Inverse Optimization: Inferring Unknown Instance Parameters from Observed Decisions

THESIS

Keith Batista, First Lieutenant, USAF

AFIT-ENS-MS-22-M-116

DEPARTMENT OF THE AIR FORCE
AIR UNIVERSITY

AIR FORCE INSTITUTE OF TECHNOLOGY

Wright-Patterson Air Force Base, Ohio

AFIT-ENS-MS-22-M-116

INVERSE OPTIMIZATION: INFERRING UNKNOWN INSTANCE PARAMETERS FROM OBSERVED DECISIONS

THESIS

Presented to the Faculty

Department of Operational Sciences

Graduate School of Engineering and Management

Air Force Institute of Technology

Air University

Air Education and Training Command

in Partial Fulfillment of the Requirements for the

Degree of Master of Science in Operations Research

Keith Batista, BS

First Lieutenant, USAF

March 24, 2022

AFIT-ENS-MS-22-M-116

INVERSE OPTIMIZATION: INFERRING UNKNOWN INSTANCE PARAMETERS FROM OBSERVED DECISIONS

THESIS

Keith Batista, BS
First Lieutenant, USAF

Committee Membership:

Dr. Brian J. Lunday
Chair

Maj Phillip R. Jenkins, PhD
Member

AFIT-ENS-MS-22-M-116

Abstract

Countries continuously strive to gain and maintain a competitive advantage over their adversaries. In doing so, countries invest in a variety of technical and other developments to advance their military preparedness. Although one can observe an adversary's investments, unknown are often the manner in which the adversary views the potential investments, either individually or in aggregate. Thus, it is of interest to infer the adversary's information that informs their decisions. The objective of this research was to develop procedures that estimate selected, unknown parameters over an adversary's investment portfolio across a set of new or existing technologies. To solve for the selected unknown parameters, it is assumed that the adversary is maximizing the portfolio optimization problem and investing along the efficient frontier. These models give a deeper understanding of the individual's risk tolerance level, which is the first step to forecasting an adversary's investment strategy. Anticipating an adversary's investment strategy can give a country or corporation a strategic advantage when it comes to resources and power. To accurately infer the adversary's investment strategy, this research will conduct two unique techniques. The first technique was when an unknown risk attitude exists but all other parameters were known (i.e. expected return, variance, covariance). An adaptive line search technique that iteratively solved the portfolio optimization problem until the adversary's risk parameter was found. The size of the stock options varied, but for every experiment there was less than 1% average error between the identified risk parameter and the actual risk parameter. The level of precision that the adaptive line search was able to solve proves to be a noteworthy result. The second technique solved was when there are unknown parameters for a new investment option but all other information is known. To accurately solve this problem two strategies were used. First a system of equations was used, but the system of equations did not produce any substantial results. Alternatively, two variants of a mesh-based grid search were implemented over a three-dimensional space to visualize the feasible region yielding optimal solutions. The mesh-based search strategy produced a majority of the results on a single three-dimensional plane. Additionally, these strategies revealed that there are multiple optimal solutions for the same portfolio allocation in a subregion, which evidence shows may be bounded within a convex region.

Acknowledgements

I would like to express my sincere gratitude to Dr. Lunday for helping turn my paper from a frown to an upside down frown.

Keith Batista

Table of Contents

Page

List of Figures

List of Tables

INVERSE OPTIMIZATION: INFERRING UNKNOWN INSTANCE PARAMETERS FROM OBSERVED DECISIONS

I. Introduction

This chapter first discusses the significance of this research i.e., why this topic is pertinent to countries searching for strategic advantages over their adversaries. Following that, this chapter will discuss the overarching goals of this research, and the structure for the remaining chapters.

1.1 Motivation and Background

The United States (United States.) is struggling to maintain its competitive edge over its adversaries, and these adversaries are enhancing their capabilities militarily, economically, and technologically. The main competitors of the United States are the People's Republic of China (PRC), Russia, Iran, and the Democratic People's Republic of Korea (DPRK) (Biden, 2021). The United States. Intelligence Community further details the nature of this geopolitical competition, with emphasis on how adversaries seek to undercut the United States and advance their interests (Haines, 2021). The Interim National Security Strategic Guidance informs readers of the efforts made by antagonists to prevent the United States from supporting and defending the interest of the United States and its Allies (Biden, 2021).

Defense Department officials have noted the United States' adversaries are outspending them in selected global investments, causing acute security concerns (United States Senate, 2015). In general, potential adversaries are allocating time, money, and other resources to develop and advance capabilities that threaten the United States.

and its ability to safeguard its national security interests. The PRC has been escalating their cyber-espionage operations to extract intelligence from the United States and its allies (Haines, 2021). Russia has been developing long-range strike weapons including hypersonic missiles that, when launched by air or sea, pose a compelling first-strike capability (Missile Defense Project, 2018). Iran continues to increase the enrichment levels of their uranium in spite of global restrictions (Haines, 2021). More ever, the DPRK continues to advance its cyber capabilities and is capable of causing devastation to the United States' infrastructure.

As these countries are becoming ever more aggressive, there is ever-increasing need for the United States to anticipate their future actions by understanding their priorities, their attitude towards risk, and the manner in which they perceive potential investments. This anticipation is necessary because, although an adversaries decisions can be observed, one does not always know what data or assumptions inform those decisions. Thus, effective methods to accurately infer unknown parameters is important to assess an adversaries perspective on a problem that may affect future investment decisions.

1.2 Problem Statement

Given unknown parameters from observed financial distributions of an adversary's investment portfolio, this research seeks to identify models and accompanying solution methods to accurately and efficiently estimate the unknown parameter values.

1.3 Organization of the Thesis

The organization for this thesis is as follows. Chapter II reviews the literature related to modern and post-modern portfolio theory, as well as parametric inference techniques. Chapter III proposes solution methods to develop point or interval estimates for selected, unknown parameters for a classical Markowitz portfolio optimization problem, used as a proxy for the broader problem of adversary investment over a set of existing and in-development capabilities. Chapter IV discusses instance generation for testing, after which it presents and discusses the efficacy and efficiency of solving for the unknown risk levels. Chapter V utilizes two separate techniques to solve for an optimal portfolio with limited or unknown stock information. Chapter VI concludes the thesis with a summary of results and proposed extensions for future research.

II. Literature Review

This chapter first discusses modern and post-modern theories of portfolio optimization. For the former category (i.e., the focus of this research), it subsequently introduces Sharpe Ratio maximization as the foundational approach to classic portfolio investment strategy, presents the equivalent mathematical programming approach, and discusses alternative nonlinear math programming formulations that seek to approximate such a strategy. Thereafter, this chapter discusses the methods used to identify the parameters for such mathematical programs.

2.1 Portfolio Optimization Frameworks

Understanding modern portfolio theory (MPT) is critical to addressing the problem introduced in Chapter I. Markowitz is considered the father of modern portfolio theory, and the fundamentals following are based on and developed from his work (Markowitz, 1999). The goal of MPT is to invest along an efficient (E, V)-combination after making probabilistic estimates of future performances. That is, investors should try to maximize E, which is the expected return of the investments, while minimizing V, which is the variance of risk (Markowitz, 1952). To limit the risk, the investor needs to diversify their portfolio and decrease the effect of positive covariances between the different alternatives. In MPT, Markowitz assumes that there are no short sells, which occur when one borrows a security and sells it, planning on buying it back at a future date. A reader interested in a deeper understanding of the portfolio theory with short selling should consult either the work of Pogue (1970) and/or Jacobs and Levy (2013). Although MPT was created in the 1950's, it is still relevant today. Individuals still use Markowitz's properties to ensure their portfolios are diversified adequately. Some examples of these topics are examined by Pan (2021) and Green

(2021).

Postmodern portfolio theory (PMPT) was developed by Sortino and Ven Der Meer (1991) many years after MPT was discovered by Markowitz. The main difference is that PMPT measures negative volatility as risk, whereas MPT measures all standard deviation in the investments as risk (Swisher and Kasten, 2005). PMPT also uses minimum acceptable return (MAR), which allows additional constraints of the investor to be considered in the model (Rom and Ferguson, 1994). PMPT also utilizes downside risk which minimizes the worst case scenario for an investor. Calculating downside risk helps individuals construct a more consistent portfolio rather than have high variance between gains and losses (Sortino et al., 2001). Utilizing PMPT allows a financial analyst to make more appropriate assumptions when it comes to the investors and how they respond to losing money. Downside risk is still relevant today; an interested reader is recommended to read the works by Sortino and Van Der Meer (1991) and Delle Monache et al. (2021). This paper focuses on MPT instead of PMPT because many of the advantages that are gained from PMPT (i.e., investment options and short sales) are not present for the motivating problem of a country investing resources in the development and acquisition of new technologies (e.g., they may not be traded on an open financial market). If the reader wishes to learn more about PMPT, they can explore the works by Yildiz and Erzurumlu (2018), Sortino and Van Der Meer (1991), Rollinger and Hoffman (2013), and Sortino et al. (2001).

2.2 Sharpe Ratio

In the foundational work of modern portfolio optimization, Sharpe (1966) proposed maximizing the ratio of the expected increase of a portfolio's return (beyond a benchmark return) to its variance. Known as the Sharpe Ratio (SR), it is represented in Equation (1), wherein $E(\mathbf{x})$ is the expected return of the portfolio, E_b is the risk free rate of return (e.g., a security backed by the U.S. Treasury), and $V(\mathbf{x})$ is the variance of the portfolio's return.

$$\max_{\mathbf{x}} \ \frac{E(\mathbf{x}) - E_b}{\sqrt{V(\mathbf{x})}} \tag{1}$$

Given E_b is a constant, a common proxy is to maximize Equation (2), which is appropriate when either there is no risk-free alternative or a lack of consumer confidence perceives an absence of such an alternative.

$$\max_{\mathbf{x}} \ \frac{E(\mathbf{x})}{\sqrt{V(\mathbf{x})}} \tag{2}$$

From a mathematical programming perspective, the latter problem can be represented as a nonlinear program (NLP), given the following sets, parameters, and decision variables. Define N to be a set of possible investments and x_i to be the proportion of available funds in the portfolio allocated to investment $i \in N$. Each investment has an expected return r_i over a planned duration of portfolio, and each pair of investments $i, j \in N$ has a covariance q_{ij}. Within this context, a decision-maker solves the following problem **P1**, which is equivalent to solving Equation (2).

$$\mathbf{P1}\colon \ \max_{\mathbf{x}} \ \frac{\sum_{i \in N} r_i x_i}{\sqrt{\sum_{i \in N} \sum_{j \in N} x_i q_{ij} x_j}} \tag{3a}$$

$$\text{s.t.} \ \sum_{i \in N} x_i = 1, \tag{3b}$$

$$x_i \geq 0, \ \forall \ i \in N. \tag{3c}$$

Within P1, the objective function (3a) computes the ratio indicated in Equation (2), whereas Constraints (3b) and (3c) collectively require the allocation of available funds to be efficient and feasible. Of note, the non-negativity of allocations indicated by Constraint (3c) assumes the portfolio does not allow for short selling (Pogue, 1970).

