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The Gibbs Variational Method in Thermodynamics of Equilibrium Plasma: 5. Exact Solution for Equilibrium Configurations of Two-Component Plasma Confined between Parallel Plates

by Michael Grinfeld and Pavel Grinfeld

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14. ABSTRACT It is the fifth part of the study published under the common umbrella *The Gibbs Variational Method in Thermodynamics of Equilibrium Plasma*. In Parts 1–4, we formulated a novel approach to thermodynamics of one- and two-component heterogeneous systems completely or partially filled with a liquid substance in plasma state. The approach is based on the use of the Gibbs variational principles, and it enables efforts to address a variety of problems relating to equilibrium and stability of such systems. In this fifth part, the results of the Parts 1–4 are applied to the analysis of equilibrium configurations of a two-component charged plasma trapped between two parallel plates (the geometry often used in various applications).						
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1. Introduction

In Part 1 of this series of reports,[1–4] we formulated a novel approach to thermodynamics of heterogeneous systems completely or partially filled with a liquid or gaseous substance in a plasma state. The approach is based on the use of the Gibbs variational principles,[5] and it enables efforts to address a variety of problems relating to equilibrium and stability of such systems.

The general motivation for this series of reports is discussed in Grinfeld and Grinfeld,[1] in which we also demonstrated how the Gibbs approach[5] can be applied to the heterogeneous systems with charged gases. In the part Grinfeld and Grinfeld,[3] we developed a general thermodynamic methodology, applicable to gaseous two-component plasma with arbitrary equations of state (EOS). The general analysis is the most effective tool to elucidate the most universal features of the approach. On the other hand, it puts obvious limitations to the application of mathematically rigorous tools.

The analytical difficulties appear because of two main reasons: 1) the difficulties caused by the geometrical complexities of the problems under study, and 2) the general relationships lead to essentially nonlinear systems of partial differential equation. Therefore, further simplifying assumptions are unavoidable if one needs to proceed with exact mathematics. The exact solutions are the main tools for deeper understanding of the gross physical features of the models and for verification of the theory.

In this fifth part of our study, we consider the equilibrium configurations of a gaseous plasma confined between two infinite parallel plates, which are shown in Fig. 1.

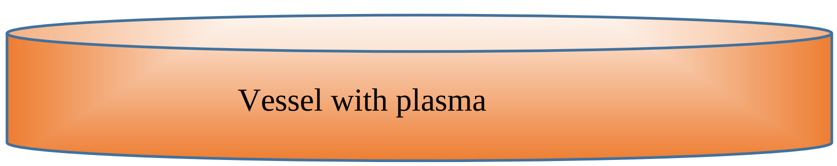

Fig. 1 Model of a charged plasma system

At the same time, the general approach does not permit further progress with the mathematically exact or the ultimate numerical analysis of the equilibrium configurations unless we choose an appropriate EOS. The essential progress in this direction was achieved in Grinfeld and Grinfeld,[2] in which we specified the EOS following the footsteps of Grinfeld.[6] For that model EOS we found the exact solution of the classical 1-D problem for the unbounded, uniformly charged plate inserted in the one-component charged gas.

In this report, we develop this EOS further and establish the exact solution for the same EOS but considering a mixture of two gases with positive and negative charges. The geometry is presented in Fig. 1.

Fortunately, the equations of electrostatics in a vacuum are linear. The only source of nonlinearity in the static problems are the EOS. To address this difficulty, we choose the EOS suggested in Grinfeld.[6] This choice results in dealing with the linear ordinary differential equations.

2. Formulation of the 1-D Boundary Value Problem for a Two-Component Charged Mixture

We follow here Grinfeld and Grinfeld[3] and Grinfeld.[6] Per these reports, the entire system of equilibrium equations includes the following three elements:

1) the condition of thermal equilibrium

$$T = T^{\circ} = const \tag{1.1}$$

throughout the whole configuration,

2) the electrostatics system for the electrostatic potential φ

$$\frac{d^2\varphi}{dz^2} = -4\pi(\sigma_e\rho_e + \sigma_i\rho_i) \tag{1.2}$$

where σ_e, σ_i are the charge densities of the components per unit mass, and ρ_e, ρ_i are the mass densities of the components,

3) the electrochemistry equations

$$(\rho e)_{\rho_I} + \sigma_I\varphi = \Lambda_I \tag{1.3}$$

where I assumes the values e and i, and Λ_I are the indefinite Lagrange multipliers.

To determine the Lagrange multipliers, we must use the equation dealing with the total charge (or mass) of the system. Let M_I be the total mass of the gas per unit cross section. This leads to the relationships

$$\int_{-H}^{H} dz\rho_I(z) = M_I \tag{1.4}$$

3. The Exact Solution of the Boundary Value Problem for Two Charged Liquids with the Canonical EOS

Differentiating Eq. 1.3, we get two equations.

$$\frac{d\varphi}{dz} = -\frac{a_I^2}{\sigma_I}\frac{d\rho_I}{dz} \tag{2.1}$$

where $a_I^2(\rho) \equiv (\rho_I e_I(\rho))_{\rho\rho}$.

We call canonical the EOS for which $a_I^2(\rho) = const$. We use the combining index I which assumes two values "e" and "i".

