited payload space. More specifically, the co-prime configuration of

hydrophones are affixed to an inflatable support structure, which can be folded and compressed. Once the stowed package is detached from the platform, the inflatable structure morphs into its final geometry, driven by a buoy on the top and an anchor at the bottom. The buoyancy and gravity straighten the array structure and maintain the hydrophones at the desired spacings. While the inflatable structure allows a compact stowed package dimension, it renders the array relatively more susceptible to structural deflection arising from ocean currents or surface waves compared to semi-stiff cables used in traditional towed arrays or moored vertical arrays [11], [12]. Irrespective of the type of support structure, the structural deflections, if unaccounted for, may adversely impact subsequent signal processing of acquired data for source estimation.

In addition to structural deflection, ambient noise brings complexity to source estimation in an underwater environment [13], [14]. More specifically, the ocean ambient noise exhibits spatial correlations [15]. As such, the typical assumption of spatially white noise is not valid generally in array processing and particularly in CA processing. There exists an abundance of signal processing algorithms for effective source estimation with CAs in the presence of white noise [7], [8], [16]–[35]. However, only a limited number of works have focused on CA processing in the presence of spatially correlated noise. More specifically, Ref. [36] examined the performance of conventional beamforming and product processors applied to the CA in terms of their detection gain and robustness against spatially correlated Gaussian noise. Spatial power spectral density estimation performance of a product processor applied to the CA in the presence of spatially correlated noise was analyzed for finite array apertures in [37] and in the limit of a large aperture in [38]. Ref. [39] examined the variance of the spatial spectral density estimate of a min processor applied to the CA under non-white Gaussian noise. Changes in CA gain under anisotropic correlated noise in shallow water environments were investigated in [40]. In [41], an iterative Fourier method was applied in the coarray domain for DOA estimation with a CA and its performance was examined in the presence of heavy-tailed colored noise. Since this iterative technique is applicable only to uniformly spaced configurations, it was applied to the uniform segment of the non-uniform coarray of a CA. All aforementioned works analyzed the performance of existing techniques and did not attempt to devise new methods to specifically address the presence of spatially correlated noise.

Multiple scattering, among other phenomena, also introduces further challenges to source DOA estimation. The signal of interest can experience multiple scatterings from schools of fish, clouds of air bubbles, and other foreign bodies in the water as well as the ocean boundaries [13], [14], [42]. Multiple scattering

in the vicinity of a point source can result in spatial spreading. Methods to address DOA estimation of spatially-spread sources have primarily focused on ULAs [43]–[46]. The transmission loss introduced by multiple reflections from the ocean boundaries was recently modeled for shallow water environments in [40] and its effects on sidelobe level, array gain, and resolution of the CA were examined therein.

Motivated by the aforementioned challenges facing underwater sensing with passive arrays in general and those with inflatable support structures in particular, in this paper, we consider direction-of-arrival (DOA) estimation of a spatially-spread source with vertical CAs in the presence of unknown spatially correlated noise under structural deflection. More specifically, we assume a large number of independent scatterers to be distributed in the vicinity of the point source of interest, resulting in a spatially-spread source. The angular spread of the source is described in terms of a Gaussian probability density function (PDF) centered around its nominal DOA. We first develop a parametric signal model under correlated noise, considering a spatially-spread source in the absence of structural deflection. The noise is modeled as a first-order autoregressive (AR) process [36], [47]. The signal and noise parameter estimation is posed as a Frobenius norm minimization problem, which is then solved using a global optimization scheme called covariance matrix adaptation evolution strategy (CMA-ES) [48], [49]. Next, the signal model is extended to consider the impact of structural deflection of the CA. Using Jacobi-Anger approximation of manifold separation [50], the array steering vector is decomposed into a matrix-vector product, where the matrix is a function of the array parameters and the vector captures the source parameters. The sensor perturbations along with the other unknown source and noise parameters are iteratively estimated using CMA-ES. Although we focus on a point source with local scattering, the signal model and the proposed algorithms are also applicable to sources with appreciable spatial extent; the latter can be assumed to be made up of infinitesimal elements with statistically independent amplitudes [43]. We note that several self-calibration approaches have been proposed in the literature to deal with location errors in linear arrays [51]–[55]. For CAs in particular, the impact of array bowing on the performance of the product processor was analyzed in [56]. The impact of sensor perturbations on the performance of coarray MUSIC in the presence of white noise was analyzed in [57] for non-uniform linear arrays, including CAs. A self-calibration approach for DOA estimation using non-uniform arrays under white noise was proposed in [28] which exploits the redundancies in the coarray to eliminate nuisance perturbation variables. However, all of these methods deal with point sources and do not take spatially-spread sources into consideration.

Numerical simulations are conducted to assess the performance of the proposed method in estimating a

spatially-spread source under both perturbed and unperturbed vertical CA. For comparison, we also apply existing beamforming approaches to the same measurements with the CA as the proposed method. The results show that the proposed method provides superior performance, especially under unfavorable conditions, such as high noise correlation, low signal-to-noise ratio (SNR), and limited number of snapshots. As a benchmark, the proposed method is also applied to a ULA containing the same number of sensors as the CA. Although the ULA has a smaller aperture and thus a lower angular resolution, having the same number of sensors is deemed to be an appropriate comparison. This is because we are simulating a single spatially-spread source and, more importantly, having the same number of sensors in both the CA and the ULA will ensure same dimensions for the array covariance matrix which will be employed for determining the unknown model parameters.

The remainder of the paper is organized as follows. Section II presents the signal model and the proposed approach for DOA estimation of a source with independent local scatterers under unknown spatially correlated noise. Supporting simulation results are also provided therein. In Section III, the signal model and the proposed method are extended to take into account array structural deflection. Section IV concludes the paper.

II. DOA ESTIMATION IN SPATIALLY CORRELATED NOISE UNDER LOCAL SCATTERING

In this section, we first describe the signal model for a CA, considering a far-field narrowband source with independent local scatterers in proximity of the source and unknown spatially correlated noise. Then, we present the proposed method for estimating the model parameters. We also briefly describe two beamforming techniques which we employ for DOA estimation performance comparison with the proposed approach. We assume perfect knowledge of array element positions in this section. The case of array perturbations is addressed in the following section.

A. Signal Model

A CA comprises two undersampled uniform linear subarrays with M and N elements, where M and N are co-prime integers [7]. The sensor locations of the M -element subarray are given by $S_M = \{Mnd, 0 \leq n \leq M - 1\}$, while those of the N -element subarray are denoted by $S_N = \{Nmd, 0 \leq m \leq N - 1\}$, where d is equal to one-half wavelength at the operating frequency. Interleaving the two subarrays, with a shared first element, yields the CA with $K = M + N - 1$ elements. We denote the element positions of

the CA by $\mathbf{k}d = [k_0 \cdots k_{K-1}]^\top d$, where $k_i d \in (S_M \cup S_N)$ for $i = 0, 1, \dots, K-1$, are sorted in ascending order with $k_0 = 0$ and the superscript ‘ $\top$ ’ denotes matrix transpose.

We consider a narrowband point source in the far-field of the CA and P local independent scatterers at the source. Then, the signal measured at time t with the CA can be expressed as

$$\mathbf{x}(t) = \sum_{p=0}^P \alpha_p(t) \mathbf{a}(\theta_p) + \mathbf{n}(t) = \mathbf{s}(t) + \mathbf{n}(t) \in \mathbb{C}^K, \quad (1)$$

where $\alpha_0(t)$ is the source signal, $\alpha_p(t), p = 1, 2, \dots, P$, is the signal from the p th scatterer, θ_p is the p th signal direction measured relative to broadside, $\mathbf{a}(\theta_p) \in \mathbb{C}^K$ is the array steering vector corresponding to the p th signal, $\mathbf{s}(t) \in \mathbb{C}^K$ denotes the vector of signal contributions, and $\mathbf{n}(t) \in \mathbb{C}^K$ is the noise vector. The i th element of the steering vector $\mathbf{a}(\theta_p)$ is given by $[\mathbf{a}(\theta_p)]_i = \exp(j\pi k_i \sin \theta_p)$. Assuming the signals and the noise to be uncorrelated, the array covariance matrix is

$$\mathbf{R}_x \triangleq \mathbb{E}\{\mathbf{x}(t)\mathbf{x}^H(t)\} = \mathbf{R}_s + \mathbf{R}_n, \quad (2)$$

where $\mathbb{E}\{\cdot\}$ is the expectation operator, $\mathbf{R}_s = \mathbb{E}\{\mathbf{s}(t)\mathbf{s}^H(t)\} \in \mathbb{C}^{K \times K}$ is the signal covariance matrix, $\mathbf{R}_n = \mathbb{E}\{\mathbf{n}(t)\mathbf{n}^H(t)\} \in \mathbb{C}^{K \times K}$ is the noise covariance matrix, and the superscript ‘ H ’ denotes conjugate transpose. The (k, l) th element of $\mathbf{R}_s$ is

$$[\mathbf{R}_s]_{k,l} \triangleq \mathbb{E}\{\mathbf{s}_k(t)\mathbf{s}_l^*(t)\} = \sum_{p=0}^P \sum_{h=0}^P \mathbb{E}\{\alpha_p(t)\alpha_h^*(t)\} e^{j\pi(k_k \sin \theta_p - k_l \sin \theta_h)}, \quad (3)$$

where ‘ $*$ ’ denotes complex conjugation. Assuming that all signals have power equal to σ_s^2 , we have

$$\mathbb{E}\{\alpha_p(t)\alpha_h^*(t)\} = \begin{cases} \sigma_s^2 & p = h \\ 0 & \text{otherwise} \end{cases}. \quad (4)$$

As a result, (3) can be rewritten as

$$[\mathbf{R}_s]_{k,l} = \sum_{p=0}^P \sigma_s^2 e^{j\pi(k_k - k_l) \sin \theta_p}. \quad (5)$$

For sufficiently large P , the source with local scattering can be approximated as a spread source with its direction following a Gaussian distribution, i.e., $\theta \sim \mathcal{N}(\theta_0, \sigma_\theta^2)$, where the mean, θ_0 , is the nominal

DOA and σ_θ^2 is the variance of the angular spread [46]. As such, $[\mathbf{R}_s]_{k,l}$ can be approximated as

$$[\mathbf{R}_s]_{k,l} \approx \frac{\sigma_s^2}{\sqrt{2\pi\sigma_\theta^2}} \int_{-\infty}^{\infty} e^{-\frac{(\theta-\theta_0)^2}{2\sigma_\theta^2}} e^{j\pi(k_k-k_l)\sin\theta} d\theta. \quad (6)$$

For small angular spread, we can model the source direction as a perturbation of the nominal DOA, i.e., $\theta = \theta_0 + \delta_\theta$, with δ_θ denoting the deviation from θ_0 . The parameter δ_θ follows a zero-mean Gaussian distribution, i.e., $\delta_\theta \sim \mathcal{N}(0, \sigma_\theta^2)$. As a result, (6) assumes the form

$$\begin{aligned} [\mathbf{R}_s]_{k,l} &= \frac{\sigma_s^2}{\sqrt{2\pi\sigma_\theta^2}} \int_{-\infty}^{\infty} e^{-\frac{\delta_\theta^2}{2\sigma_\theta^2}} e^{j\pi(k_k-k_l)\sin(\theta_0+\delta_\theta)} d\delta_\theta \\ &\approx \sigma_s^2 e^{j\pi(k_k-k_l)\sin\theta_0} \frac{1}{\sqrt{2\pi\sigma_\theta^2}} \int_{-\infty}^{\infty} e^{-\frac{\delta_\theta^2}{2\sigma_\theta^2} + j\pi(k_k-k_l)\delta_\theta \cos\theta_0} d\delta_\theta \\ &= \sigma_s^2 e^{j\pi(k_k-k_l)\sin\theta_0} e^{-0.5(\pi(k_k-k_l))^2 \sigma_\theta^2 \cos^2\theta_0}, \end{aligned} \quad (7)$$

where we have made use of the small angle approximation and the integral formula $\int_{-\infty}^{\infty} e^{-ax^2+bx+c} dx = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}+c}$ to obtain the final expression [58].

Using (7), the signal covariance matrix can be expressed in the form

$$\mathbf{R}_s = \sigma_s^2 \mathbf{a}(\theta_0) \mathbf{a}^H(\theta_0) \odot \mathbf{B}(\theta_0, \sigma_\theta^2), \quad (8)$$

where ‘ $\odot$ ’ denotes the Hadamard product and the (k, l) th element of matrix $\mathbf{B}$ is given by

$$[\mathbf{B}(\theta_0, \sigma_\theta^2)]_{k,l} = e^{-0.5(\pi(k_k-k_l))^2 \sigma_\theta^2 \cos^2\theta_0}. \quad (9)$$

The noise $\mathbf{n}$ in (1) is assumed to be spatially correlated across the sensors. We model the noise as a first-order AR process driven by independent and identically distributed Gaussian noise, $\mathbf{w} \sim \mathcal{CN}(\mathbf{0}, \sigma_w^2 \mathbf{I})$ [36],

$$[\mathbf{n}]_k \triangleq n_k(t) = \sqrt{1-\beta^2} w_k(t) + \beta n_{k-1}(t), \quad (10)$$

where w_k is the k th element of $\mathbf{w}$ and β is the noise spatial correlation between two sensors separated by one-half wavelength. We note that $\beta = 0$ corresponds to the special case of spatially uncorrelated noise. Under the AR model of (10), the (k, l) th element of the noise covariance matrix is given by

$$[\mathbf{R}_n]_{kl} = \sigma_w^2 \beta^{|k_k-k_l|}. \quad (11)$$

Thus, the noise correlation decreases exponentially as the separation between sensor pairs increases.

The developed signal model can be readily extended to the case of multiple uncorrelated spread sources. In this case, $s(t)$ in (1) is a superposition of signals from all spread sources and, consequently, the signal covariance matrix in (8) will be a summation of the individual covariance matrices corresponding to each spread source.

