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Reduced Order Modeling for Hypersonic Aeroelasticity

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SOME THEMES IN NONLINEAR MODEL REDUCTION WITH APPLICATIONS TO STRUCTURAL DYNAMICS

A thesis submitted to attain the degree of
DOCTOR OF SCIENCES of ETH ZURICH
(Dr. sc. ETH Zurich)

presented by
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Summary

Computer simulations have become an integral part of modeling and response prediction of any physical process. The ever-increasing demand for realistic simulations leads to detailed geometries, inclusion of nonlinear effects and multi-disciplinary modeling, resulting in complex models with very high dimensionality. Despite the tremendous increase in computational power over the past decades, the time required to perform dynamic simulation of such high-dimensional models remains a bottleneck towards efficient design and optimization. *Model reduction* aims to reduce the computational efforts in solving such large problems by detecting low-dimensional features. In this work, we explore a few themes in model reduction with special attention to structural dynamics applications featuring geometric nonlinearities.

Most of the established reduction techniques involve Galerkin projection of full system of governing equations onto linear subspaces for the reduction of dimensionality. Moreover, such reduced-order models (ROM) are often obtained with the help of *training* via full system simulations that are computationally expensive to begin with, if not unaffordable. In contrast, this work promotes the use of nonlinear manifolds and proposes *simulation-free* approaches for model reduction that are applicable in the present context.

The main contributions of this work include a method for obtaining ROMs of geometrically nonlinear structures via a quadratic mapping (Chapter 2). This quadratic manifold is constructed using dominant vibration modes and modal derivatives, which are considered as essential, simulation-free tools for any structural dynamicist. More importantly, the obtained ROM contains an order-of-magnitude fewer variables compared to that obtained via the conventional approach based on Galerkin projection. At the same time, a comparable accuracy is obtained on realistic numerical models, including one of an aircraft wing.

Exploring further the notions of nonlinear manifolds and simulation-free reduction, the recent techniques of slow-fast decomposition [50] and spectral submanifolds [49] are applied to the finite-element model of a von Kármán beam (Chapter 3). In contrast to most of the existing methods, these techniques allow for nonlinear model reduction in a mathematically justifiable manner and result in a single-degree-of-freedom yet *exact* ROM for the beam.

Further contributions in this work pertain to the common notion of *hyper-reduction* that is essential to achieve the necessary computational speed in large finite-element models once the prerequisite dimensionality-reduction is ensured. One issue is that the existing hyper-reduction methods have been proposed in the context of the conventional linear-mapping-based reduction only. To allow for hyper-reduction using the quadratic manifold, we propose a nonlinear extension to the popular energy conserving sampling and weighing (ECSW) method [41] that leads to significant performance gains (Chapter 4).

On another front, hyper-reduction methods usually operate by a fast approximation of the expensive nonlinear terms in the ROM, which is made possible, once again, through full-simulation training. Carrying forward the notion of simulation-free reduction, we also propose a heuristic method that avoids the need for full simulations in the training stage (Chapter 5). In our context, this method works by lifting the amplitudes of bending-dominated modes obtained the quadratic manifold. This results in physically-meaningful training vectors that capture the geometrically nonlinear behavior and we obtain a much higher computational speed-up, overall.

As a first step towards multi-disciplinary modeling, we include the effects of temperature in the nonlinear structural dynamics equations. The dynamic variation in temperature parameters, however, renders the conventional parametric model reduction techniques either inapplicable or extremely inefficient. Thanks to the slowness of thermal dynamics in comparison to the structural dynamics, we propose reduction via the

method of multiple scales, where the reduction basis efficiently adapts to the instantaneous temperature configuration of the structure (Chapter 6).

Throughout this work, we apply the proposed methods on a range of examples varying in complexity from those of academic interest to those of industrial relevance.

Sommario

Le simulazioni computerizzate sono diventate parte integrante nella modellazione e nella previsione del comportamento di molti processi fisici. La richiesta sempre crescente per simulazioni sempre più realistiche comporta la discretizzazione accurata di dettagli geometrici, l'inclusione di effetti nonlineari e la modellazione multidisciplinare. Tutto ciò comporta modelli complessi e dimensionalmente onerosi. Nonostante il significativo aumento della potenza di calcolo negli ultimi decenni, il tempo richiesto per portare a termine una simulazione dinamica di tali modelli rimane un ostacolo per attività di dimensionamento e ottimizzazione. A tal fine, i modelli ridotti puntano a ridurre l'onere computazionale relativo alla soluzione del modello di grandi dimensioni attraverso l'identificazione di caratteristiche salienti della risposta di quest'ultimo. Tali caratteristiche possono essere descritte in uno spazio di dimensioni notevolmente inferiori rispetto a quello del modello originario.

La maggior parte delle tecniche di riduzione si basa sulla proiezione *à la Galerkin* delle equazioni del sistema originario su un sottospazio di dimensioni ridotte. Inoltre, tali modelli ridotti sono spesso ricavati attraverso *training* ottenuti da simulazioni del modello originario, che, come sopra descritto, risultano estremamente onerose o addirittura intrattabili numericamente. Per superare questa limitazione, questo contributo propone l'utilizzo di sottospazi nonlineari (curvi) combinati con approcci *simulation-free* per la generazione del *training*.

I contributi principali di questo lavoro includono un metodo per ottenere un modello ridotto di strutture caratterizzate da nonlinearietà geometrica attraverso una varietà differenziabile quadratica tra coordinate modali e gradi di libertà nodali (Capitolo 2). Tale varietà è costruita utilizzando i modi di vibrare dominanti e le rispettive derivate modali. Tali oggetti sono considerati strumenti essenziali per la sintesi della risposta della dinamica strutturale, e sono ricavati senza ricorrere alla simulazione del modello a grandi dimensioni (ovvero, sono ottenuti mediante un approccio *simulation-free*).

Elaborando ulteriormente le nozioni di varietà differenziabile e riduzione *simulation-free*, applichiamo tecniche recenti di riduzione, denominate *slow-fast decomposition* [50] e *spectral submanifolds* [49] ad un modello ad elementi finiti di una trave elastica modellata con cinematica nonlineare di von Kármán (Capitolo 3). In contrasto con la maggior parte dei metodi esistenti, queste tecniche permettono la riduzione nonlineare in modo matematicamente giustificato, e portano ad un modello ridotto ad un singolo grado di libertà, ma al contempo esatto, per la trave in questione.

Ulteriori contributi di questo lavoro vertono sulla nozione di *hyper-reduction*, tecnica essenziale per ottenere significativi incrementi della velocità di soluzione di grossi modelli ad elementi finiti, una volta che il prerequisito della riduzione di dimensionalità è stato soddisfatto. Un ostacolo è costituito dal fatto che gli esistenti metodi di riduzione sono stati proposti solamente in contesti di mapping lineare tra coordinate ridotte e fisiche. Per permettere *hyper-reduction* con varietà differenziabili quadratiche, proponiamo qui una estensione nonlineare al noto algoritmo di *Energy Conserving Sampling and Weighting (ECSW)* [41]. Tale estensione permette significativi miglioramenti nella velocità di ottenimento della soluzione del modello ridotto (Capitolo 4).

Su un altro fronte, i metodi di *hyper-reduction* operano di solito costruendo una approssimazione computazionalmente poco costosa dei termini nonlineari, attraverso *training* ricavati da simulazioni del modello originario. Perseguendo la filosofia della riduzione *simulation-free*, proponiamo un metodo che elimina la necessità di simulare il modello completo per la generazione di *training* (Capitolo 5). Nel nostro caso, questo metodo comporta la mappatura delle ampiezze dei modi dominanti (tipicamente flessionali) ottenute tramite analisi lineare modale, sulla varietà differenziale quadratica. Questa procedura fornisce *training* fisicamente

plausibili, che riproducono il comportamento geometricamente nonlineare e allo stesso tempo, abbattano significativamente i tempi di calcolo.

Come primo passo verso la modellazione multi-disciplinare, introduciamo gli effetti di campi termici spazio- e tempo-variabili nelle equazioni nonlineari della dinamica strutturale. Purtroppo, la variazione dinamica del campo termico rende le tecniche convenzionali di riduzione non applicabili, o quantomeno estremamente inefficienti. Grazie alla dicotomia tra le scale temporali relative alla dinamica termica ed elastica, proponiamo una riduzione attraverso il metodo multi-scala, in cui la base di riduzione si adatta in modo efficiente alla configurazione termica corrente (Capitolo 6).

In questo lavoro, applichiamo i metodi illustrati ad una serie di esempi di complessità crescente, da puramente accademici a industrialmente rilevanti.

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I came to ETH for my doctoral studies with the idea that life as a PhD student would be idyllic. After all, not many jobs pay you well for reading and learning what interests you and pursuing research in the company of great minds. I must say, life has been a bit more hectic than that. But more importantly, it has also been fun. And this fun can be attributed to the many people I got to interact with over the past three years.

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¹Research can always lead to unexpected outcomes.

²Authors are not cute.

Contents

1	Introduction	13
1.1	General motivation	13
1.2	Model reduction: background, profits and pitfalls	14
1.3	Thesis contributions and outline	16
2	Quadratic manifolds	19
2.1	Introduction	19
2.2	Reduced dynamics	20
2.3	Linear subspace reduction	21
2.3.1	Vibration modes	22
2.3.2	Modal derivatives	23
2.3.3	Calculation of modal derivatives	23
2.3.4	The use of modal derivatives in a reduction basis	25
2.3.5	Optimal basis selection	25
2.4	Quadratic manifold	26
2.5	Applications and results	31
2.5.1	Flat structure	31
2.5.2	NACA airfoil wing structure	36
2.6	Conclusions	41
	Appendix	43
2.A	Comparison of quadratic manifold with the static condensation approach	43
3	Slow-fast decomposition & spectral submanifolds	45
3.1	Introduction	45
3.2	The discretized beam model	47
3.3	Slow-Fast Decomposition	50
3.3.1	Verification of assumptions for the application of SFD	50
3.3.2	Global reduced-order model from SFD	51
3.3.3	Specific results from SFD	52

3.4	SSM reduction	54
3.4.1	Separation in damping: spectral quotient analysis	54
3.4.2	SSM-based reduction	56
3.4.3	Results	58
3.5	Conclusions	61
	Appendix	63
3.A	$\mathcal{O}(\epsilon^2)$ terms in the slow manifold	63
3.B	Formulas for single-mode SSMs	64
4	Hyper-reduction over nonlinear manifolds	67
4.1	Introduction	67
4.2	Model reduction using nonlinear mappings	70
4.3	Hyper-reduction and Extended ECSW	71
4.3.1	Element sampling and weight selection	72
4.3.2	Preservation of Lagrangian structure and numerical stability	74
4.4	Numerical results	78
4.4.1	Setup	78
4.4.2	Flat Structure	80
4.4.3	Wing Structure	81
4.5	Conclusion	83
5	Hyper-reduction: reducing offline costs	85
5.1	Introduction	85
5.2	Setup	87
5.3	Classical hyper-reduction	88
5.3.1	Dimensionality reduction using POD	88
5.3.2	Hyper-reduction with ECSW	88
5.4	Simulation-free hyper-reduction	90
5.4.1	Modal-based reduction basis	90
5.4.2	QM-based training set generation	91
5.5	Numerical results	95
5.5.1	Setup	95
5.5.2	Shallow-arch structure	97
5.5.3	Wing structure	100
5.6	Discussion on alternative simulation-free ROM constructions	101
5.7	Conclusions	103
6	Model reduction using adaptive bases	105

<i>CONTENTS</i>	11
6.1 Introduction	105
6.2 Multiple time scale dynamics	109
6.3 Adaptive basis for model reduction	111
6.3.1 Linear systems: reduction using vibration modes	111
6.3.2 Geometrically nonlinear systems	114
6.3.3 Basis interpolation and mode veering	115
6.4 Numerical examples	115
6.4.1 Straight beam	119
6.4.2 Curved beam	120
6.5 Discussion	126
6.6 Conclusion	127
7 Conclusions & Outlook	129

Chapter 1

Introduction

1.1 General motivation

Modeling and analysis have a critical impact on the design and performance of various mechanical structures such as aircraft, wind turbines, space structures, offshore structures etc. In the past, such analyses have been performed with the help of analytical and numerical models that were often simplified in terms of the underlying assumptions and geometry (e.g., point-mass; simple geometrical shapes; discrete and massless elastic elements, dash-pots). These simplifications usually resulted in linear models, where the coupling effects due to the external environment (e.g., thermal, acoustic, electric, magnetic) were neglected to make the analysis feasible [6, 36, 126]. The need to make realistic predictions, however, involves accounting for nonlinear effects and multidisciplinary modeling.

Nonlinear analysis has become increasingly prevalent in modern engineering practices. Some systems are designed to operate in the linear range, but the linearity assumption needs to be verified against nonlinear analysis, e.g., to avoid buckling, plastic deformations. A preliminary design based on linear analysis is sufficient in such cases. In some other cases, the linear range of operation perhaps needs to be broadened, which requires the nonlinearities to be accurately modeled and further minimized by design optimization [3, 44]. In yet other cases, nonlinearity is exploited to maximize the performance [75, 76, 98, 110, 118, 125]. All such cases necessitate the inclusion of nonlinearity even in the preliminary stages of design.

The acoustic and thermal environment that a structural system is subjected to can also introduce important coupling effects. Consider, for instance, the problem of aeroelastic flutter [36] which is caused by the aerodynamic loading due to the air flowing around an aircraft wing. Temperature fluctuations can also lead to thermally-induced dynamic buckling of structures. In fact, many examples highlight the significance of thermo-mechanical coupling, including a hypersonic vehicle where structural deformations due to heating induce a shock wave [86, 130]. The prediction of the structural response to such effects is of prime interest for designers and engineers and, naturally, involves multi-disciplinary modeling.

Typically, such physical processes are mathematically modeled using partial differential equations (PDEs), which are discretized (using, e.g., finite element, finite difference, finite volume methods) to obtain problems with a finite, but often very large, number of unknowns or degrees of freedom (DOF). The mesh-refinement needed to account for extremely detailed geometric features and material distribution leads to an increase in the dimensionality of such models. The inclusion of nonlinearities and multidisciplinary effects only adds to the complexity of these models. Modern scientific computation capabilities have paved the way to model and simulate such complex systems. However,

despite the boost in computational power over the past decades, the computational demands imposed by high-dimensional discretized models impede the efficient and optimal design of structures. As a consequence, routine simulations to explore different load scenarios or parameter changes such as geometric layouts and material properties may carry prohibitive computational costs [1]. To this end, reduced-order models (ROMs) offer a reprieve from such extreme computational costs and allow effective design and optimization activities.

1.2 Model reduction: background, profits and pitfalls

Essentially, model reduction deals with identifying low-dimensional features of the response in a high-dimensional dynamical system, whose numerical simulation would otherwise be either extremely computationally-intensive or entirely unaffordable. In such cases, model reduction leads to a small number of variables that describe the system dynamics evolving over low-dimensional subspaces or manifolds, thereby facilitating feasible computation times. Before moving towards specific contributions in this thesis, we attempt to critically examine the notion of model reduction in the following.

Are all systems reducible? Model reduction relies on the inherent assumption that complex systems exhibit simple behavior that can be described (reasonably approximated) by a few variables. For general nonlinear systems, however, it is a priori unclear why such an assumption should be valid. Nevertheless, literature is replete with examples where high-dimensional systems have been successfully reduced to lower-dimensional models [13, 18, 19, 26, 37, 41, 42, 77, 84, 103, 106]. This suggests that dynamical systems of practical interest are often characterized by low-dimensional, attracting manifolds that somehow enable model reduction.

Classical approaches to model reduction involve the Galerkin projection of the full system of equations, often referred to as the high-fidelity model (HFM), onto some linear subspaces of interest (cf. Benner *et al.* [18], Besseling *et al.* [19], and Rutzmoser [106]). These approaches tacitly assume invariance of the relevant subspaces by projecting away the velocity components orthogonal to these subspaces. Indeed, linear modal subspaces are a prime example of such invariant subspaces in the context linear time-invariant dynamical systems; techniques based in linear modal analysis (see, e.g., G rardin & Rixen [45]) form the *de facto* standard for model reduction of linear structural dynamics. However, it is important to keep in mind that general nonlinear dynamical systems may not exhibit such invariant subspaces. Any attempt to reduce such systems by projecting onto linear subspaces would be justifiable, at best, only *locally*, i.e., near an equilibrium. Nonetheless, structural dynamics applications are often characterized by the so-called weak or mild nonlinearities, where such locally-valid projections seem to be meaningful.

Whether a general nonlinear system is *reducible* or not depends to a large extent on the requirements imposed by the end-user of the ROM, as we shall discuss shortly. At this point, however, it is worth to mention a characterization of reducible systems due to Haller & Ponsioen [50] that allows for a model reduction in a mathematically justifiable manner. They discuss that a reducible system must exhibit an attracting low-dimensional invariant manifold with the additional requirement that the full system trajectories are attracted towards the manifold at rates that are faster than the typical rates within the manifold.

In general, before attempting to reduce a system, it is essential to have clarity about the desired characteristics of the ROM. A given system could possibly be reduced in more than one way, each leading to the preservation of a certain property of the full system. And since there is no single method befitting all applications, the choice of a reduction technique would depend on what features of the full system are critical for the practitioner. A few examples are as follows.

Prediction of transient behavior. The transient response or the time series prediction of the structure is often sought via ROMs. Consider, for instance, an initially-straight beam buckled under uniform thermal load [96]. Under the application of forcing, the beam dynamically snaps between

two stable configurations and performs oscillations about these configurations. In this case, the transient oscillations of the beam before it settles down to one of the stable configurations are of significant interest for the designer and the accurate prediction of the transient response amplitudes would be an important requirement from the ROM. Indeed such transient snaps imply large stresses, which impact the fatigue life of the components.

Data-driven ROMs obtained via *training* using full-simulation trajectories such as the proper orthogonal decomposition method (POD) [65, 68] have been found to be effective for transient response reduction [41, 42, 79, 129]. Other projection-based techniques involving the use of vibration modes and modal derivatives that avoid full simulation runs have also found application in this context [57]. The relevance of such methods is further enhanced upon equipment with *hyper-reduction* techniques that deal with the acceleration of projection-based ROMs by a fast approximation of the nonlinearity, cf. Ryckelynck [108], the discrete empirical interpolation method (DEIM) of Chaturantabut & Sorensen [28], the energy conserving sampling and weighing (ECSW) method of Farhat *et al.* [41].

Prediction of the steady-state response. Similar to the static nonlinear analysis that provides insight into the behavior of a mechanical system towards quasistatic loading, the nonlinear *forced-response* curves or *backbone* curves are paramount to the practitioner in understanding the dynamic behavior and resonance structure of mechanical systems in response to periodic or quasi-periodic forcing. Despite the broad availability of computer packages or techniques (cf. COCO [31], MATCONT [33], AUTO [35], the shooting method [64], the harmonic balance method [32, 69, 71], SteadyStateTool [58], see also Renson *et al.* [101] for a review), the computation of nonlinear steady-state response remains a challenge for high-dimensional systems and model reduction rightfully finds application in this context.

The approximation of nonlinearities routinely performed in projection-based (hyper-) reduction techniques, however, does not guarantee the preservation of the steady-state response. For instance, the POD-DEIM [28] hyper-reduction is indeed demonstrated to be numerically unstable for geometrically nonlinear mechanical systems [42], leading to the blow-up of reduced solution corresponding to an otherwise bounded HFM trajectory. As such, it would fail to capture the steady-state response. Hence, reduction techniques which preserve the steady-state response of nonlinear systems require a more careful analysis with special attention to the persistence of the steady-state response under an approximation of the nonlinearity. To this end, recently developed *exact* reduction methods of slow-fast decomposition [50] and spectral submanifolds [49] allow for accurate approximation of the steady-state response under precisely defined mathematical conditions [21, 67, 94, 95].

Prediction of response for different parameters. ROMs are especially useful in carrying out optimization activities, which require the repeated evaluation of the objective function across a range of optimization parameters. This objective function may be dependent on the dynamic response of the model which makes its evaluation extremely expensive for each parameter value. To this end, there have been significant efforts to interpolate the reduced-order information available for a set of parameters to obtain ROMs for parameter values outside the given set [9, 38, 77]. In particular, Amsallem & Farhat [9] addressed the singularity issues resulting from the naive interpolation of reduction bases used in projection-based ROMs by performing interpolation over an appropriate Stiefel manifold to preserve orthogonality.

When using ROMs for parametric model reduction, it seems natural to expect that the ROM would reproduce the bifurcation structure with respect to the parameters. Indeed, a bifurcation represents a fundamental change in the nature of the response of the system and the knowledge of the bifurcation structure allows for the designs to be robust under perturbation of parameters (cf., e.g., Guckenheimer & Holmes [47]). However, preservation of the bifurcation structure under reduction is a rather subtle issue and by no means trivial to achieve. To the best of our knowledge, the currently available reduction techniques in the literature (including the ones introduced in this work) do not guarantee this.

Preserving stability and Lagrangian structure. Structural systems are usually characterized by an underlying Hamilton’s principle that also leads to conservation of energy in conservative systems and numerical stability (along with the choice of an energy-conserving numerical time integration scheme). The Hamilton’s principle can be used to derive the Lagrange equations that govern the system response. Special attention must be paid to ensure that the (hyper-) reduced model preserves this basic Lagrangian structure and inherits the stability properties of the response. Indeed, as mentioned before, the popular DEIM method that was not designed with specific attention towards finite-element-based structural systems does not preserve these desirable properties [42]. Among other techniques, the ECSW method [41] is specifically apt for hyper-reduction in this context.

1.3 Thesis contributions and outline

In this work, we explore and propose model reduction techniques with applications to geometrically nonlinear structural dynamics. In particular, this work encompasses the following themes in model reduction.

In Chapter 2, we discuss the use of vibration modes and the corresponding modal derivatives [54, 55] as essential tools for reducing the transient response of geometrically nonlinear structural dynamics problems, while avoiding the prohibitive HFM simulations that are often necessary for other training-based reduction approaches such as the POD method. In particular, we propose their use via a quadratic mapping [63] that minimizes the number of reduced variables for a given set of vibration modes and modal derivatives. At the same time, this approach provides accuracy comparable to the conventional Galerkin approaches that project the system of equations onto linear subspaces. With the help of realistic numerical models, we also demonstrate the general idea that nonlinear manifolds may be far more efficient for model-reduction purposes in comparison to their linear subspace counterparts. Haller & Ponsioen [50] further derived explicit conditions under which such a quadratic mapping would result in a mathematically-justifiable ROM in a slow-fast setting (cf. Proposition 1, Examples 7 and 8 in [50]). Following their work of slow-fast decomposition (SFD) [50] and reduction using spectral submanifolds (SSM) [49], we derive an exact ROM for a nonlinear von Kármán beam in Chapter 3. This constitutes the first application of these methods to a realistic model of moderate dimensionality. We show that the non-dimensionalized beam satisfies the requirements for the application of these techniques. SFD allows us to obtain a globally valid ROM, where the axial displacements are enslaved to the transverse displacements of the beam. Subsequently, we further reduce this global ROM using an SSM to obtain a single degree-of-freedom ROM that is valid locally around the equilibrium.

It is well-known that the mere identification of low-dimensional manifolds or subspaces to obtain ROMs is not tantamount to an increase in the computational speed in high-dimensional finite-element models [28, 41, 108]. To accelerate computation in such ROMs, there are mainly three approaches—namely, the intrusive/direct computation of reduced tensors for the case of polynomial nonlinearities in the HFM, the use of hyper-reduction techniques (cf. ref. [57] for a comparison between the two), and the non-intrusive/indirect computation of reduced tensors based on various heuristic techniques (cf. Mignolet *et al.* [84] for a review). In this thesis, we focus on hyper-reduction methods due to their general applicability (cf. Section 5.6 for a discussion). Acknowledging the relevance of the ECSW method [41] for hyper-reduction of finite-element models, Chapters 4 and 5 propose an extension to the method and address certain computational bottlenecks in our context.

Firstly, ECSW was essentially conceived for hyper-reduction in the context of linear Galerkin projection-based ROMs. In Chapter 4, we propose an extension to ECSW that allows for hyper-reduction over nonlinear manifolds. In particular, we demonstrate our extension on the ROMs obtained using the quadratic manifold from Chapter 2 and accelerate the computation of the corresponding examples. This leads to a remarkable online speed-up of over one thousand times relative

to the full system on a realistic aircraft wing model using our proposed extension.

Secondly, ECSW functions by training over a finite-element mesh to obtain a reduced mesh, leading to faster computations. The training stage itself, however, usually carries massive computational costs. In Chapter 5, we propose a heuristic method to obtain such training sets at negligible computational costs for geometrically nonlinear systems. We use the quadratic mapping from Chapter 2 in lifting the snapshots from an inexpensive linearized modal analysis of the system to obtain training vectors that automatically capture the geometrically nonlinear effects, which are physically relevant, such as a bending-stretching coupling. Apart from qualitative discussions to compare the proposed method with commonly used techniques in literature, we use an effective computational speed measure that accounts for offline costs involved in setting up a ROM and suggests an order of magnitude better performance of the proposed method in terms of computational speed.

Chapter 6 pertains to the reduction of the geometrically nonlinear structural dynamics equations under the influence of thermal effects, where the temperature variables of the structure are modeled as time-dependent parameters. Due to the presence of dynamically varying parameters, conventional techniques for parametric reduced-order modeling are not directly helpful in this case. However, the time-scale separation in the evolution of the thermal dynamics in comparison to the structural dynamics prompts us to use the method of multiple scales. Subsequently, we propose a reduction approach where the basis adapts to the instantaneous temperature configuration of the structure. We demonstrate the effectiveness of this method on arches, where we use temperature-dependent vibration modes and modal derivatives for reduction. Final sections of this chapter discuss the steps involved in our ongoing research that enable the extension of this work in the context of hyper-reduction.

Chapter 2

Quadratic manifolds

Abstract. We propose the use of a quadratic manifold for the model reduction of structural dynamics problems featuring geometric nonlinearities. This manifold is tangent to a subspace spanned by the most relevant vibration modes, and its curvature is provided by modal derivatives obtained by sensitivity analysis of the eigenvalue problem, or its static approximation, along with the vibration modes. The construction of the quadratic manifold requires minimal computational effort once the vibration modes are known. The reduced-order model is then obtained by projecting the discretized equations of motion onto the configuration-dependent tangent space of the manifold. This chapter closely follows the current author’s work in the original reference¹[63].

2.1 Introduction

Geometric nonlinearities, among other nonlinear effects, are a prime characteristic of thin-walled structural components that are typically employed when high stiffness-to-weight and strength-to-weight ratios must be achieved. The redirection of stresses due to non-infinitesimal deflections causes peculiar behaviors as bending and torsion-stretching coupling, buckling, snap-through, and mode jumping [102] in such structures. In this context, nonlinear ROMs are paramount to enable sound design and optimization activities.

An established method to obtain accurate ROMs by Galerkin projection is the Proper Orthogonal Decomposition [11, 51], where the reduction basis is constructed using the solution snapshots of the HFM. Albeit optimal in a sense, it bears the drawback of requiring the full solution (cf. [57] for a discussion). Nonetheless, it is meaningfully applied in the so-called *many-queries* scenarios, for which the cost of the full training simulations is justified. In a preliminary design context, however, the resources required for such an approach might not be available. In this case, it is desirable to build a ROM *not* with the reliance on full simulations, but rather using certain intrinsic characteristics of the underlying physical system, which are usually available at a very small fraction of the computational cost associated to such full simulation(s).

Modal truncation and superposition is standard practice for linear structural dynamics, as it enables the decoupling of the linear governing equation to readily assess the dynamic response. However, a reduction based solely on Vibration Modes (VMs) would perform poorly in the presence of geometric nonlinearities, as they typically do not capture the relevant bending/torsion-stretching coupling. This would require the inclusion of in-plane displacement dominated fields in the basis. An appealing enrichment to a reduction basis of few VMs is constituted by the Modal Derivatives

¹Jain, S., Tiso, P., Rutzmoser, J. B. & Rixen, D. J. A quadratic manifold for model order reduction of nonlinear structural dynamics. *Computers & Structures* **188**, 80–94. doi:[10.1016/J.COMPSTRUC.2017.04.005](https://doi.org/10.1016/J.COMPSTRUC.2017.04.005) (2017).

(MDs), which were originally proposed by Idelsohn and Cardona [54, 55]. These are computed by differentiating the eigenvalue problem associated with small, undamped vibrations with respect to the modal amplitudes. A static version of their construction (i.e. neglecting the inertial terms) enjoys computational advantages: the MDs thus obtained, are the solutions of a set of linear systems where the coefficient matrix is factorized only once and the right-hand sides are symmetric functions of the VMs.

In a reduced basis approach, the MDs could be appended to a reduction basis constituted by the dominant VMs. This approach leads to very accurate results [16, 113]. Unfortunately, the number of MDs that can be generated, grows quadratically with the size of the VMs basis used to generate them, thereby severely hampering the efficiency of the method. However, the MDs are in fact the curvature of a quadratic manifold arising from the Taylor expansion of the physical displacement into the direction of the dominant vibration modes. As such, the modal amplitudes associated with the MDs are enslaved, in a quadratic fashion, to those of the VMs, and hence do not require independent, reduced unknowns for their description. This approach can often be supported by a sound theoretical justification in examples which are characterized by a special dichotomy in time scales and corresponds to neglecting the inertial forces associated to the fast dynamics of the system at hand [107]. More specifically, the static MDs provide a second order approximation to the underlying critical manifold in such examples [50]. This leads to the notion of the solution lying on a *quadratic* manifold, parameterized by the amplitudes of the dominant VMs. This idea already appeared in a static context when evaluating the initial post-buckling response of thin-walled structures using a perturbation approach [83, 119], and, with a very similar framework, in the computation of the backbone curves for nonlinear harmonic responses [127].

The use and efficacy of such a Quadratic Manifold to construct a ROM for dynamic applications remains unexplored and is the focus of this work. In this work, we propose a unified approach to construct a ROM using a quadratic manifold comprised of VMs and MDs. The classical notion of the Galerkin projection is extended here to projection on a tangent, configuration-dependent space, which is variationally consistent with the nonlinear mapping between modal and full DOFs. Further, we test this approach on a simple, illustrative example as well as a realistic, industrial structure and compare it with established reduction techniques.

This chapter is organized as follows. The generalization of the Galerkin projection onto a non-linear manifold is sketched in Section 2.2. The construction of an MD-based linear manifold is discussed in Section 2.3. The quadratic manifold is then introduced in Section 2.4. Numerical results are presented and discussed in Section 2.5, and finally, the conclusions are given in Section 2.6.

2.2 Reduced dynamics

The dynamical response of a structure to externally applied loading is obtained by solving an Initial Value Problem (IVP). This IVP is characterized by a system of second-order Ordinary Differential Equations (ODEs) usually resulting from the FE discretization of the governing Partial Differential Equations (PDEs), and can be written in the following form:

$$\begin{aligned} \mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{C}\dot{\mathbf{u}}(t) + \mathbf{f}(\mathbf{u}(t)) &= \mathbf{g}(t) \\ \mathbf{u}(t_0) &= \mathbf{u}_0 \\ \dot{\mathbf{u}}(t_0) &= \mathbf{v}_0, \end{aligned} \tag{2.1}$$

where the solution $\mathbf{u}(t) \in \mathbb{R}^n$ is a high dimensional generalized displacement vector with the initial conditions $\mathbf{u}_0$ for displacements and $\mathbf{v}_0$ for velocities given as inputs at initial time t_0 , $\mathbf{M} \in \mathbb{R}^{n \times n}$ is the mass matrix, $\mathbf{C} \in \mathbb{R}^{n \times n}$ is the damping matrix, $\mathbf{f}(\mathbf{u}) : \mathbb{R}^n \mapsto \mathbb{R}^n$ is the nonlinear internal force and $\mathbf{g}(t) \in \mathbb{R}^n$ is the time-dependent external load vector. These ODEs are further discretized in time using a suitable time integration scheme, resulting in a high-dimensional, fully discrete,

nonlinear system of algebraic equations, to be iteratively solved at each time step with a Newton method, for instance. This full solution bears a prohibitive computational cost even for a single-query scenario, not to mention the case when the time integration needs to be performed several times, e.g., to explore different operational scenarios.

Fortunately, in structural dynamics applications, a relatively small number of "modal" coordinates are expected to govern the system response. This is to say that, in general, the solution may be assumed to evolve on a low dimensional manifold in $\mathbb{R}^n$. In other words, we seek a mapping $\mathbf{\Gamma} : \mathbb{R}^n \mapsto \mathbb{R}^m$ with $m \ll n$ such that

$$\mathbf{u}(t) \approx \mathbf{\Gamma}(\mathbf{q}(t)), \quad (2.2)$$

where $\mathbf{\Gamma}$ is a general nonlinear mapping and $\mathbf{q} \in \mathbb{R}^m$ is the reduced vector of unknowns. The semi-discrete equations for dynamic equilibrium in (2.1) can be written in the following variational or weak form (time dependency is omitted for clarity purposes):

$$[\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{f}(\mathbf{u})] \cdot \delta\mathbf{u} = \mathbf{g} \cdot \delta\mathbf{u}, \quad (2.3)$$

where $\delta\mathbf{u}$ is an admissible variation in the solution $\mathbf{u}$. By introducing the lower dimensional approximation (2.2) into (2.3), we obtain

$$[\mathbf{M}\ddot{\mathbf{\Gamma}}(\mathbf{q}) + \mathbf{C}\dot{\mathbf{\Gamma}}(\mathbf{q}) + \mathbf{f}(\mathbf{\Gamma}(\mathbf{q}))] \cdot \delta\mathbf{\Gamma}(\mathbf{q}) = \mathbf{g} \cdot \delta\mathbf{\Gamma}(\mathbf{q}). \quad (2.4)$$

Observing that the variation $\delta\mathbf{\Gamma}(\mathbf{q})$ is given by $\frac{\partial\mathbf{\Gamma}(\mathbf{q})}{\partial\mathbf{q}}\delta\mathbf{q}$, and $\delta\mathbf{q}$ being arbitrary, we finally obtain

$$\mathbf{P}_{\mathbf{\Gamma}}^T [\mathbf{M}\ddot{\mathbf{\Gamma}}(\mathbf{q}) + \mathbf{C}\dot{\mathbf{\Gamma}}(\mathbf{q}) + \mathbf{f}(\mathbf{\Gamma}(\mathbf{q}))] = \mathbf{P}_{\mathbf{\Gamma}}^T \mathbf{g}, \quad (2.5)$$

where $\mathbf{P}_{\mathbf{\Gamma}}$ denotes the tangent subspace $\frac{\partial\mathbf{\Gamma}(\mathbf{q})}{\partial\mathbf{q}}$.

If the mapping function is chosen to be linear such that $\mathbf{\Gamma}(\mathbf{q}) := \mathbf{V}\mathbf{q}$ (where $\mathbf{V} \in \mathbb{R}^{n \times m}$ is typically a basis spanning some lower dimensional subspace in $\mathbb{R}^n$ in which the solutions is assumed to live), the above treatment leads to the *Bubnov-Galerkin* or simply the *Galerkin Projection*. The reduced ODEs can then be simplified as

$$\underbrace{\mathbf{V}^T \mathbf{M} \mathbf{V}}_{\tilde{\mathbf{M}}} \ddot{\mathbf{q}}(t) + \underbrace{\mathbf{V}^T \mathbf{C} \mathbf{V}}_{\tilde{\mathbf{C}}} \dot{\mathbf{q}}(t) + \underbrace{\mathbf{V}^T \mathbf{f}(\mathbf{V}\mathbf{q}(t))}_{\tilde{\mathbf{f}}(\mathbf{q}(t))} = \mathbf{V}^T \mathbf{g}(t), \quad (2.6)$$

where $\tilde{\mathbf{M}}, \tilde{\mathbf{C}} \in \mathbb{R}^{m \times m}$ are the reduced mass and damping matrices, respectively. For a linear system, one would have $\mathbf{f}(\mathbf{u}) = \mathbf{K}\mathbf{u}$ ($\mathbf{K} \in \mathbb{R}^{n \times n}$ being the linear stiffness matrix), and a reduced stiffness matrix $\tilde{\mathbf{K}} = \mathbf{V}^T \mathbf{K} \mathbf{V} \in \mathbb{R}^{m \times m}$ is also obtained.

The choice of projection basis $\mathbf{V}$ (or the mapping $\mathbf{\Gamma}(\mathbf{q})$) is critical in determining the accuracy of the reduced solution. The size of the basis (or the reduced number of unknowns) is important in determining the speed-up in computation time. In further sections, we consider the candidates for such linear and nonlinear mappings.

2.3 Linear subspace reduction

The existence of an invariant subspace (or manifold) is a key requirement for reduction of system (2.1), as described above. Upon reduction over an invariant *linear* subspace, we refer to the reduced solution to lie on a *Linear Manifold*. Finding a suitable invariant subspace is by no means trivial, if at all possible. In the linear mappings context, the Proper Orthogonal Decomposition (POD) is a remarkably versatile and robust method. However, one of its drawbacks is the need for

training snapshots of solution vectors which are obtained from a full nonlinear run. Typically, a reduction basis constructed in such a manner is suitable only for the trained set(s) of loading(s). One natural question that arises then, is that if it is possible to obtain a Reduced Order Model (ROM) without the need of a full nonlinear run.

In structural dynamics, linear modal analysis is a powerful and insightful tool for preliminary analysis of any system on a *linear* level. The use of a truncated set of Vibration Modes (VMs) certainly goes a long way towards the reduction of a linear system without the need for a full solution. The concept of reduction using a linear basis of vibration modes can be extended to nonlinear systems by the use of *Modal Derivatives*. We briefly review these concepts in the following sections.

2.3.1 Vibration modes

The system (2.1) can be linearized around its static equilibrium position. Without the loss of generality, we assume the equilibrium configuration is $\mathbf{u}_{eq} = \mathbf{0}$ to obtain

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}|_{eq}\mathbf{u} = \mathbf{g}(t), \quad (2.7)$$

where

$$\mathbf{K}|_{eq} = \left. \frac{\partial \mathbf{f}(\mathbf{u})}{\partial \mathbf{u}} \right|_{\mathbf{u}=\mathbf{0}} \quad (2.8)$$

This linearized system would be a good approximation to the original system (2.1) for small enough displacements from the linearization point. It is well known that for such a linear system, the system response can be written as a linear combination of constant eigenvectors (also referred to as the eigenmodes or VMs) in the structural dynamics context) which form a basis of $\mathbb{R}^n$ as follows.

$$\mathbf{u}(t) = \sum_{i=1}^n \phi_i \eta_i(t) \quad (2.9)$$

where the eigenmodes $\phi_i \in \mathbb{R}^n$ are found by the solution of the generalized eigenvalue problem ²

$$(\mathbf{K}|_{eq} - \omega_i^2 \mathbf{M})\phi_i = \mathbf{0} \quad (2.10)$$

(ω_i^2 is the eigenvalue or the eigenfrequency squared). This concept of expressing the solution $\mathbf{u}(t)$ in terms of a basis of eigenvectors is referred to as the principle of *linear modal superposition*. However, if one is considering the slowly varying dynamics of the system, then it can be shown that the response can be very accurately approximated by a few low frequency modes and a modal truncation can be obtained [45].

$$\mathbf{u}(t) \approx \sum_{i=1}^m \phi_i \eta_i(t) = \Phi \boldsymbol{\eta}(t), \quad (2.11)$$

where $\Phi \in \mathbb{R}^{n \times m}$, $\boldsymbol{\eta}(t) = [\eta_1(t) \ \eta_2(t) \ \cdots \ \eta_m(t)]^T \in \mathbb{R}^m$, $m \ll n$. Thus in doing so, we introduce a mapping $\mathbf{y} : \mathbb{R}^n \mapsto \mathbb{R}^m$ such that $\mathbf{u} = \mathbf{u}(\boldsymbol{\eta}) = \Phi \boldsymbol{\eta}$. Since $m \ll n$, this reduces the number of unknowns in the system and an effective ROM is obtained for linear systems. Note that this is equivalent to performing a Galerkin projection as shown in eq. (2.6) for a linearized system, where Φ is the reduction basis and $\boldsymbol{\eta}$ are the corresponding reduced unknowns.

²Here we neglect the damping contribution in eigenvalue problem to avoid complex eigenvalues and vectors. Note that for damped linear systems with low damping or *Rayleigh* damping as explained in [45], the eigenvectors for an undamped system are a good approximation for the damped counterpart and still form a good basis for linear modal superposition. Such a damping is very popular in structural dynamics and the theory is illustrated in this context.

2.3.2 Modal derivatives

When the deviation from the linearization point increases, the response of system (2.7) can no longer be considered as a good approximation for the original nonlinear counterpart. One might still think of using the VMs obtained from the linearized model to form a reduction basis for the reduction of the nonlinear set of equations. A basis composed of a few dominant VMs, however, is typically not sufficient for reduction since it does not feature the dominant coupling effects (e.g. membrane-bending), typical of geometrically nonlinear structures.

Earlier work in [54, 55] and recent works [128, 131] discuss the use of the so called Modal Derivatives (MDs) to capture the response of the nonlinear system upon departure from the linear behavior. After replacing $\mathbf{K}_{eq}$ in (2.10) with the tangential stiffness matrix $\mathbf{K}$ and differentiating (2.10) with respect to the modal amplitude η_j (assuming $\mathbf{M}$ to be a constant mass matrix), the resulting equation evaluated at equilibrium yields

$$(\mathbf{K}|_{eq} - \omega_i^2|_{eq}\mathbf{M}) \left. \frac{\partial \phi_i}{\partial \eta_j} \right|_{eq} + \left(\left. \frac{\partial \mathbf{K}}{\partial \eta_j} \right|_{eq} - \left. \frac{\partial \omega_i^2}{\partial \eta_j} \right|_{eq} \mathbf{M} \right) \phi_i|_{eq} = \mathbf{0}, \quad (2.12)$$

where the MD $\left. \frac{\partial \phi_i}{\partial \eta_j} \right|_{eq}$ denotes the derivative of the i^{th} mode in the j^{th} modal direction. Here the tangent stiffness matrix derivative w.r.t. η_j is obtained by giving the system a displacement in the direction of $\phi_j|_{eq}$ i.e.,

$$\left. \frac{\partial \mathbf{K}}{\partial \eta_j} \right|_{eq} = \left. \frac{\partial \mathbf{K}(\mathbf{u} = \eta_j \phi_j|_{eq})}{\partial \eta_j} \right|_{\eta_j=0} \quad (2.13)$$

Physically, an MD represents the sensitivity of VM ϕ_i corresponding to a displacement given in the direction of VM ϕ_j . As will be shown, these MDs can be used as efficient tools to model the departure from the linear behavior in a nonlinear system.

Remark 2.1. Conceptually, VMs are defined only about an equilibrium point and modal amplitudes are not parameters which change the equilibrium points and hence the VMs. Here, the spectral expansion of tangent operators leads to this intuitive notion of MDs, which is not linked to the free vibration of a nonlinear system.

2.3.3 Calculation of modal derivatives

It is easy to see that $\left. \frac{\partial \phi_i}{\partial \eta_j} \right|_{eq}$ cannot be trivially obtained from (2.12) since the coefficient matrix is singular by definition (cf. (2.10)). This singularity can be dealt with by imposing a normalization condition for the eigenmodes. Reference [112] covers an extensive account of different solution techniques and introduces a generalised approach to find eigenvector derivatives for different kinds of normalizations. The popular mass normalization has been adopted here i.e.

$$\phi_i^T \mathbf{M} \phi_i = 1 \quad \forall i \in \{1, 2, \dots, m\}. \quad (2.14)$$

Differentiating the equation above w.r.t. the modal amplitude results in

$$\phi_i^T \mathbf{M} \frac{\partial \phi_i}{\partial \eta_j} + \phi_i^T \mathbf{M}^T \frac{\partial \phi_i}{\partial \eta_j} = 0 \quad \forall i, j \in \{1, 2, \dots, m\}. \quad (2.15)$$

Exploiting the symmetry of $\mathbf{M}$ and subsequent evaluation at the equilibrium position results in the following relation

$$\phi_i^T|_{eq} \mathbf{M} \left. \frac{\partial \phi_i}{\partial \eta_j} \right|_{eq} = 0 \quad \forall i, j \in \{1, 2, \dots, m\}. \quad (2.16)$$

The following *direct* approach to calculate the MDs can then be formulated using (2.12), (2.16):

$$\begin{bmatrix} [\mathbf{K}|_{eq} - \omega_i^2|_{eq}\mathbf{M}]_{n \times n} & -[\mathbf{M}\phi_i|_{eq}]_{n \times 1} \\ -[\mathbf{M}\phi_i|_{eq}]_{1 \times n}^T & 0_{1 \times 1} \end{bmatrix} \begin{bmatrix} \frac{\partial \phi_i}{\partial \eta_j}|_{eq} \\ \frac{\partial \omega_i^2}{\partial \eta_j}|_{eq} \end{bmatrix} = \begin{bmatrix} -\frac{\partial \mathbf{K}}{\partial \eta_j}|_{eq} \phi_i|_{eq} \\ 0 \end{bmatrix}. \quad (2.17)$$

The above non-singular system can be used to solve for the MDs $\frac{\partial \phi_i}{\partial \eta_j}|_{eq}$. This approach, however, is not very attractive since it destroys the band structure of the original system. Nonetheless, it is rigorous and accurate, and has been used here for the calculation of MDs. Apart from this direct approach, the pseudo inverse technique (cf. [45]) and the Nelson's method [88] are some techniques that preserve the band structure of the matrices.

Regardless of the method adopted to solve eq. (2.12), a high dimensional matrix needs to be factorized for each ω_i . When considered together for a large number of modes, the computational costs of this factorization could be significantly high. Idelsohn and Cardona [55] discuss a way to approximate the problem (2.12) by neglecting the inertial contribution, i.e. by eliminating the terms containing $\mathbf{M}$ in it to obtain

$$\mathbf{K}|_{eq} \frac{\partial \phi_i}{\partial \eta_j}|_{eq}^s = -\frac{\partial \mathbf{K}}{\partial \eta_j}|_{eq} \phi_i|_{eq}. \quad (2.18)$$

We call the MDs calculated in such manner as *Static* MDs (SMDs). It is then easy to see that the computation of SMDs is much easier than solving (6.33) for every mode, since it involves the factorization of $\mathbf{K}|_{eq}$ only once. The use of SMDs can be shown to be analogous to Static Condensation (cf. [84] for a review) in special cases. This is discussed in more details further in the chapter. The superscript s in $\frac{\partial \phi_i}{\partial \eta_j}|_{eq}^s$ stands for static.

Theorem 2.1 (Symmetry of SMDs). The Static Modal Derivatives given in (2.18) are symmetric, i.e. $\frac{\partial \phi_i}{\partial \eta_j}|_{eq}^s = \frac{\partial \phi_j}{\partial \eta_i}|_{eq}^s$.

Proof. It is easy to see that the stiffness matrix derivative given by (2.13) can be written as

$$\frac{\partial \mathbf{K}}{\partial \eta_j}|_{eq} = \frac{\partial \mathbf{K}(\mathbf{u})}{\partial \mathbf{u}}|_{\mathbf{u}=\mathbf{0}} \cdot \phi_j|_{eq} = \frac{\partial^2 \mathbf{f}(\mathbf{u})}{\partial \mathbf{u} \partial \mathbf{u}}|_{\mathbf{u}=\mathbf{0}} \cdot \phi_j|_{eq}. \quad (2.19)$$

Substituting this into (2.18), we obtain

$$\mathbf{K}|_{eq} \frac{\partial \phi_i}{\partial \eta_j}|_{eq}^s = -\left(\frac{\partial^2 \mathbf{f}(\mathbf{u})}{\partial \mathbf{u} \partial \mathbf{u}}|_{\mathbf{u}=\mathbf{0}} \cdot \phi_j|_{eq} \right) \cdot \phi_i|_{eq}. \quad (2.20)$$

But the third order tensor $\left(\frac{\partial^2 \mathbf{f}(\mathbf{u})}{\partial \mathbf{u} \partial \mathbf{u}} \right)$ which contains the second order partial derivatives, is symmetric by Shwarz' theorem ($\mathbf{f}(\mathbf{u}) \in C^2(\mathbb{R}^n, \mathbb{R}^n)$) i.e. $\left(\frac{\partial^2 \mathbf{f}(\mathbf{u})}{\partial \mathbf{u} \partial \mathbf{u}} \right)_{Iij} = \left(\frac{\partial^2 \mathbf{f}(\mathbf{u})}{\partial \mathbf{u} \partial \mathbf{u}} \right)_{Iji}$. Thus, we get

$$\frac{\partial \phi_i}{\partial \eta_j}|_{eq}^s = -(\mathbf{K}|_{eq})^{-1} \left[\left(\frac{\partial^2 \mathbf{f}(\mathbf{u})}{\partial \mathbf{u} \partial \mathbf{u}}|_{\mathbf{u}=\mathbf{0}} \cdot \phi_j|_{eq} \right) \cdot \phi_i|_{eq} \right] \quad (2.21)$$

$$= -(\mathbf{K}|_{eq})^{-1} \left[\left(\frac{\partial^2 \mathbf{f}(\mathbf{u})}{\partial \mathbf{u} \partial \mathbf{u}}|_{\mathbf{u}=\mathbf{0}} \cdot \phi_i|_{eq} \right) \cdot \phi_j|_{eq} \right] = \frac{\partial \phi_j}{\partial \eta_i}|_{eq}^s. \quad (2.22)$$

□

Remark 2.2 (Symmetry of MDs). While the SMDs have been shown to be symmetric, such a claim cannot be made for the modal derivatives as given by (6.33). These MDs are infact, *not* symmetric in general.

2.3.4 The use of modal derivatives in a reduction basis

A linear basis (Ψ) consisting of VMs augmented with these MDs could be used to effectively reduce the nonlinear system [54, 55].

$$\Psi = [\phi_1|_{eq} \ \phi_2|_{eq} \ \dots \ \phi_m|_{eq} \ \dots \ \theta_{ij}|_{eq} \ \dots], \quad (2.23)$$

where $\theta_{ij} = \left. \frac{\partial \phi_i}{\partial \eta_j} \right|_{eq}$ or $\left. \frac{\partial \phi_i}{\partial \eta_j} \right|_{eq}^s$, are the MDs or the SMDs. Using MDs, one could expect a maximum basis size of m^2 (if all MDs are linearly independent). Since SMDs are symmetric (cf. Remark 2.2), a basis $\Psi \in \mathbb{R}^{n \times M}$ can be obtained using (S)MDs, where $M = m + \frac{m(m+1)}{2}$ would be its maximum size. Indeed, in both cases inclusion of MDs in the basis is expected to increase the reduced number of unknowns quadratically with the number of VMs (m) in the basis.

Remark 2.3 (On normalization). From the chosen normalization (2.16), it is easy to see that any VM ϕ_i is $\mathbf{M}$ -orthogonal to all corresponding MDs $\left. \frac{\partial \phi_i}{\partial \eta_j} \right|_{eq} \forall j \in \{1, \dots, m\}$. However, this is not sufficient to ensure that Ψ in (2.23) possesses a full column rank, since all vectors in Ψ may not be mutually orthogonal (or linearly independent). Consequently, the condition number of Ψ might suffer. Rutzmoser et al. [105] show that the corresponding condition number decreases with an increasing number of modes and (S)MDs. Hence, an orthogonalization or a deflation of the linear basis Ψ is necessary to avoid bad conditioning or even singularities in the reduced model.

2.3.5 Optimal basis selection

As seen above, the MDs capture the essential second order non-linearities of the system. But if all the MDs corresponding to a given set of VMs are used in order to augment the basis, then the size of the basis increases with $\mathcal{O}(m^2)$. This is undesirable and in practise only a few MDs could be selected to capture the nonlinear response of the system. We propose different heuristic methods to *a priori* select (S)MDs which are expected to produce the highest contribution depending on the applied loading function $\mathbf{g}(t)$ during time integration.

2.3.5.1 Maximum modal interaction (MMI) method

Similar to the selection strategy proposed in [120], the basic idea of this method is to calculate the modal interaction between different modes during a linear run and use the SMDs corresponding to the maximum interaction in augmenting the basis. A weighting matrix $\mathbf{W}$ can be built to rank the SMDs in order of relevance

$$W_{ij} = \int_0^T |\eta_i(t)\eta_j(t)| \, dt, \quad (2.24)$$

where W_{ij} represents the weight of the SMD θ_{ij} and $\eta_i(t)$ represents the time varying amplitude of the i^{th} mode obtained in response to the applied external loading, in a linear modal superposition run over time span $[0, T]$. Since this procedure only involves a linear modal superposition run, this weighting matrix is obviously extremely cheap to obtain. By looking at the product of two modal amplitudes, one obtains the interaction between the corresponding modes in the sense that if the relevant weightage becomes high, then the corresponding nonlinearity may get triggered and that SMD becomes important. Furthermore, as the weighting matrix $\mathbf{W}$ is symmetric, this technique would be suitable for ranking only SMDs (and not MDs), since they are also symmetric. Using it rank to MDs would imply equality between $\left. \frac{\partial \phi_j}{\partial \eta_i} \right|_{eq}$ and $\left. \frac{\partial \phi_i}{\partial \eta_j} \right|_{eq}$, which is general does not hold, as stated in (2.4).

2.3.5.2 Modal virtual work (MVW) method

Here, the basic idea of assigning weights is to compute the virtual work done by the nonlinear elastic forces arising from one mode upon another mode. First, the maximum modal amplitudes for each of the m VMs in response to the applied external forcing are computed with a linear modal superposition run. They are multiplied with the corresponding modes to obtain m modal displacement vectors scaled with the maximum amplitude. Then, the nonlinear internal force corresponding to these displacement fields is computed, and projected onto each mode to obtain the resultant virtual-work. Finally, the magnitude of this work is collected as the MD-weight in the matrix $\mathbf{W} \in \mathbb{R}^{m \times m}$. This can be mathematically written as follows:

$$t_i^{max} = \arg \max_{t \in T} |\eta_i(t)|, \quad (2.25)$$

$$W_{ij} = |\phi_j^T \mathbf{f}(\eta_i(t_i^{max}) \phi_i)| \quad (\text{no summation}). \quad (2.26)$$

Physically, W_{ij} would also represent the interaction between modes ϕ_i and ϕ_j , thereby establishing importance of the corresponding MD. It is easy to see that $\mathbf{W}$ would not be symmetric in general (it would though be always diagonal when the internal force $\mathbf{f}(\mathbf{u})$ is linear). Due to this asymmetry, this makes the MVW a suitable technique for ranking MDs obtained from eq. (6.33), which are not symmetric in general.

It should be noted that the (S)MDs are only "ranked" using the weighing matrices described above. While the contribution of a low ranked MD is intuitively expected to be less than that of a high ranked one during a reduced nonlinear run, the weights cannot be taken as a quantitative measure of their relative contribution since they are obtained from a linear analysis. Furthermore, different normalizations of the VMs can lead to potentially different weights. In all the presented examples, the VMs are mass-normalized according to eq. (2.14).

2.4 Quadratic manifold

As explained before, the linear modal superposition using a few VMs is a good technique to obtain the reduced solution of a linear system. However, when the nonlinearities become significant, the modal basis can be augmented with MDs to effectively capture the response. This quickly increases the size of the basis as more VMs are used and defeats the purpose of reduction. Though an MD selection alleviates this problem to some extent, this selection is not robust for different loads. If we assume that the linearized modal subspace which was good for capturing small displacement smoothly persists for displacements in the nonlinear range but this subspace deforms into an analytic manifold around the linearization point, then, interestingly, it can be shown that the MDs capture the second order components of this analytic modal manifold. To this effect, we aim to introduce a nonlinear (quadratic) mapping for our purpose of reduction. Such a mapping can be written as

$$\mathbf{u} \approx \Gamma(\mathbf{q}) := \Phi \cdot \mathbf{q} + \frac{1}{2} (\Omega \cdot \mathbf{q}) \cdot \mathbf{q}, \quad (2.27)$$

where $\mathbf{q} \in \mathbb{R}^m$, $m \ll n$ are the reduced unknowns, $\Phi \in \mathbb{R}^{n \times m}$, and $\Omega \in \mathbb{R}^{n \times m \times m}$ is a third order tensor. The mapping (2.27) can be written using the Einstein summation convention in the indicial notation as

$$\Gamma_I = \Phi_{Ii} q_i + \frac{1}{2} \Omega_{Iij} q_i q_j \quad I \in \{1, \dots, n\}, \quad i, j \in \{1, \dots, m\}. \quad (2.28)$$

Remark 2.4 (on symmetry of Ω). It should be noted that the mapping Γ is independent of the anti-symmetric part of Ω . Indeed, Ω can be split into its symmetric and antisymmetric parts as follows.

$$\Omega = \Theta + \Lambda,$$

where

$$\Theta_{IJK} := \frac{1}{2}(\Omega_{IJK} + \Omega_{IKJ}), \quad (2.29)$$

$$\Lambda_{IJK} := \frac{1}{2}(\Omega_{IJK} - \Omega_{IKJ}), \quad I \in \{1, \dots, n\}, \quad J, K \in \{1, \dots, m\}. \quad (2.30)$$

Thus, Θ and Λ are the symmetric and anti-symmetric parts of Ω , respectively. It is then easy to see in the indicial notation that

$$\begin{aligned} \Gamma_I &= \Phi_{Ii}q_j + \frac{1}{2}\Omega_{Iij}q_iq_j = \Phi_{Ii}q_j + \frac{1}{2}\Theta_{Iij}q_iq_j + \frac{1}{2}\Lambda_{Iij}q_iq_j \\ &= \Phi_{Ii}q_j + \frac{1}{2}\Theta_{Iij}q_iq_j + \frac{1}{4}(\Omega_{Iij}q_iq_j - \Omega_{Iji}q_iq_j) = \Phi_{Ii}q_j + \frac{1}{2}\Theta_{Iij}q_iq_j + \frac{1}{4}(\Omega_{Iij}q_iq_j - \Omega_{Iij}q_jq_i) \\ &= \Phi_{Ii}q_j + \frac{1}{2}\Theta_{Iij}q_iq_j, \end{aligned}$$

is independent of Λ .

Taking the mapping in (2.27) to be a nonlinear extension of linear modal superposition, the first order derivative evaluated at equilibrium would correspond to the set of VMs at equilibrium such that

$$\left. \frac{\partial \Gamma}{\partial q_j} \right|_{\mathbf{q}=\mathbf{0}} = \phi_j|_{eq}. \quad (2.31)$$

Thus Φ would be a matrix containing m VMs (essentially the same VMs which are good for capturing the linearized system response). Intuitively this means that the tangent space of the manifold Γ at the equilibrium point is the subspace spanned by the m VMs. Rutzmoser et al. [105] show that this can be generalized to any linear subspace, and not just the subspace spanned by VMs at equilibrium.

The second order derivatives evaluated at equilibrium would then be

$$\left. \frac{\partial^2 \Gamma}{\partial \mathbf{q} \partial \mathbf{q}} \right|_{\mathbf{q}=\mathbf{0}} = \Theta, \quad (2.32)$$

where Θ is the symmetric part of Ω as defined above. This second order derivative gives information about how the tangent space of the manifold Γ changes on departure from the equilibrium.

We would now like to use the (S)MDs (as given by (2.12),(2.18)) in choosing the components for Ω such that

$$\Omega_{:ij} = \left. \frac{\partial \phi_i}{\partial q_j} \right|_{eq}^s \quad \text{or} \quad \left. \frac{\partial \phi_i}{\partial q_j} \right|_{eq}, \quad (2.33)$$

where $\Omega_{:ij} = \{\mathbf{v} \in \mathbb{R}^n : v_I = \Omega_{Iij}\}$. We attempt to justify this choice for the quadratic component in the following manner.

If Ω is chosen to consist of the SMDs, then due to Remark 2.1, we would obtain that $\left. \frac{\partial^2 \Gamma}{\partial q_i \partial q_j} \right|_{\mathbf{q}=\mathbf{0}} = \Theta_{:ij} = \left. \frac{\partial \phi_i}{\partial q_j} \right|_{eq}^s$ ³. The resulting tangent space of the manifold would then be given by

$$\left. \frac{\partial \Gamma}{\partial q_i} \right|_{eq} = \phi_i|_{eq} + \sum_j \left. \frac{\partial \phi_i}{\partial q_j} \right|_{eq}^s q_j. \quad (2.34)$$

³This is contrary to the suggestions in [54] where the author motivates the modal derivatives using a Taylor expansion which leads to an inconsistent definition of the second order components of (2.27) by a factor of $\frac{1}{2}$. This of course would not pose a problem if the MDs are used as independent components in a linear basis (which was the focus of that work). But when MDs are quadratically enslaved to the VM amplitudes (as is done here) then this factor plays a crucial role in the mapping.

Since the MDs intuitively represent how VMs change upon departure from equilibrium, (2.34) shows that the tangent space is being corrected using SMDs upon departure from equilibrium. The choice $\Omega_{:ij} = \frac{\partial \phi_i}{\partial q_j} \Big|_{eq}^s$ is thus intuitively justified.

A similar argument would hold for the choice $\Omega_{:ij} = \frac{\partial \phi_i}{\partial q_j} \Big|_{eq}$, except that one would then obtain $\frac{\partial^2 \Gamma}{\partial q_i \partial q_j} \Big|_{\mathbf{q}=\mathbf{0}} = \frac{1}{2} \left(\frac{\partial \phi_i}{\partial \eta_j} \Big|_{eq} + \frac{\partial \phi_j}{\partial \eta_i} \Big|_{eq} \right)$, since the MDs are not symmetric in general (cf. Remark 2.2). This is not an issue since due to Remark 2.4, the mapping is only dependent on the symmetric part of Ω .