Inserting Eq. 2.1 in the equation of electrostatics Eq. 1.2, we arrive at the equations

$$\frac{a_e^2}{4\pi\sigma_e}\frac{d^2\rho_e}{dz^2} = \sigma_e\rho_e + \sigma_i\rho_i \tag{2.2}$$

and

$$\frac{a_i^2}{4\pi\sigma_i}\frac{d^2\rho_i}{dz^2} = \sigma_e\rho_e + \sigma_i\rho_i \tag{2.3}$$

Looking for the solutions of Eqs. 2.2 and 2.3 in the form

$$\begin{Vmatrix}\rho_e \\ \rho_i\end{Vmatrix} = \begin{Vmatrix}A_e \\ A_i\end{Vmatrix} e^{\lambda z} \tag{2.4}$$

we get the system of two equations

$$\left(\sigma_e - \frac{a_e^2}{4\pi\sigma_e}\lambda^2\right)A_e + \sigma_i A_i = 0 \tag{2.5}$$

and

$$\sigma_e A_e + \left(\sigma_i - \frac{a_i^2}{4\pi\sigma_i}\lambda^2\right)A_i = 0 \tag{2.6}$$

The system in Eqs. 2.5 and 2.6 leads to the following secular equation

$$\left(\sigma_e - \frac{a_e^2}{4\pi\sigma_e}\lambda^2\right)\left(\sigma_i - \frac{a_i^2}{4\pi\sigma_i}\lambda^2\right) - \sigma_e\sigma_i = 0 \tag{2.7}$$

which can be rewritten as

$$\frac{a_e^2 a_i^2}{4\pi\sigma_e\sigma_i}\lambda^4 - \left(\frac{a_e^2\sigma_i}{\sigma_e} + \frac{a_i^2\sigma_e}{\sigma_i}\right)\lambda^2 = 0 \tag{2.8}$$

or

$$\lambda^4 - \Delta^2\lambda^2 = 0 \tag{2.9}$$

where we use the notation

$$\Delta^2 \equiv \frac{4\pi\sigma_i^2}{a_i^2} + \frac{4\pi\sigma_e^2}{a_e^2} \tag{2.10}$$

Thus, we arrive at the following spectrum of the eigenvalues:

$$\begin{gathered} \lambda^2 = 0, \sigma_e A_e + \sigma_i A_i = 0 \rightarrow \left\| \begin{matrix} A_e \\ A_i \end{matrix} \right\| = C_0 \left\| \begin{matrix} \sigma_i \\ -\sigma_e \end{matrix} \right\| \\ \lambda^2 = \Delta^2 \equiv \frac{4\pi\sigma_i^2}{a_i^2} + \frac{4\pi\sigma_e^2}{a_e^2},\ \frac{a_e^2}{\sigma_e}A_e - \frac{a_i^2}{\sigma_i}A_i = 0 \rightarrow \left\| \begin{matrix} A_e \\ A_i \end{matrix} \right\| = C_1 \left\| \begin{matrix} \dfrac{a_i^2}{\sigma_i} \\ \dfrac{a_e^2}{\sigma_e} \end{matrix} \right\| \end{gathered} \tag{2.11}$$

and the following general solution:

$$\left\| \begin{matrix} \rho_e \\ \rho_i \end{matrix} \right\| = C_0 \left\| \begin{matrix} \sigma_i \\ -\sigma_e \end{matrix} \right\| + C_1 \left\| \begin{matrix} \dfrac{a_i^2}{\sigma_i} \\ \dfrac{a_e^2}{\sigma_e} \end{matrix} \right\| \cosh(\Delta z) \tag{2.12}$$

The constants C_0 and C_1 can be determined from the mass balance equation (Eq. 1.4).

By elementary integration we get

$$\int_{-H}^{H} dz \cosh(\Delta z) = \frac{2}{\Delta}\sinh(\Delta H) \tag{2.13}$$

as implied by the following chain:

$$\int_{-H}^{H} dz \cosh(\Delta z) = \frac{2}{\Delta}\sinh(\Delta H)$$

$$\int_{-H}^{H} dz \cosh(\Delta z) = \frac{1}{\Delta}\int_{-H}^{H} dz\Delta \cosh(\Delta z) = \frac{1}{\Delta}\int_{-\Delta H}^{\Delta H} d\eta \cosh\eta =$$

$$\frac{1}{\Delta}\int_{-\Delta H}^{\Delta H} d\eta \cosh\eta = \frac{2}{\Delta}\sinh(\Delta H)$$

With the help of Eqs. 2.12 and 2.13, Eq. 1.4 gives us the following system of linear algebraic equations:

$$\begin{aligned} &C_0 2H\sigma_i + C_1 \frac{a_i^2}{\sigma_i}\frac{2}{\Delta}\sinh(\Delta H) = M_e \\ &-C_0 2H\sigma_e + C_1 \frac{a_e^2}{\sigma_e}\frac{2}{\Delta}\sinh(\Delta H) = M_i \end{aligned} \tag{2.14}$$

Equation 2.14 implies the following solution

$$\begin{aligned} C_0 &= \frac{1}{2H}\frac{M_e a_e^2\sigma_e^{-1} - M_i a_i^2\sigma_i^{-1}}{\sigma_i\sigma_e\left(a_i^2\sigma_i^{-2} + a_i^2\sigma_e^{-2}\right)} \\ C_1 &= \frac{1}{2H}\frac{\Delta H}{\sinh(\Delta H)}\frac{M_e\sigma_e + M_i\sigma_i}{\sigma_i\sigma_e\left(a_i^2\sigma_i^{-2} + a_e^2\sigma_e^{-2}\right)} \end{aligned} \tag{2.15}$$

