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EXPLOITING SPATIAL DIVERSITY IN MIMO AND MASSIVE MIMO RADAR SYSTEMS

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The theoretical work to analyze the effect of non coherence on multiple input multiple output (MIMO) RADAR has been concluded. The report is organized in four chapters. Chapter one describes the system model, which uses ambiguity function to determine the resolution capabilities of a MIMO system, while the Cramer Rao Lower Bound technique is used to evaluate the accuracy in the estimation of the target parameters such as position, velocity or reflection coefficient in the presence of clutter. In chapter 2 the ideal bounds are derived for spatially white and spatially correlated clutter. Chapter 3 introduces the phase mismatch for the white clutter case and Chapter 4 is where the results are presented

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EXPLOITING SPATIAL DIVERSITY IN MIMO AND MASSIVE MIMO RADAR SYSTEMS

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Contents

Chapter 1

Introduction

The topic of multistatic radars has seen a great and renewed interest in the last years thanks to the success of MIMO (Multiple-Input Multiple-Output) technologies for communications, which induced new research to apply those result to the radar field [[**1-19**]]. A MIMO radar can be described as a multistatic system where each transmitter sends a generally unique waveform. MIMO radars can then be classified as co-located, when the antennas constitute an array, or widely distributed, when the antennas are geographically distant.

The former class is the evolution of the phased-array radar, which is a SIMO (Single-Input Multiple-Output) system that can be obtained by using the same waveform for each transmitter: as proven in [1][2], the additional degrees of freedom granted by the selection of transmitted signals, from completely correlated to orthogonal, with the same number of antennas the MIMO radar can achieve a greater synthetic aperture, leading to better angular resolution and parameter identifiability. The ambiguity function for this kind of radar, already derived in [3][4][5], proves also that with orthogonal waveform there is no beamforming in transmission: the SNR (Signal-to-Noise Ratio) is lower than the phased-array, but the system can now scan simultaneously a bigger area (proportional to the radiation pattern of the single element).

On the other hand, widely distributed MIMO radars take advantage of the spatial diversity to reduce the "target fading", i.e. to see the target from different angles. While previous distributed systems tried to improve the detection capabilities by performing data fusion after the detection at each receiver, MIMO radars operate by processing the received data all together, meaning there is no loss of information.

MIMO radars can also be classified as coherent or non-coherent: coherent processing can be applied when each antenna observes the same aspect of the target (i.e. the reflection coefficient is the same for each transmitter-receiver path), meaning that the phase information in the received data can be used to perform detection. When the reflection coefficient varies between the different path, the radar is non-coherent and the phase information is lost [6].

Because these systems operate with separate oscillators at each antenna, ensuring synchronization in time, frequency and, to achieve coherent processing, phase is of critical importance, even more so for widely distributed systems: synchronization algorithms have already been explored in [6][7].

While the ambiguity function is used to determine the resolution capabilities of the system, the CRLB (Cramér-Rao Lower Bound)is used to evaluate the accuracy in the estimation of the target parameters, like position, velocity or reflection coefficient, in the presence of clutter. Extensive research has been conducted in this sense: in [8] both coherent and non-coherent cases are evaluated for widely distributed radars, but the clutter is gaussian and both spatially and temporally white and the target is considered stationary; in [9] and [10] the bound on the velocity is added. It is known in literature [11] that the gaussian model for the clutter requires a high number i.i.d. scatterers (through the central limit theorem) to be realistic, meaning that this assumption doesn't hold with high-resolution systems. In order to build a more realistic model the wide class of CES (Complex Elliptically Symmetric) distributions has been proposed in [12], including among the others the CG (Compound-Gaussian) distributions and allowing to model both thin-tailed and heavy-tailed clutter. The other assumption that will be dropped is the whiteness in space: despite being useful to derive easy estimator, it is widely known that clutter is in general correlated. To our knowledge CES (in particular K-distributed and t-distributed) clutter has been studied in MIMO radar only in [13][14], although the CRLB has been derived for the Resolution Limit and not for the target parameters, and [15], where only the estimation algorithm is proposed, and in both cases spatial correlation and time independence has been considered.

The scope of this thesis is then to derive the CRLB of the target parameters (position, velocity, reflectivity) under CES-distributed clutter (in particular complex-t-distributed), and to analyze how spatially correlated clutter influences the performance of the MIMO radar w.r.t. the ideal case of white clutter. Then the impact of phase mismatch between transmitters and receivers will also be analyzed. The thesis is organized as follows: in Chapter 1 the system model is presented, followed by a treatment of the theory behind the Cramér-Rao and Cramér-Rao-like bounds; in Chapter 2 the ideal bounds are derived for spatially white and spatially correlated clutter; in Chapter 3 the phase mismatch is introduced for the white clutter case; in Chapter 4 results are presented.

Chapter 2

System Model and Geometry

2.1 Signal Model

Consider a widely distributed narrowband MIMO radar system illuminating a point-like, isotropic moving target in a 2-D plane. Then, assuming there is only one radar pulse per CPI, the signal at each of the receivers, after down-conversion and sampling, before matched filtering, can be written as

$$r_l(n) = \sqrt{\frac{E}{M}}\zeta \sum_{k=1}^{M} e^{-\jmath 2\pi f_0 \tau_{lk}} e^{\jmath 2\pi f_{lk} n T_s} s_k(n) + z_l(n) \ , \qquad l = 1 \dots N \ , \qquad n = 0 \dots N_o - 1 \tag{2.1}$$

where

- N and M are the number of receivers and transmitters respectively
- ζ is the target complex reflectivity, assumed deterministic and constant
- E is the total transmitted energy, equally split between all transmitters
- f_0 is the carrier frequency
- τ_{lk} is the time delay of a signal that propagates from the $\mathrm{k^{th}}$ transmitter to target, and then to the $\mathrm{l^{th}}$receiver
- f_{lk} is the Doppler shift of a signal that propagates from the $\mathrm{k^{th}}$ transmitter to target, and then to the $\mathrm{l^{th}}$receiver
- $\boldsymbol{s}_k = [s_k(1) \dots s_k(N_s)]^T$, with $s_k(n) = s_k(nT_s - \tau_{lk})$ being the complex baseband $\mathrm{n^{th}}$ sample of the signal transmitted by the $\mathrm{k^{th}}$ transmitter. Each waveform has the

same energy E_s, i.e. $\sum_{n=1}^{N_s} |s_k(n)|^2 = E_s$, and the time delays are left implicit to simplify the notation. As commonly done in MIMO radar the transmitted signals are assumed to be orthogonal even for any set of time delays and Doppler shifts, i.e. $\sum_{n=1}^{N_s} s_k(nT_s - \tau_{lk}) s_{k'}^*(nT_s - \tau_{l'k'}) e^{j2\pi(f_{lk} - f_{l'k'})nT_s} = E_s\delta[k-k']$, where $\delta[k]$ is the Kronecker delta function.
In this thesis the behavior of some sets of waveforms is examined through their Cross-Ambiguity Function (CAF), in order to validate this assumption

- T_s is the sampling time, chosen to satisfy the Nyquist condition
- T is the pulse length, and $N_s = [T/T_s]$ is the number of samples per pulse
- $N_o = [T_o/T_s]$ is the number of samples collected at each receiver, with T_o being the observation time. T_o is chosen big enough to receive completely all the replicas reflected by the target
- $z_l(n)$ are the samples of the clutter at the l^{th} receiver.

Given the Cartesian coordinates of the target (x, y) and its velocity (v_x, v_y), the relation between the time lags and Doppler shifts, and the position and velocity of the target is

$$\begin{cases} \tau_{lk} = \dfrac{\sqrt{\left(x - x_{t_k}\right)^2 + \left(y - y_{t_k}\right)^2} + \sqrt{\left(x - x_{r_l}\right)^2 + \left(y - y_{r_l}\right)^2}}{c} = \dfrac{d_{t,t_k} + d_{t,r_l}}{c} \\ f_{lk} = \dfrac{f_0}{c} \dfrac{v_x\left(x_{t_k} - x\right) + v_y\left(y_{t_k} - y\right)}{d_{t,t_k}} + \dfrac{f_0}{c} \dfrac{v_x\left(x_{r_l} - x\right) + v_y\left(y_{r_l} - y\right)}{d_{t,r_l}} \end{cases}$$

where (x_{t_k}, y_{t_k}) and (x_{r_l}, y_{r_l}) are the locations of the k^{th} transmitter and l^{th} receiver respectively, as shown in Figure 2.1.

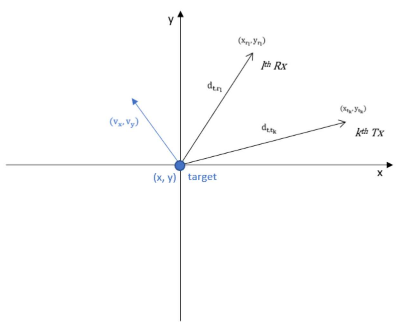

Figure 2.1: Location of transmitters and receivers with respect to the moving target.

2.2 Cramér-Rao Lower Bound

2.2.1 Formulation

In order to derive the Cramér-Rao Lower Bound (CRLB) for the estimate of the position and velocity of the target, the first step is to find the log-likelyhood function of the received data $\boldsymbol{r}$: $LL(\boldsymbol{\psi}) = \log p(\boldsymbol{r}|\boldsymbol{\psi})$, where the vector $\boldsymbol{\psi} = \left[x, \; y, \; v_x, \; v_y, \; \zeta_{Re}, \; \zeta_{Im}\right]$ contains all the target unknowns in the received data and
$\boldsymbol{r} \triangleq \left[r_1(0), r_2(0), \ldots, r_N(0), r_1(1), \ldots, r_N(1), \ldots, r_N(N_o-1)\right]$ contains all the received samples; the target reflectivity is not a parameter of interest, so ζ_{Re} and ζ_{Im} are considered nuisance parameters.
Then, the CRLB matrix is $\boldsymbol{CRLB}(\boldsymbol{\psi}) = [\boldsymbol{J}(\boldsymbol{\psi})]^{-1}$, where $\boldsymbol{J}(\boldsymbol{\psi})$ is the Fisher Information Matrix (FIM) w.r.t. $\boldsymbol{\psi}$ and
$[\boldsymbol{J}(\boldsymbol{\psi})]_{p,q} \triangleq -E\left[\frac{\partial^2 LL(\psi)}{\partial\psi_p \partial\psi_q}\right], \quad p,q = 1\ldots 6.$

In order to simplify the computation of the FIM, the chain rule for matrix derivation is applied: instead of using $\boldsymbol{\psi}$, it was used
$\boldsymbol{\theta} = \left[\tau_{11}, \; \tau_{12}, \ldots, \; \tau_{NM}, \; f_{11}, \; f_{12}, \; \ldots, \; f_{NM}, \; \zeta_{Re}, \; \zeta_{Im}\right]$ because x, y, v_x, v_y are not explicit in the received vector model, and $\boldsymbol{J}(\boldsymbol{\psi}) = \boldsymbol{P}\boldsymbol{J}(\boldsymbol{\theta})\boldsymbol{P}^T$, where $\boldsymbol{P} = \frac{\partial\boldsymbol{\theta}}{\partial\psi}$ is the Jacobian of $\boldsymbol{\theta}$ w.r.t. $\boldsymbol{\psi}$.

Since the target reflectivity is independent of the other target parameters, and since the time delays do not depend on the target velocity, the structure of $\mathbf{P}$ is as follows:

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{F}_{2\times NM} & \boldsymbol{G}_{2\times NM} & \mathbf{0}_{2\times 2} \\ \mathbf{0}_{2\times NM} & \boldsymbol{H}_{2\times NM} & \mathbf{0}_{2\times 2} \\ \mathbf{0}_{2\times NM} & \mathbf{0}_{2\times NM} & \boldsymbol{I}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{T}_{4\times 2NM} & \mathbf{0}_{4\times 2} \\ \mathbf{0}_{2\times 4} & \boldsymbol{I}_2 \end{bmatrix} \tag{2.2}$$

where

$$\begin{cases} \boldsymbol{F}(1,i) = \frac{\partial\tau_{lk}}{\partial x} = \frac{1}{c}\left[\frac{x-x_{r_l}}{d_{t,r_l}} + \frac{x-x_{t_k}}{d_{t,t_k}}\right] \\ \boldsymbol{F}(2,i) = \frac{\partial\tau_{lk}}{\partial y} = \frac{1}{c}\left[\frac{y-y_{r_l}}{d_{t,r_l}} + \frac{y-y_{t_k}}{d_{t,t_k}}\right] \\ \boldsymbol{G}(1,i) = \frac{\partial f_{lk}}{\partial x} = \frac{f_0}{c}\left[-\frac{v_x}{d_{t,t_k}} - \frac{v_x}{d_{t,r_l}} + \frac{(x_{t_k}-x)\left(v_x(x_{t_k}-x)+v_y(y_{t_k}-y)\right)}{d^3_{t,t_k}} + \frac{(x_{r_l}-x)\left(v_x(x_{r_l}-x)+v_y(y_{r_l}-y)\right)}{d^3_{t,r_l}}\right] \\ \boldsymbol{G}(2,i) = \frac{\partial f_{lk}}{\partial y} = \frac{f_0}{c}\left[-\frac{v_y}{d_{t,t_k}} - \frac{v_y}{d_{t,r_l}} + \frac{(y_{t_k}-y)\left(v_x(x_{t_k}-x)+v_y(y_{t_k}-y)\right)}{d^3_{t,t_k}} + \frac{(y_{r_l}-y)\left(v_x(x_{r_l}-x)+v_y(y_{r_l}-y)\right)}{d^3_{t,r_l}}\right] \\ \boldsymbol{H}(1,i) = \frac{\partial f_{lk}}{\partial v_x} = \frac{f_0}{c}\frac{x_{t_k}-x}{d_{t,t_k}} + \frac{f_0}{c}\frac{x_{r_l}-x}{d_{t,r_l}} \\ \boldsymbol{H}(2,i) = \frac{\partial f_{lk}}{\partial v_y} = \frac{f_0}{c}\frac{y_{t_k}-y}{d_{t,t_k}} + \frac{f_0}{c}\frac{y_{r_l}-y}{d_{t,r_l}} \end{cases}$$

$$i = (l-1)M + k, l = 1\ldots N, k = 1\ldots M. \tag{2.3}$$

The FIM w.r.t. $\boldsymbol{\theta}$ can then be subdivided as

$$\boldsymbol{J}(\boldsymbol{\theta}) = \begin{bmatrix} \boldsymbol{S}_{2NM\times 2NM} & \boldsymbol{V}_{2NM\times 2} \\ \boldsymbol{V}^T & \boldsymbol{\Lambda}_{2\times 2} \end{bmatrix} \tag{2.4}$$

where the matrix $\boldsymbol{S}$ contains the terms related only to the Doppler shifts and time delays, the matrix $\boldsymbol{\Lambda}$ contains the terms related only to the target reflectivity, and $\boldsymbol{V}$ contains the remaining cross-terms. Then, after applying the block processing rules to these matrices, we have

$$\boldsymbol{CRLB}(\boldsymbol{\psi}) = \boldsymbol{J}(\boldsymbol{\psi})^{-1} = \left(\boldsymbol{P}\boldsymbol{J}(\boldsymbol{\theta})\boldsymbol{P}^T\right)^{-1} = \begin{bmatrix} \boldsymbol{T}\boldsymbol{S}\boldsymbol{T}^T & \boldsymbol{T}\boldsymbol{V} \\ (\boldsymbol{T}\boldsymbol{V})^T & \boldsymbol{\Lambda} \end{bmatrix}^{-1}. \tag{2.5}$$

Two different kinds of signals are considered in this work: the first set is made of complex exponential signals, each at a different frequency; the second set comprises of pseudo-noise-like signals. In both cases the objective is to approximate the behavior of a set of orthogonal signals, but in the latter case the aforementioned formula for the CRLB is not easily applicable. Also, in Chapter 4 we introduce the phase mismatch between receivers' and transmitters' oscillators, introducing additional random terms. Indeed, the issue with the approach of the CRLB is that, even though the pdf of the clutter $p(\boldsymbol{z})$ is a known function, $p(\boldsymbol{r}|\boldsymbol{\theta})$ is not easy to derive because there are additional random parameters (either the pseudo-noise code sequences or the phase noises) in the form of the vector $\boldsymbol{\phi}$, that is

$$p(\boldsymbol{r}|\boldsymbol{\theta}) = \int p(\boldsymbol{r}, \boldsymbol{\phi}|\boldsymbol{\theta})\, d\boldsymbol{\phi} \tag{2.6}$$

where $p(\boldsymbol{r}, \boldsymbol{\phi}|\boldsymbol{\theta})$ is the joint probability density function between the received data and the additional nuisance random parameters. In order to simplify the analysis in such cases, several Cramér-Rao-like bounds have been proposed and studied in literature [16][17][18][19].

2.2.2 Alternative bounds

By considering $\boldsymbol{\phi}$ as an actual parameter, albeit random, the HCRB (Hybrid Cramér-Rao Bound) can be formulated in a similar fashion by modifying the FIM definition as follow:

$$\left[\boldsymbol{J}\left(\boldsymbol{\psi}^h\right)\right]_{p,q} \triangleq -E_{\boldsymbol{r},\boldsymbol{\phi}}\left\{\frac{\partial^2 \log p\left(\boldsymbol{r}, \boldsymbol{\phi}|\boldsymbol{\psi}^h\right)}{\partial\psi_p^h \partial\psi_q^h}\right\} = -E_{\boldsymbol{r},\boldsymbol{\phi}}\left\{\frac{\left[\partial^2 \log p\left(\boldsymbol{r}|\boldsymbol{\phi};\boldsymbol{\psi}^h\right) + \partial^2 \log p\left(\boldsymbol{\phi}|\boldsymbol{\psi}^h\right)\right]}{\partial\psi_p^h \partial\psi_q^h}\right\} \tag{2.7}$$

where now the joint pdf is used and the expectation is performed both w.r.t. $\boldsymbol{r}$ and $\boldsymbol{\phi}$ and $\boldsymbol{\psi}^h \triangleq \left[\boldsymbol{\psi}, \boldsymbol{\phi}\right]$: then $\boldsymbol{HCRB} = \boldsymbol{J}\left(\boldsymbol{\psi}^h\right)^{-1}$. Because the HCRB provides the bounds for the nuisance random parameters as well, the dimensions of $\boldsymbol{\psi}^h$ and the FIM are larger (dependent on $\boldsymbol{\phi}$'s dimension) meaning that using this bound is impractical for very dimensionally large parameters.

The MCRB (Modified Cramér-Rao Bound) and MCB (Miller Chang Bound) operate by considering the conditional pdf $p(\boldsymbol{r}|\boldsymbol{\phi};\boldsymbol{\psi})$ in the derivation of the FIM $\boldsymbol{J}\left(\boldsymbol{\psi}|\boldsymbol{\phi}\right)$, thus considering the random parameter as deterministic and known. Then to remove the dependency the expectation is performed:

$$\boldsymbol{MCRB} \triangleq \left[E_{\boldsymbol{\phi}}\left\{\boldsymbol{J}\left(\boldsymbol{\psi}|\boldsymbol{\phi}\right)\right\}\right]^{-1} \qquad \boldsymbol{MCB} \triangleq E_{\boldsymbol{\phi}}\left\{\boldsymbol{J}\left(\boldsymbol{\psi}|\boldsymbol{\phi}\right)^{-1}\right\}.$$

The FIM in this case is easier to derive, having a more limited number of terms and not requiring the explicit knowledge of the distribution of the additional parameters because it can be easily calculated by means of Monte Carlo simulations.

Regarding the tightness of the approximations of the CR-like bounds, it has been proven in literature, i.e. in [17], that

$$\begin{aligned} &\boldsymbol{CRLB} \geq \boldsymbol{HCRB} \geq \boldsymbol{MCRB} \\ &\boldsymbol{MCB} \geq \boldsymbol{MCRB}, \end{aligned} \tag{2.8}$$

but there is not a general inequality between CRLB and MCB. Indeed, it is important to notice that the CRLB is a lower bound on the MSE of any unbiased estimator, while the MCB applies only to a more restrictive class of estimators, i.e. those who are unbiased for all values of the nuisance parameters. Also, despite being always looser, the MCRB applies to estimators that are unbiased on the average of the nuisance parameters, which is again a looser condition. Because the derivation and implementation of an estimation algorithm is out of the scope of this thesis, the MCB is chosen for the pseudo-noise transmitted signals without concerning about the class of estimators to which it is applicable. This choice allows to derive the FIM only once, for a general set of transmitted waveforms $\boldsymbol{s}$: then, in the exponential case the CRLB formula will be applied; in the pseudo-noise-like case the MCB formula will be applied. Regarding the phase mismatch, only the exponential signals will be used and thus the HCRB will be used, since it is easy enough to derive and calculate (only adding $N+M$ terms).

Chapter 3

Cramér-Rao Lower Bounds

3.1 On CES distributions

Following [12], Complex Elliptically Symmetric (CES) distributions are defined by their symmetry center $\boldsymbol{\mu}$, their scatter matrix $\boldsymbol{\Sigma}$ and their characteristic generator function $\phi(t)$, where $\boldsymbol{\Sigma}$ is a positive semi-definite Hermitian matrix and $\phi(t)$ is related to the characteristic function of the distribution: the CES distributed random vector of dimension m is then $\boldsymbol{z} \sim CE_m(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \phi)$.
CES distributions share important properties with the complex normal distribution (which is a particular case of the CES for $\phi(t) = e^{-\frac{t}{4}}$): they are close w.r.t. affine transformations, i.e. $\boldsymbol{B}\boldsymbol{z} + \boldsymbol{b} \sim CE_k(\boldsymbol{B}\boldsymbol{\mu} + \boldsymbol{b}, \boldsymbol{B}\boldsymbol{\Sigma}\boldsymbol{B}^H, \phi)$ where $\boldsymbol{B}$ is an $k \times m$ matrix and $\boldsymbol{b}$ is a $k \times 1$ vector; any partition of $\boldsymbol{z}$ is still CES distributed with the same ϕ and the corresponding elements of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$, although it may belong to a different family given that $\phi(t)$ might be a function of the dimension of the random vector.
Only when $\boldsymbol{\Sigma}$ is a full rank matrix (a necessary, but not sufficient condition) the CES distribution admits a probability density function (pdf) described by the density generator $g(t)$ as $p(\boldsymbol{z}) = C_{m,g} |\boldsymbol{\Sigma}|^{-1} g(t), \quad t = (\boldsymbol{z} - \boldsymbol{\mu})^H \boldsymbol{\Sigma}^{-1} (\boldsymbol{z} - \boldsymbol{\mu})$, where $C_{m,g}$ is a normalizing coefficient.
An important subclass of the CES distributions are the Compound-Gaussian (CG) distributions: following the CG representation, $\boldsymbol{z} =_d \boldsymbol{\mu} + \sqrt{\tau}\boldsymbol{n}$, where τ is the texture and is independent of the speckle $\boldsymbol{n} \sim \mathbb{CN}(\boldsymbol{0}, \boldsymbol{\Sigma})$.
In this thesis the bounds are derived for the Complex-t distribution: while it admits a CG representation with an Inverse-Gamma texture, the CES representation is used to derive the bounds.

3.2 Clutter Models

Regarding the probability function of the received data, the clutter is distributed according to a 0-mean Complex-t distribution, described by its degrees of freedom ν (or scale parameter), its density generator g(t) and its scatter matrix $\mathbf{\Sigma}$. Following the definition given in [OLL12], a complex-t distribution of dimension m has the following pdf

$$p(t) = \frac{C_{m,g}}{|\mathbf{\Sigma}|}\left(1 + \frac{2t}{\nu}\right)^{-(2m+\nu)/2} \tag{3.1}$$

where $C_{m,g} = 2^m \Gamma\left(\frac{2m+\nu}{2}\right) / \left[(\pi\nu)^m \Gamma(\nu/2)\right]$ and $\Gamma(x)$ is the Gamma function.In Figure 3.1 the pdf is represented (as a function of t) for $m = 1$ and different values of ν, and it is compared against the normal distribution. Clearly, a complex-t distribution is always heavy-tailed, but when ν approaches infinity it converges to the Gaussian one.

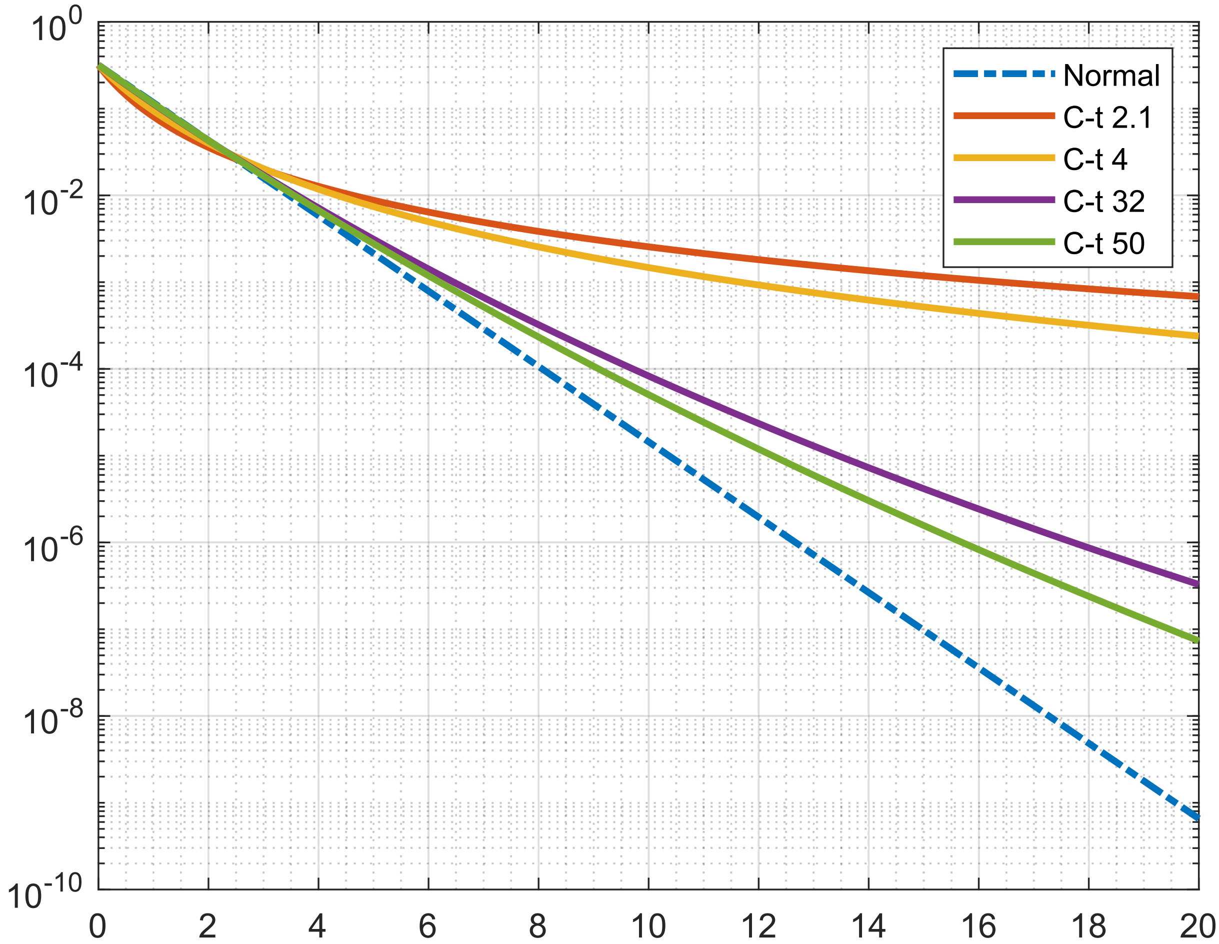

Figure 3.1

In the following derivations, to simplify the notation, the matrix $\mathbf{\Omega}(n)$ is defined as $[\mathbf{\Omega}(n)]_{p,q} = e^{-\jmath 2\pi f_0 \tau_{pq}} e^{\jmath 2\pi f_{pq} n T_s}$, and $\mathbf{\Omega}_p(n)$ is the p$^{\text{th}}$ row of this matrix: the scalar quantity

$u_p(n) \triangleq \boldsymbol{\Omega}_p(n)\boldsymbol{s}(n)$. Then, two types of clutter are examined:

Case 1 The clutter is independent in both the time and space domains, meaning that all the clutter samples $z_l(n)$ are i.i.d. ($l = 1 \dots N, \quad n = 0 \dots N_o - 1$) and that the scatter matrix is actually a scalar σ^2. Then, by recalling the received data model, it is

$$p\left(r_l(n)|\boldsymbol{\theta}\right) = \frac{C_{1,g}}{\sigma} g\left(t_l(n)\right) \tag{3.2}$$

where $t_l(n)$ is the quadratic form $t_l(n) = \frac{1}{\sigma^2}\left|r_l(n) - \sqrt{\frac{E}{M}}\zeta u_l(n)\right|^2$ and $C_{1,g} = \frac{2\Gamma\left(\frac{2+\nu}{\nu}\right)}{(\pi\nu)\Gamma\left(\frac{\nu}{2}\right)}$. After applying the independence assumption

$$\log p\left(\boldsymbol{r}|\boldsymbol{\theta}\right) = \log \prod_{n=0}^{N_o-1}\prod_{p=1}^{N} p\left(r_p(n)|\boldsymbol{\theta}\right) = C + \sum_{n=0}^{N_o-1}\sum_{p=1}^{N} \log p\left(r_p(n)|\boldsymbol{\theta}\right) = C + \sum_{n=0}^{N_o-1}\sum_{p=1}^{N} \log g(t_p(n)) \tag{3.3}$$

where C is a generic constant that does not depend on the parameters of target. Then, the generic element of the FIM is

$$\begin{aligned}\left[\boldsymbol{J}\left(\boldsymbol{\theta}\right)\right]_{i,j} &= -E\left[\frac{\partial^2 LL\left(\boldsymbol{\theta}\right)}{\partial\theta_i\partial\theta_j}\right] = \\ &= -E\left[\sum_{n=0}^{N_o-1}\sum_{p=1}^{N}\left(\frac{g''(t_p(n))}{g(t_p(n))} - \frac{g'(t_p(n))^2}{g(t_p(n))^2}\right)\frac{\partial t_p(n)}{\partial\theta_i}\frac{\partial t_p(n)}{\partial\theta_j} + \frac{g'(t_p(n))}{g(t_p(n))}\frac{\partial^2 t_p(n)}{\partial\theta_i\partial\theta_j}\right], \\ &\qquad i,j = 1\dots 2NM+2\end{aligned} \tag{3.4}$$

where, recalling the definition of $\boldsymbol{\theta}$,

$$\theta_i = \begin{cases} \zeta_{Im} & i = 2NM+2 \\ \zeta_{Re} & i = 2NM+1 \\ f_{lk} & i = NM + (l-1)M + k, \quad l = 1\dots N, \quad k = 1\dots M \\ \tau_{lk} & i = (l-1)M + k, \quad l = 1\dots N, \quad k = 1\dots M. \end{cases} \tag{3.5}$$

Case 2 The clutter is correlated in the space domain and independent in the time domain, i.e. the N_s vectors $\mathbf{z}[\mathrm{n}] = \begin{bmatrix} z_1(n) & z_2(n) & \dots & z_N(n)\end{bmatrix}^T$ are i.i.d. and $\boldsymbol{\Sigma}$ is an $N \times N$ space matrix. In this case the optimal way to express the received data is

$$\boldsymbol{r}(n) = \sqrt{\frac{E}{M}}\zeta\boldsymbol{\Omega}(n)\boldsymbol{s}(n) + \boldsymbol{z}(n), \qquad n = 1\dots N_s \tag{3.6}$$

where $\mathbf{s}[\mathrm{n}] = \begin{bmatrix} s_1(n) & s_2(n) & \dots & s_M(n) \end{bmatrix}^T$, and then

$$p\left(\boldsymbol{r}(n)|\boldsymbol{\theta}\right) = \frac{C_{N,g}}{|\boldsymbol{\Sigma}|} g\left(t(n)\right) \tag{3.7}$$

where $t(n) = \left[\boldsymbol{r}(n) - \sqrt{\frac{E}{M}}\zeta\boldsymbol{\Omega}(n)\boldsymbol{s}(n)\right]^H \boldsymbol{\Sigma}^{-1} \left[\boldsymbol{r}(n) - \sqrt{\frac{E}{M}}\zeta\boldsymbol{\Omega}(n)\boldsymbol{s}(n)\right]$ and $C_{N,g} = \frac{2^N\Gamma\left(\frac{2N+\nu}{\nu}\right)}{(\pi\nu)^N\Gamma\left(\frac{\nu}{2}\right)}$. After applying the independence assumption

$$\log p\left(\boldsymbol{r}|\boldsymbol{\theta}\right) = \log \prod_{n=1}^{N_s} p\left(\boldsymbol{r}(n)|\boldsymbol{\theta}\right) = C + \sum_{n=0}^{N_o-1} \log p\left(\boldsymbol{r}(n)|\boldsymbol{\theta}\right) = C + \sum_{n=0}^{N_o-1} \log g(t(n)) \tag{3.8}$$

where C is a generic constant. Then, the generic element of the FIM is

$$\begin{aligned}
\left[\boldsymbol{J}\left(\boldsymbol{\theta}\right)\right]_{i,j} &= -E\left[\frac{\partial^2 LL\left(\boldsymbol{\theta}\right)}{\partial\theta_i\partial\theta_j}\right] = \\
&= -E\left[\sum_{n=0}^{N_o-1}\left(\frac{g''(t(n))}{g(t(n))} - \frac{g'(t(n))^2}{g(t(n))^2}\right)\frac{\partial t(n)}{\partial\theta_i}\frac{\partial t(n)}{\partial\theta_j} + \frac{g'(t(n))}{g(t(n))}\frac{\partial^2 t(n)}{\partial\theta_i\partial\theta_j}\right], \\
&\qquad\qquad i,j = 1\dots 2NM+2
\end{aligned} \tag{3.9}$$

In the general case, the density generator of a complex-t distribution with ν degrees of freedom and dimension m is $g(t) = \left(1 + \frac{2}{\nu}t\right)^{-\frac{2m+\nu}{\nu}}$, where t is the quadratic form of the random vector. This leads to

$$\frac{g'(t)}{g(t)} = \frac{2m+\nu}{\nu}\left(1 + \frac{2}{\nu}t\right)^{-1} \tag{3.10}$$

$$\left(\frac{g''(t)}{g(t)} - \frac{g'(t)^2}{g(t)^2}\right) = -2\frac{2m+\nu}{\nu^2}\left(1 + \frac{2}{\nu}t\right)^{-2} \tag{3.11}$$

Notes about notation: in order to have more compact formulas in the derivation, the following variables are defined as $A_{lk} \triangleq e^{\jmath 2\pi f_0 \tau_{lk}} e^{\jmath 2\pi f_{lk} n T_s}$, $A_{l'k'} \triangleq e^{\jmath 2\pi f_0 \tau_{l'k'}} e^{\jmath 2\pi f_{l'k'} n T_s}$. In addition the time index may be omitted, although the final result will be given in the extended form. Additionally, to complete the FIM matrix its symmetry property can be exploited, thus only half of the elements need to be calculated.

3.3 Derivation - Case 1

In order to calculate the partial derivatives, the first step is to determine the quadratic form explicitly:

$$t_p(n) = \frac{1}{\sigma^2}\left|r_p(n) - \sqrt{\frac{E}{M}}\zeta u_p(n)\right|^2 = \frac{|r_p(n)|^2}{\sigma^2} - \frac{2}{\sigma^2}\sqrt{\frac{E}{M}}\,\mathbb{R}\mathrm{e}\left\{\zeta \mathrm{u_p(n) r_p^*(n)}\right\} + \frac{1}{\sigma^2}\frac{E}{M}|\zeta|^2|u_p(n)|^2. \tag{3.12}$$

where $u_p(n) = \sum_{q=1}^{M} e^{-\jmath 2\pi f_0 \tau_{pq}} e^{\jmath 2\pi f_{pq} n T_s} s_q(n)$. Therefore, the first-order derivatives are

$$\begin{aligned}
\frac{\partial t_p(n)}{\partial f_{lk}} &= \Bigg(\frac{4\pi n T_s}{\sigma^2}\sqrt{\frac{E}{M}}\,\mathbb{I}\mathrm{m}\left\{\zeta \mathrm{e}^{-\jmath 2\pi f_0 \tau_{lk}} \mathrm{e}^{\jmath 2\pi f_{lk} n T_s} \mathrm{s_k(n) r_p^*(n)}\right\} \\
&\quad - \frac{4\pi n T_s}{\sigma^2}\frac{E}{M}|\zeta|^2\,\mathbb{I}\mathrm{m}\left\{\mathrm{e}^{-\jmath 2\pi f_0 \tau_{lk}} \mathrm{e}^{\jmath 2\pi f_{lk} n T_s} \mathrm{s_k(n) u_p^*(n)}\right\}\Bigg)\,\delta[l-p] \\
\frac{\partial t_p(n)}{\partial \zeta_{Re}} &= -\frac{2}{\sigma^2}\sqrt{\frac{E}{M}}\,\mathbb{R}\mathrm{e}\left\{\mathrm{u_p(n) r_p^*(n)}\right\} + \frac{2}{\sigma^2}\frac{E}{M}\zeta_{Re}|u_p(n)|^2 \\
\frac{\partial t_p(n)}{\partial \zeta_{Im}} &= \frac{2}{\sigma^2}\sqrt{\frac{E}{M}}\,\mathbb{I}\mathrm{m}\left\{\mathrm{u_p(n) r_p^*(n)}\right\} + \frac{2}{\sigma^2}\frac{E}{M}\zeta_{Im}|u_p(n)|^2 \\
\frac{\partial t_p(n)}{\partial \tau_{lk}} &= \Bigg(-\frac{4\pi f_0}{\sigma^2}\sqrt{\frac{E}{M}}\,\mathbb{I}\mathrm{m}\left\{\zeta \mathrm{e}^{-\jmath 2\pi f_0 \tau_{lk}} \mathrm{e}^{\jmath 2\pi f_{lk} n T_s} \mathrm{s_k r_p^*(n)}\right\} + \\
&\quad + \frac{4\pi f_0}{\sigma^2}\frac{E}{M}|\zeta|^2\,\mathbb{I}\mathrm{m}\left\{\mathrm{e}^{-\jmath 2\pi f_0 \tau_{lk}} \mathrm{e}^{\jmath 2\pi f_{lk} n T_s} \mathrm{s_k u_p^*(n)}\right\}\Bigg)\,\delta[l-p] + \\
&\quad \Bigg(-\frac{2}{\sigma^2}\sqrt{\frac{E}{M}}\,\mathbb{R}\mathrm{e}\left\{\zeta \mathrm{e}^{-\jmath 2\pi f_0 \tau_{lk}} \mathrm{e}^{\jmath 2\pi f_{lk} n T_s} \mathrm{\dot{s}_k r_p^*(n)}\right\} + \\
&\quad + \frac{2}{\sigma^2}\frac{E}{M}|\zeta|^2\,\mathbb{R}\mathrm{e}\left\{\mathrm{e}^{-\jmath 2\pi f_0 \tau_{lk}} \mathrm{e}^{\jmath 2\pi f_{lk} n T_s} \mathrm{\dot{s}_k u_p^*(n)}\right\}\Bigg)\,\delta[l-p]
\end{aligned} \tag{3.13}$$

where $\delta[l-p] = 1$ for $l = p$ and 0 elsewhere, and $\frac{\partial u_p(n)}{\partial \tau_{lk}} = (-\jmath 2\pi f_0 s_k(n) + \dot{s}_k(n))\, e^{-\jmath 2\pi f_0 \tau_{lk}} e^{\jmath 2\pi f_{lk} n T_s} \delta[l-p]$. Then, the second-order derivatives are

$$\frac{\partial^2 t_p(n)}{\partial\zeta_{Re}\partial\zeta_{Re}} = \frac{\partial^2 t_p(n)}{\partial\zeta_{Im}\partial\zeta_{Im}} = \frac{2}{\sigma^2}\frac{E}{M}|u_p(n)|^2$$

$$\frac{\partial^2 t_p(n)}{\partial\zeta_{Re}\partial\zeta_{Im}} = 0$$

$$\frac{\partial^2 t_p(n)}{\partial\tau_{lk}\partial\zeta_{Re}} = \left(\frac{8\pi f_0}{\sigma^2}\frac{E}{M}\zeta_{Re}\,\mathbb{Im}\left\{\mathrm{e}^{-\jmath 2\pi f_0\tau_{lk}}\mathrm{e}^{\jmath 2\pi f_{lk} n T_s}\mathrm{s_k(n)u_p^*(n)}\right\} + -\frac{4\pi f_0}{\sigma^2}\sqrt{\frac{E}{M}}\,\mathbb{Im}\left\{\mathrm{e}^{-\jmath 2\pi f_0\tau_{lk}}\mathrm{e}^{\jmath 2\pi f_{lk} n T_s}\mathrm{s_k(n)r_p^*(n)}\right\} + \frac{4}{\sigma^2}\frac{E}{M}\zeta_{Re}\,\mathbb{Re}\left\{\dot{\mathrm{s}}_\mathrm{k}(\mathrm{n})\mathrm{e}^{-\jmath 2\pi f_0\tau_{lk}}\mathrm{e}^{\jmath 2\pi f_{lk} n T_s}\mathrm{u_l^*(n)}\right\} + -\frac{2}{\sigma^2}\sqrt{\frac{E}{M}}\,\mathbb{Re}\left\{\dot{\mathrm{s}}_\mathrm{k}\mathrm{e}^{-\jmath 2\pi f_0\tau_{lk}}\mathrm{e}^{\jmath 2\pi f_{lk} n T_s}\mathrm{r_l^*(n)}\right\}\right)\delta[l-p]$$

$$\frac{\partial^2 t_p(n)}{\partial\tau_{lk}\partial\zeta_{Im}} = \left(\frac{8\pi f_0}{\sigma^2}\frac{E}{M}\zeta_{Im}\,\mathbb{Im}\left\{\mathrm{e}^{-\jmath 2\pi f_0\tau_{lk}}\mathrm{e}^{\jmath 2\pi f_{lk} n T_s}\mathrm{s_k(n)u_p^*(n)}\right\} + -\frac{4\pi f_0}{\sigma^2}\sqrt{\frac{E}{M}}\,\mathbb{Re}\left\{\mathrm{e}^{-\jmath 2\pi f_0\tau_{lk}}\mathrm{e}^{\jmath 2\pi f_{lk} n T_s}\mathrm{s_k(n)r_p^*(n)}\right\} + \frac{4}{\sigma^2}\frac{E}{M}\zeta_{Im}\,\mathbb{Re}\left\{\dot{\mathrm{s}}_\mathrm{k}(\mathrm{n})\mathrm{e}^{-\jmath 2\pi f_0\tau_{lk}}\mathrm{e}^{\jmath 2\pi f_{lk} n T_s}\mathrm{u_l^*(n)}\right\} + -\frac{2}{\sigma^2}\sqrt{\frac{E}{M}}\,\mathbb{Im}\left\{\dot{\mathrm{s}}_\mathrm{k}\mathrm{e}^{-\jmath 2\pi f_0\tau_{lk}}\mathrm{e}^{\jmath 2\pi f_{lk} n T_s}\mathrm{r_l^*(n)}\right\}\right)\delta[l-p]$$

$$\frac{\partial^2 t_p(n)}{\partial f_{lk}\partial\zeta_{Re}} = \left(\frac{4\pi n T_s}{\sigma^2}\sqrt{\frac{E}{M}}\,\mathbb{Im}\left\{\mathrm{e}^{-\jmath 2\pi f_0\tau_{lk}}\mathrm{e}^{\jmath 2\pi f_{lk} n T_s}\mathrm{s_k(n)r_p^*(n)}\right\} + -\frac{8\pi n T_s}{\sigma^2}\frac{E}{M}\zeta_{Re}\,\mathbb{Im}\left\{\mathrm{e}^{-\jmath 2\pi f_0\tau_{lk}}\mathrm{e}^{\jmath 2\pi f_{lk} n T_s}\mathrm{s_k(n)u_p^*(n)}\right\}\right)\delta[l-p]$$

$$\frac{\partial^2 t_p(n)}{\partial f_{lk}\partial\zeta_{Im}} = \left(\frac{4\pi n T_s}{\sigma^2}\sqrt{\frac{E}{M}}\,\mathbb{Re}\left\{\mathrm{e}^{-\jmath 2\pi f_0\tau_{lk}}\mathrm{e}^{\jmath 2\pi f_{lk} n T_s}\mathrm{s_k(n)r_p^*(n)}\right\} + -\frac{8\pi n T_s}{\sigma^2}\frac{E}{M}\zeta_{Im}\,\mathbb{Im}\left\{\mathrm{e}^{-\jmath 2\pi f_0\tau_{lk}}\mathrm{e}^{\jmath 2\pi f_{lk} n T_s}\mathrm{s_k(n)u_p^*(n)}\right\}\right)\delta[l-p]$$