Thus, with the help of such a quadratic mapping (using (S)MDs), one can extend the classical notion of linear modal truncation to nonlinear systems while remarkably preserving the reduced number of unknowns.

For the purpose of further discussion, we treat $\Gamma(\mathbf{q})$ as a general smooth nonlinear mapping, as given in (2.27). The velocity and acceleration are then expressed as functions of the modal coordinates $\mathbf{q}$ as:

$$\dot{\Gamma} = \mathbf{P}_\Gamma \cdot \dot{\mathbf{q}} = \Phi \cdot \dot{\mathbf{q}} + (\Theta \cdot \dot{\mathbf{q}}) \cdot \mathbf{q}, \quad (2.35)$$

$$\ddot{\Gamma} = \mathbf{P}_\Gamma \cdot \ddot{\mathbf{q}} + \left[\frac{\partial \mathbf{P}_\Gamma}{\partial \mathbf{q}} \cdot \dot{\mathbf{q}} \right] \cdot \dot{\mathbf{q}} = \Phi \cdot \ddot{\mathbf{q}} + (\Theta \cdot \ddot{\mathbf{q}}) \cdot \mathbf{q} + (\Theta \cdot \dot{\mathbf{q}}) \cdot \dot{\mathbf{q}}, \quad (2.36)$$

where

$$\underbrace{\mathbf{P}_\Gamma}_{\in \mathbb{R}^{n \times m}} = \frac{\partial \Gamma(\mathbf{q})}{\partial \mathbf{q}} = \Phi + \Theta \cdot \mathbf{q}. \quad (2.37)$$

The equations above write in index notation:

$$(P_\Gamma)_{IJ} = \frac{\partial \Gamma_I}{\partial q_J} = \Phi_{IJ} + \frac{1}{2} \Omega_{IJj} q_j + \frac{1}{2} \Omega_{IjJ} q_j = \Phi_{IJ} + \Theta_{IJj} q_j, \quad I \in \{1, \dots, n\}, \quad J, j \in \{1, \dots, m\}, \quad (2.38)$$

$$\dot{\Gamma}_I = (P_\Gamma)_{Ii} \dot{q}_i = \Phi_{Ii} \dot{q}_i + \Theta_{Iij} \dot{q}_i q_j, \quad I \in \{1, \dots, n\}, \quad i, j \in \{1, \dots, m\}, \quad (2.39)$$

$$\ddot{\Gamma}_I = (P_\Gamma)_{Ii} \ddot{q}_i + \frac{\partial (P_\Gamma)_{Ii}}{\partial q_j} \dot{q}_i \dot{q}_j = \Phi_{Ii} \ddot{q}_i + \Theta_{Iij} \ddot{q}_i q_j + \Theta_{Iij} \dot{q}_i \dot{q}_j, \quad I \in \{1, \dots, n\}, \quad i, j \in \{1, \dots, m\}. \quad (2.40)$$

The expressions calculated above can be substituted into (2.5) to obtain the reduced order model in m unknowns i.e. $\mathbf{q} \in \mathbb{R}^m$ as

$$\widetilde{\mathbf{M}} \ddot{\mathbf{q}} + \widetilde{\mathbf{p}} + \widetilde{\mathbf{C}} \dot{\mathbf{q}} + \widetilde{\mathbf{f}} = \widetilde{\mathbf{g}}, \quad (2.41)$$

where $\widetilde{\mathbf{M}} = \mathbf{P}_\Gamma^T \mathbf{M} \mathbf{P}_\Gamma$ and $\widetilde{\mathbf{C}} = \mathbf{P}_\Gamma^T \mathbf{C} \mathbf{P}_\Gamma$ are the configuration-dependent reduced mass and damping matrix respectively; $\widetilde{\mathbf{f}} = \mathbf{P}_\Gamma^T \mathbf{f}(\Gamma(\mathbf{q}))$ is the reduced internal force vector; $\widetilde{\mathbf{g}} = \mathbf{P}_\Gamma^T \mathbf{g}(t)$ is the configuration-dependent, reduced external load vector; and $\widetilde{\mathbf{p}} = \mathbf{P}_\Gamma^T (\mathbf{M}(\Theta \cdot \dot{\mathbf{q}}) \cdot \dot{\mathbf{q}})$ is a convective term, quadratic in generalized velocities, which is similar to the convective terms that appear in the multibody dynamical systems in rotating frames. For the sake of completeness, we have included the steps to obtain a ROM using quadratic manifold and to perform time integration in Algorithm 1.

Remark 2.5 (Reduction Error). The quadratic manifold (2.27) is a special instance of the more general subspace defined by (2.23), i.e. where the amplitudes of the (S)MDs are taken as predefined functions of the amplitudes of the VMs. Hence it is obvious that the approximation resulting from (2.27) is never better (and in general worse) than the approximation obtained with (2.23) (in the sense of Galerkin).

Comparison of the quadratic manifold with static condensation. At this point, it is interesting to compare the proposed reduction method using the Quadratic Manifold with the idea of *static condensation*, whereby the axial modes of the structure are statically condensed out of the system leaving only the bending degrees of freedom in the ROM (see [84] for a review). The reduction mapping obtained using the quadratic manifold (when SMDs are used) is, in fact, equivalent to the one obtained from static condensation for the special case of a flat and isotropic structure. This is shown with the help of an example in Appendix 2.A. In the case of a more general layout (such as the wing example presented in this chapter), one may not be able to distinguish between transverse and axial DOFs to perform a static condensation approximation. However, for slender structures, this dichotomy is still present in modal coordinates, as low-frequency modes are indeed bending dominated. The quadratic manifold is applicable to such structures. Another distinction from the static condensation approximation lies in the fact that the ROM is obtained after projection of governing equations onto a configuration dependent tangent space, giving rise to quadratic generalized velocity terms and configuration-dependent mass matrix. Though the reduction mapping is the same in some special cases (as discussed above), the ROM from static condensation approximation misses these terms. The static-condensation-like approaches do have the advantage of being non-intrusive in nature, whereby commercial FE packages can be treated as a black box. Nonetheless, it is worth mentioning that the essential components of a quadratic manifold reduction, namely, the (S)MDs can also be computed in a non-intrusive manner using finite difference schemes from commercial packages.

Algorithm 1 Quadratic Manifold ReductionOffline Stage**Input:** VMs : $\Phi = [\phi_1, \dots, \phi_m] \in \mathbb{R}^{n \times m}$ **Output:** Quadratic manifold and tangent space mapping

```

1:  $\Omega, \Theta \leftarrow$  all empty 3-tensor  $\in \mathbb{R}^{n \times m \times m}$ 
2: for  $j \leftarrow 1$  to  $m$  do
3:   Calculate  $\left. \frac{\partial \mathbf{K}}{\partial \eta_j} \right|_{eq}$  using (2.13)
4:   for  $i \leftarrow 1$  to  $m$  do
5:      $\theta_{ij} \leftarrow \left. \frac{\partial \phi_i}{\partial \eta_j} \right|_{eq}^s$  or  $\left. \frac{\partial \phi_i}{\partial \eta_j} \right|_{eq}$   $\triangleright$  Use SMDs from (2.18) or MDs from (6.33) in the quadratic
       part
6:      $\Omega(:, i, j) \leftarrow \theta_{ij}$ 
7:   end for
8: end for
9:  $\Theta \leftarrow \text{SYMMETRIZE}(\Omega)$   $\triangleright$  SYMMETRIZE returns the symmetric part of a 3-tensor  $\Omega$  as given by
   (2.29)
10: Build QM:  $\triangleright$  Build function QM with argument  $\mathbf{q}$ , which returns  $\Gamma(\mathbf{q})$  from (2.55) and  $\mathbf{P}_\Gamma$  from
   (2.37)
    $[\Gamma(\mathbf{q}), \mathbf{P}_\Gamma(\mathbf{q})] \leftarrow \text{QM}(\mathbf{q})$ 

```

Online Stage: using implicit Newmark time integration ([89], see [45] for a review)**Input:** Initial conditions for (2.41): $\mathbf{q}_0, \dot{\mathbf{q}}_0 \in \mathbb{R}^m$; Newmark time integration parameters: α, β, γ ; time step: h ; residual error tolerance for iteration convergence: ϵ .**Output:** Reduced-order model using QM

```

11:  $[\mathbf{u}_0, \mathbf{P}_\Gamma] \leftarrow \text{QM}(\mathbf{q}_0)$ ,  $\triangleright$  Initialization
12:  $\tilde{\mathbf{M}} \leftarrow \mathbf{P}_\Gamma^T \mathbf{M} \mathbf{P}_\Gamma$ ,  $\tilde{\mathbf{C}} \leftarrow \mathbf{P}_\Gamma^T \mathbf{C} \mathbf{P}_\Gamma$ ,  $\tilde{\mathbf{f}} \leftarrow \mathbf{P}_\Gamma^T \mathbf{f}(\mathbf{u}_0)$ ,  $\tilde{\mathbf{g}} \leftarrow \mathbf{P}_\Gamma^T \mathbf{g}(t_0)$ ,  $\tilde{\mathbf{p}} \leftarrow \mathbf{P}_\Gamma^T (\mathbf{M}(\Theta \cdot \dot{\mathbf{q}}_0) \cdot \dot{\mathbf{q}}_0)$ 
13:  $\ddot{\mathbf{q}}_0 = \tilde{\mathbf{M}}^{-1} [\tilde{\mathbf{g}} - \tilde{\mathbf{f}} - \tilde{\mathbf{C}} \dot{\mathbf{q}}_0 - \tilde{\mathbf{p}}]$ 
    $k \leftarrow 0$ 
14: while  $t < t_{max}$  do  $\triangleright$  Time-marching loop
15:    $k \leftarrow k + 1$ 
16:    $t_{k+1} \leftarrow t_k + h$   $\triangleright$  Time increment
17:    $\dot{\mathbf{q}}_{k+1} \leftarrow \dot{\mathbf{q}}_k + (1 - \gamma)h\ddot{\mathbf{q}}_k$   $\triangleright$  Prediction
18:    $\mathbf{q}_{k+1} \leftarrow \mathbf{q}_k + h\dot{\mathbf{q}}_k + (0.5 - \beta)h^2\ddot{\mathbf{q}}_k$ 
19:    $\ddot{\mathbf{q}}_k \leftarrow \mathbf{0}$ 
20:   while true do  $\triangleright$  Newton-Raphson loop
21:      $[\mathbf{u}_{k+1}, \mathbf{P}_\Gamma] \leftarrow \text{QM}(\mathbf{q}_{k+1})$ ,  $\triangleright$  Full solution using QM
22:      $\tilde{\mathbf{M}} \leftarrow \mathbf{P}_\Gamma^T \mathbf{M} \mathbf{P}_\Gamma$ ,  $\tilde{\mathbf{C}} \leftarrow \mathbf{P}_\Gamma^T \mathbf{C} \mathbf{P}_\Gamma$ ,  $\tilde{\mathbf{f}} \leftarrow \mathbf{P}_\Gamma^T \mathbf{f}(\mathbf{u}_{k+1})$ ,
        $\tilde{\mathbf{g}} \leftarrow \mathbf{P}_\Gamma^T \mathbf{g}(t_{k+1})$ ,  $\tilde{\mathbf{p}} \leftarrow \mathbf{P}_\Gamma^T (\mathbf{M}(\Theta \cdot \dot{\mathbf{q}}_{k+1}) \cdot \dot{\mathbf{q}}_{k+1})$ 
23:      $\mathbf{r}_{k+1} \leftarrow \tilde{\mathbf{M}}\ddot{\mathbf{q}}_{k+1} + \tilde{\mathbf{p}} + \tilde{\mathbf{C}}\dot{\mathbf{q}}_{k+1} + \tilde{\mathbf{f}} - \tilde{\mathbf{g}}$   $\triangleright$  Residual evaluation
24:     if  $\|\mathbf{r}_{k+1}\| < \epsilon \|\tilde{\mathbf{f}}\|$  then  $\triangleright$  Convergence criterion
25:       break
26:     end if
27:      $\mathbf{S} \leftarrow \mathbf{P}_\Gamma^T \mathbf{K}(\mathbf{u}_{k+1}) \mathbf{P}_\Gamma + \frac{\gamma}{\beta h} \tilde{\mathbf{C}} + \frac{1}{\beta h^2} \tilde{\mathbf{M}}$   $\triangleright$  (Approximate) Jacobian evaluation
28:      $\Delta \mathbf{q} \leftarrow -\mathbf{S}^{-1} \mathbf{r}_{k+1}$   $\triangleright$  Calculation of correction
29:      $\mathbf{q}_{k+1} \leftarrow \mathbf{q}_{k+1} + \Delta \mathbf{q}$   $\triangleright$  Correction
30:      $\dot{\mathbf{q}}_{k+1} \leftarrow \dot{\mathbf{q}}_{k+1} + \frac{\gamma}{\beta h} \Delta \mathbf{q}$ 
31:      $\ddot{\mathbf{q}}_{k+1} \leftarrow \ddot{\mathbf{q}}_{k+1} + \frac{1}{\beta h^2} \Delta \mathbf{q}$ 
32:   end while
33: end while

```

2.5 Applications and results

The proposed reduction techniques are tested and compared on examples. Two models are considered, each being a thin-walled structure with different levels of complexity. Both structures are modeled using triangular shell elements featuring 6 DOFs per node or 18 DOFs per element. Rayleigh damping is used as structural damping in all models ⁴.

For a general nonlinear system, the evaluation of the tangent stiffness and internal forces required during the formation of the Jacobian and the residual respectively is done by element level assembly during each iteration. This is an expensive online cost apart from the linear system solution. The linear system solution cost is mitigated by projection onto a ROB. But as the system becomes larger, the mapping, nonlinearity evaluation, and projection become dominant in taking the CPU time during the time integration. An effective way to deal with this is the evaluation of nonlinearities offline using tensors or to use hyper-reduction, thereby making time integration independent of the system size (cf. ref. [57]). However, as discussed in the introduction of the chapter, we perform time integration in the presented examples without such hyper-reduction, as the focus of this work is on investigating the appropriateness and effectiveness of MDs and quadratic manifolds to approximate the response of non-linear structures. For this reason, we do not report computational speedup but only the reduction in problem size.

For each of the models, the accuracy of the results has been compared to the corresponding full nonlinear solutions. In this context, a global relative error measure shall be used, defined as

$$GRE_M = \frac{\sqrt{\sum_{t \in S} (\mathbf{u}(t) - \tilde{\mathbf{u}}(t))^T \mathbf{M} (\mathbf{u}(t) - \tilde{\mathbf{u}}(t))}}{\sqrt{\sum_{t \in S} \mathbf{u}(t)^T \mathbf{M} \mathbf{u}(t)}}} \times 100\%, \quad (2.42)$$

where $\mathbf{u}(t) \in \mathbb{R}^n$ is the vector of generalized displacements at the time t obtained from the full nonlinear solution, $\tilde{\mathbf{u}}(t) \in \mathbb{R}^n$ is the solution based on the reduced model, and S is the set of time instants at which the error is recorded. The mass matrix $\mathbf{M}$ provides a relevant normalization for the generalized displacements, which could be a combination of physical displacements and rotations, as is the case in the shell models shown here.

2.5.1 Flat structure

A flat plate simply supported on two opposite sides is considered. The Model (henceforth referred to as Model-I) sketch and parameters are shown in Figure 2.1.

A uniform pressure distribution is chosen to act normal to the plate surface as the external load. A time varying amplitude (load function) is used given by

$$\mathbf{g}(t) = p(t)\mathbf{l} \quad (2.43)$$

$$p(t) = p_0[\sin(\omega t) + \sin(\pi \omega t)] \quad (2.44)$$

where $\mathbf{l}$ is a constant load vector corresponding to a uniform pressure distribution of 1 Pa. Here, $p(t)$ can be termed as the *dynamic load function* which determines the time-dependency of the external load. Here the results are shown for a quasi periodic choice for $p(t)$, ω is a typical loading frequency

⁴A modal damping assumption is used to create a so-called *diagonal damping* matrix using weighted sum of Mass and stiffness matrices ($\mathbf{M}$ and $\mathbf{K}$ respectively). A modal damping factor of 0.4% for the first two modes is used to determine the weights (see [45] for details about this implementation). This low value is realistic and is chosen to make sure the VMs of the undamped system can be used for the reduction.

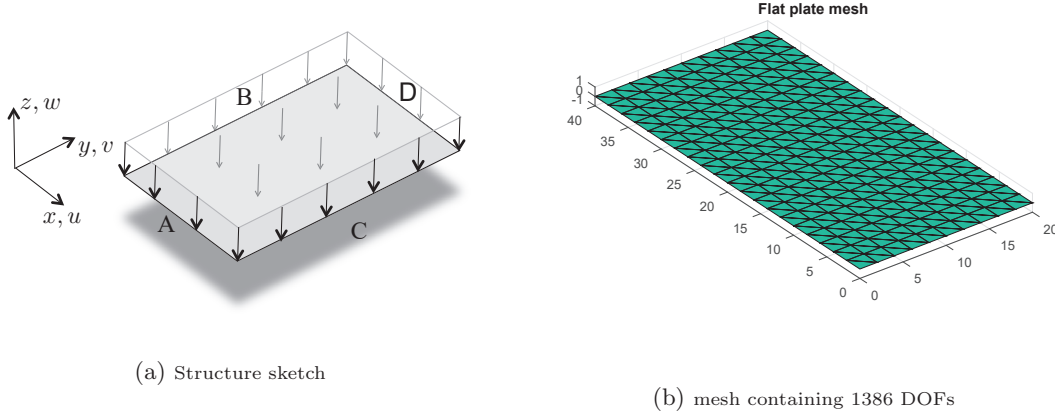


Figure 2.1: **Model - I:** Flat plate example - simply supported on sides A and D . The plate is $L = 40$ mm long, $H = 20$ mm wide, $t = 0.8$ mm thick. The Young Modulus is $E = 70$ GPa, the Poisson's ratio is $\nu = 0.33$, and the density is $\rho = 2700$ Kg/m³. For simply supported sides, we have $u = v = w = 0$. A spatially uniform pressure is applied on the plate as shown.

chosen as the first eigenfrequency of the linearized system (cf. (2.44)). The amplitude of loading is kept large enough to trigger significant nonlinear behaviour. The magnitude of p_0 is chosen such that the magnitude of the linear and nonlinear internal forces are of the same order (cf. Figure 2.2b). This is in agreement with the domain of applicability of von Kármán kinematics adopted here.

The *full* nonlinear solution was computed by updating the Jacobian of the residual at each Newton-Raphson (N-R) iteration (within every time step). No reduction is involved here and thus the linearised system solve of full size is a costly procedure. This also involves the element level calculation and assembly of the tangent-stiffness matrix at each N-R iteration which further adds to the *online* cost (at least for large systems). Different levels of mesh refinement were considered to reach an optimum number of degrees of freedom in terms of accuracy. The resulting mesh, containing 1386 DOFs and $n_e = 400$ elements, is shown in Figure 2.1b. This nonlinear solution is used as a reference for comparing the various reduction techniques. For the shown time span, the solution is computed over $n_h = 400$ time steps of equal size for all the techniques.

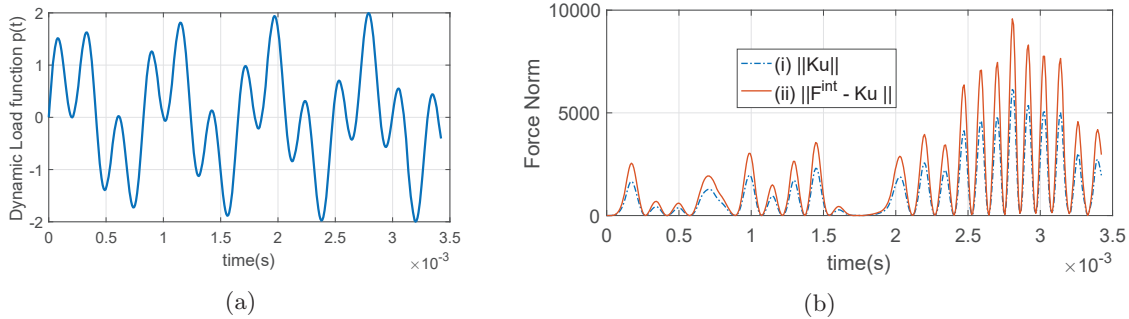


Figure 2.2: (a) Dynamic load function for a quasi-periodic loading (cf. (2.44)). (b) The comparison of the norm of the linear and the nonlinear internal forces during the full nonlinear solution for a flat plate simply supported on two opposite sides (cf. Figure 2.1).

2.5.1.1 Linear manifold

Model-I is a rather simplistic example and linear modal superposition using the first and the fifth VMs (corresponding to the first and second bending mode of the plate) is able to accurately reproduce the linear solution⁵. However, a basis containing just these modes is not good enough for capturing the nonlinear response. The Linear Manifold (LM) reduction involves the use of a VMs basis augmented by all or some of the (S)MDs, as explained in Section 2.3. The (S)MDs corresponding to the first 3 bending VMs are shown in Figure 2.4. This figure shows that the MDs inherently capture the bending-stretching coupling associated with geometrical nonlinearities in the structure. Using 2 VMs (first and fifth), 4(3) (S)MDs can be obtained, to constitute a basis of 5(6) vectors. This reduced basis was used for integration and the results (Figure 2.6, Table 2.1) are quite accurate.

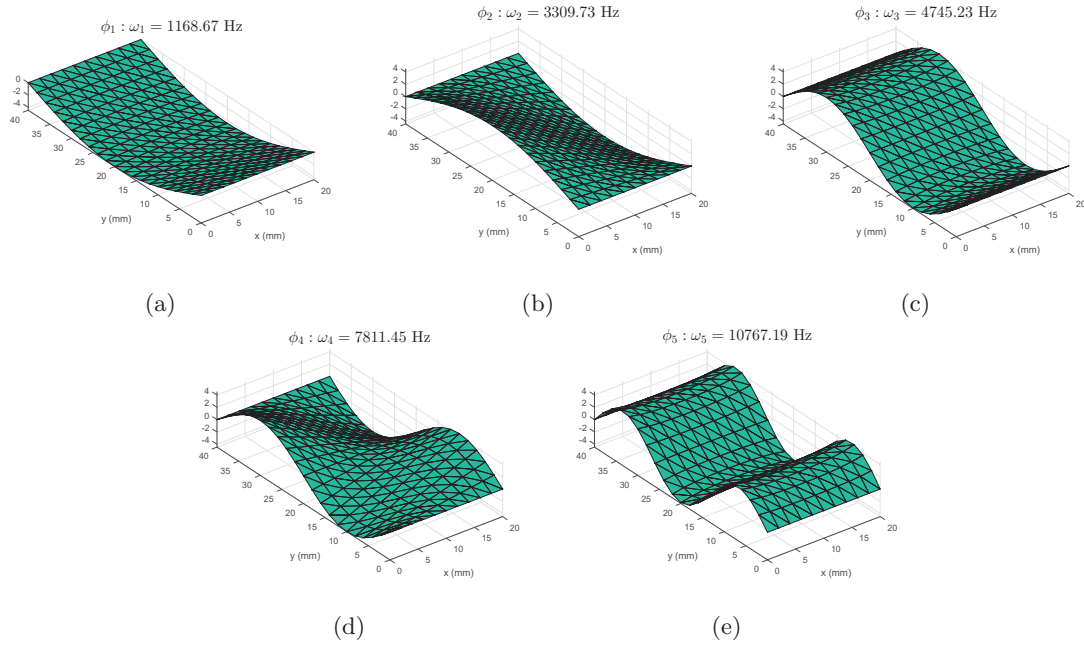


Figure 2.3: **Model - I:** Mode shapes and frequencies for first 5 vibration modes of Model-I.

MD Selection As noted earlier, the reduction basis size grows with complexity $\mathcal{O}(m^2)$ for m VMs and all corresponding (S)MDs in the basis. However, only a few of these MDs might be important for capturing the nonlinear behavior. Section 2.3.5 describes techniques to rank the (S)MDs to be considered in the basis. The ranking and the weights obtained for all (S)MDs using the corresponding techniques are shown in Figure 2.5. As a rule of thumb, a total of $n_{MD} = m$ MDs has been chosen in this work for an LM basis containing m VMs, thus making the basis size $2m$. This is done for a fair comparison and keeping the basis size linear with m .

Incidentally, the SMDs and MDs turn out to be identical for this example. Thus using a basis of size 4 (2 VMs + 2 selected (S)MDs), both techniques give results with same accuracy (Table 2.1) since the (S)MDs selected are the identical.

⁵The intermediate modes (i.e. the 2nd, 3rd and the 4th modes) do not contribute towards the linear response because these shapes are antisymmetric, see Figure 2.3, and are not excited by the uniform loading case being considered here.

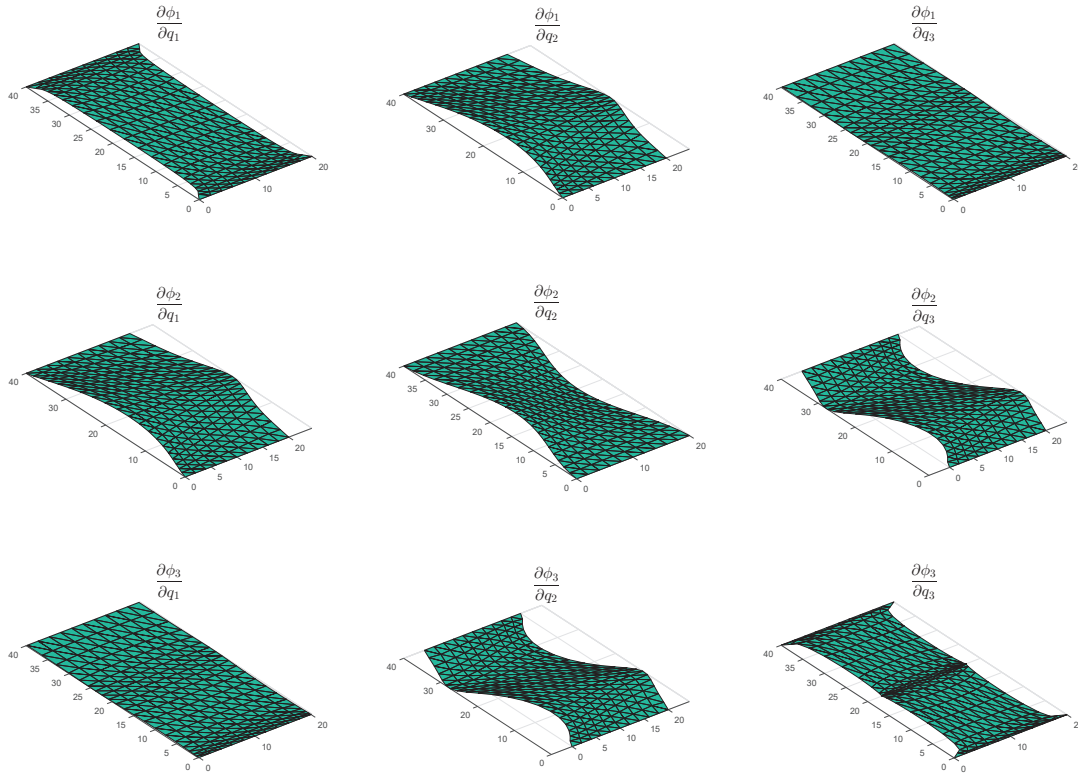


Figure 2.4: (S)MDs corresponding to first 3 VMs of Model-1. The (S)MDs for a flat plate contain only in-plane displacement contributions. They capture the inherent membrane effects associated to the bending modes of the flat structure (cf. first three VMs in Figure 2.3). Physically, an MD represents the sensitivity of VM ϕ_i corresponding to a displacement given in the direction of VM ϕ_j . For example, if the structure is given a displacement in the direction of the first VM, then we expect in-plane stretching effects due to geometric nonlinearities as given by $\frac{\partial \phi_1}{\partial q_1}$.

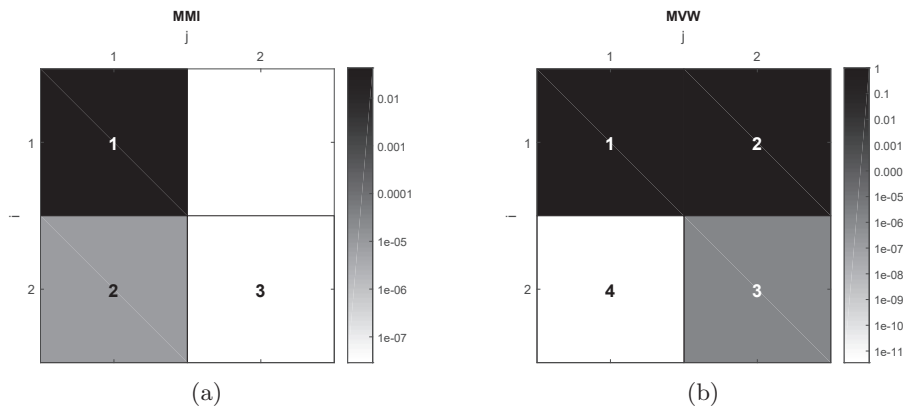


Figure 2.5: The ranking and the weights obtained for all (S)MDs using 2 VMs in Model-I. (a) shows the colour intensity plot for weights found using Maximum Modal Interaction (MMI) technique. Cell (i,j) contains the normalized weight W_{ij} for SMD_{ij} based on (2.24) (Note: part above diagonal not shown due to symmetry), (b) shows the color plot for ranking found using Modal Virtual Work (MVW) technique. Cell (i,j) contains weight for MD_{ij} .

2.5.1.2 Quadratic manifold

A quadratic manifold is constructed using the first and the fifth VMs. The size of the reduced system is therefore $m = 2$ instead of $m = 5$ as in the linear Manifold consisting of all MDs. As shown in Table 2.1 and Figure 2.6, the QM is able to reach similar accuracy for the response in this case.

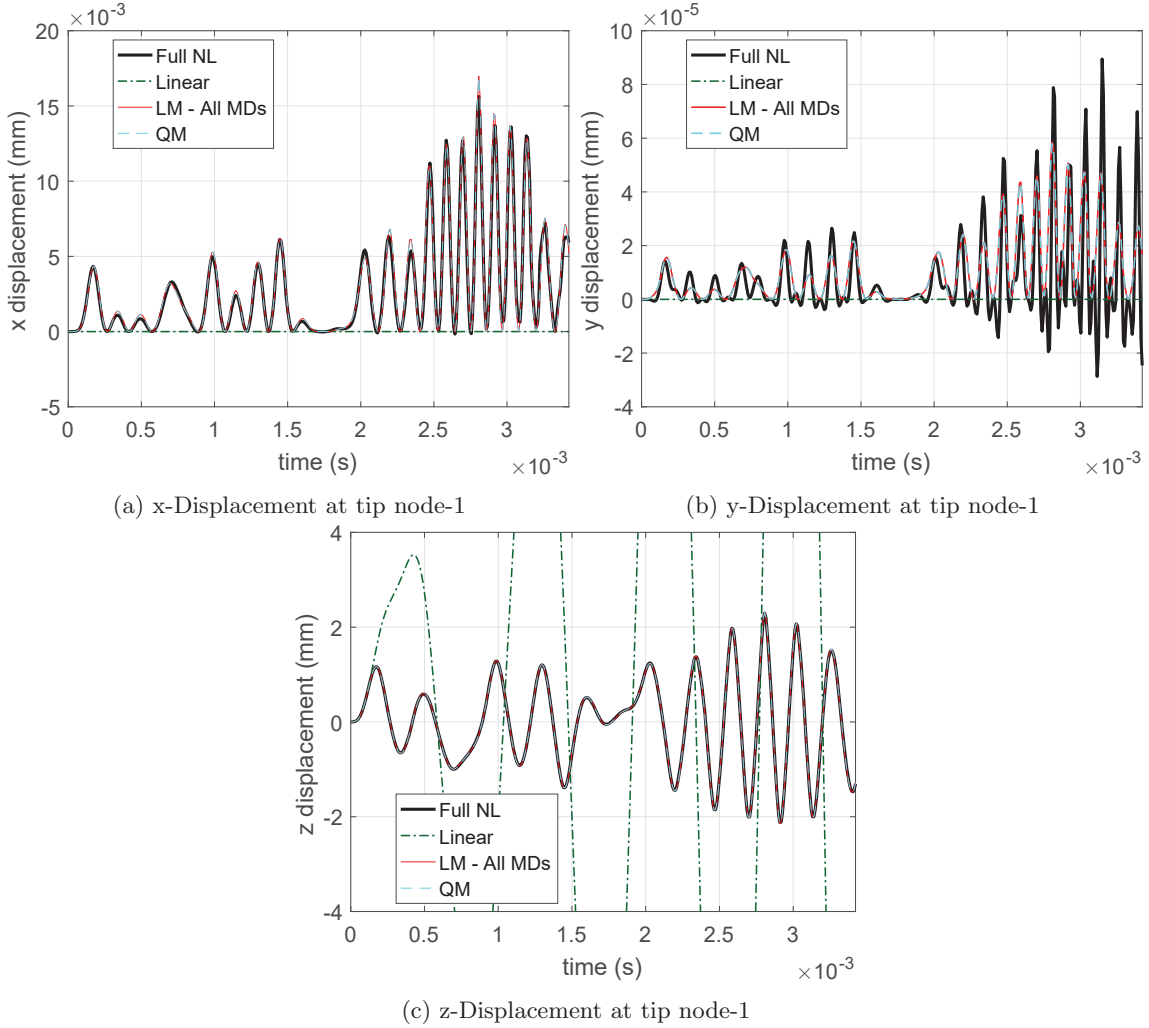


Figure 2.6: Time history of solution (displacement) in x,y,z direction at a node situated at the centroid of the Flat structure (cf. Figure 2.1) is shown. Full nonlinear solution, Linearized system solution, reduced solution on the Linear Manifold (using All MDs) and Quadratic Manifold are shown for comparison. The remarkable difference in Linearised and Full nonlinear solution shows the system being in the nonlinear range of operation. System is successfully reduced on the Linear and Quadratic Manifold with practically identical accuracy.

For the sake of comparison, a POD basis was created containing the same number of vectors (5 in case of Model-I) as the LM basis (containing all (S)MDs) and it was observed that 5 POD modes do not provide a comparable accuracy as reported in Table 2.1. It is interesting to note that the GRE_M for Quadratic Manifold reduction is lower than that for the Linear Manifold reduction using MDs. The quadratic manifold cannot give a better result as the LM (with the same MDs), as specified earlier in the Remark 2.5. This, however, holds for the error in the residual in the sense of Galerkin projection at each time step, which is not what is measured by the GRE_M here adopted.

Table 2.1: Global Relative Error in Flat structure model (Model-I) for different reduction techniques.

Reduction Technique	GRE_M (%)	# unknowns
LM (All (S)MDs)	2.29	5
LM-Selected SMDs (MMI)	2.32	4
LM-Selected MDs (MVW)	2.32	4
Quadratic Manifold -(S)MDs	1.91	2
POD	14.83	5

2.5.2 NACA airfoil wing structure

In the context of a more realistic application, a thin-walled wing structure is considered. The mesh for this structural model (referred to as Model-II hereafter) contains a realistically high number of DOFs so that the accuracy and reduction in problem size can be compared and appreciated. For

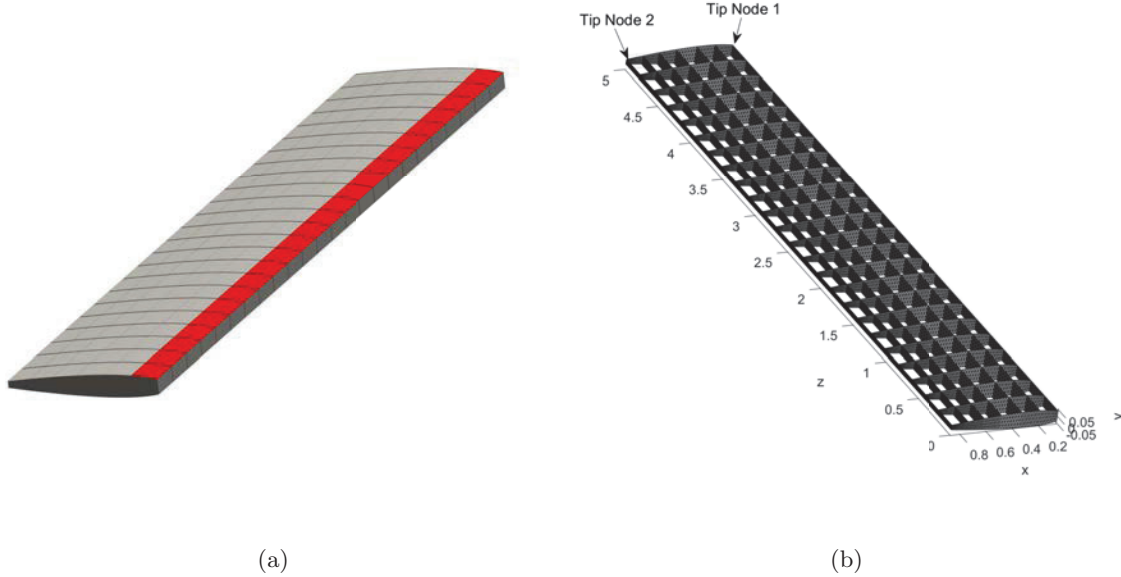


Figure 2.7: **Model - II:** A wing structure with NACA 0012 airfoil (length(L) = 5 m, Width(W) $\approx$ 0.9 m, Height(H) = 0.1 m) stiffened with ribs along the longitudinal and lateral direction. The Young's modulus is $E = 70$ GPa, the Poisson's ratio is $\nu = 0.33$, and the density is $\rho = 2700$ Kg/m³. The wing is cantilevered at one end. Uniform pressure is applied to the highlighted area (shown in (a)). The structure is meshed with triangular flat shell elements with 6 DOFs per node and each with a thickness of 1.5 mm. The mesh contains $n = 135770$ DOFs, $n_{el} = 49968$ elements. For illustration purposes, the skin panels are removed and the mesh is shown in (b).

illustration purposes, the results for a low frequency pulse load are shown here. A spatially uniform pressure load is applied locally on the structure skin at an area highlighted in Figure 2.7a. The pressure load takes the shape of a pulse in time as shown in Figure 2.8a. The dynamic load function is given as

$$p(t) = A \sin^2(\omega t) \left[H(t) - H\left(\frac{\pi}{\omega} - t\right) \right], \quad (2.45)$$

where $H(t)$ is the *heaviside* function and ω chosen as the average of the first and second natural frequency of vibration. Again the load amplitude is chosen so that the linear and nonlinear internal forces have magnitudes of similar order (see Figure 2.8b). The displacement response against time

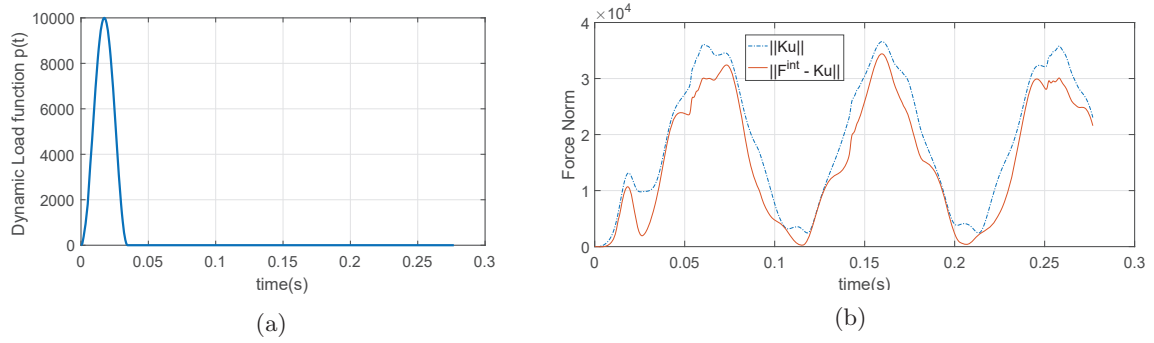


Figure 2.8: (a) Dynamic load function for pulse loading (cf. (2.45)). (b) The comparison of the norm of the linear and the nonlinear internal force during a full nonlinear solution for model-II (cf. Figure 2.7).

are shown for two nodes on the tip of the structure in Figure 2.7b.

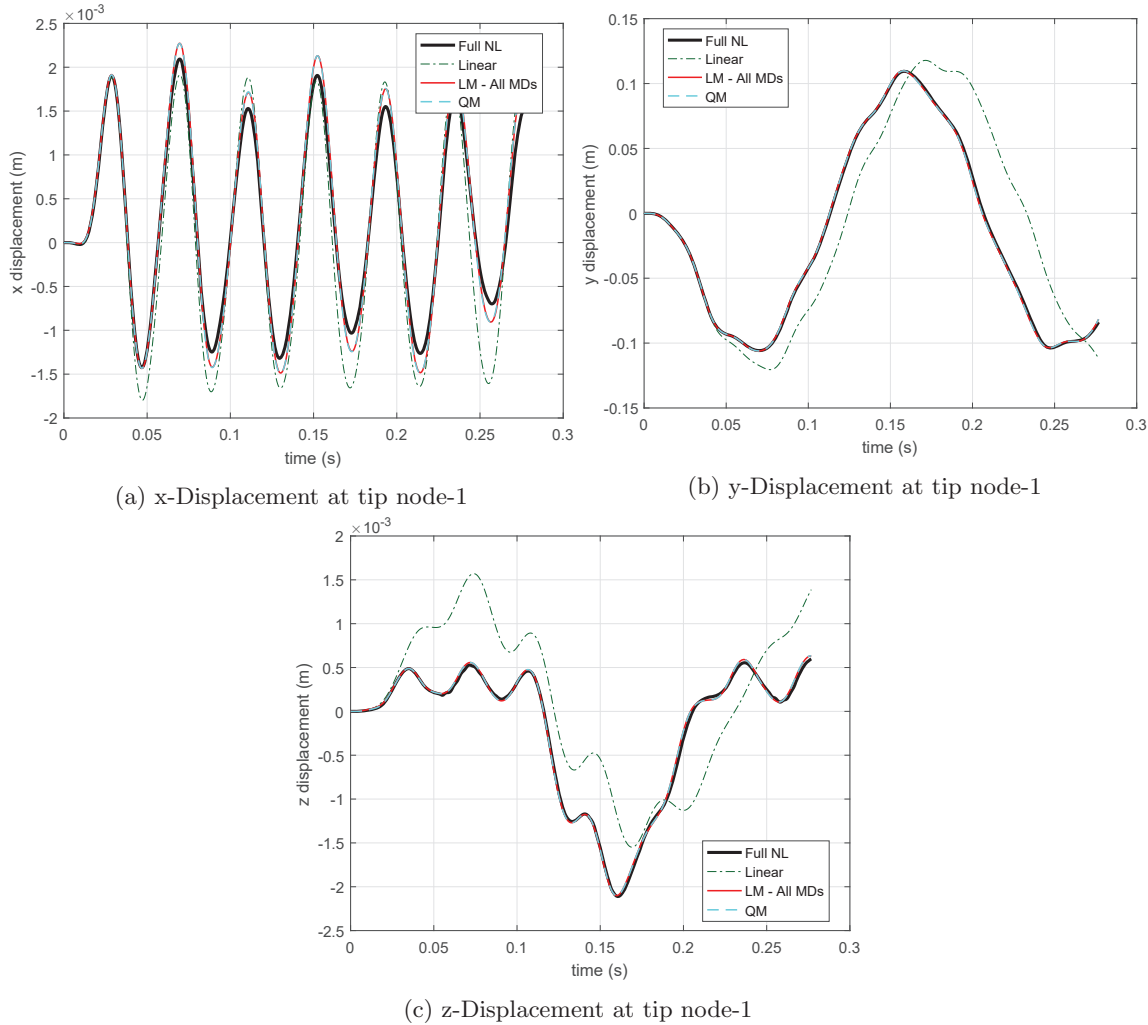


Figure 2.9: Time history of solution (displacement) in x,y,z-directions at the tip-node 1 of Model-II (see Figure 2.7) is shown. Full nonlinear solution, Linearized system solution, reduced solution using the Linear Manifold (using All MDs) and Quadratic Manifold are shown for comparison. The remarkable difference in Linearized and Full nonlinear solution shows the system being in the nonlinear range of operation. The system is successfully reduced with the Linear and Quadratic Manifold with practically identical accuracy.

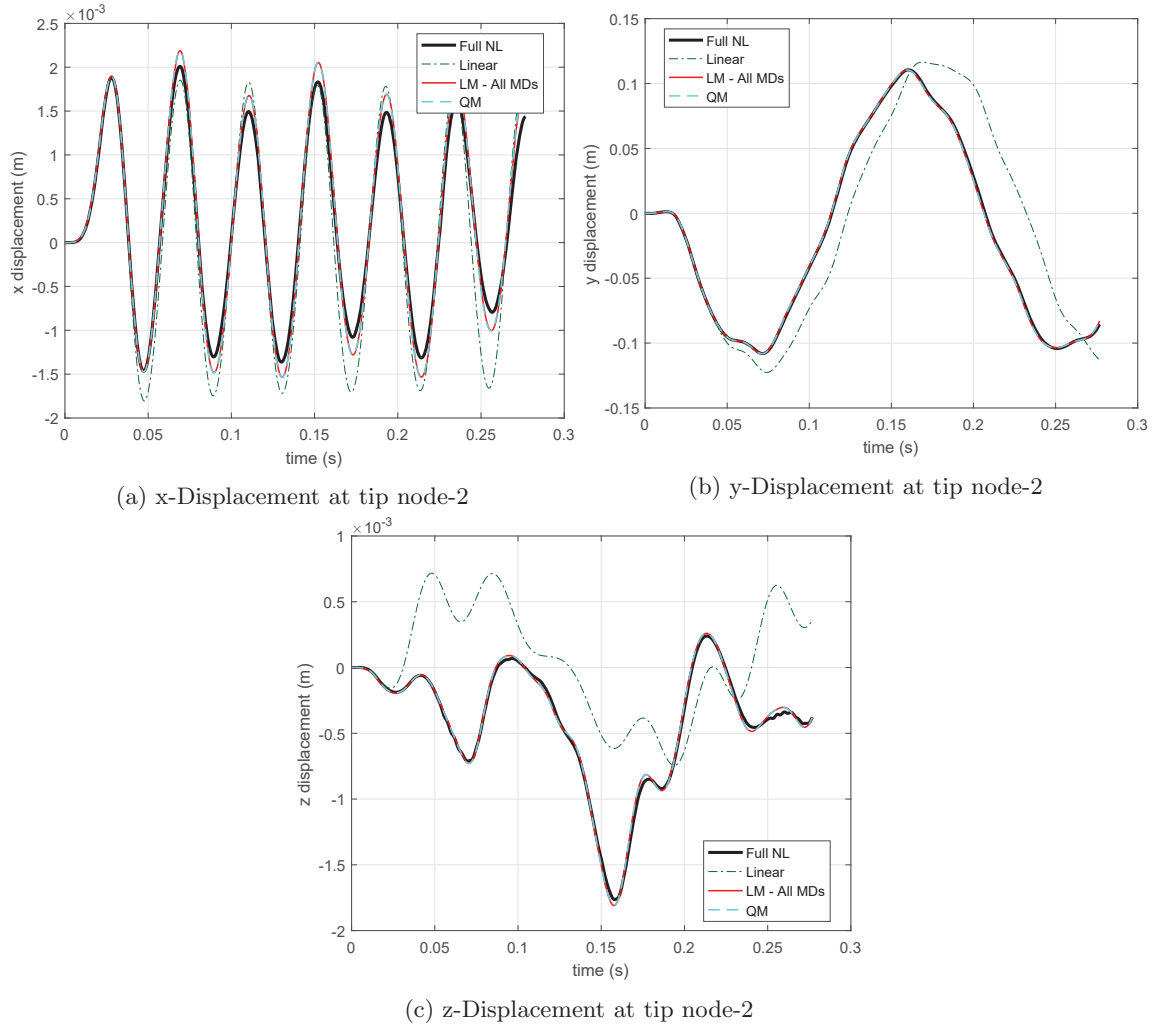


Figure 2.10: Time history of solution (displacement) in x,y,z-directions at the tip-node 2 of Model-II (see Figure 2.7) is shown. Full nonlinear solution, Linearized system solution, reduced solution over the Linear Manifold (using All MDs) and Quadratic Manifold are shown for comparison. The remarkable difference in Linearized and Full nonlinear solution shows the system being in the nonlinear range of operation. The system is successfully reduced over the Linear and Quadratic Manifold with practically identical accuracy.

2.5.2.1 Linear manifold

A linear manifold is constructed with the first 5 VMs and corresponding (S)MDs. A maximum of (15)25 (S)MDs can be obtained from 5 VMs, which makes the ROB size (20)30 if all the (S)MDs are considered. However, as discussed in Section 2.3.5, the (S)MD selection techniques (MMI for SMDs & MVW for MDs) are also tested by selecting 5 (S)MDs instead of 15 (ranking shown in Figure 2.11). It can be seen that both the methods are able to reproduce the nonlinear response with good accuracy. The results are shown in Figures 2.9 and 2.10, and Table 2.2. Note that the LM composed of all SMDs has 20 unknowns whereas the LM composed of all MDs has 30 modes, since the SMDs are symmetric and the MDs are not (cf. Theorem 1 and Remark 2). Based on $m = 5$ VMs in this case, there are $m(m+1)/2 = 15$ SMDs, and $m^2 = 25$ MDs.

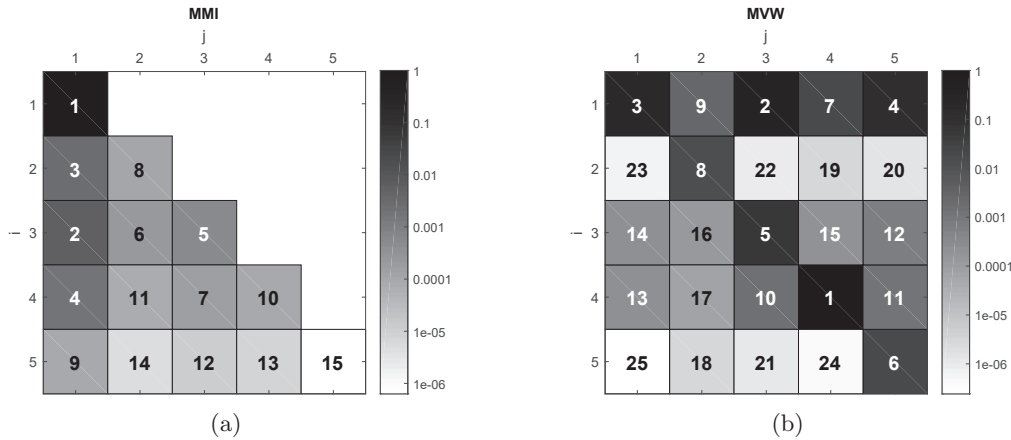


Figure 2.11: The ranking and the weights obtained for all (15)25 (S)MDs using 5 VMs in Model-II. (a) shows the colour intensity plot for weights found using Maximum Modal Interaction (MMI) technique, cell (i, j) contains rank (number) and weight (colour) for MD_{ij} . (b) shows the colour intensity plot for ranking found using Modal Virtual Work (MVW) technique, Cell (i,j) contains rank (number) and weight (colour) for MD_{ij} . (Note: part above diagonal not shown due to symmetry)

2.5.2.2 Quadratic manifold

A quadratic manifold is used with first 5 modes reducing the number of unknowns m to 5 instead of 20 as in case of LM. As shown in Figures 2.9 and 2.10, and Table 2.2, the QM is able to provide same accuracy as the LM using quarter the number of unknowns in case of SMDs (six times less in case of MDs). The quadratic manifold was constructed using MDs as well as SMDs and both show very similar accuracy (cf. Table 2.2).

It is interesting to see that a POD based approach with 5 basis vectors performs worse than the QM in this case. Indeed the performance improves significantly if more POD modes are included in the basis (cf. Table 2.2).

Table 2.2: Global Relative Error in Model-II for different reduction techniques.

Reduction Technique	# unknowns	GRE_M (%)
LM - MDs (All)	30	1.60
LM - Selected MDs (MVW)	10	2.06
LM - SMDs (All)	20	1.65
LM - Selected SMDs (MMI)	10	1.84
Quadratic Manifold (MDs)	5	1.66
Quadratic Manifold (SMDs)	5	1.65
POD	5	4.90
POD	20	0.31

2.6 Conclusions

In this work, we consider projection based Model Order Reduction (MOR) techniques in the context of thin-walled structural dynamics characterized by Von Kármán kinematics. The main focus of the research is on MOR methods which essentially do not require a full non-linear solution run to construct the reduced-order model. Though the use of Modal Derivatives (MDs) for reduction addresses this need, it quickly becomes inhibitive due to the growing number of unknowns with the number of Vibration Modes (VMs). The proposed quadratic mapping using MDs avoids this undesirable growth with a negligible loss of accuracy. Some conclusions are as follows.

Linear Manifold vs. Quadratic Manifold:

- **Accuracy:** It is easy to see that the amplitudes connected to the (S)MDs in the linear manifold are unknowns in the corresponding reduced equations and thus are free to choose any value, whereas in the quadratic manifold they are inherently constrained by the amplitudes of VMs. Indeed this makes the former more accurate in the Galerkin sense (cf. Remark 2.5), but results on tested examples show that the solution accuracy is very similar in both cases. However, it is anticipated that the quadratic enslavement of the MDs to the VM amplitudes would not, in general, hold for an arbitrary structural system. In the experience of the authors, this approach provides excellent accuracy for systems characterized by slow dynamics which is dominated by a few, separated VMs, while the (fast) nonlinear coupling effects are merely triggered quasi-statically by the slow dynamics. Systems featuring beam-like behavior (as the wing box structure shown) are excellent candidates for the advocated approach. Likewise, as the nonlinearities in the system rise, it is expected that there would be a range where (S)MDs could still be good candidates for reduction without obeying quadratic enslavement to the VM amplitudes. Rutzmoser et al. [105] discuss this further.
- **Speed:** Even after selection of important (S)MDs, it is easy to see that the number of unknowns is greater in the Linear Manifold than the Quadratic Manifold. Thus, the reduced system is smaller in case, when a Quadratic Manifold is used. Since the evaluation of nonlinearity and projection are the bottlenecks, the reduction in problem size would not be tantamount to the computational speed up in the current implementation. A hyper-reduction technique or a tensor-based approach (in case of polynomial nonlinearities) would be required in order to obtain the desirable computational speedups. These aspects are studied in Chapter 4.

MD Selection: The search for selection criteria of (S)MDs to reduce the Linear manifold basis size resulted in two possible candidates (MMI for SMDs and MVW for MDs cf. Section 2.3.5). Both of them give results with very similar accuracy when all (S)MDs are used in the reduction basis. It should be noted that both selection techniques use "weights" to rank the MDs in the order of

relative importance, but these techniques did not indicate how many of these ranked (S)MDs should be used to accurately reproduce the solution obtained using all the (S)MDs. Thus, there is a need for further work along this course. As a rule of thumb, a total of $n_{MD} = m$ (S)MDs was chosen in this work in a linear manifold basis containing m VMs. This was done for fair comparison and for keeping the basis size linear with m .

In this work, we do not specifically address problems characterized by local or global structural instabilities. However, a quadratic manifold approach was also adopted for this class of problems and shown to be able to reproduce static complex post-buckling behavior even leading to structural instability [119]. Furthermore, for transient analysis, it has been shown that the inclusion of modes and their derivatives at different linearization points could capture buckling phenomena [121]. The reduced model using a quadratic manifold, as proposed here, considers the quadratic manifold as a local extension around the equilibrium point only. Thus, its applicability to systems with mild nonlinearities is justified. The presented examples have given a good overview of the relative behavior and accuracy of the results. However, we remark that the present work does not make general claims about the accuracy of these techniques.

Appendix

2.A Comparison of quadratic manifold with the static condensation approach

We consider a 2-DOF dynamical system in variables w, v given as

$$m_1 \ddot{w} + c_1 \dot{w} + k_1 w + awv + bw^3 = g(t), \quad (2.46)$$

$$m_2 \ddot{v} + c_2 \dot{v} + k_2 v + cw^2 = 0. \quad (2.47)$$

The tangent stiffness to this system can be written as

$$\mathbf{K}(\mathbf{u}) = \begin{bmatrix} k_1 + 3bw^2 & aw \\ 2cw & k_2 \end{bmatrix}, \quad (2.48)$$

with $\mathbf{u} \in \mathbb{R}^2$ denoting the full vector of unknowns $\begin{bmatrix} w \\ v \end{bmatrix}$. The above system can be considered as a 2-DOF-variant of the FE discretized equations of the von Kármán beam (cf. [107]), where the solution variables $w(t), v(t) \in \mathbb{R}$ are analogous to the transverse and axial displacements of the beam, respectively, $m_1, m_2, k_1, k_2, a, b, c \in \mathbb{R}$ are physical parameters, and $g(t)$ correspond to the externally applied load in the transverse direction.

The Static Condensation (SC) approach applied to this example implies that the membrane variables, being stiff, are not dynamically excited and statically follow the load. This leads to the quadratic enslavement of the axial displacement variable v in (2.47) to the transverse displacement w as

$$v = k_2^{-1}(-cw^2), \quad (2.49)$$

which can be substituted into (2.46) to obtain a single DOF reduced-order model in w as

$$m_1 \ddot{w} + c_1 \dot{w} + k_1 w - ak_2^{-1}cw^3 + bw^3 = g(t). \quad (2.50)$$

Thus, the SC approach has mapped the full system of two unknowns into the single unknown w as

$$\mathbf{u} \approx \mathbf{\Gamma}_{SC} = \begin{bmatrix} w \\ -k_2^{-1}cw^2 \end{bmatrix}. \quad (2.51)$$

After linearizing the system (2.46)-(2.47) around the equilibrium position ($w = v = 0$), it is easily seen that the two VMs of the system are given by $\phi_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\phi_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Thus, the system is in modal coordinates, with the modal unknowns $q_1 = w$ and $q_2 = v$. We consider a quadratic manifold constructed using the first VM and its corresponding SMD (cf. (2.18)) given by

$$\begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \frac{\partial \phi_1}{\partial q_1} \Big|_{eq}^s = - \frac{\partial \mathbf{K}}{\partial q_1} \Big|_{eq} \phi_1, \quad (2.52)$$

where the tangent stiffness sensitivity can be calculated as

$$\left. \frac{\partial \mathbf{K}}{\partial q_1} \right|_{eq} = \left. \frac{\partial \mathbf{K}(\mathbf{u} = q_1 \phi_1)}{\partial q_1} \right|_{eq} = \begin{bmatrix} 0 & a \\ 2c & 0 \end{bmatrix}. \quad (2.53)$$

From (2.52) and (2.53), we get

$$\left. \frac{\partial \phi_1}{\partial q_1} \right|_{eq}^s = - \begin{bmatrix} 0 \\ k_2^{-1}(2c) \end{bmatrix}. \quad (2.54)$$

The single mode quadratic manifold using the SMD is then given by

$$\mathbf{\Gamma}_{QM} = \phi_1 q_1 + \frac{1}{2} \left. \frac{\partial \phi_1}{\partial q_1} \right|_{eq}^s q_1^2 = \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix} w}_{\text{bending contribution}} + \underbrace{\begin{bmatrix} 0 \\ -k_2^{-1}c \end{bmatrix} w^2}_{\text{membrane contribution}} = \begin{bmatrix} w \\ -k_2^{-1}cw^2 \end{bmatrix}. \quad (2.55)$$

The final expression in (2.55), along with (2.51), shows that for this example, the reduction mapping constructed using the quadratic manifold is exactly the same as that obtained using the static condensation approach. The ROM obtained using the QM can be written using (2.41) as

$$m_1 \ddot{w} + \underbrace{4k_2^{-2}c^2 m_2 (w^2 \ddot{w} + w \dot{w}^2)}_{\text{extra inertial terms}} + c_1 \dot{w} + \underbrace{4k_2^{-2}c^2 c_2 w^2 \dot{w}}_{\text{extra damping terms}} + k_1 w - ak_2^{-1}cw^3 + bw^3 = g(t). \quad (2.56)$$

Though the reduction mapping is the same, it is easy to see that the two ROMs (2.50) and (2.56) are different. In particular, the ROM (2.56) created using the QM differs only by the inclusion of extra inertial and damping contributions for the statically condensed axial variables. This is due to the nonlinear mapping and projection of full equations on to the tangent space in case of the QM-based ROM. For the ease of readability, we demonstrated a 2-DOF example, but it is easy to see that these conclusions hold for multi-dimensional analogs of (2.46)-(2.47), as well. A similar discussion comparing the static condensation approach with the static quadratic manifold may also be found in refs. [104, 106].

Chapter 3

Slow-fast decomposition & spectral submanifolds

Abstract. We apply two recently formulated mathematical techniques, Slow-Fast Decomposition (SFD) and Spectral Submanifold (SSM) reduction, to a von Kármán beam with geometric nonlinearities and viscoelastic damping. SFD identifies a global slow manifold in the full system which attracts solutions at rates faster than typical rates within the manifold. An SSM, the smoothest nonlinear continuation of a linear modal subspace, is then used to further reduce the beam equations within the slow manifold. This two-stage, mathematically exact procedure results in a drastic reduction of the finite-element beam model to a one-degree-of-freedom nonlinear oscillator. We also introduce the technique of spectral quotient analysis, which gives the number of modes relevant for reduction as output rather than input to the reduction process. This chapter closely follows the current author’s work in the original reference¹[62].