Given the challenging nature of efficiently and effectively solving nonlinear programs, having fractional representations of functions can decrease the efficacy of the program. It is desirable to transform Equations (3a)–(3c) into an simpler math programming representation. Defining λ as a risk parameter that corresponds to the relative priority on the two objectives of maximizing a portfolio's expected return and minimizing its variance, a common alternative approach is to solve the nonlinear program **P2**.

$$\textbf{P2}: \max_{\mathbf{x}} \ (1-\lambda)\sum_{i\in N} r_i x_i - \lambda \sum_{i\in N}\sum_{j\in N} x_i q_{ij} x_j \tag{4a}$$

$$\text{s.t.} \ \sum_{i\in N} x_i = 1, \tag{4b}$$

$$x_i \geq 0, \ \forall\, i \in N. \tag{4c}$$

Within P2, the objective function (4a) combines the two objective functions into a single objective function via the implementation of the Weighted Sum Method (Ehrgott, 2005) for a fixed value of $\lambda \in (0,1)$. For a decision-maker who is completely risk seeking (or arguably risk agnostic), solving this NLP with $\lambda = 0$ will identify a solution that strictly maximizes the expected return of the portfolio, whereas a completely risk-adverse decision-maker will minimize the portfolio's variance by solving the problem with $\lambda = 1$.

Note that solving an instance of P2 for a fixed λ is not necessarily equivalent to solving the corresponding instance of P1. However, for an optimal solution $\mathbf{x}^*$ to P1, there exists a λ-value for which $\mathbf{x}^*$ is also optimal to P2. Thus, there are two practical ways to utilize P2. As a first method, the λ-value may be affixed as a risk attitude parameter to identify an investment strategy by solving P2 once, or one may iteratively solve P2 over the range of $\lambda \in (0,1)$ to identify the λ-value yielding a maximal Sharpe Ratio. It is also possible to relax λ, treating it as a decision variable, although such an approach is not common due to the following characteristic of P2 for a fixed λ-value.

An important characteristic of P2 is the potential convexity of the mathematical program. Theoretically, the covariance matrix and the resulting problem will always be symmetric and positive semi-definite, but in a practical instance it may be possible not to have these conditions (Newey and West, 1987). Consider the covariance matrix P having entries $q_{ij}, \forall i,j \in N$. If P is positive definite (PD) for an instance of P2, the objective function (4a) is convex. Given its linear constraints (4b) and (4c), such an instance of P2 yields a convex program, and any local optimal solution is also a global optimal solution (Bazaraa et al., 2013). Thus, an instance of P2 having a PD matrix P allows the use of readily available commercial solvers for convex NLPs that are guaranteed to find an optimal solution quickly, rather than resort to either invoking a global optimization solver or developing a customized algorithm.

Alternative to solving P2, one can optimize one objective function while bounding the other objective function, as follows:

$$\max_{\mathbf{x}} \quad \sum_{i \in N} r_i x_i \tag{5a}$$

$$\text{s.t.} \quad \sum_{i \in N} \sum_{j \in N} x_i q_{ij} x_j \leq \varepsilon \tag{5b}$$

$$\sum_{i \in N} x_i = 1,$$

$$x_i \geq 0, \; \forall \, i \in N.$$

For the objective function (5a), the individual is exclusively maximizing returns using the ε-constraint Method (Ehrgott, 2005) to impose an upper bound on the portfolio's variance. Equation (5b) prevents the decision maker from choosing a portfolio of investments having insufficiently high variance. Additionally, this formulation is more intuitive and clearly outlines the worst-case scenario, given their chosen investment strategy. Such a formulation strategy would be conducted by risk neutral individuals (Hodnett et al., 2012). The reciprocal of providing an upper bound for variance is providing a lower bound for the expected return.

$$\min_{\mathbf{x}} \quad \sum_{i \in N} \sum_{j \in N} x_i q_{ij} x_j \tag{6a}$$

$$\text{s.t.} \quad \sum_{i \in N} r_i x_i \geq \varepsilon \tag{6b}$$

$$\sum_{i \in N} x_i = 1,$$

$$x_i \geq 0, \ \forall \ i \in N.$$

In the objective function (6a), variance is being minimized over the selected portfolio. The subsequent constraint (6b) is a lower bound requiring a minimum expected return for a given solution. Equation (6b) ensures the portfolio's expected return is above l_r which allows the investor's goals to be adequately reached. This formulation would generally be the approach taken by risk averse individual (Hodnett et al., 2012).

From an optimization perspective, solving Equation (6a) ,a quadratic program with linear constraints, is appealing because many leading commercial solvers (e.g., CPLEX, Gurobi) can identify a global optimal solution even if the objective function is not convex, whereas the previous formulation having a nonlinear constraint requires specially designed, global optimization solvers or customized solution procedures to identify a global optimal solution.

2.3 Parameterization of a Portfolio Optimization Instance

Relevant to solving a portfolio optimization problem instance is having accurate and useful data. Within this research, expected return was calculated using logarithmic returns. The use of logarithmic returns enhances the ability to compare the performances across different assets easier (Hudson, 2010). To calculate the returns, we used the formulations in Equations (7) and (8). Within this research, the period considered was 24 months; the period selected should be tailored to the application of interest.

$$r_{it} = \log(P_{it} - P_{i(t-1)}),\ \forall\, i \in N,\ t \in T \setminus \{1\}, \tag{7}$$

$$\mu_i = \frac{\left(\sum\limits_{t \in T \setminus \{1\}} r_i \right)}{|T| - 1},\ \forall\, i \in N \tag{8}$$

The formulation in Equation (7) takes the logarithm of the price at time t and subtracts the previous month's price. Doing this over $|T|$-month period will yield the mean monthly return price as shown in Equation (8) that will be time additive[1] and unique[2] (Hudson, 2010).

The calculation for covariance will also depend on the logarithmic approach for the expected return and will remain consistent to adequately compare assets.

[1] Time additive means that one can add them across time to get the total return over T.
[2] Unique means that every stock will have its own differing value each period.

$$q_{ij} = \frac{\sum_{t \in T \setminus \{1\}} (r_{it} - u_i)(r_{jt} - u_j)}{(T-1)}, \forall\ i, j \in N \tag{9}$$

Equation (9) multiples the difference of each stock's return for every day from their average and divides by $T - 1$. When $i = j$, q_{ij} is the variance of a given stock. This strategy is discussed in Cherewyk (2018) and this method will provide an adequate procedure for determining the similarities and differences that exist between the differing stock options.

III. Modeling and Solution Methodology

This chapter will first revisit MPT to discussing how is applied in this research. The parameters for test instances are presented, and then the formulations for calculating an unknown risk parameters λ are discussed. Lastly, we will talk through the possibility of solving for r_i and q_{ij} for additional stock options.

3.1 The Portfolio Optimization Problem

In MPT, the objective function seeks to maximize the expected return while minimizing the variance over the entire portfolio. To optimize the portfolio, the risk attitude of the investor, expected returns, and the variance of each investment opportunity are required. Hereafter, an *observed solution* means an optimal solution to the portfolio optimization problem for a subset of known parameters and a conjecture about the selected, unknown parameters.

3.2 Parameterization of the Portfolio Optimization Problem

For testing the proposed methods developed in this research, the specific instance data will be informed by the US stock market, and an adversary will be assumed to select an optimal solution from a set of different stock options indicated by N. The set of stocks will each have an expected return ($E(\mathbf{x})$), variance ($V(\mathbf{x})$), and a covariance (q_{ij}) between differing stock options. T for this problem will be 24, which is one data point every month for two years. To increase the distinction in covariance between differing stocks, time increments were monthly rather than daily. The investor will also have a risk attitude when selecting an optimal portfolio, and the different attitudes fall on a spectrum anywhere from zero to one. A zero means the investor is risk seeking, and a one means the investor is risk averse. If a risk averse investor had two portfolios that each yielded an expected return of $10,000, but portfolio A had a volatility of 8% while portfolio B had a volatility of 15%, the investor would choose portfolio A. However, a risk seeking investor would see those portfolios as identical in terms of performance.

3.2.1 Expected Value

The expected values were calculated using Equation (7) and the adjusted closing price of an individual stock once a month for two years. After obtaining the current price at time t, the value would be subracted from its price at $t-1$ to see the difference. Doing this over a 24-month period yields the mean monthly return price for this research.

3.2.2 Covariance of stocks

The covariance of a stock uses the expected return of an individual stock, and it relates the expected return to the mean monthly return of the same stock. Covariance is similar to the variance equation, but instead of squaring the equation , one multiplies the two different stock's data. The formulation can be found as Equation (9).

3.3 Risk Parameter Estimation

The risk parameter in this research enables the investor to characterize their preference of relative priorities over the two objectives in P2. If the investor has a low risk tolerance, they have the ability to increase their expected returns. However, as discussed in Chapter II, Markowitz asserts how a diversification of investments can lead to much better results for the overall portfolio. The risk parameter λ allows the individual investing to decide how important maximizing gains is compared to minimizing the risk of the portfolio. Changing the risk level does not always yield a different solution. In Equation(4a), the reader can see how λ influences the overall objective function; however, changing the risk parameter from (for example) 0.50 to 0.51 may not change the allocations of the investments. The expected values, volatility, and the risk parameter collectively affect the optimal allocation of investments.

Knowing someone's risk attitude can help predict how they will invest in the future, which gives an investor a competitive edge over their adversary. If the expected returns and the variance of the stocks are known, then the risk (λ) can be iteratively solved for. Below is a depiction of how the risk factor could be solved for using the Sharpe's Ratio, given all the other parameters are known.