B. Proposed DOA Estimation Method

Let $\boldsymbol{\eta}_s = [\theta_0, \sigma_\theta^2, \sigma_s^2]^\top \in \mathbb{R}^3$ denote the signal parameters to be estimated and $\boldsymbol{\eta}_n = [\sigma_w^2, \beta]^\top \in \mathbb{R}^2$ represent the unknown noise parameters. We propose a joint optimization approach to solve for the unknown parameters. That is, a joint minimization over the source and noise parameters can be posed as

$$\min_{\boldsymbol{\eta}_s, \boldsymbol{\eta}_n} \|\mathbf{R}_x - \mathbf{R}_s(\boldsymbol{\eta}_s) - \mathbf{R}_n(\boldsymbol{\eta}_n)\|_F^2, \quad (12)$$

where $\|\cdot\|_F$ denotes the Frobenius norm of its matrix argument and $\mathbf{R}_s$ and $\mathbf{R}_n$ are given by (8) and (11), respectively. Eq. (12) is a non-convex optimization problem and is difficult to solve directly. However, it is always possible to minimize over some parameters first and then minimize over the remaining ones [59]. Therefore, (12) can equivalently be posed as a nested optimization problem,

$$\min_{\boldsymbol{\eta}_s} \min_{\boldsymbol{\eta}_n} \|\mathbf{R}_x - \mathbf{R}_s(\boldsymbol{\eta}_s) - \mathbf{R}_n(\boldsymbol{\eta}_n)\|_F^2. \quad (13)$$

Both the outer and the inner minimization problems in (13) are non-convex and, in order to solve these problems, we use CMA-ES which is a self-adaptive evolution strategy that requires no parameter tuning. Evolutionary strategies are a class of nature-based global optimization methods in which candidate solutions or offsprings are generated stochastically and the best performing offsprings are selected to generate the next set of candidate solutions. CMA-ES, in particular, samples a new generation of offsprings from a multivariate Gaussian distribution and updates the distribution parameters sequentially using those members of the generation that provide the best performance in terms of their fitness value or score on the outer or inner optimization in (13) [49]. The iterative process is terminated when a specific criterion is met. This criterion can be a maximum number of generations or the target value for the difference in the objective function between two iterations.

We note that, in practice, $\mathbf{R}_x$ in (13) is replaced by the sample covariance matrix, $\hat{\mathbf{R}}_x = \frac{1}{T} \sum_{i=1}^T \mathbf{x}(t_i) \mathbf{x}^\top(t_i)$, with T denoting the number of time snapshots.

C. Beamforming Techniques

In this section, we briefly review two existing beamforming techniques, namely, the product processor and coarray beamforming, which have been proposed for use with CAs [7], [60]. The product processor first applies conventional beamforming to each subarray individually, and then multiplies the beamformed output of one subarray with the complex conjugate of the beamformed output of the other subarray [7]. The output of the product processor is given by

$$y_{pp}(\theta) = \mathbb{E} \{ |B_M(\theta)B_N^*(\theta)| \} = \mathbb{E} \{ |(\mathbf{a}_M^H(\theta)\mathbf{x}_M(t))(\mathbf{a}_N^H(\theta)\mathbf{x}_N(t))^*| \} \quad (14)$$

where $B_g(\cdot)$, $\mathbf{a}_g(\theta)$, and $\mathbf{x}_g(t)$ represent the respective beamformed output, steering vector, and received signal vector for the g -element subarray with $g = M, N$. Note that $\mathbf{a}_g(\theta)$ and $\mathbf{x}_g(t)$ can be readily extracted from $\mathbf{a}(\theta)$ and $\mathbf{x}(t)$, respectively, by using the values associated with each subarray. When θ matches the true DOA, the product processor has a high response. By inspecting the output for all possible $\theta \in [-\pi/2, \pi/2)$ and choosing the angles corresponding to the peaks, we obtain the DOA estimates. In practice, the expectation in (14) is computed as an average over T time snapshots, i.e., $\frac{1}{T} \sum_{i=1}^T |(\mathbf{a}_M^H(\theta)\mathbf{x}_M(t_i))(\mathbf{a}_N^H(\theta)\mathbf{x}_N(t_i))^*|$.

Unlike the product processor, the second method performs beamforming with respect to signal powers instead of signal amplitudes [60]. More specifically, the covariance matrix $\mathbf{R}_x$ in (2) is first vectorized to yield $\mathbf{z} = \text{vec}(\mathbf{R}_x) \in \mathbb{C}^{K^2}$, where $\text{vec}(\cdot)$ returns the column-wise vectorization of its matrix argument. The vector $\mathbf{z}$ has its support on the difference coarray, S_{co} , which is defined as the set of spatial lags realized by the sensor array. For the CA, there can be more than one pair of sensors that contribute to a single spatial lag. As such, either the autocorrelation value for only one of those pairs or an average value may be used for further processing [8], [61], [62]. We denote the resulting autocorrelation-vector by $\mathbf{z}_{co} \in \mathbb{C}^{K_{co}}$, with K_{co} being the cardinality of S_{co} . Thus, the vector $\mathbf{z}_{co}$ emulates measurements at a virtual array whose elements are given by the difference coarray. Denoting the coarray element positions by $\mathbf{m}d = [m_0 \cdots m_{K_{co}-1}]^\top d$, where $m_i d \in S_{co}$ for $i = 0, 1, \dots, K_{co} - 1$, the i th element of the coarray steering vector is given by

$$[\tilde{\mathbf{a}}(\theta)]_i = \exp(j\pi m_i \sin \theta). \quad (15)$$

The coarray beamformer output is obtained as the inner product of $\mathbf{z}$ with the weight vector $\tilde{\mathbf{a}}(\theta)$,

$$y(\theta) = \tilde{\mathbf{a}}^H(\theta)\mathbf{z}_{co}, \quad (16)$$

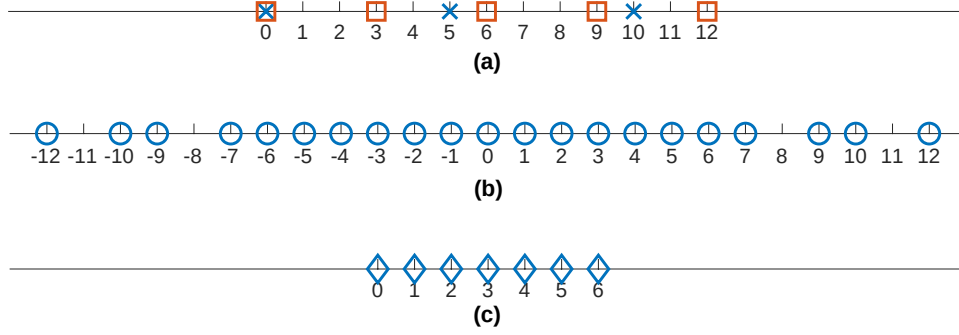


Fig. 1: (a) The 7-element CA, with $M = 3$ and $N = 5$, comprised of a 3-element subarray (indicated by crosses) and a 5-element subarray (indicated by squares). The first element is common to the two subarrays. (b) Corresponding coarray. (c) A 7-element ULA. The inter-element spacings in all subfigures are normalized by d .

and the DOA estimates are determined as the angles corresponding to the peaks of $|y(\theta)|$. Similar to the proposed method, $\mathbf{R}_x$ is replaced by the sample covariance matrix $\hat{\mathbf{R}}_x$ in practice.

D. Simulation Results

For illustration, we consider a 7-element CA with $M = 3$, $N = 5$, and $d = 0.3$ m (one-half wavelength corresponding to 2.5 kHz). The array consists of two subarrays, with $S_M = [0, 5, 10]^\top d$ and $S_N = [0, 3, 6, 9, 12]^\top d$ as the respective sensor positions of the 3-element and 5-element subarrays. The sensor positions of the CA are $\mathbf{k}d = [0, 3, 5, 6, 9, 10, 12]^\top d$. The corresponding coarray has 21 elements and extends from $-12d$ to $12d$. The physical array is shown in Fig. 1(a), where the sensor positions of the 5-element subarray are marked as squares, while those of the 3-element subarray are indicated as crosses. The corresponding coarray is depicted in Fig. 1(b), with its elements marked as circles. We consider a source with direction -30° in close proximity to 50 independent local scatterers. The DOAs of the scatterers are drawn from a Gaussian distribution with a mean of -30° and a standard deviation of 5° . In the CMA-ES algorithm, the offspring and selection sizes are set to 100 and 20, respectively.

First, we set the noise correlation as $\beta = 0.5$ and choose the sensor level input SNR of the source as 0 dB. We vary the number of snapshots from $T = 5$ to $T = 500$. For each value of T , 500 trials are used to perform DOA estimation with the proposed method for the CA. For comparison, the product processor and the coarray beamforming are also applied to the CA measurements. In addition, we provide as a benchmark example the application of the proposed method to a 7-element ULA, shown in Fig. 1(c), which has the same number of sensors as the CA. Fig. 2 compares the performance of the various

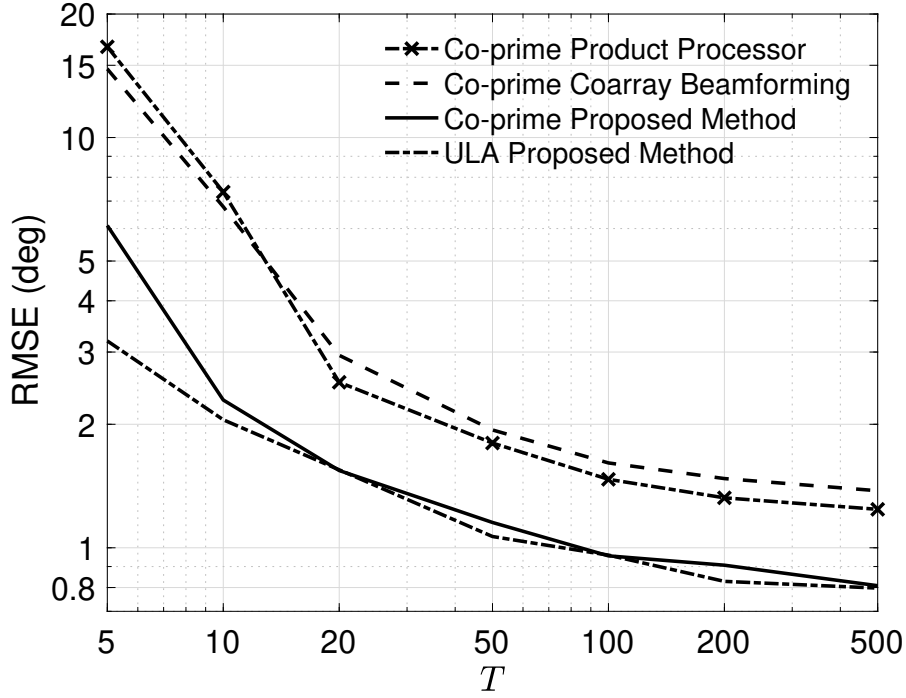


Fig. 2: RMSE of the DOA estimate vs. the number of snapshots T for a spread source at -30° .

methods in terms of the root-mean-squared-error (RMSE) of the DOA estimates for different number of snapshots. We observe that, as expected, the performance of all methods improves with increasing number of snapshots. For the CA, the proposed method consistently outperforms the two beamforming techniques. This is attributed to the fact that the proposed method incorporates the spatial spreading due to local scattering and also specifically accounts for the correlated noise. With regard to the RMSE of the proposed method applied to the CA and the ULA, we observe that, as expected, the RMSE degrades when the number of snapshots is smaller than the number of sensors, the so-called small-sample regime. The ULA and the co-prime results are similar for all values of T except in the small-sample regime, where Fig. 2 reveals the CA to be more negatively impacted than the ULA. To provide more insight into this behavior, we list in Table I the mean and the standard deviation of the noise and remaining source parameter estimates obtained with the proposed method, corresponding to $T = 5, 50$, and 500 . It is evident that the mean values of all estimates are close to the nominal values for $T = 50$ and 500 , and as the number of snapshots increases, the standard deviation of the estimates decreases. For $T = 5$, however, the estimate of β for the CA exhibits a larger bias and a relatively higher standard deviation as compared to that of the ULA. This is because the number of significant elements of $\mathbf{R}_n$ for the ULA is much higher than that for the CA, thereby enabling relatively better estimation of β with the ULA in the challenging small-sample regime. As a result, subsequent estimation of the signal parameters suffers

a higher degradation for the CA over the ULA. Further, the degraded performance of the beamforming techniques for smaller number of snapshots is because of the persistence of the unwanted cross-product terms arising due to multiple local scatterers after averaging [7], [63].

TABLE I: The mean and the standard deviation of the parameter estimates obtained with the proposed method, corresponding to $T = 5, 50, 500$.

	Snapshots (T)	5				50				500			
	Array	CA		ULA		CA		ULA		CA		ULA	
	Nominal	Mean	Std	Mean	Std	Mean	Std	Mean	Std	Mean	Std	Mean	Std
σ_s^2	1	0.97	0.45	1.03	0.47	1.00	0.16	1.00	0.15	1.00	0.06	1.00	0.05
σ_w^2	1	1.07	0.29	0.96	0.28	1.03	0.11	1.01	0.09	1.00	0.05	1.00	0.03
$\sigma_\theta(\text{deg})$	5	3.09	2.82	4.00	3.02	4.53	0.99	4.83	1.07	4.79	0.69	4.85	0.58
β	0.5	0.44	0.23	0.51	0.20	0.48	0.09	0.50	0.06	0.49	0.03	0.50	0.02

Next, we compare the performance of the proposed method and the beamforming techniques as a function of the SNR. The simulation parameters are the same as in the previous example, with the exception that the number of snapshots is fixed at $T = 500$ while the SNR is varied from -10 dB to 10 dB in 2.5 dB increments. Fig. 3 shows the RMSE of the DOA estimate vs. the SNR. Again, 500 trials are used to compute the RMSE at each SNR. We observe that the performance of all methods improves as SNR increases. As expected, at SNR values < -2.5 dB, the two beamforming methods fail to estimate the source direction. For the correlated noise model, the spatial noise power compresses towards broadside. This causes the beamforming methods to incorrectly estimate the DOA as 0° instead of $\theta = -30^\circ$ at low SNR where noise dominates the source signal. The proposed method outperforms the beamforming methods at all SNR values, significantly more so for SNR values < -2.5 dB. Both the ULA and the CA yield similar DOA estimation performance.