3.1 Introduction

The classical approach to model reduction involves a linear projection of the full system onto a set of basis vectors. This linear projection is characterized by a matrix whose columns span a suitable low-dimensional subspace. Various techniques have been applied to high-dimensional systems to obtain such a reduction basis, including the Proper Orthogonal Decomposition (POD) [7, 65, 68] (also known as Singular Value Decomposition (SVD), Karhunen-Loeve Decomposition), Linear Normal Modes (LNM) and Krylov subspace projection [14]. Once a suitable basis is chosen, the reduced-order model (ROM) is then obtained using Galerkin projection. Similar linear projection techniques have been devised for component-mode synthesis (CMS), such as the Craig-Bampton method [15]. An implicit assumption to all linear projection techniques is that the full system dynamics evolves in a lower-dimensional linear invariant *subspace* of the phase space of the system. While such linear subspaces do exist for linear systems (linear modal subspaces), they are generally non-existent in nonlinear systems. This results in a priori unknown and potentially large errors for linear projection-based reduction methods, necessitating the verification of the accuracy of the reduction procedure on a case-by-case basis.

More recent trends in model reduction account for this issue by constructing the reduced solution over nonlinear manifolds [63, 74, 105]. The seminal idea of Shaw & Pierre [111] is to construct assumed nonlinear invariant surfaces (nonlinear normal modes) that act as continuations of linear

¹Jain, S., Tiso, P. & Haller, G. Exact nonlinear model reduction for a von Kármán beam: Slow-fast decomposition and spectral submanifolds. *Journal of Sound and Vibration* **423**, 195–211. doi:[10.1016/J.JSV.2018.01.049](https://doi.org/10.1016/J.JSV.2018.01.049) (2018).

normal mode families under the addition of nonlinear terms near an equilibrium (cf. Qu [97] for a recent review). More heuristic reduction procedures in structural dynamics include the static condensation approaches, where the fast or stiff variables in the system are intuitively identified, and statically enslaved to the slow or flexible ones (cf. Mignolet *et al.* [84] for a review). Guyan reduction [48] is a classic example of this reduction philosophy at the linear level.

Most classic reduction techniques, coming from an intuitive and heuristic standpoint, do not provide an a priori estimate of their accuracy or even validity for a given system. To this effect, Haller & Ponsioen [50] proposed requirements for mathematically justifiable and robust model reduction in a non-linear mechanical system. These requirements ensure not only that the lower dimensional attracting structure (manifold) in the phase space is robust under perturbation, but also that the full system trajectories are attracted to it and synchronize with the reduced model trajectories at rates that are faster than typical rates within the manifold.

Recent advances in nonlinear dynamics enable such an *exact* model-reduction using the slow-fast decomposition (SFD) [50] and spectral submanifold (SSM) based reduction [49]. SFD is a general procedure to identify if a mechanical system exhibits a global partitioning of degrees of freedom into slow (flexible) and fast (stiff) components such that the fast variables can be enslaved to the slow ones. This results in a global ROM containing only the slow degrees of freedom of the full system. Such a global dichotomy in speed(stiffness) is modeled by a system parameter denoted by ϵ . SSMs, on the other hand, are the smoothest nonlinear extensions of the linear modal subspaces near an asymptotically stable equilibrium. Neither SFD nor SSM-based reduction has been applied to problems with high numbers of degrees of freedom. A beam model is often the first step in showing the potential of a new technique for reduction of high dimensional systems (cf. [29, 56, 93]). To this effect, we combine here, for the first time, the application of these techniques on a finite-element discretized nonlinear von Kármán beam model.

The von Kármán kinematic model is a second-order approximation of geometrically exact beam theory, accurately reproducing the bending-stretching coupling arising from bending displacements in the order of the thickness. This model is employed in various applications, including nonlinear vibratory behavior of carbon nano beams [4], micro and nano electromechanical systems [73, 122], aeroelastic response [36] and flexible multibody modeling [132]. When discretized, the von Kármán model yields a nonlinear internal elastic force term containing multi-variate, quadratic and cubic terms of the generalized nodal displacements. These features make the numerical solution of the model expensive for high levels of discretization, necessitating the use of ROMs that keep the computational time within practical limits. Several references tackle this issue by performing a Galerkin projection of the discretized equations of motion onto linear subspaces [73, 114, 122, 132].

Although SFD is not generally applicable to every mechanical system, it is interesting from a theoretical point-of-view that our beam model satisfies the assumptions required to apply SFD, as shown in Section 3. Using SFD in practice, we obtain a ROM on a slow manifold defined over the transverse degrees of freedom of the beam. Importantly, SFD leads to a globally-valid ROM in the phase space for the beam, which is in contrast with other reduction procedures in literature (e.g., [7, 14, 56, 63, 65, 93, 105]) which approximate near-equilibrium system dynamics. While applying SFD to our beam model, we also obtained general expressions for $\mathcal{O}(\epsilon^2)$ of terms in an SFD-based ROM. Along with the expressions for $\mathcal{O}(1), \mathcal{O}(\epsilon)$ terms obtained by Haller and Ponsioen [50], our expressions (given in 3.A) would enable a more accurate reduced description of any SFD system.

The SFD-reduced ROM for the von Kármán beam possesses no clear spectral gaps but a *spectral quotient analysis* nonetheless identifies a further potential for reduction to an SSM. Importantly, our spectral quotient analysis returns the number of modes relevant for reduction, instead of postulating or deducing this number from numerical experimentation. To perform SSM reduction, we use the formulas given by Szalai *et al.* [116], where these were derived in the context of nonlinear system identification of a beam from experimentally observed data.

In the end, the subsequent application of SFD and SSM reduction to our beam model results

in an exact ROM with a drastic reduction in the number of degrees of freedom of the system. In contrast with other techniques, e.g., [63, 93, 105] which also use nonlinear manifolds for reduction, the exact model reduction procedure used here results in the full system trajectories being attracted to model trajectories. This avoids the need to verify the accuracy of the reduction procedure on a case-by-case basis, without the need of any expensive training/statistical analysis, as is often the case for most reduction techniques in literature.

In the next section, we start by reviewing the main steps involved in the derivation of the governing PDEs for the von Kármán beam and the non-dimensionalization we performed upon them. The finite-element discretized equations obtained from the resulting nondimensionalized PDEs are then presented. This system of equations is then first reduced using the SFD in Section 3.3. The SSM-based reduction applied to the SFD-reduced system in Section 3.4. The conclusions, along with scope for further work, are presented in Section 3.5.

3.2 The discretized beam model

We briefly summarize the main steps leading to the derivation of the governing partial differential equations (PDEs) for the von Kármán beam (see, e.g., Reddy [99] for a detailed derivation). We consider a straight 2D beam aligned initially with the x_1 axis, as shown in Figure 3.1. The motion of the beam takes place in the $x_1 - x_3$ plane. The frame (x_1, x_2, x_3) is fixed to the reference initial configuration of the beam, and all displacements and rotations are defined with respect to this fixed frame.

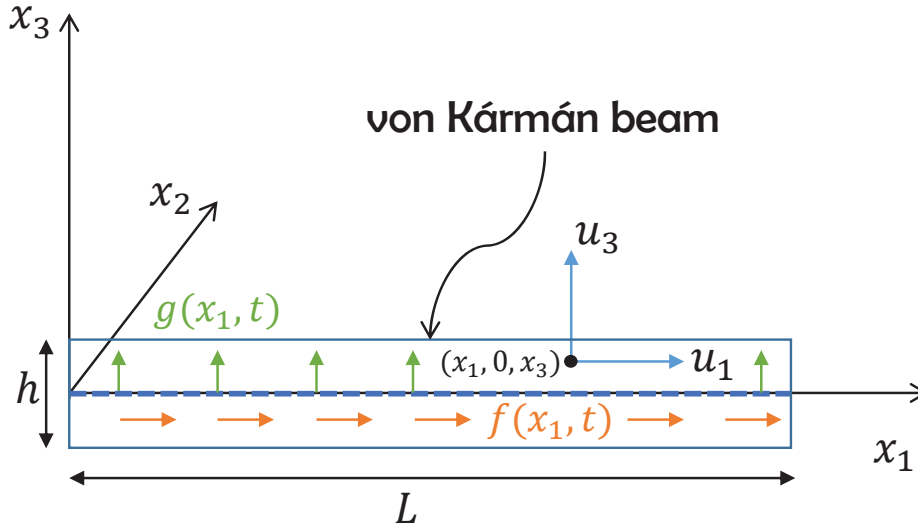


Figure 3.1: The schematic of a two-dimensional von Kármán beam with height h and length L , initially aligned with the x_1 . The x_1 and x_3 displacements of a material point with coordinates $(x_1, 0, x_3)$ are given by u_1 and u_3 , respectively. The transverse and axial load per unit length applied to the beam are given by $g(x_1, t)$ and $f(x_1, t)$, respectively.

Assuming the Euler-Bernoulli hypothesis for the kinematics of bending (i.e., that the lines initially straight and perpendicular to the beam axis remain so after deformation), we obtain the displacement field:

$$\begin{aligned}
u_1(x_1, x_3) &= u_0(x_1) - x_3 \partial_{x_1} w_0(x_1), \\
u_2(x_1, x_3) &= 0, \\
u_3(x_1, x_3) &= w_0(x_1),
\end{aligned} \tag{3.1}$$

where (u_1, u_2, u_3) denote the (x_1, x_2, x_3) displacements of a material point with coordinates $(x_1, 0, x_3)$, and $(u_0(x_1), w_0(x_1))$ are the (x_1, x_3) displacements of any material point lying on the reference line given by the x_1 axis. The von Kármán strain approximation of the Green-Lagrange strain for moderate rotations is given by

$$\varepsilon_{11} = \partial_{x_1} u_0 + \frac{1}{2}(\partial_{x_1} w_0)^2 - x_3 \partial_{x_1}^2 w_0. \tag{3.2}$$

Using the virtual work principle, we can formulate the equations of motion in terms of the primary unknowns u_0, w_0 as

$$\rho A \partial_t^2 u_0 = \partial_{x_1} \underbrace{\left[\int_A \sigma \, dA \right]}_N + f(x_1, t), \tag{3.3}$$

$$\rho A \partial_t^2 w_0 = \partial_{x_1}^2 \left[\int_A x_3 \sigma \, dA \right] + \partial_{x_1} (N \partial_{x_1} w_0) + g(x_1, t), \tag{3.4}$$

where f and g are external body forces per unit length of the beam in the x_1 and x_3 directions, respectively; A is the area of the cross-section; and σ is the σ_{11} component of the Cauchy stress. We choose the Kelvin-Voigt model for viscoelasticity as constitutive law to relate the stress σ to the von Kármán strain ε_{11} in (3.2) and to the corresponding strain rate as

$$\sigma = E \varepsilon_{11} + \kappa \dot{\varepsilon}_{11}. \tag{3.5}$$

Symbol	Meaning [unit]
t	time [s]
L	Length of beam [m]
h	Height of beam [m]
b	Width of beam [m]
A	Area of cross section ($A = bh$) [m ²]
E	Young's Modulus [Pa]
κ	Viscous damping rate of material [Pa s]
ρ	Density [kg/m ³]
$\partial(\bullet)$	$\frac{\partial}{\partial(\bullet)}$

Table 3.1: Notation

Here E denotes Young's modulus and κ is the rate of viscous damping for the material. Further assumptions include a uniform rectangular cross section with the reference x_1 axis passing through the centroid of the cross-section. We perform a non-dimensionalization of variables as summarized in Table 3.2, where α, β have been introduced as scaling factors for transverse and axial loading respectively.

Non-dimensionalized variable	Relation to physical variables
Thickness to length ratio (slenderness): ϵ	$= \frac{h}{L}$
Time: τ	$= \left(\frac{h}{L^2} \sqrt{\frac{E}{\rho}} \right) t$
Spatial variable along length: x	$= \frac{1}{L} x_1$
Transverse displacement: w	$= \frac{1}{h} w_0$
Axial displacement: u	$= \frac{1}{h} u_0$
Axial forcing: $p(x, \tau)$	$= \frac{L^4}{\beta b E h^4} f \left(x, \frac{L^2}{h} \sqrt{\frac{\rho}{E}} \tau \right)$
Transverse forcing: $q(x, \tau)$	$= \frac{L^4}{\alpha b E h^4} g \left(x, \frac{L^2}{h} \sqrt{\frac{\rho}{E}} \tau \right)$

Table 3.2: Definition of non-dimensionalized variables

This choice of non-dimensionalization leads to the following dimensionless PDEs governing the beam behavior:

$$\begin{aligned}
\ddot{w} + \frac{1}{12} \partial_x^4 w + \frac{\zeta \epsilon}{12} \partial_x^4 \dot{w} - \frac{1}{\epsilon} \partial_x (\partial_x u \partial_x w) - \zeta \partial_x (\partial_x \dot{u} \partial_x w) \\
- \frac{1}{2} \partial_x (\partial_x w)^3 - \zeta \epsilon \partial_x \left((\partial_x w)^2 \partial_x \dot{w} \right) = \alpha q(x, \tau), \\
\ddot{u} - \frac{1}{\epsilon} \partial_x^2 u - \frac{\zeta}{\epsilon} \partial_x^2 \dot{u} - \frac{1}{2\epsilon} \partial_x (\partial_x w)^2 - \zeta \partial_x (\partial_x w \partial_x \dot{w}) = \beta p(x, \tau).
\end{aligned} \tag{3.6}$$

Here $(\dot{\bullet})$ denotes $\partial_\tau(\bullet)$, $\epsilon = \frac{h}{L}$ is the length to thickness ratio and $\zeta = \frac{\kappa}{L\sqrt{E\rho}}$ is a dimensionless constant resulting from non-dimensionalization. We have chosen the displacements to be scaled with respect to the thickness h of the beam, and not the length. Moreover, the applied loading has been nondimensionalized and scaled with respect to a load which leads to transverse displacements in the order of the thickness h of the beam. This is because the von Kármán kinematic approximations are justified for displacements and forces in this range. Furthermore, the time t has been nondimensionalized with respect to a time period that is representative of natural frequency of the oscillation of the beam, since the structural response is generally studied at such time scales.

Upon finite-element discretization of the non-dimensional system (3.6) with cubic shape functions for w and linear shape functions for u (see, e.g., Crisfield [30]), we obtain the finite-dimensional discretized version of (3.6) as

$$\begin{aligned}
\mathbf{M}_1 \ddot{\mathbf{x}} + \zeta \epsilon (\mathbf{K}_1 + \mathcal{C}(\mathbf{x})) \dot{\mathbf{x}} + \zeta \mathcal{D}(\mathbf{x}) \dot{\mathbf{y}} + \mathbf{K}_1 \mathbf{x} + \frac{1}{\epsilon} \mathcal{F}(\mathbf{x}, \mathbf{y}) + \mathcal{G}(\mathbf{x}) &= \alpha \mathbf{q}(\tau), \\
\mathbf{M}_2 \ddot{\mathbf{y}} + \frac{\zeta}{\epsilon} \mathbf{K}_2 \dot{\mathbf{y}} + \zeta \mathcal{E}(\mathbf{x}) \dot{\mathbf{x}} + \frac{1}{\epsilon^2} \mathbf{K}_2 \mathbf{y} + \frac{1}{\epsilon} \mathcal{H}(\mathbf{x}) &= \beta \mathbf{p}(\tau),
\end{aligned} \tag{3.7}$$

where $\mathbf{x} \in \mathbb{R}^{n_s}$, $\mathbf{y} \in \mathbb{R}^{n_f}$ are the finite dimensional (discretized) counterparts of the unknowns w, u respectively (n_s, n_f being the number of unknowns dependent on the finite-element discretization), and $\mathbf{M}_1, \mathbf{K}_1 \in \mathbb{R}^{n_s \times n_s}$ and $\mathbf{M}_2, \mathbf{K}_2 \in \mathbb{R}^{n_f \times n_f}$ are the corresponding mass and stiffness matrices. Here, $\mathcal{F}$ (a bilinear function of the form $(\mathcal{F}(\mathbf{x}, \mathbf{y}))_i = F_{ijk} x_j y_k$, $i, j \in \{1, \dots, n_s\}$, $k \in \{1, \dots, n_f\}$), $\mathcal{G}$ (a cubic function of the form $(\mathcal{G}(\mathbf{x}))_i = G_{ijkl} x_j x_k x_l$, $i, j, k, l \in \{1, \dots, n_s\}$) and $\mathcal{H}$ (a quadratic function of the form $(\mathcal{H}(\mathbf{x}))_i = H_{ijk} x_j x_k$, $i \in \{1, \dots, n_f\}$, $j, k \in \{1, \dots, n_s\}$) correspond to the nonlinear elastic

forces in the beam; $\mathcal{C}$ (a quadratic function of the form $(\mathcal{C}(\mathbf{x}))_{ij} = C_{ijkl}x_kx_l$, $i, j, k, l \in \{1, \dots, n_s\}$), $\mathcal{D}$ (a linear function of the form $(\mathcal{D}(\mathbf{x}))_{ij} = D_{ijk}x_k$, $i, k \in \{1, \dots, n_s\}$, $j \in \{1, \dots, n_f\}$) and $\mathcal{E}$ (a linear function of the form $(\mathcal{E}(\mathbf{x}))_{ij} = E_{ijk}x_k$, $j, k \in \{1, \dots, n_s\}$, $i \in \{1, \dots, n_f\}$) correspond to the nonlinear contributions resulting from the viscoelastic material damping.

3.3 Slow-Fast Decomposition

3.3.1 Verification of assumptions for the application of SFD

In the finite-element-discretized system (3.7), the $\mathbf{y}$ variables (representing the axial displacement components) are stiffer and hence faster than the $\mathbf{x}$ variables (representing the transverse displacement components). Such a global difference of speeds indicates the possible existence of a lower-dimensional slow manifold, as described in mathematical terms by the geometric singular perturbation theory of Fenichel [43]. If such a slow manifold is robust and attracts nearby solutions, then the dynamics on this manifold provides an exact reduced-order model with which all nearby solutions synchronize exponentially fast.

For general finite-dimensional mechanical systems characterized by such a dichotomy of time scales, Haller & Ponsioen [50] deduced conditions under which positions and velocities in the fast degrees of freedom $(\mathbf{y}, \dot{\mathbf{y}})$ can be expressed as a graph over their slow counterparts $(\mathbf{x}, \dot{\mathbf{x}})$, resulting in a globally exact model reduction. If these conditions for a Slow-Fast Decomposition (SFD) are satisfied, then all trajectories of the full system (close enough to the slow manifold in the phase space) synchronize with the reduced model trajectories at rates faster than those within the slow manifold. To check these conditions, we take $0 < \epsilon \ll 1$, the thickness to length ratio of the beam, as the required non-dimensional small parameter. The system (3.7) can then be reformulated in terms of $\frac{\mathbf{y}}{\epsilon}$ as

$$\begin{aligned} \mathbf{M}_1 \ddot{\mathbf{x}} + \zeta \epsilon (\mathbf{K}_1 + \mathcal{C}(\mathbf{x})) \dot{\mathbf{x}} + \zeta \mathcal{D}(\mathbf{x}) \dot{\mathbf{y}} + \mathbf{K}_1 \mathbf{x} + \mathcal{F}\left(\mathbf{x}, \frac{\mathbf{y}}{\epsilon}\right) + \mathcal{G}(\mathbf{x}) &= \alpha \mathbf{q}(\tau), \\ \mathbf{M}_2 \ddot{\mathbf{y}} + \frac{\zeta}{\epsilon} \mathbf{K}_2 \dot{\mathbf{y}} + \zeta \mathcal{E}(\mathbf{x}) \dot{\mathbf{x}} + \frac{1}{\epsilon} \mathbf{K}_2 \frac{\mathbf{y}}{\epsilon} + \frac{1}{\epsilon} \mathcal{H}(\mathbf{x}) &= \beta \mathbf{p}(\tau). \end{aligned} \quad (3.8)$$

The mass-normalized forcing terms are defined in terms of the new variable $\boldsymbol{\eta} = \frac{\mathbf{y}}{\epsilon}$ as

$$\begin{aligned} \mathbf{P}_1(\mathbf{x}, \dot{\mathbf{x}}, \boldsymbol{\eta}, \dot{\mathbf{y}}, \tau; \epsilon) &= -\mathbf{M}_1^{-1} (\zeta \epsilon (\mathbf{K}_1 + \mathcal{C}(\mathbf{x})) \dot{\mathbf{x}} + \zeta \mathcal{D}(\mathbf{x}) \dot{\mathbf{y}} + \mathbf{K}_1 \mathbf{x} + \mathcal{F}(\mathbf{x}, \boldsymbol{\eta}) + \mathcal{G}(\mathbf{x}) - \alpha \mathbf{q}(\tau)), \\ \mathbf{P}_2(\mathbf{x}, \dot{\mathbf{x}}, \boldsymbol{\eta}, \dot{\mathbf{y}}, \tau; \epsilon) &= -\epsilon \mathbf{M}_2^{-1} \left(\frac{\zeta}{\epsilon} \mathbf{K}_2 \dot{\mathbf{y}} + \zeta \mathcal{E}(\mathbf{x}) \dot{\mathbf{x}} + \frac{1}{\epsilon} \mathbf{K}_2 \boldsymbol{\eta} + \frac{1}{\epsilon} \mathcal{H}(\mathbf{x}) - \beta \mathbf{p}(\tau) \right) \\ &= -\mathbf{M}_2^{-1} (\zeta \mathbf{K}_2 \dot{\mathbf{y}} + \epsilon \zeta \mathcal{E}(\mathbf{x}) \dot{\mathbf{x}} + \mathbf{K}_2 \boldsymbol{\eta} + \mathcal{H}(\mathbf{x}) - \epsilon \beta \mathbf{p}(\tau)). \end{aligned} \quad (3.9)$$

The main conditions of the SFD procedure, as derived by Haller & Ponsioen [50], are the following.

- (A1) **Non-singular extension to $\epsilon = 0$:** $\mathbf{P}_1$ and $\mathbf{P}_2$ should possess smooth (at least C^2) extension to their respective $\epsilon = 0$ limits, which is the case here by (3.9).
- (A2) **Existence of a critical manifold:** The algebraic equation $\mathbf{P}_2(\mathbf{x}, \dot{\mathbf{x}}, \boldsymbol{\eta}, \mathbf{0}, \tau; 0) \equiv \mathbf{0}$ should be solvable for $\boldsymbol{\eta}$ on an open, bounded domain. Indeed, for any $\mathcal{D}_0 \subset \mathbb{R}^{n_s} \times \mathbb{R}^{n_s} \times \mathcal{T}$, open and bounded, the set $\mathcal{M}_0(\tau)$ defined by

$$\boldsymbol{\eta} = \mathbf{G}_0(\mathbf{x}, \dot{\mathbf{x}}, \tau) := -\mathbf{K}_2^{-1} \mathcal{H}(\mathbf{x})$$

satisfies $\mathbf{P}_2(\mathbf{x}, \dot{\mathbf{x}}, \mathbf{G}_0(\mathbf{x}, \dot{\mathbf{x}}, \tau), \mathbf{0}, \tau; 0) \equiv \mathbf{0}$ for all $(\mathbf{x}, \dot{\mathbf{x}}, \tau) \in \mathcal{D}_0$, forming a critical manifold $\mathcal{M}_0$ in the language of singular perturbation theory. Clearly, the critical manifold is independent

of τ in our current setting. If, however, we assumed that $\beta = \mathcal{O}(\frac{1}{\epsilon})$ (i.e., the beam is excited in the axial direction by forces that are an order of magnitude larger than those in the transverse direction), then we would obtain a time-dependent critical manifold.

Notation: For clarity, we denote any expression $(\bullet)$ evaluated over the critical manifold $\mathcal{M}_0$ by $(\bullet) := (\bullet)|_{\eta=\mathbf{G}_0(\mathbf{x}, \dot{\mathbf{x}}, \tau), \dot{\mathbf{y}}=\mathbf{0}, \epsilon=0}$

(A3) Asymptotic stability of the critical manifold: With the matrices

$$\begin{aligned} \mathbf{A}(\mathbf{x}, \dot{\mathbf{x}}, \tau) &= -\partial_{\dot{\mathbf{y}}} \mathbf{P}_2(\mathbf{x}, \dot{\mathbf{x}}, \mathbf{G}_0(\mathbf{x}, \dot{\mathbf{x}}, \tau), \mathbf{0}, \tau; 0) = -\overline{\partial_{\dot{\mathbf{y}}} \mathbf{P}_2} = \zeta \mathbf{M}_2^{-1} \mathbf{K}_2, \\ \mathbf{B}(\mathbf{x}, \dot{\mathbf{x}}, \tau) &= -\partial_{\eta} \mathbf{P}_2(\mathbf{x}, \dot{\mathbf{x}}, \mathbf{G}_0(\mathbf{x}, \dot{\mathbf{x}}, \tau), \mathbf{0}, \tau; 0) = -\overline{\partial_{\eta} \mathbf{P}_2} = \mathbf{M}_2^{-1} \mathbf{K}_2, \end{aligned} \quad (3.10)$$

the equilibrium solution $\eta \equiv \mathbf{0} \in \mathbb{R}^{n_f}$ of the unforced, constant-coefficient linear system

$$\eta'' + \mathbf{A}(\mathbf{x}, \dot{\mathbf{x}}, \tau) \eta' + \mathbf{B}(\mathbf{x}, \dot{\mathbf{x}}, \tau) \eta = \mathbf{0} \Leftrightarrow \mathbf{M}_2 \eta'' + \zeta \mathbf{K}_2 \eta' + \mathbf{K}_2 \eta = \mathbf{0} \quad (3.11)$$

should be asymptotically stable for all fixed parameter values $(\mathbf{x}, \dot{\mathbf{x}}, \tau) \in \mathcal{D}_0$. This is again satisfied in our setting since $\mathbf{M}_2, \mathbf{K}_2$ are positive definite matrices and $\zeta > 0$.

3.3.2 Global reduced-order model from SFD

As shown by Haller & Ponsioen [50], assumptions (A1)-(A3) guarantee that the critical manifold $\mathcal{M}_0(\tau)$ perturbs into a nearby attracting slow manifold $\mathcal{M}_{\epsilon}(\tau)$ for $\epsilon > 0$ small enough. On this slow manifold, the discretized beam system (3.7) admits an exact reduced order model given by

$$\ddot{\mathbf{x}} - \overline{\mathbf{P}_1} - \epsilon [\overline{\partial_{\eta} \mathbf{P}_1} \mathbf{G}_1(\mathbf{x}, \dot{\mathbf{x}}, \tau) + \overline{\partial_{\dot{\mathbf{y}}} \mathbf{P}_1} \mathbf{H}_0(\mathbf{x}, \dot{\mathbf{x}}, \tau) + \overline{\partial_{\epsilon} \mathbf{P}_1}] + \mathcal{O}(\epsilon^2) = \mathbf{0}, \quad (3.12)$$

where

$$\begin{aligned} \mathbf{H}_0(\mathbf{x}, \dot{\mathbf{x}}, \tau) &= [\partial_{\mathbf{x}} \mathbf{G}_0(\mathbf{x}, \dot{\mathbf{x}}, \tau)] \dot{\mathbf{x}} + [\partial_{\dot{\mathbf{x}}} \mathbf{G}_0(\mathbf{x}, \dot{\mathbf{x}}, \tau)] \overline{\mathbf{P}_1} + \partial_{\tau} \mathbf{G}_0(\mathbf{x}, \dot{\mathbf{x}}, \tau), \\ \mathbf{G}_1(\mathbf{x}, \dot{\mathbf{x}}, \tau) &= [\overline{\partial_{\eta} \mathbf{P}_2}]^{-1} \overline{\partial_{\dot{\mathbf{y}}} \mathbf{P}_2} \mathbf{H}_0(\mathbf{x}, \dot{\mathbf{x}}, \tau) \end{aligned}$$

constitute the higher-order terms in the equations describing the slow manifold $\mathcal{M}_{\epsilon}(\tau)$ as

$$\begin{aligned} \mathbf{y} &= \epsilon \mathbf{G}_0(\mathbf{x}, \dot{\mathbf{x}}, \tau) + \epsilon^2 \mathbf{G}_1(\mathbf{x}, \dot{\mathbf{x}}, \tau) + \mathcal{O}(\epsilon^3), \\ \dot{\mathbf{y}} &= \epsilon \mathbf{H}_0(\mathbf{x}, \dot{\mathbf{x}}, \tau) + \mathcal{O}(\epsilon^2). \end{aligned} \quad (3.13)$$

In the context of the beam example considered here, the exact reduced-order model in (3.12) can be explicitly written out in the following form:

$$\begin{aligned} &\mathbf{M}_1 \ddot{\mathbf{x}} + \mathbf{K}_1 \mathbf{x} + \mathcal{F}(\mathbf{x}, \mathbf{G}_0(\mathbf{x}, \dot{\mathbf{x}}, \tau)) + \mathcal{G}(\mathbf{x}) + \\ &\epsilon \left[\underbrace{\overline{\partial_{\eta} \mathcal{F}(\mathbf{x}, \eta)} \mathbf{G}_1(\mathbf{x}, \dot{\mathbf{x}}, \tau)}_{\text{conservative correction}} + \underbrace{\zeta (\mathcal{D}(\mathbf{x}) \mathbf{H}_0(\mathbf{x}, \dot{\mathbf{x}}, \tau) + (\mathbf{K}_1 + \mathcal{C}(\mathbf{x})) \dot{\mathbf{x}})}_{\text{damping terms}} \right] + \mathcal{O}(\epsilon^2) = \alpha \mathbf{q}(\tau), \end{aligned} \quad (3.14)$$

where

$$\begin{aligned} \mathbf{G}_0(\mathbf{x}, \dot{\mathbf{x}}, \tau) &= -\mathbf{K}_2^{-1} \mathcal{H}(\mathbf{x}), \\ \mathbf{H}_0(\mathbf{x}, \dot{\mathbf{x}}, \tau) &= -\mathbf{K}_2^{-1} [\partial_{\mathbf{x}} \mathcal{H}(\mathbf{x})] \dot{\mathbf{x}}, \\ \mathbf{G}_1(\mathbf{x}, \dot{\mathbf{x}}, \tau) &= -\zeta (\mathbf{H}_0(\mathbf{x}, \dot{\mathbf{x}}, \tau) + \mathbf{K}_2^{-1} \mathcal{E}(\mathbf{x}) \dot{\mathbf{x}}) + \beta \mathbf{K}_2^{-1} \mathbf{p}(\tau). \end{aligned} \quad (3.15)$$

Note that the reduced order model (3.14) is conservative (contains only inertial and elastic force terms) at leading order, whereas the full system (3.7) is dissipative due to viscoelastic damping. The $\mathcal{O}(\epsilon)$ terms in the ROM account for these damping contributions and hence cannot be ignored. Apart from the damping contributions, there also exists a conservative correction at the $\mathcal{O}(\epsilon)$ level which includes the static response of the $\mathbf{y}$ variables to the corresponding loading $\beta \mathbf{p}(\tau)$ (cf. the expression for $\mathbf{G}_1(\mathbf{x}, \dot{\mathbf{x}}, \tau)$ in (3.15)).

3.3.3 Specific results from SFD

We now consider a specific beam with geometric and material parameters as follows: length $L = 1$ m; thickness to length ratio ϵ in the range from $10^{-4} - 10^{-2}$; Young's modulus $E = 70$ GPa; density $\rho = 2700$ Kg/m³; material viscous damping rate $\kappa = 10^8$ Pa s. The beam is discretized using 20 finite elements and is subject to simply supported boundary conditions with rotation free and displacements (u, w) fixed at both ends. We use a spatially uniform load on the beam in the axial and in the transverse direction, given by $\alpha = 1$, $\beta = 1$, $q(x, \tau) = p(x, \tau) = \sin(\Omega T_0 \tau)$ (cf. Figure 3.1). Here $T_0 = \frac{L}{\epsilon} \sqrt{\frac{\rho}{E}}$ is the constant used to nondimensionalize time and Ω is the loading frequency, chosen to be the first natural frequency of the beam in this case. Using these parameters, we obtain $\zeta \approx 7.2739$.

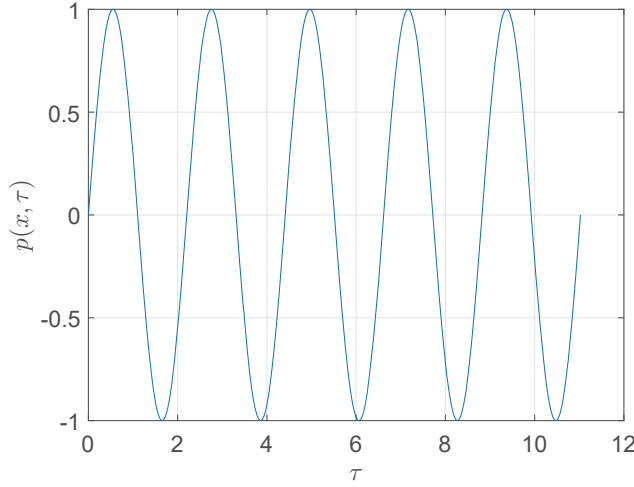
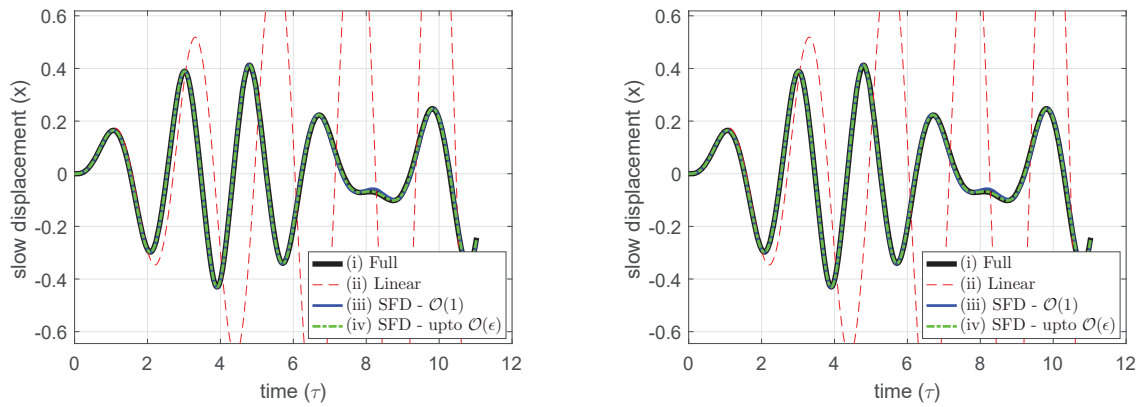


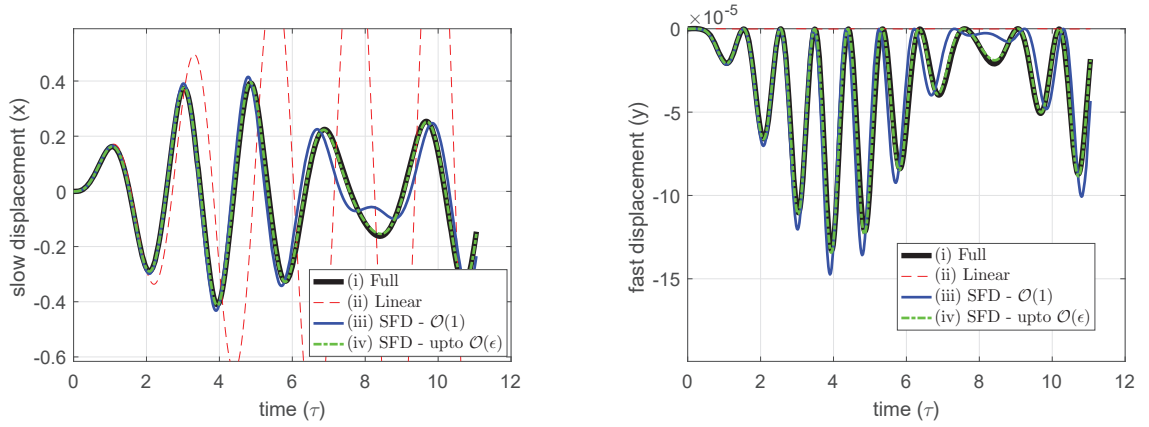
Figure 3.1: The applied loading



(a) Displacement [-] of the beam at quarter length in the transverse direction (slow x DOF).

(b) Displacement [-] of the beam at quarter length in the transverse direction (slow x DOF).

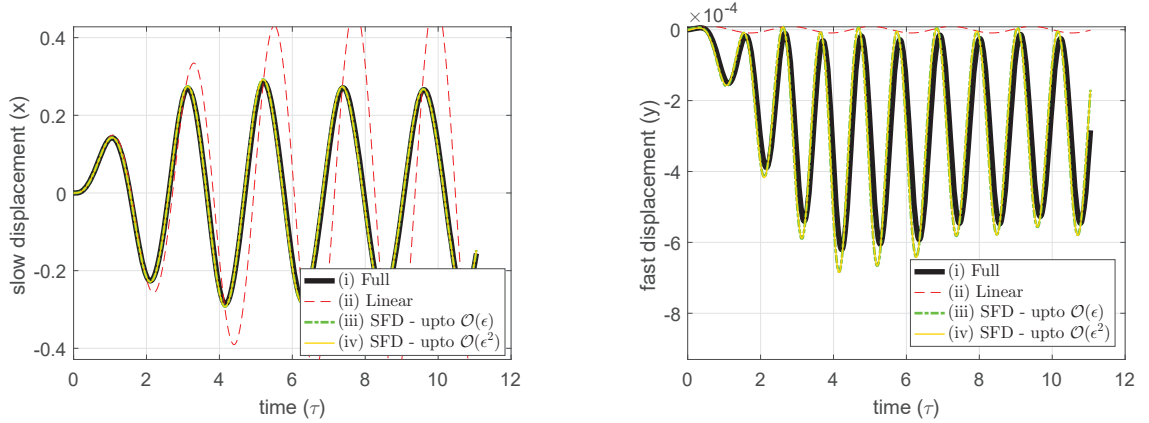
Figure 3.2: Comparison of the slow and fast components of the reduced solution with their full nonlinear and linearized counterparts for $\epsilon = 10^{-4}$. Note that for such small values of ϵ , the ROM at the leading order (containing only conservative terms) is a good enough representation of the full system, and is practically identical to ROMs obtained by inclusion of the $\mathcal{O}(\epsilon)$ and $\mathcal{O}(\epsilon^2)$ terms.



(a) Displacement [-] of the beam at quarter length in the transverse direction (slow x DOF).

(b) Displacement [-] of the beam at quarter length in the axial direction (fast y DOF).

Figure 3.3: Comparison of the slow and fast components of the reduced solution with their full nonlinear and linearized counterparts for $\epsilon = 10^{-3}$. For this larger value of ϵ , the leading-order conservative ROM is not accurate enough, and hence $\mathcal{O}(\epsilon)$ terms (which include damping contributions) are required to improve accuracy.



(a) Displacement [-] of the beam at quarter length in the transverse direction (slow x DOF).

(b) Displacement [-] of the beam at quarter length in the axial direction (fast y DOF).

Figure 3.4: Comparison of the slow and fast components of the reduced solution with their full nonlinear and linearized counterparts for $\epsilon = 10^{-2}$. The $\mathcal{O}(\epsilon^2)$ terms were added in the ROM to improve accuracy (especially in the slave variables), but only a marginal improvement was observed.

The graphs shown in Figures 3.2-3.4 compare the SFD-reduced and the full nonlinear response at a specific node of the beam in time for different values of the parameter ϵ . In particular, for $\epsilon = 10^{-4}$, we see from Figure 3.2 that the leading order approximation from SFD, is capable of reproducing the full system response. From previous discussion, we know that the reduced system is conservative at the leading order (cf. eq. (3.15)). This shows that for sufficiently thin beams, neglecting visco-elastic damping effects would be a reasonable approximation. For higher values of epsilon, this doesn't hold any more, as is clear from Figure 3.3 for $\epsilon = 10^{-2}$, where the reduction at the leading order (SFD- $\mathcal{O}(1)$) is not a good approximation of the full system response. However, adding the $\mathcal{O}(\epsilon)$ terms in the ROM, does give a good agreement with the full system response.

In order to check the performance of SFD globally, we use the following measure,

$$E = \frac{\sqrt{\sum_{\tau \in S} (\mathbf{u}(\tau) - \tilde{\mathbf{u}}(\tau))^T (\mathbf{u}(\tau) - \tilde{\mathbf{u}}(\tau))}}{\sqrt{\sum_{\tau \in S} \mathbf{u}(\tau)^T \mathbf{u}(\tau)}} \times 100\%, \quad (3.16)$$

where $\mathbf{u}(\tau) \in \mathbb{R}^{n_s+n_f}$ is the full vector of generalized displacements at time τ , obtained from the full nonlinear solution; $\tilde{\mathbf{u}}(\tau) \in \mathbb{R}^{n_s+n_f}$ is the solution based on the reduced model; and S is the set of time instants at which the error is recorded. The error recorded for different cases is shown in Table 3.1. We observe that for ϵ of order 10^{-4} , the ROM at the leading order (which is conservative) provides a close approximation for the full system. The $\mathcal{O}(\epsilon)$ terms, however, become important when ϵ is of order 10^{-3} , because they account for damping which becomes more significant in this range. We further calculated the $\mathcal{O}(\epsilon^2)$ terms in the ROM to compare changes in accuracy (see 3.A for general calculation of the $\mathcal{O}(\epsilon^2)$ terms). We observe that the $\mathcal{O}(\epsilon^2)$ terms only provide a marginal improvement in terms of accuracy in our case. A further increase in ϵ values to the order of 10^{-1} renders the reduced model highly inaccurate, thus higher-order terms in ϵ are necessary to achieve the desired accuracy. Note, however, the ϵ values around 10^{-1} are rather unphysical; the parameter ϵ , denoting the thickness-to-length ratio of a *beam*, is not expected attain such high values.

ϵ	SFD - $\mathcal{O}(1)$	SFD - up to $\mathcal{O}(\epsilon)$	SFD - up to $\mathcal{O}(\epsilon^2)$
10^{-4}	3.7078	0.3670	0.3670
10^{-3}	29.589	2.9681	2.9681
10^{-2}	> 100	5.3432	5.3425
10^{-1}	> 100	> 100	> 100

Table 3.1: Relative reduction error E calculated according to formula (3.16)

3.4 SSM reduction

Though the SFD reduction is robust and globally valid in the phase space, the reduced system on the slow manifold still contains n_s degrees of freedom, which is generally a large number. Specifically, due to the choice of the shape functions for discretization, the reduced model (3.14) obtained from SFD still contains two-thirds of the total number of unknowns of the full system (3.7). The following eigenvalue analysis of the linearized reduced system, however, reveals the existence of a further separation in time scales within the slow manifold near the origin, showing potential for a further reduction via spectral submanifolds [49].

3.4.1 Separation in damping: spectral quotient analysis

The eigenvalue problem for the linearization of the reduced system (3.14) can be formulated as

$$(\lambda_k^2 \mathbf{M}_1 + \lambda_k \zeta \epsilon \mathbf{K}_1 + \mathbf{K}_1) \phi_k = \mathbf{0}, \quad (3.17)$$

where $\lambda_k \in \mathbb{C}$ and $\phi_k \in \mathbb{C}^{n_s}$ are the k^{th} eigenvalue and eigenvector for any $k \in \{1, \dots, n_s\}$. The negative real part of the eigenvalue λ_k represents the exponential rate of decay of trajectories towards the equilibrium position along the corresponding two-dimensional eigenspace of the linearized system. The smoothest local extension of such an invariant subspace is known as the spectral submanifold (SSM), a notion introduced by Haller & Ponsioen [49]. Tangent to a slow subspace (i.e., the subspace spanned by eigenvectors corresponding to eigenvalues with the lowest-magnitude real parts), such

an SSM offers an opportunity for a further, drastic reduction from the SFD-based slow manifold to a two-dimensional invariant manifold.

Estimating the real parts of the eigenvalues arising from eq. (3.17) is sensitive to the choice of the eigensolver algorithm, especially for large n_s values. An accurate first-order approximation to the eigenvalues, however, can be obtained using the undamped eigenvalues from their defining equation

$$(\mathbf{K}_1 - \omega_{0k}^2 \mathbf{M}_1) \phi_{0k} = \mathbf{0}.$$

Specifically, for small damping and well-separated eigenvalues, a reliable first-order approximation for the real parts of the eigenvalues of system (3.17) can be obtained as follows (cf. G  radin & Rixen [45]):

$$\lambda_k = -\frac{\phi_{0k}^T (\zeta \epsilon \mathbf{K}_1) \phi_{0k}}{2 \phi_{0k}^T \mathbf{M}_1 \phi_{0k}} + i \omega_{0k}^2 + \mathcal{O}(\epsilon^2) = -\frac{\zeta \epsilon}{2} \omega_{0k}^2 + i \omega_{0k}^2 + \mathcal{O}(\epsilon^2) \quad \forall k \in \{1, \dots, n_s\}. \quad (3.18)$$

Based on this formula, the modal frequencies, real parts of the eigenvalues, and the ratios of the subsequent real parts are shown as a function of the mode number in Figure 3.1.

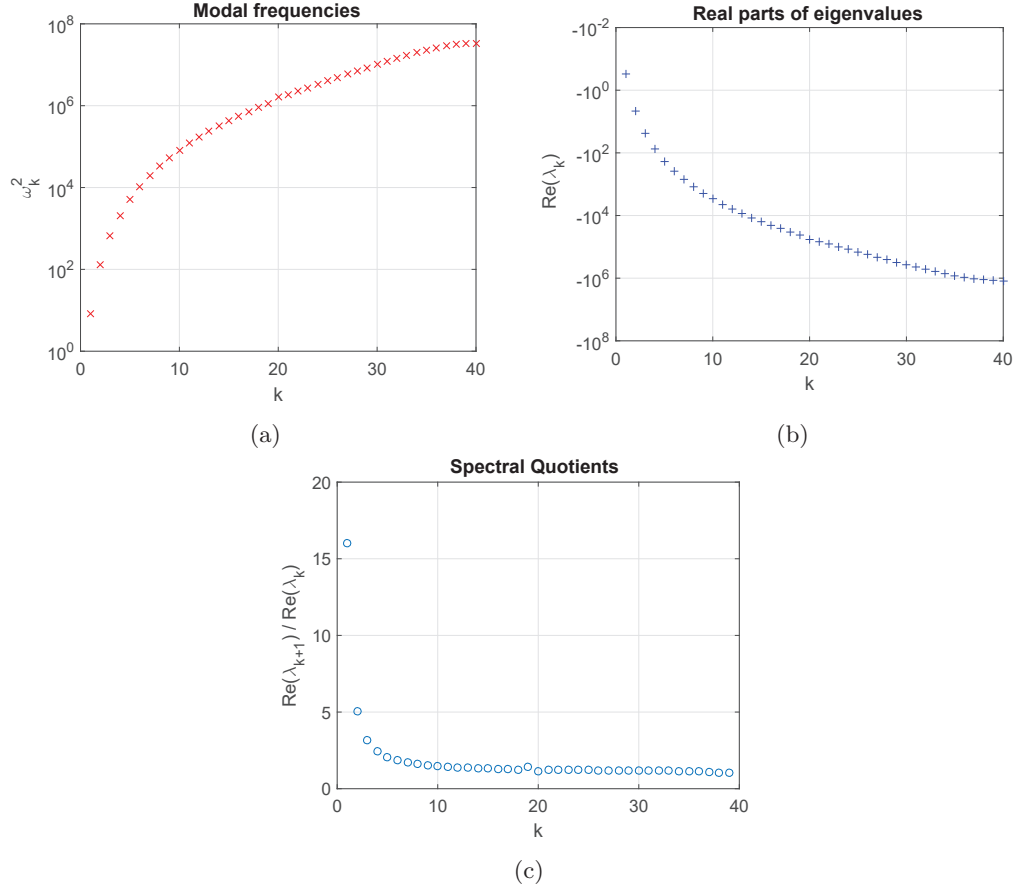


Figure 3.1: Spectral quotient analysis: the approximate ratio (3.18) between the real parts of successive eigenvalues of the system (c) reveals the spectral quotient gaps in the system. The first few eigenmodes form a subspace much slower than the rest and hence are relevant for the SSM reduction. No such large spectral gaps can be identified from the plots of the undamped natural frequencies (a) and real parts of the eigenvalues (b).

Note that the first eigenspace is about 16 times slower in terms of its exponential decay rate than the second slowest eigenspace, which renders the first mode an optimal choice for reduction using its corresponding SSM. We will, therefore, use this mode to perform a single-mode SSM reduction for the beam model, as discussed below.

3.4.2 SSM-based reduction

In order to perform a further reduction of the reduced system (3.14) obtained from SFD, we now compute the slowest single-mode SSM, whose reduced dynamics is given by a two-dimensional ODE as a final reduced-order model. Szalai *et al.* [116] have obtained general formulas for such one-mode SSMs and their associated backbone curves for autonomous dynamical systems (no external forcing in our setting). The reduced-order model obtained in this fashion is given in terms of an internal parametrization of the SSM, rather than a projection of the SSM on its underlying modal subspace. This construct allows the SSM to be recovered more globally, even if it develops a fold over the underlying modal subspace.

For this general formulation, we require the autonomous counterpart of system (3.14) to be in the phase space form

$$\underbrace{\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{x}} \end{bmatrix}}_{\dot{\tilde{\mathbf{z}}}} = \underbrace{\begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}_1^{-1}\mathbf{K}_1 & -\epsilon\zeta\mathbf{M}_1^{-1}\mathbf{K}_1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} \mathbf{x} \\ \dot{\mathbf{x}} \end{bmatrix}}_{\tilde{\mathbf{z}}} + \mathbf{t}(\mathbf{x}, \dot{\mathbf{x}}),$$

or $\dot{\tilde{\mathbf{z}}} = \mathbf{A}\tilde{\mathbf{z}} + \mathbf{t}(\tilde{\mathbf{z}}),$ (3.19)

where $\mathbf{t}(\mathbf{x}, \dot{\mathbf{x}})$ is a class C^r function representing the nonlinear terms in the ROM (3.14). By inspection of eq. (3.14), we find that $\mathbf{t}$ is a strictly cubic polynomial in $\tilde{\mathbf{z}}$ and hence belongs to the class C^a of analytic functions. The system (3.19) can be diagonalized as

$$\dot{\mathbf{z}} = \mathbf{\Lambda}\mathbf{z} + \mathcal{T}(\mathbf{z}), \quad \mathbf{z} \in \mathbb{C}^{2n_s}, \quad \mathbf{\Lambda} = \text{diag}(\lambda_1, \dots, \lambda_{2n_s}), \quad \lambda_{2i-1} = \bar{\lambda}_{2i}, \quad i \in \{1, \dots, n_s\}, \quad \mathcal{T}(\mathbf{z}) = \mathcal{O}(|\mathbf{z}|^3), \quad (3.20)$$

where $\lambda_i \in \mathbb{C}$ are the eigenvalues of the matrix $\mathbf{\Lambda}$ ordered such that $\text{Re}(\lambda_{n_s}) \leq \dots \leq \text{Re}(\lambda_1) < 0$. The matrix $\mathbf{P}$ of the linear transformation $\mathbf{z} = \mathbf{P}^{-1}\tilde{\mathbf{z}}$ leading to (3.20), contains the corresponding eigenvectors of $\mathbf{A}$.

The notion of an SSM was introduced by Haller & Ponsioen [49], more formally defined as follows:

Definition 3.1. A *spectral submanifold* (SSM), $W(\mathcal{E})$, corresponding to a spectral subspace $\mathcal{E}$ of $\mathbf{\Lambda}$ is an invariant manifold of the dynamical system (3.20) such that

- (i) $W(\mathcal{E})$ is tangent to $\mathcal{E}$ at the origin and has the same dimension as $\mathcal{E}$;
- (ii) $W(\mathcal{E})$ is strictly smoother than any other invariant manifold satisfying (i).

When $\mathcal{T}(\mathbf{z})$ is a C^r function, then under the assumptions

(B1) the *relative spectral quotient* $\sigma(\mathcal{E}) := \text{Int} \left[\frac{\min_{j \neq \ell, \ell+1} \text{Re}(\lambda_j)}{\text{Re}(\lambda_\ell)} \right]$ satisfies $\sigma(\mathcal{E}) \leq r$,

(B2) there are no resonances up to order $\sigma(\mathcal{E})$ between $\lambda_\ell, \bar{\lambda}_\ell$ and the rest of the spectrum of $\mathbf{\Lambda}$,

the existence and uniqueness of a class C^{r+1} SSM is guaranteed by the main theorem of Haller & Ponsioen [49], deduced from the more general results of Cabré *et al.* [23]. The theorem states that the SSM can be viewed as an embedding of an open set $\mathcal{U} \subset \mathbb{C}^2$ into the phase space of system

(3.20), described by a map $\mathbf{W} : \mathcal{U} \subset \mathbb{C}^2 \mapsto \mathbb{C}^{2n_s}$. Furthermore, there exists a quadratic polynomial mapping $\mathbf{R} : \mathcal{U} \mapsto \mathcal{U}$, such that the reduced dynamics on the SSM can be written as

$$\dot{\mathbf{s}} = \mathbf{R}(\mathbf{s}), \quad \mathbf{R}(\mathbf{s}) = \begin{bmatrix} \lambda_\ell z_\ell + \beta_\ell z_\ell^2 \bar{z}_\ell \\ \bar{\lambda}_\ell \bar{z}_\ell + \bar{\beta}_\ell z_\ell \bar{z}_\ell^2 \end{bmatrix}, \quad (3.21)$$

where $\mathbf{s} = (z_\ell, \bar{z}_\ell)$ are the coordinates for the mode ℓ (for which the SSM construction is performed).

In our beam setting, we obtain $\sigma(\mathcal{E}) = \frac{\omega_{0n_s}^2}{\omega_{0\ell}^2} + \mathcal{O}(\epsilon)$ from the eigenvalue approximation (3.18). This shows that the relative spectral quotient $\sigma(\mathcal{E})$ depends on the chosen discretization and monotonically increases with the number of discretization variables. This can be expected physically from a proportionally damped mechanical structure in which the damping for the modes is expected to increase monotonically with the corresponding oscillation frequency. Since $\mathbf{g} \in C^a$ here, we obtain that (B1) is satisfied. Furthermore, we assume generic parameter values under which the non-resonance requirement (B2) is satisfied by the beam system. The theorem of Haller & Ponsioen [49] then applies to the system (3.20), and the formulas obtained by Szalai *et al.* [116] yield the coefficients of $\mathbf{R}$ and the quadratic and cubic Taylor coefficients of $\mathbf{W}$. Specifically, let $\mathcal{T}$ be expanded in indicial notation as

$$(\mathcal{T}(\mathbf{z}))_i = \sum_{j=1}^{2n_s} \sum_{k=1}^{2n_s} \sum_{l=1}^{2n_s} T_{ijkl} z_j z_k z_l \quad \forall i, j, k, l \in 1, \dots, 2n_s, \quad (3.22)$$

and let $\mathbf{W}(\mathbf{s})$ be of the polynomial form

$$\mathbf{W}(\mathbf{s}) = \mathcal{W}^{(1)}(\mathbf{s}) + \mathcal{W}^{(2)}(\mathbf{s}) + \mathcal{W}^{(3)}(\mathbf{s}) + \dots, \quad (3.23)$$

where $\mathcal{W}^{(n)}(\mathbf{s})$ denotes the n^{th} order terms of $\mathbf{W}$.

The first-order terms in this expansion are given by

$$\mathcal{W}^{(1)}(\mathbf{s}) = \mathbf{W}^{(1)} \mathbf{s},$$

where $\mathbf{W}^{(1)} \in \mathbb{C}^{2n_s \times 2}$ is an all-zero matrix except for two non-zero entries given by $(\mathbf{W}^{(1)})_{1,\ell} = \lambda_\ell$, $(\mathbf{W}^{(1)})_{2,\ell+1} = \bar{\lambda}_{\ell+1}$. The i^{th} component of the quadratic terms can be written as

$$\left(\mathcal{W}^{(2)}(\mathbf{s}) \right)_i = \sum_{j=1}^2 \sum_{k=1}^2 W_{ijk}^{(2)} s_j s_k \equiv 0, \quad i \in \{1, \dots, 2n_s\}, \quad j, k \in \{1, 2\},$$

where $\mathbf{W}^{(2)} \in \mathbb{C}^{2n_s \times 2 \times 2}$ is an all-zero 3-tensor since $\mathcal{T}$ has no quadratic components. Finally, using the formulae derived by Szalai *et al.* [116], we can write the cubic terms as

$$\left(\mathcal{W}^{(3)}(\mathbf{s}) \right)_i = \sum_{j=1}^2 \sum_{k=1}^2 \sum_{l=1}^2 W_{ijkl}^{(3)} s_j s_k s_l, \quad i \in \{1, \dots, 2n_s\}, \quad j, k, l \in \{1, 2\},$$

where $\mathbf{W}^{(3)} \in \mathbb{C}^{2n_s \times 2 \times 2 \times 2}$ is a sparse 4-tensor with nonzero entries given as

$$\begin{aligned} W_{i111}^{(3)} &= \frac{T_{i\ell\ell\ell}}{3\lambda_\ell - \lambda_i}, \quad i \in \{1, \dots, 2n_s\}, \\ W_{i222}^{(3)} &= \frac{T_{i(\ell+1)(\ell+1)(\ell+1)}}{3\lambda_{\ell+1} - \lambda_i}, \quad i \in \{1, \dots, 2n_s\}, \\ W_{ijkl}^{(3)} &= (1 - \delta_{i\ell}) \frac{T_{ijkl}}{2\lambda_\ell + \bar{\lambda}_\ell - \lambda_i}, \quad i \in \{1, \dots, 2n_s\}, \quad (j, k, l) \in \{(\ell, \ell+1, \ell), (\ell, \ell, \ell+1), (\ell+1, \ell, \ell)\}, \\ W_{ijkl}^{(3)} &= (1 - \delta_{i(\ell+1)}) \frac{T_{ijkl}}{2\lambda_{\ell+1} + \bar{\lambda}_{\ell+1} - \lambda_i} \\ &\quad i \in \{1, \dots, 2n_s\}, \quad (j, k, l) \in \{(\ell+1, \ell, \ell+1), (\ell+1, \ell+1, \ell), (\ell, \ell+1, \ell+1)\}. \end{aligned}$$

Here $T_{ijkl} \in \mathbb{C}$ denotes the components of the 4-tensor in the expansion (3.22) of $\mathcal{T}$, and δ_{ij} represents the Kronecker-delta. The expression for β_ℓ used in the expansion (3.21) of $\mathbf{R}$ is given by

$$\beta_\ell = T_{\ell\ell\ell(\ell+1)} + T_{\ell\ell(\ell+1)\ell} + T_{\ell(\ell+1)\ell\ell},$$

as obtained from the general formulas of Szalai *et al.* [116], reproduced in Appendix 3.B. Note that the Einstein summation convention has *not* been followed in the above expressions. We finally express the reduced dynamics in the polar coordinates (ρ, θ) using a transformation $\mathcal{R} : \mathbb{C}^2 \mapsto \mathbb{R} \times \mathbb{S}^1$ such that $\mathbf{s} = \mathcal{R}^{-1}(\rho, \theta) = (\rho e^{i\theta}, \rho e^{-i\theta})$, given by

$$\begin{aligned} \dot{\rho} &= \rho(\operatorname{Re}\lambda_\ell + \operatorname{Re}\beta_\ell \rho^2), \\ \dot{\theta} &= \operatorname{Im}\lambda_\ell + \operatorname{Im}\beta_\ell \rho^2. \end{aligned} \quad (3.24)$$

3.4.3 Results

We now consider the SFD-reduced beam system (3.14) with the physically relevant parameter value $\epsilon = 10^{-3}$, and further reduce it to its slowest, two-dimensional SSM with $\ell = 1$. Let the modal coordinates be partitioned such that $Q_1(\mathbf{z}(\tau)) = (z_1(\tau), z_2(\tau))$ represents the displacement of master modes, and $Q_2(\mathbf{z}(\tau)) = (z_3, \dots, z_{2n_s})$ represents the rest of the slow modes obtained previously from the SFD. For the two-dimensional SSM over longer time-scales, free oscillations of the beam are expected to decay towards the origin. To illustrate this over longer time scales, we consider three sets of initial conditions.

1. **Initial condition on the SSM:** Starting with an initial condition on the two-dimensional SSM, we observe in Figure 3.2a that the full solution indeed stays on the SSM, thereby showing that the SSM is indeed invariant. Since the computed $\mathbf{W}$ is only a third-order approximation of the SSM, an initial condition far enough from the the fixed point may not stay over this approximate surface (cf. Figure 3.2b). Indeed, a higher-order approximation to $\mathbf{W}$ would be required to verify the invariance of $W(\mathcal{E})$ numerically for larger initial conditions.