$$\frac{\partial^2 t_p(n)}{\partial\tau_{lk}\partial\tau_{l'k'}} = \frac{8\pi^2 f_0^2}{\sigma^2}\sqrt{\frac{E}{M}}\,\mathbb{Re}\left\{\zeta\mathrm{s_k(n)}\mathrm{e}^{-\jmath 2\pi f_0\tau_{lk}}\mathrm{e}^{\jmath 2\pi f_{lk} n T_s}\left(\mathrm{r_p^*(n)}\text{-}\sqrt{\frac{\mathrm{E}}{\mathrm{M}}}\zeta^*\mathrm{u_p^*(n)}\right)\right\}\delta[l-p, l-l', k-k'] + + \frac{8\pi^2 f_0^2}{\sigma^2}\frac{E}{M}|\zeta|^2\,\mathbb{Re}\left\{\mathrm{s_k(n)s_{k'}^*(n)}\mathrm{e}^{-\jmath 2\pi f_0(\tau_{lk}-\tau_{l'k'})}\mathrm{e}^{\jmath 2\pi(f_{lk}-f_{l'k'}) n T_s}\right\}\delta[l-p, l'-p] + -\frac{2}{\sigma^2}\sqrt{\frac{E}{M}}\,\mathbb{Re}\left\{\zeta\mathrm{e}^{-\jmath 2\pi f_0\tau_{lk}}\mathrm{e}^{\jmath 2\pi f_{lk} n T_s}\ddot{\mathrm{s}}_\mathrm{k}(\mathrm{n})\left(\mathrm{r_l^*(n)}\text{-}\sqrt{\frac{\mathrm{E}}{\mathrm{M}}}\zeta^*\mathrm{u_l^*(n)}\right)\right\}\delta[l-p, l-l', k-k'] + -\frac{8\pi f_0}{\sigma^2}\sqrt{\frac{E}{M}}\,\mathbb{Im}\left\{\zeta\mathrm{e}^{-\jmath 2\pi f_0\tau_{lk}}\mathrm{e}^{\jmath 2\pi f_{lk} n T_s}\dot{\mathrm{s}}_\mathrm{k}(\mathrm{n})\left(\mathrm{r_l^*(n)}\text{-}\sqrt{\frac{\mathrm{E}}{\mathrm{M}}}\zeta^*\mathrm{u_l^*(n)}\right)\right\}\delta[l-p, l-l', k-k'] + + \frac{2}{\sigma^2}\frac{E}{M}|\zeta|^2\,\mathbb{Re}\left\{\dot{\mathrm{s}}_\mathrm{k}(\mathrm{n})\dot{\mathrm{s}}_{\mathrm{k'}}^*(\mathrm{n})\mathrm{e}^{-\jmath 2\pi f_0(\tau_{lk}-\tau_{l'k'})}\mathrm{e}^{\jmath 2\pi(f_{lk}-f_{l'k'}) n T_s}\right\}\delta[l-p, l-l'] + + \frac{4\pi f_0}{\sigma^2}\frac{E}{M}|\zeta|^2\,\mathbb{Im}\left\{\left(\mathrm{s_k(n)}\dot{\mathrm{s}}_{\mathrm{k'}}(\mathrm{n})\text{-}\dot{\mathrm{s}}_\mathrm{k}(\mathrm{n})\mathrm{s_{k'}^*(n)}\right)\mathrm{e}^{-\jmath 2\pi f_0(\tau_{lk}-\tau_{l'k'})}\mathrm{e}^{\jmath 2\pi(f_{lk}-f_{l'k'}) n T_s}\right\}\delta[l-p, l-l']$$

$$\frac{\partial^2 t_p(n)}{\partial f_{lk}\partial f_{l'k'}} = \frac{8\pi^2 n^2 T_s^2}{\sigma^2}\sqrt{\frac{E}{M}}\,\mathbb{R}\mathrm{e}\left\{\zeta \mathrm{s_k(n)e^{-\jmath 2\pi f_0 \tau_{lk}} e^{\jmath 2\pi f_{lk} n T_s}}\left(\mathrm{r_p^*(n)}\text{-}\sqrt{\frac{\mathrm{E}}{\mathrm{M}}}\zeta^*\mathrm{u_p^*(n)}\right)\right\}\delta[l-p, l-l', k-k']+$$
$$+\frac{8\pi^2 n^2 T_s^2}{\sigma^2}\frac{E}{M}|\zeta|^2\,\mathbb{R}\mathrm{e}\left\{\mathrm{s_k(n)s_{k'}^*(n)e^{-\jmath 2\pi f_0(\tau_{lk}-\tau_{l'k'})}e^{\jmath 2\pi(f_{lk}-f_{l'k'})nT_s}}\right\}\delta[l-p, l'-p]$$
$$\frac{\partial^2 t_p(n)}{\partial \tau_{lk}\partial f_{l'k'}} = -\frac{8\pi^2 f_0 n T_s}{\sigma^2}\sqrt{\frac{E}{M}}\,\mathbb{R}\mathrm{e}\left\{\zeta \mathrm{s_k e^{-\jmath 2\pi f_0 \tau_{lk}} e^{\jmath 2\pi f_{lk} n T_s}}\left(\mathrm{r_p^*(n)}\text{-}\sqrt{\frac{\mathrm{E}}{\mathrm{M}}}\zeta^*\mathrm{u_p^*(n)}\right)\right\}\delta[l-p, l-l', k-k']+$$
$$-\frac{8\pi^2 f_0 n T_s}{\sigma^2}\frac{E}{M}|\zeta|^2\,\mathbb{R}\mathrm{e}\left\{\mathrm{s_k s_{k'}^* e^{-\jmath 2\pi f_0(\tau_{lk}-\tau_{l'k'})}e^{\jmath 2\pi(f_{lk}-f_{l'k'})nT_s}}\right\}\delta[l-p, l'-p]+$$
$$+\frac{4\pi n T_s}{\sigma^2}\sqrt{\frac{E}{M}}\,\mathbb{I}\mathrm{m}\left\{\zeta \mathrm{\dot{s}_k(n)e^{-\jmath 2\pi f_0 \tau_{lk}} e^{\jmath 2\pi f_{lk} n T_s}}\left(\mathrm{r_l^*(n)}\text{-}\sqrt{\frac{\mathrm{E}}{\mathrm{M}}}\zeta^*\mathrm{u_l^*(n)}\right)\right\}\delta[l-p, l-l', k-k']+$$
$$+\frac{4\pi n T_s}{\sigma^2}\frac{E}{M}|\zeta|^2\,\mathbb{I}\mathrm{m}\left\{\mathrm{\dot{s}_k(n)s_{k'}^*(n)e^{-\jmath 2\pi f_0(\tau_{lk}-\tau_{l'k'})}e^{\jmath 2\pi(f_{lk}-f_{l'k'})nT_s}}\right\}\delta[l-p, l-l']$$
$$(3.14)$$

where $\delta[a_1-b_1, a_2-b_2, \ldots] = \delta[a_1-b_1]\delta[a_2-b_2]\ldots$ is defined as a multivariate Kronecker Delta function. Given the function g(t) and the previous derivatives, the elements of the FIM can be calculated as

$$[\boldsymbol{J}(\boldsymbol{\theta})]_{i,j} = -2\frac{2+\nu}{\nu^2}\sum_{n=0}^{N_o-1}\sum_{p=1}^{N} E\left\{\left(1+\frac{2t_p(n)}{\nu}\right)^{-2}\frac{\partial t_p(n)}{\partial\theta_i}\frac{\partial t_p(n)}{\partial\theta_j}\right\}+$$
$$+\frac{2+\nu}{\nu}\sum_{n=0}^{N_o-1}\sum_{p=1}^{N} E\left\{\left(1+\frac{2t_p(n)}{\nu}\right)^{-1}\frac{\partial^2 t_p(n)}{\partial\theta_i\partial\theta_j}\right\}. \quad (3.15)$$

MATRIX $\boldsymbol{\Lambda}_{2x2}$

- $[\boldsymbol{J}(\boldsymbol{\theta})]_{2NM+1,2NM+1}$

$$\frac{\partial t_p(n)}{\partial \zeta_{Re}}\frac{\partial t_p(n)}{\partial \zeta_{Re}} = \frac{4}{\sigma^2}\frac{E^2}{M^2}\zeta_{Re}^2|u_p(n)|^4 - \frac{8}{\sigma^2}\sqrt{\frac{E^3}{M^3}}\zeta_{Re}|u_p(n)|^2\,\mathbb{R}\mathrm{e}\left\{\mathrm{u_p(n)r_p^*(n)}\right\}+$$
$$+\frac{4}{\sigma^2}\frac{E}{M}\,\mathbb{R}\mathrm{e}\left\{\mathrm{u_p(n)r_p^*(n)}\right\}^2 = \frac{4}{\sigma^2}\frac{E^2}{M^2}\zeta_{Re}^2|u_p(n)|^4 - \frac{8}{\sigma^2}\sqrt{\frac{E^3}{M^3}}\zeta_{Re}|u_p(n)|^2\,\mathbb{R}\mathrm{e}\left\{\mathrm{u_p(n)r_p^*(n)}\right\}+$$
$$+\frac{2}{\sigma^2}\frac{E}{M}\left(|u_p(n)|^2|r_p(n)|^2 + \mathbb{R}\mathrm{e}\left\{\mathrm{u_p^2(n)r_p^*(n)r_p^*(n)}\right\}\right). \quad (3.16)$$

Then, from the **APPENDIX**, the expectation is

$$
E\left\{\left(1+\frac{2t_p(n)}{\nu}\right)^{-2}\frac{\partial t_p(n)}{\partial\zeta_{Re}}\frac{\partial t_p(n)}{\partial\zeta_{Re}}\right\}=
$$
$$
=\frac{4}{\sigma^2}\frac{E^2}{M^2}\zeta_{Re}^2|u_p(n)|^4\alpha_1-\frac{8}{\sigma^2}\frac{E^2}{M^2}\zeta_{Re}|u_p(n)|^2\,\mathbb{R}\mathrm{e}\left\{\zeta^*\mathrm{u_p(n)u_p^*(n)}\right\}\alpha_1+
$$
$$
+\frac{2}{\sigma^2}\frac{E}{M}\left(\frac{E}{M}|\zeta|^2|u_p(n)|^4+|u_p(n)|^2\sigma^2\frac{\nu}{\nu+2}+\mathbb{R}\mathrm{e}\left\{\frac{\mathrm{E}}{\mathrm{M}}\zeta^*\zeta^*|\mathrm{u_p(n)}|^4\right\}\right)\alpha_1=\frac{2}{\sigma^2}\frac{E}{M}|u_p(n)|^2\alpha_1\frac{\nu}{\nu+2}.
\tag{3.17}
$$

where $\alpha_1=\frac{\Gamma\left(\frac{2+\nu}{2}\right)\Gamma\left(\frac{4+\nu}{2}\right)(\pi\nu+4\pi)}{\Gamma\left(\frac{6+\nu}{2}\right)\Gamma\left(\frac{\nu}{2}\right)(\pi\nu)}\left(\frac{\nu}{\nu+4}\right)$. Regarding the second order derivative,

$$
E\left\{\left(1+\frac{2t_p(n)}{\nu}\right)^{-1}\frac{\partial^2 t_p(n)}{\partial\zeta_{Re}\partial\zeta_{Re}}\right\}=\frac{2}{\sigma^2}\frac{E}{M}|u_p(n)|^2\alpha_2.
\tag{3.18}
$$

where $\alpha_2=\frac{\Gamma\left(\frac{2+\nu}{2}\right)\Gamma\left(\frac{2+\nu}{2}\right)(\pi\nu+2\pi)}{\Gamma\left(\frac{4+\nu}{2}\right)\Gamma\left(\frac{\nu}{2}\right)(\pi\nu)}\left(\frac{\nu}{\nu+2}\right)$. Finally, after applying the orthogonality and calling $\beta\triangleq\left(\frac{-2\alpha_1+2\alpha_2}{\nu}+\alpha_2\right)$,

$$
\begin{aligned}
&[\boldsymbol{J}(\boldsymbol{\theta})]_{2NM+1,2NM+1}=-2\frac{2+\nu}{\nu^2}\frac{2}{\sigma^2}\frac{E}{M}\alpha_1\frac{\nu}{\nu+2}\sum_{n=0}^{N_o-1}\sum_{p=1}^{N}|u_p(n)|^2+\\
&+\frac{2+\nu}{\nu}\frac{2}{\sigma^2}\frac{E}{M}\alpha_2\sum_{n=0}^{N_o-1}\sum_{p=1}^{N}|u_p(n)|^2=\\
&=\beta\frac{E}{M}\sum_{p=1}^{N}\sum_{n=0}^{N_o-1}\sum_{q=1}^{M}\sum_{q'=1}^{M}e^{-\jmath 2\pi f_0\left(\tau_{pq}-\tau_{pq'}\right)}e^{\jmath 2\pi\left(f_{pq}-f_{pq'}\right)nT_s}s_q(n)s_{q'}^*(n)=\\
&=\beta\frac{E}{M}\sum_{p=1}^{N}\sum_{q=1}^{M}\sum_{n=0}^{N_o-1}|s_q(nT_s-\tau_{pq})|^2=\left(\frac{-2\alpha_1+2\alpha_2}{\nu}+\alpha_2\right)\frac{E}{M}NME_s=\\
&=\beta ENE_s.
\end{aligned}
\tag{3.19}
$$

- $[\boldsymbol{J}(\boldsymbol{\theta})]_{2NM+2,2NM+2}$

$$
\frac{\partial t_p(n)}{\partial\zeta_{Im}}\frac{\partial t_p(n)}{\partial\zeta_{Im}}=\frac{4}{\sigma^2}\frac{E^2}{M^2}\zeta_{Im}^2|u_p(n)|^4+\frac{8}{\sigma^2}\sqrt{\frac{E^3}{M^3}}\zeta_{Im}|u_p(n)|^2\,\mathbb{I}\mathrm{m}\left\{\mathrm{u_p(n)r_p^*(n)}\right\}+
$$
$$
+\frac{4}{\sigma^2}\frac{E}{M}\,\mathbb{I}\mathrm{m}\left\{\mathrm{u_p(n)r_p^*(n)}\right\}^2=\frac{4}{\sigma^2}\frac{E^2}{M^2}\zeta_{Im}^2|u_p(n)|^4+\frac{8}{\sigma^2}\sqrt{\frac{E^3}{M^3}}\zeta_{Im}|u_p(n)|^2\,\mathbb{I}\mathrm{m}\left\{\mathrm{u_p(n)r_p^*(n)}\right\}+
$$
$$
+\frac{2}{\sigma^2}\frac{E}{M}\left(|u_p(n)|^2|r_p(n)|^2-\mathbb{R}\mathrm{e}\left\{\mathrm{u_p^2(n)r_p^*(n)r_p^*(n)}\right\}\right).
\tag{3.20}
$$

By exploiting the similarities with the previous case, plus the fact that the second order derivative is the same, we obtain

$$[\boldsymbol{J}(\boldsymbol{\theta})]_{2NM+2,2NM+2} = [\boldsymbol{J}(\boldsymbol{\theta})]_{2NM+1,2NM+1} . \tag{3.21}$$

- $[\boldsymbol{J}(\boldsymbol{\theta})]_{2NM+1,2NM+2}$

$$\begin{aligned}\frac{\partial t_p(n)}{\partial \zeta_{Re}}\frac{\partial t_p(n)}{\partial \zeta_{Im}} &= \frac{4}{\sigma^2}\frac{E^2}{M^2}\zeta_{Re}\zeta_{Im}|u_p(n)|^4 - \frac{4}{\sigma^2}\sqrt{\frac{E^3}{M^3}}\zeta_{Im}|u_p(n)|^2\,\mathbb{R}\mathrm{e}\left\{\mathrm{u_p(n)r_p^*(n)}\right\}+\\ &+\frac{4}{\sigma^2}\sqrt{\frac{E^3}{M^3}}\zeta_{Re}|u_p(n)|^2\,\mathbb{I}\mathrm{m}\left\{\mathrm{u_p(n)r_p^*(n)}\right\}-\frac{4}{\sigma^2}\frac{E}{M}\,\mathbb{R}\mathrm{e}\left\{\mathrm{u_p(n)r_p^*(n)}\right\}\mathbb{I}\mathrm{m}\left\{\mathrm{u_p(n)r_p^*(n)}\right\}=\\ &= \frac{4}{\sigma^2}\frac{E^2}{M^2}\zeta_{Re}\zeta_{Im}|u_p(n)|^4 - \frac{4}{\sigma^2}\sqrt{\frac{E^3}{M^3}}\zeta_{Im}|u_p(n)|^2\,\mathbb{R}\mathrm{e}\left\{\mathrm{u_p(n)r_p^*(n)}\right\}+\\ &+\frac{4}{\sigma^2}\sqrt{\frac{E^3}{M^3}}\zeta_{Re}|u_p(n)|^2\,\mathbb{I}\mathrm{m}\left\{\mathrm{u_p(n)r_p^*(n)}\right\}-\frac{2}{\sigma^2}\frac{E}{M}\,\mathbb{I}\mathrm{m}\left\{\mathrm{u_p^2(n)r_p^*(n)r_p^*(n)}\right\}.\end{aligned} \tag{3.22}$$

After the expectation,it can be easily shown that all the terms cancel out. Then, recalling that the secon order derivative is also 0,

$$[\boldsymbol{J}(\boldsymbol{\theta})]_{2NM+1,2NM+2} = 0. \tag{3.23}$$

MATRIX $\boldsymbol{V}_{2NMx2}$

- $[\boldsymbol{J}(\boldsymbol{\theta})]_{i,NM+1}, \quad i=(l-1)M+k, \quad l=1\dots N, \quad k=1\dots M$

$$\begin{aligned}\frac{\partial t_p(n)}{\partial \tau_{lk}}\frac{\partial t_p(n)}{\partial \zeta_{Re}} &= \frac{8\pi f_0}{\sigma^4}\frac{E}{M}\,\mathbb{I}\mathrm{m}\left\{\zeta\mathrm{A_{lk}r_l^*s_k}\right\}\mathbb{R}\mathrm{e}\left\{\mathrm{u_pr_p^*}\right\}\delta[l-p]+\\ &-\frac{8\pi f_0}{\sigma^4}\sqrt{\frac{E^3}{M^3}}\zeta_{Re}\,\mathbb{I}\mathrm{m}\left\{\zeta\mathrm{A_{lk}r_l^*s_k}\right\}|u_p|^2\delta[l-p]+\\ &+\frac{4}{\sigma^4}\frac{E}{M}\,\mathbb{R}\mathrm{e}\left\{\zeta\mathrm{A_{lk}r_l^*\dot{s}_k}\right\}\mathbb{R}\mathrm{e}\left\{\mathrm{u_pr_p^*}\right\}\delta[l-p]-\frac{4}{\sigma^4}\sqrt{\frac{E^3}{M^3}}\zeta_{Re}\,\mathbb{R}\mathrm{e}\left\{\zeta\mathrm{A_{lk}r_l^*\dot{s}_k}\right\}|u_p|^2\delta[l-p]+\\ &-\frac{8\pi f_0}{\sigma^4}\sqrt{\frac{E^3}{M^3}}|\zeta|^2\,\mathbb{I}\mathrm{m}\left\{\mathrm{A_{lk}u_l^*s_k}\right\}\mathbb{R}\mathrm{e}\left\{\mathrm{u_pr_p^*}\right\}\delta[l-p]+\frac{8\pi f_0}{\sigma^4}\frac{E^2}{M^2}|\zeta|^2\zeta_{Re}\,\mathbb{I}\mathrm{m}\left\{\mathrm{A_{lk}u_l^*s_k}\right\}|u_p|^2\delta[l-p]+\\ &-\frac{4}{\sigma^4}\sqrt{\frac{E^3}{M^3}}|\zeta|^2\,\mathbb{R}\mathrm{e}\left\{\mathrm{A_{lk}u_l^*\dot{s}_k}\right\}\mathbb{R}\mathrm{e}\left\{\mathrm{u_pr_p^*}\right\}\delta[l-p]+\frac{4}{\sigma^4}\frac{E^2}{M^2}|\zeta|^2\zeta_{Re}\,\mathbb{R}\mathrm{e}\left\{\mathrm{A_{lk}u_l^*\dot{s}_k}\right\}|u_p|^2\delta[l-p],\end{aligned} \tag{3.24}$$

where the first and third terms can be expanded as

$$\frac{4\pi f_0}{\sigma^4}\frac{E}{M}\left[\mathbb{I}\mathrm{m}\left\{\zeta \mathrm{A_{lk}s_k u_l r_l^* r_l^*}\right\} + \mathbb{I}\mathrm{m}\left\{\zeta \mathrm{A_{lk}s_k u_l^* |r_l|^2}\right\}\right]\delta[l-p]+$$
$$+\frac{2}{\sigma^4}\frac{E}{M}\left[\mathbb{R}\mathrm{e}\left\{\zeta \mathrm{A_{lk}\dot{s}_k u_l r_l^* r_l^*}\right\} + \mathbb{R}\mathrm{e}\left\{\zeta \mathrm{A_{lk}\dot{s}_k u_l^* |r_l|^2}\right\}\right]\delta[l-p]. \tag{3.25}$$

Then it can be shown that after the expectation the terms directly proportional to E^2/M^2 disappear, leaving

$$E\left\{\left(1+\frac{2t_p(n)}{\nu}\right)^{-2}\frac{\partial t_p(n)}{\partial \tau_{lk}}\frac{\partial t_p(n)}{\partial \zeta_{Re}}\right\} = \frac{4\pi f_0}{\sigma^2}\frac{E}{M}\frac{\nu}{\nu+2}\alpha_1\,\mathbb{I}\mathrm{m}\left\{\zeta \mathrm{A u_l^* s_k}\right\}\delta[l-p]+$$
$$+\frac{2}{\sigma^2}\frac{E}{M}\frac{\nu}{\nu+2}\alpha_1\,\mathbb{R}\mathrm{e}\left\{\zeta \mathrm{A_{lk} u_l^* \dot{s}_k}\right\}\delta[l-p] \tag{3.26}$$

Regarding the second order derivative,

$$E\left\{\left(1+\frac{2t_p(n)}{\nu}\right)^{-1}\frac{\partial^2 t_p(n)}{\partial \tau_{lk}\partial \zeta_{Re}}\right\} = \frac{4\pi f_0}{\sigma^2}\frac{E}{M}\alpha_2\,\mathbb{I}\mathrm{m}\left\{\zeta \mathrm{A_{lk} u_l^* s_k}\right\}\delta[l-p]+$$
$$+\frac{2}{\sigma^2}\frac{E}{M}\alpha_2\,\mathbb{R}\mathrm{e}\left\{\zeta \mathrm{A_{lk} u_l^* \dot{s}_k}\right\}\delta[l-p] = \frac{\nu+2}{\nu}\frac{\alpha_2}{\alpha_1}E\left\{\left(1+\frac{2t_p(n)}{\nu}\right)^{-2}\frac{\partial t_p(n)}{\partial \tau_{lk}}\frac{\partial t_p(n)}{\partial \zeta_{Re}}\right\} \tag{3.27}$$

Finally, after applying orthogonality, it is

$$
\begin{aligned}
&[\boldsymbol{J}(\boldsymbol{\theta})]_{i,2NM+1} = \left[-2\frac{2+\nu}{\nu^2}+\frac{2+\nu}{\nu}\frac{2+\nu}{\nu}\frac{\alpha_2}{\alpha_1}\right]\sum_{p=1}^{N}\sum_{n=0}^{N_o-1}\left(\frac{4\pi f_0}{\sigma^2}\frac{E}{M}\frac{\nu}{\nu+2}\alpha_1\,\mathbb{Im}\left\{\zeta \mathrm{A u_l^* s_k}\right\}\delta[l-p]+\right.\\
&\left.+\frac{2}{\sigma^2}\frac{E}{M}\frac{\nu}{\nu+2}\alpha_1\,\mathbb{Re}\left\{\zeta \mathrm{A_{lk} u_l^* \dot{s}_k}\right\}\delta[l-p]\right) =\\
&=\beta\left[\frac{4\pi f_0}{\sigma^2}\frac{E}{M}\,\mathbb{Im}\left\{\zeta\sum_{\mathrm{n=0}}^{\mathrm{N_o\text{-}1}}\mathrm{A u_l^* s_k}\right\}+\frac{2}{\sigma^2}\frac{E}{M}\,\mathbb{Re}\left\{\zeta\sum_{\mathrm{n=0}}^{\mathrm{N_o\text{-}1}}\mathrm{A u_l^* \dot{s}_k}\right\}\right] =\\
&=\beta\frac{4\pi f_0}{\sigma^2}\frac{E}{M}\,\mathbb{Im}\left\{\zeta\sum_{\mathrm{q=1}}^{\mathrm{M}}\mathrm{e}^{-\jmath 2\pi \mathrm{f_0}\left(\tau_{\mathrm{lk}}-\tau_{\mathrm{lq}}\right)}\sum_{\mathrm{n=0}}^{\mathrm{N_o\text{-}1}}\mathrm{e}^{\jmath 2\pi\left(\mathrm{f_{lk}}-\mathrm{f_{lq}}\right)\mathrm{nT_s}}\mathrm{s_k(nT_s\text{-}}\tau_{\mathrm{lk}})\mathrm{s_q^*(nT_s\text{-}}\tau_{\mathrm{lq}})\right\}+\\
&+\beta\frac{2}{\sigma^2}\frac{E}{M}\,\mathbb{Re}\left\{\zeta\sum_{\mathrm{q=1}}^{\mathrm{M}}\mathrm{e}^{-\jmath 2\pi \mathrm{f_0}\left(\tau_{\mathrm{lk}}-\tau_{\mathrm{lq}}\right)}\sum_{\mathrm{n=0}}^{\mathrm{N_o\text{-}1}}\mathrm{e}^{\jmath 2\pi\left(\mathrm{f_{lk}}-\mathrm{f_{lq}}\right)\mathrm{nT_s}}\frac{\partial \mathrm{s_k(nT_s\text{-}}\tau_{\mathrm{lk}})}{\partial\tau_{\mathrm{lk}}}\mathrm{s_q^*(nT_s\text{-}}\tau_{\mathrm{lq}})\right\} =\\
&=\beta\frac{E}{M}\frac{4\pi f_0}{\sigma^2}E_s\zeta_{Im}+\beta\frac{2}{\sigma^2}\frac{E}{M}\,\mathbb{Re}\left\{\zeta\sum_{\mathrm{n=0}}^{\mathrm{N_o\text{-}1}}\frac{\partial \mathrm{s_k(nT_s\text{-}}\tau_{\mathrm{lk}})}{\partial\tau_{\mathrm{lk}}}\mathrm{s_k^*(nT_s\text{-}}\tau_{\mathrm{lk}})\right\}
\end{aligned}
\tag{3.28}
$$

The analytical solution of the second summation, together with the summations left unresolved in this chapter, is presented in the APPENDIX II. Orthogonality is maintained even in presence of the derivative because if $\tau_{lk} \neq \tau_{lq}$, $\partial\tau_{lk}$ can be moved out of the summation over n.

- $[\boldsymbol{J}(\boldsymbol{\theta})]_{i,NM+2}$, $\quad i=(l-1)M+k, \quad l=1\ldots N, \quad k=1\ldots M$

$$
\begin{aligned}
&\frac{\partial t_p(n)}{\partial\tau_{lk}}\frac{\partial t_p(n)}{\partial\zeta_{Im}} = -\frac{8\pi f_0}{\sigma^4}\frac{E}{M}\,\mathbb{Im}\left\{\zeta \mathrm{A_{lk} r_l^* s_k}\right\}\mathbb{Im}\left\{\mathrm{u_p r_p^*}\right\}\delta[l-p]+\\
&-\frac{8\pi f_0}{\sigma^4}\sqrt{\frac{E^3}{M^3}}\zeta_{Im}\,\mathbb{Im}\left\{\zeta \mathrm{A_{lk} r_l^* s_k}\right\}|u_p|^2\delta[l-p]-\frac{4}{\sigma^4}\frac{E}{M}\,\mathbb{Re}\left\{\zeta \mathrm{A_{lk} r_l^* \dot{s}_k}\right\}\mathbb{Im}\left\{\mathrm{u_p r_p^*}\right\}\delta[l-p]+\\
&-\frac{4}{\sigma^4}\sqrt{\frac{E^3}{M^3}}\zeta_{Im}\,\mathbb{Re}\left\{\zeta \mathrm{A_{lk} r_l^* \dot{s}_k}\right\}|u_p|^2\delta[l-p]+\frac{8\pi f_0}{\sigma^4}\sqrt{\frac{E^3}{M^3}}|\zeta|^2\,\mathbb{Im}\left\{\mathrm{A_{lk} u_l^* s_k}\right\}\mathbb{Im}\left\{\mathrm{u_p r_p^*}\right\}\delta[l-p]+\\
&+\frac{8\pi f_0}{\sigma^4}\frac{E^2}{M^2}|\zeta|^2\zeta_{Im}\,\mathbb{Im}\left\{\mathrm{A_{lk} u_l^* s_k}\right\}|u_p|^2\delta[l-p]+\frac{4}{\sigma^4}\sqrt{\frac{E^3}{M^3}}|\zeta|^2\,\mathbb{Re}\left\{\mathrm{A_{lk} u_l^* \dot{s}_k}\right\}\mathbb{Im}\left\{\mathrm{u_p r_p^*}\right\}\delta[l-p]+\\
&+\frac{4}{\sigma^4}\frac{E^2}{M^2}|\zeta|^2\zeta_{Im}\,\mathbb{Re}\left\{\mathrm{A_{lk} u_l^* \dot{s}_k}\right\}|u_p|^2\delta[l-p]
\end{aligned}
\tag{3.29}
$$

where the first and second term can be expanded as

$$
\begin{aligned}
&+\frac{4\pi f_0}{\sigma^2}\frac{E}{M}\left[\mathbb{R}\mathrm{e}\left\{\mathrm{u_p r_p^* \zeta A s_k r_p^*}\right\} - \mathbb{R}\mathrm{e}\left\{\mathrm{u_p^* r_p \zeta A s_k r_p^*}\right\}\right]\delta[l-p]+\\
&+\frac{2}{\sigma^4}\frac{E}{M}\left[-\mathbb{I}\mathrm{m}\left\{\zeta \mathrm{A_{lk}\dot{s}_k u_l r_l^* r_l^*}\right\} + \mathbb{I}\mathrm{m}\left\{\zeta \mathrm{A_{lk}\dot{s}_k u_l^* |r_l|^2}\right\}\right]\delta[l-p].
\end{aligned}
\tag{3.30}
$$

By considering the similarities with the previous case, the expectations can be derived easily, and lead to

$$
\begin{aligned}
E\left\{\left(1+\frac{2t_p(n)}{\nu}\right)^{-2}\frac{\partial t_p(n)}{\partial \tau_{lk}}\frac{\partial t_p(n)}{\partial \zeta_{Im}}\right\} &= -\frac{4\pi f_0}{\sigma^2}\frac{E}{M}\frac{\nu}{\nu+2}\alpha_1\,\mathbb{R}\mathrm{e}\left\{\zeta \mathrm{A u_l^* s_k}\right\}\delta[l-p]+\\
&+\frac{2}{\sigma^2}\frac{E}{M}\frac{\nu}{\nu+2}\alpha_1\,\mathbb{I}\mathrm{m}\left\{\zeta \mathrm{A_{lk} u_l^* \dot{s}_k}\right\}\delta[l-p]\\
E\left\{\left(1+\frac{2t_p(n)}{\nu}\right)^{-1}\frac{\partial^2 t_p(n)}{\partial \tau_{lk}\partial \zeta_{Im}}\right\} &= \frac{\nu+2}{\nu}\frac{\alpha_2}{\alpha_1}E\left\{\left(1+\frac{2t_p(n)}{\nu}\right)^{-2}\frac{\partial t_p(n)}{\partial \tau_{lk}}\frac{\partial t_p(n)}{\partial \zeta_{Im}}\right\}
\end{aligned}
\tag{3.31}
$$

meaning that

$$
\begin{aligned}
&[\boldsymbol{J}(\boldsymbol{\theta})]_{i,2NM+2} = \left[-2\frac{2+\nu}{\nu^2} + \frac{2+\nu}{\nu}\frac{2+\nu}{\nu}\frac{\alpha_2}{\alpha_1}\right]\times\\
&\times\sum_{p=1}^{N}\sum_{n=0}^{N_o-1}\left(\frac{2}{\sigma^2}\frac{E}{M}\frac{\nu}{\nu+2}\alpha_1\,\mathbb{I}\mathrm{m}\left\{\zeta \mathrm{A_{lk} u_l^* \dot{s}_k}\right\} - \frac{4\pi f_0}{\sigma^2}\frac{E}{M}\frac{\nu}{\nu+2}\alpha_1\,\mathbb{R}\mathrm{e}\left\{\zeta \mathrm{A u_l^* s_k}\right\}\right)\delta[l-p] =\\
&=\beta\left[-\frac{4\pi f_0}{\sigma^2}\frac{E}{M}\,\mathbb{R}\mathrm{e}\left\{\zeta\sum_{\mathrm{n=0}}^{\mathrm{N_o\text{-}1}}\mathrm{A u_l^* s_k}\right\} + \frac{2}{\sigma^2}\frac{E}{M}\,\mathbb{I}\mathrm{m}\left\{\zeta\sum_{\mathrm{n=0}}^{\mathrm{N_o\text{-}1}}\mathrm{A u_l^* \dot{s}_k}\right\}\right] =\\
&=\beta\frac{2}{\sigma^2}\frac{E}{M}\,\mathbb{I}\mathrm{m}\left\{\zeta\sum_{\mathrm{q=1}}^{\mathrm{M}}\mathrm{e}^{\text{-}\jmath 2\pi \mathrm{f_0}(\tau_{\mathrm{lk}}\text{-}\tau_{\mathrm{lq}})}\sum_{\mathrm{n=0}}^{\mathrm{N_o\text{-}1}}\mathrm{e}^{\jmath 2\pi\left(\mathrm{f_{lk}\text{-}f_{lq}}\right)\mathrm{nT_s}}\frac{\partial \mathrm{s_k(nT_s\text{-}}\tau_{\mathrm{lk}})}{\partial \tau_{\mathrm{lk}}}\mathrm{s_q^*(nT_s\text{-}}\tau_{\mathrm{lq}})\right\}+\\
&-\beta\frac{4\pi f_0}{\sigma^2}\frac{E}{M}\,\mathbb{R}\mathrm{e}\left\{\zeta\sum_{\mathrm{q=1}}^{\mathrm{M}}\mathrm{e}^{\text{-}\jmath 2\pi \mathrm{f_0}\left(\tau_{\mathrm{lk}}\text{-}\tau_{\mathrm{lq}}\right)}\sum_{\mathrm{n=0}}^{\mathrm{N_o\text{-}1}}\mathrm{e}^{\jmath 2\pi\left(\mathrm{f_{lk}\text{-}f_{lq}}\right)\mathrm{nT_s}}\mathrm{s_k(nT_s\text{-}}\tau_{\mathrm{lk}})\mathrm{s_q^*(nT_s\text{-}}\tau_{\mathrm{lq}})\right\} =\\
&=-\beta\left(\frac{4\pi f_0}{\sigma^2}\frac{E}{M}E_s\zeta_{Re} - \frac{2}{\sigma^2}\frac{E}{M}\,\mathbb{I}\mathrm{m}\left\{\zeta\sum_{\mathrm{n=0}}^{\mathrm{N_o\text{-}1}}\frac{\partial \mathrm{s_k(nT_s\text{-}}\tau_{\mathrm{lk}})}{\partial \tau_{\mathrm{lk}}}\mathrm{s_k^*(nT_s\text{-}}\tau_{\mathrm{lk}})\right\}\right).
\end{aligned}
\tag{3.32}
$$

- $[\boldsymbol{J}(\boldsymbol{\theta})]_{i,2NM+1}, \quad i=(N+l-1)M+k, \quad l=1\dots N, \quad k=1\dots M$

$$\begin{aligned}\frac{\partial t_p(n)}{\partial \zeta_{Re}}\frac{\partial t_p(n)}{\partial f_{lk}} &= -\frac{8\pi n T_s}{\sigma^2}\frac{E^2}{M^2}|\zeta|^2\zeta_{Re}|u_p(n)|^2\,\mathbb{Im}\left\{\mathrm{e}^{-\jmath 2\pi f_0\tau_{lk}}\mathrm{e}^{\jmath 2\pi f_{lk} n T_s}\mathrm{s_k(n)u_p^*(n)}\right\}\delta[l-p]+\\ &+\frac{8\pi n T_s}{\sigma^2}\sqrt{\frac{E^3}{M^3}}|\zeta|^2\,\mathbb{Re}\left\{\mathrm{u_p(n)r_p^*(n)}\right\}\mathbb{Im}\left\{\mathrm{e}^{-\jmath 2\pi f_0\tau_{lk}}\mathrm{e}^{\jmath 2\pi f_{lk} n T_s}\mathrm{s_k(n)u_p^*(n)}\right\}\delta[l-p]+\\ &+\frac{8\pi n T_s}{\sigma^2}\sqrt{\frac{E^3}{M^3}}\zeta_{Re}|u_p(n)|^2\,\mathbb{Im}\left\{\zeta\mathrm{e}^{-\jmath 2\pi f_0\tau_{lk}}\mathrm{e}^{\jmath 2\pi f_{lk} n T_s}\mathrm{s_k(n)r_p^*(n)}\right\}\delta[l-p]+\\ &-\frac{8\pi n T_s}{\sigma^2}\frac{E}{M}\,\mathbb{Re}\left\{\mathrm{u_p(n)r_p^*(n)}\right\}\mathbb{Im}\left\{\zeta\mathrm{e}^{-\jmath 2\pi f_0\tau_{lk}}\mathrm{e}^{\jmath 2\pi f_{lk} n T_s}\mathrm{s_k(n)r_p^*(n)}\right\}\delta[l-p]\end{aligned} \tag{3.33}$$

where the last term can be expanded as

$$\begin{aligned}&-\frac{4\pi n T_s}{\sigma^2}\frac{E}{M}\,\mathbb{Im}\left\{\mathrm{u_p(n)r_p^*(n)}\zeta\mathrm{e}^{-\jmath 2\pi f_0\tau_{lk}}\mathrm{e}^{\jmath 2\pi f_{lk} n T_s}\mathrm{s_k(n)r_p^*(n)}\right\}\delta[l-p]+\\ &-\frac{4\pi n T_s}{\sigma^2}\frac{E}{M}\,\mathbb{Im}\left\{\mathrm{u_p^*(n)r_p(n)}\zeta\mathrm{e}^{-\jmath 2\pi f_0\tau_{lk}}\mathrm{e}^{\jmath 2\pi f_{lk} n T_s}\mathrm{s_k(n)r_p^*(n)}\right\}\delta[l-p].\end{aligned} \tag{3.34}$$

Then, by considering the similarities with the delay parameter,

$$\begin{aligned}E\left\{\left(1+\frac{2t_p(n)}{\nu}\right)^{-2}\frac{\partial t_p(n)}{\partial \zeta_{Re}}\frac{\partial t_p(n)}{\partial f_{lk}}\right\} &= -\frac{4\pi n T_s}{\sigma^2}\frac{E}{M}\frac{\nu}{\nu+2}\alpha_1\,\mathbb{Im}\left\{\zeta\mathrm{e}^{-\jmath 2\pi f_0\tau_{lk}}\mathrm{e}^{\jmath 2\pi f_{lk} n T_s}\mathrm{s_k(n)u_p^*(n)}\right\}\delta[l-p]\\ E\left\{\left(1+\frac{2t_p(n)}{\nu}\right)^{-1}\frac{\partial^2 t_p(n)}{\partial \zeta_{Re}\partial\tau_{lk}}\right\} &= \frac{\nu+2}{\nu}\frac{\alpha_2}{\alpha_1}E\left\{\left(1+\frac{2t_p(n)}{\nu}\right)^{-2}\frac{\partial t_p(n)}{\partial \zeta_{Re}}\frac{\partial t_p(n)}{\partial f_{lk}}\right\}\end{aligned} \tag{3.35}$$

and the final result is

$$\begin{aligned}[\boldsymbol{J}(\boldsymbol{\theta})]_{i,2NM+1} &= \beta\frac{8\pi T_s}{\nu\sigma^2}\frac{E}{M}\alpha_1\sum_{n=0}^{N_o-1}\mathbb{Im}\left\{\zeta\mathrm{e}^{-\jmath 2\pi f_0\tau_{lk}}\mathrm{e}^{\jmath 2\pi f_{lk} n T_s}\mathrm{n s_k(n)u_l^*(n)}\right\}=\\ &=\beta\frac{8\pi T_s}{\sigma^2}\frac{E}{M}\sum_{n=0}^{N_o-1}\sum_{q=1}^{M}\mathbb{Im}\left\{\zeta\mathrm{e}^{-\jmath 2\pi f_0(\tau_{lk}-\tau_{lq})}\mathrm{e}^{\jmath 2\pi (f_{lk}-f_{pq}) n T_s}\mathrm{n s_k(nT_s\text{-}\tau_{lk})s_q^*(nT_s\text{-}\tau_{lq})}\right\}=\\ &=\beta\frac{8\pi T_s}{\sigma^2}\frac{E}{M}\,\mathbb{Im}\left\{\zeta\sum_{\mathrm{n=0}}^{\mathrm{N_o\text{-}1}}\mathrm{n}|\mathrm{s_k(nT_s\text{-}\tau_{lk})}|^2\right\}.\end{aligned} \tag{3.36}$$

- $[\boldsymbol{J}(\boldsymbol{\theta})]_{i,2NM+2}$, $\quad i=(N+l-1)M+k, \quad l=1\dots N, \quad k=1\dots M$

By considering the similarities with the previous case, it can be shown that it is

$$[\boldsymbol{J}(\boldsymbol{\theta})]_{i,2NM+2} = -\beta \frac{8\pi T_s}{\sigma^2}\frac{E}{M}\,\mathbb{Re}\left\{\zeta \sum_{\mathrm{n=0}}^{\mathrm{N_o\text{-}1}} \mathrm{n}|\mathrm{s_k}(\mathrm{nT_s}\text{-}\tau_{\mathrm{lk}})|^2\right\}. \tag{3.37}$$

Although the summation over N_o is not the scalar product of two transmitted signals due to the presence of n, it will be shown in Chapter 4 that for the chosen classes of signals orthogonality is still maintained in this case, and also in the case of n^2.