3.3.1 Adaptive Line Search Technique

Algorithm 1 is the pseudo code for adaptive line search to solve for the unknown λ values. The algorithm calculates a Sharpes Ratio (SR) and seeks the λ-value that yields the observed SR, once it calculates the SR^{obs} in Line *3*, the program will iterate through the different stock options and find a SR^{inf} for any given allocation x_i. The algorithm will compare the two Sharpe Ratios. If the SR^{inf} is significantly smaller than the SR^{obs}, the algorithm will increase the λ-value by δ_λ and resolve the problem at the new λ-value. Since the feasible region of λ is (0-1), λ must stay within the region. If the next increment to λ would yield a value greater than or equal to 1, lines *8-10* artificially increase the SR^{inf} to ensure that the program maintains a feasible λ-value via Lines *11-14*. Likewise, if the actual SR^{inf} in Line *11-14* is at least ϵ greater than SR^{obs}, the algorithm will decrement λ to the previously tested value, reduce the δ_λ by an order of magnitude, and continue solving the problem until the $|SR^{inf} - SR^{obs}| < \epsilon$.

Algorithm 1 Adaptive Line Search on λ

Given and observed solution $\boldsymbol{x}^*$ and known parameters $\boldsymbol{r}$, $\boldsymbol{q}$
Set $\lambda = 0$, $\delta_\lambda = 0.1$, and user-defined precision parameters $\epsilon = 0.0001$, $N = 10000$
Calculate $SR^{obs} = \left(\sum_{i\in N} r_i x_i^*\right) / \sqrt{\sum_{i\in N}\sum_{j\in N} x_i^* q_{ij} x_j^*}$
for $k = 1$ to M **do**
  Solve P2 to identify $\bar{\boldsymbol{x}}$
  Calculate $SR^{inf} = \left(\sum_{i\in N} r_i \bar{x}_i\right) / \sqrt{\sum_{i\in N}\sum_{j\in N} \bar{x}_i q_{ij} \bar{x}_j}$
  if $|SR^{inf} - SR^{obs}| < \epsilon$, **break**
  if $\lambda \geq 1 - \delta_\lambda$ **then**
    $SR^{inf} = SR^{obs} + 1$
  end if
  if $SR^{inf} > SR^{obs}$ **then**
    $\lambda = \lambda - \delta_\lambda$
    $\delta_\lambda = \delta_\lambda / 10$
  end if
  Increment $\lambda = \lambda + \delta_\lambda$
end for
Return the point estimate for λ.

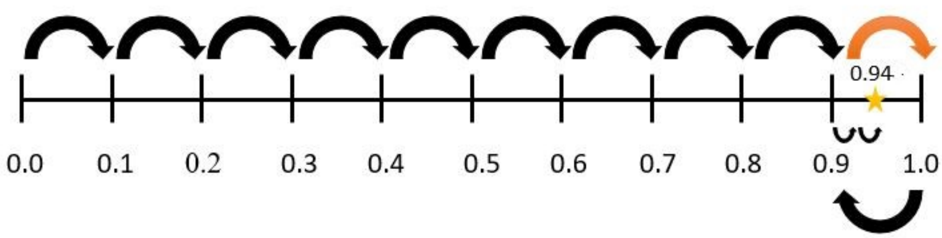

Figure 1. Illustration of Adaptive Line Search

Figure 1 provides a visual representation of how the Adaptive Line Search will continue to iterate through λ until an optimal solution given a tolerance of ϵ is found. It is worth noting that an adversary's risk attitude need not be static (Mallpress et al., 2015). As circumstances occur that may alter one's risk attitude, the inference on λ attained via this procedure must be updated by observing new behavior and reapplying the algorithm.

As an aside, the convexity is tested for each instance via a preprocessing step to verify that the covariance matrix is positive definite, to accurately identify an optimal solution.

3.4 Parameter Estimation For a 'New' Investment Option

Using Algorithm 1, it is possible to solve for a single unknown λ-value. However, what if a new investment option is introduced and its parametric values for the portfolio optimization problem are unknown? Given an adversary's observed optimal portfolio investments, their risk attitude parameter λ, expected returns, and variances for all but the new investment option, is it possible to solve for the expected return, covariances, and variance of a new investment option? This research considers three inference procedures: the first considers a system of equations to identify unique and exact values for each of the unknown variables, and the second and third utilize mesh-based search techniques.

3.4.1 System of Equations

This procedure assumes the optimal objective function value for the adversary is known. Setting Equation (4a) equal to that value, the x-variables are known but, even for an instance having only two investment options – one with known parameters and one with unknown parameters – the result is an underdetermined system. That is, this approach will have three unknown values (i.e., $r_2, q_{12} = q_{21}, q_{22}$) for a single equation. As such, even with bounds on the covariance values, we expect this solution method to fail, and preliminary experiments confirmed this intuition. As such, it is not discussed further.

3.4.2 Mesh-based Search Techniques

There are two differing search techniques that will be tested in Chapter V. Both are limited within this study to problems having $|N| = 2$.

Algorithm 2 searches over the three unknown parameters (i.e., $r_2, q_{12} = q_{21}, q_{22}$) and solves Problem P2, searching for nearly identical x_i-values to the adversary's observed portfolio investments. As an alternative approach, Algorithm 3 fixes the x_i-variables and searches over the unknown covariance parameter space, iteratively solving for the r_2-parameter. Since the only unknown parameter is r_2, this problem evolves from a NLP to a LP which locates the local optimal solutions instead of global optimal solutions.

3.4.2.1 Solving for x_i-values Searching Over q_{22}, q_{12}, and r_2

As presented in pseudocode, Algorithm 2 does require some user-determined parameters. The first parameter utilized is δ, which is the search granularity for iterating through the three unknown variables. The search granularity was altered to compare the results of different levels. The next parameter is ϵ which is the tolerance allowed between the observed solution's allocations and the inferred allocations. Lastly, r_2^{UB} limits the computation time of this algorithm. It is reasonable to assume an upper bound on the expected return of the new stock because if the value is too much higher then the other stock option then it would never be chosen and therefore would never be selected by the portfolio optimization algorithm.

Algorithm 2 starts each variable (r_2, q_{12}, and q_{22}) at its lower bound and solves Equation (4a). If at least ones of stock's portfolio allocations are within the user defined tolerance level (ϵ), then the three variables are recorded. If one or both of the stock's allocations are higher than ϵ then the program continues to iterate through the three unknown parameters. The parameter r_2 has a range from (0,1) because if

Algorithm 2 Plane search for r_2, q_{12}, q_{21}, and q_{22}, given $N = 2$

```
Given an observed solution x* and known parameters λ, r_1, q_11.
For a search granularity δ, an assumed upper bound r_2^UB on r_2, and a user-defined
accuracy tolerance ϵ:
for q_22 = 0 to 1 by δ do
    for q_12 = −1 to 1 by δ do
        for r_2 = 0 to r_2^UB by δ do
            Solve Problem P2 with q_21 = q_12 to identify x^inf
            if |x_i* − x_i^inf| ≤ ϵ_i then
                Store the values for q_22, q_12, and r_2
            end if
        end for
    end for
end for
```

the expected return was higher than one then it would allocate all of the resources to that investment. On the contrary, if r_2 is lower then zero the alternative investment would get allocated all of the portfolio. Next, q_{22} has a range from (0,1) because the variance of a stock must fall somewhere on that region. Lastly, q_{12} was given a range from (-1,1) because all of the covariances tested in this research fell within this range. Depending on those covariance values the problem being solved may or may not be globally optimal. As with Algorithm 1, the convexity of each instance is tested to verify that the covariance matrix is positive definite.

3.4.2.2 Fix x_i-values, Iterate Over q-values, and Search for r_2

As presented in pseudocode, Algorithm 3 does require some user-determined parameters. The first parameter utilized is δ which is a the search granularity for iterating through the three unknown variables. The search granularity was altered to compare the results of different levels. The next parameter is ϵ which is the tolerance allowed between the observed solution's allocations and the inferred allocations. Algorithm 3 communicates how the code iterates through the differing q-values until it finds an optimal solution. This is done by solving Equation (4a) with the x_i-values fixed and then maximizes the r_2-value.

Algorithm 3 Plane search for r_2 with fixed x_i values

```
Given an observed solution x* and known parameters λ, r_1, q_11.
For a search granularity δ:
for q_22 = 0 to 1 by δ do
    for q_12 = −1 to 1 by δ do
        Solve Problem P2 with q_21 = q_12, fixed x-values, and a relaxed r_2 ≥ 0
        to identify r_2*
        if a local maximum is identified then
            Store the values for q_22, q_12, and r_2
        end if
    end for
end for
```

IV. Testing, Results, and Analysis for Unknown Risk Parameter Identification

This chapter discusses the specific MPT problem instances examined in this research, and the values associated with their parameters. Then it discusses how subsetting can affect the problem, and whether subsetting can be used to increase the computational burden of solving problems without degrading the algorithm's efficacy. Lastly, this chapter discusses the significant outcomes of their results and how they could be further used with larger data sets.

4.1 Problem Setup

This thesis analyzes randomly selected stocks that are listed on the National Association of Securities Dealers Automated Quotations (NASDAQ), a computerized system for trading in securities. The testing is designed to examine how accurately an investor's risk can be analyzed, given a set of investment options with known parameters. There will be three different sets of stocks to test the validity of our formulation, and to measure the accuracy as the set sizes increase. Each problem will be solved over 10 iterations with differing unknown λ-values and investment allocations. The program will solve Problem P2 for all 10 instances to make inferences for the given λ-value and its appropriate allocation of investments. Table 1 provides the original models results for portfolio allocation.

Table 1. Allocation of stocks over 10 instances

Instance	1	2	3	4	5	6	7	8	9	10
KTOS (%)	0	0	0	0	0	0	0	0	0	0
VOO (%)	27.32	0	0	0	10.79	15.43	0	0	0	0.78
CBRL (%)	0	0	0	0	0	0	0	0	0	0
WMT (%)	0	0	0	0	0	0	0	0	0	0
ATVI (%)	58.28	37.96	64.83	36.13	62.99	61.67	51.54	26.27	57.49	65.85
BABA (%)	0	0	0	0	0	0	0	0	0	0
BA (%)	0	0	0	0	0	0	0	0	0	0
GME (%)	14.39	62.04	35.17	63.87	26.22	22.90	48.46	73.72	42.51	33.37
LMT (%)	0	0	0	0	0	0	0	0	0	0
λ-values	0.9942	0.9752	0.9859	0.9745	0.9895	0.9908	0.9806	0.9706	0.9829	0.9866

The baseline results reveal a slight dominance with Gamestop (GME) and Activision (ATVI). These stocks are consistently used in every model which indicates they are more ideal stocks. A stock is more ideal in a portfolio if they have a higher expected return and a lower variance. Now we will show the baseline results for sets of 20 and 30 stock options. The number of stock options are increased to analyze trends within the program as well as test the accuracy of the model as the size of data increases.