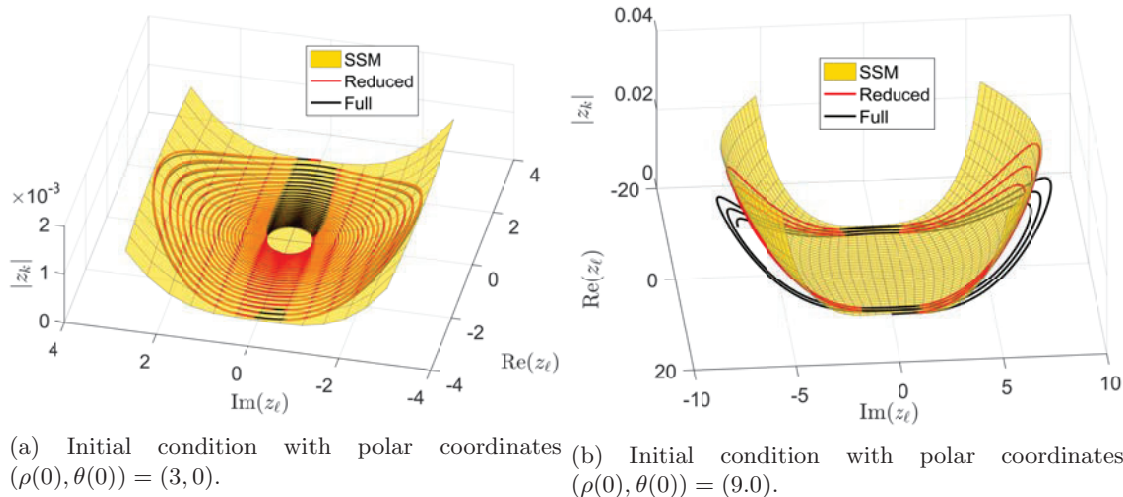


Figure 3.2: The SSM plotted in the modal coordinates $\mathbf{z}$. Full solution trajectory is shown for an initial condition taken on the SSM. Modal amplitude $|z_k|$ of the k^{th} mode ($k = 3$) is plotted against the master modal variables $(\operatorname{Re}(z_\ell), \operatorname{Im}(z_\ell))$ with $\ell = 1$. The full solution trajectory (a) stays approximately on the computed SSM, (b) does not stay exactly on the computed SSM for an initialization far enough from the fixed point.

2. **Initial conditions not lying on the SSM but still contained in the slow manifold:** We illustrate that the SSM is attracting nearby trajectories within the SFD-based slow manifold by launching a trajectory away from the SSM but still inside the slow manifold $\mathcal{M}_\epsilon(\tau)$. As shown in Figure 3.3, a trajectory of the full solution which starts at an arbitrarily chosen point in the phase space of system (3.19), close enough to the computed SSM, quickly converges towards the SSM and synchronizes with the flow on the SSM. Figure 3.4 shows that this rate of attraction towards the SSM is indeed faster than the typical decay rate within the SSM.

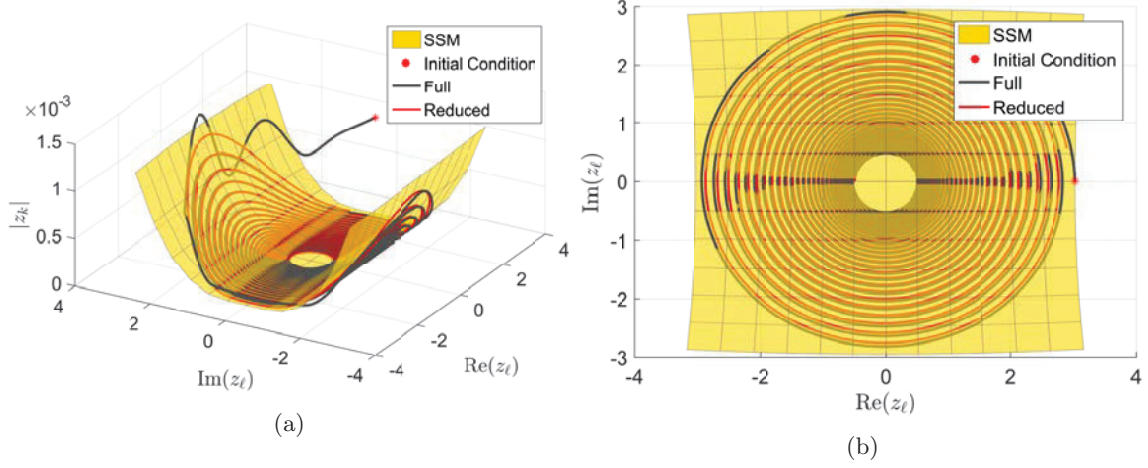


Figure 3.3: The SSM plotted in the modal coordinates $\mathbf{z}$. Full solution trajectory is shown for an initial condition taken off the SSM but still inside the slow manifold. Modal amplitude $|z_k|$ of the k^{th} mode ($k = 3$) is plotted against the master modal unknowns ($\text{Re}(z_\ell)$, $\text{Im}(z_\ell)$). The full solution trajectory (a) quickly decays onto the SSM, and (b) synchronizes with the dynamics on the SSM.

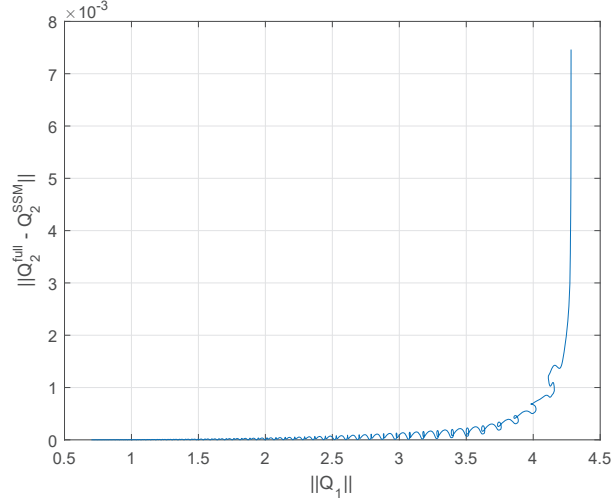


Figure 3.4: The exponential decay of trajectories towards the SSM within the slow manifold in the modal phase space: $|Q_2(\mathbf{z}) - Q_2(\mathbf{W}(\mathbf{s}))|$ vs. $|Q_1(\mathbf{z})|$. The decay rate transverse to the SSM is exponentially faster than that the typical rates along the SSM.

3. **Initial condition off the slow manifold:** Although the SSM is computed for the SFD-reduced system (3.14), the reduced dynamics on the SSM also captures the asymptotic behavior

of the full system by construction. Indeed, as seen in Figure 3.5, a trajectory initialized off the slow manifold in the phase space of system (3.7) (thus off the SSM as well) is also attracted towards the SSM, and synchronizes with the corresponding flow on the SSM. This synchronization can be seen in more detail in Figure 3.6, where the time histories of slow and fast variables are compared between the slow and full solution. As expected, the response of the fast degree of freedom, in particular, shows that the full solution performs rapid oscillations around, and stabilizes on, the reduced solution obtained from SFD. Finally, the full solution approaches the SSM-reduced solution.

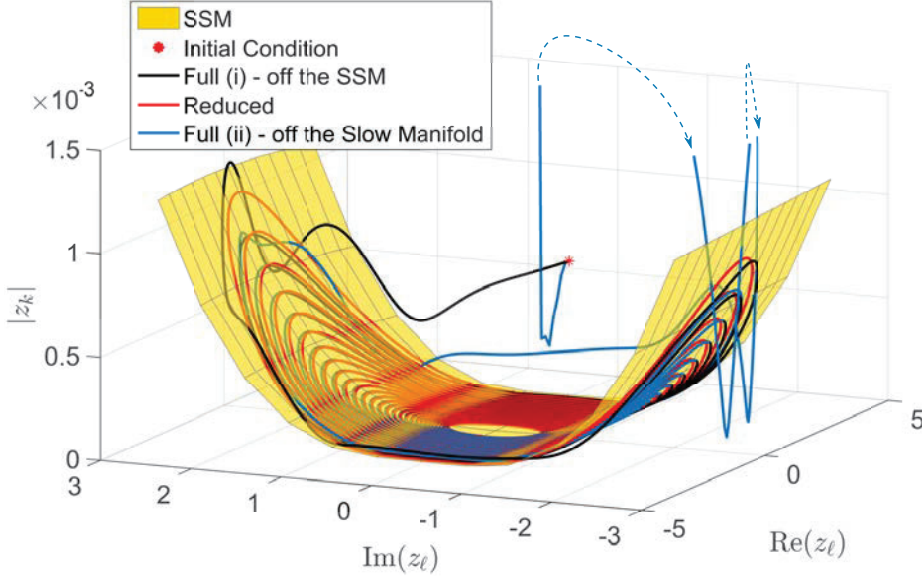


Figure 3.5: Modal amplitude $|z_k|$ of the k^{th} mode ($k = 3$) is plotted against the master modal variables ($\text{Re}(z_\ell)$, $\text{Im}(z_\ell)$). The full (ii) solution trajectory (blue) initialized off the slow manifold synchronizes with the reduced dynamics on the SSM. The full (i) solution trajectory (black) is initialized on the slow manifold by projecting the original initial condition onto the slow manifold. Note that in this plot, the two initial conditions appear to coincide, since the axes feature only the slow (modal) DOFs. The dashed lines are a schematic representation of the trajectory leaving and returning to the plot region.

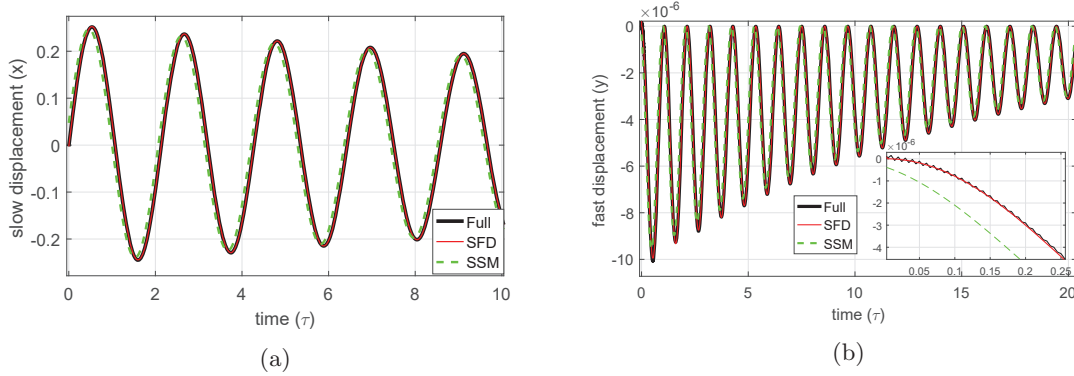


Figure 3.6: The comparison of full and different reduced solutions for (a) slow and (b) fast degrees of freedom. Note the two timescales in the zoom-in for the dynamics of the fast DOF. The full solution quickly decays to the slow manifold (with SFD-reduced solutions) after which it decays to the SSM.

Next, we study the rate of decay of the enslaved variables towards the computed SSM. From Figure 3.7, we see that the initial decay rate of a trajectory towards the SSM is approximately $e^{-0.4607\tau}$ which is actually dominated by the decay rate suggested by the second slowest eigenvalue, i.e., $\text{Re}(\lambda_2) \approx -0.4724$. The slow dynamics within the SSM is expected to occur at the rate suggested by the first eigenvalue, i.e., $\text{Re}(\lambda_1) \approx -0.0295$. The final decay-rate for the full solution is, however, a decay rate of $e^{-0.0891\tau}$ which is between the slowest and second slowest rates, still being an order of magnitude faster than the dynamics within the SSM, as expected from the underlying theory.

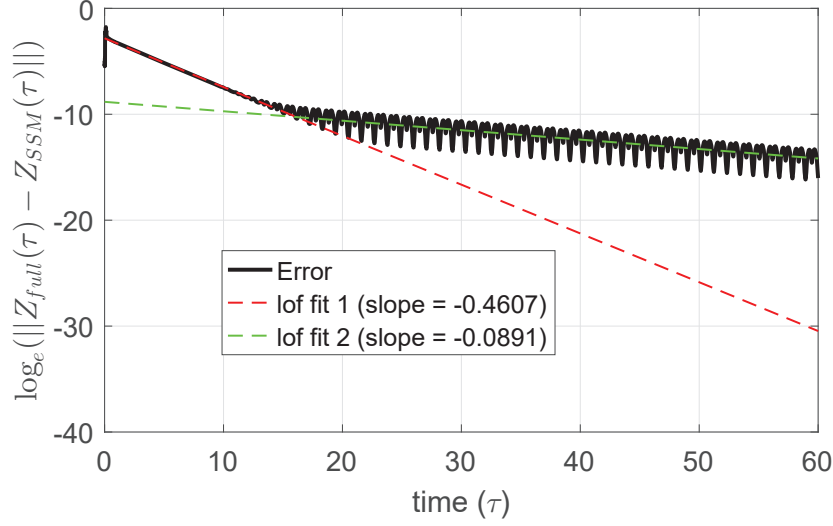


Figure 3.7: The logarithmic plot shows $\|\mathbf{z}(\tau) - \mathbf{W}(\mathbf{s}(\tau))\|$ vs. time. The two fits suggest initial and final decay rates of $e^{-0.4607\tau}$ and $e^{-0.0891\tau}$ approximately, respectively. The approximate decay rates along and transverse to the SSM near the equilibrium are given by $e^{-0.0295\tau}$ and $e^{-0.4724\tau}$, respectively (based on the eigenvalues of the linearized system).

3.5 Conclusions

We have demonstrated two exact model reduction techniques, the slow-fast decomposition (SFD) and spectral-submanifold (SSM)-based reduction, on the finite-element model of a von Kármán beam. The SFD enabled us to express the fast axial variables of the beam as a graph over the slow transverse variables, yielding a reduced-order model only in terms of the slow variables. This SFD-reduced model possesses a gap in its spectral quotient distribution, indicating an opportunity for further reduction using the SSM. Subsequently, we carried out a single-mode SSM-based reduction on the SFD-reduced model. This two-stage, exact reduction procedure resulted in a drastic reduction of the model dimension, and ensured that the full system trajectories synchronized with the reduced model trajectories at rates much faster than typical decay rates within the manifold.

The application of the above mentioned exact reduction techniques to our beam model confirms their potential for truly high-dimensional systems. However, significant work remains to be done to enable application of these techniques to simulations of realistic structures, especially on the computational implementation side. In particular, these techniques, in their present form, require the nonlinear coefficients of the system to be known a priori. For finite-element-based applications, often these nonlinear coefficients are embedded in the software at the element level and their full counterparts are never explicitly calculated during a simulation (for reasons of computational efficiency). This is certainly a computational challenge for the application of SFD/SSM-based reduction

techniques to real-world applications.

The current work features a single-mode SSM. This was justified in our beam example, because decay rates in the first mode were exceptionally slower than the rest. For more general damped-mechanical systems, however, we expect higher-dimensional SSMs (i.e., smoothest nonlinear continuations of high-dimensional linear modal subspaces) to be more relevant for reduction. Furthermore, the polynomial expansion of the SSM mapping contains terms only up to the third order in the current work. As also discussed in Section 4.3, higher-order terms would play an important role in capturing trajectories initialized farther from the fixed point. A MATLAB-based computational package SSMTTool [94] allows for the automatic computation of the higher-order terms in a single-mode SSM expansion. The capability to compute multi-mode SSMs with terms of up to arbitrary order is an important direction for future development.

This chapter covers the autonomous (non-forced) beam model. Existence and uniqueness results for SSMs are also available for non-autonomous, damped mechanical systems as discussed by Haller & Ponsioen [49]. For (quasi-periodically) forced systems, the SSM appears as a quasi-periodically deforming invariant surface in the phase space, which is again an ideal tool for model reduction. The computation of such non-autonomous SSMs will find useful application in non-autonomous mechanical systems for the computation of forced-response curves, and is subject to ongoing work [21, 95].

Appendix

3.A $\mathcal{O}(\epsilon^2)$ terms in the slow manifold

As shown by Haller & Ponsioen [50], the enslavement of the fast variables to the slow ones along the slow manifold is given by the functions

$$\begin{aligned}\mathbf{y} &= \epsilon \mathbf{G}_0(\mathbf{x}, \dot{\mathbf{x}}, \tau) + \epsilon^2 \mathbf{G}_1(\mathbf{x}, \dot{\mathbf{x}}, \tau) + \epsilon^3 \mathbf{G}_2(\mathbf{x}, \dot{\mathbf{x}}, \tau) + \mathcal{O}(\epsilon^4), \\ \dot{\mathbf{y}} &= \epsilon \mathbf{H}_0(\mathbf{x}, \dot{\mathbf{x}}, \tau) + \epsilon^2 \mathbf{H}_1(\mathbf{x}, \dot{\mathbf{x}}, \tau) + \epsilon^3 \mathbf{H}_2(\mathbf{x}, \dot{\mathbf{x}}, \tau) + \mathcal{O}(\epsilon^4),\end{aligned}$$

where $\mathbf{G}_0$, $\mathbf{G}_1$, $\mathbf{H}_0$ are as shown in eq. (3.15) and $\mathbf{H}_1$ is given by

$$\mathbf{H}_1(\mathbf{x}, \dot{\mathbf{x}}, \tau) = [\partial_{\mathbf{x}} \mathbf{G}_1] \dot{\mathbf{x}} + [\partial_{\dot{\mathbf{x}}} \mathbf{G}_1] \overline{\mathbf{P}_1} + \partial_{\tau} \mathbf{G}_1 + \partial_{\mathbf{x}} \mathbf{G}_0 (\overline{\partial_{\eta} \mathbf{P}_1} \mathbf{G}_1 + \overline{\partial_{\dot{\mathbf{y}}} \mathbf{P}_1} \mathbf{H}_0 + \overline{\partial_{\epsilon} \mathbf{P}_1}). \quad (3.25)$$

In order to obtain the $\mathcal{O}(\epsilon^2)$ terms in the reduced model, we have calculated the general expressions for $\mathbf{G}_2$ as

$$\mathbf{G}_2(\mathbf{x}, \dot{\mathbf{x}}, \tau) = [\overline{\partial_{\eta} \mathbf{P}_2}]^{-1} \left([\partial_{\mathbf{x}} \mathbf{H}_0] \dot{\mathbf{x}} + [\partial_{\dot{\mathbf{x}}} \mathbf{H}_0] \overline{\mathbf{P}_1} + \partial_{\tau} \mathbf{H}_0 - \left(\mathbf{J} \mathbf{G}_1 + \mathbf{L} \mathbf{H}_0 + \overline{\partial_{\epsilon}^2 \mathbf{P}_2} + \overline{\partial_{\dot{\mathbf{y}}} \mathbf{P}_2} \mathbf{H}_1 \right) \right), \quad (3.26)$$

where

$$\begin{aligned}\mathbf{J} &= \frac{1}{2} \left(\overline{\partial_{\eta}^2 \mathbf{P}_2} \mathbf{G}_1 + \overline{\partial_{\dot{\mathbf{y}}} \partial_{\eta} \mathbf{P}_2} \mathbf{H}_0 + \overline{\partial_{\epsilon} \partial_{\eta} \mathbf{P}_2} \right), \\ \mathbf{L} &= \frac{1}{2} \left(\overline{\partial_{\dot{\mathbf{y}}}^2 \mathbf{P}_2} \mathbf{H}_0 + \overline{\partial_{\eta} \partial_{\dot{\mathbf{y}}} \mathbf{P}_2} \mathbf{G}_1 + \overline{\partial_{\epsilon} \partial_{\dot{\mathbf{y}}} \mathbf{P}_2} \right).\end{aligned}$$

The reduced-order model including the $\mathcal{O}(\epsilon^2)$ terms is then given as

$$\ddot{\mathbf{x}} - \overline{\mathbf{P}_1} - \epsilon [\overline{\partial_{\eta} \mathbf{P}_1} \mathbf{G}_1 + \overline{\partial_{\dot{\mathbf{y}}} \mathbf{P}_1} \mathbf{H}_0 + \overline{\partial_{\epsilon} \mathbf{P}_1}] - \epsilon^2 [\mathbf{N} \mathbf{G}_1 + \overline{\partial_{\eta} \mathbf{P}_1} \mathbf{G}_2 + \mathbf{R} \mathbf{H}_0 + \overline{\partial_{\dot{\mathbf{y}}} \mathbf{P}_1} \mathbf{H}_1 + \overline{\partial_{\epsilon}^2 \mathbf{P}_1}] + \mathcal{O}(\epsilon^3) = \mathbf{0},$$

where

$$\begin{aligned}\mathbf{N} &= \frac{1}{2} \left(\overline{\partial_{\eta}^2 \mathbf{P}_1} \mathbf{G}_1 + \overline{\partial_{\dot{\mathbf{y}}} \partial_{\eta} \mathbf{P}_1} \mathbf{H}_0 + \overline{\partial_{\epsilon} \partial_{\eta} \mathbf{P}_1} \right), \\ \mathbf{R} &= \frac{1}{2} \left(\overline{\partial_{\dot{\mathbf{y}}}^2 \mathbf{P}_1} \mathbf{H}_0 + \overline{\partial_{\eta} \partial_{\dot{\mathbf{y}}} \mathbf{P}_1} \mathbf{G}_1 + \overline{\partial_{\epsilon} \partial_{\dot{\mathbf{y}}} \mathbf{P}_1} \right).\end{aligned}$$

In our beam framework, the general expressions in (3.25,3.26) become

$$\begin{aligned}\mathbf{H}_1(\mathbf{x}, \dot{\mathbf{x}}, \tau) &= \mathbf{K}_2^{-1} \left(\zeta [\partial_{\mathbf{x}} \mathcal{H}(\mathbf{x}) - \mathcal{E}(\mathbf{x})] \overline{\mathbf{P}_1} + \zeta [\partial_{\mathbf{x}}^2 \mathcal{H}(\mathbf{x}) - \partial_{\mathbf{x}} \mathcal{E}(\mathbf{x})] : (\dot{\mathbf{x}} \otimes \dot{\mathbf{x}}) + \beta \dot{\mathbf{p}} \right), \\ \mathbf{G}_2(\mathbf{x}, \dot{\mathbf{x}}, \tau) &= [\mathbf{K}_2 \mathbf{M}_2^{-1} \mathbf{K}_2]^{-1} \left([\partial_{\mathbf{x}}^2 \mathcal{H}(\mathbf{x})] : (\dot{\mathbf{x}} \otimes \dot{\mathbf{x}}) + [\partial_{\mathbf{x}} \mathcal{H}(\mathbf{x})] \overline{\mathbf{P}_1} \right) - \zeta \mathbf{H}_1.\end{aligned} \quad (3.27)$$

For the assumed visco-elastic material damping, the expressions for $\mathbf{G}_1(\mathbf{x}, \dot{\mathbf{x}}, \tau)$ in (3.15) and $\mathbf{H}_1(\mathbf{x}, \dot{\mathbf{x}}, \tau)$ in (3.27) can be further simplified to

$$\begin{aligned}\mathbf{G}_1(\mathbf{x}, \dot{\mathbf{x}}, \tau) &= \beta \mathbf{K}_2^{-1} \mathbf{p}(\tau), \\ \mathbf{H}_1(\mathbf{x}, \dot{\mathbf{x}}, \tau) &= \beta \mathbf{K}_2^{-1} \dot{\mathbf{p}}(\tau),\end{aligned}$$

and the reduced-order model can be simplified as

$$\begin{aligned}& \mathbf{M}_1 \ddot{\mathbf{x}} + \mathbf{K}_1 \mathbf{x} + \mathcal{F}(\mathbf{x}, \mathbf{G}_0(\mathbf{x}, \dot{\mathbf{x}}, \tau)) + \mathcal{G}(\mathbf{x}) + \\ & \epsilon \left[\overline{\partial_{\boldsymbol{\eta}} \mathcal{F}(\mathbf{x}, \boldsymbol{\eta})} \mathbf{G}_1(\mathbf{x}, \dot{\mathbf{x}}, \tau) + \zeta(\mathcal{D}(\mathbf{x}) \mathbf{H}_0(\mathbf{x}, \dot{\mathbf{x}}, \tau) + (\mathbf{K}_1 + \mathcal{C}(\mathbf{x})) \dot{\mathbf{x}} \right] + \\ & \epsilon^2 \left[\overline{\partial_{\boldsymbol{\eta}} \mathcal{F}(\mathbf{x}, \boldsymbol{\eta})} \mathbf{G}_2(\mathbf{x}, \dot{\mathbf{x}}, \tau) + \zeta(\mathcal{D}(\mathbf{x}) \mathbf{H}_1(\mathbf{x}, \dot{\mathbf{x}}, \tau)) \right] + \mathcal{O}(\epsilon^3) = \alpha \mathbf{q}(\tau).\end{aligned}\tag{3.28}$$

3.B Formulas for single-mode SSMs

We express the general formulas of Szalai *et al.* [116] for Taylor coefficients of the SSM mapping $\mathbf{W}(\mathbf{s})$ in our present notation. Let $\mathcal{T}(\mathbf{z})$ from (3.20) be a general polynomial of the form

$$\mathcal{T}(\mathbf{z}) = \mathcal{T}^{(2)}(\mathbf{z}) + \mathcal{T}^{(3)}(\mathbf{z}) + \dots,\tag{3.29}$$

where $\mathcal{T}^{(n)}(\mathbf{z})$ denotes the n^{th} order terms of $\mathcal{T}$. The first order terms in the expansion (3.23) are given by

$$\mathcal{W}^{(1)}(\mathbf{s}) = \mathbf{W}^{(1)} \mathbf{s},$$

where $\mathbf{W}^{(1)} \in \mathbb{C}^{2n_s \times 2}$ is an all-zero matrix except for two non-zero entries given by $(\mathbf{W}^{(1)})_{1,\ell} = \lambda_{\ell}$, $(\mathbf{W}^{(1)})_{2,\ell+1} = \lambda_{\ell+1}$. The i^{th} component of the quadratic terms can be written as

$$\left(\mathcal{W}^{(2)}(\mathbf{s}) \right)_i = \sum_{j=1}^2 \sum_{k=1}^2 W_{ijk}^{(2)} s_j s_k, \quad i \in \{1, \dots, 2n_s\}, \quad j, k \in \{1, 2\},$$

where $\mathbf{W}^{(2)} \in \mathbb{C}^{2n_s \times 2 \times 2}$ is a sparse 3-tensor with nonzero entries given as

$$\begin{aligned}W_{i11}^{(2)} &= \frac{T_{i\ell\ell}^{(2)}}{2\lambda_{\ell} - \lambda_i}, \quad i \in \{1, \dots, 2n_s\}, \\ W_{i22}^{(2)} &= \frac{T_{i(\ell+1)(\ell+1)}^{(2)}}{2\lambda_{\ell+1} - \lambda_i}, \quad i \in \{1, \dots, 2n_s\}, \\ W_{ijk}^{(2)} &= \frac{T_{ijk}^{(2)}}{\lambda_{\ell} + \bar{\lambda}_{\ell} - \lambda_i}, \quad i \in \{1, \dots, 2n_s\}, \quad (j, k) \in \{(\ell, \ell+1), (\ell+1, \ell)\}.\end{aligned}$$

Finally, the cubic terms can be written as

$$\left(\mathcal{W}^{(3)}(\mathbf{s}) \right)_i = \sum_{j=1}^2 \sum_{k=1}^2 \sum_{l=1}^2 W_{ijkl}^{(3)} s_j s_k s_l, \quad i \in \{1, \dots, 2n_s\}, \quad j, k, l \in \{1, 2\},$$

where $\mathbf{W}^{(3)} \in \mathbb{C}^{2n_s \times 2 \times 2 \times 2}$ is a sparse 4-tensor with nonzero entries given as

$$\begin{aligned} W_{i111}^{(3)} &= \frac{\sum_{j=1}^{2n_s} \left[(1 + \delta_{\ell j}) \left(T_{ij\ell}^{(2)} + T_{i\ell j}^{(2)} \right) W_{j11}^{(2)} \right] + T_{i\ell\ell\ell}^{(3)}}{3\lambda_\ell - \lambda_i}, \quad i \in \{1, \dots, 2n_s\}, \\ W_{i222}^{(3)} &= \frac{\sum_{j=1}^{2n_s} \left[(1 + \delta_{(\ell+1)j}) \left(T_{ij(\ell+1)}^{(2)} + T_{i(\ell+1)j}^{(2)} \right) W_{j22}^{(2)} \right] + T_{i(\ell+1)(\ell+1)(\ell+1)}^{(3)}}{3\lambda_{\ell+1} - \lambda_i}, \quad i \in \{1, \dots, 2n_s\}, \\ W_{ijk\ell}^{(3)} &= (1 - \delta_{i\ell}) \frac{V_i + T_{ijk\ell}^{(3)}}{2\lambda_\ell + \bar{\lambda}_\ell - \lambda_i}, \quad i \in \{1, \dots, 2n_s\}, \quad (j, k, \ell) \in \{(\ell, \ell+1, \ell), (\ell, \ell, \ell+1), (\ell+1, \ell, \ell)\}, \\ W_{ijk\ell}^{(3)} &= (1 - \delta_{i(\ell+1)}) \frac{U_i + T_{ijk\ell}^{(3)}}{2\lambda_{\ell+1} + \bar{\lambda}_{\ell+1} - \lambda_i}, \\ &\quad i \in \{1, \dots, 2n_s\}, \quad (j, k, \ell) \in \{(\ell+1, \ell, \ell+1), (\ell+1, \ell+1, \ell), (\ell, \ell+1, \ell+1)\}, \end{aligned}$$

with

$$\begin{aligned} V_i &= \sum_{j=1}^{2n_s} \left[(1 + \delta_{\ell j}) \left(T_{ij\ell}^{(2)} + T_{i\ell j}^{(2)} \right) \left(W_{j21}^{(2)} + W_{j12}^{(2)} \right) + (1 + \delta_{(\ell+1)j}) \left(T_{ij(\ell+1)}^{(2)} + T_{i(\ell+1)j}^{(2)} \right) W_{j11}^{(2)} \right], \quad i \in \{1, \dots, 2n_s\}, \\ U_i &= \sum_{j=1}^{2n_s} \left[(1 + \delta_{(\ell+1)j}) \left(T_{ij(\ell+1)}^{(2)} + T_{i(\ell+1)j}^{(2)} \right) \left(W_{j21}^{(2)} + W_{j12}^{(2)} \right) + (1 + \delta_{\ell j}) \left(T_{ij\ell}^{(2)} + T_{i\ell j}^{(2)} \right) W_{j22}^{(2)} \right], \quad i \in \{1, \dots, 2n_s\}. \end{aligned}$$

Here $T_{ijkl}^{(3)}, T_{ijk}^{(2)} \in \mathbb{C}$ denote the components of the 4-tensor and 3-tensor, respectively, in the expansion (3.29) of $\mathcal{T}$, and δ_{ij} represents the Kronecker-delta. Furthermore, the general expression for β_ℓ used in the expansion for $\mathbf{R}$ as in eq. (3.21), as derived by Szalai *et al.* [116], can be written in our notation as

$$\beta_\ell = V_\ell + T_{\ell\ell\ell(\ell+1)} + T_{\ell\ell(\ell+1)\ell} + T_{\ell(\ell+1)\ell\ell}.$$

Note that the Einstein summation convention has *not* been followed in any of the above expressions.

Chapter 4

Hyper-reduction over nonlinear manifolds

Abstract. Recent trends involve the use of nonlinear manifolds for model reduction. While there are many methods to identify such nonlinear manifolds, hyper-reduction techniques to accelerate computation in such ROMs are rare. In this chapter, we propose an extension to the popular hyper-reduction method, ECSW that was originally conceived in the context of linear-projection-based ROMs to allow for hyper-reduction using nonlinear mappings. At the same, we show that our extension retains the desirable stability and structure-preserving properties of ECSW. We demonstrate our method over models of a flat plate and a realistic wing structure, whose dynamics have been shown to evolve over a nonlinear (quadratic) manifold in Chapter 2. An online speed-up of over one thousand times relative to the full system was obtained for the wing structure using the proposed method, which is higher than its linear counterpart using the ECSW. This chapter closely follows the current author’s work in the original reference¹[60].

4.1 Introduction

Brief review of hyper-reduction

Despite assuming granted accuracy, linear projection-based ROMs are not able to deliver significant computational speedups, when applied to nonlinear systems. This is due to the fact that the computation of the reduced nonlinear terms at each solver iteration scales with the dimensionality of the HFM rather than with that of the reduction subspace. In the case of polynomial nonlinearities, one might consider pre-computing the reduced nonlinear term [16, 57, 84], leading to higher-order tensors. Typically, FE models may also contain rotational DOFs, where the associated nonlinearity cannot be pre-computed and stored in the form of tensors. Furthermore, for high polynomial degrees, the computations with higher-order tensors can become intractable, even more so than the HFM simulations [57]. For such cases, in the most general setting, the above-mentioned scalability issue is successfully alleviated by a range of *hyper-reduction* techniques, which aim at significantly reducing the cost of reduced nonlinear term evaluation by computing them only on a few, selected elements or nodes of the FE models, and cheaply approximating the missing information.

The Gappy POD method [39], first introduced for image reconstruction and later applied in dynamical systems [24, 25], was one of the first instances of hyper-reduction. However, as observed

¹Jain, S. & Tiso, P. Hyper-reduction over nonlinear manifolds for large nonlinear mechanical systems. *Journal of Computational and Nonlinear Dynamics* **14**, 081008. doi:[10.1115/1.4043450](https://doi.org/10.1115/1.4043450) (2019).

by Farhat *et al.* [41], the term hyper-reduction was introduced much later in [108], where a selection of the integration point in an FE model is performed in an a priori manner. The empirical interpolation method (EIM) of Barrault *et al.* [17] approximates a non-affine function using a set of basis functions over a continuous, bounded domain, and hence finds applications for reduced basis representations of parametrized partial differential equations (PDEs). The discrete empirical interpolation method (DEIM) of Chaturantabut & Sorensen [28] achieves this on a discrete state space. For FE applications, the unassembled DEIM (UDEIM) by Tiso & Rixen [123] was shown to be more efficient, where the unassembled element-level nonlinear force vector is used in the DEIM algorithm to return a small set of elements (instead of nodes) over which the nonlinearity is evaluated. For conservative/port-Hamiltonian systems, however, the DEIM (or UDEIM) leads to a loss of numerical stability during time integration [42]. Though a symmetrized version of DEIM has been proposed recently by Chaturantabut *et al.* [27] to avoid this issue, its potential for FE-based applications remains unexplored. To this end, the energy-conserving sampling and weighting (ECSW) hyper-reduction method [41] is remarkable. The ECSW method directly approximates the *reduced* nonlinear term and is particularly well-suited for FE-based structural dynamics applications, as it preserves the symmetry of the operators involved and leads to numerical stability.

The main idea of ECSW is simple yet powerful: only a few elements are interrogated to compute the nonlinear forces, and weights (obtained using training) are attached to them to match the virtual work done by their full counterparts onto virtual displacements given by the linear reduction basis vectors. Albeit being efficient and relevant for FE-based mechanical systems, the ECSW hyper-reduction method is proposed in the context of *linear* mappings from the space of reduced variables. To the best of our knowledge, this is a common feature of all other hyper-reduction techniques in the literature, as well.

Nonlinear manifolds in model reduction

It is quite natural, however, for the dynamics of general nonlinear systems to evolve on inherently nonlinear manifolds, and not over linear subspaces (see Chapter 1 for a discussion). Often, a nonlinear manifold relevant for system dynamics may not be embeddable in a low-dimensional linear subspace. It is then easy to see that the linear-mapping-based ROM would either fail to capture the system response or require frequent basis-updates (online) to account for the nonlinearity of the manifold, making the reduction process cumbersome and possibly redundant due to high computational costs in setting up such a ROM. Many methods allow for model reduction using attracting nonlinear manifolds in nonlinear dynamical systems – from a theoretical, as well as a data-driven, statistical perspective.

The approximate inertial manifold (AIM) provides an approximation of the solution of a PDE over sufficiently long time scales, using the theory of inertial manifolds. This is done by enslaving the amplitudes of fast modes in the system as a smooth graph over the amplitudes of a few selected modes while assuming that the corresponding inertial forces are not excited (see [34, 82], for instance). The so-called static condensation approach (cf. Mignolet *et al.* [84] for a review) achieves the same objective without the projection of the governing equations on to eigenfunctions. This is usually done for systems exhibiting a dichotomy of unknowns, such as the axial (in-plane) and transverse (out-of-plane) unknowns in a beam or a plate. From the perspective of nonlinear dynamics, a good overview of nonlinear model reduction techniques can be found in the paper by Rega & Troger [100] and the references therein. In flexible multibody dynamics, the global parametrization method (see, e.g., [2, 22, 52]) also serves an indirect usage of nonlinear manifolds for model reduction. Indeed, the system is reduced using linear subspaces that are in turn nonlinearly parametrized by rigid-body kinematic variables that are dynamically varying.

Over the past decade or so, the use of nonlinear manifolds in model reduction has received a significant boost by many data-driven techniques, broadly referred to as *nonlinear dimensionality*

reduction (NLDR) or *manifold learning*, see the book by Lee & Verleysen [74], for instance. These techniques aim at constructing low dimensional manifolds from high-dimensional data snapshots. In a ROM perspective, these snapshots may be extracted from simulations of the underlying HFM, and the chosen NLDR is deemed to provide a nonlinear mapping with the required smoothness [85]. In essence, these methods provide a nonlinear generalization of the Proper Orthogonal Decomposition (POD), a well-known technique which delivers an optimal linear subspace for the dimensionality reduction.

The use of nonlinear manifolds in model reduction is expected to not only further reduce the dimensionality of the ROM in contrast to its linear mapping counterpart, but also shed light on the underlying system behavior. This is especially true for the case of thin-walled structures, which exhibit a time scale separation between slow, out-of-plane; and fast, in-plane dynamics. The fast dynamics in such systems can often be enslaved statically to the slow one, in a nonlinear fashion [62]. Recent efforts [63, 105] have shown that the geometrically nonlinear dynamics of structures, characterized by bending-stretching coupling (as for instance, thin-walled components) can be effectively captured using a quadratic manifold (QM), constructed using model properties rather than HFM snapshots.

The nonlinear-manifold-based ROMs are usually obtained by projecting the HFM on the tangent space of the manifold, in a virtual work sense (cf. Chapter 2). This generates configuration-dependent inertial terms for the ROM in addition to the already existing projected nonlinear terms. For general nonlinear systems, the evaluation and projection of the nonlinearity remains a bottleneck for computations, much like the linear mapping counterpart. As discussed above, the state-of-the-art hyper-reduction techniques allow for fast approximation of the (projected) nonlinearity in the case of linear-mapping-based ROMs. This functionality, however, has been missing in the context of nonlinear-manifold-based ROMs.

Main contribution in this chapter

In this chapter, we generalize the application of the ECSW hyper-reduction method to nonlinear-manifold-based ROMs. This work is essentially inspired from the ideas in the seminal work of Farhat *et al.* [41, 42], where ECSW method was introduced in the context of linear mappings. Following the work of Farhat *et al.* [42], we further show that our extension of the ECSW (hereafter referred to as *extended* ECSW: EECSW, for comparison purposes) preserves the Lagrangian structure associated with the Hamilton's principle. A natural consequence of this property is numerical stability during time integration, as shown by Farhat *et al.* [42]. The proposed method is tested, first on an academically simple example, and then on a realistic model of a wing using the recently proposed QM (cf. Chapter 2) for reduction. Our numerical results indicate that better speed ups can be achieved, as compared to the ones obtained with a linear POD subspace reduction equipped with ECSW. Interestingly, EECSW selects fewer elements for the hyper-reduction as compared to its linear counterpart, and more than compensates the extra computational costs associated to the configuration dependent inertial terms, which are not present in the case of linear subspace reduction.

This chapter is organized as follows. The concept of projection-based ROM on nonlinear mappings is briefly reviewed in Section 4.2. The extension of ECSW for nonlinear mappings is the tackled in Section 4.3, where also the preservation of the Lagrangian structure of the thus-obtained ROM is shown. Numerical results showing the effectiveness of the method are reported in Section 4.4, and finally, the conclusions are given in Section 4.5.

4.2 Model reduction using nonlinear mappings

Similar to the basic setup of Chapter 2, we write the initial value problem governing the structural system response as

$$\begin{aligned} \mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{C}\dot{\mathbf{u}}(t) + \mathbf{f}(\mathbf{u}(t)) &= \mathbf{f}^{ext}(t), \\ \mathbf{u}(t_0) &= \mathbf{u}_0, \dot{\mathbf{u}}(t_0) = \mathbf{v}_0, \end{aligned} \quad (4.1)$$

where the solution $\mathbf{u}(t) \in \mathbb{R}^n$ is a high-dimensional generalized displacement vector, $\mathbf{M} \in \mathbb{R}^{n \times n}$ is the mass matrix; $\mathbf{C} \in \mathbb{R}^{n \times n}$ is the damping matrix; $\mathbf{f} : \mathbb{R}^n \mapsto \mathbb{R}^n$ gives the nonlinear elastic internal force as a function of the displacement $\mathbf{u}$ of the structure; and $\mathbf{f}^{ext}(t) \in \mathbb{R}^n$ is the time-dependent external load vector. The system (4.1), referred to as the high-fidelity model (HFM), exhibits prohibitive computational costs for large values of the dimension n of the system. The classical notion of model reduction aims to reduce this dimensionality by introducing a linear mapping onto a suitable low-dimensional invariant subspace. However, as discussed in Section 4.1, more recent advances in model reduction allow for the detection of system dynamics evolving over nonlinear manifolds. The idea of a projection-based ROM using nonlinear mappings was presented in ref. [63] (cf. Chapter 2), using a nonlinear mapping such as

$$\mathbf{u}(t) \approx \mathbf{\Gamma}(\mathbf{q}(t)), \quad (4.2)$$

where $\mathbf{\Gamma} : \mathbb{R}^m \rightarrow \mathbb{R}^n$ is the nonlinear mapping. In general the mapping (4.2), when substituted into the governing equations (4.1), results in a non-zero residual $\mathbf{r}$ as

$$\mathbf{M}\ddot{\mathbf{\Gamma}}(\mathbf{q}(t)) + \mathbf{C}\dot{\mathbf{\Gamma}}(\mathbf{q}(t)) + \mathbf{f}(\mathbf{\Gamma}(\mathbf{q}(t))) - \mathbf{f}^{ext}(t) = \mathbf{r}.$$

The virtual work principle for a kinematically admissible displacement, $\delta\mathbf{u} = \frac{\partial \mathbf{\Gamma}(\mathbf{q})}{\partial \mathbf{q}} \delta\mathbf{q}$, over the nonlinear manifold leads to

$$\begin{aligned} \mathbf{r} \cdot \delta\mathbf{u} &= 0 \\ \implies \left[\mathbf{M}\ddot{\mathbf{\Gamma}}(\mathbf{q}(t)) + \mathbf{C}\dot{\mathbf{\Gamma}}(\mathbf{q}(t)) + \mathbf{f}(\mathbf{\Gamma}(\mathbf{q}(t))) \right] \cdot \delta\mathbf{u} &= \mathbf{f}^{ext}(t) \cdot \delta\mathbf{u}. \end{aligned}$$

Physically, this means that the residual force is orthogonal to the tangent space of the manifold. This gives the corresponding reduced-order model as

$$\mathbf{P}_{\mathbf{\Gamma}}^T(\mathbf{q}) \left[\mathbf{M}\ddot{\mathbf{\Gamma}}(\mathbf{q}(t)) + \mathbf{C}\dot{\mathbf{\Gamma}}(\mathbf{q}(t)) + \mathbf{f}(\mathbf{\Gamma}(\mathbf{q}(t))) \right] = \mathbf{P}_{\mathbf{\Gamma}}^T(\mathbf{q}) \mathbf{f}^{ext}(t), \quad (4.3)$$

where $\mathbf{P}_{\mathbf{\Gamma}}(\mathbf{q}) := \frac{\partial \mathbf{\Gamma}(\mathbf{q})}{\partial \mathbf{q}} \in \mathbb{R}^{n \times m}$ represents the tangent space of the nonlinear manifold in which the system dynamics is expected to evolve. Hereafter, we shall omit the dependence of $\mathbf{P}_{\mathbf{\Gamma}}$ on $\mathbf{q}$. The acceleration and velocity vectors over the manifold can be evaluated using the chain rule as

$$\begin{aligned} \dot{\mathbf{\Gamma}}(\mathbf{q}(t)) &= \frac{d}{dt} \mathbf{\Gamma}(\mathbf{q}(t)) = \frac{\partial \mathbf{\Gamma}(\mathbf{q})}{\partial \mathbf{q}} \cdot \dot{\mathbf{q}}, \\ \ddot{\mathbf{\Gamma}}(\mathbf{q}(t)) &= \frac{d}{dt} \dot{\mathbf{\Gamma}}(\mathbf{q}(t)) = \frac{\partial \mathbf{\Gamma}(\mathbf{q})}{\partial \mathbf{q}} \cdot \ddot{\mathbf{q}} + \left[\frac{\partial^2 \mathbf{\Gamma}(\mathbf{q})}{\partial \mathbf{q} \partial \mathbf{q}} \cdot \dot{\mathbf{q}} \right] \cdot \dot{\mathbf{q}}. \end{aligned}$$

Using the above relations, (4.3) can also be expressed as

$$\mathbf{M}_r(\mathbf{q}) \ddot{\mathbf{q}} + \mathbf{C}_r(\mathbf{q}, \dot{\mathbf{q}}) \dot{\mathbf{q}} + \mathbf{f}_r(\mathbf{q}) = \mathbf{P}_{\mathbf{\Gamma}}^T \mathbf{f}^{ext}(t), \quad (4.4)$$

where $\mathbf{M}_r(\mathbf{q}) = \mathbf{P}_{\mathbf{\Gamma}}^T \mathbf{M} \mathbf{P}_{\mathbf{\Gamma}}$, $\mathbf{C}_r(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{P}_{\mathbf{\Gamma}}^T \mathbf{C} \mathbf{P}_{\mathbf{\Gamma}} + \mathbf{P}_{\mathbf{\Gamma}}^T \mathbf{M} \left(\frac{\partial^2 \mathbf{\Gamma}(\mathbf{q})}{\partial \mathbf{q} \partial \mathbf{q}} \cdot \dot{\mathbf{q}} \right)$ and $\mathbf{f}_r(\mathbf{q}) = \mathbf{P}_{\mathbf{\Gamma}}^T \mathbf{f}(\mathbf{\Gamma}(\mathbf{q}))$.

Note that for $\mathbf{\Gamma}$ being a linear function, such that $\mathbf{\Gamma}(\mathbf{q}) = \mathbf{V}\mathbf{q}$ (for some $\mathbf{V} \in \mathbb{R}^{n \times m}$ spanning a suitable low dimensional subspace), the ROM (4.3) is equivalent to the familiar Galerkin-projection-based ROM given as

$$\mathbf{V}^T \mathbf{M} \mathbf{V} \ddot{\mathbf{q}}(t) + \mathbf{V}^T \mathbf{C} \mathbf{V} \dot{\mathbf{q}}(t) + \mathbf{V}^T \mathbf{f}(\mathbf{V} \mathbf{q}(t)) = \mathbf{V}^T \mathbf{f}^{ext}(t). \quad (4.5)$$

During time integration—usually referred to as the *online* phase—of (4.5), the computational cost associated to the evaluation of the linear terms in (4.5) scales only with the number of reduced variables m , since the corresponding reduced operators can be pre-computed *offline*. However, this is not the case for the computation of the reduced nonlinear term $\mathbf{V}^T \mathbf{f}(\mathbf{V}\mathbf{q}(t))$. Specifically, for FE-based applications, this evaluation is usually carried out *online*, in the following manner:

$$\hat{\mathbf{f}}(\mathbf{q}) := \mathbf{V}^T \mathbf{f}(\mathbf{V}\mathbf{q}) = \sum_{e=1}^{n_e} \mathbf{V}_e^T \mathbf{f}_e(\mathbf{V}_e \mathbf{q}), \quad (4.6)$$

where $\mathbf{f}_e(\mathbf{u}_e) \in \mathbb{R}^{N_e}$ is the contribution of the element e towards the vector $\mathbf{f}(\mathbf{u})$ (N_e being the number of DOFs for the element e), $\mathbf{V}_e$ is the restriction of $\mathbf{V}$ to the rows indexed by the DOFs corresponding to e , and n_e is the total number of elements in the structure. It is then easy to see from eq. (4.6), that the cost associated to the computation of the reduced nonlinear force $\hat{\mathbf{f}}(\mathbf{q})$ scales linearly with the total number of elements n_e in the structure, which is potentially high for large systems. Thus, despite the reduction in dimensionality achieved in (4.5), the evaluation of the reduced nonlinear term $\hat{\mathbf{f}}(\mathbf{q})$ emerges as a new bottleneck for the fast prediction of system response using the ROM (4.5). Similar conclusions regarding computational bottlenecks hold for the nonlinear-reduction-based ROM (4.3), as well.

4.3 Hyper-reduction and Extended ECSW

Hyper-reduction techniques help mitigate the above-stated high computational costs by approximation of the reduced-nonlinear term in a computationally affordable manner. As discussed in the Introduction, the ECSW [41] was conceived for the hyper-reduction of linear-mapping-based ROMs. After briefly summarizing ECSW, we propose here a natural extension to ECSW, which facilitates hyper-reduction of nonlinear-manifold-based ROM, such as in (4.3). Much like the ECSW, we further show in Section 4.3.2 that the proposed EECSW preserves the Lagrangian structure associated to the underlying mechanical system and, hence, leads to numerical stability in time integration.

In ECSW [41], the computationally expensive reduced nonlinear forces such as $\hat{\mathbf{f}}(\mathbf{q})$ (cf. eq. (4.6)) in the Galerkin-projection-based ROMs, e.g., (4.5) is approximated as

$$(\text{ECSW}) \quad \hat{\mathbf{f}}(\mathbf{q}) \approx \tilde{\mathbf{f}}(\mathbf{q}) := \sum_{e \in E} \xi_e \mathbf{V}_e^T \mathbf{f}_e(\mathbf{V}_e \mathbf{q}), \quad (4.7)$$

where $\xi_e \in \mathbb{R}^+$ is a positive weight given to every element $e \in E$. Clearly, the evaluation of $\tilde{\mathbf{f}}(\mathbf{q})$ in (4.7) comes at a fraction of the computational cost associated to (4.6), for $|E| \ll n_e$. This is made possible with weights which are chosen using *training* to ensure a good approximation of the original sum over all n_e elements, while keeping the element sample small using sparse algorithms in literature. The element-set E is sometimes referred to as the *reduced mesh* due to its low cardinality relative to the total number of elements n_e . Farhat *et al.* [42] further show that hyper-reduction done in this manner preserves Lagrangian structure of the linear Galerkin-projection-based ROMs.

We observe that analogous to summation (4.6), the projected nonlinear force (possibly a combination of inertial, damping or elastic forces) in the nonlinear ROM (4.3) can be expressed as a summation of the element-level contributions as

$$\mathbf{h}_r(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}) := \mathbf{P}_\Gamma^T \mathbf{h} \left(\Gamma(\mathbf{q}), \dot{\Gamma}(\mathbf{q}), \ddot{\Gamma}(\mathbf{q}) \right) = \sum_{e=1}^{n_e} (\mathbf{P}_\Gamma(\mathbf{q}))_e^T \mathbf{h}_e \left(\Gamma_e(\mathbf{q}), \dot{\Gamma}_e(\mathbf{q}), \ddot{\Gamma}_e(\mathbf{q}) \right), \quad (4.8)$$

where $\mathbf{h}_r(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}})$ is the part of the ROM (4.3) which is potentially unaffordable to compute (cost scales with n or n_e), $\mathbf{h}$ is its full counterpart, with the $\mathbf{h}_e$ having the usual meaning for the element-level contribution from the element e . Just like in ECSW, the summation (4.8) may be approximated

as

$$(\text{EECSW}) \quad \mathbf{h}_r(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}) \approx \tilde{\mathbf{h}}_r(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}) = \sum_{e \in E} \xi_e (\mathbf{P}_r(\mathbf{q}))_e^T \mathbf{h}_e \left(\mathbf{\Gamma}_e(\mathbf{q}), \dot{\mathbf{\Gamma}}_e(\mathbf{q}), \ddot{\mathbf{\Gamma}}_e(\mathbf{q}) \right), \quad (4.9)$$

where $\xi_e \in \mathbb{R}^+$ is again a positive weight given to every element e in the reduced mesh E . The element sampling and weighing are discussed in Section 4.3.1

One easily observable distinction of EECSW from its linear mapping counterpart is that in ECSW, it is possible to precompute the reduced linear operators in the hyper-reduced model, such as the reduced mass, damping and stiffness matrices $\mathbf{V}^T \mathbf{M} \mathbf{V}$, $\mathbf{V}^T \mathbf{C} \mathbf{V}$ and $\mathbf{V}^T \mathbf{K} \mathbf{V}$, respectively. Due to the nonlinear mapping and projection on the tangent space, we see that EECSW doesn't feature reduced-linear operators, even for the linear mass, stiffness and damping matrices in the HFM. Thus, in principle, one could include all the terms on the left-hand-side of (4.3) in $\mathbf{h}_r$, i.e.,

$$\mathbf{h}_r(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}) = \mathbf{P}_r^T \left[\mathbf{M} \ddot{\mathbf{\Gamma}}(\mathbf{q}(t)) + \mathbf{C} \dot{\mathbf{\Gamma}}(\mathbf{q}(t)) + \mathbf{f}(\mathbf{\Gamma}(\mathbf{q}(t))) \right].$$

It is easy to see that this idea is a straight-forward generalization of ECSW to nonlinear-manifold-based ROMs with projection to the tangent space, as discussed in Chapter 2. Indeed, for $\mathbf{\Gamma}(\mathbf{q}) = \mathbf{V} \mathbf{q}$, the hyper-reduction of ROM (4.5) leads to the same approximation (4.6) of the projected internal force.

4.3.1 Element sampling and weight selection

The elements and weights are determined to approximate virtual work over chosen training sets which generally come from full solution run(s). We assume that $\{\mathbf{u}^{(i)}, \dot{\mathbf{u}}^{(i)}, \ddot{\mathbf{u}}^{(i)}\}$, $i \in \{1, \dots, n_t\}$ represents a set of available training vectors. If the solution to the HFM, indeed, evolves over a nonlinear manifold as given by (4.2), then we must be able to find reduced unknowns $\{\mathbf{q}^{(i)}, \dot{\mathbf{q}}^{(i)}, \ddot{\mathbf{q}}^{(i)}\}$ such that

$$\left. \begin{aligned} \mathbf{u}^{(i)} &= \mathbf{\Gamma}(\mathbf{q}^{(i)}) \\ \dot{\mathbf{u}}^{(i)} &= \mathbf{P}_r(\mathbf{q}^{(i)}) \dot{\mathbf{q}}^{(i)} \\ \ddot{\mathbf{u}}^{(i)} &= \mathbf{P}_r^T(\mathbf{q}^{(i)}) \ddot{\mathbf{q}}^{(i)} + \left[\frac{\partial^2 \mathbf{\Gamma}(\mathbf{q})}{\partial \mathbf{q} \partial \mathbf{q}} \bigg|_{\mathbf{q}=\mathbf{q}^{(i)}} \cdot \dot{\mathbf{q}}^{(i)} \right] \cdot \dot{\mathbf{q}}^{(i)} \end{aligned} \right\} \quad \forall i \in \{1, \dots, n_t\}. \quad (4.10)$$

As will be clear later in this section, the optimal elements and weights corresponding to a reduced mesh needs the set of reduced vectors $\{\mathbf{q}^{(i)}, \dot{\mathbf{q}}^{(i)}, \ddot{\mathbf{q}}^{(i)}\}$ during the training stage. In case these are available from a previous ROM simulation, these would be directly useful. Furthermore, if the nonlinear manifold is obtained from solution snapshots using manifold-learning procedures such as the local maximum entropy approximants, as done in, e.g., [85], then the reduced vectors would be directly available from the manifold-learning procedure. However, if the manifold is obtained from simulation-free approaches (cf. [63, 105]), then a nonlinear system of algebraic equations (4.10) can be solved for the reduced training vectors $\mathbf{q}^{(i)}$, $\dot{\mathbf{q}}^{(i)}$, and $\ddot{\mathbf{q}}^{(i)}$. This system is under-constrained ($m \ll n$) and, hence, the sought $\mathbf{q}^{(i)}$ can be obtained from the solution to a nonlinear least-squares procedures, e.g., the gradient descent method, the Gauss-Newton methods, the Levenberg-Marquardt method (which is also employed in the standard LSQNONLIN routine in MATLAB); we refer the reader to Lines and Treitel [80] for a review of standard methods for nonlinear least squares problems.

Geometrically, the first equation in the system (4.10) gives the location of the data point on the manifold, whereas as the second and the third equations give the trajectory information (tangent

and curvature along the trajectory). Thus, the reduced coordinates of the point on the manifold can be identified independent of the velocity and acceleration information by solving the equation

$$\mathbf{q}^{(i)} = \arg \min_{\mathbf{q} \in \mathbb{R}^m} \left\| \mathbf{\Gamma}(\mathbf{q}) - \mathbf{u}^{(i)} \right\|_2^2. \quad (4.11)$$

Though the nonlinear least squares problem (4.11) would add to the offline costs, note that this procedure is embarrassingly parallelizable for each training snapshot. Once the reduced coordinates $\mathbf{q}^{(i)}$ are obtained on the manifold, the second equation in system (4.10) can be solved in a least squares sense for $\dot{\mathbf{q}}^{(i)}$ as

$$\dot{\mathbf{q}}^{(i)} = \left[\mathbf{P}_{\mathbf{\Gamma}}^T(\mathbf{q}^{(i)}) \mathbf{P}_{\mathbf{\Gamma}}(\mathbf{q}^{(i)}) \right]^{-1} \mathbf{P}_{\mathbf{\Gamma}}^T(\mathbf{q}^{(i)}) \dot{\mathbf{u}}^{(i)}. \quad (4.12)$$

Finally, we can obtain the corresponding reduced accelerations as

$$\ddot{\mathbf{q}}^{(i)} = \left[\mathbf{P}_{\mathbf{\Gamma}}^T(\mathbf{q}^{(i)}) \mathbf{P}_{\mathbf{\Gamma}}(\mathbf{q}^{(i)}) \right]^{-1} \mathbf{P}_{\mathbf{\Gamma}}^T(\mathbf{q}^{(i)}) \left(\ddot{\mathbf{u}}^{(i)} - \left[\frac{\partial^2 \mathbf{\Gamma}(\mathbf{q})}{\partial \mathbf{q} \partial \mathbf{q}} \right]_{\mathbf{q}=\mathbf{q}^{(i)}} \cdot \dot{\mathbf{q}}^{(i)} \cdot \dot{\mathbf{q}}^{(i)} \right). \quad (4.13)$$

Alternatively, one might consider solving a single least squares problems involving all the reduced vectors for displacements, velocities and accelerations, resulting in a system which is three times in dimensionality. However, the sequential procedure proposed in (4.11)-(4.13) is computationally more effective. Note that for $\mathbf{\Gamma}$ being a linear function, such as $\mathbf{\Gamma}(\mathbf{q}) = \mathbf{V}\mathbf{q}$, the system (4.11) reduces to the familiar least squares solution of problem (4.10), that is available in closed form as

$$\begin{aligned} \mathbf{q}^{(i)} &= (\mathbf{V}^T \mathbf{V})^{-1} \mathbf{V}^T \mathbf{u}^{(i)}, \\ \dot{\mathbf{q}}^{(i)} &= (\mathbf{V}^T \mathbf{V})^{-1} \mathbf{V}^T \dot{\mathbf{u}}^{(i)}, \\ \ddot{\mathbf{q}}^{(i)} &= (\mathbf{V}^T \mathbf{V})^{-1} \mathbf{V}^T \ddot{\mathbf{u}}^{(i)}. \end{aligned}$$

Once the reduced training vectors $\{\mathbf{q}^{(i)}, \dot{\mathbf{q}}^{(i)}, \ddot{\mathbf{q}}^{(i)}\}$ are obtained, the element-level contribution of projected internal force for each of the training vectors can be assembled in a matrix $\mathbf{G}$, as follows:

$$\begin{aligned} \mathbf{G} &= \begin{bmatrix} \mathbf{g}_{11} & \cdots & \mathbf{g}_{1n_e} \\ \vdots & \ddots & \vdots \\ \mathbf{g}_{n_t 1} & \cdots & \mathbf{g}_{n_t n_e} \end{bmatrix} \in \mathbb{R}^{mn_t \times n_e}, \quad \mathbf{b} = \begin{bmatrix} \mathbf{b}_1 \\ \vdots \\ \mathbf{b}_{n_t} \end{bmatrix} \in \mathbb{R}^{mn_t}, \\ \mathbf{g}_{ie} &= \left(\mathbf{P}_{\mathbf{\Gamma}}(\mathbf{q}^{(i)}) \right)_e^T \mathbf{h}_e \left(\mathbf{\Gamma}_e(\mathbf{q}^{(i)}), \dot{\mathbf{\Gamma}}_e(\mathbf{q}^{(i)}), \ddot{\mathbf{\Gamma}}_e(\mathbf{q}^{(i)}) \right), \quad \mathbf{b}_i = \mathbf{h}_r(\mathbf{q}^{(i)}) = \sum_{e=1}^{n_e} \mathbf{g}_{ie} \\ &\quad \forall i \in \{1, \dots, n_t\}, \quad e \in \{1, \dots, n_e\}. \end{aligned} \quad (4.14)$$

Analogous to ECSW, the set of elements and weights is then obtained by a solution to the following problem

$$(P1) : \tilde{\boldsymbol{\xi}} = \arg \min_{\tilde{\boldsymbol{\xi}} \in \Theta} \|\tilde{\boldsymbol{\xi}}\|_0,$$

with

$$\Theta = \{ \tilde{\boldsymbol{\xi}} \in \mathbb{R}^{n_e}, \tilde{\boldsymbol{\xi}} \geq \mathbf{0} : \|\mathbf{G}\tilde{\boldsymbol{\xi}} - \mathbf{b}\|_2 \leq \tau \|\mathbf{b}\|_2 \}, \quad (4.15)$$

where $0 < \tau \ll 1$ is a user-defined tolerance, and $\|\cdot\|_0$ denotes the zero-norm that gives the number of non-zero elements of its argument vector.