MATRIX $\boldsymbol{S}_{2NMx2NM}$

- $[\boldsymbol{J}(\boldsymbol{\theta})]_{i,i'}$, $\quad i=(l-1)M+k, \quad i'=(l'-1)M+k', \quad l,l'=1\dots N, \quad k,k'=1\dots M$

$$\begin{aligned}
&\frac{\partial t_p(n)}{\partial \tau_{lk}}\frac{\partial t_p(n)}{\partial \tau_{l'k'}} = \\
&= \frac{16\pi^2 f_0^2}{\sigma^4}\frac{E}{M}\,\mathbb{Im}\left\{\zeta \mathrm{A_{lk}r_l^*s_k}\right\}\mathbb{Im}\left\{\zeta \mathrm{r_{l'}^*s_{k'}}\right\}\delta + \frac{8\pi f_0}{\sigma^4}\frac{E}{M}\,\mathbb{Im}\left\{\zeta \mathrm{A_{lk}r_l^*s_k}\right\}\mathbb{Re}\left\{\zeta \mathrm{A_{l'k'}r_{l'}^*\dot{s}_{k'}}\right\}\delta + \\
&- \frac{16\pi^2 f_0^2}{\sigma^4}\sqrt{\frac{E^3}{M^3}}|\zeta|^2\,\mathbb{Im}\left\{\zeta \mathrm{A_{lk}r_l^*s_k}\right\}\mathbb{Im}\left\{\mathrm{A_{l'k'}u_{l'}^*s_{k'}}\right\}\delta + \frac{4}{\sigma^4}\frac{E}{M}\,\mathbb{Re}\left\{\zeta \mathrm{A_{lk}r_l^*\dot{s}_k}\right\}\mathbb{Re}\left\{\zeta \mathrm{A_{l'k'}r_{l'}^*\dot{s}_{k'}}\right\}\delta + \\
&+ \frac{8\pi f_0}{\sigma^4}\frac{E}{M}\,\mathbb{Re}\left\{\zeta \mathrm{A_{lk}r_l^*\dot{s}_k}\right\}\mathbb{Im}\left\{\zeta \mathrm{A_{l'k'}r_{l'}^*s_{k'}}\right\}\delta - \frac{8\pi f_0}{\sigma^4}\sqrt{\frac{E^3}{M^3}}|\zeta|^2\,\mathbb{Im}\left\{\zeta \mathrm{A_{lk}r_l^*s_k}\right\}\mathbb{Re}\left\{\mathrm{A_{l'k'}u_{l'}^*\dot{s}_{k'}}\right\}\delta + \\
&- \frac{8\pi f_0}{\sigma^4}\sqrt{\frac{E^3}{M^3}}|\zeta|^2\,\mathbb{Re}\left\{\zeta \mathrm{A_{lk}r_l^*\dot{s}_k}\right\}\mathbb{Im}\left\{\mathrm{A_{l'k'}u_{l'}^*s_{k'}}\right\}\delta - \frac{4}{\sigma^4}\sqrt{\frac{E^3}{M^3}}|\zeta|^2\,\mathbb{Re}\left\{\zeta \mathrm{A_{lk}r_l^*\dot{s}_k}\right\}\mathbb{Re}\left\{\mathrm{A_{l'k'}u_{l'}^*\dot{s}_{k'}}\right\}\delta + \\
&- \frac{16\pi^2 f_0^2}{\sigma^4}\sqrt{\frac{E^3}{M^3}}|\zeta|^2\,\mathbb{Im}\left\{\mathrm{A_{lk}u_l^*s_k}\right\}\mathbb{Im}\left\{\zeta \mathrm{A_{l'k'}r_{l'}^*s_{k'}}\right\}\delta + \frac{4}{\sigma^4}\frac{E^2}{M^2}|\zeta|^4\,\mathbb{Re}\left\{\mathrm{A_{lk}u_l^*\dot{s}_k}\right\}\mathbb{Re}\left\{\mathrm{A_{l'k'}u_{l'}^*\dot{s}_{k'}}\right\}\delta + \\
&+ \frac{16\pi^2 f_0^2}{\sigma^4}\frac{E^2}{M^2}|\zeta|^4\,\mathbb{Im}\left\{\mathrm{A_{lk}u_l^*s_k}\right\}\mathbb{Im}\left\{\mathrm{A_{l'k'}u_{l'}^*s_{k'}}\right\}\delta + \frac{8\pi f_0}{\sigma^4}\frac{E^2}{M^2}|\zeta|^4\,\mathbb{Im}\left\{\mathrm{A_{lk}u_l^*s_k}\right\}\mathbb{Re}\left\{\mathrm{A_{l'k'}u_{l'}^*\dot{s}_{k'}}\right\}\delta + \\
&- \frac{8\pi f_0}{\sigma^4}\sqrt{\frac{E^3}{M^3}}|\zeta|^2\,\mathbb{Re}\left\{\mathrm{A_{lk}u_l^*\dot{s}_k}\right\}\mathbb{Im}\left\{\zeta \mathrm{A_{l'k'}r_{l'}^*s_{k'}}\right\}\delta - \frac{4}{\sigma^4}\sqrt{\frac{E^3}{M^3}}|\zeta|^2\,\mathbb{Re}\left\{\mathrm{A_{lk}u_l^*\dot{s}_k}\right\}\mathbb{Re}\left\{\zeta \mathrm{A_{l'k'}r_{l'}^*\dot{s}_{k'}}\right\}\delta + \\
&+ \frac{8\pi f_0}{\sigma^4}\frac{E^2}{M^2}|\zeta|^4\,\mathbb{Re}\left\{\mathrm{A_{lk}u_l^*\dot{s}_k}\right\}\mathbb{Im}\left\{\mathrm{A_{l'k'}u_{l'}^*s_{k'}}\right\}\delta - \frac{8\pi f_0}{\sigma^4}\sqrt{\frac{E^3}{M^3}}|\zeta|^2\,\mathbb{Im}\left\{\mathrm{A_{lk}u_l^*s_k}\right\}\mathbb{Re}\left\{\zeta \mathrm{A_{l'k'}r_{l'}^*\dot{s}_{k'}}\right\}\delta
\end{aligned} \tag{3.38}$$

where $\delta \triangleq \delta[l-p, l'-p]$. As seen before, the first, second, fourth and fifth term can be expanded as

$$
\begin{aligned}
&+\frac{8\pi^2 f_0^2}{\sigma^4}\frac{E}{M}\left[-\mathbb{R}\mathrm{e}\left\{\zeta \mathrm{A s_k r_l^*} \zeta \mathrm{A_{lk}' s_{k'} r_l^*}\right\} + \mathbb{R}\mathrm{e}\left\{\zeta \mathrm{A s_k r_l^*} \zeta^* \mathrm{A^*_{l'k'} s^*_{k'} r_l}\right\}\right]\delta[l-p, l'-p]+\\
&+\frac{4\pi f_0}{\sigma^4}\frac{E}{M}\left[\mathbb{I}\mathrm{m}\left\{\zeta \mathrm{A_{lk} s_k r_l^*} \zeta \mathrm{A_{l'k'} \dot{s}_{k'} r_l^*}\right\} + \mathbb{I}\mathrm{m}\left\{\zeta \mathrm{A_{lk} s_k r_l^*} \zeta^* \mathrm{A^*_{l'k'} \dot{s}^*_{k'} r_l}\right\}\right]\delta[l-p, l'-p]+\\
&+\frac{4\pi f_0}{\sigma^4}\frac{E}{M}\left[\mathbb{I}\mathrm{m}\left\{\zeta \mathrm{A_{lk} \dot{s}_k r_l^*} \zeta \mathrm{A_{l'k'} s_{k'} r_l^*}\right\} - \mathbb{I}\mathrm{m}\left\{\zeta \mathrm{A_{lk} \dot{s}_k r_l^*} \zeta^* \mathrm{A^*_{l'k'} s^*_{k'} r_l}\right\}\right]\delta[l-p, l'-p]+\\
&+\frac{2}{\sigma^4}\frac{E}{M}\left[\mathbb{R}\mathrm{e}\left\{\zeta \mathrm{A_{lk} \dot{s}_k r_l^*} \zeta \mathrm{A_{lk}' \dot{s}_{k'} r_l^*}\right\} + \mathbb{R}\mathrm{e}\left\{\zeta \mathrm{A_{lk} \dot{s}_k r_l^*} \zeta^* \mathrm{A^*_{l'k'} \dot{s}^*_{k'} r_l}\right\}\right]\delta[l-p, l'-p].
\end{aligned}
\tag{3.39}
$$

It can then be shown that, after performing the expectation, the terms proportional to $\frac{E^2}{M^2}$ cancel out, leaving

$$
\begin{aligned}
&E\left\{\left(1+\frac{2t_p(n)}{\nu}\right)^{-2}\frac{\partial t_p(n)}{\partial \tau_{lk}}\frac{\partial t_p(n)}{\partial \tau_{l'k'}}\right\} = \frac{8\pi^2 f_0^2}{\sigma^2}\frac{E}{M}|\zeta|^2\frac{\nu}{\nu+2}\alpha_1\,\mathbb{R}\mathrm{e}\left\{\mathrm{AA'^* s_k s^*_{k'}}\right\}\delta[l-p, l'-p]+\\
&+\frac{2}{\sigma^2}\frac{E}{M}|\zeta|^2\frac{\nu}{\nu+2}\alpha_1\,\mathbb{R}\mathrm{e}\left\{\mathrm{AA'^* \dot{s}_k \dot{s}^*_{k'}}\right\}\delta[l-p, l'-p]+\\
&+\frac{4\pi f_0}{\sigma^2}\frac{E}{M}|\zeta|^2\frac{\nu}{\nu+2}\alpha_1\left[\mathbb{I}\mathrm{m}\left\{\mathrm{AA'^* s_k \dot{s}^*_{k'}}\right\} - \mathbb{I}\mathrm{m}\left\{\mathrm{AA'^* \dot{s}_k s^*_{k'}}\right\}\right]\delta[l-p, l'-p].
\end{aligned}
\tag{3.40}
$$

Regarding the second order derivative,

$$
\begin{aligned}
&E\left\{\left(1+\frac{2t_p(n)}{\nu}\right)^{-1}\frac{\partial^2 t_p(n)}{\partial \tau_{lk}\partial \tau_{l'k'}}\right\} = \frac{8\pi^2 f_0^2}{\sigma^2}\frac{E}{M}|\zeta|^2\alpha_2\,\mathbb{R}\mathrm{e}\left\{\mathrm{AA'^* s_k s^*_{k'}}\right\}\delta[l-p, l'-p]+\\
&+\frac{2}{\sigma^2}\frac{E}{M}|\zeta|^2\alpha_2\,\mathbb{R}\mathrm{e}\left\{\mathrm{AA'^* \dot{s}_k \dot{s}^*_{k'}}\right\}\delta[l-p, l'-p] + \frac{4\pi f_0}{\sigma^2}\frac{E}{M}|\zeta|^2\alpha_2\left[\mathbb{I}\mathrm{m}\left\{\mathrm{AA'^* s_k \dot{s}^*_{k'}}\right\}+\right.\\
&\left.-\,\mathbb{I}\mathrm{m}\left\{\mathrm{AA'^* \dot{s}_k s^*_{k'}}\right\}\right]\delta[l-p, l'-p] = \frac{\nu+2}{\nu}\frac{\alpha_2}{\alpha_1}E\left\{\left(1+\frac{2t_p(n)}{\nu}\right)^{-2}\frac{\partial t_p(n)}{\partial \tau_{lk}}\frac{\partial t_p(n)}{\partial \tau_{l'k'}}\right\}.
\end{aligned}
\tag{3.41}
$$

Then, by using the orthogonality assumption it is

$$
\begin{aligned}
\left[\boldsymbol{J}\left(\boldsymbol{\theta}\right)\right]_{i,i'} &= \left[-2\frac{2+\nu}{\nu^2}+\frac{2+\nu}{\nu}\frac{2+\nu}{\nu}\frac{\alpha_2}{\alpha_1}\right]\times \\
&\times\sum_{p=1}^{N}\sum_{n=0}^{N_o-1}\left(\frac{8\pi^2 f_0^2}{\sigma^2}\frac{E}{M}|\zeta|^2\frac{\nu}{\nu+2}\alpha_1\,\mathbb{Re}\left\{\mathrm{AA'^*s_k s_{k'}^*}\right\}+\frac{2}{\sigma^2}\frac{E}{M}|\zeta|^2\frac{\nu\alpha_1}{\nu+2}\,\mathbb{Re}\left\{\mathrm{AA'^*\dot{s}_k\dot{s}_{k'}^*}\right\}+\right. \\
&+\left.\frac{4\pi f_0}{\sigma^2}\frac{E}{M}|\zeta|^2\frac{\nu\alpha_1}{\nu+2}\left[\mathbb{Im}\left\{\mathrm{AA'^*s_k\dot{s}_{k'}^*}\right\}-\mathbb{Im}\left\{\mathrm{AA'^*\dot{s}_k s_{k'}^*}\right\}\right]\right)\delta[l-p,l'-p]= \\
&=\beta\frac{8\pi^2 f_0^2}{\sigma^2}\frac{E}{M}|\zeta|^2\,\mathbb{Re}\left\{\mathrm{e}^{-\jmath 2\pi \mathrm{f_0}(\tau_{\mathrm{lk}}-\tau_{\mathrm{lk'}})}\sum_{\mathrm{n=0}}^{\mathrm{N_o}}\mathrm{e}^{\jmath 2\pi(\mathrm{f_{lk}}-\mathrm{f_{lk'}})\mathrm{nT_s}}\mathrm{s_k(nT_s}-\tau_{\mathrm{lk}})\mathrm{s_{k'}^*(nT_s}-\tau_{\mathrm{lk'}}\right\}\delta[l-l']+ \\
&+\beta\frac{2}{\sigma^2}\frac{E}{M}|\zeta|^2\,\mathbb{Re}\left\{\mathrm{e}^{-\jmath 2\pi \mathrm{f_0}(\tau_{\mathrm{lk}}-\tau_{\mathrm{lk'}})}\sum_{\mathrm{n=0}}^{\mathrm{N_o}-1}\mathrm{e}^{\jmath 2\pi(\mathrm{f_{lk}}-\mathrm{f_{lk'}})\mathrm{nT_s}}\frac{\partial\mathrm{s_k(nT_s}-\tau_{\mathrm{lk}})}{\partial\tau_{\mathrm{lk}}}\frac{\partial\mathrm{s_{k'}^*(nT_s}-\tau_{\mathrm{lk'}}}{\partial\tau_{\mathrm{lk'}}}\right\}\delta[l-l']+ \\
&+\beta\frac{4\pi f_0}{\sigma^2}\frac{E}{M}|\zeta|^2\,\mathbb{Im}\left\{\mathrm{e}^{-\jmath 2\pi \mathrm{f_0}(\tau_{\mathrm{lk}}-\tau_{\mathrm{lk'}})}\sum_{\mathrm{n=0}}^{\mathrm{N_o}-1}\mathrm{e}^{\jmath 2\pi(\mathrm{f_{lk}}-\mathrm{f_{lk'}})\mathrm{nT_s}}\mathrm{s_k(nT_s}-\tau_{\mathrm{lk}})\frac{\partial\mathrm{s_{k'}^*(nT_s}-\tau_{\mathrm{lk'}})}{\partial\tau_{\mathrm{lk'}}}\right\}\delta[l-l']+ \\
&-\beta\frac{4\pi f_0}{\sigma^2}\frac{E}{M}|\zeta|^2\,\mathbb{Im}\left\{\mathrm{e}^{-\jmath 2\pi \mathrm{f_0}(\tau_{\mathrm{lk}}-\tau_{\mathrm{lk'}})}\sum_{\mathrm{n=0}}^{\mathrm{N_o}-1}\mathrm{e}^{\jmath 2\pi(\mathrm{f_{lk}}-\mathrm{f_{lk'}})\mathrm{nT_s}}\frac{\partial\mathrm{s_k(nT_s}-\tau_{\mathrm{lk}})}{\partial\tau_{\mathrm{lk}}}\mathrm{s_{k'}^*(nT_s}-\tau_{\mathrm{lk'}})\right\}\delta[l-l']= \\
&=\beta\left(\frac{8\pi^2 f_0^2}{\sigma^2}\frac{E}{M}|\zeta|^2 E_s+\frac{2}{\sigma^2}\frac{E}{M}|\zeta|^2\,\mathbb{Re}\left\{\sum_{\mathrm{n=0}}^{\mathrm{N_o}-1}\left|\frac{\partial\mathrm{s_k(nT_s}-\tau_{\mathrm{lk}}}{\partial\tau_{\mathrm{lk}}}\right|\right\}\right)\delta[l-l',k-k'].
\end{aligned}
\tag{3.42}
$$

since the last two terms become equal and opposite.

- $\left[\boldsymbol{J}\left(\boldsymbol{\theta}\right)\right]_{i,i'}$, $i=(N+l-1)M+k$, $i'=(N+l'-1)M+k'$, $l,l'=1\dots N$, $k,k'=1\dots M$

The same considerations made for the cross term between the imaginary part of the target reflectivity and the Doppler shifts apply here, w.r.t. the previous case.

$$
\left[\boldsymbol{J}\left(\boldsymbol{\theta}\right)\right]_{i,i'}=\beta\frac{8\pi^2 T_s^2}{\sigma^2}\frac{E}{M}|\zeta|^2\,\mathbb{Re}\left\{\sum_{\mathrm{n=0}}^{\mathrm{N_o}-1}\mathrm{n}^2\left|\mathrm{s_k(nT_s}-\tau_{\mathrm{lk}})\right|^2\right\}\delta[l-l',k-k'].
\tag{3.43}
$$

- $\left[\boldsymbol{J}\left(\boldsymbol{\theta}\right)\right]_{i,i'}$, $i=(l-1)M+k$, $i'=(N+l'-1)M+k'$, $l,l'=1\dots N$, $k,k'=1\dots M$

$$
\begin{aligned}
\frac{\partial t_p(n)}{\partial \tau_{lk}}\frac{\partial t_p(n)}{\partial f_{l'k'}} &= -\frac{16\pi^2 f_0 n T_s}{\sigma^4}\frac{E}{M}\,\mathbb{Im}\left\{\zeta \mathrm{A_{lk}} \mathrm{r_l^*} \mathrm{s_k}\right\}\mathbb{Im}\left\{\zeta \mathrm{A_{lk}}\text{'}\mathrm{s_{k'}}\mathrm{r_{l'}^*}\right\}\delta[l-p,l'-p]+\\
&+\frac{16\pi^2 f_0 n T_s}{\sigma^4}\sqrt{\frac{E^3}{M^3}}|\zeta|^2\,\mathbb{Im}\left\{\zeta \mathrm{A_{lk}}\mathrm{r_l^*}\right\}\mathbb{Im}\left\{\mathrm{A_{l'k'}}\mathrm{s_{k'}}\mathrm{u_{l'}^*}\right\}\delta[l-p,l'-p]+\\
&-\frac{8\pi n T_s}{\sigma^4}\frac{E}{M}\,\mathbb{Re}\left\{\zeta \mathrm{A_{lk}}\mathrm{r_l^*}\dot{\mathrm{s}}_\mathrm{k}\right\}\mathbb{Im}\left\{\zeta \mathrm{A_{l'k'}}\mathrm{s_{k'}}\mathrm{r_{l'}^*}\right\}\delta[l-p,l'-p]+\\
&+\frac{8\pi n T_s}{\sigma^4}\sqrt{\frac{E^3}{M^3}}\,\mathbb{Re}\left\{\zeta \mathrm{A_{lk}}\mathrm{r_l^*}\dot{\mathrm{s}}_\mathrm{k}\right\}\mathbb{Im}\left\{\mathrm{A_{l'k'}}\mathrm{s_{k'}}\mathrm{u_{l'}^*}\right\}\delta[l-p,l'-p]+\\
&+\frac{16\pi^2 f_0 n T_s}{\sigma^4}\sqrt{\frac{E^3}{M^3}}|\zeta|^2\,\mathbb{Im}\left\{\mathrm{A_{lk}}\mathrm{u_l^*}\mathrm{s_k}\right\}\mathbb{Im}\left\{\zeta \mathrm{A_{l'k'}}\mathrm{s_{k'}}\mathrm{r_{l'}^*}\right\}\delta[l-p,l'-p]+\\
&-\frac{16\pi^2 f_0 n T_s}{\sigma^4}\frac{E^2}{M^2}|\zeta|^4\,\mathbb{Im}\left\{\mathrm{A_{lk}}\mathrm{u_l^*}\mathrm{s_k}\right\}\mathbb{Im}\left\{\mathrm{A_{l'k'}}\mathrm{s_{k'}}\mathrm{u_{l'}^*}\right\}\delta[l-p,l'-p]+\\
&+\frac{8\pi n T_s}{\sigma^4}\sqrt{\frac{E^3}{M^3}}|\zeta|^2\,\mathbb{Re}\left\{\mathrm{A}\mathrm{u_l^*}\dot{\mathrm{s}}_\mathrm{k}\right\}\mathbb{Im}\left\{\zeta \mathrm{A_{lk}}\text{'}\mathrm{s_{k'}}\mathrm{r_{l'}^*}\right\}\delta[l-p,l'-p]+\\
&-\frac{8\pi n T_s}{\sigma^4}\frac{E^2}{M^2}|\zeta|^4\,\mathbb{Re}\left\{\mathrm{A}\mathrm{u_l^*}\dot{\mathrm{s}}_\mathrm{k}\right\}\mathbb{Im}\left\{\mathrm{A_{lk}}\text{'}\mathrm{s_{k'}}\mathrm{y_{l'}^*}\right\}\delta[l-p,l'-p]
\end{aligned}
\tag{3.44}
$$

where the first and the third terms can be expanded as

$$
\begin{aligned}
&+\frac{8\pi^2 f_0 T_s}{\sigma^4}\frac{E}{M}\left[\mathbb{Re}\left\{\zeta \mathrm{A_{lk}}\mathrm{s_k}\mathrm{r_l^*}\zeta \mathrm{A_{l'k'}}\mathrm{s_{k'}}\mathrm{r_l^*}\right\}-\mathbb{Re}\left\{\zeta \mathrm{A_{lk}}\mathrm{s_k}\mathrm{r_l^*}\zeta^*\mathrm{A_{l'k'}^*}\mathrm{s_{k'}^*}\mathrm{r_l}\right\}\right]\delta[l-p,l'-p]+\\
&-\frac{4\pi n T_s}{\sigma^4}\frac{E}{M}\left[\mathbb{Im}\left\{\zeta \mathrm{A_{lk}}\dot{\mathrm{s}}_\mathrm{k}\mathrm{s_{k'}}\mathrm{r_l^*}\mathrm{r_l^*}\right\}-\mathbb{Im}\left\{\zeta \mathrm{A_{lk}}\dot{\mathrm{s}}_\mathrm{k}\mathrm{r_l^*}\zeta^*\mathrm{A_{l'k'}^*}\mathrm{s_{k'}^*}\mathrm{r_l}\right\}\right]\delta[l-p,l'-p].
\end{aligned}
\tag{3.45}
$$

Then

$$
\begin{aligned}
&E\left\{\left(1+\frac{2t_p(n)}{\nu}\right)^{-2}\frac{\partial t_p(n)}{\partial \tau_{lk}}\frac{\partial t_p(n)}{\partial f_{l'k'}}\right\}=\\
&=\left(-\frac{8\pi^2 f_0 n T_s}{\sigma^2}\frac{E}{M}|\zeta|^2\frac{\nu}{\nu+2}\alpha_1\,\mathbb{Re}\left\{\mathrm{A}\mathrm{A}\text{'}^*\mathrm{s_k}\mathrm{s_{k'}}\right\}+\right.\\
&\left.+\frac{4\pi n T_s}{\sigma^2}\frac{E}{M}|\zeta|^2\frac{\nu}{\nu+2}\alpha_1\,\mathbb{Im}\left\{\mathrm{A}\mathrm{A}\text{'}^*\dot{\mathrm{s}}_\mathrm{k}\mathrm{s_{k'}^*}\right\}\right)\delta[l-p,l'-p].
\end{aligned}
\tag{3.46}
$$

Regarding the second order derivative

$$E\left\{\left(1+\frac{2t_p(n)}{\nu}\right)^{-1}\frac{\partial^2 t_p(n)}{\partial\tau_{lk}\partial f_{l'k'}}\right\}=$$
$$=\left(-\frac{8\pi^2 f_0 nT_s}{\sigma^2}\frac{E}{M}|\zeta|^2\alpha_2\,\mathbb{R}\mathrm{e}\left\{\mathrm{AA'^*s_k s_{k'}}\right\}+\frac{4\pi nT_s}{\sigma^2}\frac{E}{M}|\zeta|^2\alpha_2\,\mathbb{I}\mathrm{m}\left\{\mathrm{AA'^*\dot{s}_k s^*_{k'}}\right\}\right)\delta[l-p,l'-p]=$$
$$=\frac{\nu+2}{\nu}\frac{\alpha_2}{\alpha_1}E\left\{\left(1+\frac{2t_p(n)}{\nu}\right)^{-2}\frac{\partial t_p(n)}{\partial\tau_{lk}}\frac{\partial t_p(n)}{\partial f_{l'k'}}\right\} \tag{3.47}$$

leading, with the use of orthogonality,to

$$[\boldsymbol{J}(\boldsymbol{\theta})]_{i,i'}=\left[-2\frac{2+\nu}{\nu^2}+\frac{2+\nu}{\nu}\frac{2+\nu}{\nu}\frac{\alpha_2}{\alpha_1}\right]\sum_{p=1}^{N}\sum_{n=0}^{N_o-1}\left(-\frac{8\pi^2 f_0 nT_s}{\sigma^2}\frac{E}{M}|\zeta|^2\frac{\nu}{\nu+2}\alpha_1\,\mathbb{R}\mathrm{e}\left\{\mathrm{AA'^*s_k s_{k'}}\right\}+\right.$$
$$\left.+\frac{4\pi nT_s}{\sigma^2}\frac{E}{M}|\zeta|^2\frac{\nu}{\nu+2}\alpha_1\,\mathbb{I}\mathrm{m}\left\{\mathrm{AA'^*\dot{s}_k s^*_{k'}}\right\}\right)\delta[l-p,l'-p]=$$
$$=-\beta\frac{8\pi^2 f_0}{\sigma^2}\frac{E}{M}|\zeta|^2\,\mathbb{R}\mathrm{e}\left\{\mathrm{e}^{\jmath 2\pi \mathrm{f_0}(\tau_{\mathrm{lk}}\text{-}\tau_{\mathrm{lk'}})}\sum_{\mathrm{n=0}}^{\mathrm{N_o}}\mathrm{nT_s}\mathrm{e}^{\jmath 2\pi(\mathrm{f_{lk}\text{-}f_{lk'}})\mathrm{nT_s}}\mathrm{s_k(nT_s\text{-}\tau_{lk})s^*_{k'}(nT_s\text{-}\tau_{lk'}}\right\}\delta[l-l']+$$
$$+\beta\frac{4\pi}{\sigma^2}\frac{E}{M}|\zeta|^2\,\mathbb{I}\mathrm{m}\left\{\mathrm{e}^{\jmath 2\pi \mathrm{f_0}(\tau_{\mathrm{lk}}\text{-}\tau_{\mathrm{lk'}})}\sum_{\mathrm{n=0}}^{\mathrm{N_o}}\mathrm{nT_s}\mathrm{e}^{\jmath 2\pi(\mathrm{f_{lk}\text{-}f_{lk'}})\mathrm{nT_s}}\frac{\partial\mathrm{s_k(nT_s\text{-}\tau_{lk})}}{\partial\tau_{\mathrm{lk}}}\mathrm{s^*_{k'}(nT_s\text{-}\tau_{lk'})}\right\}\delta[l-l']=$$
$$=-\beta\frac{8\pi^2 f_0 T_s}{\sigma^2}\frac{E}{M}|\zeta|^2\,\mathbb{R}\mathrm{e}\left\{\sum_{\mathrm{n=0}}^{\mathrm{N_o\text{-}1}}\mathrm{n}\,|\mathrm{s_k(nT_s\text{-}\tau_{lk}}|^2\right\}\delta[l-l',k-k']+$$
$$+\beta\frac{4\pi}{\sigma^2}\frac{E}{M}|\zeta|^2\,\mathbb{I}\mathrm{m}\left\{\sum_{\mathrm{n=0}}^{\mathrm{N_o\text{-}1}}\mathrm{nT_s}\frac{\partial\mathrm{s_k(nT_s\text{-}\tau_{lk})}}{\partial\tau_{\mathrm{lk}}}\mathrm{s^*_k(nT_s\text{-}\tau_{lk})}\right\}\delta[l-l',k-k']. \tag{3.48}$$

3.4 Derivation - Case 2

In order to calculate the partial derivatives, the first step is to determine the quadratic form explicitly:

$$t(n)=\left[\boldsymbol{r}(n)-\sqrt{\frac{E}{M}}\zeta\boldsymbol{\Omega}(n)\boldsymbol{s}(n)\right]^H\boldsymbol{\Sigma}^{-1}\left[\boldsymbol{r}(n)-\sqrt{\frac{E}{M}}\zeta\boldsymbol{\Omega}(n)\boldsymbol{s}(n)\right]=$$
$$=\sum_{p=1}^{N}\sum_{p'=1}^{N}\eta_{pp'}r_p^*(n)r_{p'}(n)-2\frac{E}{M}\,\mathbb{R}\mathrm{e}\left\{\zeta\sum_{\mathrm{p=1}}^{\mathrm{N}}\sum_{\mathrm{p'=1}}^{\mathrm{N}}\eta_{\mathrm{pp'}}\mathrm{r^*_p(n)u_{p'}(n)}\right\}+\frac{E}{M}|\zeta|^2\sum_{p=1}^{N}\sum_{p'=1}^{N}\eta_{pp'}u_p^*(n)u_{p'}(n) \tag{3.49}$$

where $\eta_{pq} \triangleq \left[\boldsymbol{\Sigma}^{-1}\right]_{p,q}$. To keep the notation compact, the dependency on the time index n will be left implicit until the final result. Additionally, A_{lk} is defined as in the previous case. Then the first-order derivatives are

$$
\begin{aligned}
\frac{\partial t(n)}{\partial f_{lk}} &= 4\pi n T_s \sqrt{\frac{E}{M}}\, \mathbb{Im}\left\{\zeta \mathrm{A_{lk} s_k} \sum_{\mathrm{p=1}}^{\mathrm{N}} \eta_{\mathrm{pl}} \mathrm{r_p^*}\right\} - 4\pi n T_s \frac{E}{M} |\zeta|^2\, \mathbb{Im}\left\{\mathrm{A_{lk} s_k} \sum_{\mathrm{p=1}}^{\mathrm{N}} \eta_{\mathrm{pl}} \mathrm{u_p^*}\right\} \\
\frac{\partial t(n)}{\partial \zeta_{Re}} &= -2\sqrt{\frac{E}{M}}\, \mathbb{Re}\left\{\sum_{\mathrm{p=1}}^{\mathrm{N}} \sum_{\mathrm{p'=1}}^{\mathrm{N}} \eta_{\mathrm{pp'}} \mathrm{r_p^* u_{p'}}\right\} + 2\frac{E}{M} \zeta_{Re} | \sum_{p=1}^{N} \sum_{p'=1}^{N} \eta_{pp'} u_p^* u_{p'} \\
\frac{\partial t(n)}{\partial \zeta_{Im}} &= +2\sqrt{\frac{E}{M}}\, \mathbb{Im}\left\{\sum_{\mathrm{p=1}}^{\mathrm{N}} \sum_{\mathrm{p'=1}}^{\mathrm{N}} \eta_{\mathrm{pp'}} \mathrm{r_p^* u_{p'}}\right\} + 2\frac{E}{M} \zeta_{Im} | \sum_{p=1}^{N} \sum_{p'=1}^{N} \eta_{pp'} u_p^* u_{p'} \\
\frac{\partial t(n)}{\partial \tau_{lk}} &= -4\pi f_0 \sqrt{\frac{E}{M}}\, \mathbb{Im}\left\{\zeta \mathrm{A_{lk} s_k} \sum_{\mathrm{p=1}}^{\mathrm{N}} \eta_{\mathrm{pl}} \mathrm{r_p^*}\right\} - 2\sqrt{\frac{E}{M}}\, \mathbb{Re}\left\{\zeta \mathrm{A_{lk} \dot{s}_k} \sum_{\mathrm{p=1}}^{\mathrm{N}} \eta_{\mathrm{pl}} \mathrm{r_p^*}\right\} + \\
&\quad + 4\pi f_0 \frac{E}{M} |\zeta|^2\, \mathbb{Re}\left\{\mathrm{A_{lk} s_k} \sum_{\mathrm{p=1}}^{\mathrm{N}} \eta_{\mathrm{pl}} \mathrm{r_u^*}\right\} + 2\frac{E}{M} |\zeta|^2\, \mathbb{Re}\left\{\mathrm{A_{lk} \dot{s}_k} \sum_{\mathrm{p=1}}^{\mathrm{N}} \eta_{\mathrm{pl}} \mathrm{u_p^*}\right\}.
\end{aligned}
\tag{3.50}
$$

Then, the second-order derivatives are

$$
\begin{aligned}
&\frac{\partial^2 t(n)}{\partial\zeta_{Re}\partial\zeta_{Re}} = \frac{\partial^2 t_p(n)}{\partial\zeta_{Im}\partial\zeta_{Im}} = 2\frac{E}{M}\sum_{p=1}^{N}\sum_{p'=1}^{N}\eta_{pp'}u_p^* u_{p'}\\
&\frac{\partial^2 t(n)}{\partial\zeta_{Re}\partial\zeta_{Im}} = 0\\
&\frac{\partial^2 t(n)}{\partial\tau_{lk}\partial\zeta_{Re}} = -4\pi f_0\sqrt{\frac{E}{M}}\,\mathbb{Im}\left\{\mathrm{A_{lk}s_k}\sum_{\mathrm{p=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\mathrm{r_p^*}\right\} -2\sqrt{\frac{E}{M}}\,\mathbb{Re}\left\{\mathrm{A_{lk}\dot{s}_k}\sum_{\mathrm{p=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\mathrm{r_p^*}\right\}+\\
&\qquad + 8\pi f_0\frac{E}{M}\zeta_{Im}\,\mathbb{Im}\left\{\mathrm{A_{lk}s_k}\sum_{\mathrm{p=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\mathrm{u_p^*}\right\} +4\frac{E}{M}\zeta_{Im}\,\mathbb{Re}\left\{\mathrm{A_{lk}\dot{s}_k}\sum_{\mathrm{p=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\mathrm{u_p^*}\right\}\\
&\frac{\partial^2 t(n)}{\partial\tau_{lk}\partial\zeta_{Im}} = -4\pi f_0\sqrt{\frac{E}{M}}\,\mathbb{Re}\left\{\mathrm{A_{lk}s_k}\sum_{\mathrm{p=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\mathrm{r_p^*}\right\} +2\sqrt{\frac{E}{M}}\,\mathbb{Im}\left\{\mathrm{A_{lk}\dot{s}_k}\sum_{\mathrm{p=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\mathrm{r_p^*}\right\}+\\
&\qquad + 8\pi f_0\frac{E}{M}\zeta_{Re}\,\mathbb{Im}\left\{\mathrm{A_{lk}s_k}\sum_{\mathrm{p=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\mathrm{u_p^*}\right\} +4\frac{E}{M}\zeta_{Re}\,\mathbb{Re}\left\{\mathrm{A_{lk}\dot{s}_k}\sum_{\mathrm{p=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\mathrm{u_p^*}\right\}\\
&\frac{\partial^2 t(n)}{\partial f_{lk}\partial\zeta_{Re}} = 4\pi nT_s\sqrt{\frac{E}{M}}\,\mathbb{Im}\left\{\mathrm{A_{lk}s_k}\sum_{\mathrm{p=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\mathrm{r_p^*}\right\} -8\pi nT_s\frac{E}{M}\zeta_{Re}\,\mathbb{Im}\left\{\mathrm{A_{lk}s_k}\sum_{\mathrm{p=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\mathrm{u_p^*}\right\}\\
&\frac{\partial^2 t(n)}{\partial f_{lk}\partial\zeta_{Im}} = 4\pi nT_s\sqrt{\frac{E}{M}}\,\mathbb{Re}\left\{\mathrm{A_{lk}s_k}\sum_{\mathrm{p=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\mathrm{r_p^*}\right\} -8\pi nT_s\frac{E}{M}\zeta_{Re}\,\mathbb{Im}\left\{\mathrm{A_{lk}s_k}\sum_{\mathrm{p=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\mathrm{u_p^*}\right\}\\
&\frac{\partial^2 t(n)}{\partial\tau_{lk}\partial\tau_{l'k'}} = \delta[l-l',k-k']\times\\
&\times\left(8\pi^2 f_0^2\sqrt{\frac{E}{M}}\,\mathbb{Re}\left\{\zeta\mathrm{A_{lk}s_k}\sum_{\mathrm{p=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\mathrm{r_p^*}\right\} -4\pi f_0\sqrt{\frac{E}{M}}\,\mathbb{Im}\left\{\zeta\mathrm{A_{lk}\dot{s}_k}\sum_{\mathrm{p=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\mathrm{r_p^*}\right\}+\right.\\
&- 4\pi f_0\sqrt{\frac{E}{M}}\,\mathbb{Im}\left\{\zeta\mathrm{A_{lk}\dot{s}_k}\sum_{\mathrm{p=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\mathrm{r_p^*}\right\} -2\sqrt{\frac{E}{M}}\,\mathbb{Re}\left\{\zeta\mathrm{A_{lk}\ddot{s}_k}\sum_{\mathrm{p=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\mathrm{r_p^*}\right\}+\\
&- 8\pi^2 f_0^2\frac{E}{M}|\zeta|^2\,\mathbb{Re}\left\{\mathrm{A_{lk}s_k}\sum_{\mathrm{p=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\mathrm{u_p^*}\right\} +4\pi f_0\frac{E}{M}|\zeta|^2\,\mathbb{Im}\left\{\mathrm{A_{lk}\dot{s}_k}\sum_{\mathrm{p=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\mathrm{u_p^*}\right\}+\\
&\left.+4\pi f_0\frac{E}{M}|\zeta|^2\,\mathbb{Im}\left\{\mathrm{A_{lk}\dot{s}_k}\sum_{\mathrm{p=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\mathrm{u_p^*}\right\} +2\frac{E}{M}|\zeta|^2\,\mathbb{Re}\left\{\mathrm{A_{lk}\ddot{s}_k}\sum_{\mathrm{p=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\mathrm{u_p^*}\right\}\right)+\\
&- 8\pi^2 f_0^2\frac{E}{M}|\zeta|^2\,\mathbb{Re}\left\{\eta_{\mathrm{l'l}}\mathrm{A_{lk}A^*_{l'k'}s_k s^*_{k'}}\right\} +4\pi f_0\frac{E}{M}|\zeta|^2\,\mathbb{Im}\left\{\eta_{\mathrm{l'l}}\mathrm{A_{lk}A^*_{l'k'}s_k\dot{s}^*_{k'}}\right\}+\\
&- 4\pi f_0\frac{E}{M}|\zeta|^2\,\mathbb{Im}\left\{\eta_{\mathrm{l'l}}\mathrm{A_{lk}A^*_{l'k'}\dot{s}_k s^*_{k'}}\right\} +2\frac{E}{M}|\zeta|^2\,\mathbb{Re}\left\{\eta_{\mathrm{l'l}}\mathrm{A_{lk}A^*_{l'k'}\dot{s}_k\dot{s}^*_{k'}}\right\}
\end{aligned}
$$

$$\frac{\partial^2 t(n)}{\partial f_{lk}\partial f_{l'k'}} = -8\pi^2 f_0^2 \frac{E}{M}|\zeta|^2\, \mathbb{R}\mathrm{e}\left\{\eta_{\mathrm{l'l}}\mathrm{A_{lk}}\mathrm{A}^*_{\mathrm{l'k'}}\mathrm{s_k}\mathrm{s}^*_{\mathrm{k'}}\right\} +$$
$$+\left(8\pi^2 n^2 T_s^2\sqrt{\frac{E}{M}}\, \mathbb{R}\mathrm{e}\left\{\zeta \mathrm{A_{lk}s_k}\sum_{\mathrm{p=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\mathrm{r}^*_{\mathrm{p}}\right\} - 8\pi^2 n^2 T_s^2 \frac{E}{M}|\zeta|^2\, \mathbb{R}\mathrm{e}\left\{\mathrm{A_{lk}s_k}\sum_{\mathrm{p=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\mathrm{u}^*_{\mathrm{p}}\right\}\right)\delta[l-l',k-k']$$
$$\frac{\partial^2 t(n)}{\partial \tau_{lk}\partial f_{l'k'}} = \left(-8\pi^2 f_0 n T_s\sqrt{\frac{E}{M}}\, \mathbb{R}\mathrm{e}\left\{\zeta \mathrm{A_{lk}s_k}\sum_{\mathrm{p=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\mathrm{r}^*_{\mathrm{p}}\right\} + 8\pi^2 f_0 n T_s \frac{E}{M}|\zeta|^2\, \mathbb{R}\mathrm{e}\left\{\mathrm{A_{lk}s_k}\sum_{\mathrm{p=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\mathrm{u}^*_{\mathrm{p}}\right\}\right.$$
$$\left.4\pi n T_s\sqrt{\frac{E}{M}}\, \mathbb{I}\mathrm{m}\left\{\zeta \mathrm{A_{lk}\dot{s}_k}\sum_{\mathrm{p=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\mathrm{r}^*_{\mathrm{p}}\right\} - 4\pi n T_s \frac{E}{M}|zeta|^2\, \mathbb{I}\mathrm{m}\left\{\mathrm{A_{lk}\dot{s}_k}\sum_{\mathrm{p=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\mathrm{u}^*_{\mathrm{p}}\right\}\right)\delta[l-l',k-k'] +$$
$$-8\pi^2 f_0^2 \frac{E}{M}|\zeta|^2\, \mathbb{R}\mathrm{e}\left\{\eta_{\mathrm{l'l}}\mathrm{A_{lk}}\mathrm{A}^*_{\mathrm{l'k'}}\mathrm{s_k}\mathrm{s}^*_{\mathrm{k'}}\right\} + 4\pi n T_s \frac{E}{M}|\zeta|^2\, \mathbb{I}\mathrm{m}\left\{\eta_{\mathrm{l'l}}\mathrm{A_{lk}}\mathrm{A}^*_{\mathrm{l'k'}}\mathrm{\dot{s}_k}\mathrm{s}^*_{\mathrm{k'}}\right\}. \tag{3.51}$$

To easily evaluate the terms related to the Doppler shifts, it can be noted that the dependencies on τ_{lk} and f_{lk} are similar, differing only for the coefficient ($-f_0$ and nT_s, respectively), and also for the fact that the signal waveforms are independent of the Doppler shifts. Then, by removing the terms related to $\dot{s}_k$ and changing the coefficients, moving from τ to f is immediate.