Table 2. Allocation for 20 stock set over 10 instances (only reporting stocks having allocation values $\geq 0.10\%$)

Instance	1	2	3	4	5	6	7	8	9	10
GILD (%)	3.87	0.00	0.00	0.00	1.66	4.08	0.00	0.00	0.00	0.01
AMAT (%)	12.78	31.71	47.07	31.23	35.58	30.95	39.67	24.78	43.55	46.29
ZM (%)	23.70	68.29	51.74	68.76	40.69	36.45	60.33	75.22	56.45	49.93
ABNB (%)	20.00	0.00	1.19	0.00	9.08	10.77	0.00	0.00	0.00	3.31
COST (%)	30.30	0.00	0.00	0.00	12.99	17.60	0.00	0.00	0.00	0.25
TMUS (%)	9.35	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
NFLX (%)	0.00	0.00	0.00	0.01	0.00	0.00	0.00	0.00	0.00	0.14
λ-**values**	0.9837	0.9749	0.9861	0.9745	0.9893	0.9907	0.9803	0.9706	0.9829	0.9865

Table 3. Allocation for 30 stock set over 10 instances (only reporting stocks having allocation values $\geq 0.10\%$)

Instance	1	2	3	4	5	6	7	8	9	10
SHOP (%)	1.72	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
T (%)	8.30	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
NVDA (%)	11.15	0.00	0.00	0.00	0.00	2.09	0.00	0.00	0.00	0.00
TSLA (%)	28.53	100.00	73.61	100.00	56.32	49.40	94.40	100.00	87.78	70.67
TWTR (%)	1.17	0.00	0.00	0.00	0.25	2.91	0.00	0.00	0.00	0.00
AMN (%)	30.84	0.00	9.73	0.00	23.08	26.19	0.00	0.00	0.00	12.08
ETSY (%)	18.29	0.00	16.66	0.00	20.35	19.41	5.60	0.00	12.22	17.25
λ-**values**	0.9670	0.9750	0.9859	0.9742	0.9852	0.9835	0.9800	0.9700	0.9829	0.9865

Tables 2 and 3 respectfully provide the results and allocations of portfolios for the sets of 20 and 30 stocks. These results suggest there are superior performing stock options that consistently are selected within an optimal portfolio, and these stocks have a good ratio of return and risk.

4.2 Analysis of Adaptive Line Search λ-values

To test the accuracy of the model, the adaptive line search will identify an optimal allocation of stocks and their respective λ-value. To test the accuracy, the inferred stock allocations and respective λ-values will attempt to infer the observed allocations and λ-values from Tables 1, 2, and 3. First, the λ-values between the actual and identified models will be compared.

Table 4. Comparing λ's – observed vs inferred

Instance	**10 Stocks**		**20 Stocks**		**30 Stocks**	
	$\lambda_{observed}$	$\lambda_{inferred}$	$\lambda_{observed}$	$\lambda_{inferred}$	$\lambda_{observed}$	$\lambda_{inferred}$
1	0.994180	0.994180	0.994180	0.98370	0.994180	0.966245
2	0.975185	0.975181	0.975185	0.97492	0.975185	0.975100
3	0.985886	0.985884	0.985886	0.98590	0.985886	0.986302
4	0.974466	0.974470	0.974466	0.97494	0.974466	0.974378
5	0.989448	0.989447	0.989448	0.98940	0.989448	0.985000
6	0.990772	0.990771	0.990772	0.99080	0.990772	0.982310
7	0.980567	0.980570	0.980567	0.98030	0.980567	0.980600
8	0.970605	0.970600	0.970605	0.97060	0.970605	0.970730
9	0.982939	0.982940	0.982939	0.98276	0.982939	0.983000
10	0.986607	0.986600	0.986607	0.98660	0.986607	1.000000
Average error	-	0.00000	-	0.00118	-	0.00551
Absolute error	-					

Table 4 displays the difference in λ values from the observed optimal risk values compared to the inferred optimal solutions for each of the stock sizes. The 10 stock options solved nearly perfect for every iteration, while the 20 and 30 stock available sets under performed in terms of accuracy they still had noteworthy results. The 20 stock set had adequate results and would most likely meet be usable given the λ value was necessary, however, the results for the set of 30 stocks did have accurate results (less than 1% error). Due to the increasing complexity of the problem, the reduction in accuracy does make sense. To mitigate any computational or accuracy issues, a dynamic subset will be added later in this thesis to improve the results. To get a more complete scope of the correctness that the identified results had, the number of stocks in the actual model was compared to the identified stocks.

Table 5. Comparing number of stocks in observed solution versus inferred via Adapative Line Search on λ

Instance	**10 Stocks**		**20 Stocks**		**30 Stocks**	
	obs	inf	obs	inf	obs	inf
1	3	3	6	2	5	5
2	2	2	9	4	4	5
3	2	2	4	9	5	6
4	2	2	4	4	5	5
5	3	3	6	5	5	5
6	3	3	8	7	7	5
7	2	2	3	4	5	7
8	2	2	3	3	5	4
9	2	2	3	9	7	6
10	3	3	10	5	5	5
Avg Correct (%)	100		72.3		72.3	

Table 5 reveals that, as the number of stock options increase, the relative number of stocks identified in an optimal portfolio for an inferred λ-value decreases. Having more stocks available to chose from inevitably causes the absolute error between observed and inferred λ-values to increase. Additionally, increasing the number of stock options exponentially increases the computational time to solve the problem. The total time to solve the 10, 20, and 30 stock sets were 47 seconds, 3426 seconds, and 8281 seconds respectfully. This time issue needs to be resolved to improve the scalability of the problem. To combat the increase in time, a dynamic subset will be used to solve the problem.

4.3 Dynamic Subset to Exclude some Stock Options

The two key aspects of solving these problems are accuracy and time. To improve accuracy and decrease the computational time to solve the problem, a dynamic subsetting technique will be implemented. The main objective is to reduce the overall time to solve the models while maintaining an acceptable risk value for each iteration. If the subsetting is done correctly, it will reduce the alternatives available to the program which should reduce the total computational time. Tests were run to see if creating a subset of stocks while removing only stocks with an $x_i = 0$ has any effect of the model, but removing those stocks does not effect the optimal results because they were dominated by other investments. This result is not only intuitive, but holds in general. Eliminating the consideration of stocks for which $x_i = 0$ is observed when applying the Adaptive Line Search on λ is equivalent reducing the feasible region of the observed problem without eliminating the optimal solution; the optimal objective function value will not change. Reducing the feasible region of this problem drastically reduced the computation time of this problem.

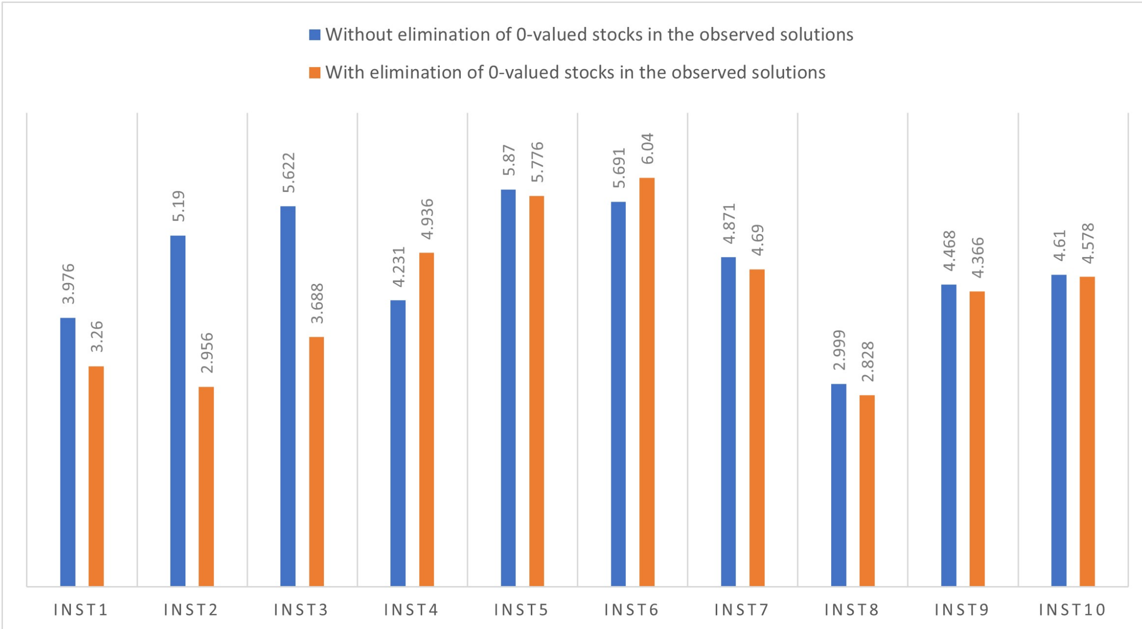

Figure 2. Computational effort (seconds) required by the Adaptive Line Search on λ for 10-stock Instances 1-10, without and with elimination of 0-valued stocks in the observed solutions

Figure 2 illuminates a slight decrease in the overall computational time. The total time went from 4.76 seconds to 4.31 seconds. For a majority of the instances there is a significant reduction, however for two instances, the time after applying the subset increased slightly. Overall, the time was reduced by not momentously.