The set Θ is clearly non-empty since it contains the weights vector $\mathbf{1} = [1, \dots, 1] \in \mathbb{R}^{n_e}$. According to (5.6), the vector $\mathbf{1}$ satisfies

$$\mathbf{G}\mathbf{1} - \mathbf{b} = \mathbf{0} \implies \|\mathbf{G}\mathbf{1} - \mathbf{b}\|_2 = 0. \quad (4.16)$$

and $\mathbf{1} \in \Theta$. Indeed, the vector $\mathbf{1} \in \Theta$ corresponds to each element in the mesh being assigned a unit weight. However, by weakly satisfying the relation (4.16) in the form given by definition (4.15) of the feasible set Θ , we expect to obtain a sparse solution $\tilde{\xi}$ to (P1). The non-zero entries of $\tilde{\xi}$ form the reduced mesh E used in (5.5) as

$$E = \{e : \zeta_e > 0\}.$$

An optimally sparse solution to (P1) is NP-hard to obtain [8]. However, a greedy-approach-based algorithm [90], which finds a sub-optimal solution $\xi \in \mathbb{R}^{n_e}$ by solving a sparse non-negative least squares (sNNLS) variant of (P1), has been found to deliver an effective reduced mesh E [41]. We have included its pseudo-code in Algorithm 3, where ζ_E and $\mathbf{G}_E$ denote, respectively, the restriction of $\zeta \in \mathbb{R}^{n_e}$ and column-wise restriction of $\mathbf{G}$ to the elements in the *active* subset E . The set Z is the disjoint *inactive* subset which contains the zero entry-indices of ξ . For further discussion on the problem (P1), we refer the reader to the original work of Farhat *et al.* [41, 42] in the context of linear mappings. More recent work [26] proposes and discusses different alternatives to accelerate the sampling procedure during hyper-reduction, including its parallel implementation to reduce offline costs.

Algorithm 2 Training for EECSW

Input: $\{\mathbf{u}^{(i)}, i \in \{1, \dots, n_t\}, \tau$ (sparse NNLS tolerance)

Output: $\xi \in \mathbb{R}^{n_e}$ sparse, $E \subset \{1, \dots, n_e\}$

```

1: for  $i \leftarrow 1$  to  $n_t$  do
2:    $\mathbf{q}^{(i)} \leftarrow \text{LSQNLIN}(\mathbf{u}^{(i)})$   $\triangleright$  LSQNLIN: solves the nonlinear least squares problem (4.11)
3: end for
4:  $[\mathbf{G}, \mathbf{b}] \leftarrow \text{Construct\_G\_b}(\{\mathbf{q}^{(i)}\}_1^{n_t})$   $\triangleright$  Construct_G_b: assembles  $\mathbf{G}, \mathbf{b}$ , as given by (4.14)
5:  $E \leftarrow \emptyset, Z \leftarrow \{1, \dots, n_e\}; \zeta, \xi \leftarrow \mathbf{0} \in \mathbb{R}^{n_e}$ 
6: while  $\|\mathbf{G}\xi - \mathbf{b}\|_2 > \tau \|\mathbf{b}\|_2$  do  $\triangleright$  sparse NNLS solver
7:    $\mu \leftarrow \mathbf{G}^T(\mathbf{b} - \mathbf{G}\xi)$ 
8:    $[\nu, e] \leftarrow \max(\mu)$   $\triangleright$  max: returns maximum value in a vector followed by its location (index)
9:    $E \leftarrow E \cup \{e\}, Z \leftarrow Z \setminus \{e\}$ 
10:  while true do
11:     $\zeta_E \leftarrow \mathbf{G}_E^\dagger \mathbf{b}$   $\triangleright \dagger$  represents pseudo-inverse
12:     $\zeta_Z \leftarrow \mathbf{0}$ 
13:    if  $\zeta_E > \mathbf{0}$  then
14:       $\xi \leftarrow \zeta$ 
15:      break
16:    end if
17:     $\eta \leftarrow \min_{k \in E} \xi_k / (\xi_k - \zeta_k)$ 
18:     $\xi \leftarrow \xi + \eta(\zeta - \xi)$ 
19:     $Z \leftarrow \{i | \xi_i = 0\}$ 
20:     $E \leftarrow \{1, \dots, n_e\} \setminus Z$ 
21:  end while
22: end while

```

4.3.2 Preservation of Lagrangian structure and numerical stability

4.3.2.1 Conservative systems

Following the work of Farhat *et al.* [42], we show here that the hyper-reduced model preserves the Lagrangian structure, leading to numerical stability of the hyper-reduced model. For this purpose, we restrict our discussion to a conservative system, where the Hamiltonian principle represents the

conservation of total mechanical energy as follows:

$$\mathcal{E} = \mathcal{T}(\dot{\mathbf{u}}) + \mathcal{V}(\mathbf{u}) = \text{constant}, \quad (4.17)$$

where $\mathcal{T}$ and $\mathcal{V}$ are the functionals associated to the kinetic energy and the potential energy of the system, respectively. The kinetic energy is given by

$$\mathcal{T}(\dot{\mathbf{u}}) = \frac{1}{2} \dot{\mathbf{u}}^T \mathbf{M} \dot{\mathbf{u}} = \sum_{e=1}^{n_e} \underbrace{\frac{1}{2} \dot{\mathbf{u}}_e^T \mathbf{M}_e \dot{\mathbf{u}}_e}_{\mathcal{T}_e(\dot{\mathbf{u}}_e)}, \quad (4.18)$$

where $\mathbf{M}_e \in \mathbb{R}^{N_e \times N_e}$ is the contribution of the element e towards the mass matrix and $\mathcal{T}_e$ is the corresponding contribution towards the total kinetic energy. Furthermore, the potential energy associated to the elastic internal force $\mathbf{f}$ and the conservative external force $\mathbf{f}_c^{ext}$ can be written as

$$\mathcal{V}(\mathbf{u}) = \underbrace{\int \mathbf{f}(\mathbf{u}) \cdot d\mathbf{u}}_{\mathcal{V}^{int}(\mathbf{u})} + \underbrace{\int -\mathbf{f}_c^{ext} \cdot d\mathbf{u}}_{\mathcal{V}^{ext}(\mathbf{u})}, \quad (4.19)$$

such that

$$\mathbf{f}(\mathbf{u}) = \frac{\partial \mathcal{V}^{int}(\mathbf{u})}{\partial \mathbf{u}}, \quad (4.20)$$

$$\mathbf{f}_c^{ext}(\mathbf{u}) = -\frac{\partial \mathcal{V}^{ext}(\mathbf{u})}{\partial \mathbf{u}}, \quad (4.21)$$

and $\mathcal{V}^{int}$ and $\mathcal{V}^{ext}$ are the potential energies associated to internal and external forces, respectively. The conservative counterpart of the equations of motion (4.1) can then be expressed using Lagrange equations as:

$$\frac{d}{dt} \left(\frac{\partial \mathcal{T}(\dot{\mathbf{u}})}{\partial \dot{\mathbf{u}}} \right) + \frac{\partial \mathcal{V}^{int}(\mathbf{u})}{\partial \mathbf{u}} = \mathbf{f}_c^{ext}. \quad (4.22)$$

As shown in (4.3), the reduced-order model, obtained by introduction of a nonlinear mapping $\mathbf{\Gamma}$ and projection of the resulting equations to the tangent space $\mathbf{P}_{\mathbf{\Gamma}}$ of the manifold, is given by

$$\mathbf{P}_{\mathbf{\Gamma}}^T \left[\frac{d}{dt} \left(\frac{\partial \mathcal{T}(\dot{\mathbf{\Gamma}}(\mathbf{q}))}{\partial \dot{\mathbf{\Gamma}}(\mathbf{q})} \right) + \frac{\partial \mathcal{V}^{int}(\mathbf{\Gamma}(\mathbf{q}))}{\partial \mathbf{\Gamma}(\mathbf{q})} \right] = \mathbf{P}_{\mathbf{\Gamma}}^T \mathbf{f}_c^{ext}, \quad (4.23)$$

where $\mathbf{\Gamma} : \mathbb{R}^n \mapsto \mathbb{R}^m$ describes a general nonlinear mapping which approximates the solution $\mathbf{u}$ using a reduced set of unknowns $\mathbf{q}$. It is easy to see that (4.23) is equivalent to ROM (4.3) in the conservative setting, i.e.,

$$\mathbf{P}_{\mathbf{\Gamma}}^T \left[\mathbf{M} \ddot{\mathbf{\Gamma}}(\mathbf{q}) + \frac{\partial \mathcal{V}^{int}(\mathbf{\Gamma}(\mathbf{q}))}{\partial \mathbf{\Gamma}(\mathbf{q})} \right] = \mathbf{P}_{\mathbf{\Gamma}}^T \mathbf{f}_c^{ext}. \quad (4.24)$$

Recall that the ROM obtained using a linear Galerkin projection is a special case of the general setting in (4.24) when a linear mapping $\mathbf{\Gamma}(\mathbf{q}) = \mathbf{V}\mathbf{q}$ is used. It is well known that such a linear Galerkin projection is structure-preserving (cf., e.g., [42, 72]). We now show that this is also true for the ROM (4.24), obtained by projection on to the tangent space of general nonlinear manifolds.

We can define the reduced potential and kinetic energy associated to the reduced system (4.24) as

$$\mathcal{V}_r(\mathbf{q}) = \mathcal{V}(\mathbf{\Gamma}(\mathbf{q})), \quad (4.25)$$

$$\mathcal{T}_r(\dot{\mathbf{q}}, \mathbf{q}) = \mathcal{T}(\dot{\mathbf{\Gamma}}(\mathbf{q})). \quad (4.26)$$

Then, the Hamiltonian associated to ROM (4.24) is given by

$$\begin{aligned}
\mathcal{E}_r(\mathbf{q}, \dot{\mathbf{q}}) &= \mathcal{V}_r(\mathbf{q}) + \mathcal{T}_r(\mathbf{q}), \\
\Rightarrow \dot{\mathcal{E}}_r(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}) &= \dot{\mathcal{T}}(\dot{\mathbf{q}}(\mathbf{q})) + \dot{\mathcal{V}}(\mathbf{q}(\mathbf{q})) \\
&= \frac{\partial \mathcal{T}(\dot{\mathbf{q}}(\mathbf{q}))}{\partial \dot{\mathbf{q}}(\mathbf{q})} \ddot{\mathbf{q}}(\mathbf{q}) + \frac{\partial \mathcal{V}^{int}(\mathbf{q}(\mathbf{q}))}{\partial \mathbf{q}(\mathbf{q})} \dot{\mathbf{q}}(\mathbf{q}) + \frac{\partial \mathcal{V}^{ext}(\mathbf{q}(\mathbf{q}))}{\partial \mathbf{q}(\mathbf{q})} \dot{\mathbf{q}}(\mathbf{q}) \\
&= (\dot{\mathbf{q}}(\mathbf{q}))^T \mathbf{M} \ddot{\mathbf{q}}(\mathbf{q}) + (\dot{\mathbf{q}}(\mathbf{q}))^T \frac{\partial \mathcal{V}^{int}(\mathbf{q}(\mathbf{q}))}{\partial \mathbf{q}(\mathbf{q})} - (\dot{\mathbf{q}}(\mathbf{q}))^T \mathbf{f}_c^{ext} \\
&= \dot{\mathbf{q}}^T \left(\mathbf{P}_{\mathbf{r}}^T \left[\mathbf{M} \ddot{\mathbf{q}}(\mathbf{q}) + \frac{\partial \mathcal{V}^{int}(\mathbf{q}(\mathbf{q}))}{\partial \mathbf{q}(\mathbf{q})} - \mathbf{f}_c^{ext} \right] \right) \quad (\because \dot{\mathbf{q}}(\mathbf{q}) = \mathbf{P}_{\mathbf{r}} \dot{\mathbf{q}}) \\
&= 0 \quad (\text{Using (4.24)})
\end{aligned} \tag{4.27}$$

Thus, $\mathcal{E}_r = \text{constant}$, and the total mechanical energy is conserved for the ROM (4.24). It is also easy to see that the reduced equations can be directly obtained using the reduced Lagrangian $\mathcal{L}_r := \mathcal{T}_r - \mathcal{V}_r$. This proves that the projection of the equations to the tangent space of the manifold leads to preservation of this Lagrangian structure associated to the corresponding Hamiltonian principle. Note that the projection of the full system on to any subspace other than the tangent space of the manifold would not lead to these desirable properties, in general.

In the hyper-reduced model, the reduced internal force from (4.23) is approximated using EECSW (cf. (4.9)) as

$$\mathbf{P}_{\mathbf{r}}^T \frac{\partial \mathcal{V}^{int}(\mathbf{q}(\mathbf{q}))}{\partial \mathbf{q}(\mathbf{q})} \approx \frac{\partial \tilde{\mathcal{V}}_r^{int}(\mathbf{q})}{\partial \mathbf{q}} = \sum_{e \in E} \xi_e (\mathbf{P}_{\mathbf{r}}^T)_e \frac{\partial \mathcal{V}_e^{int}(\mathbf{q}_e(\mathbf{q}))}{\partial \mathbf{q}_e(\mathbf{q})}, \tag{4.28}$$

where

$$\tilde{\mathcal{V}}_r^{int}(\mathbf{q}) = \sum_{e \in E} \xi_e \mathcal{V}_e^{int}(\mathbf{q}_e(\mathbf{q})) \tag{4.29}$$

can be identified as the potential energy associated to the hyper-reduced internal force. Furthermore, the inertial force can also be approximated using EECSW as

$$\mathbf{P}_{\mathbf{r}}^T \frac{d}{dt} \left(\frac{\partial \mathcal{T}(\dot{\mathbf{q}}(\mathbf{q}))}{\partial \dot{\mathbf{q}}(\mathbf{q})} \right) \approx \frac{d}{dt} \left(\frac{\partial \tilde{\mathcal{T}}_r(\dot{\mathbf{q}}, \mathbf{q})}{\partial \dot{\mathbf{q}}} \right) = \sum_{e \in E} \xi_e (\mathbf{P}_{\mathbf{r}}^T)_e \frac{d}{dt} \frac{\partial \mathcal{T}_e(\dot{\mathbf{q}}_e(\mathbf{q}))}{\partial \dot{\mathbf{q}}_e(\mathbf{q})} = \sum_{e \in E} \xi_e (\mathbf{P}_{\mathbf{r}}^T)_e \mathbf{M}_e \ddot{\mathbf{q}}_e(\mathbf{q}), \tag{4.30}$$

where

$$\tilde{\mathcal{T}}_r(\dot{\mathbf{q}}, \mathbf{q}) = \sum_{e \in E} \xi_e \mathcal{T}_e(\dot{\mathbf{q}}_e(\mathbf{q})) \tag{4.31}$$

can be interpreted as the kinetic energy associated to the reduced mesh. To summarize, the hyper-reduced equations, using EECSW, for the ROM (4.23) are given as

$$\sum_{e \in E} \xi_e (\mathbf{P}_{\mathbf{r}}^T)_e \mathbf{M}_e \ddot{\mathbf{q}}_e(\mathbf{q}) + \sum_{e \in E} \xi_e (\mathbf{P}_{\mathbf{r}}^T)_e \frac{\partial \mathcal{V}_e^{int}(\mathbf{q}_e(\mathbf{q}))}{\partial \mathbf{q}_e(\mathbf{q})} = \mathbf{P}_{\mathbf{r}}^T \mathbf{f}_c^{ext}. \tag{4.32}$$

We now show that the hyper-reduced model (4.32), obtained using EECSW, is also structure-preserving. Analogous to the previous discussion, the Hamiltonian associated to the hyper-reduced

system becomes

$$\begin{aligned}
\tilde{\mathcal{E}}_r(\mathbf{q}, \dot{\mathbf{q}}) &= \tilde{\mathcal{T}}_r(\dot{\mathbf{q}}, \mathbf{q}) + \tilde{\mathcal{V}}_r^{int}(\mathbf{q}) + \mathcal{V}^{ext}(\Gamma(\mathbf{q})) \\
\Rightarrow \dot{\tilde{\mathcal{E}}}_r(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}) &= \dot{\tilde{\mathcal{T}}}_r(\dot{\mathbf{q}}, \mathbf{q}) + \dot{\tilde{\mathcal{V}}}_r^{int}(\mathbf{q}) + \dot{\mathcal{V}}^{ext}(\Gamma(\mathbf{q})) \\
&= \dot{\mathbf{q}}^T \left(\sum_{e \in E} \xi_e (\mathbf{P}_\Gamma^T)_e \mathbf{M}_e \ddot{\Gamma}_e(\mathbf{q}) + \sum_{e \in E} \xi_e (\mathbf{P}_\Gamma^T)_e \frac{\partial \mathcal{V}_e^{int}(\Gamma_e(\mathbf{q}))}{\partial \Gamma_e(\mathbf{q})} - \mathbf{P}_\Gamma^T \mathbf{f}_c^{ext} \right) \\
&= 0 \quad (\text{Using (4.32)}).
\end{aligned}$$

This shows that hyper-reduced model (4.32) also conserves energy and preserves the Lagrangian structure associated to the corresponding Hamilton's principle. The numerical stability of the hyper-reduced model is a direct consequence of this structure-preserving property as noted by Farhat *et al.* [42], as well.

4.3.2.2 Dissipative, non-conservative systems

To include dissipation effects which, typically, generate forces opposite to the generalized velocity vector $\dot{\mathbf{u}}$ in the system, a dissipation functional $\mathcal{D}$ can be defined, such that the corresponding dissipative forces are given by

$$\mathbf{f}^{dis}(\dot{\mathbf{u}}) = -\frac{\partial \mathcal{D}(\dot{\mathbf{u}})}{\partial \dot{\mathbf{u}}}.$$

Assuming that the functional $\mathcal{D}$ has the same degree of homogeneity, d , in the generalized velocities $(\dot{\mathbf{u}})$ for all elements in the mesh, we have

$$\dot{\mathbf{u}}^T \frac{\partial \mathcal{D}(\dot{\mathbf{u}})}{\partial \dot{\mathbf{u}}} = \sum_{e=1}^{n_e} \dot{\mathbf{u}}_e^T \frac{\partial \mathcal{D}_e(\dot{\mathbf{u}}_e)}{\partial \dot{\mathbf{u}}_e} = d \sum_{e=1}^{n_e} \mathcal{D}_e(\dot{\mathbf{u}}_e), \quad (4.33)$$

where $\mathcal{D}_e$ is the usual notation for the contribution of element e towards the total dissipation $\mathcal{D}$, which would be dependent on the nodal velocities $\dot{\mathbf{u}}_e$ only. For the special case of linear viscous damping, i.e., $d = 2$, we have

$$\mathcal{D}(\dot{\mathbf{u}}) = \frac{1}{2} \dot{\mathbf{u}}^T \mathbf{C} \dot{\mathbf{u}}.$$

Other values of d lead to different types of dissipation models, e.g., $d = 1$ leads to dry Coulomb friction and $d = 3$ leads to a viscous drag in fluids.

The Hamilton's principle in this non-conservative setting leads to the power balance (cf. G  rardin & Rixen [45], Chapter 1)

$$\dot{\mathcal{T}} + \dot{\mathcal{V}} = \left(-\frac{\partial \mathcal{D}(\dot{\mathbf{u}})}{\partial \dot{\mathbf{u}}} + \mathbf{f}_{nc}^{ext} \right) \cdot \dot{\mathbf{u}} = \dot{\mathbf{u}}^T \mathbf{f}_{nc}^{ext} - d \sum_{e=1}^{n_e} \mathcal{D}_e(\dot{\mathbf{u}}_e), \quad (4.34)$$

where $\mathbf{f}_{nc}^{ext}$ is a non-conservative, time-dependent external force. The corresponding Lagrange equations of motion can then be expressed, equivalently, as

$$\mathbf{M} \ddot{\mathbf{u}} + \frac{\partial \mathcal{D}(\dot{\mathbf{u}})}{\partial \dot{\mathbf{u}}} + \frac{\partial \mathcal{V}^{int}(\mathbf{u})}{\partial \mathbf{u}} = \mathbf{f}_c^{ext} + \mathbf{f}_{nc}^{ext}. \quad (4.35)$$

Now, the nonlinear ROM associated to (4.35) is given by

$$\mathbf{P}_\Gamma^T \left[\mathbf{M} \ddot{\Gamma}(\mathbf{q}) + \frac{\partial \mathcal{D}(\dot{\Gamma}(\mathbf{q}))}{\partial \dot{\Gamma}(\mathbf{q})} + \frac{\partial \mathcal{V}^{int}(\Gamma(\mathbf{q}))}{\partial \Gamma(\mathbf{q})} \right] = \mathbf{P}_\Gamma^T [\mathbf{f}_c^{ext} + \mathbf{f}_{nc}^{ext}]. \quad (4.36)$$

The approximation of the additional nonlinear term in (4.36) using EECSW is given by

$$\mathbf{P}_\Gamma^T \frac{\partial \mathcal{D}(\dot{\Gamma}(\mathbf{q}))}{\partial \dot{\Gamma}(\mathbf{q})} = \sum_{e=1}^{n_e} (\mathbf{P}_\Gamma^T)_e \frac{\partial \mathcal{D}_e(\dot{\Gamma}_e(\mathbf{q}))}{\partial \dot{\Gamma}_e(\mathbf{q})} \approx \frac{\partial \tilde{\mathcal{D}}_r(\dot{\mathbf{q}}, \mathbf{q})}{\partial \dot{\mathbf{q}}} = \sum_{e \in E} (\mathbf{P}_\Gamma^T)_e \frac{\partial \mathcal{D}_e(\dot{\Gamma}_e(\mathbf{q}))}{\partial \dot{\Gamma}_e(\mathbf{q})}, \quad (4.37)$$

where

$$\tilde{\mathcal{D}}_r(\dot{\mathbf{q}}, \mathbf{q}) = \sum_{e \in E} \xi_e \mathcal{D}_e(\dot{\Gamma}_e(\mathbf{q})), \quad (4.38)$$

can be thought of as the dissipation associated to the hyper-reduced model, given in final form as

$$\sum_{e \in E} \xi_e (\mathbf{P}_\Gamma^T)_e \left[\mathbf{M}_e \ddot{\Gamma}_e(\mathbf{q}) + \frac{\partial \mathcal{D}_e(\dot{\Gamma}_e(\mathbf{q}))}{\partial \dot{\Gamma}_e(\mathbf{q})} + \frac{\partial \mathcal{V}_e^{int}(\Gamma_e(\mathbf{q}))}{\partial \Gamma_e(\mathbf{q})} \right] = \mathbf{P}_\Gamma^T [\mathbf{f}_c^{ext} + \mathbf{f}_{nc}^{ext}]. \quad (4.39)$$

Following the same procedure as shown above for conservative systems, it is easy to see that the Lagrangian structure is preserved using EECSW for non-conservative systems such as (4.39), as well. Indeed, the rate of change of the sum of kinetic energy ($\tilde{T}_r$) and potential energy ($\tilde{\mathcal{V}}_r = \tilde{\mathcal{V}}_r^{int} + \tilde{\mathcal{V}}_r^{ext}$) associated to the hyper-reduced model (4.39) as

$$\begin{aligned} \frac{d}{dt} (\tilde{T}_r + \tilde{\mathcal{V}}_r) &= \dot{\tilde{T}}_r(\dot{\mathbf{q}}, \mathbf{q}) + \dot{\tilde{\mathcal{V}}}_r^{int}(\mathbf{q}) + \dot{\mathcal{V}}_r^{ext}(\Gamma(\mathbf{q})) \\ &= \dot{\mathbf{q}}^T \left(\sum_{e \in E} \xi_e (\mathbf{P}_\Gamma^T)_e \left[\mathbf{M}_e \ddot{\Gamma}_e(\mathbf{q}) + \frac{\partial \mathcal{V}_e^{int}(\Gamma_e(\mathbf{q}))}{\partial \Gamma_e(\mathbf{q})} \right] - \mathbf{P}_\Gamma^T \mathbf{f}_c^{ext} \right) \\ &= - \sum_{e \in E} \xi_e \dot{\mathbf{q}}^T (\mathbf{P}_\Gamma^T)_e \frac{\partial \mathcal{D}_e(\dot{\Gamma}_e(\mathbf{q}))}{\partial \dot{\Gamma}_e(\mathbf{q})} + \dot{\mathbf{q}}^T \mathbf{P}_\Gamma^T \mathbf{f}_{nc}^{ext} \quad (\text{Using (4.39)}) \\ &= - \sum_{e \in E} \xi_e (\dot{\Gamma}_e(\mathbf{q}))^T \frac{\partial \mathcal{D}_e(\dot{\Gamma}_e(\mathbf{q}))}{\partial \dot{\Gamma}_e(\mathbf{q})} + (\dot{\Gamma}(\mathbf{q}))^T \mathbf{f}_{nc}^{ext} \\ &= -d\tilde{\mathcal{D}}_r(\dot{\mathbf{q}}, \mathbf{q}) + (\dot{\Gamma}(\mathbf{q}))^T \mathbf{f}_{nc}^{ext} \quad (\text{Using (4.33), (4.38)}) \end{aligned} \quad (4.40)$$

When compared with (4.34) and (4.38), Equation (4.40) proves that EECSW preserves the Lagrangian structure associated to the Hamiltonian principle for a non-conservative system, as well.

The conditional/unconditional numerical stability properties of any chosen time integration scheme are carried over to the hyper-reduced model as a direct consequence of the energy-conserving/structure-preserving properties of EECSW. We refer the reader to Section 4.3 in [42] for the discussion on how these properties lead to numerical stability.

4.4 Numerical results

4.4.1 Setup

To test and illustrate the performance of the EECSW method, proposed here, we require a high-dimensional system which can be reduced using a suitable nonlinear mapping. To this end, we use the quadratic manifold, introduced in [63], for model reduction. In the quadratic manifold approach, the full set of unknowns $\mathbf{u}$ are mapped to a lower dimensional set of variables $\mathbf{q}$ using a quadratic mapping as

$$\mathbf{u} \approx \Gamma(\mathbf{q}) := \Phi \mathbf{q} + \frac{1}{2} \Omega : (\mathbf{q} \otimes \mathbf{q}), \quad (4.41)$$

where the columns of $\Phi \in \mathbb{R}^{n \times M}$ contains M significant vibration modes (VMs) extracted at equilibrium $\mathbf{u} = \mathbf{0}$, and $\Omega \in \mathbb{R}^{n \times M \times M}$ is a third-order tensor containing the corresponding static modal derivatives (SMDs). The symbol $\otimes$ denotes the dyadic product, while $:$ indicates contraction over the last two indices ².

For consistency, we use the same examples used in [63]: Model-I, a flat plate, simply-supported on two opposite sides, and Model-II, a NACA-airfoil-based wing model. The details of both models are shown in Figures 2.1 and 2.7. Time integration is performed using the implicit Newmark scheme with a time step constant across all techniques for a fair comparison. Both structures are discretized using flat, triangular-shell elements with 6 DOFs per node, i.e., 18 DOFs per element. For both models, the accuracy of the results is compared to the corresponding full nonlinear solutions using a mass-normalized global relative error (GRE) measure, defined as

$$GRE_M = \frac{\sqrt{\sum_{t \in S} (\mathbf{u}(t) - \tilde{\mathbf{u}}(t))^T \mathbf{M} (\mathbf{u}(t) - \tilde{\mathbf{u}}(t))}}{\sqrt{\sum_{t \in S} \mathbf{u}(t)^T \mathbf{M} \mathbf{u}(t)}}} \times 100,$$

where $\mathbf{u}(t) \in \mathbb{R}^n$ is the vector of generalized displacements at the time t , obtained from the HFM solution, $\tilde{\mathbf{u}}(t) \in \mathbb{R}^n$ is the solution based on the (hyper)reduced model, and S is the set of time instants at which the error is recorded. The mass matrix $\mathbf{M}$ provides a relevant normalization for the generalized displacements, which could be a combination of physical displacements and rotations, as is the case in the shell models shown here.

All computations were performed using MATLAB on a single node of the Euler cluster of ETH Zürich with a requested RAM of 8GB. Since the success of reduction techniques is often reported in terms of savings in simulation time, we define an online speedup S^* computed according to the following simple formula:

$$S^* = \frac{T_{full}}{T_{sim}^*},$$

where T_{full} and T_{sim}^* represent the CPU time taken during the time integration of *full* and (hyper)reduced solution respectively. The superscript $\star$ denotes the reduction technique being used.

We compare the hyper-reduction using ECSW (where the ROMs are obtained from linear mappings), with the EECSW method. For both ECSW and EECSW, the convergence tolerance τ for problem (P1), as required in Algorithm 2, is chosen to be 0.01. This is within the practically recommended range as reported by Farhat *et al.* [41]. A total of $n_t = 200$ training vectors, chosen uniformly from the solution time-span, are used in sparse NNLS routine. The following ROMs are considered in our comparison:

- *Linear-mapping-based (Hyper-)Reduction:*

POD: Here, the first most significant m Proper Orthogonal Decomposition (POD) modes obtained from the HFM run are used to construct $\mathbf{V}$.

ECSW-POD: Hyper-reduction using ECSW is performed, for the ROM obtained in the POD approach. This is the classic ECSW [41] implementation.

- *Quadratic-manifold-based (Hyper-)Reduction:*

²The mapping (4.41) can be written using the Einstein summation convention in the indicial notation as: $\Gamma_I = \Phi_{Ii} q_i + \frac{1}{2} \Omega_{Iij} q_i q_j$ $I \in \{1, \dots, n\}$, $i, j \in \{1, \dots, M\}$

QM: Here, the quadratic mapping (4.41) is used to construct the ROM as given by (4.3).

EECSW-QM: The ROM obtained in QM approach using the quadratic manifold is hyper-reduced using the EECSW.

4.4.2 Flat Structure

We consider the rectangular flat plate structure shown in Figure 2.1. A uniform pressure distribution is chosen to act normal to the plate surface as the external load. A time varying amplitude (load function) is used given by

$$\begin{aligned} \mathbf{g}(t) &= p(t)\mathbf{l}, \\ p(t) &= \sin(\omega t), \end{aligned} \quad (4.42)$$

where $\mathbf{l}$ is a constant load vector corresponding to a uniform pressure distribution of 1 N/mm². Here, $p(t)$, termed as the dynamic load function, determines the time-dependency of the external load. The results are shown for a periodic choice for $p(t)$ as given by eq. (4.42), where ω is chosen to be the first natural frequency of the linearized system. Due to the resonant nature of the dynamics, it is easy to excite geometrically nonlinear behavior in the system, as shown by Figure 4.1. Furthermore, Figure 4.1 shows that the full nonlinear solution is reduced using the QM technique and, subsequently, hyper-reduced using EECSW with good accuracy. Here, we use the first and the fifth bending modes and the corresponding modal derivatives to construct the QM. In this example, these two modes were found to contribute significantly to the linearized solution of the system and hence are used for reduction.

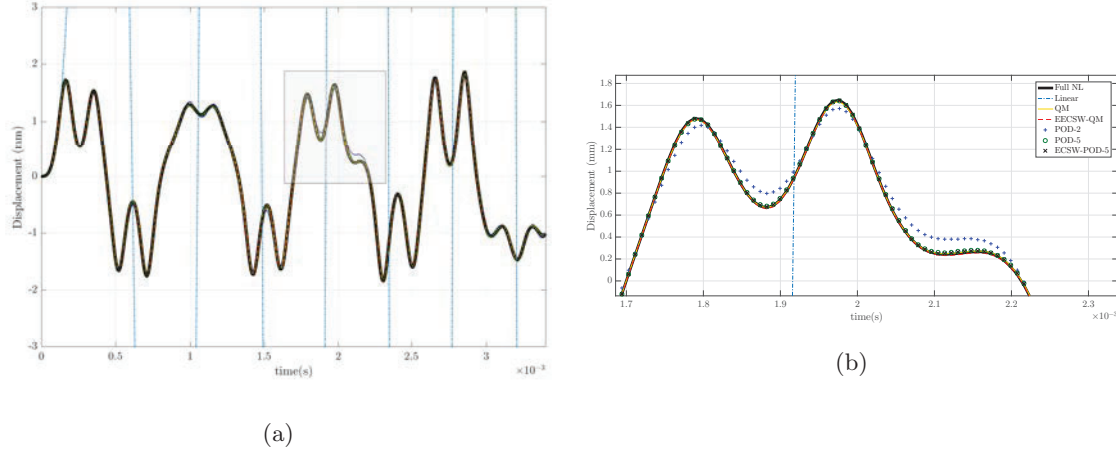


Figure 4.1: The linear solution (dashed-blue) grows quickly to high amplitudes for the choice of resonant loading; the full solution (thick black) shows much lower amplitudes due to the geometrical nonlinearities; the ROM obtained using quadratic manifold (yellow) and hyper-reduced solution for this ROM using EECSW (dashed-red) capture the full solution accurately. Subfigure (b) highlights the inaccuracy of the POD ROM with $m = 2$.

It can also be confirmed from the global relative error GRE_M and speed-up S^* , shown in Table 4.1, that the QM based nonlinear ROM reproduces the results of Model-I with good accuracy but, as expected, does not deliver a good speed-up. The EECSW-based hyper-reduction, on the other hand, gives results with a GRE_M of 1.37% but delivers a much higher speed-up of about 100. Note also that a linear-mapping-based ROM with POD of the same size ($m = 2$) as the adopted nonlinear mapping does not deliver a good accuracy.

Reduction Technique	Basis size (m)	# elements	$GRE_M(\%)$	S^*
POD	2	400	5.62	2.96
POD	5	400	1.81	2.84
ECSW-POD	5	12	2.52	77.7
QM	2	400	1.32	3.03
EECSW-QM	2	6	1.37	99.93

Table 4.1: Comparison of global relative error GRE_M and speed-ups in Model-I for linear-mapping-based ROMs and their hyper-reduced counterparts on one hand, and for quadratic-manifold-based ROMs and their hyper-reduced counterparts using EECSW on the other hand. Time for a full solution run $T_{full} = 74.4$ s.

4.4.3 Wing Structure

For the application of the proposed extension of EECSW to more realistic models, we consider the NACA-airfoil-based wing structure model of Chapter 2, introduced in ref. [63]. This model, referred to as Model-II, contains truly high number of DOFs ($n = 135770$), thereby, allowing for the appreciation of obtained accuracy and computational speed-ups. We simulate the response of the structure to a low frequency pulse load, applied as a spatially uniform pressure load on the highlighted area on the structure skin (cf. Figure 2.7). The dynamic load function, shown in Figure 4.2a, is given as

$$p(t) = A \sin^2(\omega t) \left[H(t) - H\left(\frac{\pi}{\omega} - t\right) \right], \quad (4.43)$$

where $H(t)$ is the Heaviside function and ω chosen as the mean of the first and second natural frequency of vibration. We refer the reader to [63] for the detailed linearized, full nonlinear and reduced nonlinear response plots of this model.

Figure 4.2b and Table 4.2 show that the hyper-reduction using EECSW for the quadratic-manifold based ROM reproduce the full system response with a very good accuracy ($GRE_M \approx 1.71\%$) and is more than a thousand times faster than the full response computation. As expected, we obtain a much higher speed-up during hyper-reduction in comparison with Model-I. The speed-up without hyper-reduction, however, are fairly similar between the two models, despite the striking difference in the number of DOFs. Thus, the numerical results confirm the need and effectiveness of EECSW for hyper-reduction of large nonlinear systems over nonlinear manifolds.

Figure 4.2 and Table 4.2 compare the response of Model-II to the applied load. For a given number of reduced unknowns ($m = 5$), Figure 4.2 further shows that the response of the full system is captured more effectively using the quadratic manifold as compared to the POD (POD-5). This underscores the importance of nonlinear manifolds in model reduction of nonlinear systems. In terms of speed as well, we see in Table 4.2 that the EECSW method performs better than its linear counterpart.

It is interesting that in this case, EECSW-QM ends up sampling a much smaller set of elements as compared to its linear counterpart EECSW-POD, leading to a higher speed-up. This is not to say that nonlinear-manifold-based hyper-reduction would always outperform its linear mapping counterpart. In fact, for a given basis size (m) and same number of elements in the sampled mesh, the EECSW method equipped with nonlinear mapping can be expected to be slower than ECSW-POD approach on two accounts:

1. The additional convective terms in the nonlinear-mapping-based ROM (4.4) adds to the computational burden in comparison with its linear counterpart (4.5)
2. The evaluation of the nonlinear mapping (quadratic in this case), is expected to be more ex-

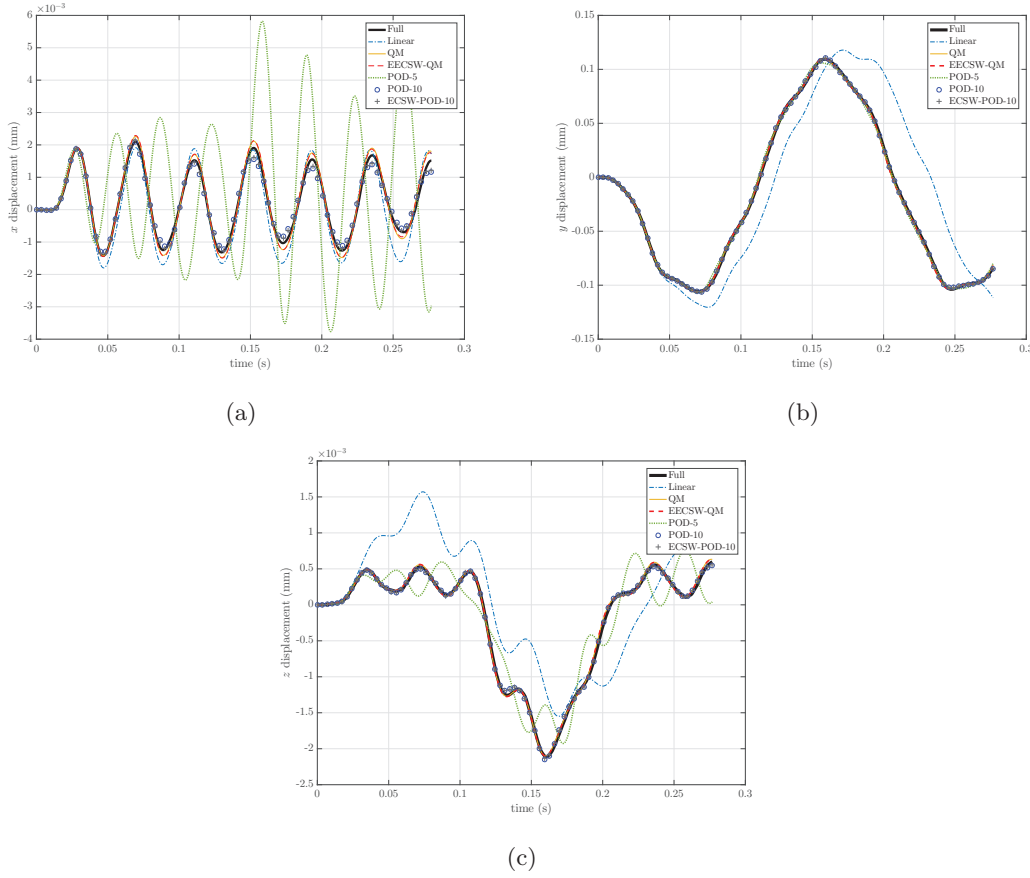


Figure 4.2: (a), (b), (c): Comparison of full, linear and various reduced responses at tip node 2 in x, y and z directions. Note that the QM-based reduced solution (with 5 reduced unknowns) accurately captures the full system response, where as the POD-based ROM with 5 modes fails to reproduce the nonlinear system response (POD-5).

pensive that a linear mapping, which boils down to a standard matrix-vector product. Though MATLAB is extremely efficient at performing such matrix-vector products, and hence, in the evaluation of linear mappings, the same cannot be said about our sub-optimal implementation for evaluation of the quadratic mapping at hand.

However, for large nonlinear systems, we expect nonlinear manifolds would require much lesser variables to capture the nonlinear behavior in comparison with linear mappings in order to achieve the desired accuracy (cf. discussion in Introduction). We have attempted to demonstrate this using a simple quadratic manifold on a large system, where only 5 unknowns were required to match the accuracy obtained using 10 POD modes (cf. Table 4.2 and Figure 4.2). In our experience, a smaller size reduced system usually leads to a smaller size of reduced mesh during training phase as well. This is intuitively clear since the sNNLS problem according to Algorithm 2 has fewer constraints to satisfy. Thus, despite the extra costs mentioned in the two points above, EECSW is capable of producing better computational speed-up when equipped with the proper nonlinear manifold (as also seen in Tables 4.1 and 4.2).

Reduction Technique	Basis size (m)	# elements	$GRE_M(\%)$	S^*
POD	5	49968	5.01	3.07
POD	10	49968	0.98	2.77
ECSW-POD	10	152	1.05	794.6
QM	5	49968	1.65	2.92
EECSW-QM	5	44	1.71	1098

Table 4.2: Reduction techniques as described in Section 4.4.1 are formulated. Global Relative Error and speed-ups in Model-II for these reduction techniques are tabulated. A total of $n_t = 200$ training vectors, chosen uniformly from the solution time-span, are used to setup hyper-reduction in ECSW-POD and EECSW-QM. Time for a full solution run $T_{full} = 3.808 \times 10^4$ s.

4.5 Conclusion

Nonlinear behavior of large structural systems can often be more effectively reduced using nonlinear manifolds in comparison with linear mappings. Though nonlinear manifolds help in reducing the number of unknowns in a large system, hyper-reduction is further needed to achieve the corresponding computational speed up using such ROMs. In this chapter, we extended the recently developed ECSW technique to allow for hyper-reduction using smooth nonlinear mappings into the reduced system of unknowns. In doing so, we also showed that the ROM obtained by projection onto the tangent space of the manifold leads to the preservation of structure and stability properties of the ECSW method.

We demonstrated the EECSW method on two structural dynamics examples by hyper-reduction of a ROM based on the quadratic manifold, whereby better speed and accuracy were obtained using the EECSW method in comparison with ECSW-POD based hyper-reduced models. Since these models were based on von Kármán nonlinearities, the degree of nonlinearity in the examples is expected to be much lower than the nominal thin-walled structures featuring geometrically nonlinear behavior. A quadratic manifold was, therefore, effective in capturing the nonlinear behavior effectively for these structures (cf. Chapter 2). However, nonlinear manifolds – possibly constructed from solution databases – would play a crucial role in the effective reduction of models with a higher degree of nonlinearity (cf. Millán & Arroyo [85]). EECSW in such cases is expected to play a vital role during the hyper-reduction stage by providing the necessary computational speed-ups.

Furthermore, it is easy to see that the proposed extension is capable of handling parameter-dependence in an analogous manner as the original ECSW. However, before applying hyper-reduction using EECSW, we require parameter-dependent nonlinear manifolds in the construction of such ROMs. This is certainly not as straight forward as in the linear mapping context, where reduction bases are sometimes even linearly interpolated. Nonetheless, in data-based approaches, parameter dependence can be included in the manifold learning phase. In particular, the parameters can be explicitly modeled as the known features, apart from the reduced variables which would be identified in the manifold learning procedure. This would directly result in parameter-dependent nonlinear manifolds obtained from data. This underscores the need for manifold learning in model reduction with parameter dependence. Thus, using the EECSW on ROMs based on nonlinear manifolds constructed using manifold-learning techniques forms an essential part of our current and future efforts.

Chapter 5

Hyper-reduction: reducing offline costs

Abstract. In this chapter, we propose an efficient method to significantly reduce the offline cost associated with the construction of training sets for hyper-reduction of geometrically nonlinear, finite element discretized, structural dynamics problems. The ROM is obtained by projecting the governing equation onto a basis formed by vibration modes and corresponding modal derivatives, thus avoiding cumbersome manual selection of high-frequency modes to represent nonlinear coupling effects. Cost-effective hyper-reduction is then achieved by lifting inexpensive linear modal transient analysis to a quadratic manifold constructed with dominant modes and related modal derivatives. The training forces are then computed from the so obtained representative displacement sets. In this manner, the need for full simulations required by traditional, POD-based projection and training is completely avoided. In addition to significantly reducing the offline cost, our numerical examples show that this technique selects a smaller hyper-reduced mesh as compared to POD-based training and therefore leads to larger online speedups. Hence, the proposed method constitutes a solid alternative to direct methods for the construction of the reduced order model, which suffer from either high intrusiveness into the finite element code or expensive offline nonlinear evaluations for the determination of the nonlinear coefficients. This chapter closely follows the current author's work in the original reference¹[59].

5.1 Introduction

The classical projection-based reduced-order models are obtained via a linear projection of the full system of equations onto a reduction basis, which has been precomputed *offline*. This basis spans a low-dimensional subspace suitable for capturing the HFM solution. As a result, the ROM is characterized by only a few DOFs. The construction cost of the reduced operators in the projected equations, however, scales with the size of the HFM and not with that of the ROM. This constitutes a serious bottleneck for achieving significant speed-ups, as the assembly of the nonlinear reduced operators dominates the cost of the time integration. Hence, such ROMs are effective only if these reduced operators can be precomputed offline, which is not the case for nonlinear systems in general. To this end, various techniques have been developed over the recent years, which have made the *online* computation of the reduced nonlinear operators in ROMs much cheaper [17, 27, 28, 41, 123].

¹Jain, S. & Tiso, P. Simulation-free hyper-reduction for geometrically nonlinear structural dynamics: a quadratic manifold lifting approach. *Journal of Computational and Nonlinear Dynamics* **13**, 071003. doi:[10.1115/1.4040021](https://doi.org/10.1115/1.4040021) (2018).

These are commonly referred to as *hyper-reduction* techniques (cf. Section 4.1 for a brief review).

Generally, hyper-reduction techniques alleviate the computational costs of the nonlinear terms by optimally selecting a small set of nodes (or elements) in the mesh over which the nonlinearity is evaluated. The nonlinearity is then cheaply interpolated over the rest of the mesh. This selection process is performed with the help of *training* vectors, which are usually obtained from the solution of the HFM. Such *full*-solution vectors are often also used for the construction of the reduction basis in projection-based ROMs, for example, using the proper orthogonal decomposition (POD) [7, 65, 68]. The use of these full-solution *snapshots* for training and reduction purposes is a computationally expensive affair, which can be unaffordable, especially at the preliminary design stage of structures.

Many techniques enable the construction of a ROM or a reduction basis for nonlinear problems without the need of a full solution (cf. [14, 62, 84]). Indeed, for a certain class of problems—characterized by mild geometric nonlinearities—the use of vibration modes (VMs) and modal derivatives (MDs) has been shown to be effective for the construction of a modal-based reduction basis [55, 63, 105]. Indeed, the robustness and effectiveness of full-simulation vectors in general nonlinear reduction scenarios (e.g. using POD) cannot be discounted. Nonetheless, such modal-based reduction techniques, albeit being sub-optimal, find use when the available time and computational resources do not allow for creating a database of full simulation runs. By avoiding full solutions, these techniques go a long way in reducing offline costs incurred during construction of an effective reduction basis.

The generation of training vectors for hyper-reduction of nonlinear terms in the ROM, however, still leads to tremendous offline costs in the form of full-simulations. To this effect we propose a modal-based training-set generation technique, which completely avoids the HFM simulations, thereby, making the reduction procedure truly *simulation-free*. As an alternative to hyper-reduction of nonlinearity, techniques which precompute the reduced nonlinear operators in the form of polynomial tensors up to cubic degree are also common in literature [5, 70, 73, 87, 124]. These techniques, however, are usually based upon an ad-hoc selection of VMs and are limited in their scope (see Section 5.6 for a discussion).

The modal derivatives have been classically used to form a reduction basis along with VMs in order to systematically capture the geometrically nonlinear behavior. Recently, these were also used for reduction via a quadratic manifold (QM) [63], where a linear subspace, formed by a truncated set of VMs, captures the linearized dynamics near the equilibrium and the corresponding MDs provide the necessary nonlinear (quadratic) extension to this subspace. In this work, we use this notion of a QM to generate meaningful training vectors from a linear modal superposition of the underlying linearized system. This unified approach builds upon the linear modal signature, which is rather inexpensively available and essential for the analysis of any structural system. Similar to Chapters 2 and 4, we have tested this approach on a simple, illustrative example of a curved plate as well as on a realistic structural model of an aircraft wing. To demonstrate effectiveness of the approach, we have compared the offline costs involved in reduction and hyper-reduction of these examples with that of other established techniques. This is done by taking into account all the effort needed to construct such a simulation-free hyper-reduced ROM.

This chapter is organized as follows. In the next section, we start by reviewing the concepts of projection-based model reduction, which results in the reduction of the dimensionality of the HFM. The concept of hyper-reduction is reviewed in Section 5.3.2, where we propose the use of a stability-preserving, FE-based hyper-reduction technique, known as energy conserving sampling and weighing (ECSW) [41]. The main idea of modal-based training-set generation using the QM, is presented in Section 5.4. The results for the numerical examples, along with comparisons with other techniques, are presented in Section 5.5. Section 5.6 discusses the state-of-the-art alternatives in the spirit of simulation-free reduction. Finally, the conclusions are given in Section 5.7.

5.2 Setup

Consider the initial value problem (IVP) governing the structural system response introduced in Chapter 2 as

$$\begin{aligned} \mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{C}\dot{\mathbf{u}}(t) + \mathbf{f}(\mathbf{u}(t)) &= \mathbf{g}(t), \\ \mathbf{u}(t_0) &= \mathbf{u}_0, \quad \dot{\mathbf{u}}(t_0) = \mathbf{v}_0, \end{aligned} \quad (5.1)$$

where the solution $\mathbf{u}(t) \in \mathbb{R}^n$ is a high-dimensional generalized displacement vector, $\mathbf{M} \in \mathbb{R}^{n \times n}$ is the mass matrix, $\mathbf{C} \in \mathbb{R}^{n \times n}$ is the damping matrix, $\mathbf{f} : \mathbb{R}^n \mapsto \mathbb{R}^n$ gives the nonlinear elastic internal force as a function of the displacement $\mathbf{u}$ of the structure, and $\mathbf{g}(t) \in \mathbb{R}^n$ is the time dependent external load vector. In our case, the nonlinear term $\mathbf{f}(\mathbf{u})$ models the effect of geometric nonlinearities, arising from large deflections and rotations. Though the techniques presented in this work are general, von Kármán kinematics has been used to model geometric nonlinearities for shell-based structures.

In this work, to allow for hyper-reduction in the standard ECSW setting, we assume that the response of the HFM (5.1) is reduced using the classical notion Galerkin projection onto a suitable low-dimensional subspace $\mathcal{V}$. To this end, we introduce a linear mapping in to the space of reduced variables as

$$\mathbf{u}(t) \approx \mathbf{V}\mathbf{q}(t), \quad \mathbf{V} \in \mathbb{R}^{n \times m},$$

where $\mathbf{q}(t) \in \mathbb{R}^m$ ($m \ll n$) is the low-dimensional vector of reduced variables, and $\mathbf{V}$ is known as the reduction basis since its columns form a basis for $\mathcal{V}$. The ROM is then obtained using Galerkin projection as

$$\underbrace{\mathbf{V}^T \mathbf{M} \mathbf{V}}_{\tilde{\mathbf{M}}} \ddot{\mathbf{q}}(t) + \underbrace{\mathbf{V}^T \mathbf{C} \mathbf{V}}_{\tilde{\mathbf{C}}} \dot{\mathbf{q}}(t) + \mathbf{V}^T \mathbf{f}(\mathbf{V}\mathbf{q}(t)) = \mathbf{V}^T \mathbf{g}(t),$$

where $\tilde{\mathbf{M}}, \tilde{\mathbf{C}} \in \mathbb{R}^{m \times m}$ are the reduced mass and damping matrices, respectively. Often, the internal force can be split into its linear and nonlinear contributions as $\mathbf{f}(\mathbf{u}) = \mathbf{K}\mathbf{u} + \mathbf{f}^{nl}(\mathbf{u})$, to obtain a reduced stiffness matrix as well:

$$\tilde{\mathbf{M}}\ddot{\mathbf{q}}(t) + \tilde{\mathbf{C}}\dot{\mathbf{q}}(t) + \underbrace{\mathbf{V}^T \mathbf{K} \mathbf{V}}_{\tilde{\mathbf{K}}} \mathbf{q}(t) + \underbrace{\mathbf{V}^T \mathbf{f}^{nl}(\mathbf{V}\mathbf{q}(t))}_{\tilde{\mathbf{f}}(\mathbf{q}(t))} = \mathbf{V}^T \mathbf{g}(t). \quad (5.2)$$

It is easy to see that each of the reduced matrices $\tilde{\mathbf{M}}, \tilde{\mathbf{C}}$ and $\tilde{\mathbf{K}}$ can be precomputed in the offline stage prior to time integration. Hence, during time integration (online phase), the computational cost associated to the evaluation of the linear terms in eq. (5.2) scales only with the number of reduced variables m . This is, however, not the case for the computation of the nonlinear term $\tilde{\mathbf{f}}(\mathbf{q}(t))$. For FE-based applications, this evaluation is usually carried out online in the following manner:

$$\tilde{\mathbf{f}}(\mathbf{q}) = \mathbf{V}^T \mathbf{f}^{nl}(\mathbf{V}\mathbf{q}) = \sum_{e=1}^{n_e} \mathbf{V}_e^T \mathbf{f}_e(\mathbf{V}_e \mathbf{q}), \quad (5.3)$$

where $\mathbf{f}_e(\mathbf{u}_e) \in \mathbb{R}^{N_e}$ is the contribution of the element e towards the vector $\mathbf{f}^{nl}(\mathbf{u})$ (N_e being the number of DOFs for the element e), $\mathbf{V}_e$ is the restriction of $\mathbf{V}$ to the rows indexed by the DOFs corresponding to e , and n_e is the total number of elements in the structure. Since the reduced nonlinear term $\tilde{\mathbf{f}}(\mathbf{q})$ is evaluated in the space of full variables, the computational cost associated to its evaluation does not scale with m alone. Indeed, eq. (5.3) shows that this cost scales linearly with the number of elements in the structure, and can hence be high for large systems. Thus, despite the reduction in dimensionality achieved in ROM (5.2), the evaluation of the reduced nonlinear term $\tilde{\mathbf{f}}(\mathbf{q})$ emerges as a new bottleneck for the fast prediction of system response using the ROM. Hyper-reduction techniques such as the ECSW [41] help mitigate these high computational costs by approximation of the reduced nonlinear term in a computationally affordable manner.

5.3 Classical hyper-reduction

We briefly review the classical notion of reduction and hyper-reduction with the use of HFM solution vectors.

5.3.1 Dimensionality reduction using POD

Let $\mathbf{u}_i \in \mathbb{R}^n$, $i \in \{1, \dots, n_t\}$ be training vectors obtained from the full solution of system (5.1). Let $\mathbf{U} := [\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_{n_t}] \in \mathbb{R}^{n \times n_t}$ be the ensemble of snapshots obtained from the full solution. A lower dimensional POD basis $\mathbf{V} = [\mathbf{v}_1 \ \mathbf{v}_2 \ \dots \ \mathbf{v}_m] \in \mathbb{R}^{n \times m}$ containing $m \ll n_t$ orthogonal vectors which *best* spans the vectors in this ensemble can be obtained by the solution to the following minimization problem.

$$\min_{\mathbf{v}_i \in \mathbb{R}^n} \sum_{j=1}^{n_s} \left\| \mathbf{u}_j - \sum_{i=1}^m (\mathbf{u}_j^T \mathbf{v}_i) \mathbf{v}_i \right\|_2^2.$$

This is a least squares problem and the vectors in $\mathbf{V}$ are the left singular vectors of $\mathbf{U}$, obtained by the singular value decomposition (SVD) of $\mathbf{U}$ as²

$$\mathbf{U} = \mathbf{L} \mathbf{S} \mathbf{R}^T, \quad (5.4)$$

where $\mathbf{U}$ is factorized into unitary matrices $\mathbf{L} = [\mathbf{l}_1, \mathbf{l}_2, \dots, \mathbf{l}_n] \in \mathbb{R}^{n \times n}$ (containing the left singular vectors) and $\mathbf{R} \in \mathbb{R}^{n_t \times n_t}$ (containing the right singular vectors); and the diagonal (rectangular) matrix $\mathbf{S} \in \mathbb{R}^{n \times n_t}$ (containing corresponding singular values on the diagonal). These singular values represent the relative importance of corresponding vectors of $\mathbf{L}$ in forming the basis $\mathbf{V}$. If the singular values (and the corresponding singular vectors) are arranged in a descending order $S_{11} \geq S_{22} \geq \dots \geq 0$, it can be shown that

$$\sum_{j=1}^{n_s} \left\| \mathbf{u}_j - \sum_{i=1}^m (\mathbf{u}_j^T \mathbf{l}_i) \mathbf{l}_i \right\|_2^2 = \sum_{i=m+1}^r S_{ii}^2.$$

Thus, the left singular vectors $\mathbf{l}_i$ corresponding to the highest singular values are the most relevant for constructing a reduction basis using POD. Since an SVD can be performed over any solution snapshots of any general nonlinear problem, POD is seen as a versatile method. It should be noted, however, that such a reduction basis is optimal only for capturing a solution which is characteristic of the applied loading, and the procedure needs to be repeated (or a database of full solutions need to be created in the first place) to take other types of loading into account.

Once a reduction basis is obtained, the same ensemble of full solution vectors can be used as training vectors for hyper-reduction (e.g. using ECSW) to accelerate the computation of nonlinearity during the online stage.

5.3.2 Hyper-reduction with ECSW

Essentially, ECSW aims to identify a small set of elements E of the structure ($|E| \ll n_e$) to cheaply approximate $\tilde{\mathbf{f}}(\mathbf{q})$ as (cf. (5.3))

$$\tilde{\mathbf{f}}(\mathbf{q}) = \sum_{e=1}^{n_e} \mathbf{v}_e^T \mathbf{f}_e(\mathbf{V}_e \mathbf{q}) \approx \sum_{e \in E} \xi_e \mathbf{v}_e^T \mathbf{f}_e(\mathbf{V}_e \mathbf{q}), \quad (5.5)$$

²Without loss of generality, we assume a zero mean of the training samples in $\mathbf{U}$.

where $\xi_e \in \mathbb{R}^+$ is a positive weight attached to each element $e \in E$, which is empirically chosen to ensure a good approximation of the summation in (5.3). Clearly, if $|E| \ll n_e$, then the evaluation of the approximation in (5.5) would come at a fraction of the original computational cost associated to (5.3). In doing so, ECSW approximates the virtual work done by the internal forces on the set of vectors in the basis $\mathbf{V}$. As a consequence, the ECSW preserves the structure and the stability properties of the underlying full model [42].

The elements and weights are determined to approximate virtual work over the chosen training sets, which generally come from full solution run(s). If there are n_t training vectors in the set with $\mathbf{u}^{(i)}$ representing the i^{th} vector, then corresponding reduced unknowns $\mathbf{q}^{(i)}$ can be easily calculated using least squares as

$$\mathbf{q}^{(i)} = (\mathbf{V}^T \mathbf{V})^{-1} \mathbf{V}^T \mathbf{u}^{(i)},$$

and element level contribution of projected internal force for each of the training vectors can be assembled in a matrix $\mathbf{G}$ as follows:

$$\mathbf{G} = \begin{bmatrix} \mathbf{g}_{11} & \cdots & \mathbf{g}_{1n_e} \\ \vdots & \ddots & \vdots \\ \mathbf{g}_{n_t 1} & \cdots & \mathbf{g}_{n_t n_e} \end{bmatrix} \in \mathbb{R}^{mn_t \times n_e}, \quad \mathbf{b} = \begin{bmatrix} \mathbf{b}_1 \\ \vdots \\ \mathbf{b}_{n_t} \end{bmatrix} \in \mathbb{R}^{mn_t}, \quad (5.6)$$

$$\mathbf{g}_{ie} = \mathbf{V}_e^T \mathbf{f}_e(\mathbf{V}_e \mathbf{q}^{(i)}), \quad \mathbf{b}_i = \tilde{\mathbf{f}}(\mathbf{q}^{(i)}) = \sum_{e=1}^{n_e} \mathbf{g}_{ie}, \quad \forall i \in \{1, \dots, n_t\}, \quad e \in \{1, \dots, n_e\}.$$

We skip the discussion of the sparse element sampling that essentially follows the same lines as given in Section 4.3.1 but generalized to the context of nonlinear mappings. For the sake completeness, however, we reproduce in Algorithm 3 the sparse nonnegative least squares algorithm [90] used for element sampling and weighing in the ECSW method [41]

Algorithm 3 Sparse NNLS for ECSW

Input: $\mathbf{G}, \mathbf{b}, \tau$
Output: $\boldsymbol{\xi} \in \mathbb{R}^{n_e}$ sparse, $E \subset \{1, \dots, n_e\}$

```

1:  $E \leftarrow \emptyset, Z \leftarrow \{1, \dots, n_e\}, \boldsymbol{\xi} \leftarrow \mathbf{0} \in \mathbb{R}^{n_e}$ 
2: while  $\|\mathbf{G}\boldsymbol{\xi} - \mathbf{b}\|_2 > \tau\|\mathbf{b}\|_2$  do
3:    $\boldsymbol{\mu} \leftarrow \mathbf{G}^T(\mathbf{b} - \mathbf{G}\boldsymbol{\xi})$ 
4:    $[\nu, e] = \max(\boldsymbol{\mu}) \triangleright \max$ : returns maximum value in a vector followed by its location (index)
5:    $E \leftarrow E \cup \{e\}, Z \leftarrow Z \setminus \{e\}$ 
6:   while true do
7:      $\boldsymbol{\zeta}_E \leftarrow \mathbf{G}_E^\dagger \mathbf{b}$   $\triangleright \dagger$  represents pseudo-inverse
8:      $\boldsymbol{\zeta}_Z \leftarrow \mathbf{0}$ 
9:     if  $\boldsymbol{\zeta}_E > \mathbf{0}$  then
10:       $\boldsymbol{\xi} \leftarrow \boldsymbol{\zeta}$ 
11:      break
12:     end if
13:      $\eta = \min_{k \in E} \xi_k / (\xi_k - \zeta_k)$ 
14:      $\boldsymbol{\xi} \leftarrow \boldsymbol{\xi} + \eta(\boldsymbol{\zeta} - \boldsymbol{\xi})$ 
15:      $Z \leftarrow \{i | \xi_i = 0\}$ 
16:      $E \leftarrow \{1, \dots, n_e\} \setminus Z$ 
17:   end while
18: end while
```

We summarize the previous discussion in Algorithm 4 outlining the steps involved in a ECSW-based hyper-reduction with POD, using full-solution snapshots for training.