Given the function g(t) and the previous derivatives, the elements of the FIM can be calculated as

$$[\boldsymbol{J}(\boldsymbol{\theta})]_{i,j} = -2\frac{2N+\nu}{\nu^2}\sum_{n=0}^{N_o-1} E\left\{\left(1+\frac{2t(n)}{\nu}\right)^{-2}\frac{\partial t(n)}{\partial\theta_i}\frac{\partial t(n)}{\partial\theta_j}\right\} +$$
$$+\frac{2N+\nu}{\nu}\sum_{n=0}^{N_o-1} E\left\{\left(1+\frac{2t(n)}{\nu}\right)^{-1}\frac{\partial^2 t(n)}{\partial\theta_i\partial\theta_j}\right\}. \tag{3.52}$$

MATRIX $\boldsymbol{\Lambda}_{2x2}$

- $[\boldsymbol{J}(\boldsymbol{\theta})]_{2NM+1,2NM+1}$

$$\frac{\partial t(n)}{\partial \zeta_{Re}}\frac{\partial t(n)}{\partial \zeta_{Re}} = 4\frac{E}{M}\,\mathbb{R}\mathrm{e}\left\{\sum_{\mathrm{p=1}}^{\mathrm{N}}\sum_{\mathrm{p'=1}}^{\mathrm{N}}\eta_{\mathrm{pp'}}\mathrm{r}_{\mathrm{p}}^{*}\mathrm{u}_{\mathrm{p'}}\right\}^{2} + 4\frac{E^2}{M^2}\zeta_{Re}^2|\sum_{p=1}^{N}\sum_{p'=1}^{N}\eta_{pp'}u_p^*u_{p'}+$$
$$-8\sqrt{\frac{E^3}{M^3}}\zeta_{Re}\,\mathbb{R}\mathrm{e}\left\{\sum_{\mathrm{p=1}}^{\mathrm{N}}\sum_{\mathrm{p'=1}}^{\mathrm{N}}\eta_{\mathrm{pp'}}\mathrm{r}_{\mathrm{p}}^{*}\mathrm{u}_{\mathrm{p'}}\right\}\sum_{p=1}^{N}\sum_{p'=1}^{N}\eta_{pp'}u_p^*u_{p'} =$$
$$= 2\frac{E}{M}\left[\mathbb{R}\mathrm{e}\left\{\sum_{\mathrm{p=1}}^{\mathrm{N}}\sum_{\mathrm{p'=1}}^{\mathrm{N}}\sum_{\mathrm{h=1}}^{\mathrm{N}}\sum_{\mathrm{h'=1}}^{\mathrm{N}}\eta_{\mathrm{pp'}}\eta_{\mathrm{hh'}}\mathrm{u}_{\mathrm{p'}}\mathrm{u}_{\mathrm{h'}}\mathrm{r}_{\mathrm{p}}^{*}\mathrm{r}_{\mathrm{h}}^{*}\right\} + \mathbb{R}\mathrm{e}\left\{\sum_{\mathrm{p=1}}^{\mathrm{N}}\sum_{\mathrm{p'=1}}^{\mathrm{N}}\sum_{\mathrm{h=1}}^{\mathrm{N}}\sum_{\mathrm{h'=1}}^{\mathrm{N}}\eta_{\mathrm{pp'}}\eta_{\mathrm{hh'}}^{*}\mathrm{u}_{\mathrm{p'}}\mathrm{u}_{\mathrm{h'}}^{*}\mathrm{r}_{\mathrm{p}}^{*}\mathrm{r}_{\mathrm{h}}\right\}\right] +$$
$$+4\frac{E^2}{M^2}\zeta_{Re}^2|\sum_{p=1}^{N}\sum_{p'=1}^{N}\eta_{pp'}u_p^*u_{p'} - 8\sqrt{\frac{E^3}{M^3}}\zeta_{Re}\,\mathbb{R}\mathrm{e}\left\{\sum_{\mathrm{p=1}}^{\mathrm{N}}\sum_{\mathrm{p'=1}}^{\mathrm{N}}\eta_{\mathrm{pp'}}\mathrm{r}_{\mathrm{p}}^{*}\mathrm{u}_{\mathrm{p'}}\right\}\sum_{p=1}^{N}\sum_{p'=1}^{N}\eta_{pp'}u_p^*u_{p'}. \tag{3.53}$$

Then, again from the **APPENDIX I**, the expectation is

$$E\left\{\left(1+\frac{2t(n)}{\nu}\right)^{-2}\frac{\partial t(n)}{\partial \zeta_{Re}}\frac{\partial t(n)}{\partial \zeta_{Re}}\right\} = 2\frac{E}{M}\frac{\nu}{\nu+2}\alpha_1\,\mathbb{R}\mathrm{e}\left\{\sum_{\mathrm{p'=1}}^{\mathrm{N}}\sum_{\mathrm{h'=1}}^{\mathrm{N}}\mathrm{u}_{\mathrm{p'}}\mathrm{u}_{\mathrm{h'}}^{*}\sum_{\mathrm{p=1}}^{\mathrm{N}}\sum_{\mathrm{h=1}}^{\mathrm{N}}\eta_{\mathrm{pp'}}\eta_{\mathrm{h'h}}\sigma_{\mathrm{hp}}\right\} \tag{3.54}$$

where $\alpha_1 = \frac{\Gamma\left(\frac{4+2N+\nu}{2}\right)\Gamma\left(\frac{\nu}{2}\right)(\pi\nu)^N}{\Gamma\left(\frac{2N+\nu}{2}\right)\Gamma\left(\frac{4+\nu}{2}\right)(\pi\nu+4\pi)^N}\left(\frac{\nu}{\nu+4}\right)^N$ and $\sigma_{ab} \triangleq [\boldsymbol{\Sigma}]_{a,b}$. Regarding the second order derivative,

$$E\left\{\left(1+\frac{2t(n)}{\nu}\right)^{-1}\frac{\partial^2 t(n)}{\partial \zeta_{Re}\partial \zeta_{Re}}\right\} = 2\frac{E}{M}\sum_{p=1}^{N}\sum_{p'=1}^{N}\eta_{pp'}u_p^*u_{p'} \tag{3.55}$$

where $\alpha_2 = \frac{\Gamma\left(\frac{2+2N+\nu}{2}\right)\Gamma\left(\frac{\nu}{2}\right)(\pi\nu)^N}{\Gamma\left(\frac{2N+\nu}{2}\right)\Gamma\left(\frac{2+\nu}{2}\right)(\pi\nu+2\pi)^N}\left(\frac{\nu}{\nu+2}\right)^N$. Finally, after applying orthogonality,

$$[\boldsymbol{J}(\boldsymbol{\theta})]_{2NM+1,2NM+1} = -2\frac{2N+\nu}{\nu^2}\sum_{n=0}^{N_o-1}2\frac{E}{M}\frac{\nu}{\nu+2}\alpha_1\,\mathbb{R}\mathrm{e}\left\{\sum_{\mathrm{p'=1}}^{\mathrm{N}}\sum_{\mathrm{h'=1}}^{\mathrm{N}}\mathrm{u}_{\mathrm{p'}}\mathrm{u}_{\mathrm{h'}}^{*}\sum_{\mathrm{p=1}}^{\mathrm{N}}\sum_{\mathrm{h=1}}^{\mathrm{N}}\eta_{\mathrm{pp'}}\eta_{\mathrm{h'h}}\sigma_{\mathrm{hp}}\right\} +$$
$$+\frac{N+\nu}{\nu}\sum_{n=0}^{N_o-1}2\frac{E}{M}\sum_{p=1}^{N}\sum_{p'=1}^{N}\eta_{pp'}u_p^*u_{p'} = +2\frac{N+\nu}{\nu}\frac{E}{M}\sum_{p=1}^{N}\sum_{p'=1}^{N}\eta_{pp'}\sum_{n=0}^{N_o-1}u_p^*u_{p'}+$$
$$-4\frac{2N+\nu}{\nu^2}\frac{E}{M}\frac{\nu}{\nu+2}\alpha_1\,\mathbb{R}\mathrm{e}\left\{\sum_{\mathrm{p'=1}}^{\mathrm{N}}\sum_{\mathrm{h'=1}}^{\mathrm{N}}\sum_{\mathrm{p=1}}^{\mathrm{N}}\sum_{\mathrm{h=1}}^{\mathrm{N}}\eta_{\mathrm{pp'}}\eta_{\mathrm{h'h}}\sigma_{\mathrm{hp}}\sum_{\mathrm{n=0}}^{\mathrm{N_o\text{-}1}}\mathrm{u}_{\mathrm{p'}}\mathrm{u}_{\mathrm{h'}}^{*}\right\} \tag{3.56}$$

where

$$\sum_{n=0}^{N_o-1} u_p^* u_{p'} = \sum_{q=1}^{N}\sum_{q'=1}^{N} e^{-\jmath 2\pi f_0(\tau_{pq}-\tau_{p'q'})} \sum_{n=0}^{N_o-1} e^{\jmath 2\pi (f_{pq}-f_{p'q'})nT_s} s_q(nT_s-\tau_{pq}) s_{q'}^*(nT_s-\tau_{p'q'}) = \sum_{q=1}^{N} e^{-\jmath 2\pi f_0(\tau_{pq}-\tau_{p'q})} \sum_{n=0}^{N_o-1} e^{\jmath 2\pi (f_{pq}-f_{p'q})nT_s} s_q(nT_s-\tau_{pq}) s_q^*(nT_s-\tau_{p'q}) \quad (3.57)$$

- $[\boldsymbol{J}(\boldsymbol{\theta})]_{2NM+2,2NM+2}$

$$\frac{\partial t(n)}{\partial \zeta_{Re}}\frac{\partial t(n)}{\partial \zeta_{Re}} = 4\frac{E}{M}\,\mathbb{Im}\left\{\sum_{\mathrm{p}=1}^{\mathrm{N}}\sum_{\mathrm{p'}=1}^{\mathrm{N}} \eta_{\mathrm{pp'}}\mathrm{r}_{\mathrm{p}}^*\mathrm{u}_{\mathrm{p'}}\right\}^2 + 4\frac{E^2}{M^2}\zeta_{Im}^2 | \sum_{p=1}^{N}\sum_{p'=1}^{N} \eta_{pp'} u_p^* u_{p'} + 8\sqrt{\frac{E^3}{M^3}}\zeta_{Im}\,\mathbb{Im}\left\{\sum_{\mathrm{p}=1}^{\mathrm{N}}\sum_{\mathrm{p'}=1}^{\mathrm{N}} \eta_{\mathrm{pp'}}\mathrm{r}_{\mathrm{p}}^*\mathrm{u}_{\mathrm{p'}}\right\}\sum_{p=1}^{N}\sum_{p'=1}^{N} \eta_{pp'} u_p^* u_{p'} =$$
$$= 2\frac{E}{M}\left[-\mathbb{Re}\left\{\sum_{\mathrm{p}=1}^{\mathrm{N}}\sum_{\mathrm{p'}=1}^{\mathrm{N}}\sum_{\mathrm{h}=1}^{\mathrm{N}}\sum_{\mathrm{h'}=1}^{\mathrm{N}} \eta_{\mathrm{pp'}}\eta_{\mathrm{hh'}}\mathrm{u}_{\mathrm{p'}}\mathrm{u}_{\mathrm{h'}}\mathrm{r}_{\mathrm{p}}^*\mathrm{r}_{\mathrm{h}}^*\right\} + \mathbb{Re}\left\{\sum_{\mathrm{p}=1}^{\mathrm{N}}\sum_{\mathrm{p'}=1}^{\mathrm{N}}\sum_{\mathrm{h}=1}^{\mathrm{N}}\sum_{\mathrm{h'}=1}^{\mathrm{N}} \eta_{\mathrm{pp'}}\eta_{\mathrm{hh'}}^*\mathrm{u}_{\mathrm{p'}}\mathrm{u}_{\mathrm{h'}}^*\mathrm{r}_{\mathrm{p}}^*\mathrm{r}_{\mathrm{h}}\right\}\right] +$$
$$+ 4\frac{E^2}{M^2}\zeta_{Im}^2 | \sum_{p=1}^{N}\sum_{p'=1}^{N} \eta_{pp'} u_p^* u_{p'} + 8\sqrt{\frac{E^3}{M^3}}\zeta_{Im}\,\mathbb{Im}\left\{\sum_{\mathrm{p}=1}^{\mathrm{N}}\sum_{\mathrm{p'}=1}^{\mathrm{N}} \eta_{\mathrm{pp'}}\mathrm{r}_{\mathrm{p}}^*\mathrm{u}_{\mathrm{p'}}\right\}\sum_{p=1}^{N}\sum_{p'=1}^{N} \eta_{pp'} u_p^* u_{p'}. \quad (3.58)$$

By exploiting the similarities with the previous case, plus the fact that the second order derivative is the same, we obtain

$$[\boldsymbol{J}(\boldsymbol{\theta})]_{2NM+2,2NM+2} = [\boldsymbol{J}(\boldsymbol{\theta})]_{2NM+1,2NM+1}\,. \quad (3.59)$$

- $[\boldsymbol{J}(\boldsymbol{\theta})]_{2NM+1,2NM+2}$

$$\frac{\partial t(n)}{\partial \zeta_{Re}}\frac{\partial t(n)}{\partial \zeta_{Im}} = -4\frac{E}{M}\,\mathbb{Re}\left\{\sum_{\mathrm{p}=1}^{\mathrm{N}}\sum_{\mathrm{p'}=1}^{\mathrm{N}} \eta_{\mathrm{pp'}}\mathrm{r}_{\mathrm{p}}^*\mathrm{u}_{\mathrm{p'}}\right\}\mathbb{Im}\left\{\sum_{\mathrm{p}=1}^{\mathrm{N}}\sum_{\mathrm{p'}=1}^{\mathrm{N}} \eta_{\mathrm{pp'}}\mathrm{r}_{\mathrm{p}}^*\mathrm{u}_{\mathrm{p'}}\right\} + $$
$$+ 4\frac{E^2}{M^2}\left(\sum_{p=1}^{N}\sum_{p'=1}^{N} \eta_{pp'} r_p^* u_{p'}\right)^2 - 4\sqrt{\frac{E^3}{M^3}}\zeta_{Im}\,\mathbb{Re}\left\{\sum_{\mathrm{p}=1}^{\mathrm{N}}\sum_{\mathrm{p'}=1}^{\mathrm{N}} \eta_{\mathrm{pp'}}\mathrm{r}_{\mathrm{p}}^*\mathrm{u}_{\mathrm{p'}}\right\}\sum_{p=1}^{N}\sum_{p'=1}^{N} \eta_{pp'} r_p^* u_{p'} +$$
$$+ 4\sqrt{\frac{E^3}{M^3}}\zeta_{Re}\,\mathbb{Im}\left\{\sum_{\mathrm{p}=1}^{\mathrm{N}}\sum_{\mathrm{p'}=1}^{\mathrm{N}} \eta_{\mathrm{pp'}}\mathrm{r}_{\mathrm{p}}^*\mathrm{u}_{\mathrm{p'}}\right\}\sum_{p=1}^{N}\sum_{p'=1}^{N} \eta_{pp'} r_p^* u_{p'} \quad (3.60)$$

After the expectation,it can be easily shown that all the terms cancel out. Then, recalling that the second order derivative is also 0,

$$[\boldsymbol{J}(\boldsymbol{\theta})]_{2NM+1,2NM+2} = 0. \tag{3.61}$$

MATRIX $\boldsymbol{V}_{2NMx2}$

- $[\boldsymbol{J}(\boldsymbol{\theta})]_{i,NM+1}, \quad i=(l-1)M+k, \quad l=1\dots N, \quad k=1\dots M$

$$\begin{aligned}
\frac{\partial t(n)}{\partial \tau_{lk}}\frac{\partial t(n)}{\partial \zeta_{Re}} &= 8\pi f_0 \frac{E}{M}\,\mathbb{Im}\left\{\zeta \mathrm{A_{lk}s_k}\sum_{\mathrm{p=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\mathrm{r_p^*}\right\}\mathbb{Re}\left\{\sum_{\mathrm{p=1}}^{\mathrm{N}}\sum_{\mathrm{p'=1}}^{\mathrm{N}}\eta_{\mathrm{pp'}}\mathrm{r_p^*u_{p'}}\right\}+\\
&- 8\pi f_0\sqrt{\frac{E^3}{M^3}}\zeta_{Re}\,\mathbb{Im}\left\{\zeta \mathrm{A_{lk}s_k}\sum_{\mathrm{p=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\mathrm{r_p^*}\right\}\sum_{p=1}^{N}\sum_{p'=1}^{N}\eta_{pp'}u_p^*u_{p'}+\\
&+ 4\frac{E}{M}\,\mathbb{Re}\left\{\zeta \mathrm{A_{lk}\dot{s}_k}\sum_{\mathrm{p=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\mathrm{r_p^*}\right\}\mathbb{Re}\left\{\sum_{\mathrm{p=1}}^{\mathrm{N}}\sum_{\mathrm{p'=1}}^{\mathrm{N}}\eta_{\mathrm{pp'}}\mathrm{r_p^*u_{p'}}\right\}+\\
&- 4\sqrt{\frac{E^3}{M^3}}\zeta_{Re}\,\mathbb{Re}\left\{\zeta \mathrm{A_{lk}\dot{s}_k}\sum_{\mathrm{p=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\mathrm{r_p^*}\right\}\sum_{p=1}^{N}\sum_{p'=1}^{N}\eta_{pp'}u_p^*u_{p'}+\\
&- 8\pi f_0\frac{E^2}{M^2}|\zeta|^2\,\mathbb{Im}\left\{\mathrm{A_{lk}s_k}\sum_{\mathrm{p=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\mathrm{u_p^*}\right\}\mathbb{Re}\left\{\sum_{\mathrm{p=1}}^{\mathrm{N}}\sum_{\mathrm{p'=1}}^{\mathrm{N}}\eta_{\mathrm{pp'}}\mathrm{r_p^*u_{p'}}\right\}+\\
&+ 8\pi f_0\frac{E^2}{M^2}\zeta_{Re}\,\mathbb{Im}\left\{\zeta \mathrm{A_{lk}s_k}\sum_{\mathrm{p=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\mathrm{u_p^*}\right\}\sum_{p=1}^{N}\sum_{p'=1}^{N}\eta_{pp'}u_p^*u_{p'}+\\
&- 4\sqrt{\frac{E^3}{M^3}}\,\mathbb{Re}\left\{\zeta \mathrm{A_{lk}\dot{s}_k}\sum_{\mathrm{p=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\mathrm{u_p^*}\right\}\mathbb{Re}\left\{\sum_{\mathrm{p=1}}^{\mathrm{N}}\sum_{\mathrm{p'=1}}^{\mathrm{N}}\eta_{\mathrm{pp'}}\mathrm{r_p^*u_{p'}}\right\}+\\
&+ 4\frac{E^2}{M^2}\zeta_{Re}\,\mathbb{Re}\left\{\zeta \mathrm{A_{lk}\dot{s}_k}\sum_{\mathrm{p=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\mathrm{u_p^*}\right\}\sum_{p=1}^{N}\sum_{p'=1}^{N}\eta_{pp'}u_p^*u_{p'}
\end{aligned} \tag{3.62}$$

where the first and third terms can be expanded as

$$\begin{aligned}
&4\pi f_0\frac{E}{M}\left[\mathbb{Im}\left\{\zeta \mathrm{A_{lk}s_k}\sum_{\mathrm{p=1}}^{\mathrm{N}}\sum_{\mathrm{p'=1}}^{\mathrm{N}}\sum_{\mathrm{h=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\eta_{\mathrm{p'h}}\mathrm{u_h r_p^* r_{p'}^*}\right\}+\mathbb{Im}\left\{\zeta \mathrm{A_{lk}s_k}\sum_{\mathrm{p=1}}^{\mathrm{N}}\sum_{\mathrm{p'=1}}^{\mathrm{N}}\sum_{\mathrm{h=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\eta_{\mathrm{hp'}}\mathrm{u_h^* r_p^* r_{p'}}\right\}\right]+\\
&+2\frac{E}{M}\left[\mathbb{Re}\left\{\zeta \mathrm{A_{lk}\dot{s}_k}\sum_{\mathrm{p=1}}^{\mathrm{N}}\sum_{\mathrm{p'=1}}^{\mathrm{N}}\sum_{\mathrm{h=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\eta_{\mathrm{p'h}}\mathrm{u_h r_p^* r_{p'}^*}\right\}+\mathbb{Re}\left\{\zeta \mathrm{A_{lk}\dot{s}_k}\sum_{\mathrm{p=1}}^{\mathrm{N}}\sum_{\mathrm{p'=1}}^{\mathrm{N}}\sum_{\mathrm{h=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\eta_{\mathrm{hp'}}\mathrm{u_h^* r_p^* r_{p'}}\right\}\right].
\end{aligned} \tag{3.63}$$

Then it can be shown that after the expectation the terms directly proportional to E^2/M^2 disappear, leaving

$$E\left\{\left(1+\frac{2t_p(n)}{\nu}\right)^{-2}\frac{\partial t_p(n)}{\partial \tau_{lk}}\frac{\partial t_p(n)}{\partial \zeta_{Re}}\right\} = 4\pi f_0\frac{E}{M}\frac{\nu}{\nu+2}\alpha_1\,\mathbb{Im}\left\{\zeta \mathrm{A_{lk}s_k}\sum_{\mathrm{p=1}}^{\mathrm{N}}\sum_{\mathrm{p'=1}}^{\mathrm{N}}\sum_{\mathrm{h=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\eta_{\mathrm{hp'}}\mathrm{u_h^*}\sigma_{\mathrm{p'p}}\right\}+$$
$$2\frac{E}{M}\frac{\nu}{\nu+2}\alpha_1\,\mathbb{Re}\left\{\zeta \mathrm{A_{lk}\dot{s}_k}\sum_{\mathrm{p=1}}^{\mathrm{N}}\sum_{\mathrm{p'=1}}^{\mathrm{N}}\sum_{\mathrm{h=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\eta_{\mathrm{hp'}}\mathrm{u_h^*}\sigma_{\mathrm{p'p}}\right\} \tag{3.64}$$

Regarding the second order derivative,

$$E\left\{\left(1+\frac{2t_p(n)}{\nu}\right)^{-1}\frac{\partial^2 t_p(n)}{\partial \tau_{lk}\partial \zeta_{Re}}\right\} =$$
$$= 4\pi f_0\frac{E}{M}\alpha_2\,\mathbb{Im}\left\{\zeta \mathrm{A_{lk}s_k}\sum_{\mathrm{p=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\mathrm{u_p^*}\right\}+2\frac{E}{M}\alpha_2\,\mathbb{Re}\left\{\zeta \mathrm{A_{lk}\dot{s}_k}\sum_{\mathrm{p=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\mathrm{u_p^*}\right\} \tag{3.65}$$

Finally, after applying orthogonality, it is

$$[\boldsymbol{J}(\boldsymbol{\theta})]_{i,2NM+1} = -2\frac{2N+\nu}{\nu^2}\sum_{n=0}^{N_o-1}4\pi f_0\frac{E}{M}\frac{\nu}{\nu+2}\alpha_1\,\mathbb{Im}\left\{\zeta \mathrm{A_{lk}s_k}\sum_{\mathrm{p=1}}^{\mathrm{N}}\sum_{\mathrm{p'=1}}^{\mathrm{N}}\sum_{\mathrm{h=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\eta_{\mathrm{hp'}}\mathrm{u_h^*}\sigma_{\mathrm{p'p}}\right\}+$$
$$-2\frac{2N+\nu}{\nu^2}\sum_{n=0}^{N_o-1}2\frac{E}{M}\frac{\nu}{\nu+2}\alpha_1\,\mathbb{Re}\left\{\zeta \mathrm{A_{lk}\dot{s}_k}\sum_{\mathrm{p=1}}^{\mathrm{N}}\sum_{\mathrm{p'=1}}^{\mathrm{N}}\sum_{\mathrm{h=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\eta_{\mathrm{hp'}}\mathrm{u_h^*}\sigma_{\mathrm{p'p}}\right\}+$$
$$+\frac{N+\nu}{\nu}\left(4\pi f_0\frac{E}{M}\alpha_2\,\mathbb{Im}\left\{\zeta \mathrm{A_{lk}s_k}\sum_{\mathrm{p=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\mathrm{u_p^*}\right\}+2\frac{E}{M}\alpha_2\,\mathbb{Re}\left\{\zeta \mathrm{A_{lk}\dot{s}_k}\sum_{\mathrm{p=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\mathrm{u_p^*}\right\}\right)=$$
$$=-8\pi f_0\frac{2N+\nu}{\nu^2}\frac{E}{M}\frac{\nu}{\nu+2}\alpha_1\,\mathbb{Im}\left\{\zeta\sum_{\mathrm{h=1}}^{\mathrm{N}}\sum_{\mathrm{n=0}}^{\mathrm{N_o\text{-}1}}\mathrm{A_{lk}s_k u_h^*}\sum_{\mathrm{p=1}}^{\mathrm{N}}\sum_{\mathrm{p'=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\eta_{\mathrm{hp'}}\sigma_{\mathrm{p'p}}\right\}+$$
$$-4\frac{2N+\nu}{\nu^2}\frac{E}{M}\frac{\nu}{\nu+2}\alpha_1\,\mathbb{Re}\left\{\zeta\sum_{\mathrm{h=1}}^{\mathrm{N}}\sum_{\mathrm{n=0}}^{\mathrm{N_o\text{-}1}}\mathrm{A_{lk}\dot{s}_k u_h^*}\sum_{\mathrm{p=1}}^{\mathrm{N}}\sum_{\mathrm{p'=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\eta_{\mathrm{hp'}}\sigma_{\mathrm{p'p}}\right\}+$$
$$+\frac{N+\nu}{\nu}\left(4\pi f_0\frac{E}{M}\alpha_2\,\mathbb{Im}\left\{\zeta\sum_{\mathrm{p=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\sum_{\mathrm{n=0}}^{\mathrm{N_o\text{-}1}}\mathrm{s_k u_p^*}\right\}+2\frac{E}{M}\alpha_2\,\mathbb{Re}\left\{\zeta\sum_{\mathrm{p=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\sum_{\mathrm{n=0}}^{\mathrm{N_o\text{-}1}}\mathrm{A_{lk}\dot{s}_k u_p^*}\right\}\right) \tag{3.66}$$

where

$$\sum_{n=0}^{N_o-1} A_{lk}s_k u_p^* = e^{\jmath 2\pi f_0(\tau_{lk}-\tau_{pk})} \sum_{n=0}^{N_o-1} e^{\jmath 2\pi(f_{lk}-f_{pk})nT_s} s_k(nT_s-\tau_{lk})s_k^*(nT_s-\tau_{pk})$$
$$\sum_{n=0}^{N_o-1} A_{lk}\dot{s}_k u_p^* = e^{\jmath 2\pi f_0(\tau_{lk}-\tau_{pk})} \sum_{n=0}^{N_o-1} e^{\jmath 2\pi(f_{lk}-f_{pk})nT_s} \frac{\partial s_k(nT_s-\tau_{lk})}{\partial \tau_{lk}} s_k^*(nT_s-\tau_{pk}) \tag{3.67}$$

- $[\boldsymbol{J}(\boldsymbol{\theta})]_{i,NM+2}, \quad i=(l-1)M+k, \quad l=1\ldots N, \quad k=1\ldots M$

$$\begin{aligned}
\frac{\partial t(n)}{\partial \tau_{lk}}\frac{\partial t(n)}{\partial \zeta_{Im}} &= -8\pi f_0 \frac{E}{M}\,\mathbb{Im}\left\{\zeta \mathrm{A_{lk}s_k}\sum_{\mathrm{p=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\mathrm{r_p^*}\right\}\mathbb{Im}\left\{\sum_{\mathrm{p=1}}^{\mathrm{N}}\sum_{\mathrm{p'=1}}^{\mathrm{N}}\eta_{\mathrm{pp'}}\mathrm{r_p^*u_{p'}}\right\}+\\
&- 8\pi f_0\sqrt{\frac{E^3}{M^3}}\zeta_{Im}\,\mathbb{Im}\left\{\zeta \mathrm{A_{lk}s_k}\sum_{\mathrm{p=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\mathrm{r_p^*}\right\}\sum_{p=1}^{N}\sum_{p'=1}^{N}\eta_{pp'}u_p^*u_{p'}+\\
&- 4\frac{E}{M}\,\mathbb{Re}\left\{\zeta \mathrm{A_{lk}\dot{s}_k}\sum_{\mathrm{p=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\mathrm{r_p^*}\right\}\mathbb{Im}\left\{\sum_{\mathrm{p=1}}^{\mathrm{N}}\sum_{\mathrm{p'=1}}^{\mathrm{N}}\eta_{\mathrm{pp'}}\mathrm{r_p^*u_{p'}}\right\}+\\
&- 4\sqrt{\frac{E^3}{M^3}}\zeta_{Im}\,\mathbb{Re}\left\{\zeta \mathrm{A_{lk}\dot{s}_k}\sum_{\mathrm{p=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\mathrm{r_p^*}\right\}\sum_{p=1}^{N}\sum_{p'=1}^{N}\eta_{pp'}u_p^*u_{p'}+\\
&+ 8\pi f_0\frac{E^2}{M^2}|\zeta|^2\,\mathbb{Im}\left\{\mathrm{A_{lk}s_k}\sum_{\mathrm{p=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\mathrm{u_p^*}\right\}\mathbb{Im}\left\{\sum_{\mathrm{p=1}}^{\mathrm{N}}\sum_{\mathrm{p'=1}}^{\mathrm{N}}\eta_{\mathrm{pp'}}\mathrm{r_p^*u_{p'}}\right\}+\\
&+ 8\pi f_0\frac{E^2}{M^2}\zeta_{Im}\,\mathbb{Im}\left\{\zeta \mathrm{A_{lk}s_k}\sum_{\mathrm{p=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\mathrm{u_p^*}\right\}\sum_{p=1}^{N}\sum_{p'=1}^{N}\eta_{pp'}u_p^*u_{p'}+\\
&+ 4\sqrt{\frac{E^3}{M^3}}\,\mathbb{Re}\left\{\zeta \mathrm{A_{lk}\dot{s}_k}\sum_{\mathrm{p=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\mathrm{u_p^*}\right\}\mathbb{Im}\left\{\sum_{\mathrm{p=1}}^{\mathrm{N}}\sum_{\mathrm{p'=1}}^{\mathrm{N}}\eta_{\mathrm{pp'}}\mathrm{r_p^*u_{p'}}\right\}+\\
&+ 4\frac{E^2}{M^2}\zeta_{Im}\,\mathbb{Re}\left\{\zeta \mathrm{A_{lk}\dot{s}_k}\sum_{\mathrm{p=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\mathrm{u_p^*}\right\}\sum_{p=1}^{N}\sum_{p'=1}^{N}\eta_{pp'}u_p^*u_{p'}
\end{aligned} \tag{3.68}$$

where the first and third terms can be expanded as

$$4\pi f_0 \frac{E}{M}\left[\mathbb{R}\mathrm{e}\left\{\zeta \mathrm{A_{lk}s_k}\sum_{\mathrm{p=1}}^{\mathrm{N}}\sum_{\mathrm{p'=1}}^{\mathrm{N}}\sum_{\mathrm{h=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\eta_{\mathrm{p'h}}\mathrm{u_h r_p^* r_{p'}^*}\right\} - \mathbb{R}\mathrm{e}\left\{\zeta \mathrm{A_{lk}s_k}\sum_{\mathrm{p=1}}^{\mathrm{N}}\sum_{\mathrm{p'=1}}^{\mathrm{N}}\sum_{\mathrm{h=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\eta_{\mathrm{hp'}}\mathrm{u_h^* r_p^* r_{p'}}\right\}\right] +$$
$$+2\frac{E}{M}\left[-\mathbb{I}\mathrm{m}\left\{\zeta \mathrm{A_{lk}\dot{s}_k}\sum_{\mathrm{p=1}}^{\mathrm{N}}\sum_{\mathrm{p'=1}}^{\mathrm{N}}\sum_{\mathrm{h=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\eta_{\mathrm{p'h}}\mathrm{u_h r_p^* r_{p'}^*}\right\} + \mathbb{I}\mathrm{m}\left\{\zeta \mathrm{A_{lk}\dot{s}_k}\sum_{\mathrm{p=1}}^{\mathrm{N}}\sum_{\mathrm{p'=1}}^{\mathrm{N}}\sum_{\mathrm{h=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\eta_{\mathrm{hp'}}\mathrm{u_h^* r_p^* r_{p'}}\right\}\right]. \tag{3.69}$$

Then, by considering the similarities with the previous parameter the final result can be reached immediately:

$$[\boldsymbol{J}(\boldsymbol{\theta})]_{i,2NM+2} = 8\pi f_0 \frac{2N+\nu}{\nu^2}\frac{E}{M}\frac{\nu}{\nu+2}\alpha_1\,\mathbb{R}\mathrm{e}\left\{\zeta\sum_{\mathrm{h=1}}^{\mathrm{N}}\sum_{\mathrm{n=0}}^{\mathrm{N_o\text{-}1}}\mathrm{A_{lk}s_k u_h^*}\sum_{\mathrm{p=1}}^{\mathrm{N}}\sum_{\mathrm{p'=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\eta_{\mathrm{hp'}}\sigma_{\mathrm{p'p}}\right\} +$$
$$-4\frac{2N+\nu}{\nu^2}\frac{E}{M}\frac{\nu}{\nu+2}\alpha_1\,\mathbb{I}\mathrm{m}\left\{\zeta\sum_{\mathrm{h=1}}^{\mathrm{N}}\sum_{\mathrm{n=0}}^{\mathrm{N_o\text{-}1}}\mathrm{A_{lk}\dot{s}_k u_h^*}\sum_{\mathrm{p=1}}^{\mathrm{N}}\sum_{\mathrm{p'=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\eta_{\mathrm{hp'}}\sigma_{\mathrm{p'p}}\right\} +$$
$$+\frac{N+\nu}{\nu}\left(-4\pi f_0\frac{E}{M}\alpha_2\,\mathbb{R}\mathrm{e}\left\{\zeta\sum_{\mathrm{p=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\sum_{\mathrm{n=0}}^{\mathrm{N_o\text{-}1}}\mathrm{s_k u_p^*}\right\} + 2\frac{E}{M}\alpha_2\,\mathbb{I}\mathrm{m}\left\{\zeta\sum_{\mathrm{p=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\sum_{\mathrm{n=0}}^{\mathrm{N_o\text{-}1}}\mathrm{A_{lk}\dot{s}_k u_p^*}\right\}\right). \tag{3.70}$$

- $[\boldsymbol{J}(\boldsymbol{\theta})]_{i,2NM+1}, \quad [\boldsymbol{J}(\boldsymbol{\theta})]_{i,2NM+2}, \quad i=(N+l-1)M+k, \quad l=1\ldots N, \quad k=1\ldots M$

By using the same approach of the second order derivatives, the bounds for the Doppler shifts can be calculated from the previous ones, and it is

$$[\boldsymbol{J}(\boldsymbol{\theta})]_{i,2NM+1} = 8\pi\frac{2N+\nu}{\nu^2}\frac{E}{M}\frac{\nu}{\nu+2}\alpha_1\,\mathbb{I}\mathrm{m}\left\{\zeta\sum_{\mathrm{h=1}}^{\mathrm{N}}\sum_{\mathrm{n=0}}^{\mathrm{N_o\text{-}1}}\mathrm{nT_s A_{lk}s_k u_h^*}\sum_{\mathrm{p=1}}^{\mathrm{N}}\sum_{\mathrm{p'=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\eta_{\mathrm{hp'}}\sigma_{\mathrm{p'p}}\right\} +$$
$$-\frac{N+\nu}{\nu}4\pi\frac{E}{M}\alpha_2\,\mathbb{I}\mathrm{m}\left\{\zeta\sum_{\mathrm{p=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\sum_{\mathrm{n=0}}^{\mathrm{N_o\text{-}1}}\mathrm{nT_s s_k u_p^*}\right\}$$

$$[\boldsymbol{J}(\boldsymbol{\theta})]_{i,2NM+2} = -8\pi\frac{2N+\nu}{\nu^2}\frac{E}{M}\frac{\nu}{\nu+2}\alpha_1\,\mathbb{R}\mathrm{e}\left\{\zeta\sum_{\mathrm{h=1}}^{\mathrm{N}}\sum_{\mathrm{n=0}}^{\mathrm{N_o\text{-}1}}\mathrm{nT_s A_{lk}s_k u_h^*}\sum_{\mathrm{p=1}}^{\mathrm{N}}\sum_{\mathrm{p'=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\eta_{\mathrm{hp'}}\sigma_{\mathrm{p'p}}\right\} +$$
$$+\frac{N+\nu}{\nu}4\pi\frac{E}{M}\alpha_2\,\mathbb{R}\mathrm{e}\left\{\zeta\sum_{\mathrm{p=1}}^{\mathrm{N}}\eta_{\mathrm{pl}}\sum_{\mathrm{n=0}}^{\mathrm{N_o\text{-}1}}\mathrm{nT_s s_k u_p^*}\right\}. \tag{3.71}$$

where, as before,

$$\sum_{n=0}^{N_o-1} nT_s A_{lk} s_k u_p^* = e^{\jmath 2\pi f_0(\tau_{lk}-\tau_{pk})} \sum_{n=0}^{N_o-1} e^{\jmath 2\pi (f_{lk}-f_{pk})nT_s} nT_s s_k(nT_s-\tau_{lk}) s_k^*(nT_s-\tau_{pk}) \tag{3.72}$$

MATRIX $\boldsymbol{S}_{2NMx2NM}$

- $[\boldsymbol{J}(\boldsymbol{\theta})]_{i,i'}$, $\quad i=(l-1)M+k, \quad i'=(l'-1)M+k', \quad l,l'=1\ldots N, \quad k,k'=1\ldots M$

$$\begin{aligned}
\frac{\partial t(n)}{\partial \tau_{lk}}\frac{\partial t(n)}{\partial \tau_{l'k'}} &= 16\pi^2 f_0^2 \frac{E}{M}\,\mathbb{Im}\left\{\zeta \mathrm{A_{lk}s_k}\sum_{\mathrm{p=1}}^{\mathrm{N}} \eta_{\mathrm{pl}}{}^*\mathrm{r_p^*}\right\}\mathbb{Im}\left\{\zeta \mathrm{A_{l'k'}s_{k'}}\sum_{\mathrm{p'=1}}^{\mathrm{N}} \eta_{\mathrm{p'l'}}\mathrm{r_{p'}^*}\right\}+\\
&+ 8\pi f_0 \frac{E}{M}\,\mathbb{Im}\left\{\zeta \mathrm{A_{lk}s_k}\sum_{\mathrm{p=1}}^{\mathrm{N}} \eta_{\mathrm{pl}}{}^*\mathrm{r_p^*}\right\}\mathbb{Re}\left\{\zeta \mathrm{A_{l'k'}s_{k'}}\sum_{\mathrm{p'=1}}^{\mathrm{N}} \eta_{\mathrm{p'l'}}\mathrm{r_{p'}^*}\right\}+\\
&- 16\pi^2 f_0^2 \sqrt{\frac{E^3}{M^3}}|\zeta|^2\,\mathbb{Im}\left\{\zeta \mathrm{A_{lk}s_k}\sum_{\mathrm{p=1}}^{\mathrm{N}} \eta_{\mathrm{pl}}{}^*\mathrm{r_p^*}\right\}\mathbb{Im}\left\{\mathrm{A_{l'k'}s_{k'}}\sum_{\mathrm{p'=1}}^{\mathrm{N}} \eta_{\mathrm{p'l'}}\mathrm{u_{p'}^*}\right\}+\\
&- 8\pi f_0 \sqrt{\frac{E^3}{M^3}}|\zeta|^2\,\mathbb{Im}\left\{\zeta \mathrm{A_{lk}s_k}\sum_{\mathrm{p=1}}^{\mathrm{N}} \eta_{\mathrm{pl}}{}^*\mathrm{r_p^*}\right\}\mathbb{Re}\left\{\mathrm{A_{l'k'}\dot{s}_{k'}}\sum_{\mathrm{p'=1}}^{\mathrm{N}} \eta_{\mathrm{p'l'}}\mathrm{u_{p'}^*}\right\}+\\
&+ 8\pi f_0 \frac{E}{M}\,\mathbb{Re}\left\{\zeta \mathrm{A_{lk}\dot{s}_k}\sum_{\mathrm{p=1}}^{\mathrm{N}} \eta_{\mathrm{pl}}\mathrm{r_p^*}\right\}\mathbb{Im}\left\{\zeta \mathrm{A_{l'k'}s_{k'}}\sum_{\mathrm{p'=1}}^{\mathrm{N}} \eta_{\mathrm{p'l'}}\mathrm{r_{p'}^*}\right\}+\\
&+ 4\frac{E}{M}\,\mathbb{Re}\left\{\zeta \mathrm{A_{lk}\dot{s}_k}\sum_{\mathrm{p=1}}^{\mathrm{N}} \eta_{\mathrm{pl}}\mathrm{r_p^*}\right\}\mathbb{Re}\left\{\zeta \mathrm{A_{l'k'}s_{k'}}\sum_{\mathrm{p'=1}}^{\mathrm{N}} \eta_{\mathrm{p'l'}}\mathrm{r_{p'}^*}\right\}+\\
&- 8\pi f_0 \sqrt{\frac{E^3}{M^3}}|\zeta|^2\,\mathbb{Re}\left\{\zeta \mathrm{A_{lk}\dot{s}_k}\sum_{\mathrm{p=1}}^{\mathrm{N}} \eta_{\mathrm{pl}}\mathrm{r_p^*}\right\}\mathbb{Im}\left\{\mathrm{A_{l'k'}s_{k'}}\sum_{\mathrm{p'=1}}^{\mathrm{N}} \eta_{\mathrm{p'l'}}\mathrm{u_{p'}^*}\right\}+\\
&- 4\sqrt{\frac{E^3}{M^3}}|\zeta|^2\,\mathbb{Re}\left\{\zeta \mathrm{A_{lk}\dot{s}_k}\sum_{\mathrm{p=1}}^{\mathrm{N}} \eta_{\mathrm{pl}}\mathrm{r_p^*}\right\}\mathbb{Re}\left\{\mathrm{A_{l'k'}\dot{s}_{k'}}\sum_{\mathrm{p'=1}}^{\mathrm{N}} \eta_{\mathrm{p'l'}}\mathrm{u_{p'}^*}\right\}+\\
&- 16\pi^2 f_0^2 \sqrt{\frac{E^3}{M^3}}|\zeta|^2\,\mathbb{Im}\left\{\mathrm{A_{lk}s_k}\sum_{\mathrm{p=1}}^{\mathrm{N}} \eta_{\mathrm{pl}}{}^*\mathrm{u_p^*}\right\}\mathbb{Im}\left\{\zeta \mathrm{A_{l'k'}s_{k'}}\sum_{\mathrm{p'=1}}^{\mathrm{N}} \eta_{\mathrm{p'l'}}\mathrm{r_{p'}^*}\right\}+\\
&- 8\pi f_0 \sqrt{\frac{E^3}{M^3}}|\zeta|^2\,\mathbb{Im}\left\{\mathrm{A_{lk}s_k}\sum_{\mathrm{p=1}}^{\mathrm{N}} \eta_{\mathrm{pl}}{}^*\mathrm{u_p^*}\right\}\mathbb{Re}\left\{\zeta \mathrm{A_{l'k'}s_{k'}}\sum_{\mathrm{p'=1}}^{\mathrm{N}} \eta_{\mathrm{p'l'}}\mathrm{r_{p'}^*}\right\}+\\
&+ 16\pi^2 f_0^2 \frac{E^2}{M^2}|\zeta|^4\,\mathbb{Im}\left\{\mathrm{A_{lk}s_k}\sum_{\mathrm{p=1}}^{\mathrm{N}} \eta_{\mathrm{pl}}{}^*\mathrm{u_p^*}\right\}\mathbb{Im}\left\{\mathrm{A_{l'k'}s_{k'}}\sum_{\mathrm{p'=1}}^{\mathrm{N}} \eta_{\mathrm{p'l'}}\mathrm{u_{p'}^*}\right\}+
\end{aligned}$$

$$+ 8\pi f_0 \frac{E^2}{M^2}|\zeta|^4 \,\mathbb{I}\mathrm{m}\left\{\mathrm{A_{lk}s_k}\sum_{\mathrm{p=1}}^{\mathrm{N}} \eta_{\mathrm{pl}}{*}\mathrm{u_p^*}\right\} \mathbb{R}\mathrm{e}\left\{\mathrm{A_{l'k'}\dot{s}_{k'}}\sum_{\mathrm{p'=1}}^{\mathrm{N}} \eta_{\mathrm{p'l'}}\mathrm{u_{p'}^*}\right\} +$$
$$- 8\pi f_0\sqrt{\frac{E^3}{M^3}}|\zeta|^2 \,\mathbb{R}\mathrm{e}\left\{\mathrm{A_{lk}\dot{s}_k}\sum_{\mathrm{p=1}}^{\mathrm{N}} \eta_{\mathrm{pl}}\mathrm{u_p^*}\right\} \mathbb{I}\mathrm{m}\left\{\zeta\mathrm{A_{l'k'}s_{k'}}\sum_{\mathrm{p'=1}}^{\mathrm{N}} \eta_{\mathrm{p'l'}}\mathrm{r_{p'}^*}\right\} +$$
$$- 4\sqrt{\frac{E^3}{M^3}}|\zeta|^2 \,\mathbb{R}\mathrm{e}\left\{\mathrm{A_{lk}\dot{s}_k}\sum_{\mathrm{p=1}}^{\mathrm{N}} \eta_{\mathrm{pl}}\mathrm{u_p^*}\right\} \mathbb{R}\mathrm{e}\left\{\zeta\mathrm{A_{l'k'}s_{k'}}\sum_{\mathrm{p'=1}}^{\mathrm{N}} \eta_{\mathrm{p'l'}}\mathrm{r_{p'}^*}\right\} +$$
$$+ 8\pi f_0 \frac{E^2}{M^2}|\zeta|^4 \,\mathbb{R}\mathrm{e}\left\{\mathrm{A_{lk}\dot{s}_k}\sum_{\mathrm{p=1}}^{\mathrm{N}} \eta_{\mathrm{pl}}\mathrm{u_p^*}\right\} \mathbb{I}\mathrm{m}\left\{\mathrm{A_{l'k'}s_{k'}}\sum_{\mathrm{p'=1}}^{\mathrm{N}} \eta_{\mathrm{p'l'}}\mathrm{u_{p'}^*}\right\} +$$
$$+ 4\frac{E^2}{M^2}|\zeta|^4 \,\mathbb{R}\mathrm{e}\left\{\mathrm{A_{lk}\dot{s}_k}\sum_{\mathrm{p=1}}^{\mathrm{N}} \eta_{\mathrm{pl}}\mathrm{u_p^*}\right\} \mathbb{R}\mathrm{e}\left\{\mathrm{A_{l'k'}\dot{s}_{k'}}\sum_{\mathrm{p'=1}}^{\mathrm{N}} \eta_{\mathrm{p'l'}}\mathrm{u_{p'}^*}\right\}$$
(3.73)