Inserting the constants from the Eq. 2.15 in Eq. 2.12, we get eventually

$$\begin{aligned} \left\|\begin{matrix}\rho_e \\ \rho_i\end{matrix}\right\| &= \frac{1}{2H}\frac{M_e a_e^2\sigma_e^{-1} - M_i a_i^2\sigma_i^{-1}}{\sigma_i\sigma_e\left(a_i^2\sigma_i^{-2} + a_e^2\sigma_e^{-2}\right)}\left\|\begin{matrix}\sigma_i \\ -\sigma_e\end{matrix}\right\| + \\ &\frac{1}{2H}\frac{\Delta H}{\sinh(\Delta H)}\frac{M_e\sigma_e + M_i\sigma_i}{\sigma_i\sigma_e\left(a_i^2\sigma_i^{-2} + a_e^2\sigma_e^{-2}\right)}\left\|\begin{matrix}a_i^2\sigma_i^{-1} \\ a_e^2\sigma_e^{-1}\end{matrix}\right\|\cosh(\Delta z) \end{aligned} \tag{2.16}$$

Using elementary transformation, we can rewrite Eq. 2.16 as

$$\begin{aligned} &2H\sigma_i\sigma_e\left(a_i^2\sigma_i^{-2} + a_e^2\sigma_e^{-2}\right)\left\|\begin{matrix}\rho_e \\ \rho_i\end{matrix}\right\| = \left(M_e a_e^2\sigma_e^{-1} - M_i a_i^2\sigma_i^{-1}\right)\left\|\begin{matrix}\sigma_i \\ -\sigma_e\end{matrix}\right\| + \\ &\cosh(\Delta z)\frac{\Delta H}{\sinh(\Delta H)}\left(M_e\sigma_e + M_i\sigma_i\right)\left\|\begin{matrix}a_i^2\sigma_i^{-1} \\ a_e^2\sigma_e^{-1}\end{matrix}\right\| \end{aligned} \tag{2.17}$$

or else

$$\sigma_i\sigma_e\left(a_i^2\sigma_i^{-2}+a_e^2\sigma_e^{-2}\right)\left\|\begin{matrix}\rho_e\\ \rho_i\end{matrix}\right\| = \left(\bar{\rho}_e a_e^2\sigma_e^{-1}-\bar{\rho}_i a_i^2\sigma_i^{-1}\right)\left\|\begin{matrix}\sigma_i\\ -\sigma_e\end{matrix}\right\| + \cosh(\Delta z)\frac{\Delta H}{\sinh(\Delta H)}\left(\bar{\rho}_e\sigma_e+\bar{\rho}_i\sigma_i\right)\left\|\begin{matrix}a_i^2\sigma_i^{-1}\\ a_e^2\sigma_e^{-1}\end{matrix}\right\| \tag{2.18}$$

where we use the following notation

$$\Delta^2 \equiv \frac{4\pi\sigma_i^2}{a_i^2}+\frac{4\pi\sigma_e^2}{a_e^2},\quad \bar{\rho}_e \equiv \frac{\rho_e}{2H},\ \ \bar{\rho}_i \equiv \frac{\rho_i}{2H} \tag{2.19}$$

The solution Eq. 2.16 implies

$$\sigma_e\rho_e+\sigma_i\rho_i = \frac{Q}{2H}\frac{\Delta H}{\sinh(\Delta H)}\cosh(\Delta z) \tag{2.20}$$

where Q is the full charge of the plasma

$$Q \equiv M_e\sigma_e + M_i\sigma_i \tag{2.21}$$

Equation 2.16 implies the following relationships for the spatial distributions the volumetric charge densities

$$\begin{aligned}
&\left\|\begin{matrix}\sigma_e\rho_e\\ \sigma_i\rho_i\end{matrix}\right\| = \frac{1}{2H}\frac{M_e a_e^2\sigma_e^{-1}-M_i a_i^2\sigma_i^{-1}}{\sigma_i\sigma_e\left(a_i^2\sigma_i^{-2}+a_e^2\sigma_e^{-2}\right)}\left\|\begin{matrix}\sigma_i\sigma_e\\ -\sigma_i\sigma_e\end{matrix}\right\| + \\
&\frac{1}{2H}\frac{\Delta H}{\sinh(\Delta H)}\frac{M_e\sigma_e+M_i\sigma_i}{\sigma_i\sigma_e\left(a_i^2\sigma_i^{-2}+a_e^2\sigma_e^{-2}\right)}\left\|\begin{matrix}a_i^2\sigma_e\sigma_i^{-1}\\ a_e^2\sigma_i\sigma_e^{-1}\end{matrix}\right\|\cosh(\Delta z) = \\
&\frac{\bar{\rho}_e a_e^2\sigma_e^{-1}-\bar{\rho}_i a_i^2\sigma_i^{-1}}{a_e^2\sigma_e^{-2}+a_i^2\sigma_i^{-2}}\left\|\begin{matrix}1\\ -1\end{matrix}\right\| + \\
&\frac{\Delta H}{\sinh(\Delta H)}\frac{\bar{\rho}_e\sigma_e+\bar{\rho}_i\sigma_i}{a_e^2\sigma_e^{-2}+a_i^2\sigma_i^{-2}}\left\|\begin{matrix}a_i^2\sigma_i^{-2}\\ a_e^2\sigma_e^{-2}\end{matrix}\right\|\cosh(\Delta z) = \\
&\frac{\bar{\rho}_e a_e^2\sigma_e^{-1}-\bar{\rho}_i a_i^2\sigma_i^{-1}}{a_e^2\sigma_e^{-2}+a_i^2\sigma_i^{-2}}\left\|\begin{matrix}1\\ -1\end{matrix}\right\| + \\
&\frac{\Delta H}{\sinh(\Delta H)}\frac{\bar{\rho}_e\sigma_e+\bar{\rho}_i\sigma_i}{a_e^2\sigma_e^{-2}+a_i^2\sigma_i^{-2}}\left\|\begin{matrix}a_i^2\sigma_i^{-2}\\ a_e^2\sigma_e^{-2}\end{matrix}\right\|\cosh(\Delta z)
\end{aligned} \tag{2.22}$$