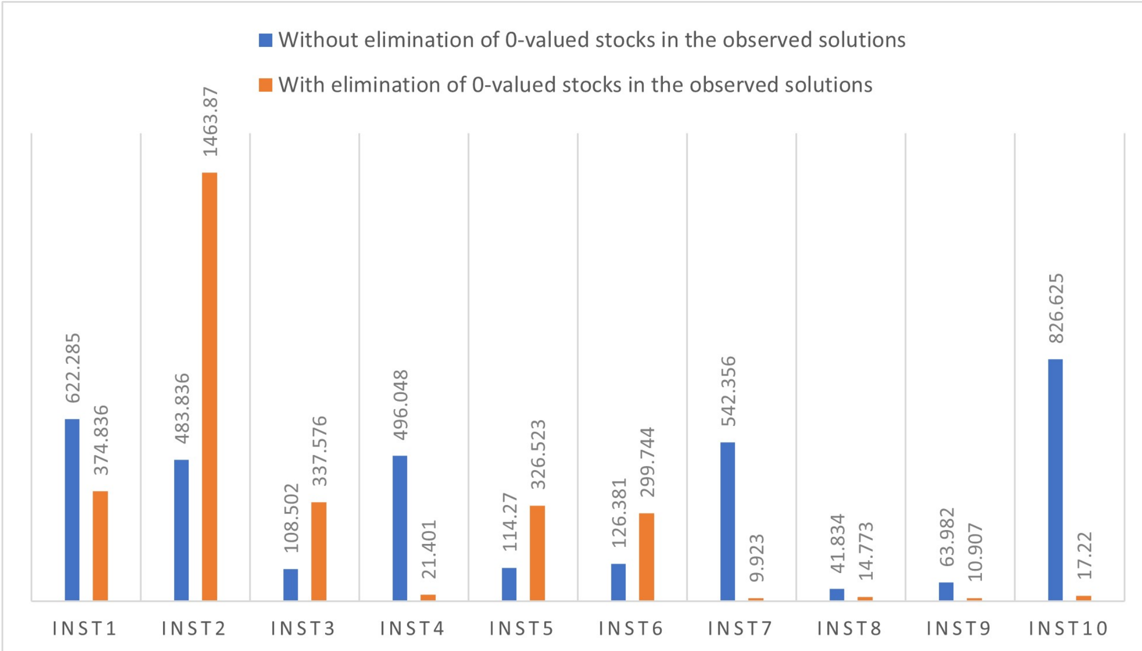

Figure 3. Computational effort (seconds) required by the Adaptive Line Search on λ for 20-stock Instances 1-10, without and with elimination of 0-valued stocks in the observed solutions

Figure 3 displays a much more varied distribution when it comes to the success in the reduction of time for each instance. Using the dynamic subset for the 20 stock set did reduce the computational time from 3426 seconds to 2826 seconds which is a reduction by roughly 16%, on average. This is a significant reduction, even though it was not as consistent for each individual performance.

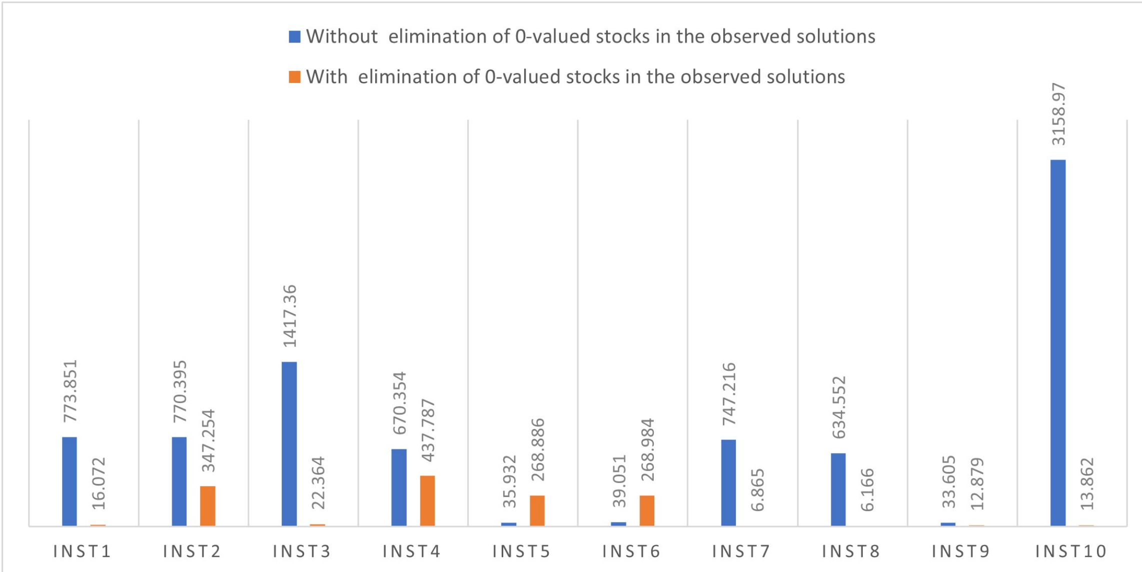

Figure 4. Computational effort (seconds) required by the Adaptive Line Search on λ for 30-stock Instances 1-10, without and with elimination of 0-valued stocks in the observed solutions

Figure 4 shows a reduction from 8221 seconds to 1401 seconds which is approximately an 83% reduction in average computational time for the 30 stock instances. The subset narrows the best potential investment options, and it limits the number of stocks that the program must iterate through every time. The time reduction was not as significant for the 10 and 20 stock options, but the reduction in time was still noteworthy and provided similar results. Table 6 visualizes the overall computational time before an after including the dynamic subsetting. Along with those values, the overall percent change was included to see the increase in efficiency of the dynamic subset. Figures 2-4 above show that using a dynamic subset can drastically improve the computational time, but how does it effect the results for λ? Table 7 displays the total difference in λ over 10 separate iterations before and after the subset was applied.

Table 6. Comparing the computational time before and after dynamic subsetting automatically removing $x_i = 0$

	10 Stocks		20 Stocks		30 Stocks	
	N	$N_{x_i>0}$	N	$N_{x_i>0}$	N	$N_{x_i>0}$
Average Time (seconds)	4.75	4.31	342.61	287.68	828.13	140.11
Change in Time	-	-9.26%	-	-16.03%	-	-83.08%

Table 7. Comparing observed solution versus identified for λ via dynamic subsetting automatically removing $x_i = 0$

	10 Stocks		20 Stocks		30 Stocks	
Target λ	N	$N_{x_i>0}$	N	$N_{x_i>0}$	N	$N_{x_i>0}$
0.994180	0.99418	0.99418	0.98310	0.98400	0.96625	0.96701
0.975185	0.97518	0.97518	0.97590	0.97557	0.97510	0.97504
0.985886	0.98588	0.98588	0.98581	0.98609	0.98630	0.98590
0.974466	0.97447	0.97447	0.97480	0.97446	0.97438	0.97421
0.989448	0.98945	0.98945	0.98940	0.98532	0.98500	0.98518
0.990772	0.99077	0.99077	0.99100	0.98418	0.98231	0.98354
0.980567	0.98057	0.98056	0.98030	0.98000	0.98060	0.98000
0.970605	0.97060	0.97060	0.97060	0.97060	0.97073	0.97000
0.982939	0.98294	0.98294	0.98290	0.98200	0.98300	0.98290
0.986607	0.98661	0.98661	0.98660	0.98650	1.00000	0.98650
Average error	0.00000	0.00000	0.00128	0.00231	0.00551	0.00404

Figure 7 illustrates the importance of this dynamic subset on larger sets of stock options. The average computation time was substantially reduced and in some instances the λ-value was more accurately inferred. Limiting the options of the program may have substantial results, but this theory should be tested further to test the validity of these results.

4.4 Removing Significant Stock Options

To reduce time even further, it is imperative to test if there are any patterns that exist when reducing the set of available stocks. After obtaining the baseline results, a subset of the original model was made by removing one of the active investment options ($x_i \geq 0$). Activision (ATVI) was removed to identify if there is proportionality given to the remaining investment options. Knowing if there is an exact distribution given to each remaining stock would be useful because it would enable the investor to accurately allocate a different subset by distributing the removed stocks in a predictable pattern.

Table 8. Portfolio in the absence of ATVI - optimal stock allocations and relative change (%) over original, optimal allocations.

Instance	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
VOO (%)	73.85	36.65	64.11	34.78	73.13	76.47	50.54	24.70	56.61	65.95
	170%	*	*	*	578%	396%	*	*	*	8370%
WMT (%)	11.42	0	0	0	0	0	0	0	0	0
	*	-	-	-	-	-	-	-	-	-
GME (%)	14.72	63.35	35.88	65.22	26.87	23.53	49.46	75.30	43.39	34.05
	-	-	2%	2%	3%	2%	2%	2%	2%	2%

* A relative percent change (%) cannot be computed; cannot divide by 0

However, Table 8 indisputably reveals there is no consistency in the percent change increase from the reduction of stock with an $x_i \geq 0$. Adding Walmart (WMT) is the leading indicator that there is uncertainty in the new distribution of investments after removing ATVI from the original set. Therefore, the model will need to be completely solved again if certain investment options are no longer available to choose from.

4.5 Results and Analysis

The expected results are that the unknown parameters can either be predicted correctly or very close to what adversaries have chosen. Originally the problem was solved while keeping every stock option available for all 10 iterations. This led to noteworthy results that showed the λ-value can be accurately identified. To expedite the computational time and strive for better results, the dynamic subsetting strategy that removed unused stocks was tested. After implementing the dynamic programming, the average error between the observed and inferred solution was less than 0.404%. Although it does not always infer the exact risk level, inferring someone's risk attitude within 0.404% is a notable result. Additionally, the computational time for solving the 30 stock set problem was reduced by 83%. For the 30 stock instance, the computational time was decreased from 138 minutes to 23 minutes. The reduction in time may be an important for time sensitive decision making that requires the adversary's risk attitude. These results are promising when it comes to utilizing this process for inferring our adversaries risk attitude while observing their previous spending habits. This is the first step to the United States being able to infer the spending habits of its enemies. The United States will be able to allocated enough money to surpass their adversaries while not overspending in any particular area which creates a more efficient portfolio. Having this advantage will allow the United States to deter or defeat potential adversaries in a resource-efficient manner.

V. Testing, Results, and Analysis for a New Stock with Unknown Performance Data

This chapter will start by describing the toy problem and all of the accompanying variables. Following that, we will discuss significant results and show accompanying visuals for our system of equations and mesh-based search techniques.

5.1 Toy Problem Setup

In this toy problem example, there will be two stocks ($N = 2$) having a unique, optimal allotment of x_i-values equal to ${x_1}^* = 0.452$, and ${x_2}^* = 0.548$. The goal is to solve for r_2, q_{12}, q_{21}, and q_{22} given $r_1 = 0.377$, $q_{11} = 0.0097$, and $\lambda = 0.978$. To acquire the observed x_i-values, Problem P2 was solved using the values $r_2 = 0.469$, $q_{12} = q_{21} = 0.0058$, and $q_{22} = 0.0109$.

5.2 Mesh-based Search Techniques

The first technique solves for x_i-values within an allowed tolerance level (ϵ) while adjusting q_{22}, q_{12}, and r_2. The second technique holds the x_i-values to the optimal solution and iterates over q_{22} and q_{12} to find local and optimal solutions.