As seen before, the first, second, fifth and sixth term can be expanded as

$$8\pi^2 f_0^2 \frac{E}{M}\left[\mathbb{R}\mathrm{e}\left\{|\zeta|^2\mathrm{A_{lk}A_{l'k'}^*s_k s_{k'}^*}\sum_{\mathrm{p=1}}^{\mathrm{N}}\sum_{\mathrm{p'=1}}^{\mathrm{N}} \eta_{\mathrm{pl}}\eta_{\mathrm{p'l'}}^*\mathrm{r_p^* r_{p'}}\right\} - \mathbb{R}\mathrm{e}\left\{\zeta^2\mathrm{A_{lk}A_{l'k'}s_k s_{k'}}\sum_{\mathrm{p=1}}^{\mathrm{N}}\sum_{\mathrm{p'=1}}^{\mathrm{N}} \eta_{\mathrm{pl}}\eta_{\mathrm{p'l'}}\mathrm{r_p^* r_{p'}^*}\right\}\right] +$$
$$+4\pi f_0 \frac{E}{M}\left[\mathbb{I}\mathrm{m}\left\{|\zeta|^2\mathrm{A_{lk}A_{l'k'}^*s_k\dot{s}_{k'}^*}\sum_{\mathrm{p=1}}^{\mathrm{N}}\sum_{\mathrm{p'=1}}^{\mathrm{N}} \eta_{\mathrm{pl}}\eta_{\mathrm{p'l'}}^*\mathrm{r_p^* r_{p'}}\right\} + \mathbb{I}\mathrm{m}\left\{\zeta^2\mathrm{A_{lk}A_{l'k'}s_k\dot{s}_{k'}}\sum_{\mathrm{p=1}}^{\mathrm{N}}\sum_{\mathrm{p'=1}}^{\mathrm{N}} \eta_{\mathrm{pl}}\eta_{\mathrm{p'l'}}\mathrm{r_p^* r_{p'}^*}\right\}\right] +$$
$$-4\pi f_0 \frac{E}{M}\left[\mathbb{I}\mathrm{m}\left\{|\zeta|^2\mathrm{A_{lk}A_{l'k'}^*\dot{s}_k s_{k'}^*}\sum_{\mathrm{p=1}}^{\mathrm{N}}\sum_{\mathrm{p'=1}}^{\mathrm{N}} \eta_{\mathrm{pl}}\eta_{\mathrm{p'l'}}^*\mathrm{r_p^* r_{p'}}\right\} - \mathbb{I}\mathrm{m}\left\{\zeta^2\mathrm{A_{lk}A_{l'k'}\dot{s}_k s_{k'}}\sum_{\mathrm{p=1}}^{\mathrm{N}}\sum_{\mathrm{p'=1}}^{\mathrm{N}} \eta_{\mathrm{pl}}\eta_{\mathrm{p'l'}}\mathrm{r_p^* r_{p'}^*}\right\}\right] +$$
$$+2\frac{E}{M}\left[\mathbb{R}\mathrm{e}\left\{|\zeta|^2\mathrm{A_{lk}A_{l'k'}^*\dot{s}_k\dot{s}_{k'}^*}\sum_{\mathrm{p=1}}^{\mathrm{N}}\sum_{\mathrm{p'=1}}^{\mathrm{N}} \eta_{\mathrm{pl}}\eta_{\mathrm{p'l'}}^*\mathrm{r_p^* r_{p'}}\right\} - \mathbb{R}\mathrm{e}\left\{\zeta^2\mathrm{A_{lk}A_{l'k'}\dot{s}_k\dot{s}_{k'}}\sum_{\mathrm{p=1}}^{\mathrm{N}}\sum_{\mathrm{p'=1}}^{\mathrm{N}} \eta_{\mathrm{pl}}\eta_{\mathrm{p'l'}}\mathrm{r_p^* r_{p'}^*}\right\}\right].$$

(3.74)

It can then be shown that, after performing the expectation, the terms proportional to $\frac{E^2}{M^2}$ cancel out, leaving

$$E\left\{\left(1+\frac{2t(n)}{\nu}\right)^{-2}\frac{\partial t(n)}{\partial\tau_{lk}}\frac{\partial t(n)}{\partial\tau_{l'k'}}\right\} = 8\pi^2 f_0^2\frac{E}{M}|\zeta|^2\frac{\nu}{\nu+2}\alpha_1\,\mathbb{R}\mathrm{e}\left\{\mathrm{A_{lk}A_{l'k'}^*s_k s_{k'}^*}\sum_{\mathrm{p=1}}^{\mathrm{N}}\sum_{\mathrm{p'=1}}^{\mathrm{N}} \eta_{\mathrm{pl}}\eta_{\mathrm{l'p'}}\sigma_{\mathrm{p'p}}\right\} +$$
$$+ 2\frac{E}{M}|\zeta|^2\frac{\nu}{\nu+2}\alpha_1\,\mathbb{R}\mathrm{e}\left\{\mathrm{A_{lk}A_{l'k'}^*\dot{s}_k\dot{s}_{k'}^*}\sum_{\mathrm{p=1}}^{\mathrm{N}}\sum_{\mathrm{p'=1}}^{\mathrm{N}} \eta_{\mathrm{pl}}\eta_{\mathrm{l'p'}}\sigma_{\mathrm{p'p}}\right\} +$$
$$+ 4\pi f_0\frac{E}{M}|\zeta|^2\frac{\nu}{\nu+2}\alpha_1\,\mathbb{I}\mathrm{m}\left\{\mathrm{A_{lk}A_{l'k'}^*}\left(\mathrm{s_k\dot{s}_{k'}^*\text{-}\dot{s}_k s_{k'}^*}\right)\sum_{\mathrm{p=1}}^{\mathrm{N}}\sum_{\mathrm{p'=1}}^{\mathrm{N}} \eta_{\mathrm{pl}}\eta_{\mathrm{p'l'}}\sigma_{\mathrm{p'p}}\right\}$$
(3.75)

Regarding the second order derivative,

$$E\left\{\left(1+\frac{2t(n)}{\nu}\right)^{-1}\frac{\partial^2 t(n)}{\partial\tau_{lk}\partial\tau_{l'k'}}\right\} = 8\pi^2 f_0^2\frac{E}{M}|\zeta|^2\alpha_2\,\mathbb{Re}\left\{\mathrm{A_{lk}A^*_{l'k'}s_k s^*_{k'}}\eta_{\mathrm{l'l}}\right\}+$$
$$+\,2\frac{E}{M}|\zeta|^2\alpha_2\,\mathbb{Re}\left\{\mathrm{A_{lk}A^*_{l'k'}\dot{s}_k\dot{s}^*_{k'}}\eta_{\mathrm{l'l}}\right\}+$$
$$+\,4\pi f_0\frac{E}{M}|\zeta|^2\frac{\nu}{\nu+2}\alpha_2\,\mathbb{Im}\left\{\mathrm{A_{lk}A^*_{l'k'}}\left(\mathrm{s_k\dot{s}^*_{k'}\text{-}\dot{s}_k s^*_{k'}}\right)\eta_{\mathrm{l'l}}\right\}. \tag{3.76}$$

Then, by using the orthogonality assumption and highlighting the similarities between the two parts of the FIM and naming $\gamma_{ll'} \triangleq \left(\alpha_2\eta_{l'l} - \frac{2}{\nu+2}\alpha_1\sum_{p=1}^{N}\sum_{p'=1}^{N}\eta_{pl}\eta_{l'p'}\sigma_{p'p}\right)$, it is

$$[\boldsymbol{J}(\boldsymbol{\theta})]_{i,i'} = 8\pi^2 f_0^2\frac{E}{M}|\zeta|^2\,\mathbb{Re}\left\{\frac{2\mathrm{N}+\nu}{\nu}\gamma_{\mathrm{ll'}}\sum_{\mathrm{n=0}}^{\mathrm{N_o\text{-}1}}\mathrm{A_{lk}A^*_{l'k'}s_k s^*_{k'}}\right\}+$$
$$+\,4\pi f_0\frac{E}{M}|\zeta|^2\,\mathbb{Im}\left\{\frac{2\mathrm{N}+\nu}{\nu}\gamma_{\mathrm{ll'}}\sum_{\mathrm{n=0}}^{\mathrm{N_o\text{-}1}}\mathrm{A_{lk}A^*_{l'k'}}\left(\mathrm{s_k\dot{s}^*_{k'}\text{-}\dot{s}_k s^*_{k'}}\right)\right\}+$$
$$+\,2\frac{E}{M}|\zeta|^2\,\mathbb{Re}\left\{\frac{2\mathrm{N}+\nu}{\nu}\gamma_{\mathrm{ll'}}\sum_{\mathrm{n=0}}^{\mathrm{N_o\text{-}1}}\mathrm{A_{lk}A^*_{l'k'}\dot{s}_k\dot{s}^*_{k'}}\right\}=$$
$$=\left(8\pi^2 f_0^2\frac{E}{M}|\zeta|^2\frac{2N+\nu}{\nu}\,\mathbb{Re}\left\{\gamma_{\mathrm{ll'}}\sum_{\mathrm{n=0}}^{\mathrm{N_o\text{-}1}}\mathrm{e}^{\text{-}\jmath 2\pi \mathrm{f_0}(\tau_{\mathrm{lk}}\text{-}\tau_{\mathrm{l'k}})}\mathrm{e}^{\jmath 2\pi(\mathrm{f_{lk}\text{-}f_{l'k}})\mathrm{nT_s}}\mathrm{s_k}(\mathrm{nT_s}\text{-}\tau_{\mathrm{l'k}})\mathrm{s^*_k}(\mathrm{nT_s}\text{-}\tau_{\mathrm{l'k}})\right\}+\right.$$
$$+4\pi f_0\frac{E}{M}|\zeta|^2\frac{2N+\nu}{\nu}\,\mathbb{Im}\left\{\gamma_{\mathrm{ll'}}\sum_{\mathrm{n=0}}^{\mathrm{N_o\text{-}1}}\mathrm{e}^{\text{-}\jmath 2\pi \mathrm{f_0}(\tau_{\mathrm{lk}}\text{-}\tau_{\mathrm{l'k}})}\mathrm{e}^{\jmath 2\pi(\mathrm{f_{lk}\text{-}f_{l'k}})\mathrm{nT_s}}\left(\mathrm{s_k}(\mathrm{nT_s}\text{-}\tau_{\mathrm{lk}})\mathrm{\dot{s}^*_{k'}}(\mathrm{nT_s}\text{-}\tau_{\mathrm{l'k}})\right)\right\}+$$
$$-4\pi f_0\frac{E}{M}|\zeta|^2\frac{2N+\nu}{\nu}\,\mathbb{Im}\left\{\gamma_{\mathrm{ll'}}\sum_{\mathrm{n=0}}^{\mathrm{N_o\text{-}1}}\mathrm{e}^{\text{-}\jmath 2\pi \mathrm{f_0}(\tau_{\mathrm{lk}}\text{-}\tau_{\mathrm{l'k}})}\mathrm{e}^{\jmath 2\pi(\mathrm{f_{lk}\text{-}f_{l'k}})\mathrm{nT_s}}\left(\mathrm{\dot{s}_k}(\mathrm{nT_s}\text{-}\tau_{\mathrm{lk}})\mathrm{s^*_k}(\mathrm{nT_s}\text{-}\tau_{\mathrm{l'k}})\right)\right\}+$$
$$\left.+2\frac{E}{M}|\zeta|^2\frac{2N+\nu}{\nu}\,\mathbb{Re}\left\{\gamma_{\mathrm{ll'}}\sum_{\mathrm{n=0}}^{\mathrm{N_o\text{-}1}}\mathrm{e}^{\text{-}\jmath 2\pi \mathrm{f_0}(\tau_{\mathrm{lk}}\text{-}\tau_{\mathrm{l'k}})}\mathrm{e}^{\jmath 2\pi(\mathrm{f_{lk}\text{-}f_{l'k}})\mathrm{nT_s}}\mathrm{\dot{s}_k}(\mathrm{nT_s}\text{-}\tau_{\mathrm{lk}})\mathrm{\dot{s}^*_k}(\mathrm{nT_s}\text{-}\tau_{\mathrm{l'k}})\right\}\right)\delta[k-k']. \tag{3.77}$$

By changing coefficients and removing the parts related to $\dot{s}_k$, the terms related to $f_{lk}, f_{l'k'}$ and $\tau_{lk}, f_{l'k'}$ can be derived directly from above.

- $[\boldsymbol{J}(\boldsymbol{\theta})]_{i,i'}$, $i = (N+l-1)M+k$, $i' = (N+l'-1)M+k'$, $l, l' = 1 \ldots N$, $k, k' = 1 \ldots M$

$$[\boldsymbol{J}(\boldsymbol{\theta})]_{i,i'} = \\ = \frac{8\pi^2 E}{M}|\zeta|^2 \frac{2N+\nu}{\nu} \mathbb{R}\mathrm{e}\left\{\gamma_{\mathrm{ll'}} \sum_{\mathrm{n=0}}^{\mathrm{N_o\text{-}1}} \mathrm{n^2 T_s^2 e^{\text{-}\jmath 2\pi f_0(\tau_{lk}\text{-}\tau_{l'k})} e^{\jmath 2\pi (f_{lk}\text{-}f_{l'k}) nT_s} s_k(nT_s\text{-}\tau_{l'k}) s_k^*(nT_s\text{-}\tau_{l'k})}\right\} \delta[k-k']. \tag{3.78}$$

- $[\boldsymbol{J}(\boldsymbol{\theta})]_{i,i'}$, $i = (l-1)M+k$, $i' = (N+l'-1)M+k'$, $l, l' = 1 \ldots N$, $k, k' = 1 \ldots M$

$$[\boldsymbol{J}(\boldsymbol{\theta})]_{i,i'} = \\ = 4\pi\frac{E}{M}|\zeta|^2 \frac{2N+\nu}{\nu} \mathbb{I}\mathrm{m}\left\{\gamma_{\mathrm{ll'}} \sum_{\mathrm{n=0}}^{\mathrm{N_o\text{-}1}} \mathrm{nT_s e^{\text{-}\jmath 2\pi f_0(\tau_{lk}\text{-}\tau_{l'k})} e^{\jmath 2\pi (f_{lk}\text{-}f_{l'k}) nT_s} \dot{s}_k(nT_s\text{-}\tau_{lk}) s_k^*(nT_s\text{-}\tau_{l'k})}\right\} \delta[k-k'] + \\ -8\pi^2 f_0 \frac{E}{M}|\zeta|^2 \frac{2N+\nu}{\nu} \mathbb{R}\mathrm{e}\left\{\gamma_{\mathrm{ll'}} \sum_{\mathrm{n=0}}^{\mathrm{N_o\text{-}1}} \mathrm{nT_s e^{\text{-}\jmath 2\pi f_0(\tau_{lk}\text{-}\tau_{l'k})} e^{\jmath 2\pi (f_{lk}\text{-}f_{l'k}) nT_s} s_k(nT_s\text{-}\tau_{l'k}) s_k^*(nT_s\text{-}\tau_{l'k})}\right\} \delta[k-k']. \tag{3.79}$$

Chapter 4

Hybrid Cramer-Rao Lower Bound

Following the signal model proposed in [20],

$$r_l(n) = \sqrt{\frac{E}{M}}\zeta \sum_{k=1}^{M} e^{-\jmath 2\pi f_0 \tau_{lk}} e^{\jmath 2\pi f_{lk} n T_s} e^{\jmath(\phi_{t_k} - \phi_{r_l})} s_k(n) + z_l(n) \ , \qquad l = 1 \dots N \ , \quad n = 0 \dots N_o - 1 \tag{4.1}$$

where ϕ_{t_k} and ϕ_{r_l} are the phase errors introduced at transmitter k^{th} and receiver l^{th}, respectively. These additional terms are assumed i.i.d and normally distributed, with mean value 0 and variance σ_ϕ^2: then it is $\boldsymbol{\phi} = \left[\phi_{r_1}, \phi_{r_2}, \dots, \phi_{r_N}, \phi_{t_1}, \phi_{t_2}, \dots, \phi_{t_M}\right] \sim \mathcal{CN}\left(\mathbf{0}_{N+M}, \sigma_\phi^2 \boldsymbol{I}_{N+M}\right)$.

Following a similar approach to that of Ch. 2, the new parameter vector is $\boldsymbol{\psi}^h = \left[\boldsymbol{\psi}, \boldsymbol{\phi}\right]$, where obviously the additional phase terms are nuisance random parameters. Because the chain rule for matrix derivation still applies, it is preferable to work with $\boldsymbol{\theta}^h = \left[\boldsymbol{\theta}, \boldsymbol{\phi}\right]$ with a new Jacobian matrix

$$\boldsymbol{P}^h = \begin{bmatrix} \boldsymbol{P}_{6 \times 2NM+2} & \mathbf{0}_{6 \times N+M} \\ \mathbf{0}_{N+M \times 2NM+2} & \boldsymbol{I}_{N+M} \end{bmatrix} \tag{4.2}$$

such that

$$\boldsymbol{J}(\boldsymbol{\psi}^h) = \boldsymbol{P}^h \boldsymbol{J}(\boldsymbol{\theta}^h) \left(\boldsymbol{P}^h\right)^T . \tag{4.3}$$

Recalling the definition given in equation (2.7), the Hybrid FIM can be decomposed in two terms as follows

$$\begin{aligned}
\left[\boldsymbol{J}\left(\boldsymbol{\theta}^h\right)\right]_{p,q} &= -E_{\boldsymbol{r},\phi}\left\{\frac{\left[\partial^2 \log p\left(\boldsymbol{r}|\boldsymbol{\phi};\boldsymbol{\theta}\right) + \partial^2 \log p\left(\boldsymbol{\phi}|\boldsymbol{\theta}\right)\right]}{\partial\theta_p^h\partial\theta_q^h}\right\} = \\
&= -E_{\boldsymbol{\phi}}\left\{E_{\boldsymbol{r}|\boldsymbol{\phi}}\left\{\frac{\partial^2 \log p\left(\boldsymbol{r}|\boldsymbol{\phi};\boldsymbol{\theta}\right)}{\partial\theta_p^h\partial\theta_q^h}\right\}\right\} - E_{\boldsymbol{\phi}}\left\{\frac{\partial^2 \log p\left(\boldsymbol{\phi}|\boldsymbol{\theta}\right)}{\partial\theta_p^h\partial\theta_q^h}\right\} = \\
&= \left[\boldsymbol{J}_D\right]_{p,q} + \left[\boldsymbol{J}_P\right]_{p,q}, \qquad\qquad p,q = 1\ldots 2NM+N+M+2
\end{aligned} \tag{4.4}$$

where $\boldsymbol{J}_D$ represented the contribution of the data, while $\boldsymbol{J}_P$ contains the prior information. In order to simplify the notation, here $L \triangleq N + M$ and $Q \triangleq N \times M$. In addition, while the following derivations are generally valid for any kind of orthogonal signals, the final result presented will be limited to the frequency spaced pulses.

$\boldsymbol{J}_P$ derivation

Given the assumptions on the phase errors, from [20] it is

$$\begin{aligned}
\log p\left(\boldsymbol{\phi}|\boldsymbol{\theta}\right) = \log p\left(\boldsymbol{\phi}\right) &= \log\left[\frac{1}{\left(\sqrt{2\pi\sigma_\phi^2}\right)^L}\prod_{l=1}^{N} e^{-\frac{\phi_{r_l}^2}{2\sigma_\phi^2}}\prod_{k=1}^{M} e^{-\frac{\phi_{t_k}^2}{2\sigma_\phi^2}}\right] = \\
&= K - \sum_{l=1}^{N}\frac{\phi_{r_l}^2}{2\sigma_\phi^2} - \sum_{k=1}^{M}\frac{\phi_{t_k}^2}{2\sigma_\phi^2} = K - \frac{1}{2\sigma_\phi^2}\left(\sum_{l=1}^{N}\phi_{r_l}^2 + \sum_{k=1}^{M}\phi_{t_k}^2\right)
\end{aligned} \tag{4.5}$$

where K is a generic constant independent of the phase terms. Then, it can be easily shown that

$$\frac{\partial^2 \log p\left(\boldsymbol{\phi}\right)}{\partial\phi_{r_l}^2} = \frac{\partial^2 \log p\left(\boldsymbol{\phi}\right)}{\partial\phi_{t_k}^2} = -\frac{1}{\sigma_\phi^2} \qquad\qquad \frac{\partial^2 \log p\left(\boldsymbol{\phi}\right)}{\partial\phi_{r_l}\partial\phi_{t_k}} = 0 \tag{4.6}$$

and that the derivatives w.r.t. $\boldsymbol{\theta}$ are also 0, since the a-priori pdf does not depend on them. Then it is

$$\boldsymbol{J}_P = \begin{bmatrix} \mathbf{0}_{2Q+2,2Q+2} & \mathbf{0}_{2Q+2,L} \\ \mathbf{0}_{L,2Q+2} & \frac{1}{\sigma_\phi^2}\boldsymbol{I}_L \end{bmatrix} \tag{4.7}$$

$\boldsymbol{J}_D$ derivation

Similarly to the derivation of the ideal bounds, the following variables are defined:
$u_p^h(n) = \sum_{q=1}^{M} e^{-\jmath 2\pi f_0 \tau_{pq}} e^{\jmath 2\pi f_{pq} n T_s} e^{\jmath(\phi_{t_q} - \phi_{r_p})} s_q(nT_s - \tau_{pq})$,
$A^h = e^{-\jmath 2\pi f_0 \tau_{lk}} e^{\jmath 2\pi f_{lk} n T_s} e^{\jmath(\phi_{t_k} - \phi_{r_l})}$,
$A'^h = e^{-\jmath 2\pi f_0 \tau_{l'k'}} e^{\jmath 2\pi f_{l'k'} n T_s} e^{\jmath(\phi_{t_{k'}} - \phi_{r_{l'}})}$.
Then it can be shown easily that the results already found in case 1 of Ch. 3 still stand: on one hand, the additional phase terms have a similar structure of the terms related to delays and Doppler shifts, which cancel out at the end; on the other hand, when clutter is independent there is no mutual information between different transmitter-receiver paths, so the phase noise has no impact. This result already provides some information on the final HCRB: by partitioning the $\boldsymbol{J}(\boldsymbol{\theta}^h)$ matrix as

$$\boldsymbol{J}(\boldsymbol{\theta}^h) = \boldsymbol{J}_D + \boldsymbol{J}_P = \begin{bmatrix} \boldsymbol{J}(\boldsymbol{\theta}) & \boldsymbol{B}^h \\ (\boldsymbol{B}^h)^T & \boldsymbol{D}^h \end{bmatrix}, \tag{4.8}$$

and proceeding by the means of block processing, it is

$$\boldsymbol{HCRLB}(\boldsymbol{\psi}^h) = \left[\boldsymbol{P}^h \boldsymbol{J}(\boldsymbol{\theta}^h) \left(\boldsymbol{P}^h\right)^T\right]^{-1} = \begin{bmatrix} \boldsymbol{P}\boldsymbol{J}(\boldsymbol{\theta})\,(\boldsymbol{P})^T & \boldsymbol{P}\boldsymbol{B} \\ (\boldsymbol{P}\boldsymbol{B})^t & \boldsymbol{D} \end{bmatrix}^{-1} = \begin{bmatrix} \boldsymbol{J}(\boldsymbol{\psi}) & \boldsymbol{P}\boldsymbol{B} \\ (\boldsymbol{P}\boldsymbol{B})^T & \boldsymbol{D} \end{bmatrix}^{-1}$$

$$\begin{aligned}\boldsymbol{HCRLB}(\boldsymbol{\psi}) &= \boldsymbol{J}(\boldsymbol{\psi})^{-1} + \boldsymbol{J}(\boldsymbol{\psi})^{-1}\boldsymbol{P}\boldsymbol{B}\left(D - (\boldsymbol{P}\boldsymbol{B})^T \boldsymbol{J}(\boldsymbol{\psi})^{-1}\boldsymbol{P}\boldsymbol{B}\right)^{-1}(\boldsymbol{P}\boldsymbol{B})^T \boldsymbol{J}(\boldsymbol{\psi})^{-1} = \\ &= \boldsymbol{CRLB}(\boldsymbol{\psi}) + \boldsymbol{\Delta CRLB}\end{aligned} \tag{4.9}$$

where $\boldsymbol{CRLB}(\boldsymbol{\psi})$ is the bound on the target parameters without the phase mismatch, i.e. the bound derived in Ch.3 - Case 1. It should be noted that $\boldsymbol{D}$ contains the only non-0 terms of $\boldsymbol{J}_P$, in addition to the corresponding terms of $\boldsymbol{J}_D$.
Similarly to what has been done in Ch. 3, $\boldsymbol{J}_D$ can be formulated as

$$\begin{aligned}[\boldsymbol{J}_D]_{i,j} = E_\phi &\left\{-2\frac{2+\nu}{\nu^2}\sum_{n=0}^{N_o-1}\sum_{p=1}^{N} E_{\boldsymbol{r}|\phi}\left\{\left(1+\frac{2t_p(n)}{\nu}\right)^{-2}\frac{\partial t_p(n)}{\partial\theta_i}\frac{\partial t_p(n)}{\partial\theta_j}\right\}\right\} + \\ + E_\phi &\left\{\frac{2+\nu}{\nu}\sum_{n=0}^{N_o-1}\sum_{p=1}^{N} E_{\boldsymbol{r}|\phi}\left\{\left(1+\frac{2t_p(n)}{\nu}\right)^{-1}\frac{\partial^2 t_p(n)}{\partial\theta_i\partial\theta_j}\right\}\right\}.\end{aligned} \tag{4.10}$$

Then the first step in the derivation of the Hybrid FIM is to calculate the first and second order derivatives: given the similarities between $u_p(n)$ and $u_p^h(n)$, it is

$$\begin{aligned}
\frac{\partial t_p(n)}{\partial f_{lk}} &= \left(\frac{4\pi n T_s}{\sigma}\sqrt{\frac{E}{M}}\,\mathbb{Im}\left\{\zeta \mathrm{A^h s_k(n) r_p^*(n)}\right\} - \frac{4\pi n T_s}{\sigma}\frac{E}{M}|\zeta|^2\,\mathbb{Im}\left\{\mathrm{A^h s_k(n) u_p^*(n)}\right\}\right)\delta[l-p] \\
\frac{\partial t_p(n)}{\partial \zeta_{Re}} &= -\frac{2}{\sigma}\sqrt{\frac{E}{M}}\,\mathbb{Re}\left\{\mathrm{u_p^h(n) r_p^*(n)}\right\} + \frac{2}{\sigma}\frac{E}{M}\zeta_{Re}|u_p^h(n)|^2 \\
\frac{\partial t_p(n)}{\partial \zeta_{Im}} &= \frac{2}{\sigma}\sqrt{\frac{E}{M}}\,\mathbb{Im}\left\{\mathrm{u_p^h(n) r_p^*(n)}\right\} + \frac{2}{\sigma}\frac{E}{M}\zeta_{Im}|u_p^h(n)|^2 \\
\frac{\partial t_p(n)}{\partial \tau_{lk}} &= \left(-\frac{4\pi f_0}{\sigma}\sqrt{\frac{E}{M}}\,\mathbb{Im}\left\{\zeta \mathrm{A^h s_k r_p^*(n)}\right\} + \frac{4\pi f_0}{\sigma}\frac{E}{M}|\zeta|^2\,\mathbb{Im}\left\{\mathrm{A^h s_k u_p^*(n)}\right\}\right)\delta[l-p] + \\
&\quad \left(-\frac{2}{\sigma}\sqrt{\frac{E}{M}}\,\mathbb{Re}\left\{\zeta \mathrm{A^h \dot{s}_k r_p^*(n)}\right\} + \frac{2}{\sigma}\frac{E}{M}|\zeta|^2\,\mathbb{Re}\left\{\mathrm{A^h \dot{s}_k u_p^*(n)}\right\}\right)\delta[l-p] \\
\frac{\partial t_p(n)}{\partial \phi_{r_l}} &= -\frac{2}{\sigma}\sqrt{\frac{E}{M}}\,\mathbb{Im}\left\{\zeta \mathrm{u_p^h(n) r_p^*(n)}\right\}\delta[l-p] \\
\frac{\partial t_p(n)}{\partial \phi_{t_k}} &= +\frac{2}{\sigma}\sqrt{\frac{E}{M}}\,\mathbb{Im}\left\{\zeta \mathrm{e}^{-\jmath 2\pi \mathrm{f_0} \tau_{\mathrm{pk}}}\mathrm{e}^{\jmath 2\pi \mathrm{f_{pk} n T_s}}\mathrm{e}^{\jmath(\phi_{\mathrm{t_k}} - \phi_{\mathrm{r_p}})}\mathrm{r_p^*(n) s_k(nT_s} - \tau_{\mathrm{pk}})\right\} + \\
&- \frac{2}{\sigma}\frac{E}{M}|\zeta|^2\,\mathbb{Im}\left\{\mathrm{e}^{-\jmath 2\pi \mathrm{f_0} \tau_{\mathrm{pk}}}\mathrm{e}^{\jmath 2\pi \mathrm{f_{pk} n T_s}}\mathrm{e}^{\jmath(\phi_{\mathrm{t_k}} - \phi_{\mathrm{r_p}})}\mathrm{u_p^*(n) s_k(nT_s} - \tau_{\mathrm{pk}})\right\}.
\end{aligned} \tag{4.11}$$

$$
\begin{aligned}
\frac{\partial^2 t_p(n)}{\partial \zeta_{Re}\partial \phi_{r_l}} &= -\frac{2}{\sigma}\sqrt{\frac{E}{M}}\,\mathbb{Im}\left\{\mathrm{u_l^h(n)r_l^*(n)}\right\}\delta[l-p]\\
\frac{\partial^2 t_p(n)}{\partial \zeta_{Im}\partial \phi_{r_l}} &= -\frac{2}{\sigma}\sqrt{\frac{E}{M}}\,\mathbb{Re}\left\{\mathrm{u_l^h(n)r_l^*(n)}\right\}\delta[l-p]\\
\frac{\partial^2 t_p(n)}{\partial \zeta_{Re}\partial \phi_{t_k}} &= +\frac{2}{\sigma}\sqrt{\frac{E}{M}}\,\mathbb{Im}\left\{\mathrm{e^{-\jmath 2\pi f_0\tau_{pk}}e^{\jmath 2\pi f_{pk}nT_s}e^{\jmath(\phi_{t_k}-\phi_{r_p})}r_p^*(n)s_k}\right\}+\\
&\quad -\frac{4}{\sigma}\frac{E}{M}\zeta_{Re}\,\mathbb{Im}\left\{\mathrm{e^{-\jmath 2\pi f_0\tau_{pk}}e^{\jmath 2\pi f_{pk}nT_s}e^{\jmath(\phi_{t_k}-\phi_{r_p})}u_p^*(n)s_k}\right\}\\
\frac{\partial^2 t_p(n)}{\partial \zeta_{Im}\partial \phi_{t_k}} &= +\frac{2}{\sigma}\sqrt{\frac{E}{M}}\,\mathbb{Re}\left\{\mathrm{e^{-\jmath 2\pi f_0\tau_{pk}}e^{\jmath 2\pi f_{pk}nT_s}e^{\jmath(\phi_{t_k}-\phi_{r_p})}r_p^*(n)s_k}\right\}+\\
&\quad -\frac{4}{\sigma}\frac{E}{M}\zeta_{Im}\,\mathbb{Im}\left\{\mathrm{e^{-\jmath 2\pi f_0\tau_{pk}}e^{\jmath 2\pi f_{pk}nT_s}e^{\jmath(\phi_{t_k}-\phi_{r_p})}u_p^*(n)s_k}\right\}\\
\frac{\partial^2 t_p(n)}{\partial \tau_{lk}\partial \phi_{r_{l'}}} &= \frac{4\pi f_0}{\sigma}\sqrt{\frac{E}{M}}\,\mathbb{Re}\left\{\zeta\mathrm{A^h s_k}\left(\mathrm{r_p^*(n)}\text{-}\sqrt{\frac{\mathrm{E}}{\mathrm{M}}}\zeta^*\left(\mathrm{u_p^h(n)}\right)^*\right)\right\}\delta[l-p,l-l']+\\
&\quad +\left(\frac{4\pi f_0}{\sigma}\frac{E}{M}|\zeta|^2\,\mathbb{Re}\left\{\mathrm{A^h s_k}\left(\mathrm{u_p^h(n)}\right)^*\right\}-\frac{2}{\sigma}\frac{E}{M}|\zeta|^2\,\mathbb{Im}\left\{\mathrm{A^h \dot{s}_k}\left(\mathrm{u_p^h(n)}\right)^*\right\}\right)\delta[l-p,l-l']+\\
&\quad -\frac{2}{\sigma}\sqrt{\frac{E}{M}}\,\mathbb{Im}\left\{\zeta\mathrm{A^h \dot{s}_k}\left(\mathrm{r_p^*(n)}\text{-}\sqrt{\frac{\mathrm{E}}{\mathrm{M}}}\zeta^*\left(\mathrm{u_p^h(n)}\right)^*\right)\right\}\delta[l-p,l-l']\\
\frac{\partial^2 t_p(n)}{\partial f_{lk}\partial \phi_{r_{l'}}} &= -frac4\pi nT_s\sigma\sqrt{\frac{E}{M}}\,\mathbb{Re}\left\{\zeta\mathrm{A^h s_k}\left(\mathrm{r_p^*(n)}\text{-}\sqrt{\frac{\mathrm{E}}{\mathrm{M}}}\zeta^*\left(\mathrm{u_p^h(n)}\right)^*\right)\right\}\delta[l-p,l-l']+\\
&\quad -\frac{4\pi nT_s}{\sigma}\frac{E}{M}|\zeta|^2\,\mathbb{Re}\left\{\mathrm{A^h s_k}\left(\mathrm{u_p^h(n)}\right)^*\right\}\delta[l-p,l-l']\\
\frac{\partial^2 t_p(n)}{\partial \tau_{lk}\partial \phi_{t_{k'}}} &= -\frac{4\pi f_0}{\sigma}\sqrt{\frac{E}{M}}\,\mathbb{Re}\left\{\zeta\mathrm{A^h s_k}\left(\mathrm{r_p^*(n)}\text{-}\sqrt{\frac{\mathrm{E}}{\mathrm{M}}}\zeta^2\left(\mathrm{u_p^h(n)}\right)\right)\right\}\delta[l-p,k-k']+\\
&\quad -\left(\frac{4\pi f_0}{\sigma}\frac{E}{M}|\zeta|^2\,\mathbb{Re}\left\{\mathrm{A^h s_k}\left(\mathrm{A'^h s_{k'}}\right)^*\right\}-\frac{2}{\sigma}\frac{E}{M}|\zeta|^2\,\mathbb{Im}\left\{\mathrm{A^h \dot{s}_k}\left(\mathrm{A'^h s_{k'}}\right)^*\right\}\right)\delta[l-p]+\\
&\quad +\frac{2}{\sigma}\sqrt{\frac{E}{M}}\,\mathbb{Im}\left\{\zeta\mathrm{A^h \dot{s}_k}\left(\mathrm{r_p^*(n)}\text{-}\sqrt{\frac{\mathrm{E}}{\mathrm{M}}}\zeta^*\left(\mathrm{u_p^h(n)}\right)^*\right)\right\}\delta[l-p]\\
\frac{\partial^2 t_p(n)}{\partial f_{lk}\partial \phi_{t_{k'}}} &= \frac{4\pi nT_s}{\sigma}\sqrt{\frac{E}{M}}\,\mathbb{Re}\left\{\zeta\mathrm{A^h s_k}\left(\mathrm{r_p^*}\text{-}\sqrt{\frac{\mathrm{E}}{\mathrm{M}}}\zeta^2\left(\mathrm{u_p^h(n)}\right)\right)\right\}\delta[l-p,k-k']+\\
&\quad +\frac{4\pi nT_s}{\sigma}\frac{E}{M}|\zeta|^2\,\mathbb{Re}\left\{\mathrm{A^h s_k}\left(\mathrm{A'^h s_{k'}}\right)^*\right\}\delta[l-p]\\
\frac{\partial^2 t_p(n)}{\partial \phi_{r_l}\partial \phi_{r_{l'}}} &= \frac{2}{\sigma}\sqrt{\frac{E}{M}}\,\mathbb{Re}\left\{\zeta\mathrm{u_p^h(n)r_p^*(n)}\right\}\delta[l-p,l-l']\\
\frac{\partial^2 t_p(n)}{\partial \phi_{t_k}\partial \phi_{t_{k'}}} &= \frac{2}{\sigma}\sqrt{\frac{E}{M}}\,\mathbb{Re}\left\{\zeta\mathrm{e^{\jmath 2\pi f_0\tau_{pk}}e^{\jmath 2\pi f_{pk}nT_s}e^{\jmath(\phi_{t_k}-\phi_{r_l})}s_k}\left(\mathrm{r_p^*(n)}\text{-}\frac{\mathrm{E}}{\mathrm{M}}\zeta^*\left(\mathrm{u_p^h(n)}\right)^*\right)\right\}\delta[k-k']+\\
&\quad +\frac{2}{\sigma}\frac{E}{M}|\zeta|^2\,\mathbb{Re}\left\{\mathrm{e^{\jmath 2\pi f_0(\tau_{pk}-\tau_{p'k})}e^{\jmath 2\pi(f_{pk}-f_{p'k})nT_s}e^{\jmath(\phi_{t_k}-\phi_{t_{k'}})}s_k s_{k'}^*}\right\}\\
\frac{\partial^2 t_p(n)}{\partial \phi_{r_l}\partial \phi_{t_k}} &= -\frac{2}{\sigma}\sqrt{\frac{E}{M}}\,\mathbb{Re}\left\{\zeta\mathrm{r_p^* A^h s_k}\right\}\delta[l-p].
\end{aligned}
\tag{4.12}
$$

MATRIX $\boldsymbol{B}^h_{2Q+2\times L}$

- $[\boldsymbol{J}_D]_{2Q+1,i}\,, \qquad i = 2Q+3\dots 2Q+2+N$

$$
\begin{aligned}
&\frac{\partial t_p(n)}{\partial \zeta_{Re}}\frac{\partial t_p(n)}{\partial \phi_{r_l}} = \frac{4}{\sigma^2}\frac{E}{M}\,\mathbb{R}\mathrm{e}\left\{\mathrm{u}^{\mathrm{h}}_{\mathrm{p}}(\mathrm{n})\mathrm{r}^*_{\mathrm{p}}(\mathrm{n})\right\}\mathbb{I}\mathrm{m}\left\{\zeta\mathrm{u}^{\mathrm{h}}_{\mathrm{p}}(\mathrm{n})\mathrm{r}^*_{\mathrm{p}}(\mathrm{n})\right\}\delta[l-p]+\\
&-\frac{4}{\sigma^2}\sqrt{\frac{E^3}{M^3}}\zeta_{Re}|u^h_p(n)|^2\,\mathbb{I}\mathrm{m}\left\{\zeta\mathrm{u}^{\mathrm{h}}_{\mathrm{p}}(\mathrm{n})\mathrm{r}^*_{\mathrm{p}}(\mathrm{n})\right\}\delta[l-p] =\\
&=\frac{2}{\sigma^2}\frac{E}{M}\,\mathbb{I}\mathrm{m}\left\{\zeta\left(\mathrm{u}^{\mathrm{h}}_{\mathrm{p}}(\mathrm{n})\mathrm{r}^*_{\mathrm{p}}(\mathrm{n})\right)^2\right\}\delta[l-p]+\frac{2}{\sigma^2}\frac{E}{M}\,\mathbb{I}\mathrm{m}\left\{\zeta|\mathrm{u}^{\mathrm{h}}_{\mathrm{p}}(\mathrm{n})|^2|\mathrm{r}_{\mathrm{p}}(\mathrm{n})|^2\right\}\delta[l-p]+\\
&-\frac{4}{\sigma^2}\sqrt{\frac{E^3}{M^3}}\zeta_{Re}|u^h_p(n)|^2\,\mathbb{I}\mathrm{m}\left\{\zeta\mathrm{u}^{\mathrm{h}}_{\mathrm{p}}(\mathrm{n})\mathrm{r}^*_{\mathrm{p}}(\mathrm{n})\right\}\delta[l-p].
\end{aligned}
\tag{4.13}
$$

The results derived in APPENDIX I are still valid, since the received vector maintains a Complex-t distribution when deriving the HCRB. Then

$$
\begin{aligned}
&E_{\boldsymbol{r}|\phi}\left\{\left(1+\frac{2t_p(n)}{\nu}\right)^{-2}\frac{\partial t_p(n)}{\partial \zeta_{Re}}\frac{\partial t_p(n)}{\partial \phi_{r_l}}\right\} = -\alpha_1\frac{2}{\sigma^2}\frac{E^2}{M^2}|\zeta|^2\zeta_{Im}|u^h_l(n)|^4\delta[l-p]+\\
&+\alpha_1\frac{2}{\sigma^2}\frac{E^2}{M^2}|\zeta|^2\zeta_{Im}|u^h_l(n)|^4\delta[l-p]+\frac{2}{\sigma^2}\frac{E}{M}|u^h_l(n)|^2\zeta_{Im}\alpha_1\frac{\nu}{\nu+2}\sigma\delta[l-p]+\\
&-\alpha_1\frac{4}{\sigma^2}\frac{E^2}{M^2}|\zeta|^2\zeta_{Re}|u^h_l(n)|^2\,\mathbb{I}\mathrm{m}\left\{|\mathrm{u}^{\mathrm{h}}_{\mathrm{l}}(\mathrm{n})|^2\right\}\delta[l-p] =\\
&=\frac{2}{\sigma}\frac{\nu}{\nu+2}\frac{E}{M}\zeta_{Im}\alpha_1|u^h_l(n)|^2\delta[l-p]
\end{aligned}
\tag{4.14}
$$

where $\alpha_1 = \frac{\Gamma\left(\frac{6+\nu}{2}\right)\Gamma\left(\frac{\nu}{2}\right)(\pi\nu)}{\Gamma\left(\frac{2+\nu}{2}\right)\Gamma\left(\frac{4+\nu}{2}\right)(\pi\nu+4\pi)}\left(\frac{\nu}{\nu+4}\right)$. Regarding the second order derivative, it is

$$
E_{\boldsymbol{r}|\phi}\left\{\left(1+\frac{2t_p(n)}{\nu}\right)^{-1}\frac{\partial^2 t_p(n)}{\partial \zeta_{Re}\partial \phi_{r_l}}\right\} = \frac{2}{\sigma}\frac{E}{M}\zeta_{Im}\alpha_2|u^h_l(n)|^2\delta[l-p]
\tag{4.15}
$$

where $\alpha_2 = \frac{\Gamma\left(\frac{4+\nu}{2}\right)\Gamma\left(\frac{\nu}{2}\right)(\pi\nu)}{\Gamma\left(\frac{2+\nu}{2}\right)\Gamma\left(\frac{2+\nu}{2}\right)(\pi\nu+2\pi)}\left(\frac{\nu}{\nu+2}\right)$. Finally, after applying orthogonality, it is

$$\begin{aligned}
[\boldsymbol{J}_D]_{2Q+1,i} &= E_\phi\left\{\left(-2\frac{2+\nu}{\nu^2}\frac{\nu}{\nu+2}\alpha_1 + \frac{2+\nu}{\nu}\alpha_2\right)\frac{2}{\sigma}\frac{E}{M}\zeta_{Im}\sum_{n=0}^{N_o-1}\left|u_l^h(n)|^2\right|\right\} = \\
&= E_\phi\left\{\left(-2\frac{1}{\nu}\alpha_1 + \frac{2+\nu}{\nu}\alpha_2\right)\frac{2}{\sigma}\frac{E}{M}\zeta_{Im}\sum_{q=1}^{M}\sum_{q'=1}^{M}\sum_{n=0}^{N_o-1}A_{lq}^h{A_{lq'}^h}^* s_q(nT_s-\tau_{lq})s_q^*(nT_s-\tau_{lq'})\right\} = \\
&= E_\phi\left\{\left(-2\frac{1}{\nu}\alpha_1 + \frac{2+\nu}{\nu}\alpha_2\right)\frac{2}{\sigma}\frac{E}{M}\zeta_{Im}\sum_{q=1}^{M}\sum_{q'=1}^{M}E_s\delta[q-q']\right\} = \\
&= \left(-2\frac{1}{\nu}\alpha_1 + \frac{2+\nu}{\nu}\alpha_2\right)\frac{2}{\sigma}\frac{E}{M}\zeta_{Im}M = \left(-2\frac{1}{\nu}\alpha_1 + \frac{2+\nu}{\nu}\alpha_2\right)\frac{2}{\sigma}E\zeta_{Im}.
\end{aligned} \tag{4.16}$$