where $\bar{\rho}_e \equiv M_e / H$ and $\bar{\rho}_i \equiv M_i / H$ are the mean densities of the charges.

It is sometimes convenient to transform Eq. 2.22 as follows:

$$\begin{Vmatrix} \sigma_e \rho_e \\ \sigma_i \rho_i \end{Vmatrix} = \frac{\bar{\rho}_e a_e^2 \sigma_e^{-1} - \bar{\rho}_i a_i^2 \sigma_i^{-1}}{a_e^2 \sigma_e^{-2} + a_i^2 \sigma_i^{-2}} \begin{Vmatrix} 1 \\ -1 \end{Vmatrix} +$$

$$\frac{\Delta H}{\sinh(\Delta H)} \frac{\bar{\rho}_e \sigma_e + \bar{\rho}_i \sigma_i}{a_e^2 \sigma_e^{-2} + a_i^2 \sigma_i^{-2}} \begin{Vmatrix} a_i^2 \sigma_i^{-2} \\ a_e^2 \sigma_e^{-2} \end{Vmatrix} \cosh(\Delta z) =$$

$$\frac{\bar{\rho}_e \sigma_e}{a_e^2 \sigma_e^{-2} + a_i^2 \sigma_i^{-2}} \left(a_e^2 \sigma_e^{-2} \begin{Vmatrix} 1 \\ -1 \end{Vmatrix} + \frac{\Delta H}{\sinh(\Delta H)} \begin{Vmatrix} a_i^2 \sigma_i^{-2} \\ a_e^2 \sigma_e^{-2} \end{Vmatrix} \cosh(\Delta z) \right) + \tag{2.23}$$

$$\frac{\bar{\rho}_i \sigma_i}{a_e^2 \sigma_e^{-2} + a_i^2 \sigma_i^{-2}} \left(a_i^2 \sigma_i^{-2} \begin{Vmatrix} -1 \\ 1 \end{Vmatrix} + \frac{\Delta H}{\sinh(\Delta H)} \begin{Vmatrix} a_i^2 \sigma_i^{-2} \\ a_e^2 \sigma_e^{-2} \end{Vmatrix} \cosh(\Delta z) \right)$$

4. The Asymptotic Case of "Neutral Ionic" Liquid

In the case of $\sigma_i = 0$, the solutions to Eqs. 2.16 and 2.17 read

$$\begin{Vmatrix} \rho_e \\ \rho_i \end{Vmatrix} = \bar{\rho}_i \begin{Vmatrix} 0 \\ 1 \end{Vmatrix} + \bar{\rho}_e \begin{Vmatrix} 1 \\ 0 \end{Vmatrix} \frac{\tilde{\Delta} H}{\sinh(\tilde{\Delta} H)} \cosh(\tilde{\Delta} z) = \begin{Vmatrix} \bar{\rho}_e \dfrac{\tilde{\Delta} H}{\sinh(\tilde{\Delta} H)} \cosh(\tilde{\Delta} z) \\ \bar{\rho}_i \end{Vmatrix} \tag{3.1}$$

where

$$\tilde{\Delta}^2 \equiv \frac{4\pi \sigma_e^2}{a_e^2} \tag{3.2}$$

5. The Asymptotics of Macroscopically Neutral System

For verification purposes it is instructive to consider the case of the overall neutral plasma. By the natural physical definition, in this case, the net charge of plasma Q vanishes:

$$M_e \sigma_e + M_i \sigma_i = 0 \tag{4.1}$$

or

$$\Upsilon \equiv -\frac{M_i \sigma_i}{M_e \sigma_e} = 1 \tag{4.2}$$

Inserting Eq. 4.1 in Eq. 2.16, we arrive at the relation

$$\begin{Vmatrix} \rho_e \\ \rho_i \end{Vmatrix} = \frac{1}{2H} \begin{Vmatrix} M_e \\ M_i \end{Vmatrix} \tag{4.3}$$

as implied by the following chain:

$$\left\|\begin{matrix}\rho_e\\ \rho_i\end{matrix}\right\| = \frac{1}{2H}\frac{M_e a_e^2\sigma_e^{-1} - M_i a_i^2\sigma_i^{-1}}{\sigma_i\sigma_e\left(a_i^2\sigma_i^{-2}+a_e^2\sigma_e^{-2}\right)}\left\|\begin{matrix}\sigma_i\\ -\sigma_e\end{matrix}\right\| = \frac{1}{2H_e}\frac{M_e\sigma a_e^2\sigma_e^{-2} - M_i\sigma_i a_i^2\sigma_i^{-2}}{\sigma_i\sigma_e\left(a_i^2\sigma_i^{-2}+a_e^2\sigma_e^{-2}\right)}\left\|\begin{matrix}\sigma_i\\ -\sigma_e\end{matrix}\right\| =$$