5.2.1 Solving for x_i-values searching over q_{22}, q_{12}, and r_2

For the initial setup we iterated over all three of the parameters with a δ-value of 0.1. We are setting $r_2^{UB} = 1$ to bound the overall computational time, and the upper bound is consistent with the maximum returns observed in the data sampled to generate problems for this research. Another method to limit the computational time of each problem, the maximum number of solutions allowed is 100. After solving P2 with the iterated parameters, the x_i-values will be compared to the observed, and if they within the specified ϵ of 0.01, they will be stored. Table 9 shows the results for the 21 solutions that came from the model. The visual depiction of this technique is shown in Figure 5.

Table 9. Solving for x_i-values with unknown parameters over a $\delta = 0.1$ and $\epsilon = 0.01$

Solution	r_2	q_{12}	q_{22}	x_1	x_2
1	0.0	-0.6	1.0	0.45424	0.54575
2	0.1	-0.6	1.0	0.45322	0.54678
3	0.2	-0.6	1.0	0.45219	0.54780
4	0.3	-0.6	1.0	0.45117	0.54882
5	0.4	-0.6	1.0	0.45015	0.54984
6	0.5	-0.6	1.0	0.44913	0.55086
7	0.6	-0.6	1.0	0.44810	0.55189
8	0.7	-0.6	1.0	0.44708	0.55291
9	0.8	-0.6	1.0	0.44606	0.55393
10	0.9	-0.6	1.0	0.44504	0.55495
11	1.0	-0.6	1.0	0.44402	0.55597
12	0.0	-0.5	0.8	0.44407	0.55592
13	0.1	-0.5	0.8	0.44282	0.55717
14	1.0	-0.5	0.9	0.46149	0.53850
15	0.4	-0.4	0.7	0.46022	0.53977
16	0.5	-0.4	0.7	0.45872	0.54127
17	0.6	-0.4	0.7	0.45722	0.54277
18	0.7	-0.4	0.7	0.45572	0.54427
19	0.8	-0.4	0.7	0.45422	0.54577
20	0.9	-0.4	0.7	0.45272	0.54727
21	1	-0.4	0.7	0.45122	0.54877

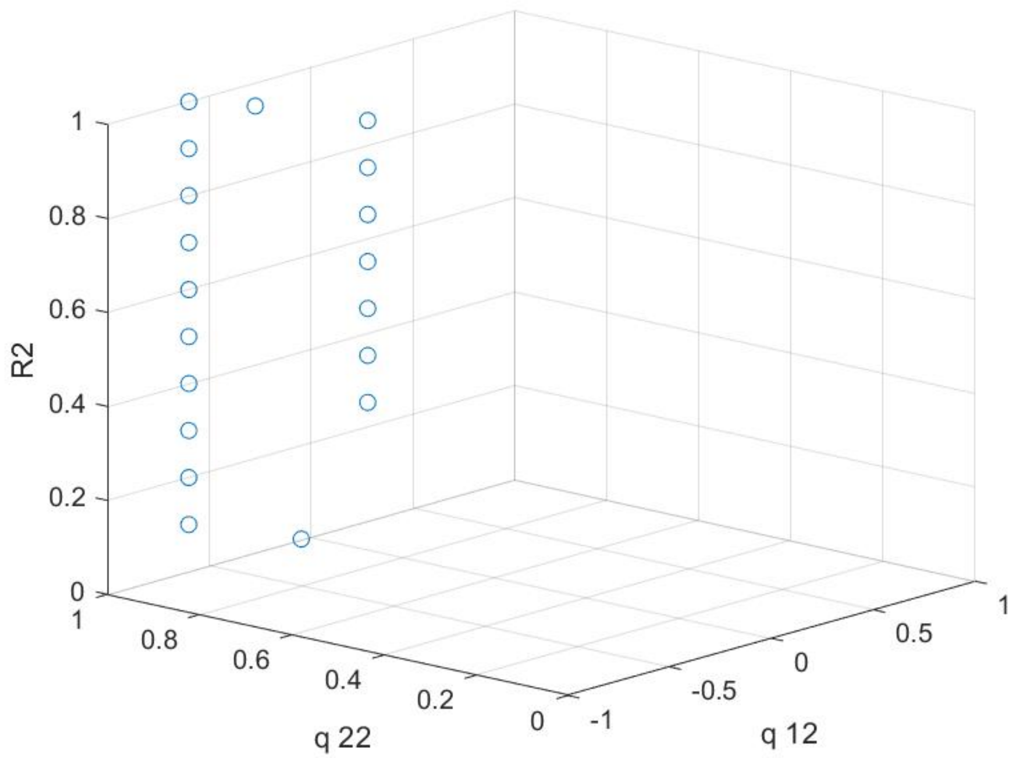

Figure 5. Solutions for Unknown x with a $\delta = 0.1$ and $\epsilon = 0.01$

As is visible in Table 8 and Figure 5, there were multiple solutions that resulted in nearly the same x_i-values as the original solution (i.e., they were within the tolerance ϵ). This shows that multiple near-optimal solutions exist, and if we want to accurately return the exact x_i-values, the δ-value needs to be reduced and a finer granularity of iterates needs to be examined. Implied from Figure 5 is that the solutions are confined to a subregion of the possible parameter values and, in fact, may be a convex set within that region. Additionally, Figure 5 indicates the solutions for this problem may be coplanar, and the granularity will need to be reduced to accurately asses the validity of this conjecture.

5.2.2 Fix x_i-values, iterate over q-values, and search for r_2

For the initial setup we iterated over q_{12} and q_{22} with a $\delta = 0.1$. We are setting $r_2^{UB} = 1$ to bound the overall computational time, and the upper bound is consistent with the maximum returns observed in the data sampled to generate problems for this research. Another method to limit the computational time of each problem, the maximum number of solutions allowed is 1000. Once a 1000 solutions are found, the model automatically stops solving for additional solutions. After solving P2 with the iterated parameters, the optimal solutions will be stored. Table 10 and Figure 6 show the results for the four solutions that came from the model.

Table 10. Solving for r_2-values with unknown parameters with $\delta = 0.1$

Solution	r_2	q_{12}	q_{22}
1	0.249379	-0.6	1
2	0.968874	-0.4	0.7
3	0.109885	-0.3	0.5
4	0.829379	-0.1	0.2

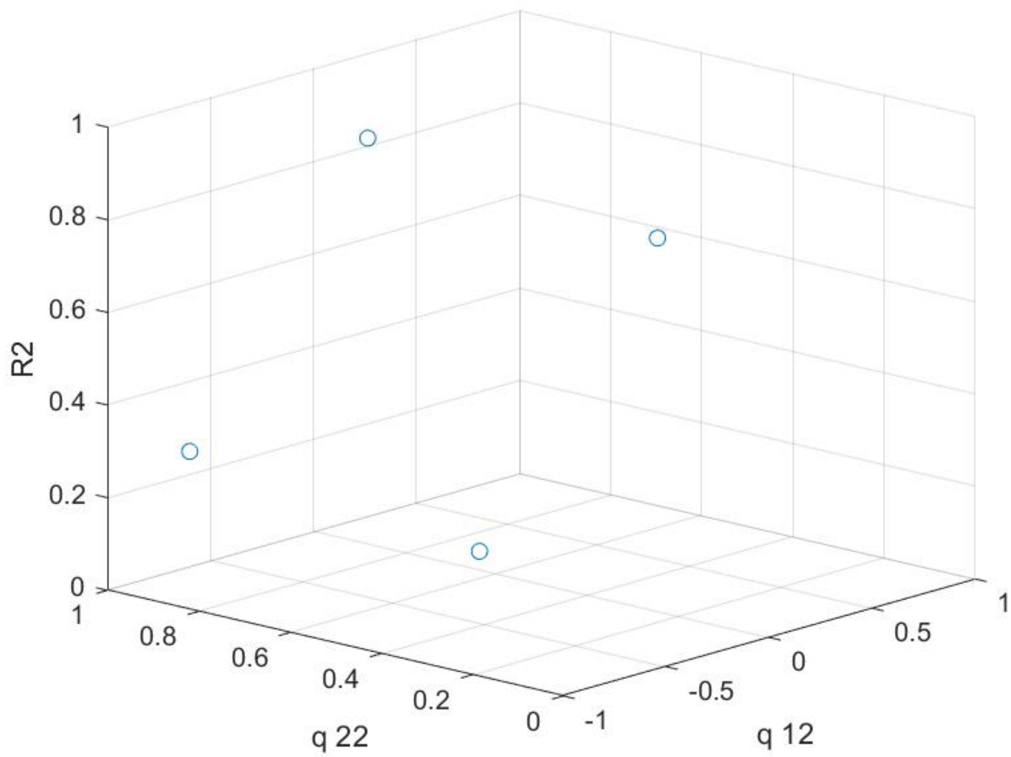

Figure 6. Solutions for Unknown r2 with $\delta = 0.1$

After running the first grid search, there were four optimal solutions for this problem. This shows that having two unknown parameters can inhibit the model from having exactly one optimal solution. Additionally, Figure 6's results are much less consequential because they encapsulate a larger region and have fewer optimal solutions. This could reduce the importance of finding optimal solutions with this approach. To test the theory that all the optimal solutions are coplanar, Figure 7 combines the optimal solutions from Figures 5 and 6. Figure 6's optimal solutions are indicated by the red points on the representation.

Within Figure 7, it is evident that there are no overlapping points but there are points that fall into the same plane.