Finally, we compare in Fig. 4 the RMSE of the proposed method and the beamforming techniques as a function of β . The SNR is set to 0 dB, and $T = 500$ snapshots are used. The value of β is varied from 0 to 1 in increments of 0.1. All other simulation parameters are the same as in the previous examples. We observe that, for the CA, the proposed method outperforms the beamforming techniques for all β , with significant difference in DOA estimation performance for $\beta > 0.7$. The beamforming techniques fail at high values of β , again due to the noise spatial power concentrating towards broadside. The results for the proposed method applied to the ULA and the CA are similar and essentially remain unchanged with varying β .

A few remarks are in order for the proposed method. First, we observe that the RMSE saturates at slightly under 1° under favorable conditions in the presented examples. This irreducible error stems from

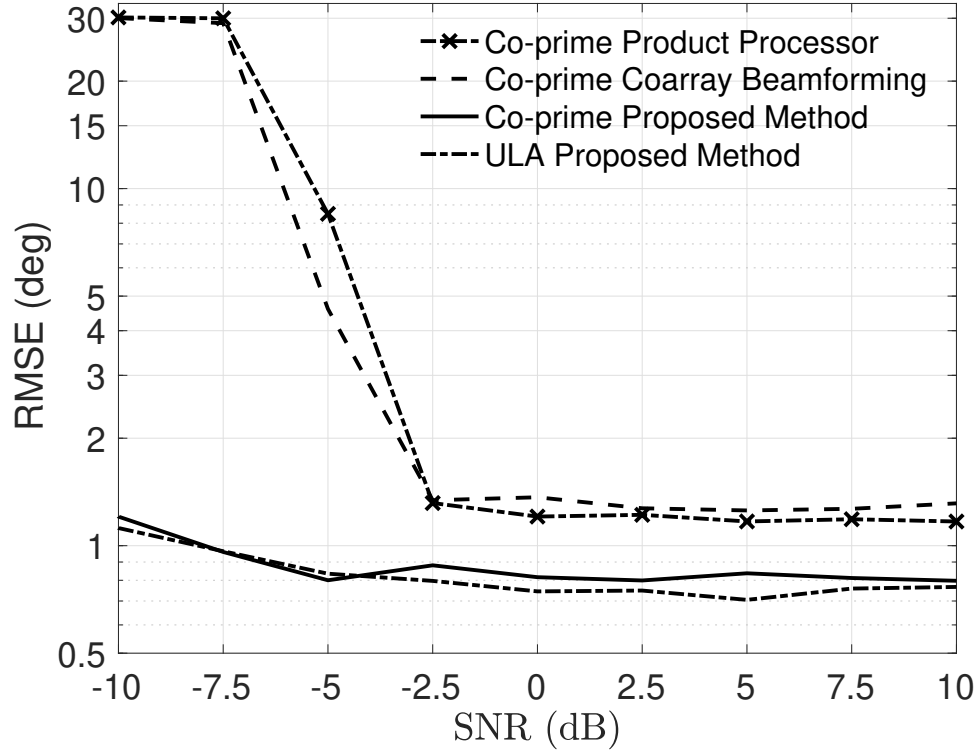


Fig. 3: RMSE of the DOA estimate vs. SNR for a spread source at -30° .

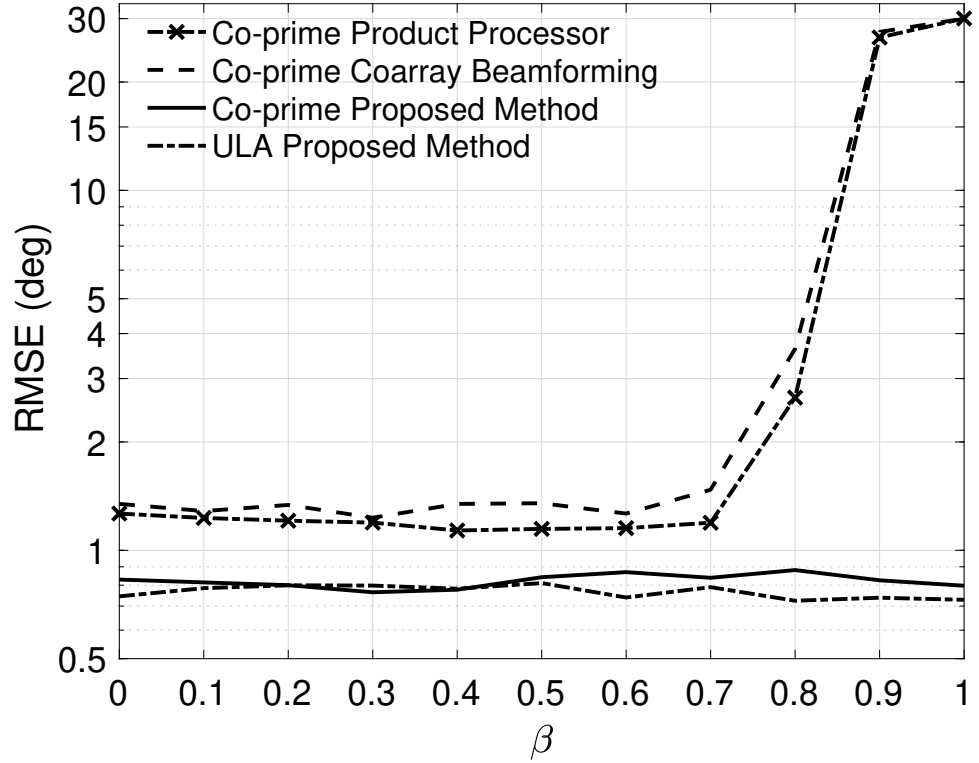


Fig. 4: RMSE of the DOA estimate vs. β for a spread source at -30° .

approximating the continuous angular spread in (6) by a finite number (50 in this simulation) of discrete scatterers. Using a higher number of scatterers would yield a lower RMSE saturation level. Second, the

performance of the proposed method can be further improved in all presented examples by increasing the offspring size or lowering the termination tolerance in the CMA-ES. However, such a gain comes at the cost of longer computation time.

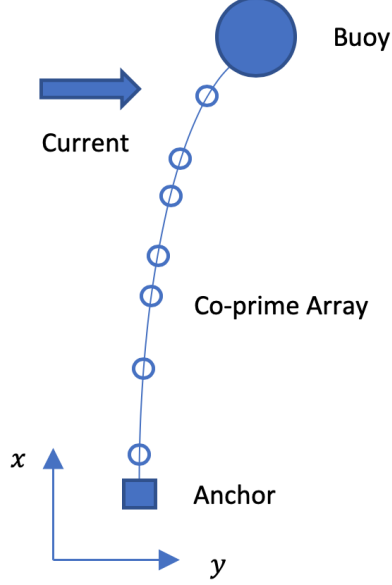


Fig. 5: Structural deflection of the co-prime sonar array under constant current conditions.

III. DOA ESTIMATION IN SPATIALLY CORRELATED NOISE UNDER LOCAL SCATTERING AND SENSOR PERTURBATIONS

In this section, we consider DOA estimation in the presence of array structural deflection arising due to ocean currents. For currents along the y -axis, i.e., direction orthogonal to the array baseline (see Fig. 5), it was shown in [11] for an inflatable CA that i) the array undergoes relatively larger drifts for sensors farther away from the anchor, and ii) the ocean current has limited influence on the sensor spacing along the x -axis. We, therefore, assume the sensor locations to be perturbed along the y -axis, with the locations of the first two sensors closer to the anchor to be error-free. The latter assumption also guarantees parameter identifiability for source estimation under sensor perturbations [64].

A. Signal Model

We denote sensor perturbations as

$$\Delta = [\Delta_0, \Delta_1, \dots, \Delta_{K-1}]^T \in \mathbb{C}^K, \quad (17)$$

where Δ_i is the deflection of the i th sensor along the positive y -axis and $\Delta_0 = \Delta_1 = 0$. The perturbations Δ are assumed to be the same across all T snapshots. The i th element of the steering vector now also depends on Δ , and can be expressed as

$$[\mathbf{a}(\theta, \Delta)]_i = e^{j\pi(k_i \sin \theta + \Delta_i \cos \theta / d)} = e^{j\frac{\pi}{d} r_i \sin(\theta + \phi_i)}, \quad (18)$$

with $r_i = \sqrt{(k_i d)^2 + \Delta_i^2}$ and $\phi_i = \tan^{-1}(\frac{\Delta_i}{k_i d})$. Using the Jacobi-Anger expansion [50], [65], we can rewrite (18) as

$$[\mathbf{a}(\theta, \Delta)]_i = \sum_{m=-\infty}^{\infty} J_m\left(\frac{\pi}{d} r_i\right) e^{jm(\theta + \phi_i)} = \sum_{m=-\infty}^{\infty} \left(J_m\left(\frac{\pi}{d} r_i\right) e^{jm\phi_i} \right) e^{jm\theta}. \quad (19)$$

where $J_m(\cdot)$ is the Bessel function of the first kind of order m . In practice, the infinite summation in (19) can be truncated to $2\mathcal{M} + 1$ terms with negligible error, where [50], [66]

$$\mathcal{M} \geq \frac{2\pi}{\lambda} \max_i r_i, \quad (20)$$

and λ is the wavelength of operation. As such, (19) can be approximated as

$$[\mathbf{a}(\theta, \Delta)]_i \approx \sum_{m=-\mathcal{M}}^{\mathcal{M}} \left(J_m\left(\frac{\pi}{d} r_i\right) e^{jm\phi_i} \right) e^{jm\theta} = \mathbf{g}_i^T(\Delta_i) \mathbf{d}(\theta). \quad (21)$$

where

$$\mathbf{g}_i(\Delta_i) = [J_{-\mathcal{M}}\left(\frac{\pi}{d} r_i\right) e^{-j\mathcal{M}\phi_i}, \dots, J_0\left(\frac{\pi}{d} r_i\right), \dots, J_{\mathcal{M}}\left(\frac{\pi}{d} r_i\right) e^{j\mathcal{M}\phi_i}]^T, \quad (22)$$

and

$$\mathbf{d}(\theta) = [e^{-j\mathcal{M}\theta}, \dots, e^{j\mathcal{M}\theta}]^T. \quad (23)$$

Here, $\mathbf{g}_i(\Delta_i)$ is a function of the array parameters and sensor location error Δ_i , whereas $\mathbf{d}(\theta)$ depends on source directions only. Therefore, the steering vector of the CA assumes the form

$$\mathbf{a}(\theta, \Delta) = \mathbf{G}(\Delta) \mathbf{d}(\theta) \quad (24)$$

where the matrix $\mathbf{G}(\Delta)$ collects the location error at each sensor as

$$\mathbf{G}(\Delta) = [\mathbf{g}_1(\Delta_1), \mathbf{g}_2(\Delta_2), \dots, \mathbf{g}_K(\Delta_K)]^T. \quad (25)$$

Now, the received signal from the source with P local scatterers can be expressed as

$$\mathbf{s}(t) = \sum_{p=0}^P \alpha_p(t) \mathbf{a}(\theta_p, \Delta) = \mathbf{G}(\Delta) \sum_{p=0}^P \alpha_p(t) \mathbf{d}(\theta_p), \quad (26)$$

and the signal covariance matrix is given by

$$\mathbf{R}_s = \mathbf{G}(\Delta) \left(\sigma_s^2 \sum_{p=0}^P \mathbf{d}(\theta_p) \mathbf{d}^H(\theta_p) \right) \mathbf{G}^H(\Delta). \quad (27)$$

We note from (23) that the vector $\mathbf{d}(\theta)$ has a Vandermonde structure containing the unknown direction θ , thus emulating the steering vector of a virtual ULA with $2\mathcal{M} + 1$ elements [50], [66]. As such, the term in parentheses in (27) is the equivalent $(2\mathcal{M} + 1) \times (2\mathcal{M} + 1)$ signal covariance matrix corresponding to the virtual ULA. Therefore, similar to (7), we obtain the approximation for sufficiently large P as

$$\mathbf{R}_s \approx \mathbf{G}(\Delta) \mathbf{D}(\theta_0, \sigma_s^2) \mathbf{G}^H(\Delta) \quad (28)$$

where

$$\begin{aligned} [\mathbf{D}(\theta_0, \sigma_s^2, \sigma_\theta^2)]_{kl} &\approx \sigma_s^2 e^{j([\tilde{\mathbf{k}}]_k - [\tilde{\mathbf{k}}]_l)\theta_0} \frac{1}{\sqrt{2\pi\sigma_\theta^2}} \int_{-\infty}^{\infty} e^{-\frac{(\delta_\theta)^2}{2\sigma_\theta^2}} e^{j([\tilde{\mathbf{k}}]_k - [\tilde{\mathbf{k}}]_l)\delta_\theta} d\delta_\theta \\ &= \sigma_s^2 e^{j([\tilde{\mathbf{k}}]_k - [\tilde{\mathbf{k}}]_l)\theta_0} e^{-0.5([\tilde{\mathbf{k}}]_k - [\tilde{\mathbf{k}}]_l)^2 \sigma_\theta^2} \end{aligned} \quad (29)$$

with $\tilde{\mathbf{k}} = [-\mathcal{M}, \dots, \mathcal{M}]^\top$. In matrix form, we have

$$\mathbf{D}(\theta_0, \sigma_s^2, \sigma_\theta^2) = \sigma_s^2 \mathbf{d}(\theta_0) \mathbf{d}^H(\theta_0) \odot \tilde{\mathbf{B}}(\sigma_\theta^2) \quad (30)$$

where

$$[\tilde{\mathbf{B}}(\sigma_\theta^2)]_{kl} = e^{-0.5([\tilde{\mathbf{k}}]_k - [\tilde{\mathbf{k}}]_l)^2 \sigma_\theta^2} \quad (31)$$

B. DOA Estimation

In the presence of sensor perturbations, we split the unknown parameters into three groups, namely, the signal parameters $\boldsymbol{\eta}_s$, the noise parameters $\boldsymbol{\eta}_n$, and the sensor location errors Δ . Each group of parameters is optimized when the rest of the parameters are kept fixed. The nested optimization problem is, therefore, posed as

$$\min_{\Delta} \min_{\boldsymbol{\eta}_s} \min_{\boldsymbol{\eta}_n} \left\| \mathbf{R}_x - \mathbf{G}(\Delta) \mathbf{D}(\theta_0, \sigma_s^2, \sigma_\theta^2) \mathbf{G}^H(\Delta) - \mathbf{R}_n(\sigma_w^2, \beta) \right\|_F^2 \quad (32)$$

In practice, $\mathbf{R}_x$ is replaced by the sample covariance matrix $\hat{\mathbf{R}}_x$. Again, we use CMA-ES to solve all three subproblems due to their non-convexity.