Algorithm 4 ECSW-POD**Input:** $\mathbf{u}^{(i)}, i \in \{1, \dots, n_t\}$ snapshots from *full* solution**Output:** $\xi \in \mathbb{R}^{n_e}$ sparse, $E \subset \{1, \dots, n_e\}$ for ECSW-POD

- 1: $\mathbf{U} \leftarrow [\mathbf{u}^{(1)}, \dots, \mathbf{u}^{(n_t)}]$
- 2: $[\mathbf{V}] \leftarrow \text{POD}(\mathbf{U}, M)$
- 3: $[\mathbf{G}, \mathbf{b}] \leftarrow \text{CONSTRUCTGb}(\mathbf{U}, \mathbf{V})$ ^a
- 4: $[\xi, E] \leftarrow \text{SNNLS}(\mathbf{G}, \mathbf{b})$ ^b

^aCONSTRUCTGb constructs matrix $\mathbf{G}$ and vector $\mathbf{b}$ using the training sets in $\mathbf{U}$ and reduction basis $\mathbf{V}$, according to (5.6)

^bSNNLS performs sparse non-negative least squares solution according to Algorithm 3

5.4 Simulation-free hyper-reduction

In this section, we propose a systematic procedure for obtaining a reduction basis, as well as the necessary training vectors for hyper-reduction, while completely avoiding full simulation runs.

5.4.1 Modal-based reduction basis

For geometrically nonlinear problems with separation in time scales between in-plane and out-of-plane behavior (beams, shells, thin plates, and their assemblage), it is well known that the VMs and MDs form a meaningful basis to represent the nonlinear displacement field [16, 55, 63, 113]. We propose their use in construction of the required reduction basis $\mathbf{V}$ as

$$\Psi = [\phi_1, \dots, \phi_M, \dots, \theta_{ij}, \dots], \quad i, j \in \{1, \dots, M\}, \quad (5.7)$$

$$\mathbf{V} = \text{orth}(\Psi), \quad (5.8)$$

where $\mathbf{V}$ is obtained after an orthogonalization of the matrix Ψ in (5.7), orth represents any routine to orthogonalize any general matrix such that the result is a orthonormal matrix with a full column rank (e.g., the Gram-Schmidt orthogonalization). The matrix Ψ contains the relevant VMs and MDs, $\phi_1, \dots, \phi_M$ represent a truncated set of VMs obtained from the solution of the undamped eigenvalue problem³

$$(\mathbf{K} - \omega_i^2 \mathbf{M}) \phi_i = \mathbf{0} \quad \forall i \in \{1, \dots, n\}, \quad (5.9)$$

and θ_{ij} are the MDs, obtained from the solution to the following problem

$$(\mathbf{K} - \omega_i^2 \mathbf{M}) \theta_{ij} + \left(\frac{\partial^2 \mathbf{f}(\mathbf{u})}{\partial \mathbf{u} \partial \mathbf{u}} \bigg|_{\mathbf{u}=\mathbf{0}} \phi_j - \frac{\partial \omega_i^2}{\partial \eta_j} \mathbf{M} \right) \phi_i = \mathbf{0}, \quad i, j \in \{1, \dots, M\}. \quad (5.10)$$

This represents a derivative of the eigenvalue problem (5.9) with respect to the modal amplitude η_j of the VM ϕ_j , after $\mathbf{K}$ is replaced by the tangent stiffness matrix $\frac{\partial \mathbf{f}(\mathbf{u})}{\partial \mathbf{u}}$. The singular system (5.10)

³Here we neglect the damping contribution in eigenvalue problem to avoid complex eigenvalues and vectors. Note that for damped linear systems with low damping or *modal/Rayleigh* damping as explained in [45], the eigenvectors for an undamped system are a good approximation for the damped counterpart and still form a good basis for linear modal superposition. Such a damping is very popular in structural dynamics and the theory is illustrated in this context. Under large damping, the real and imaginary parts of the most significant complex eigenvectors would be added to the matrix Ψ .

requires an additional normalization condition to solve for $\boldsymbol{\theta}_{ij}$. We use the mass normalization of the VMs, which results in the following normalization constraint:

$$\boldsymbol{\phi}_i^T \mathbf{M} \boldsymbol{\theta}_{ij} = \mathbf{0} \quad \forall i, j \in \{1, \dots, M\}.$$

Various techniques exist to obtain MDs (cf. [63] for a brief review). These include a somehow effective static approximation of the MDs, obtained by neglecting the mass terms in (5.10) as

$$\boldsymbol{\theta}_{ij}^{static} = -\mathbf{K}^{-1} \left[\frac{\partial^2 \mathbf{f}(\mathbf{u})}{\partial \mathbf{u} \partial \mathbf{u}} \bigg|_{\mathbf{u}=\mathbf{0}} \boldsymbol{\phi}_j \right] \boldsymbol{\phi}_i, \quad (5.11)$$

where $\boldsymbol{\theta}_{ij}^{static}$ were first introduced in [55]), and termed as static MDs (SMDs) in [63].

It is also easy to see that the size of the matrix $\boldsymbol{\Psi}$ is $\mathcal{O}(M^2)$, if all MDs corresponding to M VMs are included. This could potentially increase the reduction basis significantly as M increases. Techniques to effectively select MDs (cf. [63, 120]) relevant for a given load scenario are useful in such situations. The SMDs have been used in this work for construction of the QM. We have used the following simple criterion (called maximum modal interaction (MMI) in [63]) to assign weight W_{ij} to an SMD $\boldsymbol{\theta}_{ij}^{static}$:

$$W_{ij} = \int_0^T |\eta_i(t) \eta_j(t)| dt, \quad i, j \in \mathcal{M} \quad (5.12)$$

where $\eta_i(t)$ and $\eta_j(t)$ are the modal amplitudes for VMs $\boldsymbol{\phi}_i$ and $\boldsymbol{\phi}_j$, respectively, obtained during linear modal superposition for the applied loading. Along with the M VMs, a total of M most significant SMDs have been used to construct the reduction basis. Thus, after orthogonalization (5.8), an effective reduction basis with size $m \leq 2M$ can be obtained with minimal offline costs using VMs and MDs.

5.4.2 QM-based training set generation

Starting with the idea of modal superposition, we propose a QM-based approach [63] to obtain relevant training vectors without the need of these full solution vectors. The linear modal superposition using a few significant VMs is a well-established technique to obtain the reduced solution of a linear dynamical system. However, when the nonlinearities become significant, the modal basis can be augmented with MDs to effectively capture the response.

For proportionally damped systems, the linearized solution $\mathbf{u}_{lin}(t)$ of the HFM in response to the applied external loading $\mathbf{g}(t)$ can be spectrally decomposed using linear modal superposition as (cf. [45])

$$\ddot{\eta}_i + \beta_i \dot{\eta}_i(t) + \omega_i^2 \eta_i = \gamma_i(t), \quad (5.13)$$

$$\mathbf{u}_{lin}(t) = \sum_{i=1}^n \boldsymbol{\phi}_i \eta_i(t), \quad (5.14)$$

where $\eta_i(t)$ are the time varying modal amplitudes corresponding to the $\mathbf{M}$ -normalized VM $\boldsymbol{\phi}_i$; $\beta_i := \boldsymbol{\phi}_i^T \mathbf{C} \boldsymbol{\phi}_i$ is the damping coefficient resulting from proportional damping assumption; and $\gamma_i(t) := \boldsymbol{\phi}_i^T \mathbf{g}(t)$ denotes the modal forcing. Note that (5.13) is a system of decoupled linear ODEs which can be very easily integrated. Depending on the spatial and temporal properties of the external load $\mathbf{g}(t)$, the summation in (5.14) can be truncated to obtain

$$\mathbf{u}_{lin}(t) \approx \sum_{i \in \mathcal{M}} \boldsymbol{\phi}_i \eta_i(t) = \boldsymbol{\Phi} \boldsymbol{\eta}(t), \quad (5.15)$$

where $\mathcal{M} \subset \{1, \dots, n\}$ is the set of indices corresponding to the most significant modes in modal superposition 5.14.

Clearly, the snapshots of the linearized solution $\mathbf{u}_{lin}(t)$ would not be effective in training for nonlinear hyper-reduction. Indeed, consider, for example, a flat thin walled shell structure with some externally applied out-of-plane loading. The modal truncation would feature bending dominated modes which are out of plane for a linearized solution. The nonlinear response, however, is expected to feature in-plane effects due to bending/torsion-stretching coupling. These essential features would be missing in snapshots of $\mathbf{u}_{lin}(t)$, making them poor for training nonlinear hyper-reduction. The modal amplitudes, however, still contain important bending information of the structure and can be useful in generating effective training vectors by using the notion of a QM, which was introduced for nonlinear model reduction in [63]. There, the full unknowns are mapped to a lower dimensional set of variables $\mathbf{z}$ using a quadratic mapping as

$$\mathbf{u}(t) \approx \Gamma(\mathbf{z}(t)) := \Phi \mathbf{z}(t) + \frac{1}{2} \Omega : (\mathbf{z}(t) \otimes \mathbf{z}(t)), \quad (5.16)$$

where $\Phi \in \mathbb{R}^{n \times M}$ containing a truncated set of $M(= |\mathcal{M}|)$ significant VMs, resulting from the linear modal superposition; and $\Omega \in \mathbb{R}^{n \times M \times M}$ is a third order tensor containing the corresponding MDs⁴. An arbitrary vector of modal coordinates is denoted by $\mathbf{z} \in \mathbb{R}^M$. As discussed in [63], the tangent space of the QM at equilibrium ($\mathbf{u} = \mathbf{0}$) is the modal subspace represented by Φ , which is corrected using the MD information to account for nonlinear behavior upon departure from the equilibrium. Furthermore, the MDs capture the inherent bending/torsion-stretching coupling arising from geometric nonlinearities. These components arise naturally as second components of the QM. This provides a straight-forward method to generate physically-meaningful training vectors using the snapshots of the modal amplitudes $\boldsymbol{\eta}(t)$ in (5.16), as

$$\mathbf{u}^{(i)} = \Gamma(\boldsymbol{\eta}(t_i)), \quad i \in \{1, \dots, n_t\},$$

where $t_i \in \mathcal{T}$ (with $\mathcal{T}$ being the simulation time-span) are the time instants at which the modal snapshots are captured; n_t is the number of such snapshots chosen for training. Thus, the idea involves the lifting of the linear modal snapshot to the QM to account for the nonlinearity, as illustrated in Figure 5.1.

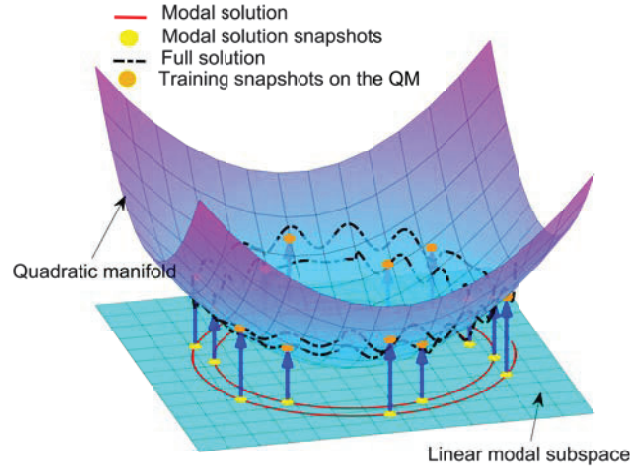


Figure 5.1: A schematic of the QM-based training set generation technique. Snapshots of the modal superposition solution, which lies in the linear modal subspace, are “lifted” on to the QM using (5.16).

This simple criterion implicitly assumes that the linear behavior is correct up to a first order and the essential nonlinear bending/torsion-stretching coupling effects (captured using the QM)

⁴The mapping (5.16) can be written using the Einstein summation convention in the indicial notation as: $\Gamma_I = \Phi_{Ii} z_j + \frac{1}{2} \Omega_{Iij} z_i z_j \quad I \in \{1, \dots, n\}, \quad i, j \in \{1, \dots, M\}$.

are of second order. This does not necessarily imply that the nonlinear forces are small. In fact, the range of deflections we are interested in makes the linear and nonlinear forces comparable. This is particularly in accordance with the von Kármán kinematics. By construction, however, this approach is not restricted to von Kármán nonlinearities. The main advantage of this method lies in the fact that the linear modal solution is available for any given system at practically no cost, resulting in physically-relevant training vectors without the need of expensive high-fidelity simulation. The training vectors thus obtained could, in principle, be used for training any hyper-reduction technique. However, for our application and testing purposes, we use the ECSW-based hyper-reduction technique. We have shown the essential steps for a simulation-free hyper-reduction using the ECSW in Algorithm 5.

Algorithm 5 Simulation-free hyper-reduction using ECSW**Input:** HFM `model` as described by (5.1), tolerance for ECSW-based training τ **Output:** $\xi \in \mathbb{R}^{n_e}$ sparse, $E \subset \{1, \dots, n_e\}$ for ECSW-Modal

- 1: $[\Phi, \eta(t)] \leftarrow \text{ModalSuperposition}(\text{model}, \mathbf{g}(t), \mathcal{T})^a$
- 2: $M \leftarrow \text{size}(\Phi, 2)^b$

Calculation of (S)MDs to construct Ω in (5.16)

- 3: $\Omega \leftarrow$ all-empty 3-tensor $\in \mathbb{R}^{n \times M \times M}$, $\Psi \leftarrow \Phi$
- 4: **for** $j \leftarrow 1$ to m **do**
- 5: Compute $\left. \frac{\partial^2 \mathbf{f}(\mathbf{u})}{\partial \mathbf{u} \partial \mathbf{u}} \right|_{\mathbf{u}=\mathbf{0}} \phi_j$
- 6: **for** $i \leftarrow 1$ to M **do**
- 7: $\omega \leftarrow \theta_{ij}$ or $\theta_{ij}^{\text{static}}^c$
- 8: $\Omega(:, i, j) \leftarrow \omega$, $\Psi \leftarrow [\Psi, \omega]$ ▷ MATLAB notation
- 9: **end for**
- 10: **end for**

Perform selection of MDs in Ψ , if needed

- 11: **if** MD selection = **true** **then**
- 12: $\Psi \leftarrow \text{MDSelection}(\Psi, \eta(t))^d$
- 13: **end if**
- 14: $[\mathbf{V}] \leftarrow \text{orth}(\Psi)^e$

Generate training vectors

- 15: Build function `QM`: $\Gamma(\mathbf{z}) \leftarrow \text{QM}(\mathbf{z})^f$
- 16: **for** $i \in \{1, 2, \dots, n_t\}$ **do**
- 17: $\mathbf{u}^{(i)} = \text{QM}(\eta(t_i))^g$
- 18: **end for**
- 19: $\mathbf{U} \leftarrow [\mathbf{u}^{(1)}, \dots, \mathbf{u}^{(n_t)}]$

Perform training

- 20: $[\mathbf{G}, \mathbf{b}] \leftarrow \text{CONSTRUCTGb}(\mathbf{U}, \mathbf{V})^h$
- 21: $[\xi, E] \leftarrow \text{SNNLS}(\mathbf{G}, \mathbf{b})^i$

^a`ModalSuperposition` performs modal superposition on the model using (5.14) and return the truncated set of $M = |\mathcal{M}|$ significant VMs in Φ and their corresponding amplitudes $\eta(t)$ according to (5.15)

^b`size`($\Phi, 2$) returns the number of columns in the matrix Φ (same as MATLAB `size` function)

^cUse MDs from (5.10) or SMDs from (5.11) in the quadratic part

^dUse MD selection techniques (cf. (5.12), [63, 120]) to retain few relevant MDs in Ψ

^e`orth` computes a full column rank orthonormal matrix $\mathbf{V}$ from Ψ shown in (same as MATLAB `orth` function)

^fUsing VMs and MDs, build function `QM` with argument $\mathbf{z}$, which returns Γ according to (5.16)

^g`QM` produces training vectors from modal snapshots $\eta(t_i)$ using (5.16)

^h`CONSTRUCTGb` constructs matrix $\mathbf{G}$ and vector $\mathbf{b}$ using the training sets in $\mathbf{U}$ and reduction basis $\mathbf{V}$, according to (5.6)

ⁱ`SNNLS` performs sparse non-negative least squares solution according to Algorithm 3

5.5 Numerical results

5.5.1 Setup

We illustrate the performance of the proposed hyper-reduction techniques on two examples of thin-walled structures. One of them is an academically-simple model of a shallow-arched rectangular plate, named Model-I. The other example is the model of a NACA-airfoil-based wing model, with realistically high number of DOFs, introduced in [63], and referred to as Model-II. The details of both models are shown in Figures 5.1 and 2.1, respectively. Both structures are modeled using flat, triangular-shell elements with 6 DOFs per node (i.e., 18 DOFs per element). For both the models, the accuracy of the results was compared to the corresponding full nonlinear solutions using a mass-normalized global relative error (GRE) measure, defined as

$$GRE_M = \frac{\sqrt{\sum_{t \in S} (\mathbf{u}(t) - \tilde{\mathbf{u}}(t))^T \mathbf{M} (\mathbf{u}(t) - \tilde{\mathbf{u}}(t))}}{\sqrt{\sum_{t \in S} \mathbf{u}(t)^T \mathbf{M} \mathbf{u}(t)}}} \times 100,$$

where $\mathbf{u}(t) \in \mathbb{R}^n$ is the vector of generalized displacements at the time t , obtained from the HFM solution, $\tilde{\mathbf{u}}(t) \in \mathbb{R}^n$ is the solution based on the (hyper) reduced model, and S is the set of time instants at which the error is recorded. The mass matrix $\mathbf{M}$ provides a relevant normalization for the generalized displacements, which could be a combination of physical displacements and rotations, as is the case in the shell models shown here.

All computations were performed in MATLAB on the Euler cluster of ETH Zürich. Since the success of reduction techniques is often reported in terms of savings in simulation time, we define an online speedup S^* computed according to the following simple formula:

$$S^* = \frac{T_{full}}{T_{sim}^*},$$

where T_{full} and T_{sim}^* represent the CPU time taken during the time integration of *full* and (hyper)reduced solution respectively. The superscript $\star$ denotes the reduction technique being used. The speed-up defined in this manner takes only the online costs into account. For a fairer comparison, we take offline costs into account by defining an *effective* speed-up as

$$S_{eff}^* = \frac{c_{on} T_{full}}{c_{off} T_{off}^* + c_{on} T_{sim}^*}, \quad (5.17)$$

where T_{off}^* represents the computational time spent *offline* for setting up a (hyper-)reduced model; $c_{on}, c_{off} \in [0, 1]$ represent the relative weights given to the online and offline costs, respectively, such that $c_{on} + c_{off} = 1$. The HFM simulation is assumed to carry zero offline costs and, thus, $S_{eff} = S = 1$ for a full HFM solution. A high value of S_{eff} would make a given (hyper-) reduction method favorable, with $S_{eff} < 1$ implying that the corresponding reduction technique is effectively more expensive than a full solution run.

One method of choosing c_{on} and c_{off} may be based on the number of simulations n_{sim} performed using a ROM as $c_{on} = \frac{n_{sim}}{n_{sim}+1}$, $c_{off} = \frac{1}{n_{sim}+1}$. In this work, we have given an equal weightage to the online and offline costs, i.e., $c_{on} = c_{off} = 0.5$ which leads to a rather conservative estimate of the effective speed-up.

During hyper-reduction using ECSW, the convergence tolerance τ for finding a sparse solution to the NNLS problem, as required in Algorithm 3, is chosen to be 0.01. This is within the practically recommended range as reported in [41]. The results from the following reduction techniques are compared in both examples considered in this work (Model-I and Model-II):

- *Simulation-free reduction* (No hyper-reduction): These technique involve the simulation of the Galerkin ROM (5.2), where the reduction basis $\mathbf{V}$ is obtained without the use of full simulation snapshots. We consider the following two sub-types:

L1: Here, a few significant VMs (say M in number) of the structure are selected based on linear modal superposition for the given load. These VMs, along with the corresponding SMDs (which would be $\frac{M(M+1)}{2}$ in number), are used to construct $\mathbf{V}$. Thus, the size of the basis would be $m = \frac{M(M+3)}{2}$.

L2: A similar basis as that in L1 is used here, except for the fact that M of most significant SMDs are selected according to the MMI criterion [63], originally proposed in [120] (cf. (5.12)). Thus, the basis contains M VMs as well as SMDs, making the basis size $m = 2M$.

- *Classical reduction* (No hyper-reduction): Here, the nonlinear system using is reduced using a POD basis.

POD-1: The reduction basis of the same size as in L1 is constructed through the SVD of the full-solution-snapshots-matrix, as explained in Section 5.3.1. Thus, $m = \frac{M(M+3)}{2}$ left singular vectors with the highest singular values are included in $\mathbf{V}$.

POD-2: Same as POD-1, except the basis size is matched with that in L2. Thus, $m = 2M$ most significant singular vectors are used.

- *Classical hyper-reduction*

ECSW-POD: The ECSW method is used with the same basis in POD-1, to hyper-reduce the nonlinearity in the ROM. The HFM simulation snapshots are used in the offline stage for training to obtain the reduced mesh.

- *Simulation-free hyper-reduction:* The proposed QM-based training set generation is used to reduce offline costs during hyper-reduction. Thus, a truly simulation-free reduction is demonstrated with the following approaches:

Modal-ECSW-L1: Here, the previously described ROM in L1 is hyper-reduced using ECSW with the help of QM-based training vectors as described in Algorithm 5.

Modal-ECSW-L2: Same as Modal-ECSW-L1, except the ROM from L2 is hyper-reduced, instead of that from L1.

5.5.2 Shallow-arch structure

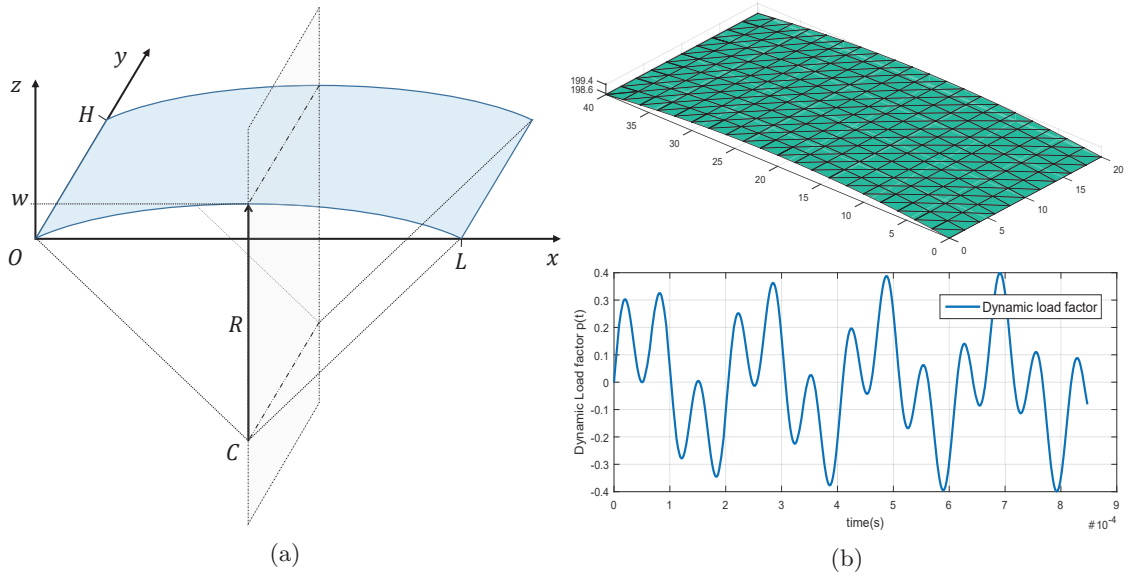


Figure 5.1: **Model - I:** (a) Schematic of a slightly curved plate. The plate is $L = 40$ mm long, $H = 20$ mm wide, $t = 0.8$ mm thick and curved with a Radius $R = 200.5$ mm and height $w = 1$ mm. The Young Modulus is $E = 70$ GPa, the Poisson's ratio is $\nu = 0.33$, and the density is $\rho = 2700$ Kg/mm³. (b) The curved plate is discretized using flat, triangular shell elements. The FE mesh contains 400 elements and 1386 DOFs. The two opposite sides parallel to the y -axis are simply supported, a uniform pressure is applied on the curved surface, according to the time history $p(t) = P[\sin(\omega t) + \sin(\pi\omega t)]$, where $P = 0.2$ N/mm² and $\omega = 2.965 \times 10^4$ rad/s.

We consider a simple arch structure, as shown in Figure 5.1. A uniform pressure distribution is chosen to act normal to the plate surface as the external load. A time varying amplitude (load function) is used given by

$$\begin{aligned} \mathbf{g}(t) &= p(t)\mathbf{l}, \\ p(t) &= p_0[\sin(\omega t) + \sin(\pi\omega t)], \end{aligned} \quad (5.18)$$

where $\mathbf{l}$ is a constant load vector corresponding to a uniform pressure distribution of 1 N/mm². Here, $p(t)$ —termed as the dynamic load factor—determines the time-dependency of the external load. The results are shown for a quasi periodic choice for $p(t)$ as given by (5.18), where ω is a typical loading frequency chosen to be the second eigenfrequency of the linearized system. The amplitude of loading is kept large enough to trigger nonlinear behavior, as shown in Figure 5.2.

A linear modal analysis, shows that the first and the fifth modes of the linearized model are sufficient for capturing the linear dynamics using modal superposition. We consider these modes along with the corresponding SMDs for reduction in L1⁵. Thus, number of modes $M = 2$ and the reduction basis size $m = \frac{M(M+3)}{2} = 5$ for the technique L1. The L2 approach uses $M = 2$ most significant SMDs in the reduction basis along with the 2 VMs, introducing $m = 2M = 4$ reduced unknowns. POD-1 and POD-2 techniques involve the use of full-solution vectors for generating

⁵Note that due to the spatially uniform nature of the loading, the second, third and fourth VMs are not excited (since they are anti-symmetric bending and twisting modes), even though the applied loading is at the second natural frequency of the model. Thus, the loading frequency is not a resonant frequency for the ROM.

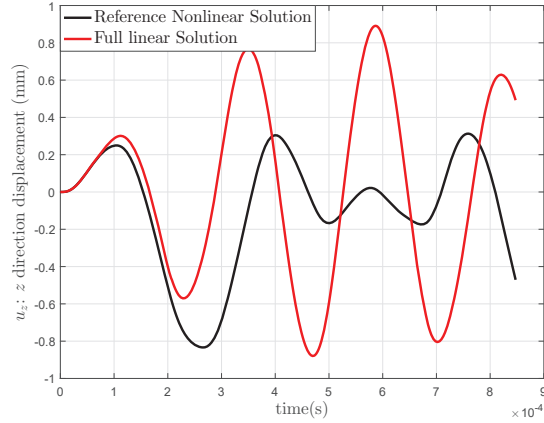


Figure 5.2: The full nonlinear solution in comparison with the linearized solution shows that the applied loading invokes nonlinear behavior. The out-of-plane displacement (in the z direction) at the middle node of the mesh in Figure 5.1 is shown.

reduction bases of size same as L1 and L2, respectively. As described in Section 5.5.1, the ROM POD-1 is hyper-reduced using the classical ECSW-POD method, while the ROMs L1 and L2 are hyper-reduced using the simulation-free hyper-reduction techniques Modal-ECSW-L1 and Modal-ECSW-L2, respectively. The results for hyper-reduction, compared with their non-hyper-reduced counterparts, are shown in Figures 5.3-5.4. The relative error estimates, and (effective) speed-ups for different reduction techniques are reported in Table 5.1.

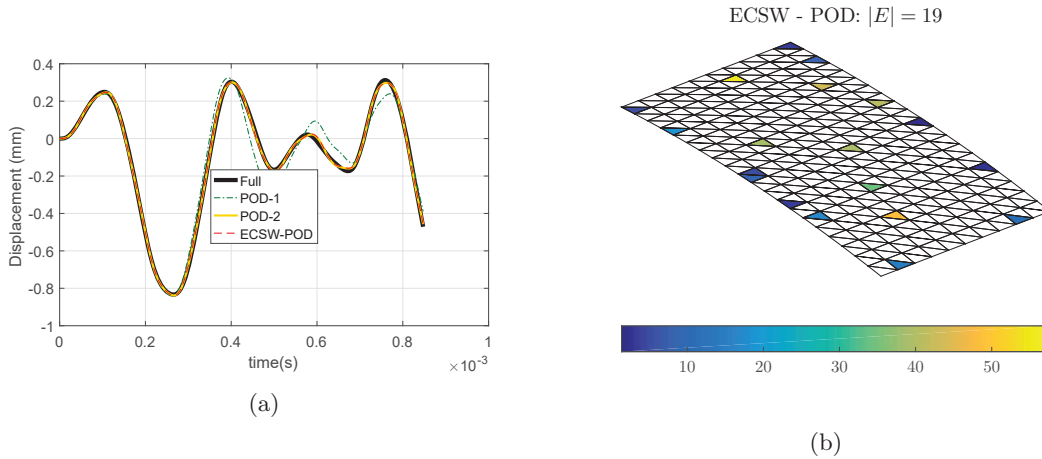


Figure 5.3: Classical (hyper-)reduction: (a) The POD-1, POD-2, ECSW-POD responses and (b) the corresponding elements and their weights selected by the sNNLS Algorithm 3.

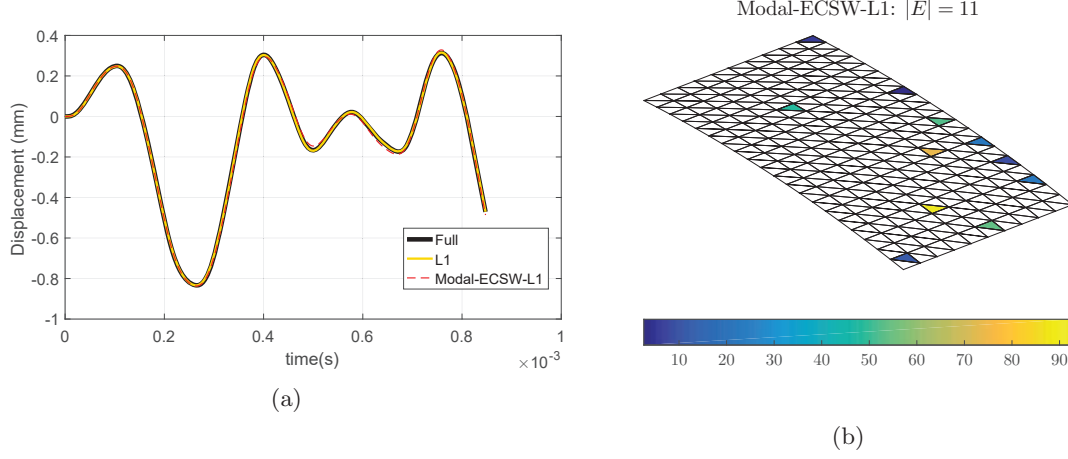


Figure 5.4: *Simulation-free* (hyper-)reduction: (a), (c) show that the Modal-based reduction techniques L1 and L2, and their Modal-based hyper-reduced counterparts are able to accurately capture the nonlinear response. (b), (d) A small fraction of the total number elements was selected for hyper-reduction by the sNNLS algorithm, resulting in high online speed-ups as reported in Table 5.1.

Table 5.1: Starting with a linear modal superposition with $M = 2$ modes (first and the fifth), reduction techniques as described in Section 5.5.1 are formulated. Global Relative Error, speed-ups and effective speed-ups (for $c_{on} = c_{off} = 0.5$) in Model-I for these reduction techniques are tabulated. A total of $n_t = 200$ training vectors, chosen uniformly from the solution time-span, are used to setup hyper-reduction in Modal-ECSW-L1/L2 as well as ECSW-POD. Computational time for the full solution run for Model-I is given by $T_{full} = 66.34$ s.

Type	Reduction Technique	Basis size (m)	# elements	$GRE_M(\%)$	S^*	S_{eff}^*
Classical reduction	POD-1	5	400	2.86	2.85	0.74
Classical hyper-reduction	ECSW-POD	5	19	3.04	50.4	0.94
<i>Simulation-free</i> reduction	L1	5	400	3.36	2.88	2.85
<i>Simulation-free</i> hyper-reduction	Modal-ECSW-L1	5	11	4.95	79.9	15.9

We report the comparison of accuracy and computational speed gains for different methods in Table 5.1. The computations were performed for POD-1, POD-2, L1, L2, Modal-ECSW-L1 and Modal-ECSW-L2. Out of these, we report here only the details for POD-1, L1 and Modal-ECSW-L1. For this small problem, the left out cases give practically the same as the ones reported in Table 5.1, which suffice in demonstrating the main point. The other cases are relevant for the wing example discussed in the next section.

Here, the POD-1 and POD-2 technique show an effective speed-up $S_{eff}^* < 1$, as expected from the very definition of S_{eff}^* (cf. (5.17)). Furthermore, the hyper-reduction of the POD-based ROMs (ECSW-POD) increases the value the effective speed-up, but these values are still poor. Indeed, this is due to the use of full solution vectors, whether for construction of the reduction basis $\mathbf{V}$, or for generation of training sets for hyper-reduction. The simulation-free reduction techniques (L1 and L2), on the other hand, show $S_{eff}^* > 1$. The simulation-free hyper-reduction techniques (Modal-ECSW-L1 and Modal-ECSW-L2) significantly increase S_{eff}^* , making them an ideal choice for effective reduction with minimal offline costs. It is interesting to note that the POD-2 approach leads to a loss of accuracy in the reduced solution, where as the simulation free L2 approach with the same size of the reduction basis performs much better in terms of the accuracy (cf. Figures 5.3 and 5.4).

5.5.3 Wing structure

For the application of the proposed simulation-free hyper-reduction methods to more realistic models, we consider the wing structure model of Chapter 2, introduced in [63]. This model (referred to as Model-II hereafter) contains truly high number of DOFs, thereby allowing for the appreciation of obtained accuracy and computational speed-ups. We simulate the response of the structure to the same low frequency pulse load as given in Section 2.5.2.

As discussed in Chapter 2, the solution to the linearized system can be accurately approximated using modal superposition with the first five VMs of the structure in this case. These VMs ($M = 5$), along with the corresponding SMDs are used to construct reduction basis to perform simulation-free reduction in L1 ($m = 20$) and L2 ($m = 10$), as discussed in Section 5.5.1. The ROMs L1 and L2 are further equipped with the QM-based training-set generation technique to perform simulation-free hyper-reduction using ECSW. The results for these techniques are compared with the classical (hyper-)reduction approaches and reported in Table 5.2.

Table 5.2: Starting with a linear modal superposition with $M = 5$ modes (first 5 VMs), reduction techniques as described in Section 5.5.1 are formulated. Global Relative Error, speed-ups and effective speed-ups (for $c_{on} = c_{off} = 0.5$) in Model-II for these reduction techniques are tabulated. A total of $n_t = 200$ training vectors, chosen uniformly from the solution time-span, are used to setup hyper-reduction in Modal-ECSW-L1/L2 as well as ECSW-POD. Time for a full solution run $T_{full} = 3.808 \times 10^4$ s.

Type	Reduction Technique	Basis size (m)	# elements	$GRE_M(\%)$	S^*	S_{eff}^*
Classical reduction	POD-1 /POD-2	20 / 10	49968	0.31 / 0.98	2.77 / 2.51	0.73/ 0.71
Classical hyper-reduction	ECSW-POD	20	278	0.38	394.6	0.97
<i>Simulation-free</i> reduction	L1 / L2	20 / 10	49968	1.65 / 1.84	2.72 / 2.74	2.69 / 2.72
<i>Simulation-free</i> hyper-reduction	Modal-ECSW-L1	20	156	1.40	638.7	38.48
	Modal-ECSW-L2	10	70	1.61	1355	44.38

The following observations can be made from the results in Table 5.2:

- The general need for hyper-reduction is apparent from the results for classical reduction as well as simulation free reduction approaches. Even for a model with a truly high number of DOFs (as Model-II here), the online speed-up (S^*) shows a value between 2 and 3 for these techniques. Indeed, the evaluation and projection of nonlinearity is a major bottleneck for saving computational time.
- For the conservative choice of equal weights assigned to online and offline costs, the Classical reduction methods (POD-1/2) show an effective speed-up $S_{eff}^* < 1$, even when equipped with hyper-reduction (ECSW-POD). This makes such techniques heavily dependent on an expensive database of full solution runs to obtain any reasonable effective speed-up for a range of load cases.
- The simulation-free reduction techniques, on the other hand, lead to effective speed-ups $S_{eff}^* > 1$. Indeed, the reduction basis constructed using modes and SMDs is much cheaper to obtain than a POD basis which requires the HFM solution vectors.
- The simulation-free hyper-reduction using the QM-based training-set generation, when applied to the ROMs L1 and L2, results in effective speed-ups which are orders of magnitude higher than any other methods. Specifically, the online speed-up for Modal-ECSW-L2 is approximately four times that of ECSW-POD. This is due to the fact that the sNNLS routine selected a quarter the number of elements for Modal-ECSW-L2 in comparison with ECSW-POD (online speed-up S^* , for ECSW is inversely proportional to the number of elements selected in sNNLS routine). Similar arguments can be made for Modal-ECSW-L1, indicating that the QM-based training set generation results in a smaller (and, hence, effective) set of elements for ECSW.

5.6 Discussion on alternative simulation-free ROM constructions

Model reduction based on vibration modes for geometrically nonlinear systems has been a topic of extensive research. As an alternative to simulation-free hyper-reduction, several methods approach the problem by assuming that the reduced nonlinear force is at most a cubic function of the reduced variables, and determining the nonlinear ROM coefficients offline ([5, 70, 73, 87, 124], cf. [84] for a review). The evaluation of these reduced nonlinear coefficients can be done

1. in a non-intrusive fashion: using a so-called stiffness evaluation procedure, whereby the structure is prescribed with numerous static displacements which are carefully selected, and a large set of nonlinear algebraic equations need to be solved to be able to determine these coefficients,
2. intrusively: For low order polynomial nonlinearities, a direct method involves the evaluation of these coefficients at the element level, exactly and intrusively (see e.g., [124]).

Model reduction based on nonlinear coefficient evaluation and natural modes has some drawbacks. It often involves the selection of a few VMs in the reduction basis to capture geometrically nonlinear effects in a rather ad hoc manner. Typically, even bending dominated response of geometrically nonlinear structures requires membrane modes to be included in the reduction basis. To the best of the authors' knowledge, there are no methods which directly give a set of vibration modes relevant for the reduction of geometrically nonlinear problems. Rather, as done for instance in refs. [5, 73, 92, 124], membrane-dominated modes need to be manually selected from the high-frequency spectrum. One must, therefore, compute many modes before finding the required additional modes. More importantly, it is often very difficult to judge which modes are required to capture the nonlinear coupling, especially for structures with complex geometries for which the membrane/bending

distinction is not well defined. Implicit condensation and expansion (cf. e.g., [53]), which involves the identification of a set of membrane modes using a series of static solutions, does away with this issue to a large extent. However, selecting relevant static fields remains a challenging and cumbersome task to ensure success with this approach. To this end, the use of MDs, along with a few low-frequency modes easily identifiable from a linear analysis, is a much straightforward way of capturing the geometrically nonlinear coupling (cf. refs. [55, 57, 63, 114]).

The indirect computation of the nonlinear coefficients is prone to several numerical challenges. Consider, for instance, a nonlinear ROM constructed using 30 modes with up to cubic nonlinearities. As many as 34280 nonlinear static load cases are needed to completely specify the nonlinear ROM. As the underlying HFM is assumed to be high-dimensional, the computational burden associated to these nonlinear solutions/evaluations⁶ can be overwhelming [109]. Although a significant reduction in the number of coefficients that need to be identified was achieved in [92], by taking into account the tangent matrix of the non-linear identification problems, one still needs load cases coming from well-selected static displacement fields to identify the nonlinear coefficients. In addition, appropriate modal amplitudes that trigger the nonlinearity to the right amount need to be prescribed, making the entire procedure somewhat difficult.

The direct computation of tensor coefficients in an intrusive manner does away with much of these numerical challenges since the tensors can be calculated exactly in a systematic manner (see e.g. [57, 124]). However, it is strictly applicable for polynomial nonlinearities in the full system. In particular, direct tensor computation cannot be applied to other types of elements where analytical polynomial expressions for internal force are not available, e.g. corotational finite elements.

Furthermore, nonlinear coefficients evaluation restricts the nonlinearities in reduced equations to low degree polynomials, usually at most cubic. One reason for this is the computation of higher-order tensors becoming intractable very quickly with the size of the reduction basis, both in terms of memory requirements as well as speed [57]. This restriction is acceptable when the full model has only up to cubic nonlinearities, e.g. in the case of von Kármán nonlinearities. Thus, for other cases with higher degrees of nonlinearity: while it is convenient to look only at lower order terms, useful physical effects might be ignored. Hyper-reduction methods address all the above issues to a large extent.

In a broad sense, Hyper-reduction is a semi-intrusive⁷ method, which facilitates very fast approximation of arbitrary nonlinearities. In ECSW-based hyper-reduction, this is done by evaluating the nonlinear terms over a reduced element set and assigning appropriate weights to the elements in such a set. Training vectors are required to obtain this reduced set of elements and corresponding weights in a systematic manner. Hence, hyper-reduction does not face the same issues encountered in a nonlinear coefficient evaluation process. However, the training vectors, which usually come from full simulations, carry huge and undesirable offline costs. We have attempted to alleviate this bottleneck in the context of geometrical nonlinearities.

Though the technique proposed here was demonstrated for von-Kármán kinematics based shell structures, the method is fairly general to include arbitrary geometrical nonlinearities and it will be a subject of future efforts. Moreover, the use of the quadratic manifold has been shown to be effective in capturing the geometrically nonlinear response of solid finite element based structures as well [105].

In the context of reducing offline costs for hyper-reduction, some very recent efforts have been made in [103], where a sequence of Krylov force vectors are used to somehow approximate the subspace spanned by the inertial forces associated to the full system. The basis vectors in this Krylov force subspace are then amplified with random Gaussian amplitudes. Finally, these force

⁶Depending on whether a force based or displacement based coefficient evaluation is used, cf. [84]

⁷We refer to hyper-reduction as semi-intrusive because although the element level contributions of nonlinear force residual and Jacobian are required for hyper-reduction, the actual expressions for these element-level nonlinearities are not needed. In other words, the element is treated as a black box.

realizations are applied to the system in order to solve for nonlinear static displacements, which are then used as training vectors for hyper-reduction. The following preliminary observations can be made about the approach in [103] in comparison with the QM-based lifting approach proposed here.

1. Since a set of random force realizations are used for training, this method is shown to be capable of handling large displacements, unlike the method proposed here.
2. It is not known (a priori or a posteriori) how many moments (or basis vectors) in the Krylov force subspace would be required to suitably approximate the inertial forces in the system. The same holds true for the standard deviation associated with the Gaussian distribution required for generating physically relevant random amplitudes and the number of random realizations needed to ensure good training. Though the approach has a wide scope, it seems that the user would need to adjust these parameters on a case-by-case basis.
3. Since the solution of nonlinear static displacements for each random realization can be a fairly expensive affair, one might incur fairly high offline costs in the process. Indeed, even after use of parallelization in the offline stage, the hyper-reduction based on the approach in [103] delivered an effective Speed-up factor of only about 2.15 on a realistic problem with a size comparable to the wing example presented in this work. Note that such an effective speed-up is even lower than that obtained using Simulation-free reduction methods without hyper-reduction (cf. Table 5.2)
4. Finally, by taking into account randomly generated amplitudes, one might over-train for hyper-reduction, resulting in a relatively large set of selected elements. This leads to limited online performance as can be deduced from the results in [103], where an online Speed-up factor of less than 5 is obtained for a realistic example with a size comparable to Model-II (cf. Table 5.2 for the corresponding online Speedups).

5.7 Conclusions

We have introduced a new method to generate training sets for hyper-reduction of geometrically nonlinear structural dynamics problems, without the need of full solution snapshot, thereby reducing offline costs significantly. This method essentially involves the projection of snapshots obtained from the (inexpensive) linear modal superposition solution, on to a quadratic manifold, which is tangent to the corresponding linear modal sub-spaces and captures the second-order nonlinear effects. As discussed in Section 5.4.2, this modal-based technique results in physically meaningful training vectors, which capture the essential bending-stretching coupling characteristic of geometrically-nonlinear thin-walled structures. The advantage of this method lies in the fact that the linear modal solution is available for any given system at practically no costs.

As remarked earlier, the QM-based training sets still rely on the underlying linear footprint of the model and correct for the relevant nonlinear effects up to the second order. It is well known that von Kármán kinematics is valid up to deflections that produce linear and nonlinear internal forces of comparable magnitude (cf. Figure 2.8). In the experience of the authors, the proposed method has been found to perform well for such a range of deflections. However, for higher displacements, the full solution may not evolve over a quadratic manifold, as also noted in [105]. The quadratic manifold is not expected to return physically relevant training vectors for hyper-reduction in such cases. Especially for resonant loading of lightly damped systems, the modal amplitudes are expected to grow significantly for a linearized system (until steady state is reached over long times). In such situations, the system response may be significantly more nonlinear than predicted by the quadratic manifold lifting.

The computational speed-ups calculated in the numerical results conservatively assume that the online and offline stages of the simulation carry equal costs. However, it should also be noted that,

in general, the full-solution vectors used for training and reduction purposes can be expected to be optimal only for the load case from which these full-solution vectors are initially obtained. Thus, practically, a database of full-solution runs for different load cases would be needed before any meaningful benefits of model-reduction could be observed. We contend the use of simulation-free reduction techniques to ease preliminary geometrically nonlinear analysis of structures, where such an expensive database of full solutions is unavailable or unaffordable. Thus, we believe that the naively defined effective speed-ups used here are still indicative.

Chapter 6

Model reduction using adaptive bases

Abstract. The thermal dynamics in thermo-mechanical systems exhibits a much slower time scale compared to the structural dynamics. In this work, we use the method of multiple scales to reduce the thermo-mechanical structural models with a slowly-varying temperature distribution in a systematic manner. In the process, we construct a reduction basis that adapts according to the instantaneous temperature distribution of the structure, facilitating an efficient reduction in the number of unknowns. As a proof of concept, we demonstrate the method on a range of linear and nonlinear beam examples and obtain a consistently better accuracy and reduction in the number of unknowns than the standard Galerkin projection using a constant basis. This chapter closely follows the current author's work in the original reference¹[61].

6.1 Introduction

Structural dynamics applications such as aerospace structures, micro-electro-mechanical systems (MEMS), civil structures, etc., often involve high-dimensional dynamical systems that also exhibit changes in mechanical behavior under the influence of external environmental effects such as thermal, acoustic etc., which manifest as coupling terms in the equations of motion. Model reduction in such coupled domains is especially challenging. In this work, we focus on structural systems featuring geometric nonlinearities and thermal stresses induced by temperature fields varying slowly in time and space, and we propose a multiple-scales-based technique to reduce the temperature-dependent equations of motion.

For modeling mechanical structures under the influence of temperature changes, a one-way coupling is often used. This means that the temperature changes affect the structural dynamics but not vice-versa. This is a reasonable assumption because roughly speaking, the thermal dynamics evolves over a much slower time-scale in comparison with the structural dynamics and though the structure can respond to temperature-changes, the same is not true for thermal response to structural motion. More specifically, this slow thermal time scale is reflected in the large time constant of temperature evolution law in comparison with the time-period of oscillation of the structure. Due to such a slow nature of temperature evolution, structural dynamics equations are sometimes even modeled with a constant temperature over a given structural time span. When the temperature does not change during a structural dynamic simulation, reduced-order models (ROM) can be constructed

¹Jain, S. & Tiso, P. Model order reduction for temperature-dependent nonlinear mechanical systems: a multiple scales approach. *Journal of Sound and Vibration*, 115022. doi:[10.1016/j.jsv.2019.115022](https://doi.org/10.1016/j.jsv.2019.115022) (2020).

using many standard techniques in literature, e.g., using vibration modes (VMs) [45], modal derivatives [55], proper orthogonal decomposition (POD) [7, 65, 68], implicit condensation and expansion (ICE) [53], among others, cf. Mignolet *et al.* [84] for a review.

As discussed in Chapter 1, projection-based nonlinear ROMs are often local in nature, i.e., these are constructed to approximate dynamical behavior locally in the vicinity of an equilibrium, as opposed to globally in the phase space. This is, especially, the case when ROMs are constructed from vibration modes and modal derivatives [55, 63], which are computed around an equilibrium and span the relevant subspaces. In the context of temperature dependence, consider a simple linear mechanical system such as

$$M\ddot{x} + C\dot{x} + K(T)x = f(t) + g(T), \quad (6.1)$$

where M, C, K^2 are mass, damping and stiffness matrices; t is the time; and T represents temperature variable(s). Usually, model reduction is performed by assuming the temperature T to be a static parameter and a reduction basis (comprising of, e.g., a few VMs) is computed for a given value of this static temperature parameter. Interestingly, in specific cases, a basis comprising of vibration modes of the unheated structure, i.e., *cold modes*, in combination with some carefully selected *dual* modes is found to be suitable for capturing the behavior of heated structures [96, 115]. It was further reported by Spottswood *et al.* [115] that the use of *hot modes* in the basis leads to an improvement in results. In general, however, a practical strategy would be to construct a database of reduced-order bases for a set of static temperature fields and perform interpolation among these precomputed bases [10, 12, 78] to come up with ROMs for other values of temperature fields.

Now, if the temperature field is indeed static for a given structural dynamics problem, these approaches may be expected to perform well. However, when the temperature varies *dynamically* during a structural simulation (e.g., due to change in thermal boundary conditions or heat source), i.e.,

$$M\ddot{x} + C\dot{x} + K(T(t))x = f(t) + g(T(t)), \quad (6.2)$$

the aforementioned techniques are not directly applicable for reducing structural dynamics equations due to two main reasons:

1. *Non-existence of an equilibrium point for local reduction:* As mentioned above, most reduction techniques usually construct relevant subspaces which are attached to an equilibrium point. However, in the case of dynamic temperature dependence, there is no such (unique) equilibrium. Even in the absence of mechanical loading, the internal force generated due to the dynamically varying temperature leads to a continuously “changing” equilibrium. Then, any local reduction technique would not be applicable since there is no unique equilibrium around which the dynamics can be approximated.
2. *No invariant subspaces for reduction:* Each temperature configuration corresponds to a new set of VMs which form relevant invariant subspaces for reduction, if temperature is kept static. However, due to a continuous change in temperature, we cannot find a fixed invariant subspace, even for the linear system (6.2).

Despite these issues, model reduction for structural equations has been successfully performed with dynamic temperature-dependence in some cases. One simple approach is to obtain a reference temperature configuration by averaging temperature variation over time for each point in space. This results in a static temperature field, around which a reduction basis can be calculated to reduce the problem where the temperature is changing dynamically. While this approach has been demonstrated to perform well in specific situations [40], it is easy to envision scenarios where this simple technique can lead to incorrect predictions. Indeed, consider a beam under the influence of

²we introduce these equations/symbols informally for the purpose of the discussion in this introductory section. A more precise treatment follows from Section 6.2 onwards.

a spatial temperature pulse which is oscillating along the length of the beam (modeling thermal effects of an oscillating shock), as examined in ref. [81]. In this case, the above-mentioned averaging approach would lead to a spatially uniform temperature distribution, around which the reduction basis would be constructed. Clearly, such an approximation would be inconsistent with the physics of the problem.

Furthermore, the above-mentioned technique of complimenting a basis of cold modes with dual modes would not be theoretically justifiable for reducing (6.2) due to the two issues above. However, these techniques have been found to work well in specific cases [81, 91] where the structural dynamics is reduced under the influence of time-varying temperature distribution.

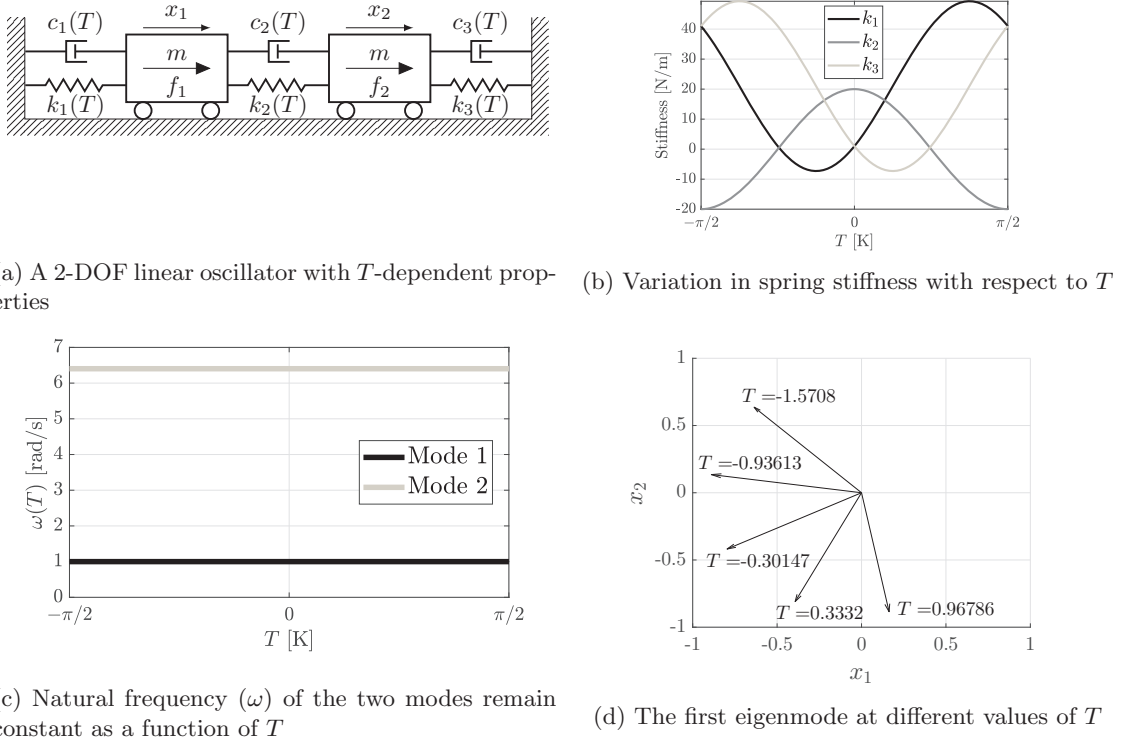


Figure 6.1: (a) A **two degree-of-freedom oscillator** with identical masses $m = 1$ Kg; $T \in [-\pi/2, \pi/2]$ represents the difference in system temperature from ambient temperature (K); $k_1(T) = a + b[1 + \cos(\alpha T) - \sin(\alpha T)]$, $k_2(T) = b \cos(\alpha T)$, $k_3(T) = a + b[1 - \cos(\alpha T) - \sin(\alpha T)]$ are the temperature-dependent spring constants, where $a = 1$ N/m, $b = 20$ N/m, $\alpha = 2$ K $^{-1}$; $c_i(T) = \beta k_i(T) \forall i \in \{1, 2, 3\}$ are the proportional damping constants with $\beta = 0.1$ s. (b) Due to the specific choice of functions, the natural frequencies of the undamped system remain constant as function of T . (c) The associated eigenvectors (VMs), however, change direction. (d) the spring constants as a function of temperature

In the context of dynamic parameter dependence, another technique [117] involves the construction of a smooth parameter-dependent basis to reduce linear problems with application to moving external loads or boundary conditions, encountered during gear meshing. In this method, a time-dependent mapping from a low-dimensional space is introduced into the full system of unknowns such that

$$u = V(p(t))q(t) \quad (6.3)$$

where V is a basis matrix, parameterized with respect to the parameters p , which are changing in time; and $q(t)$ are the reduced set of unknowns. In this setting, a set of bases which are constructed

for static parameter values are interpolated to obtain a family of bases $V(p)$, which smoothly depends on the parameters p . The time derivative of eq. (6.3) leads to convective terms in the corresponding ROM, whose evaluation can pose a numerical challenge. It is important to note that unlike the modal subspaces which are invariant for static parameters, eq. (6.3) describes an invariance relationship which does not hold in general, even for linear systems such as (6.2).

Indeed, consider a simple two-degree-of-freedom linear mechanical oscillator, as shown in Figure 6.1. The spring and damping constant in this example are assumed to be temperature-dependent such that the eigen-frequencies of the system remain unchanged and well-separated for any variation in temperature. The eigenvectors, however, change upon varying temperature. While keeping the temperature constant, we force the system near its first natural frequency. This prompts us to reduce the system using the first eigenmode, resulting in a single-degree-of-freedom ROM at any fixed temperature. The same is depicted in Figure 6.2 for a few of values of temperature.

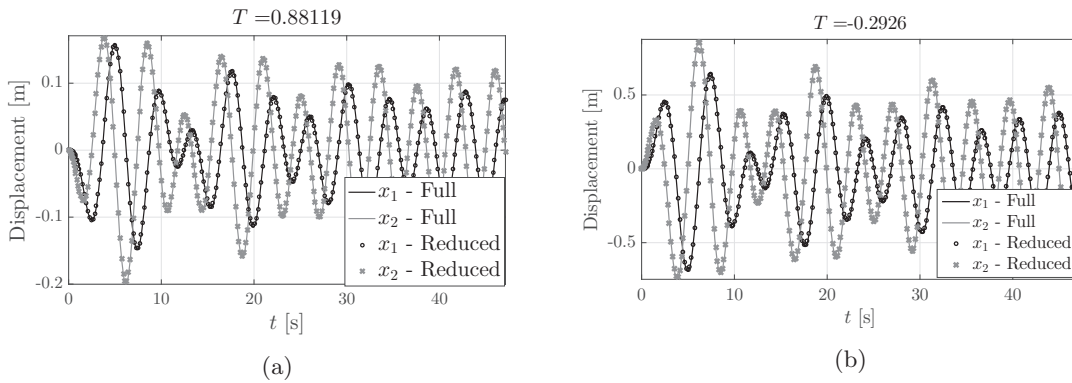


Figure 6.2: **Temperature constant:** Transient response of the two-degree-of-freedom oscillator (cf. Figure 6.1) under the harmonic forcing $[f_1(t), f_2(t)] = [0, \sin(\Omega t)]$ with forcing frequency $\Omega = 1.5$ rad/s for constant temperatures (a) $T \approx 0.88$ K, (b) $T \approx -0.29$, with initial conditions $x_1(0) = x_2(0) = 0$. A single mode reduction using the first mode computed at the respective values of T is effective in model reduction.

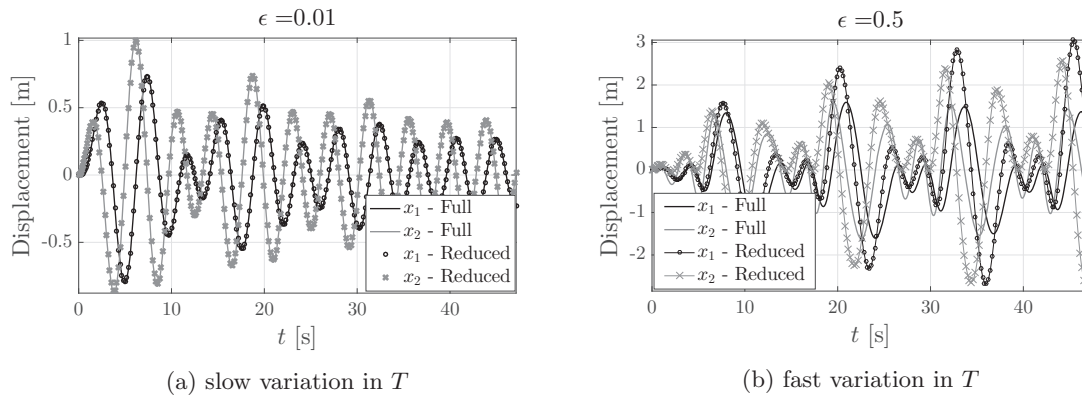


Figure 6.3: **Dynamic temperature variation:** Same as Figure 6.2 except T is dynamically varying as $T = (\pi/3)\sin(\epsilon t)$ for: (a) $\epsilon = 0.01$, i.e., when temperature is changing slowly, and (b) for $\epsilon = 0.5$, i.e., the temperature dynamics is evolving on a time-scale comparable to that of structural dynamics. (a) shows that the first VM, instantaneously adapting to temperature change is effective for reduction (cf. eq. (6.3)) when T varies slowly in comparison with the system vibration frequency. However, as shown in (b), this reduction fails when T varies relatively fast, even though the 2 modes have well-separated natural frequencies (cf. Figure 6.1c) at all values of T .

Since the first mode calculated at different temperatures is found to capture the dynamics of the system in Figure 6.1, the following questions arise in the case of dynamic temperature variation:

- Is it reasonable that a single-mode basis which adapts instantaneously to the dynamic temperature change such as in eq. (6.3) would be effective for model reduction?
- And if yes, when can we expect such a reduction approach to produce good results?