$$\frac{1}{2H_e}M_e\sigma_e\frac{a_e^2\sigma_e^{-2}+a_i^2\sigma_i^{-2}}{\sigma_i\sigma_e\left(a_i^2\sigma_i^{-2}+a_e^2\sigma_e^{-2}\right)}\left\|\begin{matrix}\sigma_i\\ -\sigma_e\end{matrix}\right\| = \frac{1}{2H}M_e\sigma_e\frac{1}{\sigma_i\sigma_e}\left\|\begin{matrix}\sigma_i\\ -\sigma_e\end{matrix}\right\| =$$

$$\frac{1}{2H}\left\|\begin{matrix}\sigma_i M_e\sigma_e\dfrac{1}{\sigma_i\sigma_e}\\ -\sigma_e M_e\sigma_e\dfrac{1}{\sigma_i\sigma_e}\end{matrix}\right\| = \frac{1}{2H}\left\|\begin{matrix}M_e\\ -\sigma_e M_e\dfrac{1}{\sigma_i}\end{matrix}\right\| = \frac{1}{2H}\left\|\begin{matrix}M_e\\ M_i\end{matrix}\right\|$$

The solution in Eq. 4.3 is in full agreement with the intuition "in the absence of external electrostatic fields the components are uniformly distributed inside the vessel."

6. The Case of Quasi-neutral Plasma

Consider the quasi-neutral case, for example, the case when

$$\Upsilon = 1 - q, \quad |q| << 1 \tag{5.1}$$

We then get

$$M_e\sigma_e + M_i\sigma_i = (1-\Upsilon)M_e\sigma_e = qM_e\sigma_e \tag{5.2}$$

In view of Eq. 5.2, we get

$$M_i\sigma_i = -(1-q)M_e\sigma_e \tag{5.3}$$

Using Eq. 5.3, we can rewrite Eq. 2.16 as follows

$$\begin{gathered}\left\|\begin{matrix}\rho_e\\ \rho_i\end{matrix}\right\| = \frac{M_e}{2H}\left\|\begin{matrix}1\\ -\sigma_e\sigma_i^{-1}\end{matrix}\right\| + \\ q\frac{M_e}{2H}\frac{1}{a_i^2\sigma_i^{-2}+a_e^2\sigma_e^{-2}}\left[\left\|\begin{matrix}-a_i^2\sigma_i^{-2}\\ a_i^2\sigma_i^{-3}\sigma_e\end{matrix}\right\| + \frac{\Delta H}{\sinh(\Delta H)}\left\|\begin{matrix}a_i^2\sigma_i^{-2}\\ a_e^2\sigma_i^{-1}\sigma_e^{-1}\end{matrix}\right\|\cosh(\Delta z)\right]\end{gathered} \tag{5.4}$$

Some regrouping in Eq. 5.4 gives us

$$\begin{gathered}\left\|\begin{matrix}\sigma_e\rho_e\\ \sigma_i\rho_i\end{matrix}\right\| = \frac{M_e\sigma_e}{2H}\left\|\begin{matrix}1\\ -1\end{matrix}\right\| + \\ q\frac{M_e\sigma_e}{2H}\frac{a_i^2\sigma_i^{-2}}{a_i^2\sigma_i^{-2}+a_e^2\sigma_e^{-2}}\left[-\left\|\begin{matrix}1\\ -1\end{matrix}\right\| + \frac{\Delta H}{\sinh(\Delta H)}\left\|\begin{matrix}1\\ \dfrac{a_e^2\sigma_e^{-2}}{a_i^2\sigma_i^{-2}}\end{matrix}\right\|\cosh(\Delta z)\right]\end{gathered} \tag{5.5}$$

as implied by the following chain:

$$\left\|\begin{matrix}\rho_e\\ \rho_i\end{matrix}\right\| = \frac{1}{2H} M_e\sigma_e \frac{a_e^2\sigma_e^{-2} + a_i^2\sigma_i^{-2} - qa_i^2\sigma_i^{-2}}{\sigma_i\sigma_e\left(a_i^2\sigma_i^{-2} + a_e^2\sigma_e^{-2}\right)}\left\|\begin{matrix}\sigma_i\\ -\sigma_e\end{matrix}\right\| +$$

$$\frac{1}{2H}\frac{\Delta H}{\sinh(\Delta H)}\frac{qM_e\sigma_e}{\sigma_i\sigma_e\left(a_i^2\sigma_i^{-2} + a_e^2\sigma_e^{-2}\right)}\left\|\begin{matrix}\frac{a_i^2}{\sigma_i}\\ \frac{a_e^2}{\sigma_e}\end{matrix}\right\|\cosh(\Delta z) = \rightarrow$$

$$\frac{M_e}{2H}\left\|\begin{matrix}1\\ -\frac{\sigma_e}{\sigma_i}\end{matrix}\right\| - q\frac{M_e}{2H}\frac{a_i^2\sigma_i^{-3}}{a_i^2\sigma_i^{-2} + a_e^2\sigma_e^{-2}}\left\|\begin{matrix}\sigma_i\\ -\sigma_e\end{matrix}\right\| +$$

$$q\frac{M_e}{2H}\frac{\Delta H}{\sinh(\Delta H)}\frac{\sigma_e}{\sigma_i\sigma_e\left(a_i^2\sigma_i^{-2} + a_e^2\sigma_e^{-2}\right)}\left\|\begin{matrix}a_i^2\sigma_i^{-1}\\ a_e^2\sigma_e^{-1}\end{matrix}\right\|\cosh(\Delta z) =$$