C. Simulation Results

For the results presented here, we use the same simulation parameters as in Section II-D. However, unlike Section II-D where the y -coordinates of all sensor locations were zero, the sensors are now perturbed along the y -axis. A total of 500 trials are used to compute the RMSE in each example below. For every trial, we draw the y -coordinate of each perturbed sensor location from a uniform distribution with a small spread, which is then kept fixed across the different snapshots. We assume the current flow to have a constant speed of 0.4 m/s, oriented from negative to positive y -direction, with uniform intensity from the ocean surface to the bottom. The resulting sensor location errors along the y -axis for the 7-element co-prime array are listed in Table II, where $\mathcal{U}(a, b)$ denotes a uniform distribution between a and b . These error values have been guided by OrcaFlex simulations of a 7-element co-prime array with an inflatable structure operating at 2.5 kHz [11]. We choose $\mathcal{M} = 50$, which satisfies the lower bound in (20), resulting in a virtual ULA of 101 elements. The offspring and selection sizes are maintained at 100 and 20, respectively, as in the case of the unperturbed array. For benchmark comparison, we again consider a 7-element ULA, with its first four sensor locations at x -coordinates from 0 to $3d$ taken to be error-free. This is to maintain consistency with the CA whose first two sensors at x -coordinates 0 and $3d$ are assumed to be error-free. The ULA sensors at $5d$ and $6d$ have the same location errors in the y -direction as the CA sensors at the corresponding x -coordinates. The perturbation for the ULA sensor with x -coordinate $4d$ is obtained by interpolating the perturbed y -locations for sensors at $3d$ and $5d$.

TABLE II: Sensor locations errors of the CA along the y -axis. Sensor 1 is closest to the anchor.

Sensor #	1	2	3	4	5	6	7
Error (cm)	0	0	$\mathcal{U}(5.90, 5.99)$	$\mathcal{U}(6.78, 6.89)$	$\mathcal{U}(8.84, 8.97)$	$\mathcal{U}(9.32, 9.47)$	$\mathcal{U}(9.99, 10.17)$

First, we set the noise correlation as $\beta = 0.5$ and the SNR is set to 0 dB. The number of snapshots is varied between 5 and 500. Fig. 6 shows the RMSE of the DOA estimates as a function of the number of snapshots, T . Similar trends are observed for all methods as in the case of the unperturbed array. While the beamforming methods fair better at higher values of T , the proposed method outperforms the beamforming results across the board. Further, the proposed method applied to the 7-element ULA has a slightly better performance compared to the CA. This is because the ULA has a higher number of error-

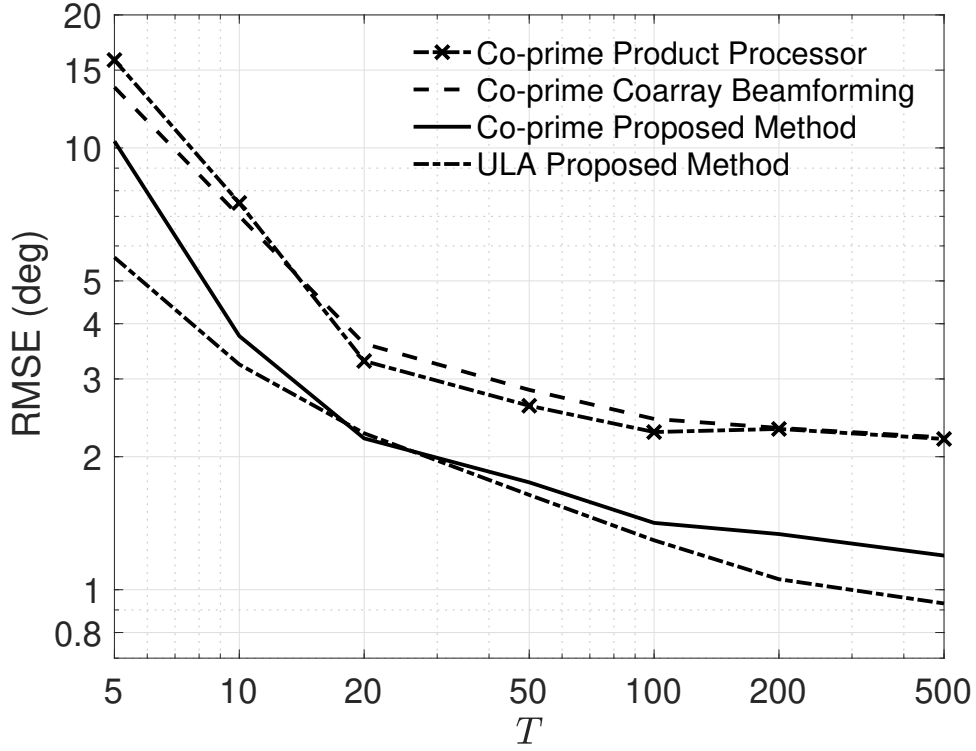


Fig. 6: Perturbed array: RMSE of the DOA estimate as a function of the number of snapshots.

free sensors and, owing to its shorter extent, smaller location error magnitudes for the farthest sensors from the anchor.

Next, we fix the number of snapshots at 500 and $\beta = 0.5$, while the SNR is varied between -10 dB and 10 dB. Figure 7 shows the corresponding RMSE of the DOA estimates as a function of the SNR. Similar to the case of the unperturbed array, we observe that the RMSE decreases as the SNR increases for all considered methods. The beamforming techniques again fail to estimate the source at low SNR values, as discussed in Section II-D. As expected, the proposed method applied to the 7-element ULA performs relatively better than the CA due to a smaller number of perturbed sensors and smaller perturbation values.

In the final example, the number of snapshots is fixed at 500 and the SNR is set to 0 dB. The noise correlation, β , is varied between 0 and 1 . The RMSE of the resulting DOA estimates obtained with the considered methods, shown in Figure 8, exhibits similar trends as observed in the corresponding case of the unperturbed array.

As stated in the unperturbed array case, further performance improvements can be achieved for the proposed method in all presented examples by increasing the offspring size or lowering the termination tolerance in the CMA-ES.

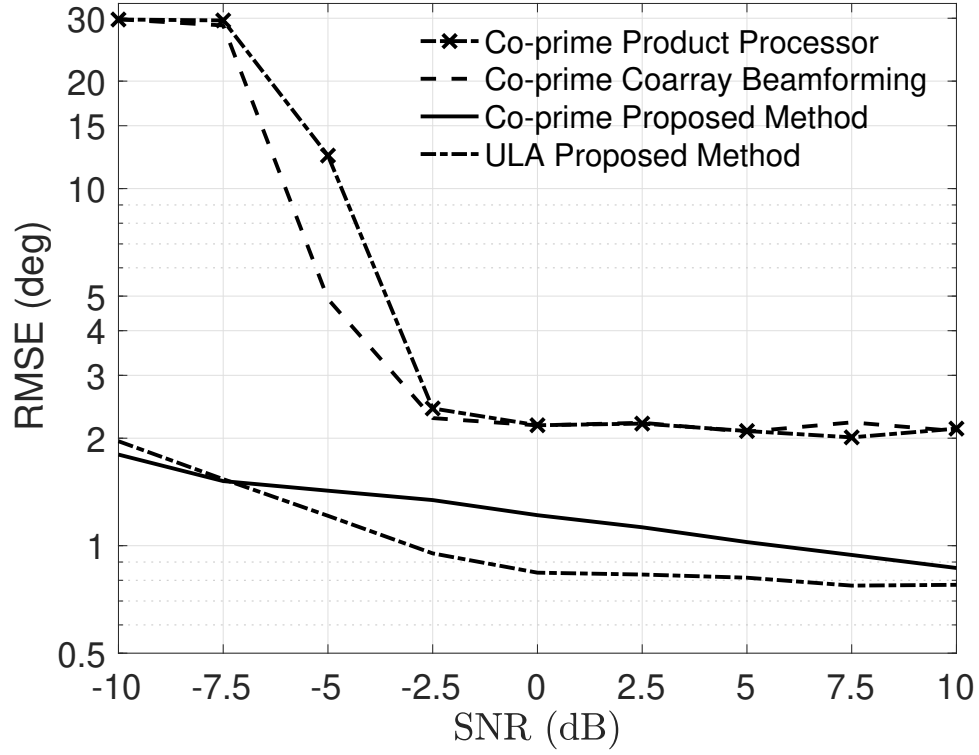


Fig. 7: Perturbed array: RMSE of the DOA estimate as a function of the SNR.

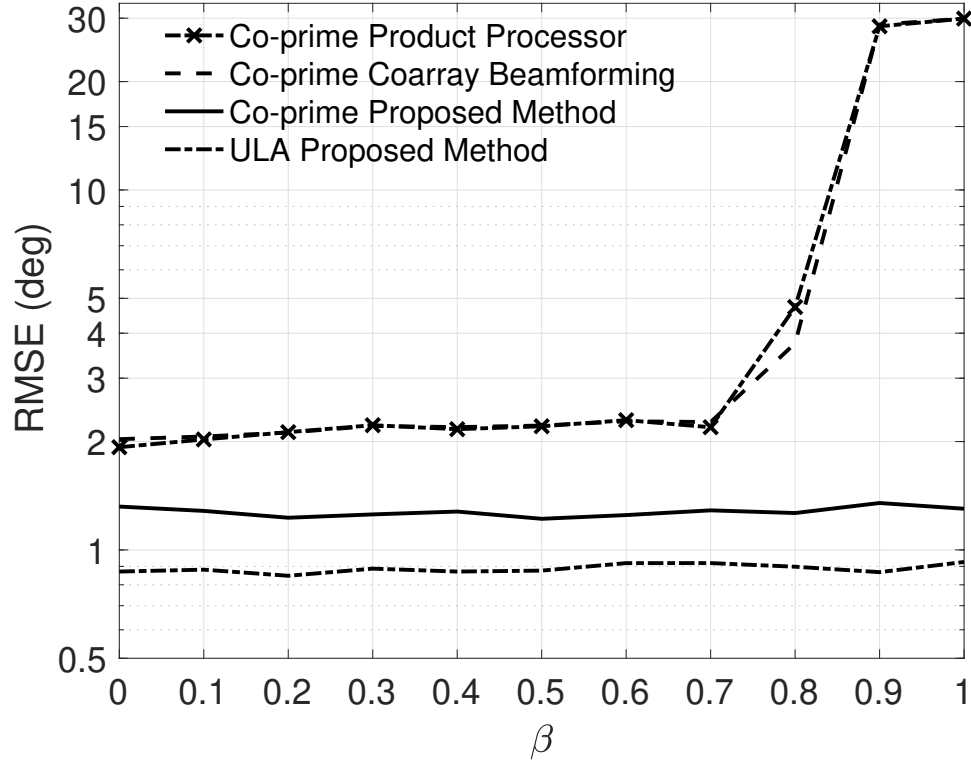


Fig. 8: Perturbed array: RMSE of the DOA estimate as a function of β .

IV. CONCLUSION

We developed a method for DOA estimation of a spatially-spread source with CAs in the presence of unknown spatially correlated noise. Modeling the angular spread of the source in terms of a Gaussian

PDF centered around its nominal DOA, the signal and noise parameters were estimated by solving a Frobenius norm minimization problem using CMA-ES. The signal model was also extended through exploitation of the Jacobi-Anger approximation to incorporate the impact of structural deflection and the resulting optimization problem was iteratively solved using CMA-ES as well. Detailed simulation analyses were performed, which demonstrated the effectiveness of the proposed DOA estimation method over beamforming approaches under unknown correlated noise.