The answer to these question lies in an interesting observation, depicted in Figure 6.3. We see that an instantaneously-adapting basis with a single mode is effective in capturing system dynamics under a slowly changing temperature. However, the same ROM is not effective when temperature varies rapidly. This shows that a slow variation of parameters is necessary (but not sufficient) to guarantee effective model reduction using interpolation of bases obtained from static parameter values in the context of dynamic parameter dependence.

To this end, we apply the method of multiple scales (MMS) (see, e.g., ref. [66]) to the temperature-dependent structural dynamics equations, where the temperature is taken to be evolving at a different, slower time scale than that of the structural motion. This results in a set of partial differential equations (PDEs) in the slow thermal time and the fast structural time. Due to this treatment, the slowly varying temperature unknowns can be treated as static parameters in a mathematically justifiable manner. Interestingly, this procedure automatically avoids any convective terms arising from the time derivative of the mappings such as (6.3), which are usually cumbersome to evaluate [117].

Using the MMS, we obtain an asymptotic set of PDEs in the structural time scale, where the temperature can be seen as a slowly varying parameter. This allows for the calculation of a parameter-dependent equilibrium around which the aforementioned local reduction techniques can be applied using the corresponding subspaces in an adaptive manner. Thus, the use of MMS resolves both the issues mentioned above in this slow-fast, thermo-mechanical setting.

After applying the MMS to temperature-dependent structural equations in Section 6.2, we first show the relevance of MMS on a simple single-degree-of-freedom thermo-mechanical oscillator. Next, we propose the use of a basis which instantaneously adapts to the slowly varying temperature in Section 6.3. In doing, we preserve minimal number of unknowns in the basis due to changing temperature. In Section 6.3, we first consider linear systems, reduced using temperature-dependent VMs, followed by geometrically nonlinear systems, reduced using VMs and modal derivatives in an adaptive basis. As a proof-of-concept, we test these propositions in Section 6.4 on straight and curved beam examples in the linear as well as the geometrically nonlinear setting. Finally, some conclusion with scope for further work are laid down in Section 6.6.

6.2 Multiple time scale dynamics

After spatial discretization using finite elements (FE) of partial differential equations (PDEs) governing temperature-dependent momentum balance, we obtain a set of ordinary differential equations (ODEs). Along with the initial conditions for generalized displacements and velocities, these ODEs govern the temperature-dependent response of the underlying structure in the form of an initial value problem as:

$$\begin{aligned} \mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{f}(\mathbf{u}, \mathbf{T}) &= \mathbf{g}(t), \\ \mathbf{u}(0) &= \mathbf{x}_0, \\ \dot{\mathbf{u}}(0) &= \mathbf{v}_0, \end{aligned} \tag{6.4}$$

where $\mathbf{u}(t) \in \mathbb{R}^n$ is the vector of generalized displacements; $\mathbf{M} \in \mathbb{R}^{n \times n}$ is the mass matrix; $\mathbf{C} \in \mathbb{R}^{n \times n}$ is the damping matrix; $\mathbf{T} \in \mathbb{R}^{n_T}$ represents the spatially discretized and possibly time-dependent

temperature field prescribed on the mechanical structure, $\mathbf{f} : \mathbb{R}^n \times \mathbb{R}^{n_T} \mapsto \mathbb{R}^n$ gives nonlinear elastic internal force as a function of the displacement $\mathbf{u}$ and temperature $\mathbf{T}$ of the structure; and $\mathbf{g}(t) \in \mathbb{R}^n$ is the time-dependent external load vector, $(\dot{\bullet}) = \frac{d(\bullet)}{dt}$ denotes the derivative with respect to time t . The system (6.4) is sometimes also referred to as the high fidelity model (HFM) in the literature.

As mentioned in the Introduction, the temperature field over structures is expected to evolve much slower as compared to the structural time scale t . The evolution of the temperature field over the structure is usually modeled using the heat equation, which upon spatial discretization (using, e.g., finite elements) gives

$$\mathbf{M}_T \mathbf{T}' + \mathbf{f}_T(\mathbf{T}) = \mathbf{h}(\tau), \quad (6.5)$$

where τ is the thermal time-scale; $(\bullet)'$ denotes the derivative with respect to τ ; $\mathbf{M}_T \in \mathbb{R}^{n_T \times n_T}$ is the thermal mass matrix; $\mathbf{f}_T : \mathbb{R}^{n_T} \mapsto \mathbb{R}^{n_T}$ gives the heat flux through the structure due to physical effects such as conduction, convection, radiation, or a combination thereof; $\mathbf{h}(\tau)$ represents the externally-applied heat source to the structure.

The slowness of the thermal dynamics with respect to the structural one can be mathematically modeled with the relation $\tau = \epsilon t$, where $0 < \epsilon \ll 1$ is a small physical parameter. In practice, ϵ can be quantified as, e.g., the ratio of the time period of the fundamental mode of oscillation of the structural system (6.4) after linearization and the thermal time constant obtained from the linearized spectral analysis of eq. (6.5). In the context of beams, the parameter ϵ may be related to the Boley non-dimensional parameter denoted by B from the seminal work of Boley [20]. Now, if the temperature field is a function of the slow time scale τ , i.e., $\mathbf{T} = \mathbf{T}(\tau)$, then we get

$$\dot{\mathbf{T}} = \epsilon \mathbf{T}'. \quad (6.6)$$

Equation (6.6) suggests that if $\mathbf{T}$ naturally evolves over the timescale τ , i.e., $\mathbf{T}$ varies at $\mathcal{O}(1)$ speed with respect to τ , then it varies at a much smaller $\mathcal{O}(\epsilon)$ speed over the structural time scale t . Physically, the solution to the HFM (6.4) can be assumed to have a fast component due structural dynamics, as well as a slow component under the influence of slowly changing temperature. In other words, the solution evolves over two timescales as

$$\mathbf{u} = \mathbf{u}(t, \tau). \quad (6.7)$$

The method of multiple scales (cf., e.g., ref. [66]) then allows us to express the solution in terms of an expansion in ϵ as

$$\mathbf{u}(t, \tau) = \mathbf{u}_0(t, \tau) + \epsilon \mathbf{u}_1(t, \tau) + \epsilon^2 \mathbf{u}_2(t, \tau) + \dots, \quad (6.8)$$

where $\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2 \dots$ are the solution components at different orders in ϵ , which can be recursively obtained in the following manner. Using chain rule for the time derivative of eq. (6.8), we obtain

$$\begin{aligned} \dot{\mathbf{u}} &= \frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{u}}{\partial \tau} \frac{\partial \tau}{\partial t} = \frac{\partial \mathbf{u}}{\partial t} + \epsilon \frac{\partial \mathbf{u}}{\partial \tau}, \\ \ddot{\mathbf{u}} &= \frac{\partial^2 \mathbf{u}}{\partial t^2} + 2\epsilon \frac{\partial^2 \mathbf{u}}{\partial t \partial \tau} + \epsilon^2 \frac{\partial^2 \mathbf{u}}{\partial \tau^2}. \end{aligned} \quad (6.9)$$

Upon substituting (6.8) and (6.9) into the governing equations (6.4), we obtain a PDE in t and τ given by

$$\mathbf{M} \left(\frac{\partial^2 \mathbf{u}}{\partial t^2} + 2\epsilon \frac{\partial^2 \mathbf{u}}{\partial t \partial \tau} + \epsilon^2 \frac{\partial^2 \mathbf{u}}{\partial \tau^2} \right) + \mathbf{C} \left(\frac{\partial \mathbf{u}}{\partial t} + \epsilon \frac{\partial \mathbf{u}}{\partial \tau} \right) + \mathbf{f}(\mathbf{u}(t, \tau), \mathbf{T}(\tau)) = \mathbf{p}(t; \epsilon), \quad (6.10)$$

where we have introduced the dependence of the external mechanical force $\mathbf{p}$ on ϵ . We note that there is no reason for the external forcing to be dependent on the physical parameter ϵ . In this perturbative setting, however, we treat ϵ as scaling constant that could be used to distinguish between components of a given forcing that have different magnitudes. Thus, $\mathbf{p}$ can be seen as an

appropriately-scaled version of $\mathbf{g}$ in (6.4), e.g., if $\mathbf{g}$ is $\mathcal{O}(\epsilon)$ small in comparison with other terms of the governing equations (6.4), then we can simply write $\mathbf{g}(t) = \mathbf{p}(t, \epsilon) = \epsilon \mathbf{l}(t)$, where $\mathbf{l}(t)$ is $\mathcal{O}(1)$. According to the MMS, eq. (6.10) must be satisfied at all orders in ϵ . Using the expansion (6.8) and after Taylor expansion of the nonlinear internal force $\mathbf{f}$ around $\epsilon = 0$, we obtain the leading order terms in the PDE 6.10 as:

$$\mathcal{O}(1) : \quad \mathbf{M} \frac{\partial^2 \mathbf{u}_0}{\partial t^2} + \mathbf{C} \frac{\partial \mathbf{u}_0}{\partial t} + \mathbf{f}(\mathbf{u}_0, \mathbf{T}(\tau)) = \mathbf{p}(t, 0). \quad (6.11)$$

The solution to eq. (6.11) gives us the leading-order component $\mathbf{u}_0$ in eq. (6.8). Equation (6.11) can be integrated in time t using usual time integration schemes adopted for eq. (6.4), e.g., Newmark's scheme, with the same initial conditions, i.e., $\mathbf{u}_0(0) = \mathbf{x}_0$, $\frac{\partial \mathbf{u}_0(0)}{\partial t} = \mathbf{v}_0$. Collecting the $\mathcal{O}(\epsilon)$ terms in eq. (6.10), we obtain

$$\mathcal{O}(\epsilon) : \quad \mathbf{M} \frac{\partial^2 \mathbf{u}_1}{\partial t^2} + \mathbf{C} \frac{\partial \mathbf{u}_1}{\partial t} + \left[\frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{u}_0, \mathbf{T}(\tau)) \right] \mathbf{u}_1 = -2\mathbf{M} \frac{\partial^2 \mathbf{u}_0}{\partial t \partial \tau} - 2\mathbf{C} \frac{\partial \mathbf{u}_0}{\partial \tau} + \frac{\partial \mathbf{p}}{\partial \epsilon}(t, 0). \quad (6.12)$$

Upon solving for $\mathbf{u}_0$ from eq.(6.11), it is easy to see that eq. (6.12) represents a linear system of equations in $\mathbf{u}_1$, the solution to which gives us an $\mathcal{O}(\epsilon)$ correction to $\mathbf{u}_0$ as given in eq. (6.8). Physically, (6.12) represents the system linearized around the leading-order solution $\mathbf{u}_0$ with a forcing term on the right-hand-side (dependent on $\mathbf{u}_0$). Since the initial conditions of the HFM (6.4) are independent of ϵ , they are included in the leading-order problem (6.11), shown above. Consequently, the initial conditions for solving eq. (6.12) should be identically zero, i.e.,

$$\mathbf{u}_1(0) = \frac{\partial \mathbf{u}_1(0)}{\partial t} = \mathbf{0}.$$

Thus, using the method of multiple scales, we obtain a series of problems which—as shall be shown later—can be used to systematically reduce the HFM at different orders in ϵ . Equations (6.11) and (6.12) give only the equations for $\mathbf{u}_0$ and $\mathbf{u}_1$. Higher-order terms in eq. (6.8) can be found by following a similar recursive procedure by collecting higher-order terms in eq. (6.10). Furthermore, as is the case with the $\mathcal{O}(\epsilon)$ problem (6.12), it easy to see that all higher problems would be linear in their corresponding unknowns.

6.3 Adaptive basis for model reduction

6.3.1 Linear systems: reduction using vibration modes

For the sake of consistency, we first briefly review the concept of modal super-position which is the classic choice for modal reduction of linear mechanical system.

6.3.1.1 Static temperature dependence

It is well known that modal subspaces (i.e., subspaces spanned by VMs) are invariant and useful in reduction of linear systems. This is also referred to as modal superposition in literature (cf., e.g., Géradin & Rixen [45]). Consider the case of linear structural dynamics independent of temperature or when the temperature configuration remains fixed during a dynamic simulation, i.e., $\mathbf{f}(\mathbf{u}, \mathbf{T}) = \mathbf{K}(\mathbf{T}_0)\mathbf{u} + \mathbf{b}(\mathbf{T}_0)$ in eq. (6.4) such that

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}(\mathbf{T}_0)\mathbf{u} + \mathbf{b}(\mathbf{T}_0) = \mathbf{g}(t), \quad (6.13)$$

where $\mathbf{T}_0$ is a fixed temperature configuration. In the absence of dynamic excitation $\mathbf{g}(t)$, the above system has a unique equilibrium given as

$$\mathbf{u}_{eq} = -\mathbf{K}(\mathbf{T}_0)^{-1}\mathbf{b}(\mathbf{T}_0).$$

Upon shifting the origin to $\mathbf{u}_{eq}$, we get

$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{C}\dot{\mathbf{y}} + \mathbf{K}(\mathbf{T}_0)\mathbf{y} = \mathbf{g}(t), \quad (6.14)$$

where $\mathbf{y} = \mathbf{u} - \mathbf{u}_{eq}$. The VMs of system (6.14) form a basis for the generalized displacements $\mathbf{y}$ around the equilibrium $\mathbf{u}_{eq}$. For proportionally damped structures, the undamped VMs ϕ_i that are obtained from the solution of the eigenvalue problem,

$$(\mathbf{K}(\mathbf{T}_0) - \omega_i^2 \mathbf{M}) \phi_i = \mathbf{0}, \quad i = 1, \dots, n \quad (6.15)$$

lead to decoupling of the system (6.14) and are used in a modal truncation approximation as

$$\mathbf{y}(t) = \sum_{i=1}^n \phi_i q_i(t) \approx \sum_{i \in \mathcal{M}} \phi_i q_i(t), \quad (6.16)$$

where $\mathcal{M} \subset \{1, 2, \dots, n\}$ is a set containing the indices of the VMs relevant for modal truncation (cf. G  rardin & Rixen [45]), such that $m := |\mathcal{M}| \ll n$.

The corresponding ROM is obtained via the classic Galerkin projection as

$$\Phi^\top \mathbf{M} \Phi \ddot{\mathbf{q}} + \Phi^\top \mathbf{C} \Phi \dot{\mathbf{q}} + \Phi^\top \mathbf{K} \Phi \mathbf{q} = \Phi^\top \mathbf{g}(t), \quad (6.17)$$

where $\Phi \in \mathbb{R}^{n \times m}$ is a matrix containing the VMs indexed by the set $\mathcal{M}$. The solution to the original system (6.13) can then be recovered as

$$\implies \mathbf{u}(t) \approx \mathbf{u}_{eq} + \sum_{i \in \mathcal{M}} \phi_i q_i(t) = \mathbf{u}_{eq} + \Phi \mathbf{q}(t). \quad (6.18)$$

6.3.1.2 Presence of dynamic temperature-dependence

If we allow temperature dependence in an arbitrary fashion, such that

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}(\mathbf{T}(t))\mathbf{u} + \mathbf{b}(\mathbf{T}(t)) = \mathbf{g}(t), \quad (6.19)$$

we see that even in the absence of external forcing $\mathbf{g}(t)$, the system (6.19) has no equilibrium point due to the presence of dynamic thermal excitation $\mathbf{b}(\mathbf{T}(t))$. Thus, due to the absence of equilibrium and dynamic temperature variation, one cannot use specific VMs to reduce the system, even in a linear system such as (6.19), as we discussed in the Introduction.

This problem is commonly tackled in the literature (cf. refs. [40, 81, 91, 96, 115]) by simply treating the temperature-dependent term $\mathbf{b}(\mathbf{T}(t))$ as yet another time-dependent forcing term similar to $\mathbf{g}(t)$. Specifically, we obtain the system

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}(\mathbf{T}(t))\mathbf{u} = \tilde{\mathbf{g}}(t), \quad (6.20)$$

where $\tilde{\mathbf{g}}(t) := \mathbf{g}(t) - \mathbf{b}(\mathbf{T}(t))$. Note that the linear system (6.20) now exhibits a unique trivial equilibrium point, i.e., $\mathbf{u}_{eq} = \mathbf{0}$ in the absence of forcing $\tilde{\mathbf{g}}(t)$. This manipulation leads to the routinely performed Galerkin projection about the origin via the mapping

$$\mathbf{u}(t) \approx \mathbf{V}\mathbf{q}(t). \quad (6.21)$$

The reduction basis $\mathbf{V}$ is usually constructed in the literature by an ad hoc selection of modes (hot/cold/dual modes) that are expected to account for the mechanical forcing as well as the dynamic temperature dependence, as discussed in the Introduction.

However, if it is a priori known that the thermal dynamics is *slow*, as is usually the case for thermo-mechanical systems, we can apply the method of multiple scales, as described in Section 6.2, to obtain the leading order problems as follows (cf. eq. (6.11)):

$$\mathbf{M} \frac{\partial^2 \mathbf{u}_0}{\partial t^2} + \mathbf{C} \frac{\partial \mathbf{u}_0}{\partial t} + \mathbf{K}(\mathbf{T}(\tau)) \mathbf{u} + \mathbf{b}(\mathbf{T}(\tau)) = \mathbf{p}(t, 0). \quad (6.22)$$

In this PDE in time t , the slow time τ can be seen as parameter, dependent upon which equilibria can be calculated for the unforced systems as

$$\mathbf{u}_{eq}(\tau) = -\mathbf{K}(\mathbf{T}(\tau))^{-1} \mathbf{b}(\mathbf{T}(\tau)). \quad (6.23)$$

Furthermore, the solution to the eigenvalue problem

$$[\mathbf{K}(\mathbf{T}(\tau)) - \omega_i^2(\tau) \mathbf{M}] \phi_i(\tau) = \mathbf{0} \quad (6.24)$$

results in the temperature-dependent VMs $\phi_i(\tau)$ around this manifold of equilibria. The solution to the leading order problem (6.22) is then approximated using a parameterized basis $\Phi(\tau)$ containing a truncated set of VMs as

$$\mathbf{u}_0(t, \tau) \approx \mathbf{u}_{eq}(\tau) + \Phi(\tau) \mathbf{q}_0(t). \quad (6.25)$$

A reduced-order model can be obtained by projecting the leading order system (6.22) on to the slowly varying basis as

$$\Phi(\tau)^\top \left[\mathbf{M} \Phi(\tau) \frac{\partial^2 \mathbf{q}_0}{\partial t^2} + \mathbf{C} \Phi(\tau) \frac{\partial \mathbf{q}_0}{\partial t} + \mathbf{K}(\mathbf{T}(\tau)) [\mathbf{u}_{eq}(\tau) + \Phi(\tau) \mathbf{q}_0(t)] + \mathbf{b}(\mathbf{T}(\tau)) \right] = \Phi(\tau)^\top \mathbf{p}(t, 0)$$

and using eq. (6.23) we obtain the following reduced order model at the leading order

$$\mathcal{O}(1): \quad \underbrace{\Phi(\tau)^\top \mathbf{M} \Phi(\tau)}_{\mathbf{M}_\Phi(\tau)} \frac{\partial^2 \mathbf{q}_0}{\partial t^2} + \underbrace{\Phi(\tau)^\top \mathbf{C} \Phi(\tau)}_{\mathbf{C}_\Phi(\tau)} \frac{\partial \mathbf{q}_0}{\partial t} + \underbrace{\Phi(\tau)^\top \mathbf{K}(\mathbf{T}(\tau)) \Phi(\tau)}_{\mathbf{K}_\Phi(\tau)} \mathbf{q}_0(t) = \Phi(\tau)^\top \mathbf{p}(t, 0) \quad (6.26)$$

Furthermore, the $\mathcal{O}(\epsilon)$ correction according to eq. (6.11) is given by

$$\mathbf{M} \frac{\partial^2 \mathbf{u}_1}{\partial t^2} + \mathbf{C} \frac{\partial \mathbf{u}_1}{\partial t} + \mathbf{K}(\mathbf{T}(t)) \mathbf{u}_1 = -2\mathbf{M} \frac{\partial^2 \mathbf{u}_0}{\partial t \partial \tau} - 2\mathbf{C} \frac{\partial \mathbf{u}_0}{\partial \tau} + \frac{\partial \mathbf{p}}{\partial \epsilon}(t, 0). \quad (6.27)$$

Using the same modal basis to reduce this linear problem and observing that origin is the unique fixed point for this system, we get

$$\mathbf{u}_1(t, \tau) \approx \Phi(\tau) \mathbf{q}_1(t), \quad (6.28)$$

which upon Galerkin projection yields the reduced-order model at $\mathcal{O}(\epsilon)$

$$\mathcal{O}(\epsilon): \quad \mathbf{M}_\Phi \frac{\partial^2 \mathbf{q}_1}{\partial t^2} + \mathbf{C}_\Phi \frac{\partial \mathbf{q}_1}{\partial t} + \mathbf{K}_\Phi(\tau) \mathbf{q}_1 = \Phi^\top \left[\frac{\partial \mathbf{p}}{\partial \epsilon}(t, 0) - 2\mathbf{M} \frac{\partial^2 \mathbf{u}_0}{\partial t \partial \tau} - 2\mathbf{C} \frac{\partial \mathbf{u}_0}{\partial \tau} \right] \quad (6.29)$$

$$= \Phi^\top \left[\frac{\partial \mathbf{p}}{\partial \epsilon}(t, 0) - 2\mathbf{M} \Phi' \frac{\partial \mathbf{q}_0}{\partial t} - 2\mathbf{C} \Phi' \mathbf{q}_0 \right]. \quad (6.30)$$

Thus, in the linear case, the reduced operators are the same for reduced problems at the leading order, as well as at $\mathcal{O}(\epsilon)$. In fact, it is easy to see that these linear operators remain the same for higher order corrections, as well.

We would like to emphasize that although this formal procedure can be carried out even if the underlying thermal dynamics is not slow, i.e., cases where ϵ is not small enough, it would not be mathematically justifiable and can very easily lead to spurious results, as shown in the example in Section 6.1.

6.3.2 Geometrically nonlinear systems

For reduction of linear systems, a basis comprising of a truncated set of VMs can be easily identified by the examination of the external forcing and the linear spectrum. However, such a basis is usually not suitable for reduction of geometrically nonlinear systems because of membrane effects, which become significant in the nonlinear regime due to the bending-stretching-torsion coupling. In such cases, modal derivatives have emerged as a simple and effective model-reduction tool (cf. refs. [54, 55, 63, 105, 128, 131]). We discuss the use of modal derivatives in the context of dynamic temperature dependence.

The temperature-dependent equilibrium $\mathbf{u}_{eq}(\tau)$ is given by the solution to static problem

$$\mathbf{f}(\mathbf{u}_{eq}(\tau), \mathbf{T}(\tau)) = \mathbf{0}. \quad (6.31)$$

We linearize the nonlinear internal force $\mathbf{f}$ around the configuration $\mathbf{u}_{eq}(\tau)$ to obtain the tangent stiffness matrix $\frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{u}_{eq}(\tau), \mathbf{T}(\tau))$ evaluated at the temperature-dependent equilibrium $\mathbf{u}_{eq}(\tau)$. Since τ is taken as a slowly varying parameter, we can compute the temperature-dependent VMs of the structure similar to the linear case as

$$\left[\frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{u}_{eq}(\tau), \mathbf{T}(\tau)) - \omega_i^2(\tau) \mathbf{M} \right] \phi_i(\tau) = \mathbf{0}. \quad (6.32)$$

The static modal derivatives (cf. ref. [63]), which are relevant for capturing the membrane effects in thin-walled structures are given as

$$\mathbf{K}_t(\mathbf{u}_{eq}(\tau), \mathbf{T}(\tau)) \boldsymbol{\theta}_{ij} = - \left[\frac{\partial^2 \mathbf{f}}{\partial \mathbf{u} \partial \mathbf{u}}(\mathbf{u}_{eq}, \mathbf{T}(\tau)) \cdot \phi_j \right] \phi_i, \quad (6.33)$$

where $\boldsymbol{\theta}_{ij}$ is the static modal derivative and physically represents the change in mode ϕ_i as the structure is perturbed in the direction of mode ϕ_j . Note that here the modal derivatives are calculated around a non-trivial, temperature-dependent equilibrium. This is in contrast with the case of no temperature dependence usually treated in literature, where the modal derivatives are simply constructed around the origin.

The reduction basis is formed by combining a truncated set of modes with the corresponding modal derivatives. As observed in ref. [63], a set of k VMs results in $k(k+1)/2$ modal derivatives. These modal derivatives may be selected to further reduce the basis size (cf. refs. [63, 120]).

$$\mathbf{V}(\tau) = [\boldsymbol{\Phi}(\tau), \quad \boldsymbol{\Theta}(\tau)], \quad (6.34)$$

where $\mathbf{V}(\tau) \in \mathbb{R}^{n \times m}$ is the reduction basis. Analogous to the linear case (cf. (6.25)), we express the leading-order reduced solution as

$$\mathbf{u}_0(t, \tau) \approx \mathbf{u}_{eq}(\tau) + \mathbf{V}(\tau) \mathbf{q}_0(t) \quad (6.35)$$

Upon substituting (6.35) into the leading order MMS expansion (6.11) and performing Galerkin projection, we get

$$\mathbf{V}(\tau)^\top \mathbf{M} \mathbf{V}(\tau) \frac{\partial^2 \mathbf{q}_0}{\partial t^2} + \mathbf{V}(\tau)^\top \mathbf{C} \mathbf{V}(\tau) \frac{\partial \mathbf{q}_0}{\partial t} + \mathbf{V}(\tau)^\top \mathbf{f}(\mathbf{u}_{eq}(\tau) + \mathbf{V}(\tau) \mathbf{q}_0(t), \mathbf{T}(\tau)) = \mathbf{V}(\tau)^\top \mathbf{p}(t, 0). \quad (6.36)$$

The $\mathcal{O}(\epsilon)$ correction $\mathbf{u}_1$ given in (6.12) can be reduced using the same adaptive basis as

$$\mathbf{u}_1(t, \tau) \approx \mathbf{V}(\tau) \mathbf{q}_1(t). \quad (6.37)$$

And again, by means of Galerkin projection, we obtain a reduced version of (6.12) which is linear in the reduced unknowns $\mathbf{q}_1$ as

$$\mathbf{M} \mathbf{V}(\tau) \frac{\partial^2 \mathbf{q}_1}{\partial t^2} + \mathbf{C} \mathbf{V}(\tau) \frac{\partial \mathbf{q}_1}{\partial t} + \left[\mathbf{V}(\tau)^\top \left(\frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{u}_0, \mathbf{T}(\tau)) \right) \mathbf{V}(\tau) \right] \mathbf{q}_1 = \mathbf{V}(\tau)^\top \left[\frac{\partial \mathbf{p}}{\partial \epsilon}(t, 0) - 2\mathbf{M} \frac{\partial^2 \mathbf{u}_0}{\partial t \partial \tau} - 2\mathbf{C} \frac{\partial \mathbf{u}_0}{\partial \tau} \right]. \quad (6.38)$$

In the special case of a small $\mathcal{O}(\epsilon)$ mechanical forcing, e.g. $\mathbf{p}(t, \epsilon) = \epsilon \mathbf{l}(t)$ we have

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{f}(\mathbf{u}, \mathbf{T}) = \epsilon \mathbf{l}(t). \quad (6.39)$$

and the leading-order ROM is given by

$$\mathcal{O}(1): \quad \mathbf{M}_{\mathbf{V}}(\tau) \frac{\partial^2 \mathbf{q}_0}{\partial t^2} + \mathbf{C}_{\mathbf{V}}(\tau) \frac{\partial \mathbf{q}_0}{\partial t} + \mathbf{V}(\tau)^\top \mathbf{f}(\mathbf{u}_{eq}(\tau) + \mathbf{V}(\tau) \mathbf{q}_0(t), \mathbf{T}(\tau)) = \mathbf{0}, \quad (6.40)$$

which is the system response to thermal loading only. This would be a good approximation of the system response for small enough ϵ as long $\mathbf{u}_{eq}$ is a stable equilibrium. Indeed, the mechanical system would be performing small-amplitude oscillations around the applied thermal loading. These can be captured by the $\mathcal{O}(\epsilon)$ ROM as

$$\mathbf{M}_{\mathbf{V}}(\tau) \frac{\partial^2 \mathbf{q}_1}{\partial t^2} + \mathbf{C}_{\mathbf{V}}(\tau) \frac{\partial \mathbf{q}_1}{\partial t} + \left[\mathbf{V}(\tau)^\top \left(\frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{u}_0, \mathbf{T}(\tau)) \right) \mathbf{V}(\tau) \right] \mathbf{q}_1 = \mathbf{V}(\tau)^\top \left[\mathbf{l}(t) - 2\mathbf{M} \frac{\partial^2 \mathbf{u}_0}{\partial t \partial \tau} - 2\mathbf{C} \frac{\partial \mathbf{u}_0}{\partial \tau} \right]. \quad (6.41)$$

6.3.3 Basis interpolation and mode veering

Though the chosen subspace spanned by basis vectors at each temperature configuration would be uniquely defined, the same is not true for the basis representation of such a subspace. Each subspace would be defined by an equivalence class of orthogonal bases. There is a possibility of mode veering and rotation between bases calculated at two different temperature configurations if the representative basis for a given temperature configuration is chosen arbitrarily. This would pose a serious issue in adaptive model reduction, as proposed in this work. Indeed, the modal amplitudes in ROM equations (6.36), (6.38) must change continuously in time. This is generally not true if the corresponding basis vectors are changing discontinuously, as would be expected when mode veering and/or rotation occurs between bases. A method to avoid this issue was proposed in ref. [12] by identifying a congruence transformation which transforms a basis such that its vectors are consistent with those in a selected reference basis. We reproduce the procedure from ref. [12] in Algorithm 6.

Algorithm 6 Setting up congruent set of bases

Input: Reference basis $\mathbf{V}_0$, other bases $\tilde{\mathbf{V}}_1, \tilde{\mathbf{V}}_2, \dots, \tilde{\mathbf{V}}_N$

Output: Bases $\mathbf{V}_1, \mathbf{V}_2, \dots, \mathbf{V}_N$, congruent to $\mathbf{V}_0$

```

1: for  $j \leftarrow 1$  to  $N$  do
2:    $\mathbf{P}_j = \tilde{\mathbf{V}}_j^\top \mathbf{V}_0$ 
3:    $[\mathbf{L}_j, \mathbf{\Sigma}_j, \mathbf{R}_j] = \text{svd}(\mathbf{P}_j)$   $\triangleright$  Singular value decomposition (SVD) of  $\mathbf{P}_j$ , i.e.,  $\mathbf{P}_j = \mathbf{L}_j \mathbf{\Sigma}_j \mathbf{R}_j^\top$ 
4:    $\mathbf{Q}_j = \mathbf{L}_j \mathbf{R}_j^\top$ 
5:    $\mathbf{V}_j = \tilde{\mathbf{V}}_j \mathbf{Q}_j$ 
6: end for

```

6.4 Numerical examples

We demonstrate the proposed method on a set of beam examples, with displacements in 2-dimensional (x, z) plane. The beam is modeled using finite elements with a linear elastic material (see Table 6.1 for geometrical parameters and physical properties). Furthermore, we consider geometric nonlinearities in the model via the von-Kármán strain approximation. The thermal effects are modelled by assuming that the temperature distribution is uniform throughout the thickness of the beam. Across all examples, the beam is clamped at both ends and discretized using 60 elements.

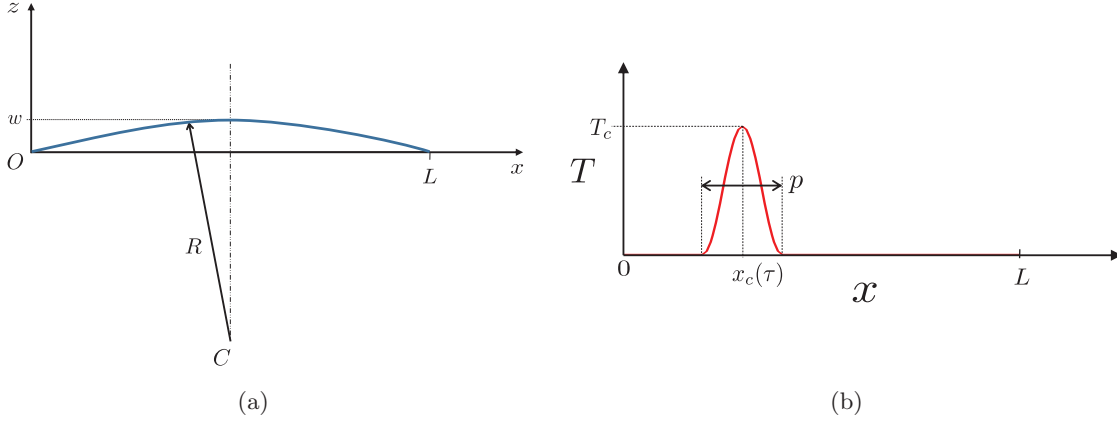


Figure 6.1: (a) An undeformed beam with length L and Radius of curvature R is shown. The beam would be straight in the undeformed configuration if the curvature parameter $w = 0$. (b) A temperature pulse with spatial distribution given in eq. (6.42)

A pulse-shaped temperature field is applied to the beam, as shown in Figure 6.1. It is modeled as using the following equation:

$$T(x) = T_c \sin^2 \left(\pi \frac{x - x_0}{p} \right) [H(x - x_0) - H((x - x_0) - p)], \quad (6.42)$$

where $x_0 = x_c - p/2$, x_c is the pulse-center location, p is the width of the pulse, T_c is the pulse height, H is the Heaviside step function. The center of the pulse x_c moves slowly across the beam according to the relation

$$x_c = x_0 + A \sin(\tau), \quad (6.43)$$

where x_0 is the initial location of the center of the pulse and A is the amplitude of temperature-pulse-oscillation along the length of the beam.

Parameters (symbol)	Value [unit]
Length of beam (L)	0.1 [m]
Thickness of beam (h)	1 [mm]
Width of beam (b)	10 [mm]
Curvature parameter for the curved-beam (w ; cf. Figure 6.1a; $w = 0$ for straight beam)	5 [mm]
Young's Modulus (E)	70 [GPa]
Material damping modulus (κ)	0.1 (GPa s)
Density (ρ)	2700 [kg/m ³]
Coefficient of linear expansion (α_T)	23.1×10^{-6} [K ⁻¹]

Table 6.1: Values for geometrical parameters and physical properties used in the beam models

We consider two geometries of the doubly-clamped beam model: one which is initially straight and the other with a small initial curvature. The FE model of the beam contains a temperature-dependent linear stiffness matrix. As a result, the VMs are also temperature-dependent. The variation in the linear VMs and natural frequencies of the beam for both models as the center of the temperature profile (x_c) changes location are depicted in Figure 6.2.

Along with the imposed thermal dynamics on the beam, we also excite the structure mechanically with a load oscillating at a much faster rate than that of the thermal distribution, the details which

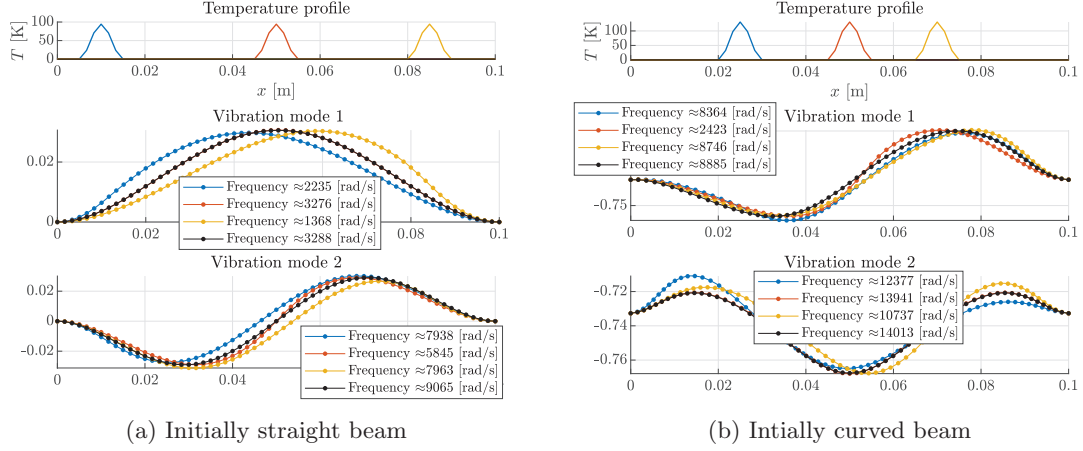


Figure 6.2: The variation in the first two VMs of the doubly-clamped (a) straight and (b) curved beams (cf. Table 6.1 for properties) with respect to the location of the temperature profile modeled according to eq. (6.42), along with unheated or *cold* configuration shown in black color. The mode shapes and corresponding frequencies change significantly across the three instances of the temperature profiles depicted.

are described in the following examples. Across all examples, we perform numerical time integration starting with the static equilibrium at the initial thermal configuration.

Model reduction strategy: During model reduction using the proposed MMS-based technique, a basis comprising of temperature-dependent VMs (and modal derivatives in case of nonlinear problems) is interpolated as the center x_c slowly changes according to eq. (6.43). In order to perform this interpolation, we use a database of $n_d = 19$ bases, each computed around the equilibrium when the thermal pulse center is at location

$$x_c^{(j)} = \frac{jL}{20}, \quad j = 1, \dots, 19. \quad (6.44)$$

We ensure that all the bases in this database are consistent using Algorithm 6. As the temperature profile moves along the length of the beam, we obtain a basis relevant for the instantaneous temperature configuration by performing linear interpolation between the bases in the database. In general, such an interpolation is not guaranteed to preserve orthogonality of the bases and can even lead to ill-conditioning. This can be avoided by interpolating the bases over a Stiefel manifold, as proposed in ref. [10]. However, in our experience, this approach is not suitable for performing interpolation online, as it can be computationally intensive and a linear interpolation is found to work just as well in structural dynamics applications.

Comparison with a stacking-based approach: A simple and robust approach for reduction would constitute stacking the modes relevant to different temperature configurations in a single basis and performing orthogonalization, e.g., using the Gram-Schmidt method or singular value decomposition (SVD) (cf. Golub & van Loan [46]). Specifically, let $\mathbf{V}^{(j)} \in \mathbb{R}^{n \times m}$ be an orthonormal basis for the j^{th} temperature configuration in the database of n_d configurations. We stack these bases in a matrix

$$\mathbf{V} := [\mathbf{V}^{(1)}, \dots, \mathbf{V}^{(n_d)}] \in \mathbb{R}^{n \times n_d m} \quad (6.45)$$

to perform SVD as

$$\mathbf{V} = \mathbf{L} \mathbf{\Sigma} \mathbf{R}^T, \quad (6.46)$$

where $\mathbf{L} \in \mathbb{R}^{n \times n}$ is an orthonormal matrix containing the left singular vectors, $\mathbf{\Sigma} \in \mathbb{R}^{n \times n_d m}$ is a diagonal matrix containing the corresponding singular values, and $\mathbf{R} \in \mathbb{R}^{n_d m \times n_d m}$ is an orthonormal matrix containing the right singular vectors. The columns of $\mathbf{L}$ corresponding to the non-zero singular

values in Σ form an orthonormal basis spanning the columns of $\mathbf{V}$. Following this procedure, the MATLAB command `orth(V)` economically computes and returns the required orthonormal basis that we have used in this work.³

Remark. The usual approach of plotting the singular value decay to cut-off the mode selection after appearance of a “knee” in such a plot is particularly justifiable in the context of POD, where a dominant subspace or principal components are identified from simulation snapshots or data. Such simulation snapshots carry trajectory information in contrast to the reduced bases that already represent dominant direction at different parameter configurations in the matrix $\mathbf{V}$. In other words, one needs a basis that spans all the columns of $\mathbf{V}$ as opposed to obtaining “dominant directions” that approximately span the data contained in $\mathbf{V}$.

The orthogonalized basis is expected to capture behavior across a range of temperature configurations in a more robust manner in comparison with other approaches mentioned in the Introduction. However, the size of such a basis can very quickly become unmanageable as more bases at different temperature configurations are added to the database, as shown by Figure 6.3. Furthermore, note that our simple examples feature a single parameter for basis description, i.e., the location x_c of the center of a temperature pulse. This simple stacking-based approach may lead to a further increase in basis size as the parameter-dimensionality increases. To avoid such high-dimensionality for a reduced-basis, we adopt the following methods to limit the size of the reduction basis used for comparison purposes, when necessary:

1. **Modal-POD:** Here, we use m modes associated to the highest singular values obtained from the SVD (6.46) of the matrix $\mathbf{V}$ given in eq. (6.45). This results in the best constant basis of size m that can be obtained from the temperature configurations in the database. Note that the size m is chosen to be the same as that of the reduction basis used in the MMS-based reduction approach to facilitate a fair comparison in terms of the number of unknowns.
2. **Modal:** One might wish to avoid the Modal-POD type reduction in view of the above-mentioned remark. In that case, one could select a subset of temperature configurations at random from the n_d configurations available in the database. Thus, lesser data is used to construct the matrix $\mathbf{V}$ and a smaller basis size is obtained after orthogonalization.

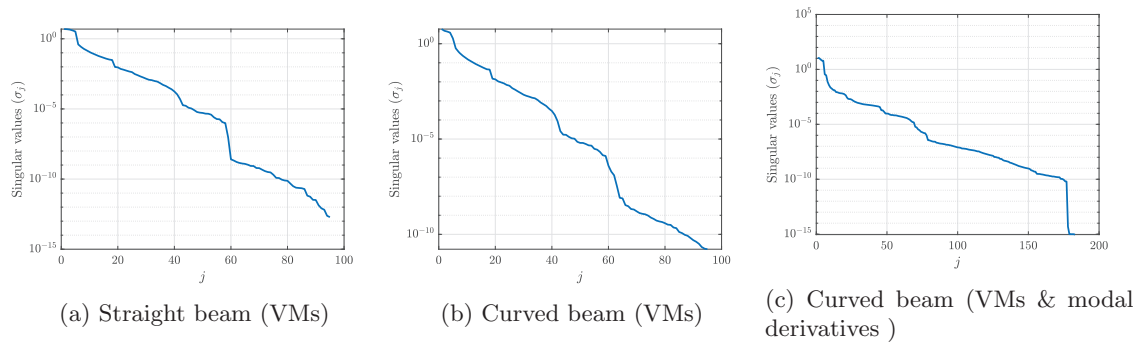


Figure 6.3: The decay of singular values of the matrix $\mathbf{V}$ (cf. eq. (6.45)) that consists reduction bases at different temperature configurations. The bases are composed of the first five vibration modes (and the corresponding modal derivatives in case of (c)) for (a) straight beam, (b) & (c) curved beam. The figure shows that temperature (parameter) change may lead to a significant change of modes that cannot be captured effectively using a low dimensional basis.

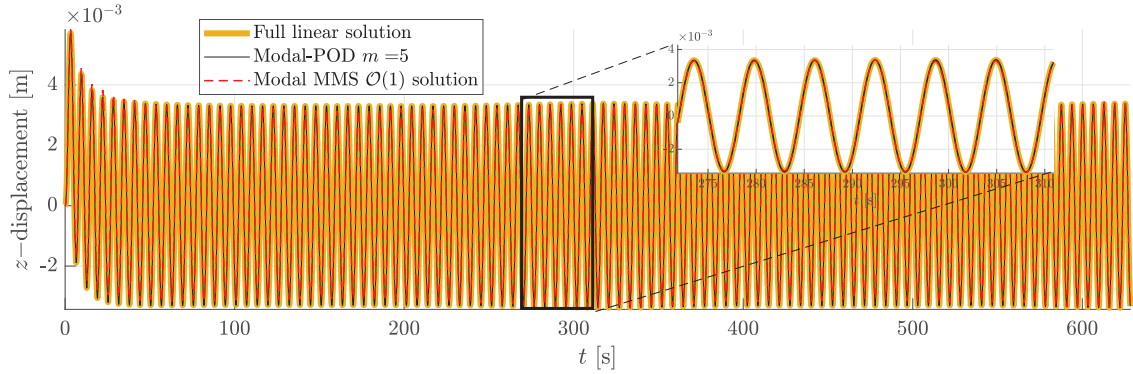
³Arguably, such a basis is conservative and might contain spurious directions due to numerical tolerances. A workaround to avoid such spurious modes would be to consider the modes associated to the non-zero singular values greater than a chosen numerical threshold, which is considerably higher than the machine zero value.

6.4.1 Straight beam

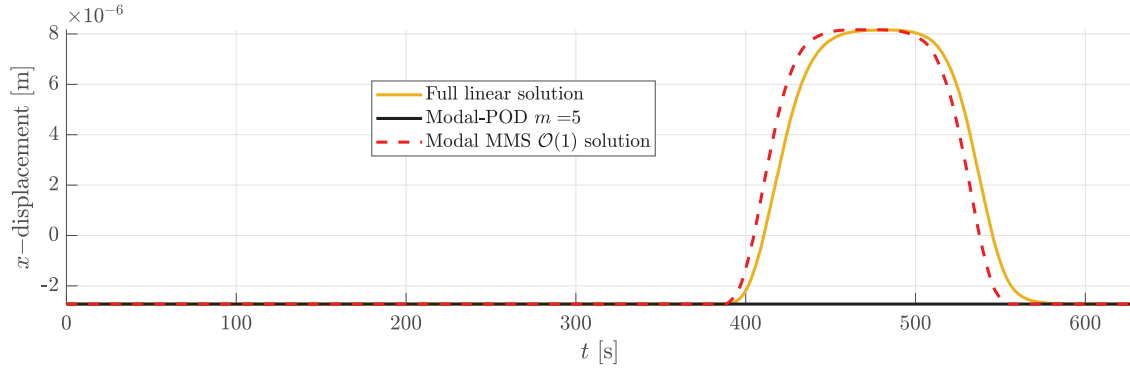
First we consider a linear model of a straight beam, whereby the nonlinear terms arising from the von-Kármán assumption are neglected. Note, however, that the model still contains a temperature-dependent linear stiffness matrix along with a thermal load vector. In addition, the beam is mechanically excited as follows:

$$\mathbf{p}(t, \epsilon) = \mathbf{l} \sin \omega_s t, \quad (6.47)$$

where ω_s is the loading frequency, chosen to be the average of the first and second natural frequency (calculated for temperature configuration $x_c = L/2$), and $\mathbf{l}$ is load amplitude vector, representing a spatially uniform load in the transverse direction with a density of 10^4N/m . The temperature profile traverses the beam span according to the relation (6.43) with $x_0 = 0.5L$, $A = 0.3L$.



(a) The reduced solution using a constant basis of 5 VMs (Modal-POD, solid black line) is just as effective in approximating transverse displacements as the leading order reduced solution (broken-red line) using the MMS.



(b) The axial displacements are expectedly missed by the reduced solution using a constant basis of Modal-POD ($m = 5$) shown by the solid black line. The MMS-based reduced solution (broken-red line) captures the qualitative behavior of the full solution.

Figure 6.4: The (a) transverse and (b) axial displacements for the node situated at $x = L/4$ in response to combined thermo-mechanical loading (6.47), (6.43) with $\epsilon = 0.01$. The simulation is performed for 100 cycles of mechanical loading when the temperature profile is traversing along the length of the beam. Note that the time on the horizontal axis is non-dimensionalized and rescaled using the time-period of loading such that each loading cycle corresponds to 2π units on the time axis.

Using the stacking approach described above that involves stacking all the modes in the database together, we obtain a basis of size (rank) 93, which is quite large for full system with 177 DOFs. It

turns out that this simple problem can be reduced using a constant basis of five modes according to the Modal-POD described above. We compare the solution using the MMS-based reduction approach proposed in this work using first five VMs which are adapted to the instantaneous temperature configuration. We test our approach for $\epsilon = 0.01, 0.001$.

The results show that the former approach using a constant basis is just as effective in capturing transverse displacements of the beam as the multiple-scales approach using an adaptive basis (cf. Figure 6.4a). Note, however, that the axial displacements are totally missed by this approach, as shown in Figure 6.4b. This is not unexpected since the first five VMs contain only low-frequency bending fields. Such a basis would thus be ineffective in capturing the membrane effects. On the other hand, the MMS-based approach systematically takes care of this issue while still using a similar basis which contains bending modes only. This is achieved by making use of the scale separation when axial components are systematically captured using a temperature-dependent equilibrium according to eq. (6.25). For $\epsilon = 0.01$, while the MMS-based approach qualitatively captures the axial response, it is not as accurate as in capturing the transverse displacements (cf. Figure 6.4). However, the full system response, including the axial displacements is expected to be captured accurately for sufficiently small ϵ . Indeed, this is the case for a smaller value of $\epsilon = 0.001$, as shown in Figure 6.5.

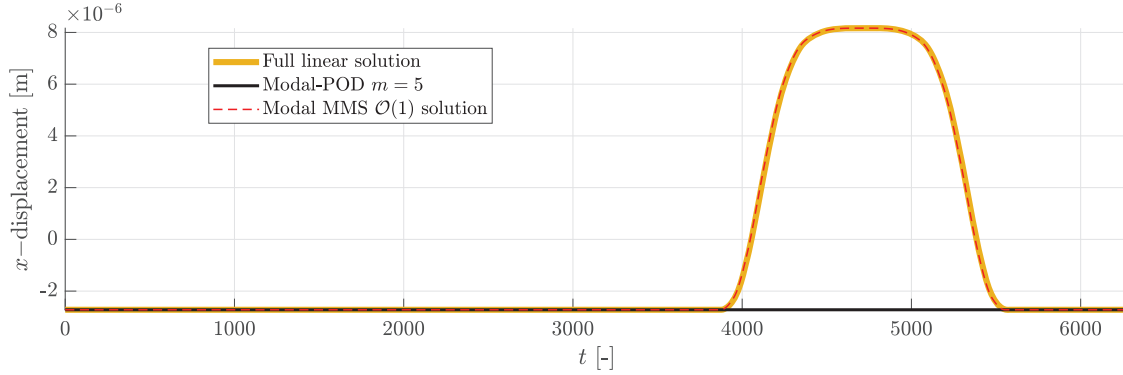


Figure 6.5: Same as Figure 6.4, except that the thermal loading is applied at a slower rate ($\epsilon = 0.001$). In contrast to Figure 6.4b, the axial displacements are very accurately captured using the MMS-based reduction for this smaller epsilon. The simulation is performed for 1000 cycles of mechanical loading.

Instead of using the proposed MMS-based reduction, the axial contributions can also be captured by appending a set of carefully selected axial modes in the original constant basis (Modal-POD). Arguably, this would be a better way to reduce the problem because even though we increase the basis size, it completely avoids the basis interpolation required in the MMS-based approach. While selecting such axial modes is relatively easy in this illustrative straight beam example, this could already be challenging if we add curvature to the beam, thereby coupling the bending and membrane DOFs. The MMS-based approach is still as effective and efficient in such situations, as we show in the next example.

6.4.2 Curved beam

Next, we consider a beam with the same properties as the previous example but curvature added (cf. Table 6.1). We consider the following mechanical excitation applied to the beam

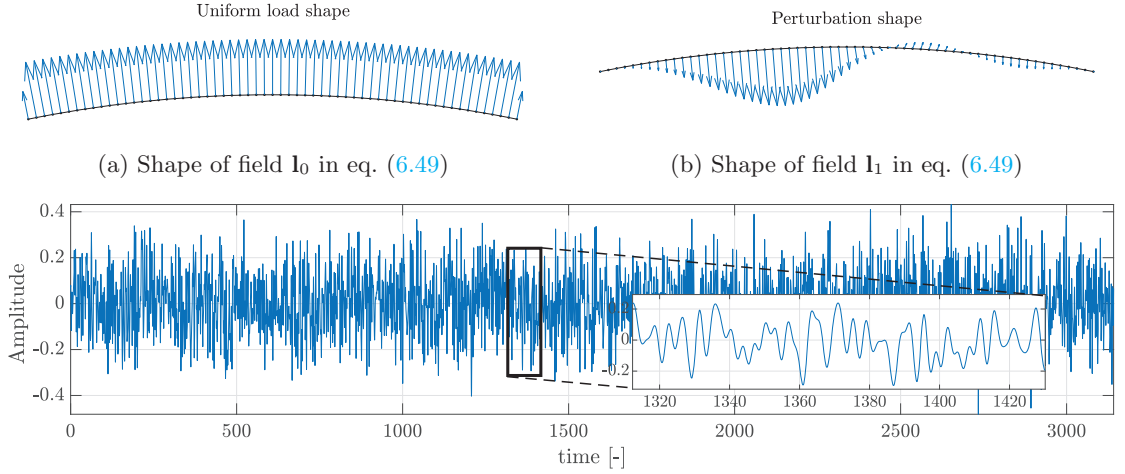
$$\mathbf{g}(t) = \mathbf{l}_0 \sin \omega_c t + \tilde{\mathbf{l}}_1 a(t), \quad (6.48)$$

where ω_c is a typical loading frequency, once again taken as the average of the first and second natural frequency of the system (calculated for the temperature configuration $x_c = L/2$); $\mathbf{l}_0$ is the

spatial load vector at the leading order whose shape represents a uniform pressure at the magnitude of 10^3 N/m (cf. Figure 6.6a for its shape). In addition to this uniform load, we have a random $\mathcal{O}(\epsilon)$ perturbation that is modeled as follows. The perturbation shape $\tilde{\mathbf{l}}_1$ (see Figure 6.6b) is constructed by a linear combination of the first five VMs of the beam (computed for $x_c = L/2$) such that the coefficients for this combination are pseudo-random values from the standard uniform distribution in the interval $(0, 1)$. The time variation of this perturbation is given by the scalar function $a(t)$ which is modeled by choosing pseudo-random values at each time instant from the uniform distribution in the interval $(0, 1)$ and thereafter filtering out the frequency contents higher than the third natural frequency of the beam. The resulting amplitude has time history as shown in Figure 6.6c.

Finally, we assume that the random perturbation $\tilde{\mathbf{l}}_1$ models noise and has small amplitude in comparison with the leading-order forcing vector $\mathbf{l}_0$. As discussed in Section 6.2, we scale this given component of the forcing as $\tilde{\mathbf{l}}_1 = \epsilon \mathbf{l}_1$, where $\mathbf{l}_1 := \tilde{\mathbf{l}}_1/\epsilon$. Our choice of loading amplitudes ensures that $\mathbf{l}_0$ and $\mathbf{l}_1$ are of similar magnitudes after this scaling, and in particular $\|\mathbf{l}_1\|_2 = \|\mathbf{l}_0\|_2$. We obtain

$$\mathbf{g}(t) = \mathbf{p}(t, \epsilon) = \mathbf{l}_0 \sin \omega_c t + \epsilon \mathbf{l}_1 a(t). \quad (6.49)$$



(c) Amplitude a in eq. (6.49) for 500 cycles of leading order loading at frequency ω_c rad/s. Note that the time on the horizontal axis is rescaled using the time-period $2\pi/\omega_c$ such that each loading cycle corresponds to 2π units on the time axis.

Figure 6.6: Description of the loading on the curved beam (cf. Figure 6.1, Table 6.1) according to eq (6.49).

6.4.2.1 Linear model

We first consider the linear example of the curved beam with the mechanical forcing (6.49) along with a time-varying temperature distribution according to relation (6.43), as before. Since the spectral content of the applied mechanical forcing is contained within the first five modes of the systems, these modes constitute an ideal choice for reduction. Thus, similar to the previous straight-beam example, we use the MMS-based reduction with a 5-mode basis that adapts instantaneously to the temperature configuration of the beam; see Figure 6.7 for results when $\epsilon = 10^{-3}$. The presence of the $\mathcal{O}(\epsilon)$ (pseudo-random) perturbation to the leading-order forcing, however, leads to more interesting results in this case. Note that the leading order ROM is uninfluenced by the $\mathcal{O}(\epsilon)$ component of the forcing and misses the corresponding fluctuating components. Nonetheless, these components are successfully captured by the $\mathcal{O}(\epsilon)$ -ROM, as shown in Figure 6.7.

On the other hand, the stacking-based approach described earlier results in a basis of size 95, which is again not feasible for reducing a 177 DOF system. To alleviate this:

1. Modal ($m = 15$): we apply the stacking approach to a subset of the sampled temperature configurations. The first five modes are available around the temperature configurations $x_c^{(j)}$ described in eq. (6.44). We select 3 of these temperature configurations at random ($j = 2, 12, 18$ in this instance) and stack the corresponding modes in a matrix and perform orthogonalization to obtain a reduction basis of size 15.
2. Modal-POD ($m = 5$): we obtain the $m = 5$ modes associated to the highest singular values of the matrix $\mathbf{V}$ which has $n_d = 19$ temperature configurations (cf. eqs. (6.44),(6.45)).

A conventional reduction is then performed via Galerkin projection using these bases. The corresponding reduced solutions show a significantly worse accuracy (see Figure 6.7) when compared to the MMS-based reduction at leading order, as well as at $\mathcal{O}(\epsilon)$.

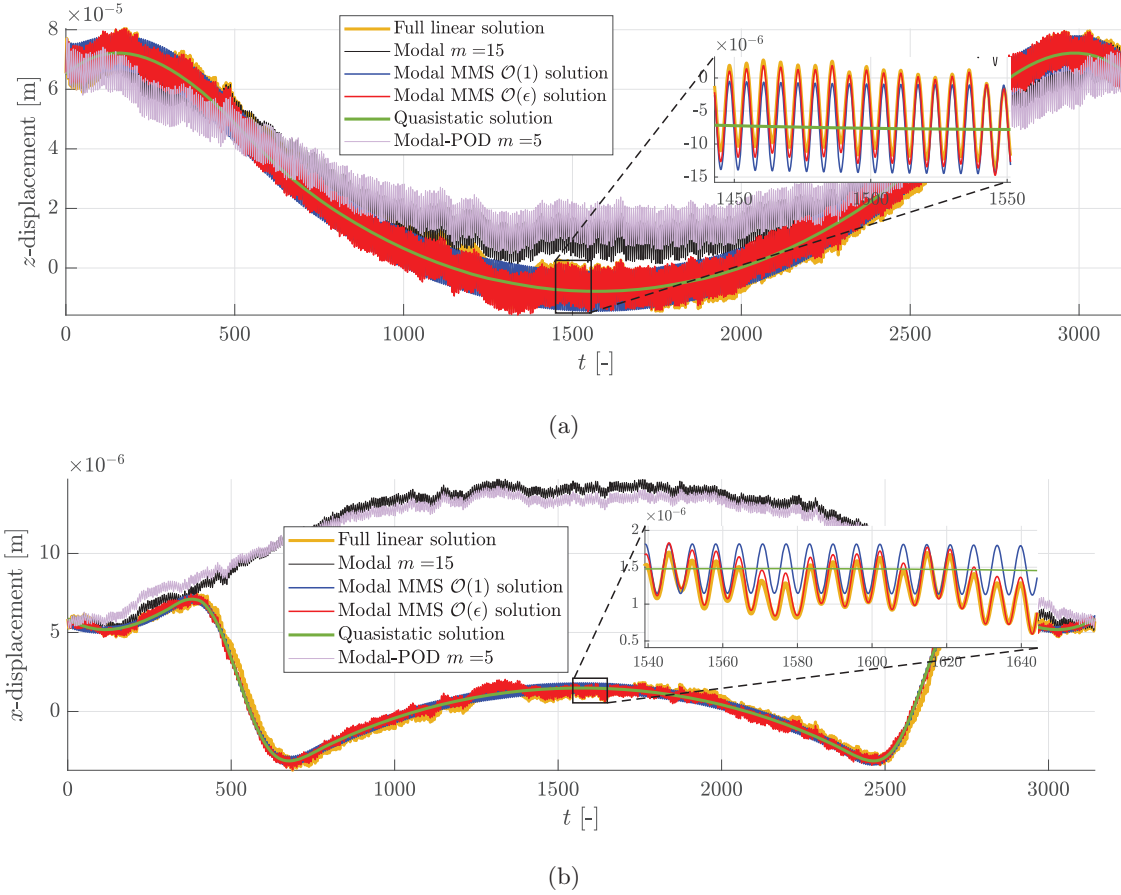


Figure 6.7: The (a) z -direction and (b) x -direction displacements for the node situated at $x = L/4$ in response to combined thermo-mechanical loading (6.47) (cf. Figure 6.6), (6.43) with $x_0 = 0.1L$, $A = 0.3L$, $\epsilon = 10^{-3}$. The Full system solution is depicted in the solid yellow line; the solid black line (Modal $m = 15$) shows the reduced solution obtained from Galerkin projection onto a constant basis containing 15 modes; the solid blue (Modal MMS $\mathcal{O}(1)$) and red (Modal MMS $\mathcal{O}(\epsilon)$) lines show the reduced solution using the MMS-based reduction upto $\mathcal{O}(1)$ and $\mathcal{O}(\epsilon)$ accuracy respectively, using a basis of $m = 5$ modes adapting to the instantaneous temperature configuration; solid pink line (Modal-POD $m = 5$) shows the reduced solution obtained from Galerkin projection onto a constant basis containing 5 modes with the highest singular values in from eq. (6.46).

Upon increasing ϵ , which quantifies the speed of thermal loading, we observe that the multiple

scales assumption starts to deviate from the true behavior of the system, as expected. The same is demonstrated in the introduction with the simple examples in Figure 6.3. In particular, for $\epsilon = 0.01$, we see in Figure 6.8 that the MMS-based reduction approach is unable to capture the Full system response, even on this linear example. The same example also demonstrates the failure of the conventional Galerkin projection-based techniques to reduce the Full system using a constant basis.