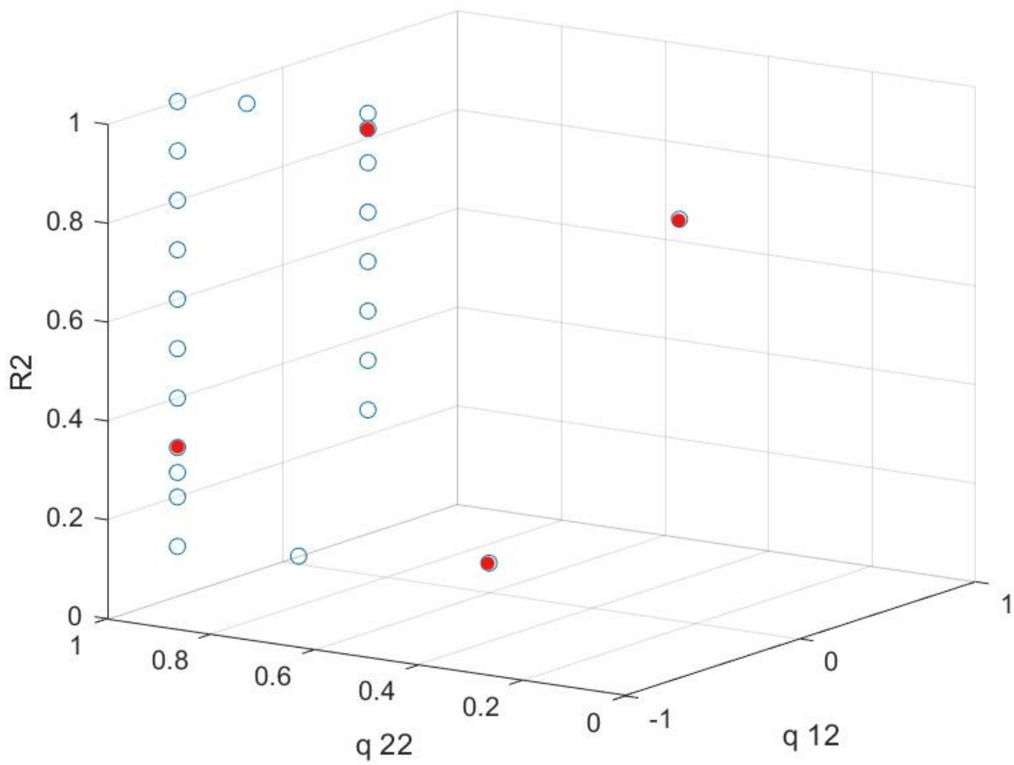

Figure 7. Combined solutions for unknown r2 and unknown x_i with $\delta = 0.1$

The two models do not yield the exact same results, but when the δ-values are reduced to 0.01, the two programs have some matching solutions. Even when the search granularity for each test is lowered to $\delta = 0.01$, the original optimal solution is not found with the exact q-values. If the maximum number of solutions were increased from 100 to ∞, then the exact solution would most likely be identified.

5.2.3 Sensitivity Analysis on δ

Additionally, we tested both of the strategies above with a granularity (δ) of 0.05 and 0.01. To limit the computational time, we limited the maximum solutions allowed to 1000. Below are the graphs of those charts:

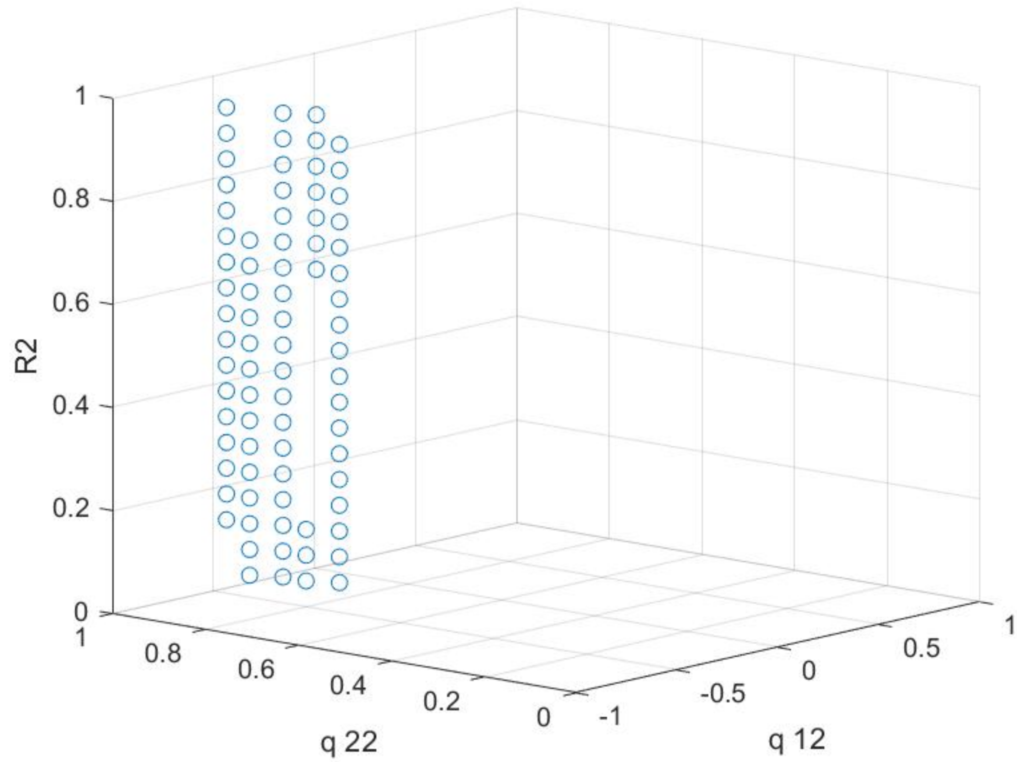

Figure 8. 82 solutions identified for unknown x_i with a $\delta = 0.05$ and $\epsilon = 0.01$

Figure 8 visualizes the solutions for solving for unknown x_i-values while searching over r_2, q_{12}, and q_{22}. The coplanar pattern that forms a rectangle shown in Figure 5 still exists when the granularity was reduced, but additional solutions were found when the granularity was decreased from $\delta = 0.10$ to $\delta = 0.05$.

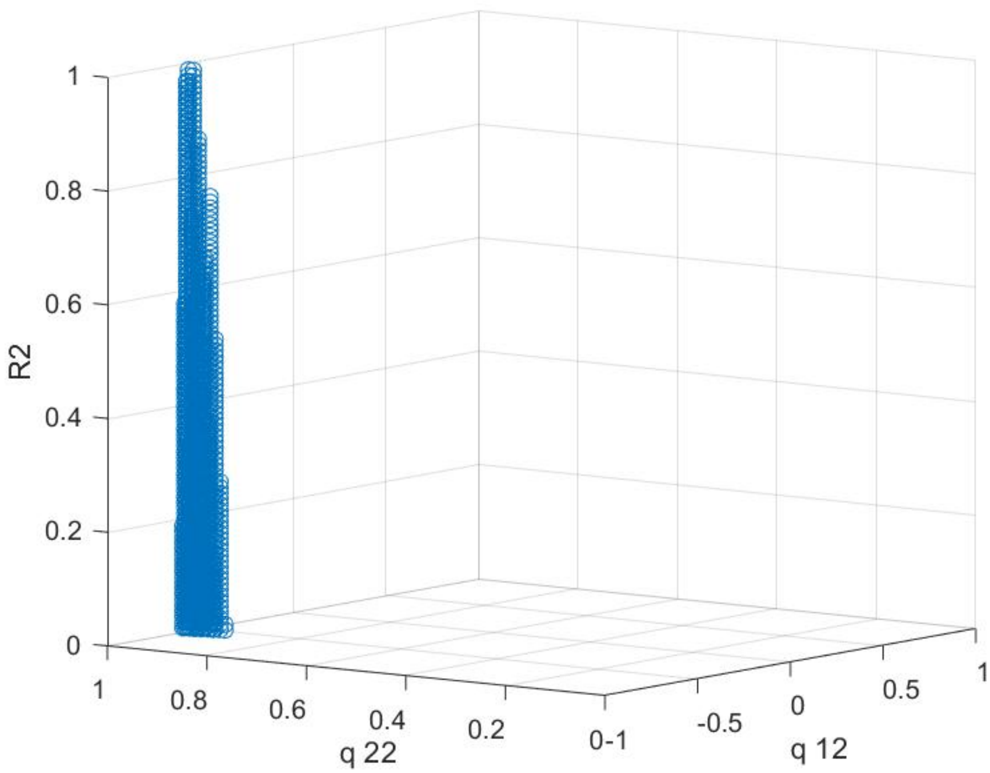

Figure 9. 1000 solutions for unknown x_i with a $\delta = 0.01$

Figure 9 illustrates differing optimal solutions from Figure 5, and if the total number of optimal solutions was increased from 1000 to a much larger number, we assume the original observed value would be included in the figure. Figure 9 does not provide great insight because it visualizes to many nearly identical solutions that would take the entire memory of the computer before providing substantial results. This information is not the most helpful at showing relevant trends for future studies. However, it can be expected that a majority of the results would exist in the same plane that was shown in the previous two figures.

After solving for the unknown r_2-values with fixed x_i-values, we used the inferred r_2, q_{12}, and q_{22}-values and resolved Problem P2 to identify if it yielded the same x_i-values. Table 10 shows the observed and inferred x_i-values resulted in differing allocations of resources, which means this process can not adequately infer all the unknown parameters.

Table 11. Solving for the unknown r_2-values with fixed x_i-values, and then using inferred r_2, q_{12}, and q_{22}-values and resolved Problem P2.

	Allocation of stock 1	Allocation of stock 2
Observed solution	0.45169	0.54831
Inferred solution 1	0.72473	0.27527
Inferred solution 2	0.72421	0.27579
Inferred solution 3	0.72362	0.27638
Inferred solution 4	0.71982	0.28018

Since the second method resulted in differing x_i-values this invalidates this solution method. Therefore the first method is the only reliable method for accurately solving the three unknown parameters. Since the first method took nearly 24 hours and outputs 1000 solutions, the first method is reliable but not necessarily practical. To be practical, the first model would need to solve iterations at a much faster rate and reduce the number of slightly vary solutions that are stored as optimal solutions. If someone has the technology or computational capacity for those underlying issues then the first method may be a viable option.

VI. Conclusions and Recommendations

Chapter VI starts by discussing the significant results from the previous two chapters. Following the results, we discuss the recommendations that can be gained from using the algorithm in this paper. Lastly, we consider future work that could extend this research, and how this is foundational for future extensions.

6.1 Conclusions

Senior leaders have consistently emphasized the rapidly increasing technological and militaristic spending from the United States' adversaries. To reduce the gap in spending and development, we attempted to infer their risk tolerance λ of an investment portfolio given their past portfolios are known. Knowing an adversary's risk tolerance is an important factor to anticipating their future investment strategy.

Calculating an adversaries' risk is the first step to identify their overall investment strategy. The investor solves to maximize the Sharpe Ratio which maximizes the ratio between expected return and volatility based off the investor's risk level. After the investor maximizes the ratio, an optimal portfolio will be generated. After having the optimal portfolio, an Adaptive Line Search technique was solved to infer the investor's risk parameter. We found that the error from the exact risk value increases as the number of stocks increase, but computation time is dependent on the stocks that are being analyzed and do not directly relate to the number. For the 10 stock instance the average error over 10 iterations was 0.0%, for 20 stocks it was 0.1%, and for 30 stocks it was 0.5%. The largest problem that was identified for that method was the computational time to solve for those results. To reduce the overall computational time of the problem, we performed a dynamic subsetting method that removed any stock option that was removed if it was not chosen in the portfolio.