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3.4 Laboratory and Field Experimental Study of Underwater Inflatable Co-prime Sonar Array (UICSA)

Abstract

We discuss the design and initial testing of a novel hydrophone array system dubbed the Underwater Inflatable Co-prime Sonar Array (UICSA). The UICSA will be a crucial component of an underwater deployable sensing network that can be rapidly deployed using compact autonomous underwater vehicles (AUVs). The UICSA initially is packed in a compact container to fit the payload space of an AUV. After deployment, the UICSA expands to its predetermined full length to acquire sensing data for source localization. More specifically, the mechanical compression of the UICSA is achieved through a non-rigid array support structure, which consists of flexible inflatable segments between adjoining hydrophones that are folded in order to package the UICSA for deployment. The system exploits compression in hydrophone layouts by utilizing a sparse array configuration, namely the co-prime array since it requires fewer hydrophones than a uniform linear array of the same length to estimate a given number of sources. With two-way compression, the storage, handling, and transportation of the compactly designed UICSA is convenient, particularly for the AUVs with limited payload space. The deployment concept and process are discussed, as well as the various UICSA designs of different support structures are described. A comparison of the various mechanical designs is presented and a novel hybrid-based expansion prototype is documented in detail. Laboratory study results of the UICSA prototype are presented that include water-swollen material tests in a pressurized environment and water tank validation of the inflation process. The UICSA prototype also has been deployed in the Harbor Branch channel to validate the performance, the related field test details and source localization results.

1. Introduction

The state-of-the-art in recent underwater networking technology has been widely accepted and examined [1]. Underwater sensing networks (UWSN) are gaining traction as an effective means of measuring and monitoring ocean properties in situ. The UWSN employs sensor nodes and marine platforms, such as Remote Operated Vehicles (ROVs), unmanned surface vehicles (USVs), and Autonomous Underwater Vehicles (AUVs). Generally, a UWSN has three components: the master node, mobile nodes, and sensor nodes [2,3]. Node deployment consists of placing a certain number of sensor nodes in a designated area to collect data. Therefore, the systematic performances of the UWSN such as coverage, connectivity, and lifetime, will

depend on this deployment. There are no general criteria when designing a sensor node deployment strategy. However, adjusting sensor detection range, communication radius, and other parameters in a suitable manner can yield a longer lifetime and greater network coverage. Master nodes are in charge of deploying the sensor nodes and controlling the network. The mobile nodes, mainly ROVs and AUVs, may carry sensor nodes and deploy or reconfigure them while simultaneously exploring and monitoring the environment. Once the nodes are implemented, the network topology is established and the routing policy and data transmission strategy of the network are set [4]. Sensor deployment is not only crucial to the functionality of the system, but it also represents one of the most significant cost inputs. The other major cost is the price of the sensor nodes. Both are more expensive than those of terrestrial sensing networks [5]. Partan et al. predicted that UWSNs would mostly comprise high-cost nodes sparsely deployed over a large area [6].

With technological advances in underwater vehicles over the past decade, sensor deployment is now possible with ROVs or AUVs, leading to cost reductions. However, when using such vehicles for deployment, a sensor node must have a dimension that does not impose any physical restrictions on the vehicle itself and/or hinder its maneuverability [7]. As a result, developing compact deployable systems is crucial to conserving resources and ensuring the design simplicity of the deployment strategy. We recently proposed an Underwater Inflatable Co-prime Sonar Array (UICSA), with a novel Two-Way Compression (TWC) concept, as a deployable system carried by autonomous vehicles [8]. TWC involves dimensional compression and algorithmic compression. The dimensional compression employs an inflatable structure to reduce the initial physical volume as the array can be packed into low-volume containers carried by AUVs. The algorithmic compression is achieved through the use of a sparse array with hydrophone placement following the co-prime configuration and associated signal processing. Unlike the linear sonar array with uniform half-wavelength spacing, the co-

prime array comprises two uniform linear subarrays having M and N hydrophones with specific inter-element spacings, with M and N being co-prime [9]. It offers $O(MN)$ degrees-of-freedom for source direction-of-arrival (DOA) estimation.

The inflatable structure morphs into its final geometry after detaching from the carrier platform, such as an AUV. The morphing process is driven by a top buoy and weight/anchor at the bottom of the stowed package. The buoyance and gravity straighten the entire array and maintain the desired hydrophone spacings. Different approaches can reinforce the stretched array and are presented in Section 2. The reinforced structure ensures the system rigidity and avoids being entangled, twisted, or bitten by marine life. The application of effective signal processing algorithms to the acquired datasets enables the localization of acoustic sources.

In this paper, we present the UICSA system concept in two narratives (mechanics and algorithm) in Section 2. The details of UICSA mechanical design are documented in Section 3. The UICSA prototype fabrication and laboratory validation experiments are outlined in Section 4. The initial field test of the UICSA conducted at the Harbor Branch Oceanographic Institute (HBOI) channel and performance evaluations of both narrowband and multi-frequency source estimation algorithms are presented in Section 5. Conclusions are drawn in Section 6.

2. UICSA System Concept

A UICSA system is the integration of a co-prime array configuration with an underwater inflatable structure [8, 10]. Both components contribute to the initial compact system volume and low-cost construction.

2.1 Co-prime array and associated signal processing

A co-prime array configuration is obtained by interleaving two uniform linear arrays (ULAs), one with M hydrophones spaced Nd units apart and the other with N hydrophones having an inter-element spacing of Md . M and N are co-prime integers, and d is usually chosen to be one-

half wavelength at the operating frequency [9]. The first hydrophone of the two ULAs coincide, resulting in a co-prime configuration with $M + N - 1$ hydrophones. Under narrowband far-field operation, the co-prime array can be used for source DOA estimation. Multi-frequency operation, on the other hand, permits localization of near-field sources.

2.1.1 Far-field narrowband DOA estimation

Let the positions of the hydrophones be denoted by $x_i = n_i d, i = 1, 2, \dots, M + N - 1$. Assume D narrowband sources are impinging on the co-prime array from directions $\theta_1, \theta_2, \dots, \theta_D$, with angles measured relative to the vertical axis. The received signal vector at snapshot t under far-field conditions can be expressed as

$$\mathbf{x}(t) = \mathbf{A}\mathbf{s}(t) + \mathbf{n}(t), \quad (1)$$

where $\mathbf{A}$ is the $(M + N - 1) \times D$ array manifold matrix whose (i, d) -th element is given by $\exp(j\omega x_i \sin(\theta_d) / c)$, ω is the frequency of operation, c is the sound speed in the water, $\mathbf{s}(t)$ is the $D \times 1$ source signal vector, and $\mathbf{n}(t)$ is the $(M + N - 1) \times 1$ noise vector. Assuming uncorrelated sources and spatially and temporally white noise, the $(M + N - 1) \times (M + N - 1)$ covariance matrix of the received signal is given by

$$\mathbf{R}_{xx} = E[\mathbf{x}(t)\mathbf{x}^H(t)] \quad (2)$$

where $E[\cdot]$ is the expectation operator and the superscript ‘ H ’ denotes conjugate transpose. We note that, in practice, the sample covariance matrix is used in place of $\mathbf{R}_{xx}$. Vectorizing the covariance matrix $\mathbf{R}_{xx}$, we obtain the coarray observation vector

$$\mathbf{z} = \text{vec}(\mathbf{R}_{xx}), \quad (3)$$

which emulates a single snapshot received by a virtual array, known as the difference coarray [11, 12], whose element positions are the pairwise differences of the physical array hydrophone positions. Therefore, we can model the vector $\mathbf{z}$ as [9]

$$\mathbf{z} = \tilde{\mathbf{A}} \mathbf{p} + \sigma_n^2 \tilde{\mathbf{I}} \quad (4)$$

where $\tilde{\mathbf{A}} = \mathbf{A}^* \odot \mathbf{A}$, with ‘ $\odot$ ’ denoting the Khatri–Rao product, is the coarray manifold matrix, $\mathbf{p}$ is a vector containing the signal powers, σ_n^2 is the noise variance, and $\tilde{\mathbf{I}}$ is the vectorized version of an identity matrix. Using $\mathbf{z}$, sparse reconstruction methods [13, 14] or subspace techniques with interpolation [15, 16, 17] can be employed to fully exploit all degrees-of-freedom for source DOA estimation. We will use the former in this paper.

For sparse reconstruction, we first discretize the angular extent of interest into Q angles with $Q \gg D$. We construct a dictionary matrix $\tilde{\mathbf{A}}$ to mimic the coarray manifold matrix corresponding to the Q discretized angles. We also define a D -sparse vector $\mathbf{y}$ of length Q , whose q -th element indicates the presence or absence of a source in the direction θ_q . Specifically, a zero-entry indicates the absence of the signal at angle θ_q ; otherwise, the q -th element of $\mathbf{y}$ is the source signal power from the corresponding direction. Then, the DOA estimation is accomplished by solving the optimization problem

$$\min_{\mathbf{y}} \|\mathbf{z} - \tilde{\mathbf{A}}\mathbf{y}\|_2^2 \quad \text{subject to} \quad \|\mathbf{y}\|_0 = D \quad (5)$$

where $\|\mathbf{y}\|_0$ denotes the number of nonzero entries in $\mathbf{y}$. Since this optimization problem is NP-hard, we use orthogonal matching pursuit (OMP), which is a widely used algorithm that can approximately solve (5) [18].

2.1.2 Near-field multi-frequency source localization

For near-field processing, we present the signal model in the frequency domain. Considering D uncorrelated wideband sources located at $\mathbf{x}_1, \dots, \mathbf{x}_D$ in the near-field of the array, the received signal vector corresponding to the q -th frequency can be expressed as

$$\mathbf{Y}(\omega_q) = \mathbf{A}(\omega_q) \mathbf{S}(\omega_q) + \mathbf{N}(\omega_q) \quad (6)$$

where the (i, d) -th element of $\mathbf{A}(\omega_q)$ is $e^{-j\omega_q \mathcal{D}_i(\mathbf{x}_d)/c}$, $\mathcal{D}_i(\mathbf{x}_k)$ is the Euclidean distance between the d -th source and the i -th hydrophone, $\mathbf{S}(\omega_q)$ is the source signal vector corresponding to the q -th frequency, and $\mathbf{N}(\omega_q)$ is the white Gaussian noise vector at frequency ω_q .

To coherently combine the received signals across different frequencies, we first pre-focus to some point $\mathbf{x}$ in the near-field of the array by premultiplying with a matrix $\mathbf{K}(\mathbf{x})$ [19, 20]

$$\mathbf{Y}(\omega_q, \mathbf{x}) = \mathbf{K}(\mathbf{x})\mathbf{Y}(\omega_q), \quad (7)$$

where $\mathbf{K}(\mathbf{x}) = \text{diag}[K_1(\mathbf{x}), K_2(\mathbf{x}), \dots, K_{N+M-1}(\mathbf{x})]$ is the pre-focusing matrix with diagonal elements $K_i(\mathbf{x}) = e^{j\omega_q \mathcal{D}_i(\mathbf{x})/c}$. The covariance matrix of the pre-focused received signal is

$$\mathbf{R}(\omega_q, \mathbf{x}) = E\{\mathbf{Y}(\omega_q, \mathbf{x})\mathbf{Y}^H(\omega_q, \mathbf{x})\} \quad (8)$$

which is replaced by the sample covariance matrix in practice. The pre-focused covariance matrices are combined across different frequencies as

$$\mathbf{R}(\mathbf{x}) = Q^{-1} \sum_{q=1}^Q \mathbf{R}(\omega_q, \mathbf{x}). \quad (9)$$

If $\mathbf{x} = \mathbf{x}_d$, then the d -th column of $\mathbf{K}(\mathbf{x})\mathbf{A}(\omega_q)$ equals $\mathbf{1}_{N+M-1}$, which is a vector of all ones and forms the steering vector of the pre-focused received signal [19]. Delay-and-sum beamforming at point $\mathbf{x}$ with $\mathbf{1}_{N+M-1}$, therefore, yields a high response. Performing the pre-focusing procedure for all points of interest followed by near-field beamforming yields the output

$$P(\mathbf{x}) = \mathbf{1}_{N+M-1}^T \mathbf{R}(\mathbf{x}) \mathbf{1}_{N+M-1}, \quad (10)$$

with the superscript ‘ T ’ denoting matrix transpose. The source locations are estimated by choosing the peaks of the beamformed output. High-resolution techniques can also be applied in lieu of beamforming for source localization [19, 21].

2.2 Vertical underwater inflatable structure

The underwater inflatable structure (UIS) is foldable and can morph into a large span, also referred to as underwater deployable structure or ocean morphing structure [8,10,22,23]. The UIS contributes to the dimensional compression of UICSA with the initial folded structure. This folded structure can be packed into a low-volume container for storage and transportation, ideal for AUVs with the limited payload space. After detaching from the carrier platform, the container packed with UICSA needs to release the entire system to guide the morphing process of the UIS. The feasibility of a horizontally-oriented UIS has been discussed in [8], where the self-contact forces drive the morphing process during UIS expansion.

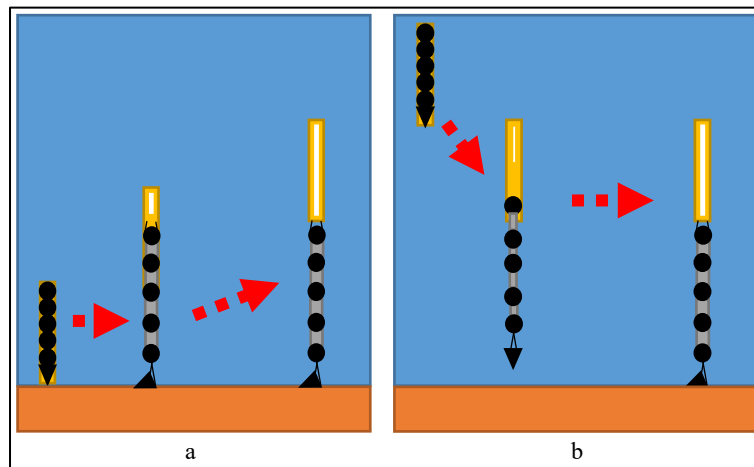


Figure 1. Vertical UIS release methods during morphing: (a) Lift method; (b) Drag method.