- $[\boldsymbol{J}_D]_{2Q+2,i}\,, \qquad i = 2Q+3\ldots 2Q+2+N$

$$\begin{aligned}
&\frac{\partial t_p(n)}{\partial\zeta_{Im}}\frac{\partial t_p(n)}{\partial\phi_{r_l}} = -\frac{4}{\sigma^2}\frac{E}{M}\,\mathbb{Im}\left\{\mathrm{u_p^h(n)r_p^*(n)}\right\}\mathbb{Im}\left\{\zeta\mathrm{u_p^h(n)r_p^*(n)}\right\}\delta[l-p]+ \\
&-\frac{4}{\sigma^2}\sqrt{\frac{E^3}{M^3}}\zeta_{Im}|u_p^h(n)|^2\,\mathbb{Im}\left\{\zeta\mathrm{u_p^h(n)r_p^*(n)}\right\}\delta[l-p] = \\
&= \frac{2}{\sigma^2}\frac{E}{M}\,\mathbb{Re}\left\{\zeta\left(\mathrm{u_p^h(n)r_p^*(n)}\right)^2\right\}\delta[l-p] - \frac{2}{\sigma^2}\frac{E}{M}\,\mathbb{Re}\left\{\zeta|\mathrm{u_p^h(n)}|^2|\mathrm{r_p(n)}|^2\right\}\delta[l-p]+ \\
&-\frac{4}{\sigma^2}\sqrt{\frac{E^3}{M^3}}\zeta_{Im}|u_p^h(n)|^2\,\mathbb{Im}\left\{\zeta\mathrm{u_p^h(n)r_p^*(n)}\right\}\delta[l-p],
\end{aligned} \tag{4.17}$$

leading to

$$\begin{aligned}
&E_{\boldsymbol{r}|\phi}\left\{\left(1+\frac{2t_p(n)}{\nu}\right)^{-2}\frac{\partial t_p(n)}{\partial\zeta_{Im}}\frac{\partial t_p(n)}{\partial\phi_{r_l}}\right\} = \alpha_1\frac{2}{\sigma^2}\frac{E^2}{M^2}|\zeta|^2\zeta_{Re}|u_l^h(n)|^4\delta[l-p]+ \\
&-\alpha_1\frac{2}{\sigma^2}\frac{E^2}{M^2}|\zeta|^2\zeta_{Re}|u_l^h(n)|^4\delta[l-p] - \frac{2}{\sigma^2}\frac{E}{M}|u_l^h(n)|^2\zeta_{Re}\alpha_1\frac{\nu}{\nu+2}\sigma\delta[l-p]+ \\
&-\alpha_1\frac{4}{\sigma^2}\frac{E^2}{M^2}|\zeta|^2\zeta_{Im}|u_l^h(n)|^2\,\mathbb{Im}\left\{|\mathrm{u_l^h(n)}|^2\right\}\delta[l-p] = \\
&= -\frac{2}{\sigma}\frac{\nu}{\nu+2}\frac{E}{M}\zeta_{Re}\alpha_1|u_l^h(n)|^2\delta[l-p].
\end{aligned} \tag{4.18}$$

Then, by exploiting the similarities with the previous case, it is

$$[\boldsymbol{J}_D]_{2Q+2,i} = -\left(-2\frac{1}{\nu}\alpha_1 + \frac{2+\nu}{\nu}\alpha_2\right)\frac{2}{\sigma}E\zeta_{Re}. \tag{4.19}$$

- $[\boldsymbol{J}_D]_{i,i'}$, $\quad i = 1\ldots Q, \quad i' = 2Q+3\ldots 2Q+2+N$

$$\begin{aligned}\frac{\partial t_p(n)}{\partial \tau_{lk}}\frac{\partial t_p(n)}{\partial \phi_{r_{l'}}} &= \frac{8\pi f_0}{\sigma^2}\frac{E}{M}\,\mathbb{I}\mathrm{m}\left\{\zeta \mathrm{A}^{\mathrm{h}}\mathrm{s}_{\mathrm{k}}\mathrm{r}_{\mathrm{p}}^*(\mathrm{n})\right\}\mathbb{I}\mathrm{m}\left\{\zeta \mathrm{u}_{\mathrm{l'}}^{\mathrm{h}}(\mathrm{n})\mathrm{r}_{\mathrm{l'}}^*(\mathrm{n})\right\}\delta[l'-p]\delta[l-p]+\\ &-\frac{2\pi f_0}{\sigma^2}\sqrt{\frac{E^3}{M^3}}|\zeta|^2\,\mathbb{I}\mathrm{m}\left\{\mathrm{A}^{\mathrm{h}}\mathrm{s}_{\mathrm{k}}\mathrm{u}_{\mathrm{p}}^*(\mathrm{n})\right\}\mathbb{I}\mathrm{m}\left\{\zeta \mathrm{u}_{\mathrm{l'}}^{\mathrm{h}}(\mathrm{n})\mathrm{r}_{\mathrm{l'}}^*(\mathrm{n})\right\}\delta[l'-p]\delta[l-p]+\\ &+\frac{4}{\sigma^2}\frac{E}{M}\,\mathbb{R}\mathrm{e}\left\{\zeta \mathrm{A}^{\mathrm{h}}\dot{\mathrm{s}}_{\mathrm{k}}\mathrm{r}_{\mathrm{p}}^*(\mathrm{n})\right\}\mathbb{I}\mathrm{m}\left\{\zeta \mathrm{u}_{\mathrm{l'}}^{\mathrm{h}}(\mathrm{n})\mathrm{r}_{\mathrm{l'}}^*(\mathrm{n})\right\}\delta[l'-p]\delta[l-p]+\\ &-\frac{4}{\sigma^2}\sqrt{\frac{E^3}{M^3}}|\zeta|^2\,\mathbb{R}\mathrm{e}\left\{\mathrm{A}^{\mathrm{h}}\dot{\mathrm{s}}_{\mathrm{k}}\mathrm{u}_{\mathrm{p}}^*(\mathrm{n})\right\}\mathbb{I}\mathrm{m}\left\{\zeta \mathrm{u}_{\mathrm{l'}}^{\mathrm{h}}(\mathrm{n})\mathrm{r}_{\mathrm{l'}}^*(\mathrm{n})\right\}\delta[l'-p]\delta[l-p]\end{aligned} \tag{4.20}$$

where the first and third terms can be expanded as

$$\begin{aligned}&\frac{4\pi f_0}{\sigma^2}\frac{E}{M}\left[|\zeta|^2\,\mathbb{R}\mathrm{e}\left\{\mathrm{A}_{\mathrm{lk}}^{\mathrm{h}}\mathrm{s}_{\mathrm{k}}\mathrm{r}_{\mathrm{l}}^*\left(\mathrm{u}_{\mathrm{l'}}^{\mathrm{h}}(\mathrm{n})\right)^*\mathrm{r}_{\mathrm{l'}}(\mathrm{n})\right\}-\mathbb{R}\mathrm{e}\left\{\zeta^2\mathrm{A}_{\mathrm{lk}}^{\mathrm{h}}\mathrm{s}_{\mathrm{k}}\mathrm{r}_{\mathrm{l}}^*\mathrm{u}_{\mathrm{l'}}^{\mathrm{h}}(\mathrm{n})\mathrm{r}_{\mathrm{l'}}^*(\mathrm{n})\right\}\right]\delta[l-p,l-l']\\ &\frac{2}{\sigma^2}\frac{E}{M}\left[\mathbb{I}\mathrm{m}\left\{\zeta^2\mathrm{A}_{\mathrm{lk}}^{\mathrm{h}}\dot{\mathrm{s}}_{\mathrm{k}}\mathrm{r}_{\mathrm{l}}^*(\mathrm{n})\mathrm{u}_{\mathrm{l'}}^{\mathrm{h}}(\mathrm{n})\mathrm{r}_{\mathrm{l'}}^*(\mathrm{n})\right\}-|\zeta|^2\,\mathbb{I}\mathrm{m}\left\{\mathrm{A}_{\mathrm{lk}}^{\mathrm{h}}\dot{\mathrm{s}}_{\mathrm{k}}\mathrm{r}_{\mathrm{l}}^*(\mathrm{n})\left(\mathrm{u}_{\mathrm{l'}}^{\mathrm{h}}(\mathrm{n})\right)^*\mathrm{r}_{\mathrm{l'}}(\mathrm{n})\right\}\right]\delta[l-p,l-l'].\end{aligned} \tag{4.21}$$

Then it can be shown that the only surviving terms are

$$\begin{aligned}&E_{\boldsymbol{r}|\phi}\left\{\left(1+\frac{2t_p(n)}{\nu}\right)^{-2}\frac{\partial t_p(n)}{\partial \tau_{lk}}\frac{\partial t_p(n)}{\partial \phi_{r_{l'}}}\right\}=\\ &=\frac{4\pi f_0}{\sigma}\frac{E}{M}|\zeta|^2\,\mathbb{R}\mathrm{e}\left\{\mathrm{A}_{\mathrm{lk}}^{\mathrm{h}}\mathrm{s}_{\mathrm{k}}\left(\mathrm{u}_{\mathrm{l}}^{\mathrm{h}}(\mathrm{n})\right)^*\right\}\alpha_1\frac{\nu}{\nu+2}\delta[l-p,l-l']+\\ &-\frac{2}{\sigma}\frac{E}{M}|\zeta|^2\,\mathbb{I}\mathrm{m}\left\{\mathrm{A}_{\mathrm{lk}}^{\mathrm{h}}\dot{\mathrm{s}}_{\mathrm{k}}\left(\mathrm{u}_{\mathrm{l}}^{\mathrm{h}}(\mathrm{n})\right)^*\right\}\alpha_1\frac{\nu}{\nu+2}\delta[l-p,l-l'].\end{aligned} \tag{4.22}$$

Regarding the second-order derivative, it is

$$E_{\boldsymbol{r}|\phi}\left\{\left(1+\frac{2t_p(n)}{\nu}\right)^{-1}\frac{\partial^2 t_p(n)}{\partial\tau_{lk}\partial\phi_{r_{l'}}}\right\}=\frac{4\pi f_0}{\sigma}\frac{E}{M}|\zeta|^2\,\mathbb{R}\mathrm{e}\left\{\mathrm{A}^{\mathrm{h}}_{\mathrm{lk}}\mathrm{s}_{\mathrm{k}}\left(\mathrm{u}^{\mathrm{h}}_{\mathrm{l}}(\mathrm{n})\right)^*\right\}\alpha_2\delta[l-p,l-l']+$$
$$-\frac{2}{\sigma}\frac{E}{M}|\zeta|^2\,\mathbb{I}\mathrm{m}\left\{\mathrm{A}^{\mathrm{h}}_{\mathrm{lk}}\dot{\mathrm{s}}_{\mathrm{k}}\left(\mathrm{u}^{\mathrm{h}}_{\mathrm{l}}(\mathrm{n})\right)^*\right\}\alpha_2\delta[l-p,l-l']. \tag{4.23}$$

Finally

$$\begin{aligned}
&[\boldsymbol{J}_D]_{i,i'}=\\
&=E_\phi\left\{\left(-2\frac{2+\nu}{\nu^2}\frac{\nu}{\nu+2}\alpha_1+\frac{2+\nu}{\nu}\alpha_2\right)\sum_{n=0}^{N_o-1}\frac{4\pi f_0}{\sigma}\frac{E}{M}|\zeta|^2\,\mathbb{R}\mathrm{e}\left\{\mathrm{A}^{\mathrm{h}}_{\mathrm{lk}}\mathrm{s}_{\mathrm{k}}\left(\mathrm{u}^{\mathrm{h}}_{\mathrm{l}}(\mathrm{n})\right)^*\right\}\delta[l-l']\right\}+\\
&-E_\phi\left\{\left(-2\frac{2+\nu}{\nu^2}\frac{\nu}{\nu+2}\alpha_1+\frac{2+\nu}{\nu}\alpha_2\right)\sum_{n=0}^{N_o-1}\frac{2}{\sigma}\frac{E}{M}|\zeta|^2\,\mathbb{I}\mathrm{m}\left\{\mathrm{A}^{\mathrm{h}}_{\mathrm{lk}}\dot{\mathrm{s}}_{\mathrm{k}}\left(\mathrm{u}^{\mathrm{h}}_{\mathrm{l}}(\mathrm{n})\right)^*\right\}\delta[l-l']\right\}=\\
&=\left(-2\frac{1}{\nu}\alpha_1+\frac{2+\nu}{\nu}\alpha_2\right)\frac{4\pi f_0}{\sigma}\frac{E}{M}|\zeta|^2\delta[l-l']\,\mathbb{R}\mathrm{e}\left\{\sum_{\mathrm{q=1}}^{\mathrm{M}}\sum_{\mathrm{n=0}}^{\mathrm{N_o\text{-}1}}\mathrm{A}^{\mathrm{h}}_{\mathrm{lk}}{\mathrm{A}^{\mathrm{h}}_{\mathrm{lq}}}^*\mathrm{s}^*_{\mathrm{q}}(\mathrm{nT_s\text{-}}\tau_{\mathrm{lq}})\mathrm{s}_{\mathrm{k}}(\mathrm{nT_s\text{-}}\tau_{\mathrm{lk}})\right\}+\\
&-\left(-2\frac{1}{\nu}\alpha_1+\frac{2+\nu}{\nu}\alpha_2\right)\frac{2}{\sigma}\frac{E}{M}|\zeta|^2\delta[l-l']\,\mathbb{I}\mathrm{m}\left\{\sum_{\mathrm{q=1}}^{\mathrm{M}}\sum_{\mathrm{n=0}}^{\mathrm{N_o\text{-}1}}\mathrm{A}^{\mathrm{h}}_{\mathrm{lk}}{\mathrm{A}^{\mathrm{h}}_{\mathrm{lq}}}^*\mathrm{s}^*_{\mathrm{q}}(\mathrm{nT_s\text{-}}\tau_{\mathrm{lq}})\frac{\partial\mathrm{s}_{\mathrm{k}}(\mathrm{nT_s\text{-}}\tau_{\mathrm{lk}})}{\partial\tau_{\mathrm{lk}}}\right\}=\\
&=\left(-2\frac{1}{\nu}\alpha_1+\frac{2+\nu}{\nu}\alpha_2\right)\frac{4\pi f_0}{\sigma}\frac{E}{M}|\zeta|^2E_s\delta[l-l']+\\
&-\left(-2\frac{1}{\nu}\alpha_1+\frac{2+\nu}{\nu}\alpha_2\right)\frac{2}{\sigma}\frac{E}{M}|\zeta|^2\delta[l-l']\,\mathbb{I}\mathrm{m}\left\{\jmath2\pi\mathrm{f_k}\right\}=\\
&=\left(-2\frac{1}{\nu}\alpha_1+\frac{2+\nu}{\nu}\alpha_2\right)\frac{4\pi f_0}{\sigma}\frac{E}{M}|\zeta|^2\delta[l-l']+\\
&-\left(-2\frac{1}{\nu}\alpha_1+\frac{2+\nu}{\nu}\alpha_2\right)\frac{4\pi f_k}{\sigma}\frac{E}{M}|\zeta|^2\delta[l-l']
\end{aligned}\tag{4.24}$$

where the results found in APPENDIX II are still valid.

- $[\boldsymbol{J}_D]_{i,i'}\,,\qquad i=Q+1\ldots2Q,\quad i'=2Q+3\ldots2Q+2+N$

As done in Ch. 3, the terms related to the Doppler shifts can be found instantly from those related to the delays. Thus it is

$$[\boldsymbol{J}_D]_{i,i'} = -E_\phi \left\{ \left(-2\frac{1}{\nu}\alpha_1 + \frac{2+\nu}{\nu}\alpha_2 \right) \frac{4\pi}{\sigma}\frac{E}{M}|\zeta|^2\delta[l-l'] \times \right.$$
$$\left. \times \, \mathbb{R}\mathrm{e}\left\{ \sum_{\mathrm{q=1}}^{\mathrm{M}} \sum_{\mathrm{n=0}}^{\mathrm{N_o\text{-}1}} \mathrm{nT_s A_{lk}^h {A_{lq}^h}^* s_q^*(nT_s\text{-}\tau_{lq}) s_k(nT_s\text{-}\tau_{lk})} \right\} \right\} =$$
$$= -\left(-2\frac{1}{\nu}\alpha_1 + \frac{2+\nu}{\nu}\alpha_2 \right) \frac{4\pi}{\sigma}\frac{E}{M}|\zeta|^2 \left(\frac{T}{2} + \tau_{lk} \right) \delta[l-l'] \tag{4.25}$$

- $[\boldsymbol{J}_D]_{2Q+1,i}$, $\quad i = 2Q+N+3 \ldots 2Q+2+L$

$$\frac{\partial t_p(n)}{\partial \zeta_{Re}}\frac{\partial t_p(n)}{\partial \phi_{t_k}} = -\frac{4}{\sigma^2}\frac{E}{M}\,\mathbb{R}\mathrm{e}\left\{ \mathrm{u_p^h(n) r_p^*(n)} \right\} \mathbb{I}\mathrm{m}\left\{ \zeta \mathrm{A_{pk}^h r_p^*(n) s_k(nT_s\text{-}\tau_{pk})} \right\} +$$
$$+ \frac{4}{\sigma^2}\sqrt{\frac{E^3}{M^3}}|\zeta|^2\,\mathbb{R}\mathrm{e}\left\{ \mathrm{u_p^h(n) r_p^*(n)} \right\} \mathbb{I}\mathrm{m}\left\{ \mathrm{A_{pk}^h u_p^*(n) s_k(nT_s\text{-}\tau_{pk})} \right\} +$$
$$+ \frac{4}{\sigma^2}\sqrt{\frac{E^3}{M^3}}\zeta_{Re}|u_p^h(n)|^2\,\mathbb{I}\mathrm{m}\left\{ \zeta \mathrm{A_{pk}^h r_p^*(n) s_k(nT_s\text{-}\tau_{pk})} \right\} +$$
$$- \frac{4}{\sigma^2}\frac{E^2}{M^2}|\zeta|^2\zeta_{Re}|u_p^h(n)|^2\,\mathbb{I}\mathrm{m}\left\{ \mathrm{A_{pk}^h u_p^*(n) s_k(nT_s\text{-}\tau_{pk})} \right\} \tag{4.26}$$

where the first term can be expanded as

$$-\frac{2}{\sigma^2}\frac{E}{M}\left[\mathbb{I}\mathrm{m}\left\{ \zeta \mathrm{A_{pk}^h s_k r_p^*(n)} \left(\mathrm{u_p^h(n)} \right)^* \mathrm{r_p(n)} \right\} + \mathbb{I}\mathrm{m}\left\{ \zeta \mathrm{A_{pk}^h s_k r_p^*(n) u_p^h(n) r_p^*(n)} \right\} \right]. \tag{4.27}$$

Then it can be shown that

$$E_{\boldsymbol{r}|\phi}\left\{ \left(1 + \frac{2t_p(n)}{\nu} \right)^{-2} \frac{\partial t_p(n)}{\partial \zeta_{Re}}\frac{\partial t_p(n)}{\partial \phi_{t_k}} \right\} = -\frac{2}{\sigma}\frac{E}{M}\,\mathbb{I}\mathrm{m}\left\{ \zeta \mathrm{A_{pk}^h s_k} \left(\mathrm{u_p^h(n)} \right)^* \right\} \alpha_1 \frac{\nu}{\nu+2}. \tag{4.28}$$

Regarding the second-order derivative, it is

$$E_{\boldsymbol{r}|\phi}\left\{\left(1+\frac{2t_p(n)}{\nu}\right)^{-1}\frac{\partial^2 t_p(n)}{\partial\zeta_{Re}\partial\phi_{t_k}}\right\} = -\frac{2}{\sigma}\frac{E}{M}\,\mathbb{I}\mathrm{m}\left\{\zeta \mathrm{A}_{\mathrm{pk}}^{\mathrm{h}}\mathrm{s}_{\mathrm{k}}\left(\mathrm{u}_{\mathrm{p}}^{\mathrm{h}}(\mathrm{n})\right)^*\right\}\alpha_2. \tag{4.29}$$

Finally,

$$\begin{aligned}
[\boldsymbol{J}_D]_{2Q+1,i} &= -E_\phi\left\{\left(-2\frac{1}{\nu}\alpha_1+\frac{2+\nu}{\nu}\alpha_2\right)\sum_{p=1}^{N}\sum_{n=0}^{N_o-1}\frac{2}{\sigma}\frac{E}{M}\,\mathbb{I}\mathrm{m}\left\{\zeta \mathrm{A}_{\mathrm{pk}}^{\mathrm{h}}\mathrm{s}_{\mathrm{k}}\left(\mathrm{u}_{\mathrm{p}}^{\mathrm{h}}(\mathrm{n})\right)^*\right\}\right\} = \\
&= -\left(-2\frac{1}{\nu}\alpha_1+\frac{2+\nu}{\nu}\alpha_2\right)\frac{2}{\sigma}\frac{E}{M}\,\mathbb{I}\mathrm{m}\left\{\zeta\sum_{\mathrm{p=1}}^{\mathrm{N}}\sum_{\mathrm{q=1}}^{\mathrm{M}}\sum_{\mathrm{n=0}}^{\mathrm{N_o\text{-}1}}\mathrm{A}_{\mathrm{pk}}^{\mathrm{h}}{\mathrm{A}_{\mathrm{pq}}^{\mathrm{h}}}^*\mathrm{s}_{\mathrm{k}}(\mathrm{nT_s}\text{-}\tau_{\mathrm{pk}})\mathrm{s}_{\mathrm{q}}^*(\mathrm{nT_s}\text{-}\tau_{\mathrm{pq}})\right\} = \\
&= -\left(-2\frac{1}{\nu}\alpha_1+\frac{2+\nu}{\nu}\alpha_2\right)\frac{2}{\sigma}\frac{E}{M}\,\mathbb{I}\mathrm{m}\left\{\zeta\sum_{\mathrm{p=1}}^{\mathrm{N}}\sum_{\mathrm{q=1}}^{\mathrm{M}}\mathrm{E_s}\delta[\mathrm{q\text{-}q'}]\right\} = \\
&= -\left(-2\frac{1}{\nu}\alpha_1+\frac{2+\nu}{\nu}\alpha_2\right)\frac{2}{\sigma}\frac{E}{M}\zeta_{Im}N.
\end{aligned} \tag{4.30}$$

- $[\boldsymbol{J}_D]_{2Q+2,i}\,, \qquad i = 2Q+N+3\ldots 2Q+2+L$

Similarities with the previous case can be used, and thus it can be proven that

$$\begin{aligned}
[\boldsymbol{J}_D]_{2Q+2,i} &= E_\phi\left\{\left(-2\frac{1}{\nu}\alpha_1+\frac{2+\nu}{\nu}\alpha_2\right)\sum_{p=1}^{N}\sum_{n=0}^{N_o-1}\frac{2}{\sigma}\frac{E}{M}\,\mathbb{R}\mathrm{e}\left\{\zeta \mathrm{A}_{\mathrm{pk}}^{\mathrm{h}}\mathrm{s}_{\mathrm{k}}\left(\mathrm{u}_{\mathrm{p}}^{\mathrm{h}}(\mathrm{n})\right)^*\right\}\right\} = \\
&= \left(-2\frac{1}{\nu}\alpha_1+\frac{2+\nu}{\nu}\alpha_2\right)\frac{2}{\sigma}\frac{E}{M}\zeta_{Re}N.
\end{aligned} \tag{4.31}$$

- $[\boldsymbol{J}_D]_{i,i'}\,, \qquad i = 1\ldots Q, \quad i' = 2Q+N+3\ldots 2Q+2+L$

$$
\begin{aligned}
\frac{\partial t_p(n)}{\partial \tau_{lk}}\frac{\partial t_p(n)}{\partial \phi_{t_{k'}}} &= -\frac{8\pi f_0}{\sigma^2}\frac{E}{M}\,\mathbb{Im}\left\{\zeta \mathrm{A}^{\mathrm{h}}\mathrm{s_k}\mathrm{r_p^*(n)}\right\}\mathbb{Im}\left\{\zeta \mathrm{A}^{\mathrm{h}}_{\mathrm{pk'}}\mathrm{r_p^*(n)s_{k'}}\right\}\delta[l-p]+ \\
&+\frac{8\pi f_0}{\sigma^2}\sqrt{\frac{E^3}{M^3}}|\zeta|^2\,\mathbb{Im}\left\{\mathrm{A}^{\mathrm{h}}\mathrm{s_k}\mathrm{u_p^*(n)}\right\}\mathbb{Im}\left\{\zeta \mathrm{A}^{\mathrm{h}}_{\mathrm{pk'}}\mathrm{r_p^*(n)s_{k'}}\right\}\delta[l-p]+ \\
&-\frac{4}{\sigma^2}\frac{E}{M}\,\mathbb{Re}\left\{\zeta \mathrm{A}^{\mathrm{h}}\dot{\mathrm{s}}_{\mathrm{k}}\mathrm{r_p^*(n)}\right\}\mathbb{Im}\left\{\zeta \mathrm{A}^{\mathrm{h}}_{\mathrm{pk'}}\mathrm{r_p^*(n)s_{k'}}\right\}\delta[l-p]+ \\
&+\frac{4}{\sigma^2}\sqrt{\frac{E^3}{M^3}}|\zeta|^2\,\mathbb{Re}\left\{\mathrm{A}^{\mathrm{h}}\dot{\mathrm{s}}_{\mathrm{k}}\mathrm{u_p^*(n)}\right\}\mathbb{Im}\left\{\zeta \mathrm{A}^{\mathrm{h}}_{\mathrm{pk'}}\mathrm{r_p^*(n)s_{k'}}\right\}\delta[l-p]+ \\
&+\frac{8pif_0}{\sigma^2}\sqrt{\frac{E^3}{M^3}}|\zeta|^2\,\mathbb{Im}\left\{\zeta \mathrm{A}^{\mathrm{h}}\mathrm{s_k}\mathrm{r_p^*(n)}\right\}\mathbb{Im}\left\{\mathrm{A}^{\mathrm{h}}_{\mathrm{pk'}}\mathrm{u_p^*(n)s_{k'}}\right\}\delta[l-p]+ \\
&-\frac{8\pi f_0}{\sigma^2}\frac{E^2}{M^2}|\zeta|^4\,\mathbb{Im}\left\{\mathrm{A}^{\mathrm{h}}\mathrm{s_k}\mathrm{u_p^*(n)}\right\}\mathbb{Im}\left\{\mathrm{A}^{\mathrm{h}}_{\mathrm{pk'}}\mathrm{u_p^*(n)s_{k'}}\right\}\delta[l-p]+ \\
&+\frac{4}{\sigma^2}\sqrt{\frac{E^3}{M^3}}|\zeta|^2\,\mathbb{Re}\left\{\zeta \mathrm{A}^{\mathrm{h}}\dot{\mathrm{s}}_{\mathrm{k}}\mathrm{r_p^*(n)}\right\}\mathbb{Im}\left\{\mathrm{A}^{\mathrm{h}}_{\mathrm{pk'}}\mathrm{u_p^*(n)s_{k'}}\right\}\delta[l-p]+ \\
&-\frac{4}{\sigma^2}\frac{E^2}{M^2}|\zeta|^4\,\mathbb{Re}\left\{\mathrm{A}^{\mathrm{h}}\dot{\mathrm{s}}_{\mathrm{k}}\mathrm{u_p^*(n)}\right\}\mathbb{Im}\left\{\mathrm{A}^{\mathrm{h}}_{\mathrm{pk'}}\mathrm{u_p^*(n)s_{k'}}\right\}\delta[l-p]
\end{aligned}
\tag{4.32}
$$

where the first and third terms can be expanded as

$$
\begin{aligned}
&-\frac{4\pi f_0}{\sigma^2}\frac{E}{M}\left[|\zeta|^2\,\mathbb{Re}\left\{\mathrm{A}^{\mathrm{h}}_{\mathrm{lk}}\mathrm{s_k}\left(\mathrm{A}^{\mathrm{h}}_{\mathrm{lk'}}\mathrm{s_{k'}}\right)^*\mathrm{r_l^*(n)r_l(n)}\right\}-\mathbb{Re}\left\{\zeta^2\mathrm{A}^{\mathrm{h}}_{\mathrm{lk}}\mathrm{s_k}\mathrm{A}^{\mathrm{h}}_{\mathrm{lk'}}\mathrm{s_{k'}}\mathrm{r_l^*(n)r_l^*(n)}\right\}\right]\delta[l-p]+ \\
&+\frac{2}{\sigma^2}\frac{E}{M}\left[|\zeta|^2\,\mathbb{Im}\left\{\mathrm{A}^{\mathrm{h}}_{\mathrm{lk}}\dot{\mathrm{s}}_{\mathrm{k}}\left(\mathrm{A}^{\mathrm{h}}_{\mathrm{lk'}}\mathrm{s_{k'}}\right)^*\mathrm{r_l^*(n)r_l(n)}\right\}-\mathbb{Im}\left\{\zeta^2\mathrm{A}^{\mathrm{h}}_{\mathrm{lk}}\dot{\mathrm{s}}_{\mathrm{k}}\mathrm{A}^{\mathrm{h}}_{\mathrm{lk'}}\mathrm{s_{k'}}\mathrm{r_l^*(n)r_l^*(n)}\right\}\right]\delta[l-p].
\end{aligned}
\tag{4.33}
$$

As before, it can be shown that only the terms proportional to $\frac{E}{M}$ survive the expectation, leaving

$$
\begin{aligned}
E_{r|\phi}\left\{\left(1+\frac{2t_p(n)}{\nu}\right)^{-2}\frac{\partial t_p(n)}{\partial \tau_{lk}}\frac{\partial t_p(n)}{\partial \phi_{t_{k'}}}\right\} &= -\frac{4\pi f_0}{\sigma}\frac{E}{M}|\zeta|^2\,\mathbb{Re}\left\{\mathrm{A}^{\mathrm{h}}_{\mathrm{lk}}\mathrm{A}^{\mathrm{h}}_{\mathrm{lk'}}{}^*\mathrm{s_k}\mathrm{s_{k'}^*}\right\}\alpha_1\frac{\nu}{\nu+2}\delta[l-p]+ \\
&+\frac{2}{\sigma}\frac{E}{M}|\zeta|^2\,\mathbb{Im}\left\{\mathrm{A}^{\mathrm{h}}_{\mathrm{lk}}\mathrm{A}^{\mathrm{h}}_{\mathrm{lk'}}{}^*\dot{\mathrm{s}}_{\mathrm{k}}\mathrm{s_{k'}^*}\right\}\alpha_1\frac{\nu}{\nu+2}\delta[l-p].
\end{aligned}
\tag{4.34}
$$

Regarding the second order derivative, it is

$$E_{\boldsymbol{r}|\phi}\left\{\left(1+\frac{2t_p(n)}{\nu}\right)^{-1}\frac{\partial^2 t_p(n)}{\partial\tau_{lk}\partial\phi_{t_{k'}}}\right\}=$$
$$=-\frac{4\pi f_0}{\sigma}\frac{E}{M}|\zeta|^2\,\mathbb{R}\mathrm{e}\left\{\mathrm{A}_{\mathrm{lk}}^{\mathrm{h}}\mathrm{A}_{\mathrm{lk'}}^{\mathrm{h}}{}^{*}\mathrm{s}_{\mathrm{k}}\mathrm{s}_{\mathrm{k'}}^{*}\right\}\alpha_2\delta[l-p]+\frac{2}{\sigma}\frac{E}{M}|\zeta|^2\,\mathbb{I}\mathrm{m}\left\{\mathrm{A}_{\mathrm{lk}}^{\mathrm{h}}\mathrm{A}_{\mathrm{lk'}}^{\mathrm{h}}{}^{*}\dot{\mathrm{s}}_{\mathrm{k}}\mathrm{s}_{\mathrm{k'}}^{*}\right\}\alpha_2\delta[l-p].$$
(4.35)

Finally it is

$$[\boldsymbol{J}_D]_{i,i'}=-E_\phi\left\{\left(-2\frac{1}{\nu}\alpha_1+\frac{2+\nu}{\nu}\alpha_2\right)\sum_{p=1}^{N}\sum_{n=0}^{N_o-1}\frac{4\pi f_0}{\sigma}\frac{E}{M}|\zeta|^2\,\mathbb{R}\mathrm{e}\left\{\mathrm{A}_{\mathrm{lk}}^{\mathrm{h}}\mathrm{A}_{\mathrm{lk'}}^{\mathrm{h}}{}^{*}\mathrm{s}_{\mathrm{k}}\mathrm{s}_{\mathrm{k'}}^{*}\right\}\delta[l-p]\right\}+$$
$$+E_\phi\left\{\left(-2\frac{1}{\nu}\alpha_1+\frac{2+\nu}{\nu}\alpha_2\right)\sum_{p=1}^{N}\sum_{n=0}^{N_o-1}\frac{2}{\sigma}\frac{E}{M}|\zeta|^2\,\mathbb{I}\mathrm{m}\left\{\mathrm{A}_{\mathrm{lk}}^{\mathrm{h}}\mathrm{A}_{\mathrm{lk'}}^{\mathrm{h}}{}^{*}\dot{\mathrm{s}}_{\mathrm{k}}\mathrm{s}_{\mathrm{k'}}^{*}\right\}\delta[l-p]\right\}=$$
$$=-\left(-2\frac{1}{\nu}\alpha_1+\frac{2+\nu}{\nu}\alpha_2\right)\frac{4\pi f_0}{\sigma}\frac{E}{M}|\zeta|^2\delta[k-k']+$$
$$+\left(-2\frac{1}{\nu}\alpha_1+\frac{2+\nu}{\nu}\alpha_2\right)\frac{4\pi f_k}{\sigma}\frac{E}{M}|\zeta|^2\delta[k-k'].$$
(4.36)

- $[\boldsymbol{J}_D]_{i,i'}$, $\quad i=Q+1\dots 2Q,\quad i'=2Q+N+3\dots 2Q+2+L$

As done before, the result can be derived directly from the previous term:

$$[\boldsymbol{J}_D]_{i,i'}=\left(-2\frac{1}{\nu}\alpha_1+\frac{2+\nu}{\nu}\alpha_2\right)\frac{4\pi}{\sigma}\frac{E}{M}|\zeta|^2\left(\frac{T}{2}+\tau_{lk}\right)\delta[k-k'].$$
(4.37)

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- $[\boldsymbol{J}_D]_{i,i'}$, $\quad i=2Q+3\dots 2Q+2+N,\quad i'=2Q+3\dots 2Q+2+N$

$$\frac{\partial t_p(n)}{\partial\phi_{r_l}}\frac{\partial t_p(n)}{\partial\phi_{r_{l'}}}=\frac{4}{\sigma^2}\frac{E}{M}\,\mathbb{I}\mathrm{m}\left\{\zeta\mathrm{u}_{\mathrm{p}}^{\mathrm{h}}(\mathrm{n})\mathrm{r}_{\mathrm{p}}^{*}(\mathrm{n})\right\}\mathbb{I}\mathrm{m}\left\{\zeta\mathrm{u}_{\mathrm{p}}^{\mathrm{h}}(\mathrm{n})\mathrm{r}_{\mathrm{p}}^{*}(\mathrm{n})\right\}\delta[l-p,l'-p]=$$
$$=\left(\frac{2}{\sigma^2}\frac{E}{M}|\zeta|^2\,\mathbb{R}\mathrm{e}\left\{\left|\mathrm{u}_{\mathrm{l}}^{\mathrm{h}}(\mathrm{n})\right|^2|\mathrm{r}_{\mathrm{l}}(\mathrm{n})|^2\right\}-\frac{2}{\sigma^2}\frac{E}{M}|\zeta|^2\,\mathbb{R}\mathrm{e}\left\{\left(\mathrm{u}_{\mathrm{l}}^{\mathrm{h}}(\mathrm{n})\right)^2\left(\mathrm{r}_{\mathrm{l}}^{*}(\mathrm{n})\right)^2\right\}\right)\delta[l-p,l-l'].$$
(4.38)

Then it can be easily shown that

$$E_{r|\phi}\left\{\left(1+\frac{2t_p(n)}{\nu}\right)^{-2}\frac{\partial t_p(n)}{\partial \phi_{r_l}}\frac{\partial t_p(n)}{\partial \phi_{r_{l'}}}\right\}=\frac{2}{\sigma}\frac{E}{M}|\zeta|^2\frac{\nu}{\nu+2}\alpha_1\left|u_l^h(n)\right|^2\delta[l-p,l-l']. \quad (4.39)$$

Regarding the second order derivative,

$$E_{r|\phi}\left\{\left(1+\frac{2t_p(n)}{\nu}\right)^{-1}\frac{\partial^2 t_p(n)}{\partial \phi_{r_l}\partial \phi_{r_{l'}}}\right\}=\frac{2}{\sigma}\frac{E}{M}|\zeta|^2\alpha_2\left|u_l^h(n)\right|^2\delta[l-p,l-l']. \quad (4.40)$$

Finally, be recalling the result of eq. 15, it is

$$\begin{aligned}[\boldsymbol{J}_D]_{i,i'} &= E_{\phi}\left\{\left(-2\frac{1}{\nu}\alpha_1+\frac{2+\nu}{\nu}\alpha_2\right)\sum_{p=1}^{N}\sum_{n=0}^{N_o-1}\frac{2}{\sigma}\frac{E}{M}|\zeta|^2\left|u_l^h(n)\right|^2\delta[l-p,l-l']\right\}= \\ &= \left(-2\frac{1}{\nu}\alpha_1+\frac{2+\nu}{\nu}\alpha_2\right)\frac{2}{\sigma}E|\zeta|^2\delta[l-l'].\end{aligned} \quad (4.41)$$

- $[\boldsymbol{J}_D]_{i,i'}$, $\quad i=2Q+N+3\ldots 2Q+2+L, \quad i'=2Q+N+3\ldots 2Q+2+L$

$$\begin{aligned}\frac{\partial t_p(n)}{\partial \phi_{t_k}}\frac{\partial t_p(n)}{\partial \phi_{t_{k'}}} &= +\frac{4}{\sigma^2}\frac{E}{M}\,\mathbb{Im}\left\{\zeta \mathrm{A}^{\mathrm{h}}_{\mathrm{pk}}\mathrm{r}^*_{\mathrm{p}}(\mathrm{n})\mathrm{s}_{\mathrm{k}}\right\}\mathbb{Im}\left\{\zeta \mathrm{A}^{\mathrm{h}}_{\mathrm{pk'}}\mathrm{r}^*_{\mathrm{p}}(\mathrm{n})\mathrm{s}_{\mathrm{k'}}\right\}+ \\ &-\frac{4}{\sigma^2}\sqrt{\frac{E^3}{M^3}}|\zeta|^2\,\mathbb{Im}\left\{\mathrm{A}^{\mathrm{h}}_{\mathrm{pk}}\mathrm{u}^*_{\mathrm{p}}(\mathrm{n})\mathrm{s}_{\mathrm{k}}\right\}\mathbb{Im}\left\{\zeta \mathrm{A}^{\mathrm{h}}_{\mathrm{pk'}}\mathrm{r}^*_{\mathrm{p}}(\mathrm{n})\mathrm{s}_{\mathrm{k'}}\right\}+ \\ &-\frac{4}{\sigma^2}\sqrt{\frac{E^3}{M^3}}|\zeta|^2\,\mathbb{Im}\left\{\zeta \mathrm{A}^{\mathrm{h}}_{\mathrm{pk}}\mathrm{r}^*_{\mathrm{p}}(\mathrm{n})\mathrm{s}_{\mathrm{k}}\right\}\mathbb{Im}\left\{\mathrm{A}^{\mathrm{h}}_{\mathrm{pk'}}\mathrm{u}^*_{\mathrm{p}}(\mathrm{n})\mathrm{s}_{\mathrm{k'}}\right\}+ \\ &+\frac{4}{\sigma^2}\frac{E^2}{M^2}|\zeta|^4\,\mathbb{Im}\left\{\mathrm{A}^{\mathrm{h}}_{\mathrm{pk}}\mathrm{u}^*_{\mathrm{p}}(\mathrm{n})\mathrm{s}_{\mathrm{k}}\right\}\mathbb{Im}\left\{\mathrm{A}^{\mathrm{h}}_{\mathrm{pk'}}\mathrm{u}^*_{\mathrm{p}}(\mathrm{n})\mathrm{s}_{\mathrm{k'}}\right\}\end{aligned} \quad (4.42)$$

where the first term can be expanded as

$$\frac{2}{\sigma^2}\frac{E}{M}\left[|\zeta|^2\,\mathbb{Re}\left\{\mathrm{A}^{\mathrm{h}}_{\mathrm{pk}}\mathrm{s}_{\mathrm{k}}\left(\mathrm{A}_{\mathrm{pk'}}\mathrm{s}_{\mathrm{k'}}\right)^*|\mathrm{r}_{\mathrm{p}}(\mathrm{n})|^2\right\}-\mathbb{Re}\left\{\zeta^2\mathrm{A}^{\mathrm{h}}_{\mathrm{pk}}\mathrm{s}_{\mathrm{k}}\mathrm{A}_{\mathrm{pk'}}\mathrm{s}_{\mathrm{k'}}\left|\mathrm{r}^*_{\mathrm{p}}(\mathrm{n})\right)^2\right\}\right]. \quad (4.43)$$

Then it can be shown that

$$E_{r|\phi}\left\{\left(1+\frac{2t_p(n)}{\nu}\right)^{-2}\frac{\partial t_p(n)}{\partial \phi_{t_k}}\frac{\partial t_p(n)}{\partial \phi_{t_{k'}}}\right\}=\frac{2}{\sigma}\frac{E}{M}|\zeta|^2\frac{\nu}{\nu+2}\alpha_1\,\mathbb{Re}\left\{\mathrm{A}^{\mathrm{h}}_{\mathrm{pk}}\mathrm{A}^{\mathrm{h}}_{\mathrm{pk'}}{}^*\mathrm{s}_{\mathrm{k}}\mathrm{s}^*_{\mathrm{k'}}\right\}. \quad (4.44)$$

Regarding the second-order derivative,

$$E_{\boldsymbol{r}|\phi}\left\{\left(1+\frac{2t_p(n)}{\nu}\right)^{-1}\frac{\partial^2 t_p(n)}{\partial\phi_{t_k}\partial\phi_{t_{k'}}}\right\}=\frac{2}{\sigma}\frac{E}{M}|\zeta|^2\frac{\nu}{\nu+2}\alpha_1\,\mathbb{R}\mathrm{e}\left\{\mathrm{A}^{\mathrm{h}}_{\mathrm{pk}}\mathrm{A}^{\mathrm{h}}_{\mathrm{pk'}}{}^{*}\mathrm{s}_{\mathrm{k}}\mathrm{s}^{*}_{\mathrm{k'}}\right\}. \quad (4.45)$$

Finally it is

$$[\boldsymbol{J}_D]_{i,i'}=E_{\phi}\left\{\left(-2\frac{1}{\nu}\alpha_1+\frac{2+\nu}{\nu}\alpha_2\right)\sum_{p=1}^{N}\sum_{n=0}^{N_o-1}\frac{2}{\sigma}\frac{E}{M}|\zeta|^2\frac{\nu}{\nu+2}\alpha_1\,\mathbb{R}\mathrm{e}\left\{\mathrm{A}^{\mathrm{h}}_{\mathrm{pk}}\mathrm{A}^{\mathrm{h}}_{\mathrm{pk'}}{}^{*}\mathrm{s}_{\mathrm{k}}\mathrm{s}^{*}_{\mathrm{k'}}\right\}\right\}=$$
$$=\left(-2\frac{1}{\nu}\alpha_1+\frac{2+\nu}{\nu}\alpha_2\right)\frac{2}{\sigma}\frac{E}{M}|\zeta|^2 N\delta[k-k']. \quad (4.46)$$

- $[\boldsymbol{J}_D]_{i,i'}$, $\quad i=2Q+3\ldots 2Q+2+N, \quad i'=2Q+N+3\ldots 2Q+2+L$

$$\frac{\partial t_p(n)}{\partial\phi_{r_l}}\frac{\partial t_p(n)}{\partial\phi_{t_k}}=-\frac{4}{\sigma^2}\frac{E}{M}\,\mathbb{I}\mathrm{m}\left\{\zeta\mathrm{u}^{\mathrm{h}}_{\mathrm{p}}(\mathrm{n})\mathrm{r}^{*}_{\mathrm{p}}(\mathrm{n})\right\}\mathbb{I}\mathrm{m}\left\{\zeta\mathrm{A}^{\mathrm{h}}_{\mathrm{pk}}\mathrm{r}^{*}_{\mathrm{p}}(\mathrm{n})\mathrm{s}_{\mathrm{k}}\right\}\delta[l-p]+$$
$$+\frac{4}{\sigma^2}\sqrt{\frac{E^3}{M^3}}|\zeta|^2\,\mathbb{I}\mathrm{m}\left\{\zeta\mathrm{u}^{\mathrm{h}}_{\mathrm{p}}(\mathrm{n})\mathrm{r}^{*}_{\mathrm{p}}(\mathrm{n})\right\}\mathbb{I}\mathrm{m}\left\{\mathrm{A}^{\mathrm{h}}_{\mathrm{pk}}\mathrm{u}^{*}_{\mathrm{p}}(\mathrm{n})\mathrm{s}_{\mathrm{k}}\right\}\delta[l-p] \quad (4.47)$$

where the first term can be expanded as

$$-\frac{2}{\sigma^2}\frac{E}{M}\left[|\zeta|^2\,\mathbb{R}\mathrm{e}\left\{\mathrm{A}^{\mathrm{h}}_{\mathrm{pk}}\mathrm{s}_{\mathrm{k}}\left(\mathrm{u}^{\mathrm{h}}_{\mathrm{p}}(\mathrm{n})\right)^{*}|\mathrm{r}_{\mathrm{p}}(\mathrm{n})|^2\right\}-\mathbb{R}\mathrm{e}\left\{\zeta^2\mathrm{A}^{\mathrm{h}}_{\mathrm{pk}}\mathrm{s}_{\mathrm{k}}(\mathrm{u}^{\mathrm{h}}_{\mathrm{p}}(\mathrm{n})\left(\mathrm{r}^{*}_{\mathrm{p}}(\mathrm{n})\right)^2\right\}\right]\delta[l-p]. \quad (4.48)$$

Then after the expectation it is

$$E_{\boldsymbol{r}|\phi}\left\{\left(1+\frac{2t_p(n)}{\nu}\right)^{-2}\frac{\partial t_p(n)}{\partial\phi_{r_l}}\frac{\partial t_p(n)}{\partial\phi_{t_k}}\right\}=-\frac{2}{\sigma}\frac{E}{M}|\zeta|^2\alpha_1\frac{\nu}{\nu+2}\,\mathbb{R}\mathrm{e}\left\{\mathrm{A}^{\mathrm{h}}_{\mathrm{lk}}\mathrm{s}_{\mathrm{k}}\left(\mathrm{u}^{\mathrm{h}}_{\mathrm{l}}(\mathrm{n})\right)^{*}\right\}\delta[l-p] \quad (4.49)$$

and similarly it is

$$E_{\boldsymbol{r}|\phi}\left\{\left(1+\frac{2t_p(n)}{\nu}\right)^{-1}\frac{\partial^2 t_p(n)}{\partial\phi_{r_l}\partial\phi_{t_k}}\right\}=-\frac{2}{\sigma}\frac{E}{M}|\zeta|^2\alpha_2\,\mathbb{R}\mathrm{e}\left\{\mathrm{A}^{\mathrm{h}}_{\mathrm{lk}}\mathrm{s}_{\mathrm{k}}\left(\mathrm{u}^{\mathrm{h}}_{\mathrm{l}}(\mathrm{n})\right)^{*}\right\}\delta[l-p]. \quad (4.50)$$

Then the final result is

$$[\boldsymbol{J}_D]_{i,i'} = -E_\phi \left\{ \left(-2\frac{1}{\nu}\alpha_1 + \frac{2+\nu}{\nu}\alpha_2\right) \sum_{p=1}^{N} \sum_{n=0}^{N_o-1} \frac{2}{\sigma} \frac{E}{M} |\zeta|^2 \, \mathbb{R}\mathrm{e} \left\{ \mathrm{A}_{\mathrm{lk}}^{\mathrm{h}} \mathrm{s}_{\mathrm{k}} \left(\mathrm{u}_{\mathrm{l}}^{\mathrm{h}}(\mathrm{n}) \right)^* \right\} \delta[l-p] \right\} =$$
$$= -\left(-2\frac{1}{\nu}\alpha_1 + \frac{2+\nu}{\nu}\alpha_2\right) \frac{2}{\sigma} \frac{E}{M} |\zeta|^2. \tag{4.51}$$

Chapter 5

Numerical Results

5.1 Transmitted signals choice

As mentioned before, orthogonality can be achieved both in the frequency and code domain.