$$\frac{M_e}{2H}\left\|\begin{matrix}1\\ -\frac{\sigma_e}{\sigma_i}\end{matrix}\right\| - q\frac{M_e}{2H}\frac{1}{a_i^2\sigma_i^{-2} + a_e^2\sigma_e^{-2}}\left[\left\|\begin{matrix}a_i^2\sigma_i^{-2}\\ -a_i^2\sigma_i^{-3}\sigma_e\end{matrix}\right\| - \frac{\Delta H}{\sinh(\Delta H)}\left\|\begin{matrix}a_i^2\sigma_i^{-2}\\ a_e^2\sigma_i^{-1}\sigma_e^{-1}\end{matrix}\right\|\cosh(\Delta z)\right]$$

At $a_i / a_e >> 1$, the Eq. 5.4 reads

$$\frac{2H}{M_e}\left\|\begin{matrix}\rho_e\\ \rho_i\end{matrix}\right\| = \left\|\begin{matrix}1\\ -\sigma_e\sigma_i^{-1}\end{matrix}\right\|(1-q) + q\frac{\Delta H}{\sinh(\Delta H)}\left\|\begin{matrix}1\\ 0\end{matrix}\right\|\cosh(\Delta z) \tag{5.6}$$

Using Eq. 5.6, we get

$$\frac{2H}{M_e}\left\|\begin{matrix}\sigma_e\rho_e\\ \sigma_i\rho_i\end{matrix}\right\| = \left\|\begin{matrix}\sigma_e\\ -\sigma_e\end{matrix}\right\|(1-q) + q\frac{\Delta H}{\sinh(\Delta H)}\left\|\begin{matrix}\sigma_e\\ 0\end{matrix}\right\|\cosh(\Delta z) \tag{5.7}$$

Using Eq. 5.7, we get in turn

$$\sigma_e\rho_e + \sigma_i\rho_i = \frac{qM_e\sigma_e}{2H}\frac{\Delta_\infty H}{\sinh(\Delta_\infty H)}\cosh(\Delta_\infty z) \tag{5.8}$$

7. The Asymptotics of Incompressible Ionic Liquid and Appearance of the Extinction Points of the Density

In this case the solution in Eq. 2.16 reads

$$\begin{Vmatrix}\rho_e\\ \rho_i\end{Vmatrix} = \frac{M_i}{2H}\begin{Vmatrix}-\dfrac{\sigma_i}{\sigma_e}\\ 1\end{Vmatrix} + \frac{M_e}{2H}\frac{\Delta_\infty H}{\sinh(\Delta_\infty H)}\begin{Vmatrix}1+\dfrac{M_i}{M_e}\dfrac{\sigma_i}{\sigma_e}\\ 0\end{Vmatrix}\cosh(\Delta z) \tag{6.1}$$

and it implies

$$\begin{Vmatrix}\sigma_e\rho_e\\ \sigma_i\rho_i\end{Vmatrix} = \frac{M_i}{2H}\begin{Vmatrix}-\sigma_i\\ \sigma_i\end{Vmatrix} + \frac{M_e}{2H}\frac{\Delta_\infty H}{\sinh(\Delta_\infty H)}\begin{Vmatrix}\sigma_e + \dfrac{M_i}{M_e}\sigma_i\\ 0\end{Vmatrix}\cosh(\Delta z) = \begin{Vmatrix}-\sigma_i\bar{\rho}_i\\ \sigma_i\bar{\rho}_i\end{Vmatrix} + \frac{\Delta_\infty H}{\sinh(\Delta_\infty H)}\begin{Vmatrix}\sigma_e\bar{\rho}_e + \sigma_-\bar{\rho}_-\\ 0\end{Vmatrix}\cosh(\Delta z) \tag{6.2}$$

as well as

$$\sigma_e\rho_e + \sigma_i\rho_i = \frac{\Delta_\infty H}{\sinh(\Delta_\infty H)}\frac{M_e\sigma_e + M_i\sigma_i}{2H}\cosh(\Delta z) \tag{6.3}$$

Using Eq. 6.1, we then get

$$2H\rho_e(z) = -M_i\frac{\sigma_i}{\sigma_e} + \left(M_e + M_i\frac{\sigma_i}{\sigma_e}\right)\frac{\Delta_\infty H}{\sinh(\Delta_\infty H)}\cosh(\Delta z) = M_e\frac{\Delta_\infty H}{\sinh(\Delta_\infty H)}\cosh(\Delta z) - M_i\frac{\sigma_i}{\sigma_e}\left(1 - \frac{\Delta_\infty H}{\sinh(\Delta_\infty H)}\cosh(\Delta z)\right) \tag{6.4}$$

or else

$$2H\sigma_e\rho_e(z) = -M_i\sigma_i + (M_i\sigma_i + M_e\sigma_e)\frac{\Delta_\infty H}{\sinh(\Delta_\infty H)}\cosh(\Delta z) \tag{6.5}$$

We see that the local electric charge disappears at the point $z = Z_{ext}$ such that

$$\Delta Z_{ext} = \cosh^{-1}\left(\frac{M_i\sigma_i}{M_i\sigma_i + M_e\sigma_e}\frac{\sinh(\Delta_\infty H)}{\Delta_\infty H}\right) \tag{6.6}$$

In a more general case, when both components are compressible, we must use the relationships in Eq. 2.21.