In this paper, we focus on the vertically-oriented tubular UIS, with the morphing process driven by buoyance at the top and gravity at the bottom. The packed UICSA container remains negatively buoyant and keeps sinking towards the seafloor. Figure 1 demonstrates two practical methods to release the UIS from a container, namely, the lift method and the drag method. In the first design, the top buoy lifts the UIS from the container after the system anchors on the

seafloor. In the second design, the weight/anchor drags the UIS out of the container during the sinking process. The lift method requires the package settlement first and then releases the buoy for morphing, which may lead to a longer morphing process. On the other hand, the drag method can complete the morphing and even can start inflation during the sinking process. However, the morphed structure is susceptible to the currents during the sinking process and can drift from the desired destination. Therefore, the lift method can cope better with strong currents, whereas the drag method is better suited for expeditious deployment in calm waters. The primary focus of this paper is the UIS system design suitable for calm shallow coastal water (around 100 to 150 m depths). The UICSA design for more challenging conditions (i.e., deep water and under strong current) will be reported in future work.

After the morphing process, UIS requires reinforcement to strengthen the array and keep the embedded hydrophones at the correct spacing under the influence of ocean currents and marine life. The reinforcement process involves swelling actions within the UIS. The UIS infill expansion provides the tension of tubular structures, secures hydrophone positions, and also ensures system resilience. The expansion process can be achieved by various approaches involving different structural designs and materials, as discussed in Section 3.

2.3 Underwater Deployable Sensing Network (UDSN)

One of the overarching goals of the UICSA project is to develop the concept of an underwater deployable sensing network (UDSN). A UDSN will consist of multiple UICSA nodes to detect and track the target within the water volume under surveillance. Figure 2 demonstrates the UDSN concept where the UICSAs are deployed from a USV, an AUV, and a helicopter. Each UICSA, as a node of the UDSN, records acoustic emissions from sources in the covered volume and relays them to the surface asset via acoustic links. The on-board processing of the surface vehicle can apply associated algorithms to the acquired signals and send the resulting source estimates to the satellite, helicopter, and/or command center. In this way, the UDSN can collect

a set of high-quality transient acoustic data over a span of locations as a sufficiently transformative UWSN.

One novelty of the UICSA design is that it realizes the TWC concept by combining the aforementioned compression methods. The sparse array requires fewer hydrophones than a linear array to provide the same resolution, which reduces the fabrication cost and initial volume. The UIS design provides the required rigidity regardless of the tension provided by the buoy and weight/anchor. By utilizing TWC, the UICSA in its deflated form can be folded and packed into reduced forms for handling, storage, and conveyance. The compact UICSA package can not only fit the limited payload capacity of AUV/USV but is also adaptable to existing watercraft as well as packed into a sonobuoy and released from airborne platforms, such as helicopters. The UICSA has advantages over towed sonar array in terms of vessel capability, and Size, Weight, Power, and Cost (SWaP-C) constraints. After the UICSA completes the morphing and reinforcing processes at the deployed location, it can start data collection as a node of the UDSN. The acquired data can be relayed to the ship-borne or onshore processing center via a hybrid of underwater acoustic links and radio links above water. Moreover, it is practical and almost effortless to increase the UDSN resolution with more UICSA-based nodes deployed from AUVs.

3. System Design and Prototyping

The UICSA needs to satisfy the initial compact volume constraint to accommodate the limited payload space of AUVs. After detachment from AUVs, the deployed UICSA needs to complete the morphing process smoothly and achieve the final stage, in a short time, to start data collection.

The fully implemented UICSA is also required to minimize the effects of structural deflections, vibrations, and entanglements due to ocean currents and marine animal activities.

In this section, we present the vertical UICSA prototype design, detailing the dimensions and applied materials.

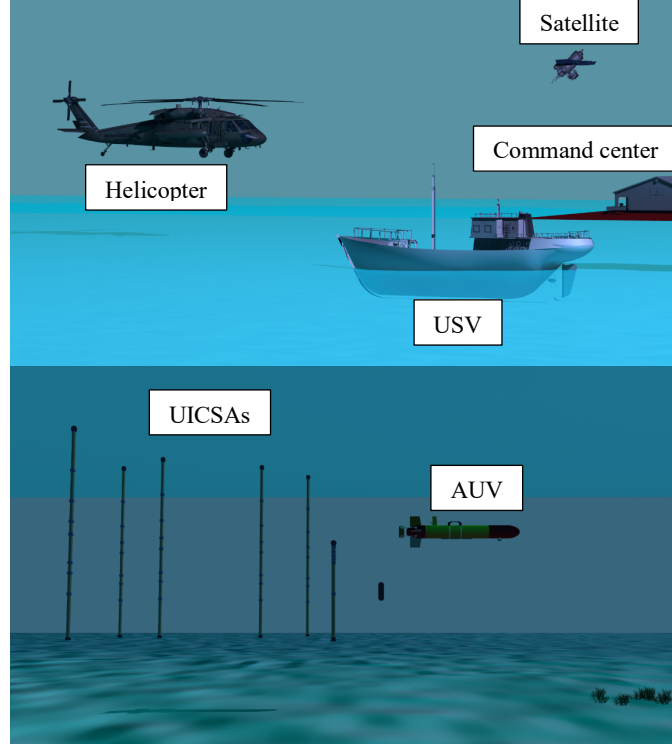
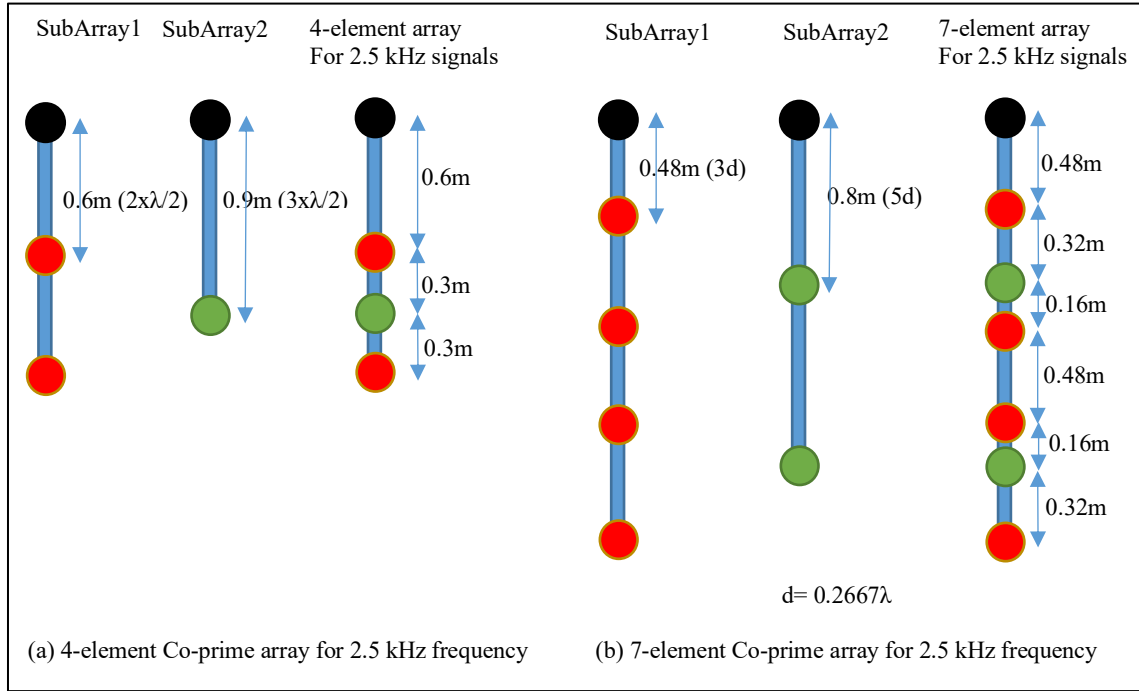


Figure 2. Underwater Deployable Sensing Network (UDSN).

3.1 Dimensions of the UICSA prototypes

The UICSA length is determined by the operating acoustic signal frequency and the desired upper limit on the number of resolvable sources. The signal frequency defines the unit length, d , whose maximum value equals half-wavelength at the operating frequency. The overall structure length is determined by the specific choice of M and N values to meet the source number constraint, together with the chosen unit length. In this study, we select 2.5 kHz as the operating frequency in the prototyping effort, which corresponds to a wavelength of $\lambda = 0.6$ m for 1500 m/s sound speed in the water. We consider two different UICSAs, namely, a 4-element ($M = 2, N = 3$) array with $d = 0.5\lambda$ and a 7-element ($M = 3, N = 5$) array with $d = 0.2667\lambda$, as shown in Figure 3. The 4-element array can resolve up to 6 acoustic sources, while

the maximum number of estimated sources increases to 15 for the 7-element array. In contrast, a ULA would require 7 and 16 sensors, respectively, to resolve the same number of sources as the corresponding co-prime arrays.



3.2 Energy-efficient UIS techniques for UICSA

The vertical UICSA is designed to be fully stretched by weight tethered to the tail and a float attached to the top. After completion of the morphing process, UICSA requires reinforcement to maintain the structural rigidity through the expansion within the array structures themselves.

3.2.1 Mechanical-based Expansion (MBE) approach

The MBE approach was initially evaluated in the UICSA design [10]. It employs the hydraulic inflation method to complete the swelling process. The MBE design requires a watertight tubular structure and water injection into the UIS by an underwater pump. The hydraulic pump can inflate the fabric in a matter of minutes and keep the UIS at appropriate stiffness to mitigate deflection under ocean currents. In the MBE design, the injection point is usually at the bottom of the UIS with a vent orifice on the top. The proposed mechanism

stabilizes the UIS during the morphing and expansion processes, as the pump provides the force to keep UIS rooted on the seafloor, and the vent permits restrained air to move along the structure during the injection process.

For the tubular UIS design, we divide the structure into sections of lengths equal to hydrophone spacings to ensure the feasibility of the prototype. The waterproof ploy film (0.15 mm thickness) is selected to fabricate the UIS. The impulse heat sealer (Metronic 400 mm manual sealer) is applied to seal the closure along the UIS axial direction. The ends are sealed by tube fittings and holding rings with NPT female threads. The top-end connects with the pressure relief valve for ventilation, while the bottom end is fitted with a barbed tube fitting for water injection. The tubular sections and connectors are cemented together using epoxy. The components applied in the prototype are built for tank test and concept verification. For deep-water deployment, the MBE UICSA requires an applicable deep-water pump and certified components.

However, the MBE design has some shortcomings. The pump needs to run continuously during expansion and then periodically to maintain the UIS stiffness. The energy consumption required to support the pump operation may be especially problematic for any long-endurance monitoring missions. The pump also requires a substantial battery pack, which increases the size and mass of the UICSA system. In addition, the pump operation generates vibrations and noise, which serve as sources of interference during data collection. For this reason, we investigated other more energy-efficient UIS approaches.

3.2.2 Chemical-based Expansion (CBE) approach

The CBE UICSA design employs a watertight tubular structure and expandable chemical resins as infilling to swell and strengthen the UIS. As the UIS stiffness is achieved by chemical reactions between the mixture synthetic resins, it only requires a smaller battery pack to power the mixing process and the expansion within a relatively shorter period. The CBE design

eliminates the bulky battery required for the MBE design. One type of material that has been evaluated for CBE design is HMI HF402 HydroFOAM [25]. The HF402 HydroFOAM is hydro-insensitive polyurethane foam, which can form high-quality foam even when submerged in water. HF402 begins with two-part pair resin and can be mixed to trigger the reaction. During a small-scale laboratory test with a 100×25 mm flat tube, the hydrofoam sustained the injection, reaction, and curing phases. The underwater expansion process took 45s and swelled seven-times compared to the initial volume. The foam can completely cure within 5 minutes in an environment at 21 to 28 °C. Since the cured hard solid tube has uniform infill and maintains its volume, there is no need for additional action once the expansion of the UIS is complete, which is an advantage over the MBE design. While the CBE bead structure can be recovered after the mission is over, the sensors and other components will need to be removed from the expanded and solidified foam and repackaged for redeployment. In this regard, the CBE design is more suitable for disposable, single-mission deployment using low-cost hydrophones and electronics. For applications requiring re-deployable UICSA, two other designs, namely, physical-based expansion (PBE) and hybrid-based expansion (HBE), are applicable, which are discussed below.

3.2.3 The PBE approach

The PBE design utilizes water swelling material (WSM) as infilling to achieve expansion after morphing. The WSM absorbs water molecules with volume expansion. One typical WSM is a hygroscopic gel (hydrogel), which can grow to over 250 times its initial volume [23]. Hydrogel, a superabsorbent polymer (polyacrylamide and polyacrylate), can shrink back to its initial volume through dehydration, which leads to a recoverable array design. With the presence of the hydrogel, the PBE design can accomplish the expansion process independently without any energy cost or additional actions after the UIS is pulled out of the packaging container.

Unlike the MBE and CBE, the current PBE design employs a permeable tube for the UIS to allow WSM contact with the surrounding water. Hydrogel beads are utilized as infilling of the nylon sleeve forming the UIS. The hydrogel beads are made of condensed superabsorbent polymer and the swelling speed is relatively slow which takes several hours to finish the expansion process. In contrast, both the MBE and CBE designs are capable of full expansion within several minutes. In general, the PBE design is practical for small diameter nylon sleeves where reasonable stiffness can be achieved within one hour. However, the rigidity of small cross-section nylon sleeves was found to be lacking for larger-span structures. To resolve these issues a hollow pipe-shaped design is employed. In this design, a set of small diameter nylon sleeves is combined to form a panel. The panel can then be rolled to form a large diameter pipe, as shown in Figure 4. Such PBE design permits more surface area to have contact with water and expands quicker than the one made of a large diameter sleeve.

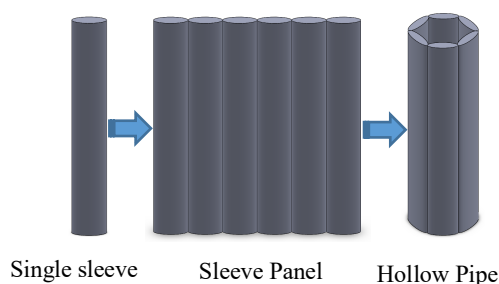


Figure 4. Nylon sleeve hollow pipe design.