- **Frequency** [9]

  Starting from a center frequency $f_0 = 10\ GHz$ and the same baseband rectangular pulse of length $T = 5.4\ ms$, the signal of k^{th} transmitter is modulated to the frequency $f_k = \Delta_f (k-1)$, i.e.

$$s_k(t) = \sqrt{\frac{T_s}{T}} e^{\left(j2\pi(k-1)\Delta_f t\right)} rect\left(\frac{t - T/2}{T}\right), \qquad k = 1 \ldots M \tag{5.1}$$

  where $\Delta_f = 1\ MHz$. Because the spectral width of each signal is $\frac{2}{T} = 3.6\ kHz < \Delta_f$ (the First-Null Band-Width is being considered), the signals generated in this way are orthogonal. Also, in order to satisfy the sampling theorem condition it must be $f_s \geq \frac{2}{T} + (M-1)\Delta_f$, which is satisfied up to $M = 8$ different transmitters with the chosen sampling frequency $f_s = 9\ MHz$. Then the sampled signal will be

$$s_k[n] = s_k(nT_s) = \sqrt{\frac{T_s}{T}} e^{\left(j2\pi(k-1)\Delta_f nT_s\right)}, \quad n = 1 \ldots N_s, \qquad k = 1 \ldots M \tag{5.2}$$

  where the scaling factor $\sqrt{\frac{T_s}{T}}$ is necessary to force the energy of the sampled signal to 1.

- **Code**

  The same parameters of the previous case are used. This time orthogonality is achieved by applying different randomly chosen complex codes of rate $1/T_c = f_c$ to each un-modulated rectangular pulse:

$$s_k(t) = \sqrt{\frac{T_s}{2T}} \sum_{i=0}^{N_c-1} c_{k,i} rect\left(\frac{t - T_c/2 - iT_c}{T_c}\right), \qquad k = 1 \dots M \tag{5.3}$$

where $c_{k,i}$ is the i$^{\text{th}}$ complex chip of the k$^{\text{th}}$ transmitted sequence, randomly chosen from the set $\{1+j, 1-j, -1+j, -1-j\}$, and $N_c = \left[\frac{T}{T_c}\right]$ is the number of chips applied to each signal. Since the chips are 0-mean and i.i.d. the power spectral density of the transmitted signals is $PSD(f) \propto sinc^2(fT_c)$. In this case the -3 dB Band-Width is $B_{-3dB} = 1/2T_c$, thus the maximum chip rate $f_c = f_s$ is chosen. Again, the scaling factor is necessary to force $E_s = 1$.

Regarding the hypothesis assumed in the system model, firstly the ratio $2B/f0 = 8e-4$ at worst, so the MIMO is truly narrowband. Regarding the orthogonality hypothesis, the Cross-Ambiguity Function (CAF) of $s_1(t)$ and $s_2(t)$ is presented in Figure 5.1 and Figure 5.2 to evaluate how good the approximation is for different delays and Doppler shifts, respectively for the frequency and the code case.

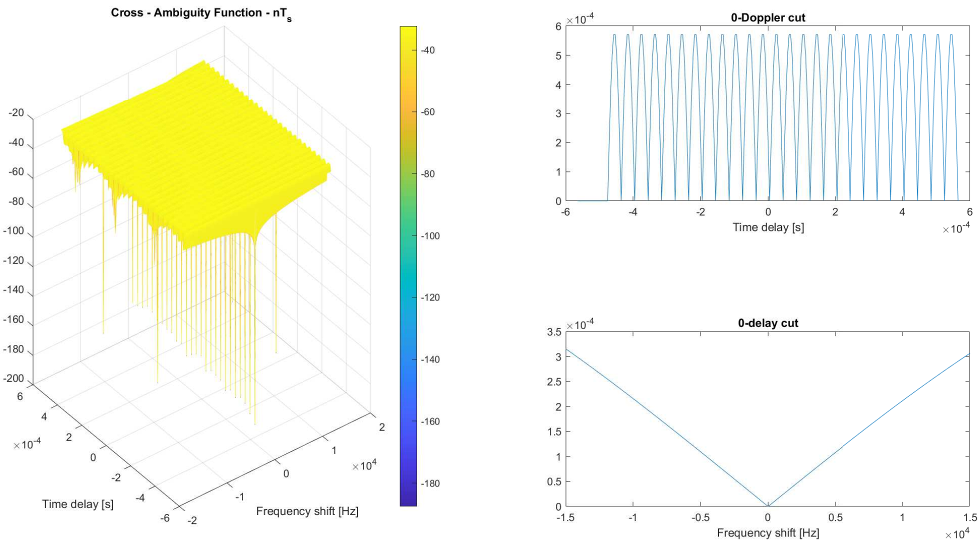


Figure 5.1: CAF for the frequency case. The complete function is scaled in dB, while the cuts are linearly scaled.

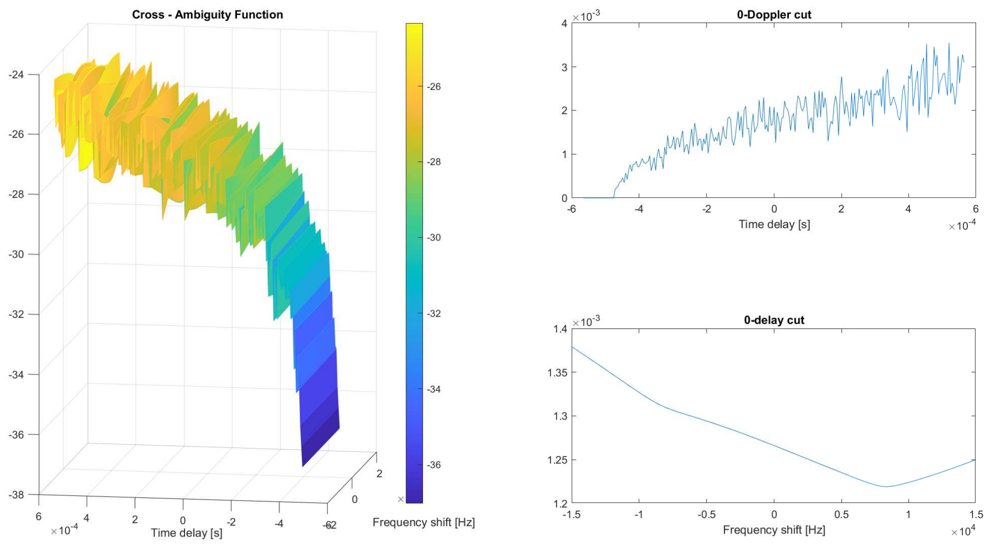
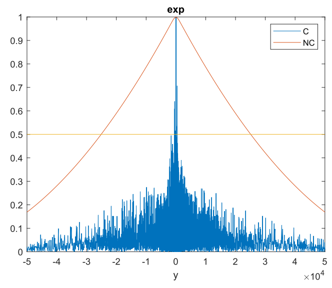


Figure 5.2: CAF for the frequency case. The complete function is scaled in dB, while the cuts are linearly scaled.

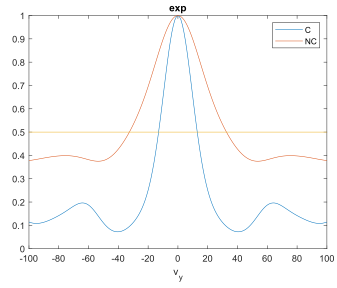
Regarding the comparison between the 2 cases, energy-wise the signals are clearly the same. Bandwidth wise, the orthogonality in frequency is achieved at the cost of spectral efficiency: of the 9 MHz of bandwidth available, only a small fraction is used since each transmitted signal has a bandwidth of 3.5 KHz and the spacing is very high. Meanwhile, each random code will occupy, on average, the whole bandwidth. This difference of bandwidth is only relevant for the resolution of the radar: in the FIM, with coherent processing, the term related to the bandwidth in the information on delay is $\propto B^2$, and get eclipsed by the term $\propto f_0^2$ related to the phase information. Because investigating the resolution of a MIMO radar is a problem already solved in literature, i.e. [4], and is out of the scope of this thesis, the difference in bandwidth will be ignored: however, in Figures 5.3 and 5.4 some cuts of the ambiguity functions of a MIMO radar are shown, both for the frequency and code cases, with and without coherent processing. Also, after showing that the bounds for the 2 cases are almost identical, the remaining graphs will be plotted from the frequency case, for which the bounds can be derived in exact form. In the case of non-coherent processing, the $f0^2$ term would disappear, and the signal bandwidth would be relevant.

Regarding the bounds on the velocity, they are poor because the information on Doppler is proportional to the pulse length, given that only pulse is being used for detection: increasing T would provide better accuracy, without losing on the bounds for position. It is important to notice, though, that the resolution of the radar is relevant when the actual Mean Square Error (MSE) of the parameters is evaluated because it sets the "Cell-Under-Test" dimensions: then the real MSE will follow the CRLB up to a certain value of

SCR, and after that it will reach a plateau because only 1 decision for CUT can be made.

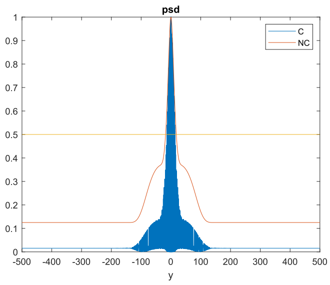


(a) Ambiguity function for y, i.e. with fixed $x = 0$, $v_x = 0$, $v_y = 0$, in the frequency case. In red the incoherent processing one is shown to be always worse. Given the high value of f_0 w.r.t. T, the peaks of the comb-like coherent AF are extremely narrow ($810^{-3}m$)

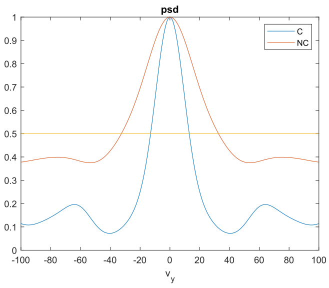


(b) Ambiguity function for v_y, i.e. with fixed $x = 0$, $y = 0$, $v_x = 0$, in the frequency case. In red the incoherent processing one is shown to be always worse. Given the low coherent integration interval, resolution in the velocity domain is poor

Figure 5.3



(a) Ambiguity function for y, i.e. with fixed $x = 0$, $v_x = 0$, $v_y = 0$, in the code case. In red the incoherent processing one is shown to be always worse. Given the high value of f_0 w.r.t. T, the peaks of the comb-like coherent AF are extremely narrow $(810^{-3}m)$



(b) Ambiguity function for v_y, i.e. with fixed $x = 0$, $y = 0$, $v_x = 0$, in the code case. In red the incoherent processing one is shown to be always worse. Given the low coherent integration interval, resolution in the velocity domain is poor

Figure 5.4

5.2 Results - Ring Topology

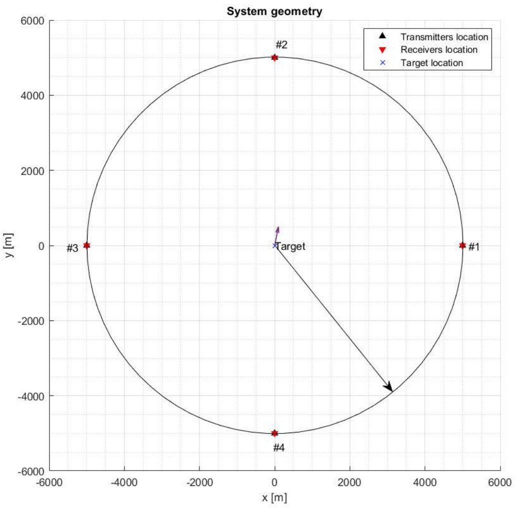


Figure 5.5: Ring Topology for the aray configuration. The target is in the center

In this scenario all the antennas are at the same distance $r = 500\ m$ from the target, which is positioned in the center of the coordinate system ($x = 0\ m$, $y = 0\ m$). At first the configuration presented in Figure 5.5 is considered, with 4 transmitters/receivers equally spaced, and a target moving with $(v_x, v_y) = (20, 20)\ m/s$ and characterized by a reflection coefficient $\zeta = e^{\jmath\frac{\pi}{4}}$. The Signal-to-Clutter Ratio (SCR) is defined as

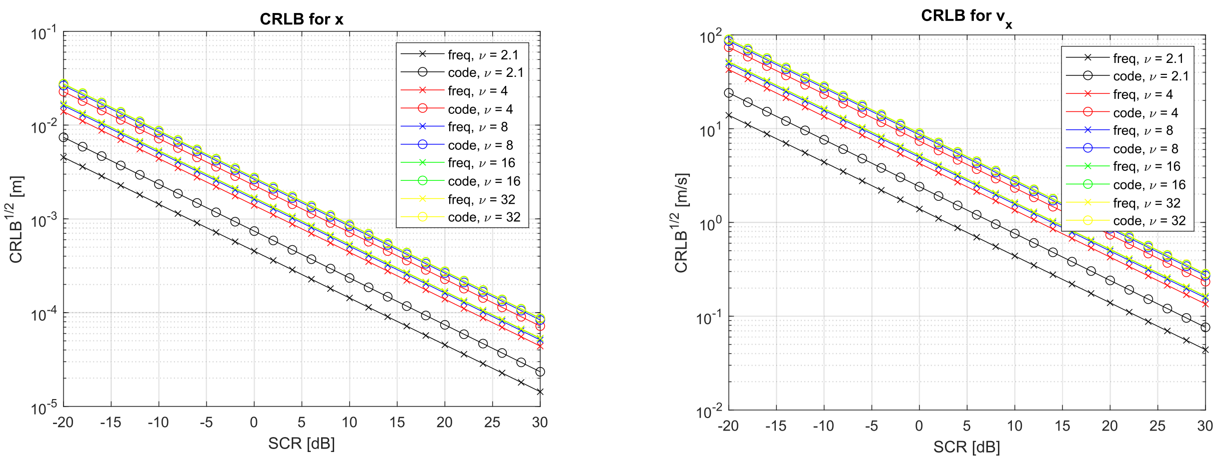
$$SCR \triangleq \frac{Transmitted\ Energy}{Clutter\ Power} = \frac{EE_s|\zeta|^2}{M\sigma^2\frac{\nu}{\nu-2}} = \frac{E}{M\sigma^2}\frac{\nu-2}{\nu} \tag{5.4}$$

where $\sigma^2\frac{\nu}{\nu-2}$ is the variance of a complex-t clutter with scatter σ^2 and ν degrees of fredom: given this relation, it is necessary that $\nu > 2$ in order to have a finite power clutter. Also, as shown in Chapter 3, for $\nu \to \infty$ the complex-t distribution converges to the gaussian distributio: thus, when evaluating the CRLB against the degrees of freedom of the distribution, the lower ν considered is 2.1, while the higher is 32, which is already a high enough value to consider the distribution Gaussian.

5.2.1 CRLB - Case 1

Firstly, the results for the spatially independent clutter are presented.
Figure 5.6 shows how the Cramér-Rao bounds behave against the degrees of freedom of the clutter distribution: only x and v_x are shown, but the other 2 bounds behave similarly given the simmetry of the distribution. As predicted, the curves converge to the same line

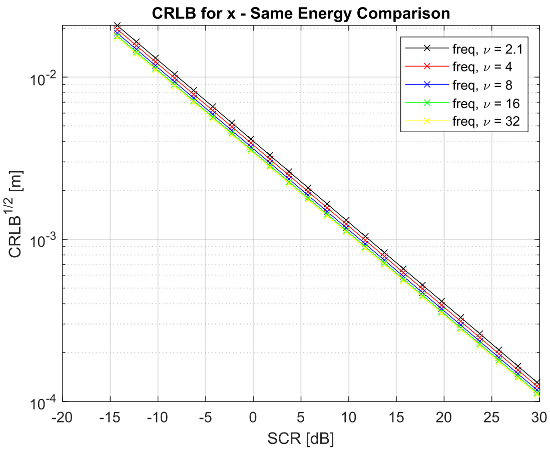
with ν increasing. Also, the bound on velocity is much larger than the one on position: this is due to the fact that only 1 pulse per CPI (Coherent Processing Interval) is being considered, which lowers the performance of the radar in the Doppler domain. It should be noted that, because the SCR depends on ν, this comparison is not fair from the p.o.v. of the total transmitted energy E: in order to have the same SCR, the lower ν is, the higher E must be to compensate (with $\sigma^2 = 1$ fixed): this explains why, when the clutter becomes spikier, the bounds appear to improve. Alternatively, it can be said that the comparison is not fair from the p.o.v. of the variance of the distribution, since it is the scatter parameter that is fixed. When the E, or the variance, is fixed, it can be seen from Figure 5.7 that, as one would expect, the bounds deteriorate with ν increasing.
In order to further validate the orthogonality assumption made at the start of this thesis, in Figure 5.8 the same bounds derived under the orthogonality assumption are compared with those derived without it: although they are slightly different, even more so when the target is not in the origin, the order of magnitude is the same of the approximate bounds, thus proving that the assumption is reasonable.



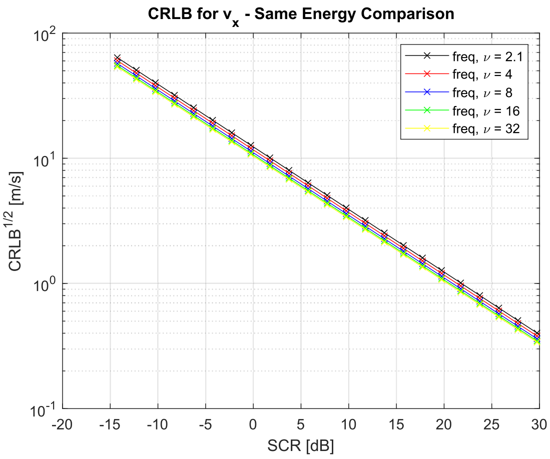
(a) Square root of the CRLB on x, both in frequency and code cases. The model is the same presented in Figure 5.5

(b) Square root of the CRLB on v_x, both in frequency and code cases. The model is the same presented in Figure 5.5

Figure 5.6

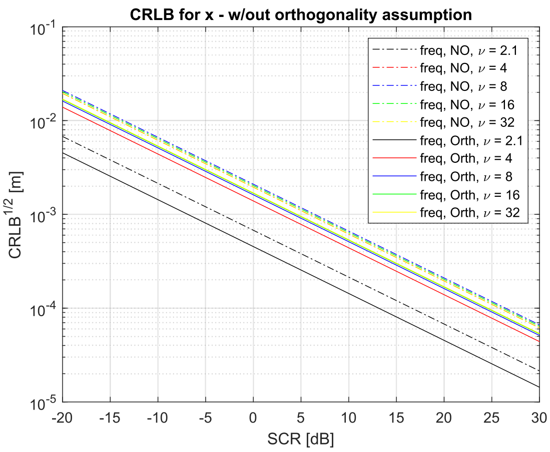


(a) Square root of the CRLB on x, for the frequency case. Same model as Figure 5.5, but this time E is fixed for different values of ν

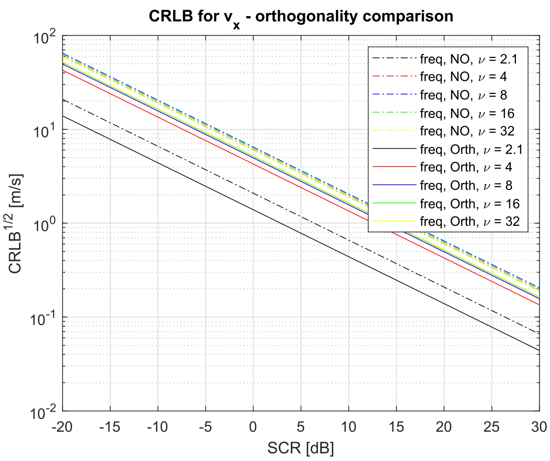


(b) Square root of the CRLB on v_x, for the frequency case. Same model as Figure 5.5, but this time E is fixed for different values of ν

Figure 5.7



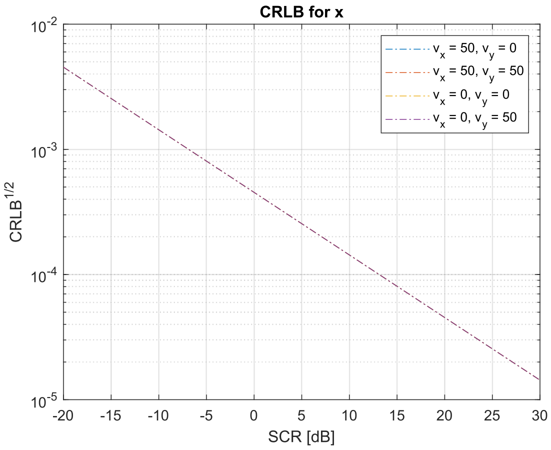
(a) Comparison of the bounds on x, both with and without the orthogonality assumption. Same mode as FIgure 5.3



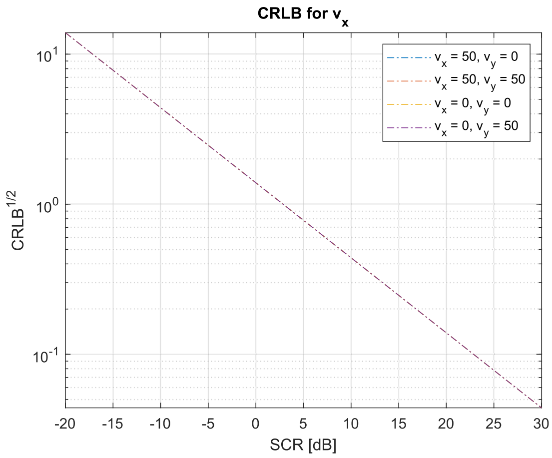
(b) Comparison of the bounds on v_x, both with and without the orthogonality assumption. Same mode as FIgure 5.3

Figure 5.8

Other interesting parameters that influence the performance of the radar are the target's position and velocity, and the number and location of the antennas. In Figure 5.9 the bounds are shown against different target velocities, with $\nu = 2.1$. Because the derived bounds on the parameter vector $\boldsymbol{\theta}$ are independent of the Doppler shifts, thanks to the orthogonality hypothesis, the only dependence is in the form of the $\boldsymbol{P}$ matrix, thus the bounds don't vary with different velocities. Even when considering the true signals, as shown in Figure 5.10, the variation over different values of v is minor, independently of the position (a target at $(x, y) = (350m, 350m)$ was considered for these figures). Again, the bounds are only slightly different from those derived with truly orthogonal signals, proving again that the assumption made was correct.

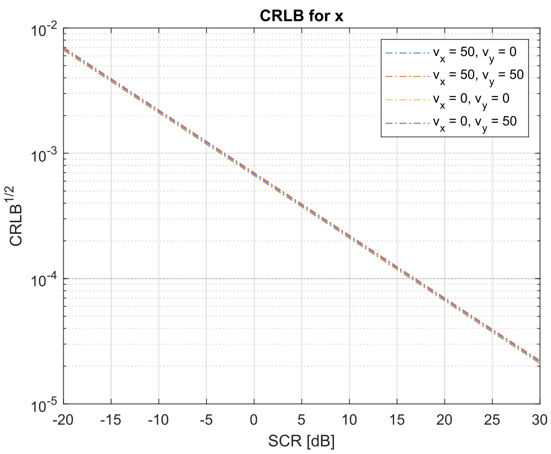


(a) Square root of the bound on x for different velocities. Same model as Figure 5.5, $\nu = 2.1$ and $SCR = 15dB$

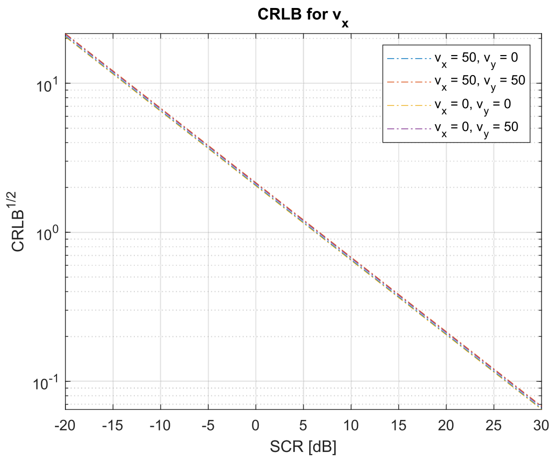


(b) Square root of the bound on x for different velocities. Same model as Figure 5.5, $\nu = 2.1$ and $SCR = 15dB$

Figure 5.9



(a) Square root of the bound on x for different velocities, without the orthogonality assumption. Same model as Figure 5.5, $\nu = 2.1$ and $SCR = 15dB$



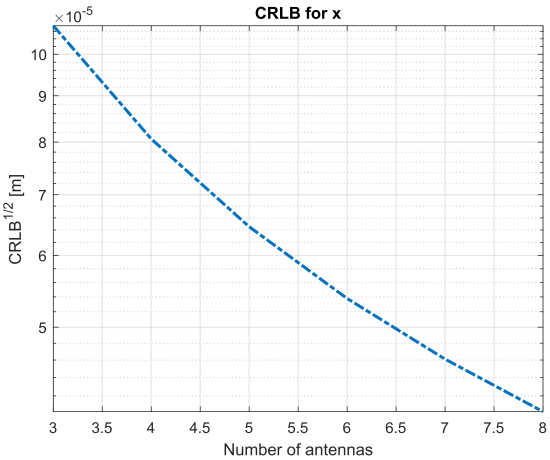
(b) Square root of the bound on x for different velocities, without the orthogonality assumption. Same model as Figure 5.5, $\nu = 2.1$ and $SCR = 15dB$

Figure 5.10

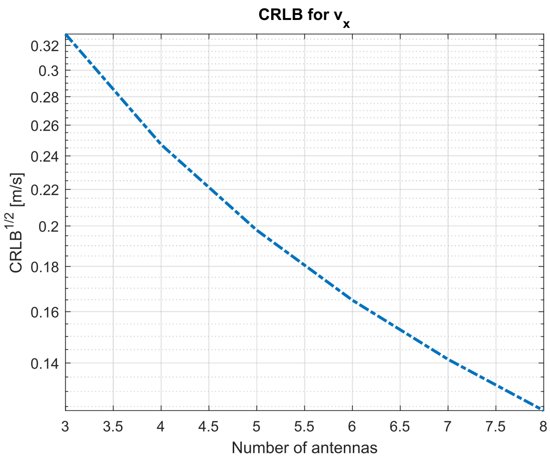
In Figure 5.10 the relation between the bounds and the number of antennas is examined, again with the assumption that the single radars are equally spaced, i.e.

$$\begin{cases} x_t = x_r = rcos(\boldsymbol{\alpha}) \\ y_t = y_r = rsin(\boldsymbol{\alpha}) \\ \boldsymbol{\alpha} = 2\pi \left[0, 1/M, \dots, 1 - 1/M\right]. \end{cases} \tag{5.5}$$

The observed dependency on the number of antennas is very minor: because the total transmitted energy doesn't vary, increasing the number of transmitters also decreases the power of each transmitted signal. Having more antennas should improve the resolution capabilities of the radar, but as already shown the bounds are mostly independent of the resolution.



(a) Square root of the bound on x for increasing number of antennas. The antennas are still uniformly distributed around the target, and each antenna acts as both a transmitter and a receiver, $\nu = 2.1$ and $SCR = 15dB$



(b) Square root of the bound on v_x for increasing number of antennas. The antennas are still uniformly distributed around the target, and each antenna acts as both a transmitter and a receiver, $\nu = 2.1$ and $SCR = 15dB$

Figure 5.11

In order to evaluate the accuracy of a given distribution of radars over the area of interest, the following functions

$$\begin{aligned} err_{xy} &\triangleq \sqrt{CRLB_x + CRLB_y} \\ err_{v_x v_y} &\triangleq \sqrt{CRLB_{v_x} + CRLB_{v_y}} \end{aligned} \tag{5.6}$$

are defined. When considering an efficient estimator, these functions represent the standard deviation of the estimation error (averaged over the 2 components for both position and velocity), and are used to give a measure of the precision of systems like GPS [GOD10]. In Figures 5.12 through 5.16 the error functions are shown for several configurations: transmitters and receivers are represented by black and red triangles, respectively; the bounds are plotted in dB, while the black lines are same-level curves. The SCR is set to $15dB$ for each figure

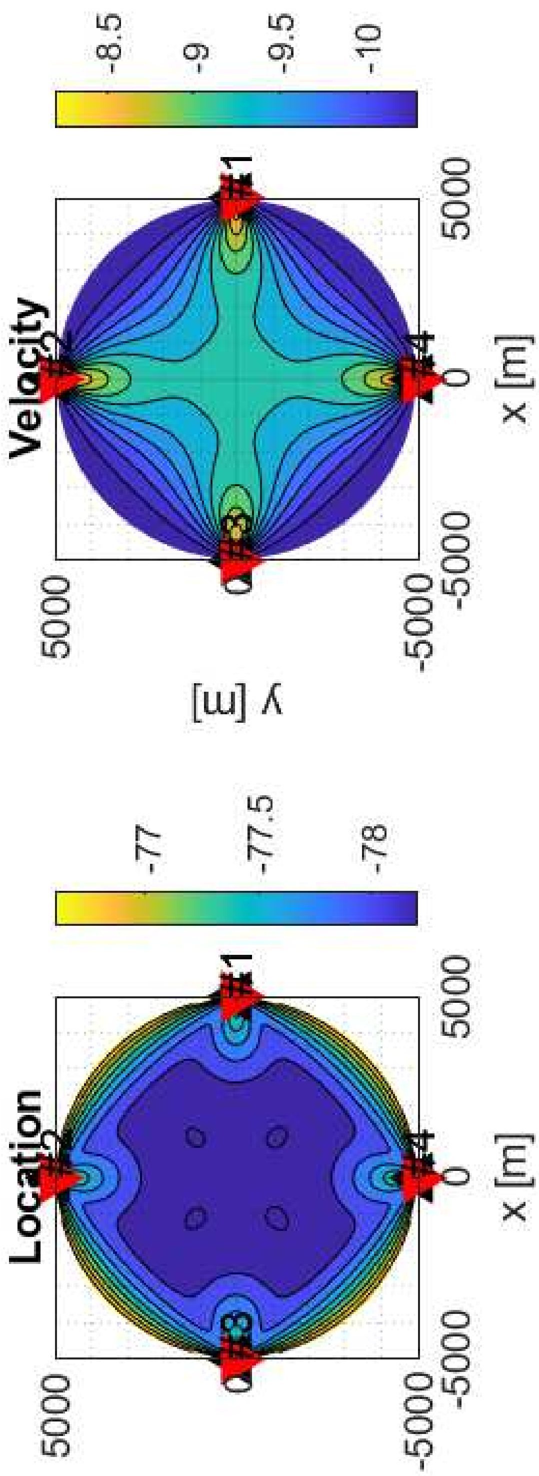


Figure 5.12

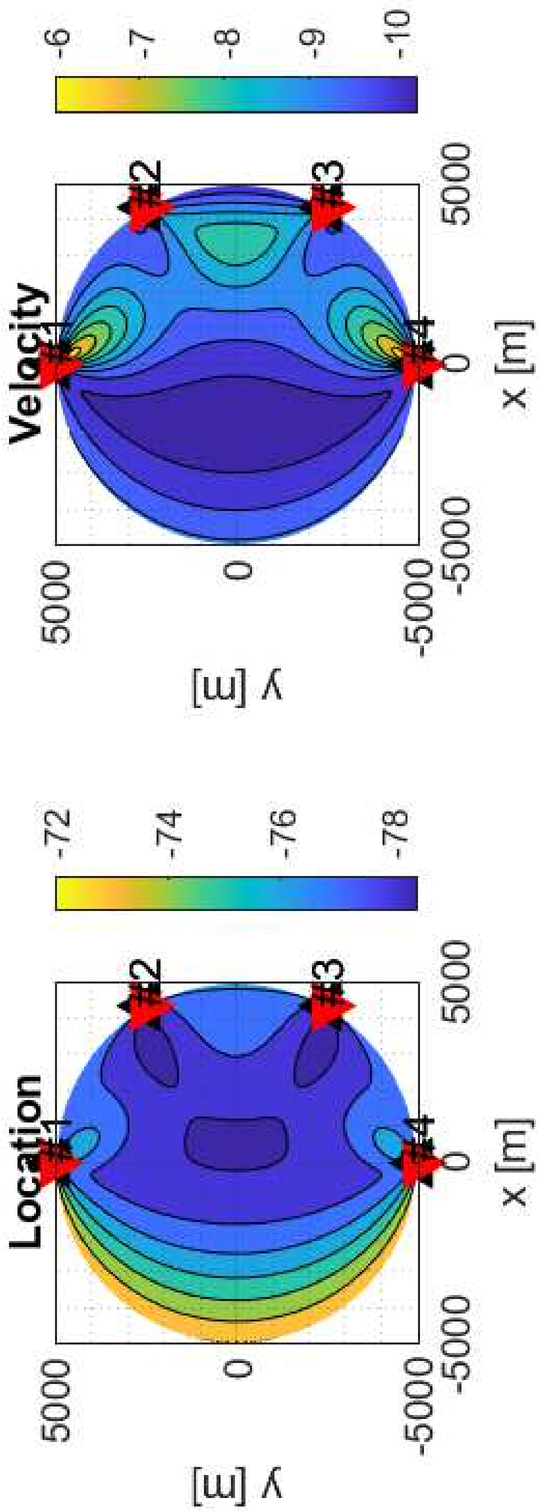


Figure 5.13

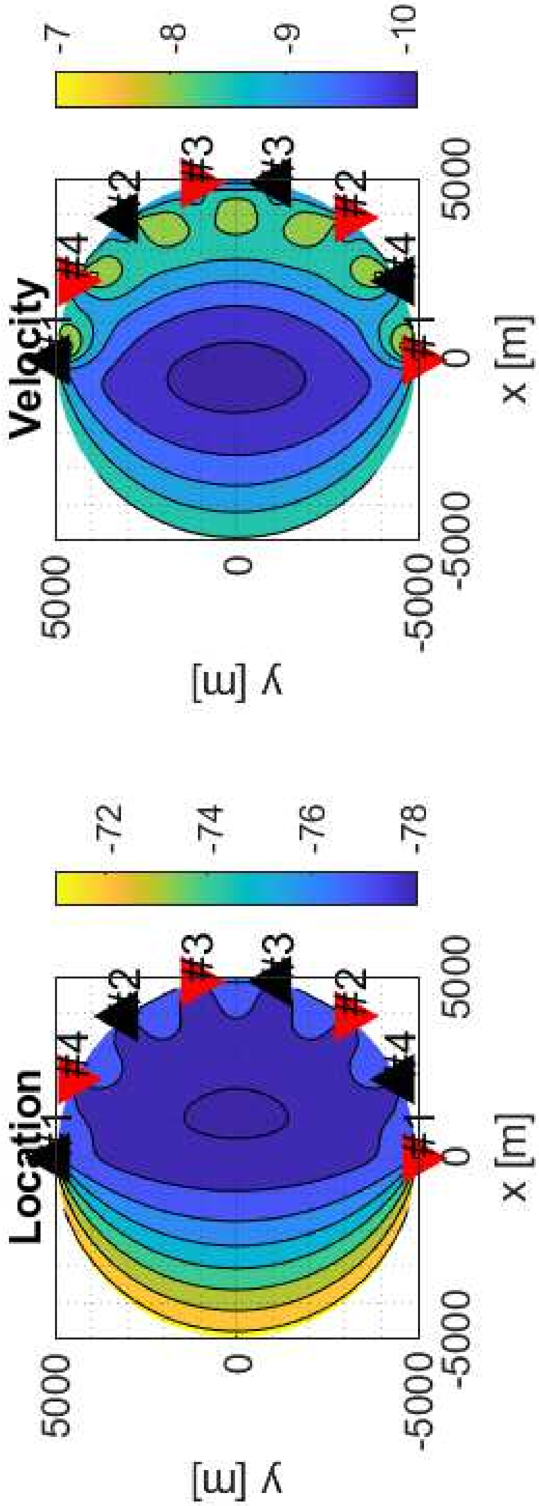


Figure 5.14

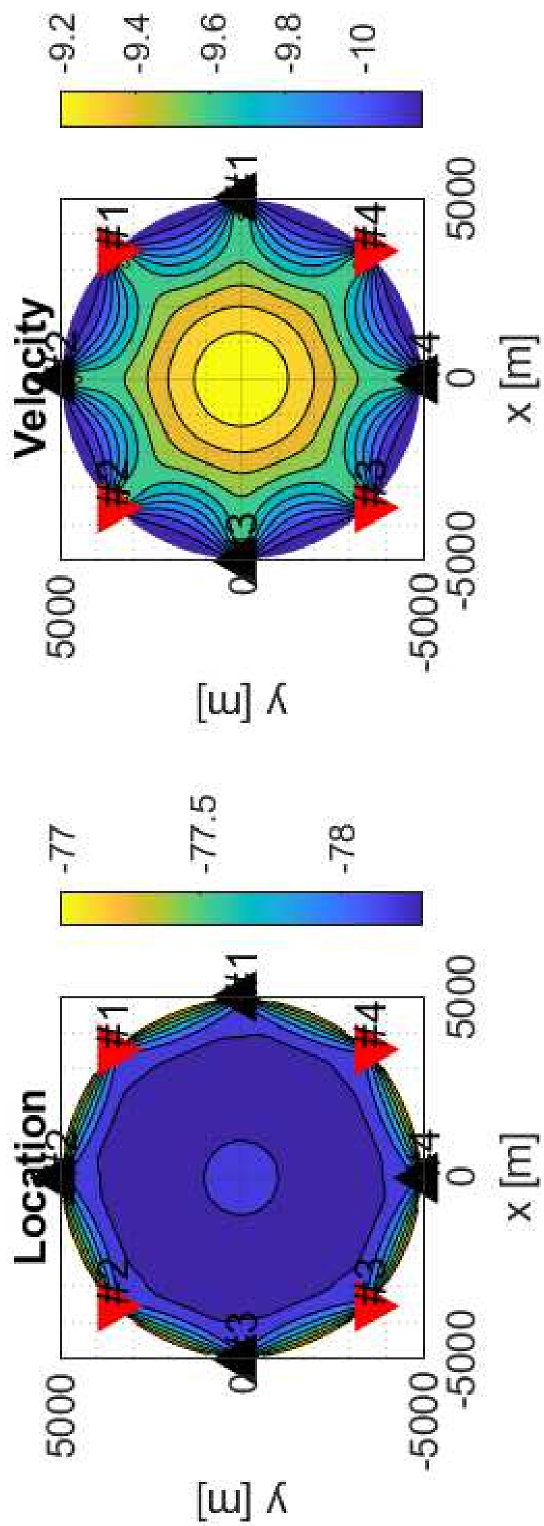

Velocity
-9.2
-9.4
-9.6
-9.8
-10
Location
-77
-77.5
-78
x [m]
y [m]
5000
0
-5000
#1
#2
#3
#4

Figure 5.15

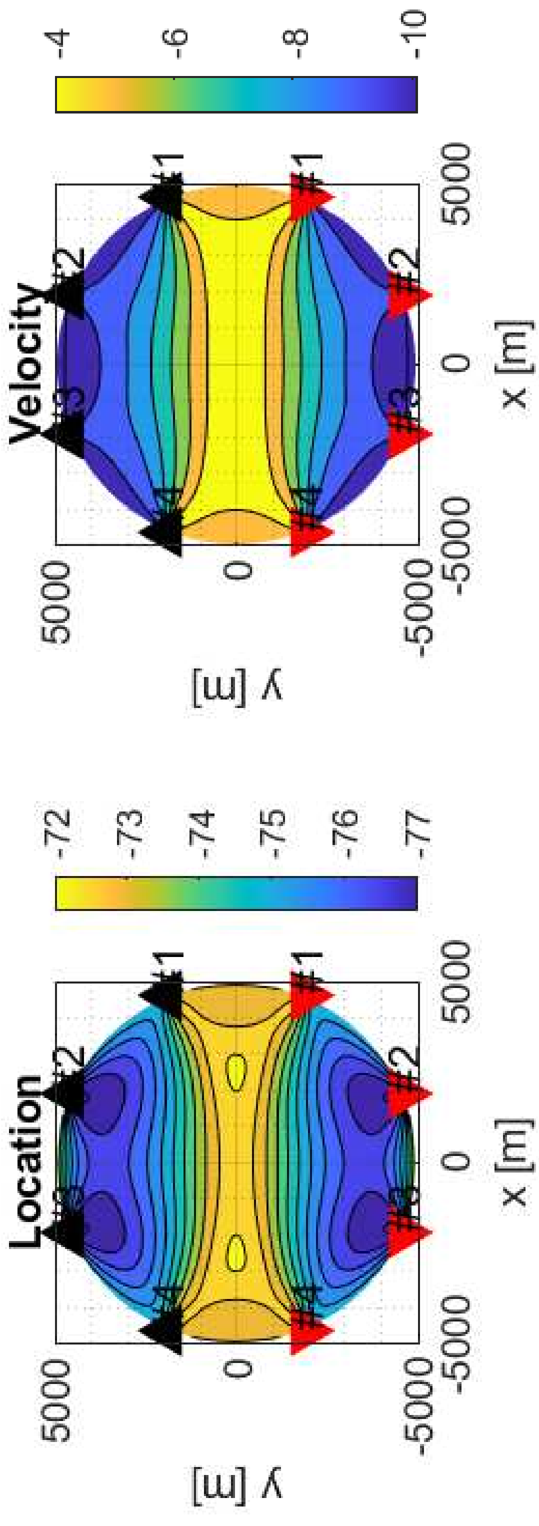

Velocity
Location
x [m]
y [m]
5000
0
-5000
-4
-6
-8
-10
-72
-73
-74
-75
-76
-77
#1
#2
#3
#4

Figure 5.16

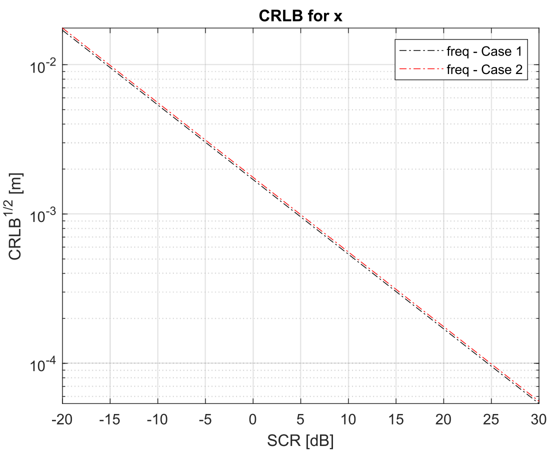
As it can be seen, the distribution of transmitters and receivers has a great impact on the performance of the radar, spatially-wise. As expected, having the antennas uniformly distributed on the ring is generally better for the position bound, that is almost the same over the whole circle; for the velocity bound, the maximum is at the center of the *constellation*. This can be explained by considering separately each of the $N \times M$ bi-static couples (counting also the degenerate ones when the tx and rx coincide): when the target is on the baseline of a bi-static radar, its Doppler is always 0 and its delay is always constant since the distance is always the length of the baseline, thus giving no information on velocity and much less information on position. Then the maximum in the center found in Figure 5.12 and Figure 5.15 is to be expected, since the origin is on 4 different baselines and near 8 different baselines, respectively. A similar reasoning can be applied to the case of Figure 5.16. When near the baseline of a bi-static radar, the Doppler shift and time delay observed don't vary much with position and velocity, and this is more true for the components parallel to the baseline: in this case the area with the higher bounds derives from the intersection of many vertical or almost vertical baselines and the almost horizontal baselines $k = 1, l = 4$ and $k = 4, l = 1$. Moving away from the horizontal axis ($y = 0$), the information provided by the 2 aforementioned bi-static couples increase, thus improving the bounds.