Then, we arrive at the following analogies of Eq. 6.6:

$$\begin{aligned}\Delta Z_{ext}^{e} &= \cosh^{-1}\left[\frac{M_i a_i^2 \sigma_i^{-1} - M_e a_e^2 \sigma_e^{-1}}{\left(M_e\sigma_e + M_i\sigma_i\right) a_i^2 \sigma_i^{-2}} \frac{\sinh\left(\Delta H\right)}{\Delta H}\right] \\ \Delta Z_{ext}^{i} &= \cosh^{-1}\left[\frac{M_e a_e^2 \sigma_e^{-1} - M_i a_i^2 \sigma_i^{-1}}{\left(M_e\sigma_e + M_i\sigma_i\right) a_e^2 \sigma_e^{-2}} \frac{\sinh\left(\Delta H\right)}{\Delta H}\right]\end{aligned} \tag{6.7}$$

In terms of $Q \equiv M_e\sigma_e + M_i\sigma_i$, the pair of equations in Eq. 6.7 can be rewritten as

$$\begin{aligned}\Delta Z_{ext}^{e} &= \cosh^{-1}\left[\frac{M_i\sigma_i\left(a_i^2\sigma_i^{-2} + a_e^2\sigma_e^{-2}\right) - Q a_e^2\sigma_e^{-2}}{Q a_i^2 \sigma_i^{-2}} \frac{\sinh\left(\Delta H\right)}{\Delta H}\right] = \\ &\cosh^{-1}\left[\frac{Q a_i^2\sigma_i^{-2} - M_e\sigma_e\left(a_i^2\sigma_i^{-2} + a_e^2\sigma_e^{-2}\right)}{Q a_i^2 \sigma_i^{-2}} \frac{\sinh\left(\Delta H\right)}{\Delta H}\right] \\ \Delta Z_{ext}^{i} &= \cosh^{-1}\left[\frac{-M_i\sigma_i\left(a_e^2\sigma_e^{-2} + a_i^2\sigma_i^{-2}\right) + Q a_e^2\sigma_e^{-2}}{Q a_e^2 \sigma_e^{-2}} \frac{\sinh\left(\Delta H\right)}{\Delta H}\right] = \\ &\cosh^{-1}\left[\frac{M_e\sigma_e\left(a_e^2\sigma_e^{-2} + a_i^2\sigma_i^{-2}\right) - Q a_i^2\sigma_i^{-2}}{Q a_e^2 \sigma_e^{-2}} \frac{\sinh\left(\Delta H\right)}{\Delta H}\right]\end{aligned} \tag{6.8}$$

At small Q, the Eq. 6.8 can be approximated with the following ones:

$$\begin{aligned}\Delta Z_{ext}^{e} &= \cosh^{-1}\left[\frac{M_i\sigma_i}{Q} \frac{a_i^2\sigma_i^{-2} + a_e^2\sigma_e^{-2}}{a_i^2\sigma_i^{-2}} \frac{\sinh\left(\Delta H\right)}{\Delta H}\right] = \\ &\cosh^{-1}\left[-\frac{M_e\sigma_e}{Q} \frac{a_i^2\sigma_i^{-2} + a_e^2\sigma_e^{-2}}{a_i^2\sigma_i^{-2}} \frac{\sinh\left(\Delta H\right)}{\Delta H}\right] \\ \Delta Z_{ext}^{i} &= \cosh^{-1}\left[-\frac{M_i\sigma_i}{Q} \frac{a_e^2\sigma_e^{-2} + a_i^2\sigma_i^{-2}}{a_e^2\sigma_e^{-2}} \frac{\sinh\left(\Delta H\right)}{\Delta H}\right] = \\ &\cosh^{-1}\left[\frac{M_e\sigma_e}{Q} \frac{a_e^2\sigma_e^{-2} + a_i^2\sigma_i^{-2}}{a_e^2\sigma_e^{-2}} \frac{\sinh\left(\Delta H\right)}{\Delta H}\right]\end{aligned} \tag{6.9}$$

The relationships in Eq. 6.9 imply that in the asymptotic case $Q \to 0$, the extinction point of the "*i*" component can appear if $\sigma_i Q > 0$; the extinction point of the "*e*" component can appear if $\sigma_e Q > 0$. The clause "can appear" means that corresponding values of the inverse hyperbolic cosine "$\cosh^{-1}$" are real. However, to be physically meaningful the corresponding values of Z_{ext}^{e} or Z_{ext}^{i} should be less than *H,* but in fact they tend to infinity at $Q \to 0$. This fact implies that for sufficiently small Q, the equilibrium configurations have no extinction points.

8. Calculation of the Pressure

If we consider the isothermal processes at $T = T_0$, where T_0 is the base temperature, we arrive at the relationships

$$\begin{aligned} \psi_e(\rho_e) - \psi_{e0} &= a_e^2 \frac{1}{2\rho_e}(\rho_e - \rho_{e0})^2 + p_{e0}\left(\frac{1}{\rho_{e0}} - \frac{1}{\rho_e}\right) \\ \psi_i(\rho_i) - \psi_{i0} &= a_i^2 \frac{1}{2\rho_i}(\rho_i - \rho_{i0})^2 + p_{i0}\left(\frac{1}{\rho_{i0}} - \frac{1}{\rho_i}\right) \end{aligned} \tag{7.1}$$

and then

$$\begin{aligned} p_e(\rho_e) &= a_e^2 \frac{1}{2}\left(\rho_e^2 - \rho_{e0}^2\right) + p_{e0}, \ (\rho\psi_e)_{\rho\rho} = a_e^2 = const \\ p_i(\rho_i) &= a_i^2 \frac{1}{2}\left(\rho_i^2 - \rho_{i0}^2\right) + p_{i0}, \ (\rho\psi_i)_{\rho\rho} = a_i^2 = const \end{aligned} \tag{7.2}$$