After implementing the subsetting, the average error over 10 iterations was 0.0%, for 20 stocks it was 0.2%, and for 30 stocks it was 0.6%. More importantly than the slight decrease in error for the 30 stock iteration, the dynamic subsetting drastically reduced the computational time from 123 minutes to 23 minutes.

After inferring the risk parameter, we worked through a toy problem solving for the unknown expected return for a stock, co-variance between two stocks, and the variance of one stock. This was to identify if it is possible to determine the unknown parameters of any new investment introduced as an available stock. We attempted to use a system of linear equations to identify these unknown parameters, which resulted in an under determined system and provided insignificant results. Following that, we used two separate mesh-based search technique which iterated through the differing unknown parameters and found multiple optimal solutions. The first technique provides noteworthy results because it resulted in multiple optimal solutions that all existed on a single plane. The main flaw of the first technique was the time it takes to find the optimal solutions. To get accurate results the search granularity needs to be small, which results in a plethora of computations that can take far to long and take most of the computers memory. To improve the practically of this method, the computational time needs to be reduced and a method for storing every near-optimal result would need to be fixed. The second technique of the mesh based search technique provided insufficient results. After solving for the inferred optimal solutions, the original observed solution and the inferred solutions had differing portfolio allocations. At this time, the second method is not a feasible alternative for solving for all of the unknown parameters.

6.2 Recommendations

A first recommendation to extend this research is to test an individual's risk level based on their past investments. After we solve for their risk level, we could give them an investor profile questionnaire that many major financial institutions use to identify risk and compare the results. If the results are similar, the broker could identify the investor's risk level without having to figure out their risk tolerance through a series of questions. This research was conducted as a foundation to identify adversarial country's risk attitude, so we would recommend procuring their allocation of resources and finding a way to determine the risk of those assets. Being able to accurately asses the risk level of different assets is the first step to identify their overall risk level. It is also imperative to track the interested parties allocation of resources to assess how often their economic strategies pivot.

To increase the effectiveness while having an unknown r_2 and q-values, we recommend considering assumptions that can bound the space of these unknown parameters. To make either of the mesh-based search techniques practical, some restrictive constraints need to be included such as limiting the covariance between stocks to include only positive variables. If there is any known information about the new investment, then some reasonable assumptions can be made that should improve the success of these models. Another recommendation would be to reduce the number of unknown parameters to accurately solve for the x_i-values, and this could lead to some discoveries in finding optimal solutions while having unknown parameters.

6.3 Future Work

There is an ample amount of work that could build on this research. The first one discussed will be utilizing supervised learning techniques and the second is scaling the size of these problems.

One of the most significant pieces of future work would be to use supervised learning techniques to accurately predict an adversaries' inferred risk level. If valid portfolio allocations and risk tolerance levels were provided, a training and testing set could be built to predict the risk tolerance of a portfolio. These results could be compared to the Adaptive Line Search approach as a validation model if the results are similar. Using supervised learning techniques may be a useful tool to solve for risk tolerance as the number of stocks available increase, and may provide a more adaptive model as time progresses and more data points are included.

This research focused on one adversary's portfolio to simply test if inferring someone's risk tolerance was possible. Since we know that it is possible to solve for the risk tolerance level, the size of the problems should be increased to incorporate large scale business decisions. This will be critical in determining the feasibility of this process for a business or agency. increasing the size and scope will allow a group to predict how government agencies or large scale businesses will utilize their assets. This will give a strategic advantage for the organizations that can predict their adversaries' investment strategies.

Bibliography

Bazaraa, M. S., Sherali, H. D. and Shetty, C. M. (2013), *Nonlinear programming: theory and algorithms*, John Wiley & Sons, Princeton, NJ.

Biden (2021), *Interim National Security Strategic Guidance*, The White House, Washington, DC.

Cherewyk, P. (2018), 'Calculating covariance for stocks', *Investopedia, Edmonton, AB* pp. 1–5.

Delle Monache, D., De Polis, A. and Petrella, I. (2021), 'Modeling and forecasting macroeconomic downside risk', *Bank of Italy Temi di Discussione (Working Paper) No* **1324**.

Ehrgott, M. (2005), *Multicriteria optimization*, Vol. 491, Springer Science & Business Media, New York, NY.

Green, A. (2021), *The Gone Fishin' Portfolio: Get Wise, Get Wealthy... and Get on With Your Life*, John Wiley & Sons, Hoboken, NJ.

Haines, A. (2021), *Annual Threat Assessment of the US Intelligence Community*, The Office of the Director of National Intelligence, Washington, DC.

Hodnett, K., Hsieh, H.-H. et al. (2012), 'Capital market theories: Market efficiency versus investor prospects', *International Business & Economics Research Journal (IBER)* **11**(8), 849–862.

Hudson, R. (2010), 'Comparing security returns is harder than you think: Problems with logarithmic returns', *Available at SSRN 1549328* .

Jacobs, B. I. and Levy, K. N. (2013), 'Leverage aversion, efficient frontiers, and the efficient region', *The Journal of Portfolio Management* **39**(3), 54–64.

Mallpress, D. E., Fawcett, T. W., Houston, A. I. and McNamara, J. M. (2015), 'Risk attitudes in a changing environment: An evolutionary model of the fourfold pattern of risk preferences.', *Psychological Review* **122**(2), 364.

Markowitz, H. M. (1952), 'Portfolio selection', *The Journal of Finance* **7**, 77–91.

Markowitz, H. M. (1999), 'Early history of portfolio theory: 1600-1960', *Financial Analysts Journal* **55**(4), 5–16.

Missile Defense Project (2018), '"Kinzhal" Missile Threat', Center for Strategic and International Studies. Available at `https://missilethreat.csis.org/missile/kinzhal/` (14 May 2021).

Newey, W. and West, K. (1987), 'A simple positive definite, het-eroskedasticity and autocorrelation consistent covariance matrix, econo-metrica 55: 703-705', *Math-SciNet MATH* .

Pan, H. (2021), Intelligent portfolio theory and strength investing in the confluence of business and market cycles and sector and location rotations, *in* 'Handbook of Financial Econometrics, Mathematics, Statistics, and machine Learning', World Scientific, pp. 1637–1674.

Pogue, G. A. (1970), 'An extension of the markowitz portfolio selection model to include variable transactions' costs, short sales, leverage policies and taxes', *The Journal of Finance* **25**(5), 1005–1027.

Rollinger, T. N. and Hoffman, S. T. (2013), 'Sortino: a 'sharper'ratio', *Chicago, Illinois: Red Rock Capital* .

Rom, B. M. and Ferguson, K. W. (1994), 'Post-modern portfolio theory comes of age', *Journal of Investing* **3**(3), 11–17.

Sharpe, W. (1966), 'Mutual fund performance', *Journal of Business* **39**, 119–138.

Sortino, F. A., Satchell, S. and Sortino, F. (2001), *Managing downside risk in financial markets*, Butterworth-Heinemann.

Sortino, F. A. and Van Der Meer, R. (1991), 'Downside risk', *Journal of Portfolio Management* **17**(4), 1–27.

Swisher, P. and Kasten, G. W. (2005), 'Post-modern portfolio theory', *Journal of Financial Planning* **18**(9), 74.

United States Senate (2015), *Committee on Armed Services United States Senate*, Washington, DC.

Yildiz, M. E. and Erzurumlu, Y. O. (2018), 'Testing postmodern portfolio theory based on global and local single factor market model: Borsa istanbul case', *Borsa Istanbul Review* **18**(4), 259–268.

REPORT DOCUMENTATION PAGE

Form Approved OMB No. 0704–0188

1. REPORT DATE *(DD–MM–YYYY)*	2. REPORT TYPE	3. DATES COVERED *(From — To)*
24–03–2022	Master's Thesis	August 2020 — March 2022

4. TITLE AND SUBTITLE

Inverse Optimization: Inferring Unknown Instance Parameters from Observed Decisions

5a. CONTRACT NUMBER

5b. GRANT NUMBER

5c. PROGRAM ELEMENT NUMBER

6. AUTHOR(S)

Batista, Keith, 1st Lt, USAF

5d. PROJECT NUMBER

5e. TASK NUMBER

5f. WORK UNIT NUMBER

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)

Air Force Institute of Technology
Graduate School of Engineering and Management (AFIT/EN)
2950 Hobson Way
WPAFB OH 45433-7765

8. PERFORMING ORGANIZATION REPORT NUMBER

AFIT-ENS-MS-22-M-116

9. SPONSORING / MONITORING AGENCY NAME(S) AND ADDRESS(ES)

Strategic Development Planning and Experimentation (SDPE)
Mr. David Panson
Air Force Research Labratory
WPAFB, OH 45433
david.panson@us.af.mil

10. SPONSOR/MONITOR'S ACRONYM(S)

SDPE

11. SPONSOR/MONITOR'S REPORT NUMBER(S)

12. DISTRIBUTION / AVAILABILITY STATEMENT

DISTRIBUTION STATEMENT A. APPROVED FOR PUBLIC RELEASE: DISTRIBUTION UNLIMITED.

13. SUPPLEMENTARY NOTES

14. ABSTRACT

The objective of this research is to develop procedures that estimate selected, unknown parameters over an adversary's investment portfolio across a set of new or existing technologies. To solve for the selected unknown parameters, it is assumed that the adversary is maximizing the portfolio optimization problem and investing along the efficient frontier. The first technique is when an unknown risk attitude exists but all other parameters were known (i.e. expected return, variance, covariance). An adaptive line search technique that iteratively solved the portfolio optimization problem until the adversary's risk parameter was found. The second problem that was solved was when there are unknown parameters for a new investment option but all other information is known. Alternatively, two variants of a mesh-based grid search were implemented over a three-dimensional space to visualize the feasible region yielding optimal solutions. Additionally, these techniques revealed that there are multiple optimal solutions for the same portfolio allocation in a subregion, which evidence shows may be bounded within a convex region.

15. SUBJECT TERMS

Nonlinear Programming, Linear Programming, Markowitz's Portfolio Theory, Adversarial Risk Analysis, Parametric Inference

16. SECURITY CLASSIFICATION OF: a. REPORT	b. ABSTRACT	c. THIS PAGE	17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
					Dr. Brian J. Lunday, AFIT/ENS
					19b. TELEPHONE NUMBER *(include area code)*
U	U	U	UU	58	(937) 255-3636, x4444; keith.batista@afit.edu

Standard Form 298 (Rev. 8–98)
Prescribed by ANSI Std. Z39.18