5.2.2 CRLB - Case 2

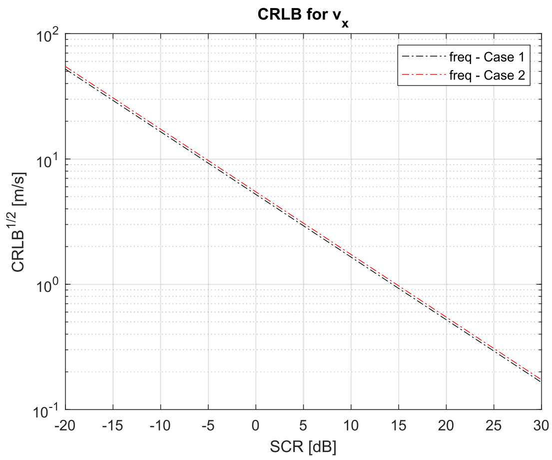
In order to study the Case 2 scenario, i.e. the spatially correlated clutter, the scatter matrix is modeled following [QIA10]:

$$[\mathbf{\Sigma}]_{m,n} \triangleq \sigma^2 \rho^{|\Delta\alpha_{mn}|} \tag{5.7}$$

where $\Delta\alpha_{mn}$ is the angular distance between receivers m and n, i.e. the angles between the m^{th} receiver, the target, the n^{th} receiver. While for phased arrays an AR(1) would be used, in this case this model better reflects the physics behind the chosen scenario: then ρ is not to be mistaken for some sort of correlation coefficient.
To check the goodness of the derivation, in Figure 5.17 the Case 1 and Case 2 bounds are compared for the same scenario described in Figure 5.5: it is chosen $\rho = 0.01$ in order to have achieve clutter orthogonality, and $\nu = 50$ in order to claim that orthogonality equals independence, since for that high number of degrees of freedom the distribution is normal. Since the curves follow each other closely, the results of Chapter 3 are consistent with each other.



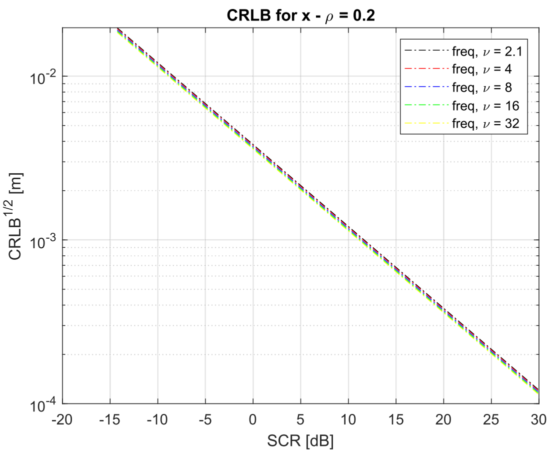
(a) Comparison of the square root of the bounds on x for the case 1 and case 2 models. $\nu = 50$ and $\rho = 0.01$ to simulate spatial white clutter. Same model as Figure 5.5



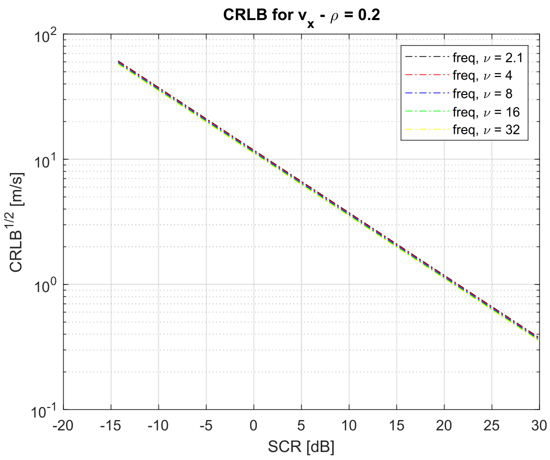
(b) Comparison of the square root of the bounds on v_x for the case 1 and case 2 models. $\nu = 50$ and $\rho = 0.01$ to simulate spatial white clutter. Same model as Figure 5.5

Figure 5.17

Then, similarly to what has been done in Case 1, other properties of the CRLB are explored, this time with $\rho = 1/5$. In Figure 5.18 the CRLB is plotted for different values of ν, and, as done in Figure 5.7, the comparison is fair for the total transmitted energy: as expected, even under correlation the less spiky clutter over performs the spikier one.



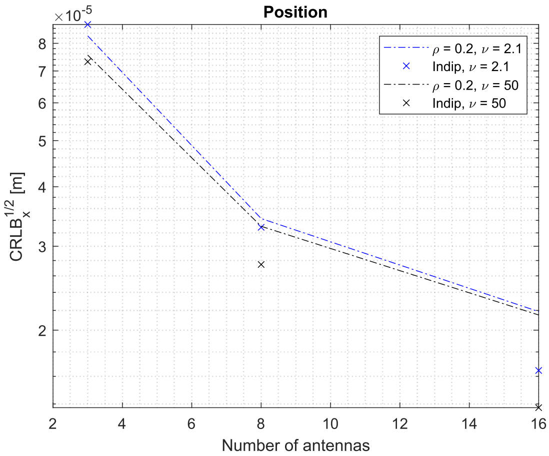
(a) Square root of the CRLB on x. The model is that of Figure 5.5, and E is fixed for different ν



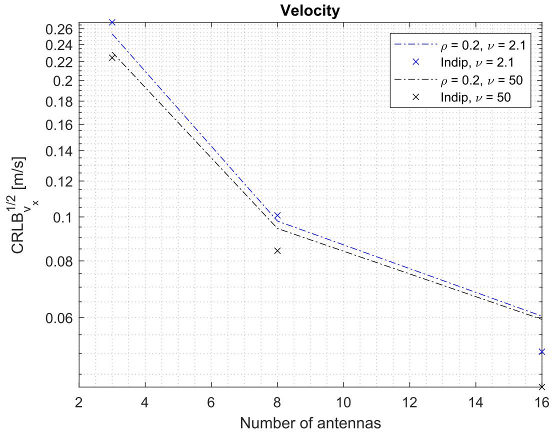
(b) Square root of the CRLB on v_x. The model is that of Figure 5.5, and E is fixed for different ν

Figure 5.18

Then the relation with the number of antennas is shown in Figure 5.19, and the result is compared both to that of case 1 and to the Gaussian equivalent, for an SCR of $30dB$. Even under correlated clutter, the bounds still improve with an increasing number of antennas: this also suggests that even for correlated clutter, CRLB performance can be achieved not only for high SCR, but also for high $N \times M$. In addition to this, while for a low number of antennas the correlated clutter seems to outperform, albeit slightly, the independent one, this is clearly not true when the number of antennas is high (which reflects to high correlation between receivers, as the angular distance reduces with increasing N), meaning that the behaviour for low $N \times M$ can be considered a result of numerical approximation (given that for case 2, formulas have been implemented in the frequency domain, while for case 1 a closed solution was found).



(a) Comparison of the square root of the bounds on x, both for the case 1 and case 2 models, with different values of ν. The antennas are still uniformly distributed around the target, and each antenna acts as both a transmitter and a receiver

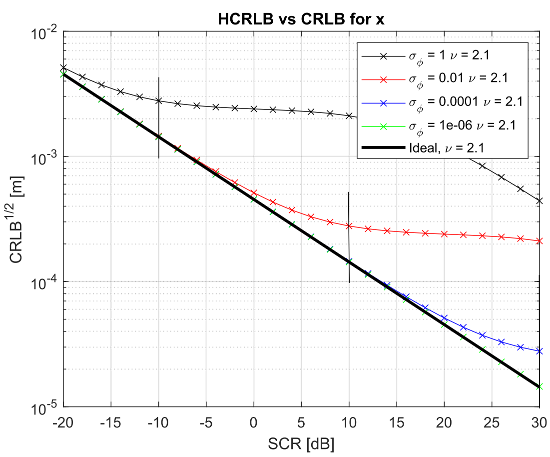


(b) Comparison of the square root of the bounds on v_x, both for the case 1 and case 2 models, with different values of ν. The antennas are still uniformly distributed around the target, and each antenna acts as both a transmitter and a receiver

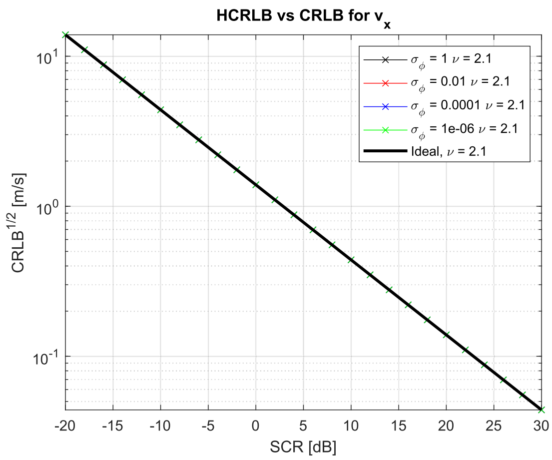
Figure 5.19

5.2.3 HCRLB

Finally, the Hybrid CRLB is presented against the ideal case. As shown in Chapter 4, the CRLB is increased by a factor $\Delta CRLB$ that depends both on the signal paramters and the phase mismatch variance. In Figure 5.20 the HCRLB is plotted against the SCR for $\nu = 2.1$ and different values of the residual phase variance σ_p^2.



(a) Comparison between the HCRLB and the CRLB on x, for different values of σ_p^2. The model is the same of Figure 5.5, and $\nu = 2.1$



(b) Comparison between the HCRLB and the CRLB on v_x, for different values of σ_p^2. The model is the same of Figure 5.5, and $\nu = 2.1$

Figure 5.20

As it can be seen, a high phase mismatch between the oscillators of the system can cause catastrophic performance degradation regarding the estimate of the target position. At low SCR, the effect of the phase mismatch is negligible even for high σ_p^2: at low SCRs the detection is already compromised, so the random phase term has less of an effect. At high SCR the effect is more destructive, and the bounds heavily deteriorate. Obviously, the higher the phase variance, the bigger the degradation of performances. On the other hand, the bounds on the velocity of the target appear largely unaffected by the phase mismatch. This can be attributed to the fact that the Doppler estimation is unaffected by the phase errors because they are constant in the observation interval, and only 1 pulse is being considered (so the phase variation is the same as the ideal case). Then the phase noise can only influence the velocity estimation through the errors on the position, following the relations given in Chapter 2. But the dependency on the velocity is generally stronger than the dependency on position, as also seen in the CRLB cross-terms, and the estimate on the velocity is already poor, given the short signal length, meaning that these effects can be considered negligible.

A parameter to represent the phase mismatch tolerance of the system could be the value of σ_p^2 for which, at the required SCR of the system, the $\Delta CRLB$ equals the ideal $CRLB$, which has been found numerically and shown in the figure: then the synchronization system must be able to achieve such phase error variance at worst, in order for the system to function properly.

Chapter 6

Conclusions

MIMO Radars are the next generation of radars. They can achieve greater performances than traditional radars, like much higher resolutions or target fading resistance thanks to spatial diversity. The price for higher performances is payed as higher costs and system complexity, in particular for widely distributed radars which require a common data fusion center to achieve optimal performances.

In this thesis the problem of the MIMO Radar performances under Complex Symmetric Clutter has been studied from the point of view of the Cramer-Rao Lower Bounds. The coherent processing and widely distributed case has been considered, and the bounds have been derived both for independent and spatially correlated complex-t distributed clutter. The relations between the bounds and several target parameters have been examined, like the operating frequency, the signal-to-clutter ratio or the number of antennas, and have been shown improving the performances even under correlated clutter. It has been shown that the bounds have a very low dependence on the signals' bandwidth, by comparing the classical rectangular pulse signals to the (higher bandwidth) code modulated ones, thanks to the coherent processing. In addition, the effect of the phase mismatch between different oscillators have been studied, and the Hybrid CRLB has been proposed.

Regarding future work, it would be interesting to add time-correlation to the clutter in order to better mimic real-world clutter behaviour and offer a complete analysis of the MIMO radar performances. In addition the colocated MIMO case is also an interesting problem to analyze, since its applications are very different from the widely distributed (like missiles or automotive) and its performances can be directly compared with those of the phased arrays. Finally, while the work has been focused on the CRLBs, the maximum likelihood estimator is not readily available: future work should focus on deriving working approximations, possibly with iterative algorithms, to then compare them to the maximum achievable bounds in order to propose viable, albeit suboptimal, receivers architectures.

Chapter 7

APPENDIX I

In this appendix it is discussed how to calculate the various expectations related to the FIM when the noise is assumed having a $\mathbb{C}$t distribution. They appear in one of the following forms:

$$
\begin{aligned}
&E\left\{\left(1+\frac{2t}{\nu}\right)^{-1}\right\}, \qquad E\left\{\left(1+\frac{2t}{\nu}\right)^{-2}\right\}, \qquad E\left\{\left(1+\frac{2t}{\nu}\right)^{-1} r_p\right\} \\
&E\left\{\left(1+\frac{2t}{\nu}\right)^{-2} r_p\right\}, \qquad E\left\{\left(1+\frac{2t}{\nu}\right)^{-2} r_p r_q\right\}, \qquad E\left\{\left(1+\frac{2t}{\nu}\right)^{-2} r_p r_q^*\right\}
\end{aligned}
\tag{7.1}
$$

where $r_p = [\boldsymbol{r}]_p$, $p = 1 \ldots m$ are the samples of a $\mathbb{C}t_{m,\nu}\left(E\left\{\boldsymbol{r}\right\}, \boldsymbol{\Sigma}\right)$, with $\boldsymbol{\Sigma}$ being the scatter matrix, m the dimension and ν the degrees of freedom. In general

$$
\begin{aligned}
E\left\{f\left(\boldsymbol{r}\right)\right\} &= \int \cdots \int_{\mathbb{C}^m} f\left(\boldsymbol{r}\right) p\left(\boldsymbol{r}\right) dr_1 \ldots dr_m = \\
&= \frac{C_{m,\nu}}{|\boldsymbol{\Sigma}|} \int \cdots \int_{\mathbb{C}^m} f\left(\boldsymbol{r}\right) \left(1+\frac{2t}{\nu}\right)^{-\frac{-2m+\nu}{\nu}} dr_1 \ldots dr_m = \\
&= \frac{C_{m,\nu}}{|\boldsymbol{\Sigma}|} \int \cdots \int_{\mathbb{C}^m} f_1\left(\boldsymbol{r}\right) \left(1+\frac{2t}{\nu}\right)^{a} \left(1+\frac{2t}{\nu}\right)^{-\frac{-2m+\nu}{\nu}} dr_1 \ldots dr_m = \\
&= \frac{C_{m,\nu}}{|\boldsymbol{\Sigma}|} \int \cdots \int_{\mathbb{C}^m} f_1\left(\boldsymbol{r}\right) \left(1+\frac{2t}{\nu}\right)^{-\frac{-2m-2a+\nu}{2}} dr_1 \ldots dr_m
\end{aligned}
\tag{7.2}
$$

where a can be either -2 or -1, and $f_1(\boldsymbol{r})$ can be either 1, r_p, $r_p r_q$ or $r_p r_q^*$. An easy way to evaluate the integral would be to change the variables of integration such that $\frac{2t}{\nu} \implies \frac{2t}{\nu-2a}$: this way, the second argument in the integral would become a scaled pdf of a $\mathbb{C}$t distribution with $\nu - 2a$ degrees of freedom. Such a transformation is $\boldsymbol{z} =$

$\sqrt{\frac{\nu-2a}{\nu}}\left(\boldsymbol{r}-E\left\{\boldsymbol{r}\right\}\right)$, and $\frac{2t}{\nu} \implies \frac{2}{\nu-2a}\boldsymbol{z}^H\boldsymbol{\Sigma}\boldsymbol{z}=\frac{2t_1}{\nu-2a}$. Then, the integral becomes

$$
\begin{aligned}
&\frac{C_{m,\nu}}{|\boldsymbol{\Sigma}|}\left(\frac{\nu}{\nu-2a}\right)^m\int\cdots\int_{\mathbb{C}^m} f_1\left(\sqrt{\frac{\nu}{\nu-2a}}\boldsymbol{z}+E\left\{\boldsymbol{r}\right\}\right)\left(1+\frac{2t_1}{\nu-2a}\right)^{-\frac{-2m-2a+\nu}{2}} dz_1\dots dz_m = \\
&=\frac{C_{m,\nu}}{C_{m,\nu-2a}}\left(\frac{\nu}{\nu-2a}\right)^m\times \\
&\quad\times\int\cdots\int_{\mathbb{C}^m} f_1\left(\sqrt{\frac{\nu}{\nu-2a}}\boldsymbol{z}+E\left\{\boldsymbol{r}\right\}\right)\frac{C_{m,\nu-2a}}{|\boldsymbol{\Sigma}|}\left(1+\frac{2t_1}{\nu-2a}\right)^{-\frac{-2m-2a+\nu}{2}} dz_1\dots dz_m
\end{aligned}
\tag{7.3}
$$

where the second term is exactly the pdf of a $\mathbb{C}t_{m,\nu-2a}\left(\boldsymbol{0},\boldsymbol{\Sigma}\right)$. Now, since $f_1(\boldsymbol{r})$ is not a function of all the elements of $\boldsymbol{z}$, the integration is performed first w.r.t. the other elements, leading to the marginal pdf w.r.t. the elements from which $f_1(\boldsymbol{r})$ depends: this sub-vector maintains a $\mathbb{C}t$ distribution with the same degrees of freedom and corresponding elements of the mean value and scatter matrix. In case $f_1(\boldsymbol{r})$, the integral equals 1 since the argument is a pdf. Thus

$$
\begin{aligned}
E\left\{\left(1+\frac{2t}{\nu}\right)^{-1}\right\} &= \frac{C_{m,\nu}}{C_{m,\nu+2}}\left(\frac{\nu}{\nu+2}\right)^m\int\cdots\int_{\mathbb{C}^m}\frac{C_{m,\nu+2}}{|\boldsymbol{\Sigma}|}\left(1+\frac{2t_1}{\nu+2}\right)^{-\frac{-2m+2+\nu}{2}} dz_1\dots dz_m = \\
&= \alpha_2
\end{aligned}
\tag{7.4}
$$

$$
\begin{aligned}
E\left\{\left(1+\frac{2t}{\nu}\right)^{-2}\right\} &= \frac{C_{m,\nu}}{C_{m,\nu+4}}\left(\frac{\nu}{\nu+4}\right)^m\int\cdots\int_{\mathbb{C}^m}\frac{C_{m,\nu+4}}{|\boldsymbol{\Sigma}|}\left(1+\frac{2t_1}{\nu+4}\right)^{-\frac{-2m+4+\nu}{2}} dz_1\dots dz_m = \\
&= \alpha_1
\end{aligned}
\tag{7.5}
$$

$$
\begin{aligned}
&E\left\{\left(1+\frac{2t}{\nu}\right)^{-1} r_p\right\} = \\
&=\alpha_2\int\cdots\int_{\mathbb{C}^m} C_{m,\nu+2}\frac{\sqrt{\frac{\nu}{\nu+2}}z_p+E\{r_p\}}{|\boldsymbol{\Sigma}|}\left(1+\frac{2t_1}{\nu-2a}\right)^{-\frac{-2m-2a+\nu}{2}} dz_1\dots dz_m = \\
&=\alpha_2 E\{r_p\}
\end{aligned}
\tag{7.6}
$$

$$
\begin{aligned}
&E\left\{\left(1+\frac{2t}{\nu}\right)^{-2} r_p\right\} = \\
&= \alpha_1 \int \cdots \int_{\mathbb{C}^m} C_{m,\nu+4} \frac{\sqrt{\frac{\nu}{\nu+4}} z_p + E\{r_p\}}{|\mathbf{\Sigma}|} \left(1+\frac{2t_1}{\nu+4}\right)^{-\frac{-2m+4+\nu}{2}} dz_1 \dots dz_m = \\
&= \alpha_1 E\{r_p\}
\end{aligned} \tag{7.7}
$$

$$
\begin{aligned}
&E\left\{\left(1+\frac{2t}{\nu}\right)^{-2} r_p r_q\right\} = \\
&= \alpha_1 \int \cdots \int_{\mathbb{C}^m} C_{m,\nu+4} \frac{\left(\sqrt{\frac{\nu}{\nu+4}} z_p + E\{r_p\}\right)\left(\sqrt{\frac{\nu}{\nu+4}} z_q + E\{r_q\}\right)}{|\mathbf{\Sigma}|} \times \\
&\qquad \times \left(1+\frac{2t_1}{\nu+4}\right)^{-\frac{-2m+4+\nu}{2}} dz_1 \dots dz_m = \\
&= \alpha_1 \int \cdots \int_{\mathbb{C}^m} C_{m,\nu+4} \frac{\frac{\nu}{\nu+4} z_p z_q + E\{r_p\}E\{r_q\} + \sqrt{\frac{\nu}{\nu+4}} z_p + \sqrt{\frac{\nu}{\nu+4}} z_q}{|\mathbf{\Sigma}|} \times \\
&\qquad \times \left(1+\frac{2t_1}{\nu+4}\right)^{-\frac{-2m+4+\nu}{2}} dz_1 \dots dz_m = \\
&= \alpha_1 E\{r_p\}E\{r_q\}
\end{aligned} \tag{7.8}
$$

$$
\begin{aligned}
&E\left\{\left(1+\frac{2t}{\nu}\right)^{-2} r_p r_q^*\right\} = \\
&= \alpha_1 \int \cdots \int_{\mathbb{C}^m} C_{m,\nu+4} \frac{\left(\sqrt{\frac{\nu}{\nu+4}} z_p + E\{r_p\}\right)\left(\sqrt{\frac{\nu}{\nu+4}} z_q^* + E\{r_q^*\}\right)}{|\mathbf{\Sigma}|} \times \\
&\qquad \times \left(1+\frac{2t_1}{\nu+4}\right)^{-\frac{-2m+4+\nu}{2}} dz_1 \dots dz_m = \\
&= \alpha_1 \int \cdots \int_{\mathbb{C}^m} C_{m,\nu+4} \frac{\frac{\nu}{\nu+4} z_p z_q^* + E\{r_p\}E\{r_q^*\} + \frac{\nu}{\nu+4} z_p + \frac{\nu}{\nu+4} z_q^*}{|\mathbf{\Sigma}|} \times \\
&\qquad \times \left(1+\frac{2t_1}{\nu+4}\right)^{-\frac{-2m+4+\nu}{2}} dz_1 \dots dz_m = \\
&= \alpha_1 E\{r_p\}E\{r_q^*\} + \alpha_1 \frac{\nu}{\nu+4}\frac{\nu+4}{\nu+2}\sigma_{pq} = \alpha_1 E\{r_p\}E\{r_q^*\} + \alpha_1 \frac{\nu}{\nu+2}\sigma_{pq}
\end{aligned} \tag{7.9}
$$

where $\sigma_{pq} = [\mathbf{\Sigma}]_{pq}$ and , from the properties of $\mathbb{C}t$ distributions and by identifying $\Gamma(x)$ as the gamma function,

$$\begin{aligned}
&E\{z_p\} = 0, \qquad E\{z_p z_q^*\} = \frac{\nu+4}{\nu+2}\sigma_{pq}, \qquad E\{z_p z_q\} = 0 \\
&\alpha_1 = \frac{C_{m,\nu}}{C_{m,\nu+4}}\left(\frac{\nu}{\nu+4}\right)^m = \frac{\Gamma\left(\frac{2m+\nu}{2}\right)\Gamma\left(\frac{4+\nu}{2}\right)(\pi\nu+4\pi)^m}{\Gamma\left(\frac{2m+4+\nu}{2}\right)\Gamma\left(\frac{\nu}{2}\right)(\pi\nu)^m}\left(\frac{\nu}{\nu+4}\right)^m \\
&\alpha_2 = \frac{C_{m,\nu}}{C_{m,\nu+2}}\left(\frac{\nu}{\nu+2}\right)^m = \frac{\Gamma\left(\frac{2m+\nu}{2}\right)\Gamma\left(\frac{2+\nu}{2}\right)(\pi\nu+2\pi)^m}{\Gamma\left(\frac{2m+2+\nu}{2}\right)\Gamma\left(\frac{\nu}{2}\right)(\pi\nu)^m}\left(\frac{\nu}{\nu+2}\right)^m
\end{aligned} \tag{7.10}$$

Chapter 8

APPENDIX II

In this appendix the summations found in the CRLBs get solved for the specific classes of transmitted signals chosen, mostly through the means of the Parseval's Theorem applied to the Fourier Transform: because the signals are band-limited, the integration interval in the frequency domain is limited to $[-1/T + f_k, 1/T + f_k]$ (First-Null Band-Width). The first case examined is the frequency modulated one, i.e. the case where

$$\begin{aligned} s_k(t) &= \sqrt{\frac{T_s}{T}} e^{\left(\jmath 2\pi (k-1)\Delta_f t\right)} rect\left(\frac{t - T/2}{T}\right), \qquad k = 1 \ldots M \\ S_k(f) &= \sqrt{T_s T} sinc\left((f - f_k)T\right) e^{-\jmath\pi(f - f_k)T} \end{aligned} \tag{8.1}$$

with $f_k = (k-1)\Delta_f$.

From equations (3.28) and (3.32)

$$\begin{aligned} &\sum_{n=0}^{N_o - 1} \frac{\partial s_k(nT_s - \tau_{lk})}{\partial \tau_{lk}} s_k^*(nT_s - \tau_{lk}) \approx \frac{1}{T_s} \int_0^{T_o} \frac{\partial s_k(t - \tau_{lk})}{\partial \tau_{lk}} s_k^*(t - \tau_{lk}) dt = \\ &= \frac{1}{T_s} \int_{\tau_{lk}}^{\tau_{lk} + T} \frac{\partial s_k(t - \tau_{lk})}{\partial \tau_{lk}} s_k^*(t - \tau_{lk}) dt = \frac{1}{T_s} \int_0^T \frac{\partial s_k(\tau)}{\partial \tau} s_k^*(\tau) d\tau = \\ &= \frac{1}{T_s} \int_{f_k - 1/T}^{f_k + 1/T} \jmath 2\pi f \left|S_k(f)\right|^2 df = \frac{\jmath 2\pi}{T_s} \int_{-1/T}^{1/T} (f + f_k) T_s T sinc^2(fT) df = \\ &= \jmath 2\pi T \int_{-1/T}^{1/T} f sinc^2(fT) df + \jmath 2\pi f_k T \int_{-1/T}^{1/T} sinc^2(fT) df \approx \jmath 2\pi f_k T \frac{1}{T} = \jmath 2\pi f_k \end{aligned} \tag{8.2}$$

because $f sinc^2(fT)$ is an odd function integrated over a symmetric interval.

From equations (3.36) and (3.37)

$$\sum_{n=0}^{N_o-1} n\left|s_k(nT_s-\tau_{lk})\right|^2 \approx \frac{1}{T_s}\int_{\tau_{lk}}^{\tau_{lk}+T}\frac{T_s}{T}\frac{t}{T_s}dt = \frac{1}{T_sT}\left[\frac{t^2}{2}\right]_{\tau_{lk}}^{\tau_{lk}+T} = \\ = \frac{1}{2T_sT}\left((\tau_{lk}+T)^2-\tau_{lk}^2\right) = \frac{T}{2T_s}+\frac{\tau_{lk}}{T_s} \tag{8.3}$$

From equation (3.42)

$$\sum_{n=0}^{N_o-1}\left|\frac{\partial s_k(nT_s-\tau_{lk})}{\partial\tau_{lk}}\right|^2 \approx \frac{1}{T_s}\int_{\tau_{lk}}^{\tau_{lk}+T}\left|\frac{\partial s_k(t-\tau_{lk})}{\partial\tau_{lk}}\right|^2 dt = \frac{1}{T_s}\int_0^T\left|\frac{\partial s_k(\tau)}{\partial\tau}\right|^2 d\tau = \\ = \frac{1}{T_s}\int_{f_k-1/T}^{f_k+1/T}4\pi^2f^2\left|S_k(f)\right|^2 df = \frac{4\pi^2}{T_s}\int_{-1/T}^{1/T}(f+f_k)^2TT_s sinc^2(fT)df = \\ = \frac{4}{T}\int_{-1/T}^{1/T}sin^2(\pi fT)df + \frac{8\pi^2Tf_k}{\pi^2T^2}\int_{-1/T}^{1/T}\frac{sin^2}{f}(fT)df + 4\pi^2Tf_k^2\int_{-1/T}^{1/T}sinc^2(fT)df = \\ = \frac{4}{T}\frac{1}{\pi T}\pi + 0 + 4\pi^2f_k^2 = \frac{4}{T^2}+4\pi^2f_k^2 \tag{8.4}$$

From equation (3.43)

$$\sum_{n=0}^{N_o-1} n^2\left|s_k(nT_s-\tau_{lk})\right|^2 \approx \frac{1}{T_s^3}\int_{\tau_{lk}}^{\tau_{lk}+T}t^2\left|s_k(t-\tau_{lk})\right|^2 dt = \frac{1}{T_s^3}\int_0^T(\tau+\tau_{lk})^2\left|s_k(\tau)\right|^2 d\tau = \\ = \frac{1}{T_s^3}\int_0^T\frac{T_s}{T}(\tau-\tau_{lk})^2d\tau = \frac{1}{T_s^2T}\left(\frac{T^3}{3}+T^2\tau_{lk}+T\tau_{lk}^2\right) = \frac{T^2}{3T_s^2}+\frac{T\tau_{lk}}{T_s^2}+\frac{\tau_{lk}^2}{T_s^2} \tag{8.5}$$

From equation (3.48)

$$\sum_{n=0}^{N_o-1} n\frac{\partial s_k(nT_s-\tau_{lk})}{\partial\tau_{lk}}s_k^*(nT_s-\tau_{lk}) \approx \frac{1}{T_s^2}\int_0^{T_o}t\frac{\partial s_k(t-\tau_{lk})}{\partial\tau_{lk}}s_k^*(t-\tau_{lk})dt = \\ = \frac{1}{T_s^2}\int_{\tau_{lk}}^{\tau_{lk}+T}t\frac{\partial s_k(t-\tau_{lk})}{\partial\tau_{lk}}s_k^*(t-\tau_{lk})dt = \frac{1}{T_s^2}\int_0^T(\tau+\tau_{lk})\frac{\partial s_k(\tau)}{\partial\tau}s_k^*(\tau)d\tau = \\ = \frac{\jmath 2\pi f_k\tau_{lk}}{T_s}+\frac{1}{T_s^2}\int_{f_k-1/T}^{f_k+1/T}\frac{\jmath}{2\pi}\frac{d}{df}\left(\jmath 2\pi fS_k(f)\right)S_k^*(f)df \tag{8.6}$$

and

$$\begin{aligned}
&\frac{1}{T_s^2}\int_{f_k-1/T}^{f_k+1/T}\frac{\jmath}{2\pi}\frac{d}{df}\left(\jmath 2\pi f S_k(f)\right)S_k^*(f)df = \\
&= -\frac{T}{T_s}\int_{f_k-1/T}^{f_k+1/T}\frac{d}{df}\left(f sinc((f-f_k)T)e^{-\jmath\pi(f-f_k)T}\right)sinc((f-f_k)T)e^{\jmath\pi(f-f_k)T}df = \\
&= -\frac{T}{T_s}\int_{f_k-1/T}^{f_k+1/T}\Big[sinc^2((f-f_k)T)+f\frac{d}{df}\left(sinc((f-f_k)T)e^{-\jmath\pi(f-f_k)T}\right)\times \\
&\qquad\qquad \times\, sinc((f-f_k)T)e^{\jmath\pi(f-f_k)T}\Big]df = \\
&= -\frac{T}{T_s}\int_{-1/T}^{1/T}sinc^2(fT)df+\frac{T}{T_s}\int_{-1/T}^{1/T}\jmath\pi T(f+f_k)sinc^2(fT)df+ \\
&-\frac{T}{T_s}\int_{-1/T}^{1/T}(f+f_k)\frac{dsinc(fT)}{df}sinc(fT)df = \\
&= -\frac{1}{T_s}+\frac{\jmath\pi f_k T}{T_s}-\frac{T}{T_s}\int_{-1/T}^{1/T}f\left[\frac{cos(\pi fT)}{f}-\frac{sinc(fT)}{f}\right]sinc(fT)df = \\
&= -\frac{1}{T_s}+\frac{\jmath\pi f_k T}{T_s}-\frac{T}{T_s}\int_{-1/T}^{1/T}\left(-sinc^2(fT)+sinc(fT)cos(\pi fT)\right)df = \\
&= -\frac{1}{T_s}+\frac{\jmath\pi f_k T}{T_s}+\frac{1}{T_s}-\frac{T}{T_s}\int_{-1/T}^{1/T}\frac{sin(\pi fT)cos(\pi fT)}{\pi fT}df = \\
&= -\frac{1}{T_s}+\frac{\jmath\pi f_k T}{T_s}+\frac{1}{T_s}-\frac{T}{T_s}\int_{-1/T}^{1/T}sinc(2fT)df = -\frac{1}{T_s}+\frac{\jmath\pi f_k T}{T_s}+\frac{1}{T_s}-\frac{1}{2T_s} = \frac{\jmath\pi f_k T}{T_s}-\frac{1}{2T_s}.
\end{aligned} \tag{8.7}$$

Then, given $\beta = \frac{-2\alpha_1+2\alpha_2}{\nu}+\alpha_2$, for the frequency modulated signals the equations derived in Ch. 3 become

- eq. (3.28): $\beta 4\pi(f_0-f_k)\zeta_{Im}\frac{E}{M\sigma^2}$
- eq. (3.32): $-\beta 4\pi(f_o-f_k)\zeta_{Re}\frac{E}{M\sigma^2}$
- eq. (3.36): $\beta 4\pi(T+2\tau_{lk})\zeta_{Im}\frac{E}{M\sigma^2}$
- eq. (3.37): $-\beta 4\pi(T+2\tau_{lk})\zeta_{Re}\frac{E}{M\sigma^2}$
- eq. (3.42): $\beta\left(8\pi^2 f_0^2+8\pi^2 f_k^2+\frac{4}{T^2}\right)|\zeta|^2\frac{E}{M\sigma^2}\delta[l-l',k-k']$
- eq. (3.43): $\beta 8\pi^2\left(\frac{T^2}{3}+T\tau_{lk}+\tau_{lk}^2\right)|\zeta|^2\frac{E}{M\sigma^2}\delta[l-l',k-k']$
- eq. (3.48): $-\beta 8\pi^2(f_0-f_k)\left(\frac{T}{2}+\tau_{lk}\right)|\zeta|^2\frac{E}{M\sigma^2}\delta[l-l',k-k']$.

Regarding the pseudo-noise signals, the aforementioned method is still valid, although the integrals can't be simplified. In this case it is:

$$
\begin{aligned}
s_k(t) &= \sqrt{\frac{T_s}{2T}} \sum_{i=0}^{N_s-1} c_{k,i} rect\left(\frac{t - T_s/2 - iT_s}{T_s}\right), \qquad k = 1 \ldots M \\
S_k(f) &= \sqrt{\frac{T_s}{2T}} T_s sinc\left(fT_s\right) e^{-\jmath\pi f(T_s)} \sum_{i=0}^{N_s-1} c_{k,i} e^{-\jmath 2\pi f i T_s}.
\end{aligned}
\tag{8.8}
$$

Regarding the derivation under correlated clutter, the steps are the same, although in this case the integrals do not simplify even for the orthogonal in frequency signals.
Finally, in order to prove that orthogonality is maintained even in the presence of frequency derivatives (caused by the nT_s and $n^2T_s^2$ terms related to the Doppler shift information), the ambiguity functions are presented for those cases as well. Numerical values are hose already used at the start of Chapter 5.

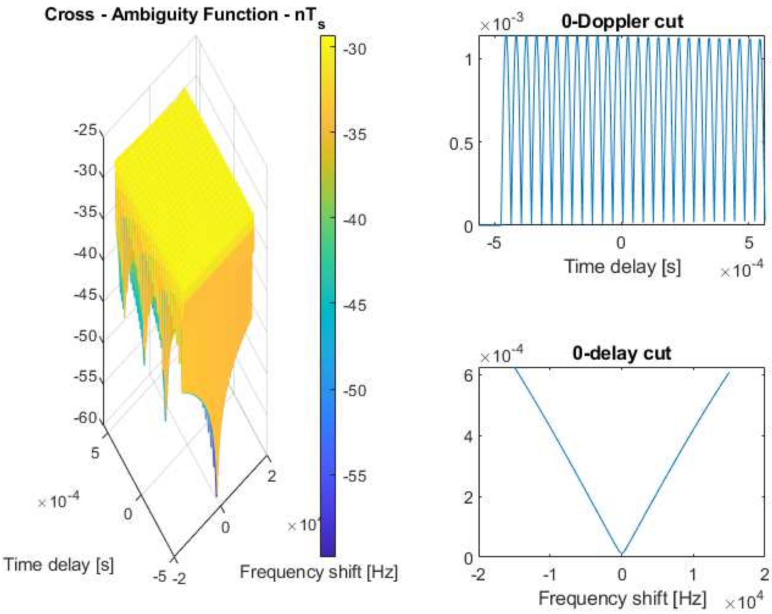


Figure 8.1: CAF for the frequency case, when one of the signals is multiplied by nT_s. The complete function is shown in dB, while the cuts are linearly scaled

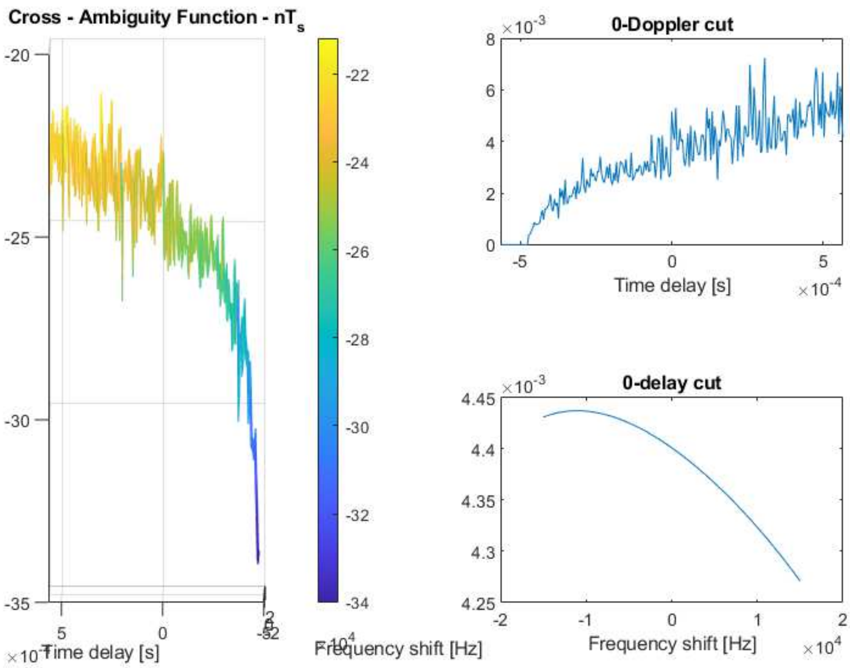


Figure 8.2: CAF for the code case, when one of the signals is multiplied by nT_s. The complete function is shown in dB, while the cuts are linearly scaled

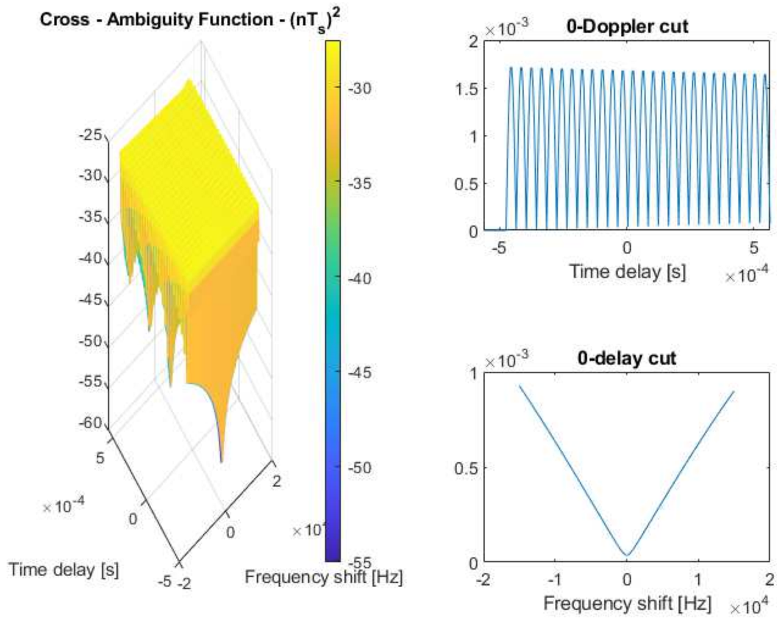


Figure 8.3: CAF for the frequency case, when one of the signals is multiplied by $(nT_s)^2$. The complete function is shown in dB, while the cuts are linearly scaled

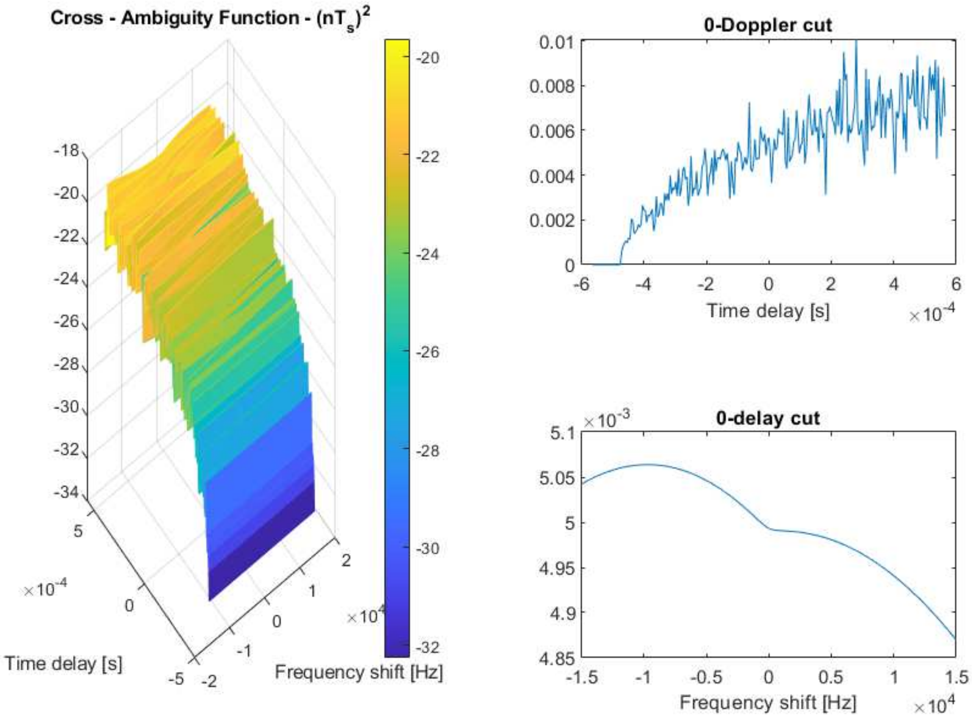


Figure 8.4: CAF for the code case, when one of the signals is multiplied by $(nT_s)^2$. The complete function is shown in dB, while the cuts are linearly scaled

Chapter 9

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