Using Eq. 2.16, we get for the case of the electrically neutral ionic substance

$$\begin{aligned} &\left\| \begin{matrix} \rho_e \\ \rho_i \end{matrix} \right\| = \frac{1}{2H}\left\| \begin{matrix} 0 \\ M_i \end{matrix} \right\| + \frac{1}{2H}\frac{\Delta H}{\sinh(\Delta H)}\left\| \begin{matrix} M_e \\ 0 \end{matrix} \right\| \cosh(\Delta z) = \\ &\frac{1}{2H}\left\| \begin{matrix} M_e \dfrac{\Delta H}{\sinh(\Delta H)}\cosh(\Delta z) \\ M_i \end{matrix} \right\| = \left\| \begin{matrix} \dfrac{M_e}{2H}\dfrac{\Delta H}{\sinh(\Delta H)}\cosh(\Delta z) \\ \dfrac{M_i}{2H} \end{matrix} \right\| \end{aligned} \tag{7.3}$$

Using Eq. 7.3, we get

$$\begin{aligned} \rho_e^2(z) &= \left(\frac{M_i}{2H}\frac{\sigma_i}{\sigma_e}\right)^2 - \frac{M_i}{2H}\frac{\sigma_i}{\sigma_e}\frac{M_e\sigma_e + M_i\sigma_i}{H\sigma_e}\frac{\Delta_\infty H\cosh(\Delta_\infty z)}{\sinh(\Delta_\infty H)} + \\ &\left(\frac{1}{2}\frac{M_e\sigma_e + M_i\sigma_i}{H\sigma_e}\frac{\Delta_\infty H\cosh(\Delta_\infty z)}{\sinh(\Delta_\infty H)}\right)^2, \\ \rho_i^2(z) &= \left(\frac{M_i}{2H}\right)^2 + \frac{M_i}{2H}\frac{\sigma_i}{\sigma_e}\frac{M_e\sigma_e + M_i\sigma_i}{H\sigma_e}\frac{a_e^2}{a_i^2}\frac{\sigma_i}{\sigma_e}\frac{\Delta_\infty H\cosh(\Delta_\infty z)}{\sinh(\Delta_\infty H)} + \\ &\left(\frac{1}{2}\frac{M_e\sigma_e + M_i\sigma_i}{H\sigma_e}\frac{a_e^2}{a_i^2}\frac{\sigma_i}{\sigma_e}\frac{\Delta_\infty H\cosh(\Delta_\infty z)}{\sinh(\Delta_\infty H)}\right)^2 \end{aligned} \tag{7.4}$$

Using Eq. 7.4, we get

$$p_i(\rho_i) - p_{i0} = a_i^2 \frac{1}{2}\left(\rho_i^2 - \rho_{i0}^2\right) \approx$$
$$\frac{1}{2} a_i^2 \left[\left(\frac{M_i}{2H}\right)^2 - \rho_{i0}^2\right] + a_e^2 \frac{M_i}{4H}\left(\frac{\sigma_i}{\sigma_e}\right)^2 \frac{M_e\sigma_e + M_i\sigma_i}{H\sigma_e} \frac{\Delta_\infty H \cosh\left(\Delta_\infty z\right)}{\sinh\left(\Delta_\infty H\right)} \tag{7.5}$$

9. Conclusion

We found an exact 1-D equilibrium solution for a two-component gaseous plasma situated between two parallel planes. We supposed that each of the plasma components has the EOS postulated in Grinfeld.[6] The equilibrium densities are described by the Eq. 2.16. This solution is physically meaningful if the corresponding densities are positive everywhere. This situation takes place if the net charge of plasma is sufficiently small. Otherwise, the solution in Eq. 2.16 should be corrected by calculating the points of extinction in the spirit of the report Grinfeld.[6]

10. References

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List of Symbols, Abbreviations, and Acronyms

1-D	1-dimensional
ARL	Army Research Laboratory
DEVCOM	US Army Combat Capabilities Development Command
EOS	equation of state

1 DEFENSE TECHNICAL
(PDF) INFORMATION CTR
DTIC OCA

DEVCOM ARL
(PDF) FCDD RLD DCI
TECH LIB

3 SANDIA NATL LAB
(PDF) J NIEDERHAUS
A ROBINSON
C SIEFERT

27 DEVCOM ARL
(PDF) FCDD RLD FC
T SANO
FCDD RLD FR
M TSCHOPP
FCDD RLW
S SCHOENFELD
FCDD RLW B
C HOPPEL
R BECKER
A TONGE
FCDD RLW MB
B LOVE
G GAZONAS
B POWERS
FCDD RLW T
M FERMEN-COKER
FCDD RLW TA
S BILYK
W UHLIG
P BERNING
M COPPINGER
C ADAMS
M GREENFIELD
C WILLIAMS
FCDD RLW TB
T WEERASOORIYA
S SATAPATHY
D CASEM
J CLAYTON
FCDD RLW TD
R DONEY
C RANDOW
FCDD RLW TE
J LLOYD
R LEAVY
G VUNNI
FCDD RLW TF
J RUNYEON