Both the single tube and hollow pipe-shaped PBE based UIS requires longitudinal uniform infilling. If the beads are placed unconstrained inside a vertical UIS, it will have a solid lower section but insufficient rigidity in the upper section. Based on this finding, a narrow column made of water-soluble paper is introduced to store the dry beads and ensure an even distribution along the length of the structure. After the structure comes into contact with water, the restriction is lifted, permitting the beads to grow while staying at a pre-set stacking location along the array length.

Table 1. UIS design benefits and detriments

UIS Designs	MBE	CBE	PBE
Initial volume	Large	Small	Small
Initial weight	Heavy	Light	Light
Initial Expansion speed	Quick (< 10minutes)	Quick (< 10 minutes)	Slow (> 1 hour)
Power consumption	Bulky battery	Not required	Not required
Maintenance After Expansion	Period inflation required	No	No
Generated Noise	Pump inflation	No	No
Expanded structure	Thin-film tube	Rigid beam	Semi-rigid beam
Recoverability	Yes	No	Yes

3.2.4 The HBE approach

We summarize the advantages and limitations of MBE, CBE, and PBE designs in Table 1, where the red blocks represent restrictions, the acceptable conditions are marked in yellow, and the advantages are shown in green. We note that all three designs have limitations to create a compact, rapidly deployable, and recoverable UICSA. In particular, the PBE design may take an hour or longer to complete the expansion, and is prone to entanglements and twisting due to ocean currents and marine life. These issues motivated us to investigate an HBE design. Specifically, we integrate MBE and PBE concepts to build an HBE UIS, which can complete the initial expansion in a short time and reduce energy consumption to maintain structural stiffness.

The proposed HBE design is a hybrid of external MBE and internal PBE configuration. The outer and inner layers are attached at both ends to ensure morphing synchronization. The PBE based UIS can achieve rigidity rapidly with water injection after fully morphed, and it maintains rigidity through the built-in expanding PBE structure.

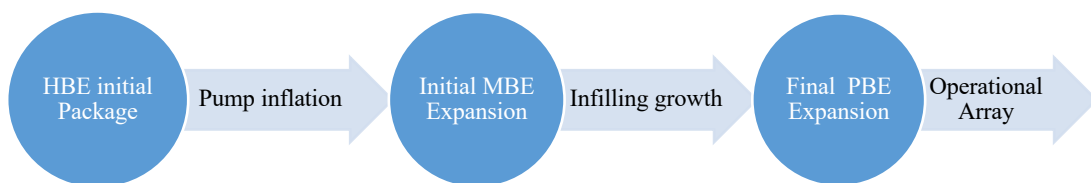


Figure 5. HBE design expansion process

Figure 5 describes the expansion process of HBE based UIS. The HBE UIS requires materials including watertight films as an external layer, permeable fabrics as an internal layer, and WSM as infill for expansion. The outer layer ensures water retention after the initial expansion through water injection. After the initial pump injection, the flat structure turns into a thin-wall tube containing internal pressurized water. It maintains certain stiffness against external forces. In the meantime, the inner layer is permeable to allow water contact with contained WSM and starts the moderate expansion. The internal PBE layer eventually turns into a solid beam after full expansion and reinforces the rigidity during data collection.

The HBE design employs a hydraulic approach to accomplish initial expansion, similar to the MBE approach. However, it eliminates the periodical inflation required in the original MBE design after the initial expansion as the internal PBE layer starts to swell. Theoretically, the WSM can absorb water within the watertight external layer and continue the absorption with given access to the surrounding water. Since the pump is only required for initial injection, a smaller battery pack may be employed to support the initial pump operation and hydrophone data acquisitions. Therefore, the initial package volume, mass, overall energy consumption, and cost can be significantly reduced compared to the original MBE design.

In addition to the reduced energy consumption, the HBE design is more robust than MBE against punched holes or leakage in the sealed layer. As the HBE's eventual rigidity is determined by the internal layer, the leaked external layer only affects expansion speed and the HBE based UIS can subsequently reach the predetermined rigidity with full expansion. The applicable depth for the HBE design depends on the applied components of the MBE design for initial expansion and the infilled WSM of the PBE design for maintaining the final geometry.

4. System Prototyping and Validation Tests in Laboratory

In this section, we report on the HBE based UICSA material evaluation, prototype fabrication details, and deployment in laboratory tanks.

4.1 Pressure test of materials

To minimize the initial volume of the HBE based UICSA, we select hydrogel beads as the WSM infillings. The hydrogel beads are made of condensed superabsorbent polymer, so the swelling speed is relatively slow compared to the powder. In preliminary tests, it took several hours for the UIS to expand and achieve a reasonable stiffness after deployment [23].

Since the UICSA is designed for deployment in shallow coastal water at around 100 to 150 m depth, we evaluated the hydrogel beads performance in a similar ambient environment with a water pressure chamber which is shown in Figure 6(a). During the test, we placed hydrogel beads of different colors in the sealed chamber as illustrated in Figure 6(b), then pressurized the chamber to reach 1241 kPa (equivalent to 125 m depth) with the help of a dial indicator, as shown in Figure 6(c). We maintained the pressure level for six hours to observe the performance of the beads through the viewport. We also placed one set of beads in a water cup under normal pressure as a controlled trial; the beads took eight hours to fully expand, as shown in Figure 6(d). Interestingly, we noticed that the beads in the pressurized tank fully expanded in about one hour, as seen in Figure 6(e). Compared to the eight hours for the submerged beads to achieve full expansion under normal pressure, an increase in the ambient pressure accelerates the swelling process. The diameters of the swollen beads were roughly 15.8 mm, as presented in Figure 6(f). We also opened and retrieved the swollen beads after six hours and compared their measurement with the dry beads as shown in Figure 6(g). The diameters of the dry beads are approximately 3.2 mm and those of the swollen beads are 14 mm roughly. Therefore, the diameters of the fully swollen beads were at the same level regardless of the applied pressure.

The results demonstrate that the hydrogel beads can swell in a shorter time under pressure, maintain integrity for a long time, and provide about 80 times volume expansion.

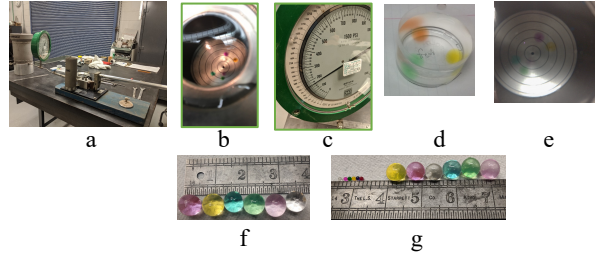


Figure 6. Hydrogel beads pressure chamber test

4.2 UICSA prototype design

We fabricated the UICSA prototypes in compliance with the designs elaborated above. Specifically, four different prototypes were developed: a 7-element MBE-based UICSA, 4-element and 7-element PBE based UICSAs, and a 7-element HBE based UICSA [10, 24]. Most components applied in the UICSA prototyping process were built through fused deposition modeling (FDM). The parts required to be watertight were coated with epoxy. In the interest of brevity, we only present the details of the 4-element HBE based UICSA prototype; the details of other prototypes are documented in [10, 24].

The tubular structure assembly of the 4-element HBE based UICSA comprised two tubes: the external tube made of watertight Polyester film, and the internal tube made of Nylon sleeve filled with WSM- dry hydrogel beads, depicted in Figure 7(a). The outer tube was attached to the inner tube at both ends using glue. The internal tube contained dry hydrogel beads stored in a string of soluble paper bags. The HBE based UICSA consisted of circular holders, tubular structure assembly, cable clamps, barbered tube fitting, and pressure relief valve, as shown in Figure 7(b). The external tube is sealed at both ends, leaving one end connected with barbered tube-fitting for water inflation and the other with a pressure relief valve for trapped gas ventilation. Figure 7(c) depicts HBE based UICSA with both ends sealed and tightened by cable

clamps, while the remaining holders are glued to the external tube. The applied circular holder is designed to carry two types of hydrophones (Teledyne RESON TC 4013 [26] and Aquarian H2a [27]). Figures 7(d) and 7(e) show the circular holder carrying Teledyne and Aquarian hydrophones, respectively.

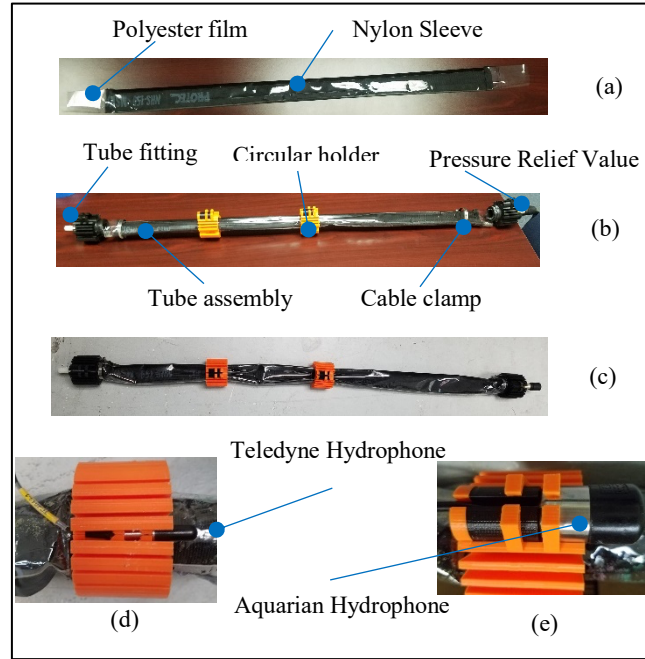


Figure 7. 4-element HBE based UICSA structural details

4.3 Validation of prototype expansion after deployment

Having validated the UICSA morphing process with the MBE-based UICSA and the expansion performance after being entirely morphed for the PBE based prototypes, as reported in [10] and [24], we conducted the feasibility test of the 4-element HBE based UICSA.

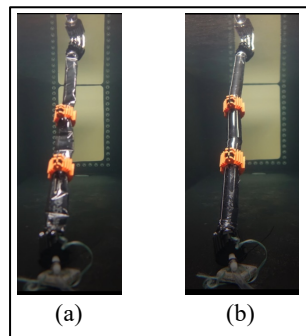


Figure 8. 4-element HBE based UICSA expansion process

Figure 8 presents the expansion process of the 4-element HBE based UICSA. The prototype required an underwater pump at the bottom end for water injection, while the top end needed to be connected with an opened pressure relief valve to allow the flow from bottom to top and to ensure that the internal nylon sleeve is submerged. The bottom end was also tethered with weight and the top-end secured with a crossbar over the water tank to keep the array stretched. Because the water tank was not deep enough, we had to tilt the UICSA to ensure that the structure remained below the surface. Figure 8(a) reveals the initial state of the structure, which is flat and slack. Once the pump started to inject water into the array, the UICSA became stiffer due to the pressure difference. Then, the hydrogel beads grew to a large volume and reinforced the structure stiffness, as seen in Figure 8(b). After complete expansion UICSAs can operate as a sensing node.

After the validation of the prototype expansion process, we also conducted a numerical study to understand the performance of the fully expanded UICSA structure under external forces, such as ocean currents. Because the HBE based UICSA stiffness is contributed by the MBE-based structure at the beginning and later by the PBE based structure, we deduce that a fully expanded HBE based structure demonstrates the same structure response as the PBE base UICSA with the same dimensions. As such, we only created numerical models of MBE, CBE, and PBE based UICSAs tethered with a buoy on the top and moored on the seafloor. Based on the simulation results in OrcaFlex [28], although all of the considered UICSAs drift along the direction of the ocean current with a slight curvature along the array length, they all maintain the hydrophones at the correct relative locations under the small current conditions. This implies that the UICSA should work properly as a sonar array for source estimation [29]. To assess the design and validate the simulation results under realistic conditions, we conducted a field deployment in the presence of underwater flow and marine animals, as described in the following section.

5. The Initial Field Validation Experiment

5.1 Prototype field deployment in HBOI Channel

In this initial field experiment, we deployed a fully expanded 7-element PBE based UICSA, because the fully expanded HBE UICSA has the same structural stiffness as the PBE design. We previously employed the same PBE based UICSA prototype in an acoustic test tank for data collections [24]. After the acoustic test tank experiment, the prototype was retrieved from the acoustic test tank and left in a dry environment for dehydration. Once the infilling shrank back to its initial volume, the UICSA was packed and stored in a container. Using the same prototype in the field experiment, thus, also provides an opportunity to test the reusability of the UICSA after dehydration of the hydrogel beads.

The total spacing from the first hydrophone to the last hydrophone is 1.92 m for the 7-element UICSA. However, the actual structure length is fabricated to be 2 m, leaving space on both ends for lift and weight installation. We selected the HBOI channel as the initial field test site instead of fully open water, such as the Indian River Lagoon or Atlantic coastal area. The primary consideration for this choice was to reduce the complexity of the field test. The Harbor Branch channel outlet connects to the Indian River Lagoon, as shown in Figure 9(a), and introduces the currents from the East to the West. To avoid multipath arising from the floor/walls, the deployment site depth needs to be over 4.5 m. The majority of the channel depth is about 4 m, which only leaves three possible locations for deployment in front of the dock area, marked as Sites A, B, and C in the hydrographic map of Figure 9(b), where the depth is marked in feet. The boat employed for array deployment could drift under the waves, currents, and wind during the field test. To mitigate such effects, the boat needed to be tied to the seawall post. As such, Site C turned out to be the best option because of its close proximity to the seawall at 8 m away, thereby permitting the HBOI research vessel (Harbor Branch Pontoon #2) of 7.5 m length to

reach this site while tied to the seawall post. The purple pentagon marked in Figure 9(b) represents the Pontoon #2 parked perpendicular to the seawall with the bow reaching Site C.