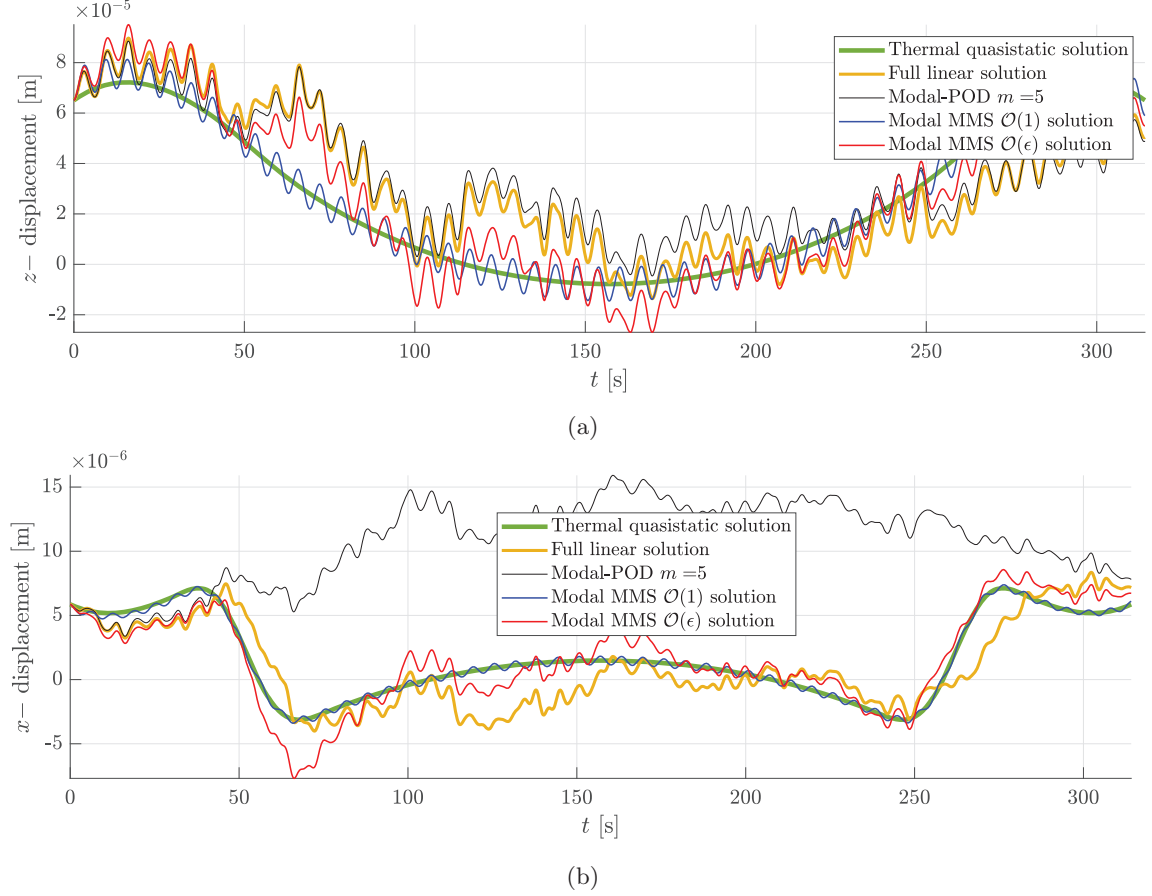


Figure 6.8: The (a) z -direction and (b) x -direction displacements for the node situated at $x = L/4$ in response to combined thermo-mechanical loading (6.47) (cf. Figure 6.6), (6.43) with $x_0 = 0.1L$, $A = 0.3L$, $\epsilon = 10^{-2}$. The Full system solution is depicted in the solid yellow line; solid pink line (Modal-POD $m = 5$) shows the reduced solution obtained from Galerkin projection onto a constant basis containing 5 modes with the highest singular values in from eq. (6.46); the solid blue (Modal MMS $\mathcal{O}(1)$) and red (Modal MMS $\mathcal{O}(\epsilon)$) lines show the reduced solution using the MMS-based reduction at $\mathcal{O}(1)$ and $\mathcal{O}(\epsilon)$ using a basis of $m = 5$ modes adapting to the instantaneous temperature configuration.

6.4.2.2 Nonlinear model

Finally, we consider the above-described curved beam model, but this time with the presence of geometric nonlinearities. As discussed in Section 6.3.2, it is well-known that a basis consisting solely of a few linear VMs is insufficient to capture the nonlinear response of a system, and the enrichment of the reduction basis with modal derivatives is an effective way to capture geometrically nonlinear response. Accordingly, we augment the database of 5-mode bases used in the linear case with the corresponding static modal derivatives. Since $k = 5$ VMs result in $k(k+1)/2 = 15$ modal derivatives,

we obtain a basis of size $m = 20$ at each temperature configuration in the database.

To compare the results with another reduction approach, we again use the stacking-based approach in the following two ways:

1. Modal ($m = 60$): Similar to the linear case, we again use a subset of randomly selected temperature configuration $(x_c^{(j)}, j = 4, 7, 13)$ from the database. In contrast to the linear case, here we obtain a reduction basis of size 60 (since each of the three temperature configurations correspond to a basis of size 20).
2. Modal-POD ($m = 20$) approach, where we use the $m = 20$ modes associated to the highest singular values of the matrix $\mathbf{V}$ which contains the first five VMs and the corresponding 15 modal derivatives at $n_d = 19$ temperature configurations (cf. eq. (6.44),(6.45)).

Figures 6.9(a) and (b) show the z and x -direction displacements at the quarter span of the beam for $\epsilon = 0.001$. To provide a more global picture, we compare the error in the vector of generalized displacements relative to the full nonlinear solution $\mathbf{u}(t)$ at each time instant (cf. Figure 6.9c). This error is defined as

$$\mathbf{e}_{red}(t) := \frac{\|\mathbf{u}(t) - \mathbf{u}_{red}(t)\|_2}{\|\mathbf{u}(t)\|_2} \quad (6.50)$$

where $\mathbf{u}_{red}(t)$ denotes the reduced solution obtain from the different approaches. It is interesting that despite have a basis size of 60, a constant basis reduction provides a worse accuracy than the MMS-based reduction with a basis of size 20 that adapts to the instantaneous temperature configuration.

The improvement in accuracy between the $\mathcal{O}(1)$ and the $\mathcal{O}(\epsilon)$ reduced solutions from the MMS is not very apparent from Figure 6.9c. For this reason, we use the following global error estimate which is uniform over time interval $[0, I]$, where I is the total time over which the simulation is performed

$$\mathbf{E}_{red} := \frac{\int_0^I \|\mathbf{u}(t) - \mathbf{u}_{red}(t)\|_2 dt}{\int_0^I \|\mathbf{u}(t)\|_2 dt} \approx \frac{\sum_{t \in \mathcal{I}} \|\mathbf{u}(t) - \mathbf{u}_{red}(t)\|_2}{\sum_{t \in \mathcal{I}} \|\mathbf{u}(t)\|_2}, \quad (6.51)$$

where the approximation is perform using a one-point numerical integration over the uniformly sampled (over $[0, I]$) time instants in the set $\mathcal{I}$. According to Table 6.2, this estimate of error shows a marginal improvement in accuracy using the $\mathcal{O}(\epsilon)$ solution over the $\mathcal{O}(1)$ solution, as expected.

Reduction method	Basis size (m)	Error
Modal	60	9.47%
Modal-POD	20	136.75%
Modal MMS $\mathcal{O}(1)$	20	3.69%
Modal MMS $\mathcal{O}(\epsilon)$	20	3.19%

Table 6.2: The uniform-in-time error estimate (6.51) for the different reduction approaches considered in the curved beam with geometric nonlinearities (cf. Figure 6.9).

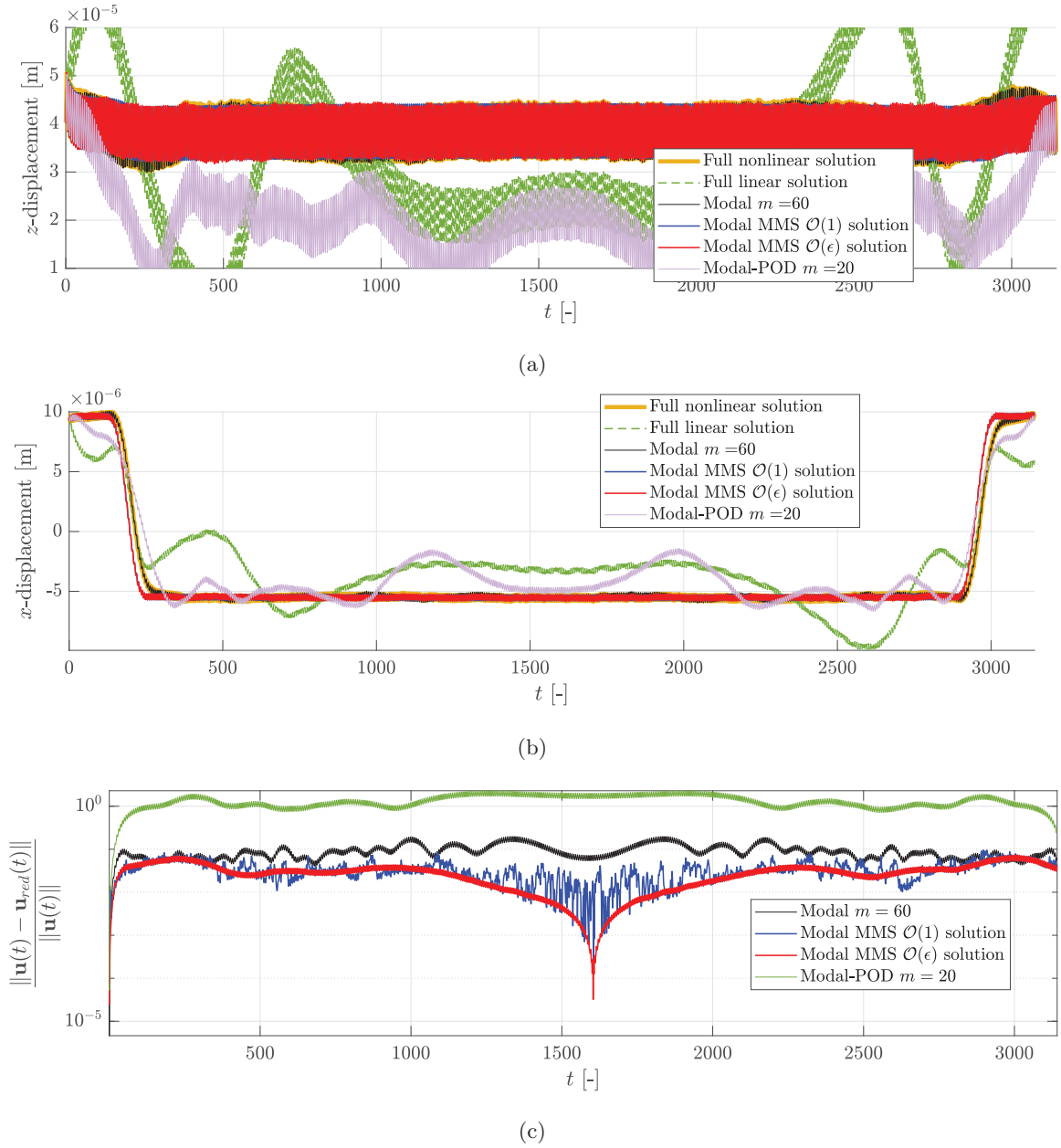


Figure 6.9: The (a) z -direction and (b) x -direction displacements for the node situated at $x = L/4$ in response to combined thermo-mechanical loading (6.47) (cf. Figure 6.6), (6.43) with $x_0 = 0.1L$, $A = 0.8L$, $\epsilon = 10^{-3}$. The Full system solution is depicted in the solid grey line; the broken green line shows the linearized response that is desirably different from the nonlinear one; the solid black line (Modal $m = 60$) shows the reduced solution obtained from Galerkin projection onto a constant basis of rank 60; the solid blue (Modal MMS $\mathcal{O}(1)$) and red (Modal MMS $\mathcal{O}(\epsilon)$) lines show the reduced solution using the MMS-based reduction upto $\mathcal{O}(1)$ and $\mathcal{O}(\epsilon)$ accuracy respectively, using a basis of size $M = 20$ adapting to the instantaneous temperature configuration. The instantaneous error (6.50) of the different reduced solutions relative to the full nonlinear solution is shown in (c).

6.5 Discussion

The numerical examples above serve as proof of concept for effective reduction using the MMS-based approach, proposed in this work. Although these examples are considerably simpler than the targeted applications of model reduction techniques, we do not envision issues with the scalability of this systematic approach. This is because the adaptive interpolation ensures that the number of reduced variables remains uniform as the temperature is dynamically changing during a simulation. Nonetheless, we discuss some general sources of computational bottleneck and possible ways to address them in the following.

Arbitrarily varying temperature fields. For simplicity, we have only considered externally-imposed temperature variation on the beam in this work. However, this variation in structural systems is usually obtained by solving the heat equation (6.5), which is given in its linearized form by the system

$$\mathbf{M}_T \mathbf{T}' + \mathbf{K}_T \mathbf{T} = \mathbf{h}(\tau), \quad (6.52)$$

where the heat source $\mathbf{h}$ is externally applied, and the temperature $\mathbf{T}$ is indeed obtained as an output. For the numerical examples treated in this work, a single parameter (x_c) was sufficient to model the variation in temperature distribution over the beam. Thus, the database of reduction bases (used in adaptive basis selection) was obtained for a range of single-parameter values (6.44). General temperatures fields, however, require n_T (number of DOFs in eq. (6.5)) parameters for description, which could be potentially large and would require a huge database of reduction bases to be computed. Thus, the memory requirements for the adaptive basis interpolation could be a potential bottleneck for adaptive basis selection in case of multi-dimensional parameters. One simple approach to reduce the number of temperature field parameters is as follows.

The temperature distribution of the structure can be spectrally decomposed into a basis of thermal modes obtained from the eigenvalue analysis of the linear operators in (6.52) as

$$(\mathbf{K}_T + \lambda_j \mathbf{M}_T) \boldsymbol{\psi}_j = \mathbf{0}, \quad j \in \{1, 2, \dots, n_T\} \quad (6.53)$$

where $\boldsymbol{\psi}_j$ is the j^{th} eigenvector related to the thermal problem (6.5) and λ_j is the corresponding eigenvalue. The temperature configuration can then be suitably approximated using a few thermal modes as

$$\mathbf{T}(\tau) = \sum_{j=1}^{n_T} \boldsymbol{\psi}_j q_{T,j}(\tau) \approx \boldsymbol{\Psi} \mathbf{q}_T(\tau), \quad (6.54)$$

where $\boldsymbol{\Psi} \in \mathbb{R}^{n_T \times m_T}$ is a matrix containing the thermal modes of the structure and the slowly varying amplitudes $\mathbf{q}_T(\tau) \in \mathbb{R}^{m_T}$ of the thermal modes determine the temperature distribution across the structure at a given time. It is then easy to see that the reduction basis for structural dynamics can be parameterized by the thermal mode amplitudes $\mathbf{q}_T$, thereby reducing the number of required parameters to $m_T \ll n_T$.

Computational time and hyper-reduction. In the case of linear systems such as eq. (6.13) with time-invariant coefficient matrices, the reduced operators can be precomputed at once when a constant reduction basis is used. However, with time-dependence in the coefficient matrices such as in eq. (6.19), the reduced operators need to be updated online at each time step during time integration. This operation poses computational bottlenecks in reduction of such systems, and a reduction in the number of unknowns does not lead to a tantamount reduction in computational time. Note that this issue affects reduction techniques using a constant basis, as well as the MMS-based reduction technique proposed here. A possible solution to this issue is to precompute the reduced operators $\mathbf{M}_\Phi$, $\mathbf{C}_\Phi$ and $\mathbf{K}_\Phi$ in eq. (6.26) along with the reduction bases at each temperature configuration in the database, during the offline stage. Thereafter, this reduced information can

be directly interpolated online (cf. ref. [12]) during time integration. This completely avoids the interpolation of reduction bases and projection at each time step at the cost of precomputation of reduced information.

For nonlinear reduced-order models, the evaluation of reduced nonlinear operators by projection onto the reduction basis can also be a computational bottleneck [28] during online time integration. Indeed, due to these extra operations, the computation of a reduced solution becomes possibly more expensive than that for a full system solution [57]. Hyper-reduction techniques provide relieve in such cases by a fast approximation of the reduced nonlinear operators using training and selection procedures. In the finite-element context, the energy conserving sampling and weighing method (ECSW) [41] is found to be effective. Based on offline training, this method provides a single reduced finite-element mesh for a given set of reduction bases and parameter values, as also discussed in ref. [26]. Furthermore, the hyper-reduction of geometrically nonlinear structural systems using ECSW with VMs and modal derivatives basis has already been affirmed in previous works [59, 60]. These features make it possible to train a ROM across a range of temperature configurations and obtain a reduced mesh so as to accelerate computation for adaptive basis reduction using the MMS-based approach proposed in this work.

6.6 Conclusion

In this work, we focused on model reduction for temperature-dependent structural dynamics equations with time varying temperature fields. We discussed that a slow variation in the structural temperature is not only physically relevant but also essential to justify model reduction using an adaptive reduction basis (cf. Section 6.1). Subsequently, we proposed the systematic use of the method of multiple scales to exploit this slow temperature dependence in the structural dynamics equations. We consistently reduced the equations of motion using a temperature-dependent basis that slowly adapts to the instantaneous temperature configuration of the structure (cf. Section 6.3).

We treated numerical examples in the linear case using an adaptive basis containing vibration modes and in the geometrically nonlinear setting, the basis was enriched using the corresponding modal derivatives. In the process, we also concluded (cf. Section 6.4) that the use constant reduction basis obtained by systematically combining reduction bases obtained from different temperature configurations is not a scalable strategy, as this results in a potentially large number of reduced variables. On the other hand, the MMS-based reduction approach turned out to be very efficient in terms of number of variables and showed consistently better accuracy.

While we addressed general computational bottlenecks in the reduction procedure and discussed possible strategies to mitigate them (cf. Section 6.5), the main focus of this work was efficient dimensionality reduction of the governing equations. Implementation of these strategies in line with the discussion in Section 6.5 is essential to obtain the necessary and sufficient computational speed. This forms part of our future efforts. In combination with these measures, we expect the MMS-based adaptive reduction would enable us to reduce realistic structures under the influence of thermal environments in a robust and efficient manner.

Chapter 7

Conclusions & Outlook

In this work, we proposed model reduction techniques with applications to geometrically nonlinear structural dynamics. In doing so, we laid emphasis, on one hand, on the use of nonlinear manifolds that act as centerpieces of the underlying nonlinear response and on the other hand, on developing simulation-free methods, i.e, methods that avoid computationally intensive full system simulations. After discussing general motivation, profits and pitfalls associated to model reduction, and setting our context of research in the introductory chapter, we focussed on some relevant themes in model reduction.

While vibration modes and modal derivatives have served as essential model reduction tools for structural dynamicists, the corresponding basis size can become prohibitive as more vibration modes are considered in the reduction basis. We proposed the use of a quadratic manifold [63] that combines vibration modes and modal derivatives in a novel fashion such that the number of reduced variables is minimized for a given set of modes. This approach preserves the simulation-free nature of the reduction process and provides accuracy comparable to the conventional Galerkin approaches that project the system of equations onto linear subspaces. With the help of realistic numerical models, we also demonstrated the general idea that nonlinear manifolds can be far more efficient for nonlinear model reduction in comparison to their linear subspace counterparts.

Apart from effectiveness, nonlinear manifolds are also desirable for model reduction on grounds of mathematical rigor as shown by the recently developed techniques of slow-fast decomposition (SFD) [50] and spectral submanifolds (SSM) [49]. We applied both these techniques on a finite-element model of a nonlinear von Kármán beam to obtain an exact reduced-order model (ROM). SFD allowed us to obtain a globally valid ROM, where the axial displacements were enslaved to the transverse displacements of the beam. A spectral quotient analysis then prompted us to further reduce this global ROM using the slowest SSM to obtain a single degree-of-freedom ROM that is locally relevant around the equilibrium.

Noting that a dimensionality reduction, though necessary, is not sufficient to guarantee computational speed-ups in the nonlinear finite-element context, we focussed on hyper-reduction methods to accelerate computation in ROMs. Owing to its relevance for hyper-reduction in the finite-element context, we laid emphasis on ECSW method [41]. In its original form, the ECSW method was proposed to hyper-reduce linear Galerkin projection-based ROM and was not suitable for nonlinear manifold-based ROMs such as the ones obtained using the quadratic manifold. We proposed an extended ECSW (EECSW) method that closes this gap and allows for the hyper-reduction of large nonlinear finite-element models over nonlinear manifolds. We demonstrated the EECSW method over the finite element model of an aircraft wing, which accelerated the ROM computations in the corresponding examples by three orders of magnitude.

The training stage that delivers a reduced mesh for faster computations in the ECSW and other hyper-reduction methods usually carries massive computational costs since it relies on full simulation data. To mitigate these costs, We further proposed a simulation-free approach for obtaining such training sets for geometrically nonlinear systems. This heuristic approach lifts inexpensive modal simulation snapshots onto the quadratic manifold to obtain training data. Using an effective computational speed measure that also takes the *offline* costs into account, we obtained an order-of-magnitude faster performance of the proposed method on the same aircraft wing model.

Finally, we proposed the use of the method of multiple scales to reduce the geometrically nonlinear structural dynamics equations under the influence of thermal effects, where the temperature variables of the structure are modeled as slowly-varying parameters. Hence, we were able to systematically use a reduction basis that adapts to the instantaneous temperature configuration of the structure, thereby outperforming conventional techniques that involve ad-hoc inclusion of hot/cold modes in the reduction basis to account for temperature effects. We demonstrated the effectiveness of this method on arches, where we used temperature-dependent vibration modes and modal derivatives for reduction.

Future research directions:

While the quadratic manifold reduction approach has shown promising results, general claims about the range of its validity were not established in this work, as also discussed in Section 2.6. To this end, heuristic estimates for the applicability of the QM approach estimates may be obtained by numerical experiments across a range of examples that compare the linear and nonlinear internal force (cf. Figures 2.2, 2.8) to assess applicability of the QM-based reduction.

The SSM-based reduction results in polynomial type ROMs that are obtained by intrusive/direct computation of reduced nonlinear coefficients or tensors, as mentioned in Section 1.3. These ROMs result in ideal speed, and in particular, do not require any hyper-reduction. However, obtaining these coefficients can be challenging in terms of memory requirements (cf. Ponsioen *et al.* [94] (Appendix), Jain [57]). This seems to be the case especially in the nonlinear finite-element context, when the modal transformations that are currently involved in SSM computations destroy the sparsity of the governing equations. Furthermore, such transformations also require the computation of *all* the eigenvectors and eigenvalues of a given system, which would be unfeasible and/or unreliable for high-dimensional systems. In this context, plausible directions for future research include the restriction to computing only the relevant eigenvector(s)/mode(s) and using sparsity-preserving methods for computing SSMs that lead to feasible memory requirements.

The EECSW proposed in Chapter 4 serves as a robust extension to ECSW. However, the simulation-free approach to obtain training vectors was tested only over ECSW with linear mappings [59] and not with EECSW [60]¹. In the context of simulation-free hyper-reduction over nonlinear manifolds, a first step would be to use the same quadratic-manifold lifting approach from Chapter 5 in combination with EECSW. Other approaches include the use of Stochastic Krylov Training Sets proposed by Rutzmoser & Rixen [103]. While a qualitative comparison between various simulation-free reduction techniques is given in Section 5.6, a quantitative comparison between different techniques on benchmark examples is another important direction for further research.

While being efficient from a dimensionality reduction perspective, the MMS-based reduction approach has several computational bottlenecks in terms of handling general dynamic temperature-dependent problems, as discussed in Section 6.5. Most importantly, such ROMs were not equipped with hyper-reduction to provide the necessary computational speed. However, hyper-reduction in the context of time-dependent parameters is currently an unexplored subject to the best of the author's knowledge. At the same time, we see no constraints in the formal application of hyper-reduction to these ROMs. This provides further direction for future development, as discussed in Section 6.5.

¹This is due to the chronology of the works in refs. [59, 60]

References

1. Aage, N., Andreassen, E., Lazarov, B. S. & Sigmund, O. Giga-voxel computational morphogenesis for structural design. *Nature* **550**, 84–86. doi:[10.1038/nature23911](https://doi.org/10.1038/nature23911) (2017).
2. Aarts, R. & Jonker, J. Dynamic simulation of planar flexible link manipulators using adaptive modal integration. *Multibody System Dynamics* **7**, 31–50. doi:[10.1023/A:1015271000518](https://doi.org/10.1023/A:1015271000518) (2002).
3. Agarwal, M., Chandorkar, S. A., Candler, R. N., Kim, B., Hopcroft, M. A., Melamud, R., Jha, C. M., Kenny, T. W. & Murmann, B. Optimal drive condition for nonlinearity reduction in electrostatic microresonators. *Applied Physics Letters* **89**, 214105. doi:[10.1063/1.2388886](https://doi.org/10.1063/1.2388886) (2006).
4. Alves, M. & Ribeiro, P. Non-linear modes of vibration of Timoshenko nanobeams under electrostatic actuation. *International Journal of Mechanical Sciences* **130**, 188–202. doi:[10.1016/j.ijmecsci.2017.06.003](https://doi.org/10.1016/j.ijmecsci.2017.06.003) (2017).
5. Amabili, M. Reduced-order models for nonlinear vibrations, based on natural modes: the case of the circular cylindrical shell. *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences* **371**, 20120474. doi:[10.1098/rsta.2012.0474](https://doi.org/10.1098/rsta.2012.0474) (2013).
6. Amabili, M. & Rivola, A. Dynamic analysis of spur gear pairs: steady-state response and stability of the sdof model with time-varying meshing damping. *Mechanical Systems and Signal Processing* **11**, 375–390. doi:[10.1006/MSSP.1996.0072](https://doi.org/10.1006/MSSP.1996.0072) (1997).
7. Amabili, M., Sarkar, A. & Païdoussis, M. Reduced-order models for nonlinear vibrations of cylindrical shells via the proper orthogonal decomposition method. *Journal of Fluids and Structures* **18**, 227–250. doi:[10.1016/J.JFLUIDSTRUCTS.2003.06.002](https://doi.org/10.1016/J.JFLUIDSTRUCTS.2003.06.002) (2003).
8. Amaldi, E. & Kann, V. On the approximability of minimizing nonzero variables or unsatisfied relations in linear systems. *Theoretical Computer Science* **209**, 237–260. doi:[10.1016/S0304-3975\(97\)00115-1](https://doi.org/10.1016/S0304-3975(97)00115-1) (1998).
9. Amsallem, D. & Farhat, C. Interpolation method for adapting reduced-order models and application to aeroelasticity. *AIAA Journal* **46**, 1803–1813. doi:[10.2514/1.35374](https://doi.org/10.2514/1.35374) (2008).
10. Amsallem, D., Cortial, J., Carlberg, K. & Farhat, C. A method for interpolating on manifolds structural dynamics reduced-order models. *International Journal for Numerical Methods in Engineering* **80**, 1241–1258. doi:[10.1002/nme.2681](https://doi.org/10.1002/nme.2681) (2009).
11. Amsallem, D., Cortial, J. & Farhat, C. Towards real-time computational-fluid-dynamics-based aeroelastic computations using a database of reduced-order information. *AIAA Journal* **48**, 2029–2037. doi:[10.2514/1.J050233](https://doi.org/10.2514/1.J050233) (2010).
12. Amsallem, D. & Farhat, C. An online method for interpolating linear parametric reduced-order models. *SIAM Journal on Scientific Computing* **33**, 2169–2198. doi:[10.1137/100813051](https://doi.org/10.1137/100813051) (2011).

13. Amsallem, D., Zahr, M. J. & Farhat, C. Nonlinear model order reduction based on local reduced-order bases. *International Journal for Numerical Methods in Engineering* **92**, 891–916. doi:[10.1002/nme.4371](https://doi.org/10.1002/nme.4371) (2012).
14. Bai, Z. Krylov subspace techniques for reduced-order modeling of large-scale dynamical systems. *Applied Numerical Mathematics* **43**, 9–44. doi:[10.1016/S0168-9274\(02\)00116-2](https://doi.org/10.1016/S0168-9274(02)00116-2) (2002).
15. Bampton, M. C. C. & Craig JR., R. R. Coupling of substructures for dynamic analyses. *AIAA Journal* **6**, 1313–1319. doi:[10.2514/3.4741](https://doi.org/10.2514/3.4741) (1968).
16. Barbič, J. & James, D. L. Real-time subspace integration for St. Venant-Kirchhoff deformable models. *ACM Transactions on Graphics* **24**, 982–990. doi:[10.1145/1073204.1073300](https://doi.org/10.1145/1073204.1073300) (2005).
17. Barrault, M., Maday, Y., Nguyen, N. C. & Patera, A. T. An ‘empirical interpolation’ method: application to efficient reduced-basis discretization of partial differential equations. *Comptes Rendus Mathématique* **339**, 667–672. doi:[10.1016/J.CRMA.2004.08.006](https://doi.org/10.1016/J.CRMA.2004.08.006) (2004).
18. Benner, P., Gugercin, S. & Willcox, K. A survey of projection-based model reduction methods for parametric dynamical systems. *SIAM Review* **57**, 483–531. doi:[10.1137/130932715](https://doi.org/10.1137/130932715) (2015).
19. Besselink, B., Tabak, U., Lutowska, A., van de Wouw, N., Nijmeijer, H., Rixen, D., Hochstenbach, M. & Schilders, W. A comparison of model reduction techniques from structural dynamics, numerical mathematics and systems and control. *Journal of Sound and Vibration* **332**, 4403–4422. doi:[10.1016/J.JSV.2013.03.025](https://doi.org/10.1016/J.JSV.2013.03.025) (2013).
20. Boley, B. A. Thermally induced vibrations of beams. *Journal of the Aeronautical Sciences* **23**, 179–181. doi:[10.2514/8.3527](https://doi.org/10.2514/8.3527) (1956).
21. Breunung, T. & Haller, G. Explicit backbone curves from spectral submanifolds of forced-damped nonlinear mechanical systems. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Science* **474**, 20180083. doi:[10.1098/rspa.2018.0083](https://doi.org/10.1098/rspa.2018.0083) (2018).
22. Brüls, O., Duysinx, P. & Golinval, J.-C. The global modal parameterization for non-linear model-order reduction in flexible multibody dynamics. *International Journal for Numerical Methods in Engineering* **69**, 948–977. doi:[10.1002/nme.1795](https://doi.org/10.1002/nme.1795) (2007).
23. Cabré, X., Fontich, E. & de la Llave, R. The parameterization method for invariant manifolds i: manifolds associated to non-resonant subspaces. *Indiana University Mathematics Journal* **52**, 283–328. <https://www.jstor.org/stable/24902854> (2003).
24. Carlberg, K., Bou-Mosleh, C. & Farhat, C. Efficient non-linear model reduction via a least-squares Petrov-Galerkin projection and compressive tensor approximations. *International Journal for Numerical Methods in Engineering* **86**, 155–181. doi:[10.1002/nme.3050](https://doi.org/10.1002/nme.3050) (2011).
25. Carlberg, K., Farhat, C., Cortial, J. & Amsallem, D. The GNAT method for nonlinear model reduction: Effective implementation and application to computational fluid dynamics and turbulent flows. *Journal of Computational Physics* **242**, 623–647. doi:[10.1016/J.JCP.2013.02.028](https://doi.org/10.1016/J.JCP.2013.02.028) (2013).
26. Chapman, T., Avery, P., Collins, P. & Farhat, C. Accelerated mesh sampling for the hyper reduction of nonlinear computational models. *International Journal for Numerical Methods in Engineering* **109**, 1623–1654. doi:[10.1002/nme.5332](https://doi.org/10.1002/nme.5332) (2017).
27. Chaturantabut, S., Beattie, C. & Gugercin, S. Structure-preserving model reduction for non-linear port-hamiltonian systems. *SIAM Journal on Scientific Computing* **38**, B837–B865. doi:[10.1137/15M1055085](https://doi.org/10.1137/15M1055085) (2016).
28. Chaturantabut, S. & Sorensen, D. C. Nonlinear model reduction via discrete empirical interpolation. *SIAM Journal on Scientific Computing* **32**, 2737–2764. doi:[10.1137/090766498](https://doi.org/10.1137/090766498) (2010).

29. Craig, R. R. & Ni, Z. Component mode synthesis for model order reduction of nonclassically-damped systems. *Journal of Guidance, Control, and Dynamics* **12**, 577–584. doi:[10.2514/3.20446](https://doi.org/10.2514/3.20446) (1989).
30. Crisfield, M. A. *Non-linear Finite Element Analysis of Solids and Structures, Volume 1* (Wiley, Chichester, 1996).
31. Dankowicz, H. & Schilder, F. *Recipes for Continuation* doi:[10.1137/1.9781611972573](https://doi.org/10.1137/1.9781611972573) (Society for Industrial and Applied Mathematics, Philadelphia, PA, 2013).
32. Detroux, T. *Performance and robustness of nonlinear systems using bifurcation analysis* PhD thesis (University of Liège, 2016). <http://hdl.handle.net/2268/195883>.
33. Dhooge, A., Govaerts, W. & Kuznetsov, Y. A. MATCONT. *ACM Transactions on Mathematical Software* **29**, 141–164. doi:[10.1145/779359.779362](https://doi.org/10.1145/779359.779362) (2003).
34. Ding, Q. & Zhang, K. Order reduction and nonlinear behaviors of a continuous rotor system. *Nonlinear Dynamics* **67**, 251–262. doi:[10.1007/s11071-011-9975-8](https://doi.org/10.1007/s11071-011-9975-8) (2012).
35. Doedel, E. J., Fairgrieve, T. F., Sandstede, B., Champneys, A. R., Kuznetsov, Y. A. & Wang, X. *Auto-07p: Continuation and bifurcation software for ordinary differential equations* tech. rep. (2007). <http://indy.cs.concordia.ca/auto/>.
36. Dowell, E. Nonlinear oscillations of a fluttering plate. *AIAA Journal* **4**, 1267–1275. doi:[10.2514/3.3658](https://doi.org/10.2514/3.3658) (1966).
37. Drohmann, M., Haasdonk, B. & Ohlberger, M. in (eds Först, J., Fürst, J., Halama, J., Herbin, R. & Hubert, F.) 369–377 (Springer, Berlin, Heidelberg, 2011). doi:[10.1007/978-3-642-20671-9_39](https://doi.org/10.1007/978-3-642-20671-9_39).
38. Epureanu, B. A parametric analysis of reduced order models of viscous flows in turbomachinery. *Journal of Fluids and Structures* **17**, 971–982. doi:[10.1016/S0889-9746\(03\)00044-6](https://doi.org/10.1016/S0889-9746(03)00044-6) (2003).
39. Everson, R. & Sirovich, L. Karhunen–Loève procedure for gappy data. *Journal of the Optical Society of America A* **12**, 1657. doi:[10.1364/JOSAA.12.001657](https://doi.org/10.1364/JOSAA.12.001657) (1995).
40. Falkiewicz, N. J., S. Cesnik, C. E., Crowell, A. R. & McNamara, J. J. Reduced-order aerothermoelastic framework for hypersonic vehicle control simulation. *AIAA Journal* **49**, 1625–1646. doi:[10.2514/1.J050802](https://doi.org/10.2514/1.J050802) (2011).
41. Farhat, C., Avery, P., Chapman, T. & Cortial, J. Dimensional reduction of nonlinear finite element dynamic models with finite rotations and energy-based mesh sampling and weighting for computational efficiency. *International Journal for Numerical Methods in Engineering* **98**, 625–662. doi:[10.1002/nme.4668](https://doi.org/10.1002/nme.4668) (2014).
42. Farhat, C., Chapman, T. & Avery, P. Structure-preserving, stability, and accuracy properties of the energy-conserving sampling and weighting method for the hyper reduction of nonlinear finite element dynamic models. *International Journal for Numerical Methods in Engineering* **102**, 1077–1110. doi:[10.1002/nme.4820](https://doi.org/10.1002/nme.4820) (2015).
43. Fenichel, N. Geometric singular perturbation theory for ordinary differential equations. *Journal of Differential Equations* **31**, 53–98. doi:[10.1016/0022-0396\(79\)90152-9](https://doi.org/10.1016/0022-0396(79)90152-9) (1979).
44. Fisher, J. W. *Tuning micro-electro-mechanical resonators for maximum dynamic range* MA thesis (Florida Institute of Technology, 2017). <http://hdl.handle.net/11141/1609>.
45. Géradin, M. & Rixen, D. *Mechanical Vibrations: Theory and Application to Structural Dynamics* Third Edition (Wiley, Chichester, 2015).
46. Golub, G. H. (H. & van Loan, C. F. *Matrix Computations* 756. <https://jhupbooks.press.jhu.edu/title/matrix-computations> (The Johns Hopkins University Press, Baltimore, 2013).

47. Guckenheimer, J. & Holmes, P. *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields* doi:[10.1007/978-1-4612-1140-2](https://doi.org/10.1007/978-1-4612-1140-2) (Springer New York, New York, NY, 1983).
48. Guyan, R. J. Reduction of stiffness and mass matrices. *AIAA Journal* **3**, 380–380. doi:[10.2514/3.2874](https://doi.org/10.2514/3.2874) (1965).
49. Haller, G. & Ponsioen, S. Nonlinear normal modes and spectral submanifolds: existence, uniqueness and use in model reduction. *Nonlinear Dynamics* **86**, 1493–1534. doi:[10.1007/s11071-016-2974-z](https://doi.org/10.1007/s11071-016-2974-z) (2016).
50. Haller, G. & Ponsioen, S. Exact model reduction by a slow-fast decomposition of nonlinear mechanical systems. *Nonlinear Dynamics* **90**, 617–647. doi:[10.1007/s11071-017-3685-9](https://doi.org/10.1007/s11071-017-3685-9) (2017).
51. Han, S. & Feeny, B. F. Enhanced proper orthogonal decomposition for the modal analysis of homogeneous structures. *Modal Analysis* **8**, 19–40. doi:[10.1177/1077546302008001518](https://doi.org/10.1177/1077546302008001518) (2002).
52. Heirman, G. H. K., Naets, F. & Desmet, W. A system-level model reduction technique for the efficient simulation of flexible multibody systems. *International Journal for Numerical Methods in Engineering* **85**, 330–354. doi:[10.1002/nme.2971](https://doi.org/10.1002/nme.2971) (2011).
53. Hollkamp, J. J. & Gordon, R. W. Reduced-order models for nonlinear response prediction: Implicit condensation and expansion. *Journal of Sound and Vibration* **318**, 1139–1153. doi:[10.1016/J.JSV.2008.04.035](https://doi.org/10.1016/J.JSV.2008.04.035) (2008).
54. Idelsohn, S. R. & Cardona, A. A load-dependent basis for reduced nonlinear structural dynamics. *Computers & Structures* **20**, 203–210. doi:[10.1016/0045-7949\(85\)90069-0](https://doi.org/10.1016/0045-7949(85)90069-0) (1985).
55. Idelsohn, S. R. & Cardona, A. A reduction method for nonlinear structural dynamic analysis. *Computer Methods in Applied Mechanics and Engineering* **49**, 253–279. doi:[10.1016/0045-7825\(85\)90125-2](https://doi.org/10.1016/0045-7825(85)90125-2) (1985).
56. Ilbeigi, S. & Chelidze, D. *Model order reduction of nonlinear Euler-Bernoulli beam in Nonlinear Dynamics, Volume 1. Conference Proceedings of the Society for Experimental Mechanics Series.* (ed Kerschen, G.) (Springer, Cham, 2016), 377–385. doi:[10.1007/978-3-319-15221-9_34](https://doi.org/10.1007/978-3-319-15221-9_34).
57. Jain, S. *Model Order Reduction for Non-Linear Structural Dynamics* MA thesis (Delft University of Technology, 2015). <http://resolver.tudelft.nl/uuid:cbid7058-2cfa-439a-bb2f-22a6b0e5bb2a>.
58. Jain, S., Breunung, T. & Haller, G. Fast Computation of Steady-State Response for Nonlinear Vibrations of High-Degree-of-Freedom Systems. *Nonlinear Dynamics*, 313–341. doi:[10.1007/s11071-019-04971-1](https://doi.org/10.1007/s11071-019-04971-1) (2019).
59. Jain, S. & Tiso, P. Simulation-free hyper-reduction for geometrically nonlinear structural dynamics: a quadratic manifold lifting approach. *Journal of Computational and Nonlinear Dynamics* **13**, 071003. doi:[10.1115/1.4040021](https://doi.org/10.1115/1.4040021) (2018).
60. Jain, S. & Tiso, P. Hyper-reduction over nonlinear manifolds for large nonlinear mechanical systems. *Journal of Computational and Nonlinear Dynamics* **14**, 081008. doi:[10.1115/1.4043450](https://doi.org/10.1115/1.4043450) (2019).
61. Jain, S. & Tiso, P. Model order reduction for temperature-dependent nonlinear mechanical systems: a multiple scales approach. *Journal of Sound and Vibration*, 115022. doi:[10.1016/j.jsv.2019.115022](https://doi.org/10.1016/j.jsv.2019.115022) (2020).
62. Jain, S., Tiso, P. & Haller, G. Exact nonlinear model reduction for a von Kármán beam: Slow-fast decomposition and spectral submanifolds. *Journal of Sound and Vibration* **423**, 195–211. doi:[10.1016/J.JSV.2018.01.049](https://doi.org/10.1016/J.JSV.2018.01.049) (2018).

63. Jain, S., Tiso, P., Rutzmoser, J. B. & Rixen, D. J. A quadratic manifold for model order reduction of nonlinear structural dynamics. *Computers & Structures* **188**, 80–94. doi:[10.1016/J.COMPSTRUC.2017.04.005](https://doi.org/10.1016/J.COMPSTRUC.2017.04.005) (2017).
64. Keller, H. B. *Numerical Methods for Two-Point Boundary-Value Problems* (Blaisdell Publishing Company, Waltham, Massachusetts, 1968).
65. Kerschen, G., Golinval, J.-c., Vakakis, A. F. & Bergman, L. A. The method of proper orthogonal decomposition for dynamical characterization and order reduction of mechanical systems: an overview. *Nonlinear Dynamics* **41**, 147–169. doi:[10.1007/s11071-005-2803-2](https://doi.org/10.1007/s11071-005-2803-2) (2005).
66. Kevorkian, J. & Cole, J. D. *Multiple Scale and Singular Perturbation Methods* doi:[10.1007/978-1-4612-3968-0](https://doi.org/10.1007/978-1-4612-3968-0) (Springer New York, New York, NY, 1996).
67. Kogelbauer, F. & Haller, G. Rigorous model reduction for a damped-forced nonlinear beam model: an infinite-dimensional analysis. *Journal of Nonlinear Science* **28**, 1109–1150. doi:[10.1007/s00332-018-9443-4](https://doi.org/10.1007/s00332-018-9443-4) (2018).
68. Kosambi, D. D. Statistics in function space. *Journal of the Indian Mathematical Society* **7**, 76–88. <http://repository.ias.ac.in/99240/> (1942).
69. Kryloff, N. & Bogoliuboff, N. *Introduction to Non-linear Mechanics* (Princeton University Press, 1949).
70. Kuether, R. J., Deaner, B. J., Hollkamp, J. J. & Allen, M. S. Evaluation of geometrically nonlinear reduced-order models with nonlinear normal modes. *AIAA Journal* **53**, 3273–3285. doi:[10.2514/1.J053838](https://doi.org/10.2514/1.J053838) (2015).
71. LaBryer, A. & Attar, P. A harmonic balance approach for large-scale problems in nonlinear structural dynamics. *Computers & Structures* **88**, 1002–1014. doi:[10.1016/J.COMPSTRUC.2010.06.003](https://doi.org/10.1016/J.COMPSTRUC.2010.06.003) (2010).
72. Lall, S., Krysl, P. & Marsden, J. E. Structure-preserving model reduction for mechanical systems. *Physica D: Nonlinear Phenomena* **184**, 304–318. doi:[10.1016/S0167-2789\(03\)00227-6](https://doi.org/10.1016/S0167-2789(03)00227-6) (2003).
73. Lazarus, A., Thomas, O. & Deü, J.-F. Finite element reduced order models for nonlinear vibrations of piezoelectric layered beams with applications to NEMS. *Finite Elements in Analysis and Design* **49**, 35–51. doi:[10.1016/j.finel.2011.08.019](https://doi.org/10.1016/j.finel.2011.08.019) (2012).
74. Lee, J. A. & Verleysen, M. *Nonlinear Dimensionality Reduction* doi:[10.1007/978-0-387-39351-3](https://doi.org/10.1007/978-0-387-39351-3) (Springer New York, New York, NY, 2007).
75. Lee, Y., Vakakis, A., Bergman, L., McFarland, D. M. & Kerschen, G. Suppression aeroelastic instability using broadband passive targeted energy transfers, part 1: theory. *AIAA Journal* **45**, 693–711. doi:[10.2514/1.24062](https://doi.org/10.2514/1.24062) (2007).
76. Li, L. L., Polunin, P. M., Dou, S., Shoshani, O., Scott Strachan, B., Jensen, J. S., Shaw, S. W. & Turner, K. L. Tailoring the nonlinear response of MEMS resonators using shape optimization. *Applied Physics Letters* **110**, 081902. doi:[10.1063/1.4976749](https://doi.org/10.1063/1.4976749) (2017).
77. Lieu, T. & Farhat, C. Adaptation of aeroelastic reduced-order models and application to an F-16 configuration. *AIAA Journal* **45**, 1244–1257. doi:[10.2514/1.24512](https://doi.org/10.2514/1.24512) (2007).
78. Lieu, T. & Lesoinne, M. *Parameter adaptation of reduced order models for three-dimensional flutter analysis* in *42nd AIAA Aerospace Sciences Meeting and Exhibit* (American Institute of Aeronautics and Astronautics, Reston, Virginia, 2004). doi:[10.2514/6.2004-888](https://doi.org/10.2514/6.2004-888).
79. Lindhorst, K., Haupt, M. & Horst, P. Reduced-order modelling of non-linear, transient aerodynamics of the HIRENASD wing. *The Aeronautical Journal* **120**, 601–626. doi:[10.1017/aer.2016.12](https://doi.org/10.1017/aer.2016.12) (2016).
80. Lines, L. & Treitel, S. A review of least-squares inversion and its application to geophysical problems. *Geophysical Prospecting* **32**, 159–186. doi:[10.1111/j.1365-2478.1984.tb00726.x](https://doi.org/10.1111/j.1365-2478.1984.tb00726.x) (1984).

81. Matney, A. *Reduced order model-based prediction of the nonlinear geometric response of a panel under thermal, aerodynamic, and acoustic loads* PhD thesis (Arizona State University, 2014). <http://hdl.handle.net/2286/R.I.27537>.
82. Matthies, H. G. & Meyer, M. Nonlinear Galerkin methods for the model reduction of nonlinear dynamical systems. *Computers & Structures* **81**, 1277–1286. doi:[10.1016/S0045-7949\(03\)00042-7](https://doi.org/10.1016/S0045-7949(03)00042-7) (2003).
83. Menken, C., Schreppers, G., Groot, W. & Petterson, R. Analyzing buckling mode interactions in elastic structures using an asymptotic approach; theory and experiments. *Computers & Structures* **64**, 473–480. doi:[10.1016/S0045-7949\(96\)00139-3](https://doi.org/10.1016/S0045-7949(96)00139-3) (1997).
84. Mignolet, M. P., Przekop, A., Rizzi, S. A. & Spottswood, S. M. A review of indirect/non-intrusive reduced order modeling of nonlinear geometric structures. *Journal of Sound and Vibration* **332**, 2437–2460. doi:[10.1016/J.JSV.2012.10.017](https://doi.org/10.1016/J.JSV.2012.10.017) (2013).
85. Millán, D. & Arroyo, M. Nonlinear manifold learning for model reduction in finite elastodynamics. *Computer Methods in Applied Mechanics and Engineering* **261-262**, 118–131. doi:[10.1016/J.CMA.2013.04.007](https://doi.org/10.1016/J.CMA.2013.04.007) (2013).
86. Miller, B., McNamara, J., Spottswood, S. & Culler, A. The impact of flow induced loads on snap-through behavior of acoustically excited, thermally buckled panels. *Journal of Sound and Vibration* **330**, 5736–5752. doi:[10.1016/J.JSV.2011.06.028](https://doi.org/10.1016/J.JSV.2011.06.028) (2011).
87. Muravyov, A. A. & Rizzi, S. A. Determination of nonlinear stiffness with application to random vibration of geometrically nonlinear structures. *Computers & Structures* **81**, 1513–1523. doi:[10.1016/S0045-7949\(03\)00145-7](https://doi.org/10.1016/S0045-7949(03)00145-7) (2003).
88. Nelson, R. B. Simplified calculation of eigenvector derivatives. *AIAA Journal* **14**, 1201–1205. doi:[10.2514/3.7211](https://doi.org/10.2514/3.7211) (1976).
89. Newmark, N. M. A method of computation for structural dynamics. *ASCE Journal of the Engineering Mechanics Division* **85**, 67–94 (1959).
90. Peharz, R. & Pernkopf, F. Sparse nonnegative matrix factorization with l0-constraints. *Neurocomputing* **80**, 38–46. doi:[10.1016/J.NEUCOM.2011.09.024](https://doi.org/10.1016/J.NEUCOM.2011.09.024) (2012).
91. Perez, R., Wang, X. Q. & Mignolet, M. P. Nonlinear reduced-order models for thermoelastodynamic response of isotropic and functionally graded panels. *AIAA Journal* **49**, 630–641. doi:[10.2514/1.J050684](https://doi.org/10.2514/1.J050684) (2011).
92. Perez, R., Wang, X. Q. & Mignolet, M. P. Nonintrusive structural dynamic reduced order modeling for large deformations: enhancements for complex structures. *Journal of Computational and Nonlinear Dynamics* **9**, 031008. doi:[10.1115/1.4026155](https://doi.org/10.1115/1.4026155) (2014).
93. Pescheck, E., Pierre, C. & Shaw, S. A new galerkin-based approach for accurate non-linear normal modes through invariant manifolds. *Journal of Sound and Vibration* **249**, 971–993. doi:[10.1006/jsvi.2001.3914](https://doi.org/10.1006/jsvi.2001.3914) (2002).
94. Ponsioen, S., Pedergrana, T. & Haller, G. Analytic prediction of isolated forced response curves from spectral submanifolds. arXiv: [1812.06664](https://arxiv.org/abs/1812.06664) (2018).
95. Ponsioen, S., Pedergrana, T. & Haller, G. Automated computation of autonomous spectral submanifolds for nonlinear modal analysis. *Journal of Sound and Vibration* **420**, 269–295. doi:[10.1016/J.JSV.2018.01.048](https://doi.org/10.1016/J.JSV.2018.01.048) (2018).
96. Przekop, A. & Rizzi, S. A. Dynamic snap-through of thin-walled structures by a reduced-order method. *AIAA Journal* **45**, 2510–2519. doi:[10.2514/1.26351](https://doi.org/10.2514/1.26351) (2007).
97. Qu, Z.-Q. *Model Order Reduction Techniques* doi:[10.1007/978-1-4471-3827-3](https://doi.org/10.1007/978-1-4471-3827-3) (Springer London, London, 2004).
98. Quinn, D. D., Triplett, A. L., Vakakis, A. F. & Bergman, L. A. Energy Harvesting From Impulsive Loads Using Intentional Essential Nonlinearities. *Journal of Vibration and Acoustics* **133**, 011004. doi:[10.1115/1.4002787](https://doi.org/10.1115/1.4002787) (2011).

99. Reddy, J. N. *An Introduction to Nonlinear Finite Element Analysis* doi:10.1093/acprof:oso/9780198525295.001.0001 (Oxford University Press, 2010).
100. Rega, G. & Troger, H. Dimension Reduction of Dynamical Systems: Methods, Models, Applications. *Nonlinear Dynamics* **41**, 1–15. doi:10.1007/s11071-005-2790-3 (2005).
101. Renson, L., Kerschen, G. & Cochelin, B. Numerical computation of nonlinear normal modes in mechanical engineering. *Journal of Sound and Vibration* **364**, 177–206. doi:10.1016/J.JSV.2015.09.033 (2016).
102. Riks, E. Buckling analysis of elastic structures: a computational approach. *Advances in Applied Mechanics* **34**, 1–76. doi:10.1016/S0065-2156(08)70319-3 (1997).
103. Rutzmoser, J. & Rixen, D. A lean and efficient snapshot generation technique for the Hyper-Reduction of nonlinear structural dynamics. *Computer Methods in Applied Mechanics and Engineering* **325**, 330–349. doi:10.1016/J.CMA.2017.06.009 (2017).
104. Rutzmoser, J., Rixen, D. & Tiso, P. *Model order reduction using an adaptive basis for geometrically nonlinear structural dynamics* in *International Conference on Noise and Vibration Engineering, ISMA, 15-17 September, 2014* (eds Sas, P., Moens, D. & Denayer, H.) (KUL, Leuven, Belgium, 2014), 2595–2603.
105. Rutzmoser, J., Rixen, D., Tiso, P. & Jain, S. Generalization of quadratic manifolds for reduced order modeling of nonlinear structural dynamics. *Computers & Structures* **192**, 196–209. doi:10.1016/J.COMPSTRUC.2017.06.003 (2017).
106. Rutzmoser, J. B. *Model order reduction for nonlinear structural dynamics* PhD thesis (Technical University of Munich, 2018). <http://nbn-resolving.de/urn/resolver.pl?urn:nbn:de:bvb:91-diss-20180313-1381943-1-3>.
107. Rutzmoser, J. B. & Rixen, D. J. *Model order reduction for geometric nonlinear structures with variable state-dependent basis* in *Dynamics of Coupled Structures, Volume 1. Conference Proceedings of the Society for Experimental Mechanics Series* (eds Allen, M., Mayes, R. & Rixen, D. J.) (Springer, Cham, 2014), 455–462. doi:10.1007/978-3-319-04501-6_43.
108. Ryckelynck, D. A priori hyperreduction method: an adaptive approach. *Journal of Computational Physics* **202**, 346–366. doi:10.1016/J.JCP.2004.07.015 (2005).
109. Schoneman, J. D., Allen, M. S. & Kuether, R. J. *Nonlinear modal substructuring of panel and stiffener assemblies via characteristic Constraint Modes* in *Dynamics of Coupled Structures, Volume 4. Conference Proceedings of the Society for Experimental Mechanics Series* (eds Allen, M. S., Mayes, R. & Rixen, D.) (Springer, Cham, 2017), 307–326. doi:10.1007/978-3-319-54930-9_27.
110. Shaw, A., Neild, S., Wagg, D., Weaver, P. & Carrella, A. A nonlinear spring mechanism incorporating a bistable composite plate for vibration isolation. *Journal of Sound and Vibration* **332**, 6265–6275. doi:10.1016/J.JSV.2013.07.016 (2013).
111. Shaw, S. & Pierre, C. Normal modes for non-linear vibratory systems. *Journal of Sound and Vibration* **164**, 85–124. doi:10.1006/jsvi.1993.1198 (1993).
112. Siddhi, V. *A generalized approach for calculation of the eigenvector sensitivity for various eigenvector normalizations* Master Thesis (University of Missouri–Columbia, 2005). <http://hdl.handle.net/10355/4244>.
113. Slaats, P., de Jongh, J. & Sauren, A. Model reduction tools for nonlinear structural dynamics. *Computers & Structures* **54**, 1155–1171. doi:10.1016/0045-7949(94)00389-K (1995).
114. Sombroek, C., Tiso, P., Renson, L. & Kerschen, G. Numerical computation of nonlinear normal modes in a modal derivative subspace. *Computers & Structures* **195**, 34–46. doi:10.1016/J.COMPSTRUC.2017.08.016 (2018).
115. Spottswood, S. M., Hollkamp, J. J. & Eason, T. G. Reduced-order models for a shallow curved beam under combined loading. *AIAA Journal* **48**, 47–55. doi:10.2514/1.38707 (2010).

116. Szalai, R., Ehrhardt, D. & Haller, G. Nonlinear model identification and spectral submanifolds for multi-degree-of-freedom mechanical vibrations. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Science* **473**, 20160759. doi:[10.1098/rspa.2016.0759](https://doi.org/10.1098/rspa.2016.0759) (2017).
117. Tamarozzi, T., Heirman, G. & Desmet, W. An on-line time dependent parametric model order reduction scheme with focus on dynamic stress recovery. *Computer Methods in Applied Mechanics and Engineering* **268**, 336–358. doi:[10.1016/J.CMA.2013.09.021](https://doi.org/10.1016/J.CMA.2013.09.021) (2014).
118. Tehrani, M. G. & Elliott, S. J. Extending the dynamic range of an energy harvester using nonlinear damping. *Journal of Sound and Vibration* **333**, 623–629. doi:[10.1016/J.JSV.2013.09.035](https://doi.org/10.1016/J.JSV.2013.09.035) (2014).
119. Tiso, P. *Finite element based reduction methods for static and dynamic analysis of thin-walled structures* PhD thesis (Delft University of Technology, 2006). <http://resolver.tudelft.nl/uuid:3affd8eb-8fbd-40da-aefc-48a96efe8afb>.
120. Tiso, P. *Optimal second order reduction basis selection for nonlinear transient analysis in Modal Analysis Topics, Volume 3. Conference Proceedings of the Society for Experimental Mechanics Series* (ed Proulx, T.) (Springer, New York, NY, 2011), 27–39. doi:[10.1007/978-1-4419-9299-4_3](https://doi.org/10.1007/978-1-4419-9299-4_3).
121. Tiso, P., Jansen, E. & Abdalla, M. Reduction method for finite element nonlinear dynamic analysis of shells. *AIAA Journal* **49**, 2295–2304. doi:[10.2514/1.J051003](https://doi.org/10.2514/1.J051003) (2011).
122. Tiso, P. & Rixen, D. J. *Reduction methods for MEMS nonlinear dynamic analysis in Nonlinear Modeling and Applications, Volume 2. Conference Proceedings of the Society for Experimental Mechanics Series* (ed Proulx, T.) (Springer, New York, NY, 2011), 53–65. doi:[10.1007/978-1-4419-9719-7_6](https://doi.org/10.1007/978-1-4419-9719-7_6).
123. Tiso, P. & Rixen, D. J. *Discrete empirical interpolation method for finite element structural dynamics in Topics in Nonlinear Dynamics, Volume 1. Conference Proceedings of the Society for Experimental Mechanics Series, vol 35.* (eds Kerschen, G., Adams, D. & Carrella, A.) (Springer, New York, NY, 2013), 203–212. doi:[10.1007/978-1-4614-6570-6_18](https://doi.org/10.1007/978-1-4614-6570-6_18).
124. Touzé, C., Vidrascu, M. & Chapelle, D. Direct finite element computation of non-linear modal coupling coefficients for reduced-order shell models. *Computational Mechanics* **54**, 567–580. doi:[10.1007/s00466-014-1006-4](https://doi.org/10.1007/s00466-014-1006-4) (2014).
125. Vakakis, A. F. Inducing passive nonlinear energy sinks in vibrating systems. *Journal of Vibration and Acoustics* **123**, 324. doi:[10.1115/1.1368883](https://doi.org/10.1115/1.1368883) (2001).
126. Warburton, G. B. The vibration of rectangular plates. *Proceedings of the Institution of Mechanical Engineers* **168**, 371–384. doi:[10.1243/PIME_PROC_1954_168_040_02](https://doi.org/10.1243/PIME_PROC_1954_168_040_02) (1954).
127. Wedel-Heinen, J. Vibration of geometrically imperfect beam and shell structures. *International Journal of Solids and Structures* **27**, 29–47. doi:[10.1016/0020-7683\(91\)90143-4](https://doi.org/10.1016/0020-7683(91)90143-4) (1991).
128. Weeger, O., Wever, U. & Simeon, B. On the use of modal derivatives for nonlinear model order reduction. *International Journal for Numerical Methods in Engineering* **108**, 1579–1602. doi:[10.1002/nme.5267](https://doi.org/10.1002/nme.5267) (2016).
129. Weickum, G., Eldred, M. S. & Maute, K. A multi-point reduced-order modeling approach of transient structural dynamics with application to robust design optimization. *Structural and Multidisciplinary Optimization* **38**, 599–611. doi:[10.1007/s00158-008-0309-5](https://doi.org/10.1007/s00158-008-0309-5) (2009).
130. Wieting, A. R., Dechaumphai, P., Bey, K. S., Thornton, E. A. & Morgan, K. Application of integrated fluid-thermal-structural analysis methods. *Thin-Walled Structures* **11**, 1–23. doi:[10.1016/0263-8231\(91\)90008-7](https://doi.org/10.1016/0263-8231(91)90008-7) (1991).

131. Witteveen, W. & Pichler, F. *Efficient model order reduction for the nonlinear dynamics of jointed structures by the use of trial vector derivatives* in *Dynamics of Coupled Structures, Volume 1. Conference Proceedings of the Society for Experimental Mechanics Series* (eds Allen, M., Mayes, R. & Rixen, D. J.) (Springer, Cham, 2014), 147–155. doi:[10.1007/978-3-319-04501-6_13](https://doi.org/10.1007/978-3-319-04501-6_13).
132. Wu, L. & Tiso, P. Nonlinear model order reduction for flexible multibody dynamics: a modal derivatives approach. *Multibody System Dynamics* **36**, 405–425. doi:[10.1007/s11044-015-9476-5](https://doi.org/10.1007/s11044-015-9476-5) (2016).