The channel also serves as a habitat for marine animals, such as dolphins, groupers, and manatees. As the UICSA is a passive array, it is necessary to deploy an active sound source during the field test operating at a center frequency of 2.5 kHz. This frequency is within the hearing spectrum of dolphins and manatees. To avoid causing any harm to marine mammals, the power of the sound source is required to be below 60 dB. As the embedded Aquarian H2a hydrophone can distinguish the sound over ambient noise within 15 m range, the speaker has to be deployed from the boat instead of a far-field location.

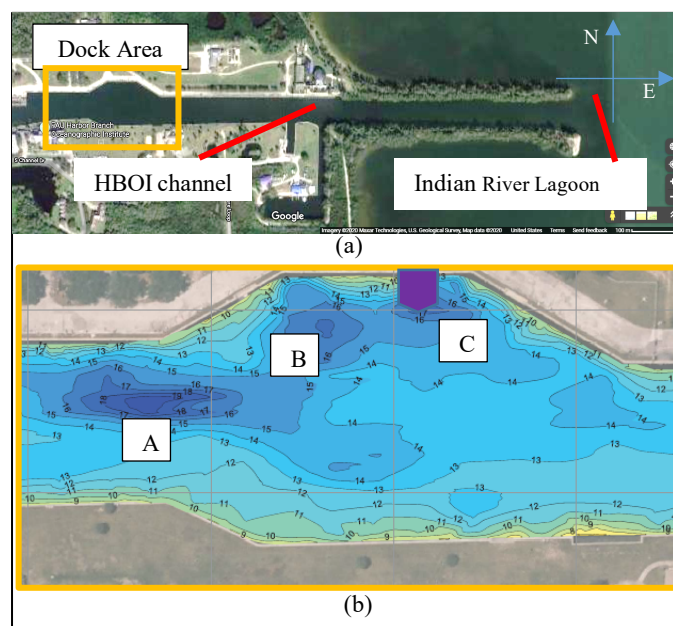


Figure 9. HBOI Channel and site selection.

We also had a time constraint since the test was to be completed in the channel within a four-hour window. Therefore, it was impractical to deploy the flat 7-element PBE based UICSA, which requires hours to reach the desired stiffness. Consequently, we pre-expanded the prototype in the Systems and Imaging Lab (SAIL) indoor tank overnight and examined it to ensure that the desired stiffness was achieved before the field test.

We loaded the re-expanded array on Pontoon #2, shown in Figure 10 (a). During the test, the boat was anchored using a three-point mooring to mitigate drifting. Two fluke anchors tethered to the bow of the boat were deployed in the channel. The stern was tied to the post next to the seawall. Figure 10 (b) shows Pontoon #2 traveling across the channel to deploy the anchors. After the front two anchors were deployed, Pontoon #2 was moved backward to reach the seawall post to fasten the stern perpendicular to the seawall. After the boat was anchored, the bow reached Site C with the 5.18 m depth. Using a marked rope, the array was deployed with the first hydrophone at 1 m depth below the surface. Figure 10(c) shows the UICSA deployed from the Pontoon #2. After the array was secured at the desired depth, an INSMY IPX7 waterproof speaker [30] was deployed from predetermined locations around the boat at a 0.5 m depth, pointing towards the array. The experimental layout for the acoustic tests is shown in Figure 10 (d). The red dots denote the speaker's locations, and the yellow dot indicates the array position. At each location, we collected two sets of measurements; a monochromatic measurement where the speaker played a looped 2.5 kHz single tone, and a multi-frequency measurement with the speaker emitting a chirp signal of bandwidth 300 Hz centered at 2.5 kHz. The data were recorded by the deployed array using two Zoom H6 data loggers [31] with a sampling frequency of 96 kHz. The current data loggers are not submersible and require manual operation for data acquisition. For comparison, we also deployed a 7-element co-prime array on a rigid structure and repeated the same experiments as that for the UICSA prototype. During the field deployment, we observed that in chirp measurements with the rigid array, the hydrophone marked in black in Figure 3(b) malfunctioned.

5.2 Acoustic performance using the field experimental data

To validate the acoustic performance of the UICSA, we processed both the single tone and the chirp measurements from the field test. The signal model detailed in Section 2.1.1 deals with far-field measurement conditions. Due to the constraints imposed by the channel

dimensions, the speaker positions 1, 2, and 5 in Figure 10 (d) are closer to the array and, thus, exhibit a much higher deviation from the far-field source approximation compared to positions 3 and 4. Therefore, these were excluded from performance evaluation for the case of 2.5 kHz single tone data. On the other hand, for near-field source localization with the chirp signals using the method discussed in Section 2.1.2, we considered speaker positions 1 and 2 only.

Table 2 shows the nominal DOA for speaker positions 3 and 4 based on the ground truth, with θ_t and θ_b denoting the respective directions of the speaker relative to the top and bottom hydrophones of the co-prime array and $\bar{\theta}$ being the average DOA. Ideally, the angular spread across the array should be zero under far-field conditions. Although the considered speaker positions do not satisfy the exact condition for the far-field source, nonetheless, we expect reasonable estimation accuracy with far-field processing. Since only a single acoustic source was present per experiment, we employed OMP with a sparsity level set to 1 for DOA estimation. For both the UICSA and the rigid array, we used measurements from speaker position 3 for calibration and retained the same calibration for processing data from position 4. The last two columns of Table 2 provide the resulting DOA estimates for the UICSA and the rigid array, respectively, while the corresponding normalized OMP spectra are depicted in Figure 11. We observe that, for each speaker position, the estimated DOA using both arrays falls within the corresponding nominal angular spread and is close to the corresponding average DOA. These results corroborate that the UICSA provides similar performance to that of a rigid co-prime array.

Table 2. Nominal and Estimated Source DOAs.

Speaker	Nominal DOA (deg)			DOA Estimate (deg)	
	θ_t	θ_b	$\bar{\theta}$	UICSA	Rigid
Position 3	3.77	17.68	10.72	10.54	10.72
Position 4	3.88	18.16	11.02	12.88	9.64

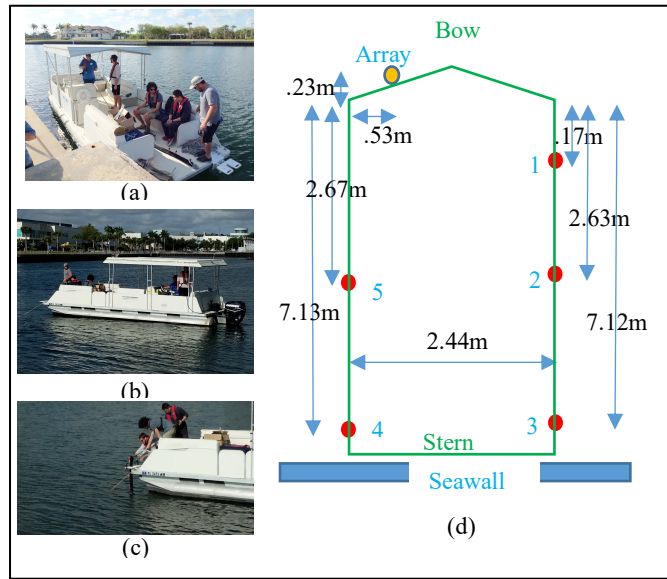


Figure10. Field test setup.

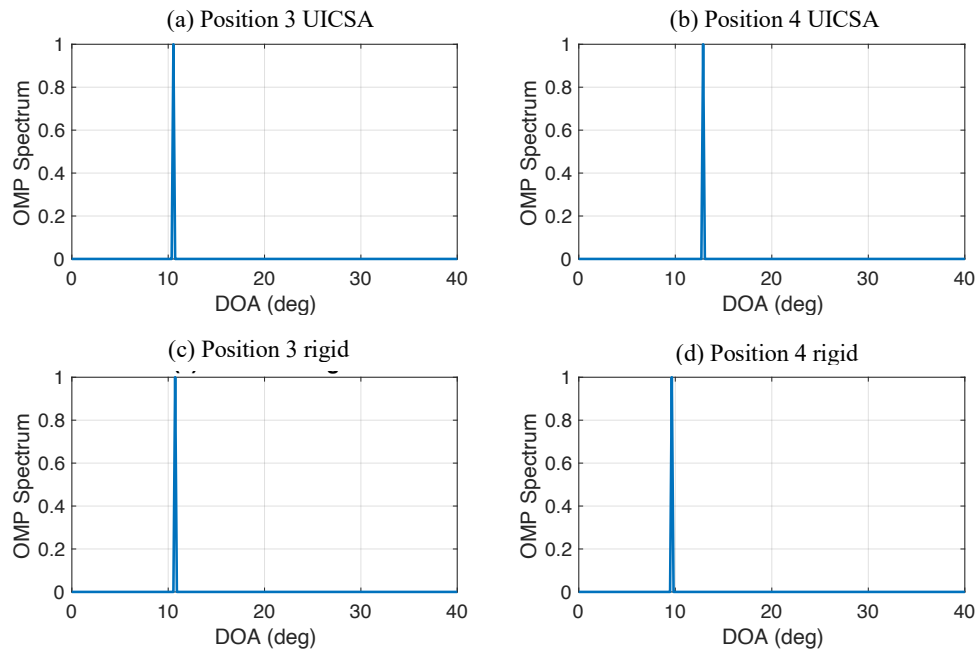


Figure 11. Normalized OMP spectra.

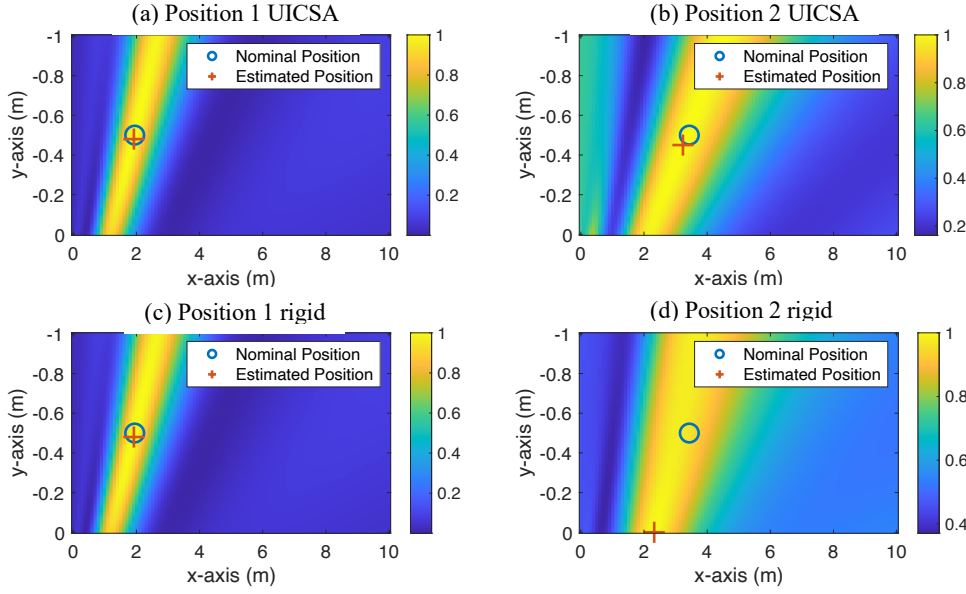


Figure 12. Normalized near-field beamforming spectra.

For near-field processing with both UICSA and the rigid array, the data from position 1 is used for calibration, and the same calibration is retained for processing the data from position 2. The normalized near-field beamforming spectra are depicted in Figure 12 where the true source positions are marked with “o”. The array is aligned along the positive y-axis. At a propagation speed of 1500 m/s for sound in water, the range resolution is 5 m for the 300 Hz bandwidth, causing the main lobe to be extended in range. However, the peak intensity value, which is the source location estimate and marked as “+” in Figure 12, is very close to the ground truth for the UICSA. For the rigid array, only position 1, which is self-calibrated, is accurately estimated. For position 2, although the direction of the source is resolved, the location estimate exhibits a large bias. This is caused by the malfunctioning of the first hydrophone. The specific hydrophone is said to be essential for reliable processing since the deletion of this hydrophone results in a reduction of the degrees of freedom offered by the co-prime array [32]. As such, its loss due to malfunctioning leads to erroneous estimate by the rigid array.

6. Discussions and Conclusion

In this paper, we investigated various aspects of UICSA, including concept validation, design, prototyping, laboratory validation tests, and field performance tests. The UICSA deployment strategy was documented and validated with tests. The UIS design concept with four different approaches, namely, MBE, CBE, PBE, and HBE, was detailed. The HBE based UICSA prototype was fabricated and deployed in SAIL tank, which validated the HBE prototype's performance as proposed. Besides, we also validated the HBE UICSA related infilling WSM's performance, which can swell and maintain its integrity at 125m depth in the pressure chamber test. Using measurements with a PBE design in field tests, we demonstrated that a UICSA could accurately estimate sources.

The current prototype was developed to validate the proposed concept. As such, some components will need further development to be ready for field tests in deeper water. For example, the data acquisition device cannot operate underwater and the 3D-printed holders are not suitable for deep water. The next-generation prototype needs to be fully submersible and configured to acquire data automatically. Further development is required in terms of the design of a new control device together with the hydrophones fully embedded within the HBE UIS to reduce the initial volume, along with other required components including battery and pump, to fit the compact deployable container. We also envision further improvements in the structural designs, enhanced processing methods to address sound speed ambiguities and hydrophone positioning errors under large current conditions, and performance validations using field tests under more challenging environments. Moreover, we plan to extend the proposed UICSA designs to two-dimensional sonar array configurations. We also aim to expand the UIS concept to other ocean engineering applications, like marine animal detection and surveillance task.

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