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Noise Characterization of Quantum Teleportation with Imperfectly Prepared States

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Entanglement swapping and distribution will be necessary components of a fully functioning quantum ecosystem and will form the backbone to a quantum network. The fundamental process that underpins these protocols is teleportation. When first conceived, these protocols assumed the deterministic generation of maximally entangled states; however, there is no known experimental method for generating such states on demand. Here we completely characterize how the noisy generation of entangled states a˙ects the fidelity of two-qubit teleportation, which is the first characterization of its kind. Additionally, we discuss extendability of this work to *n*-qubit teleportation.

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CONTENTS

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EXECUTIVE SUMMARY

The purpose of this document is to report the results and their significance for the NISE project "Noise characterization of quantum teleportation with imperfectly prepared states".

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NOISE CHARACTERIZATION OF QUANTUM TELEPORTATION WITH IMPERFECTLY PREPARED STATES

1. BACKGROUND

There are many ways to encode information in a quantum state; for example, one could represent a classical bit with the polarization of photon or even the spin of an electron. Furthermore, one can use a quantum state to represent a piece of quantum, or non-classical, information like entanglement. Each of these choices induces different noise dynamics. To completely characterize this noise, we build a model of the evolution of all possible states called a *quantum channel*. In general, quantum states are fragile and easily couple to their environment, causing decoherence. Due to this fragility, quantum states are difficult to prepare and measure. Similarly, imperfect preparation and measurement of states can inject noise into the system as well. In this chapter, we will introduce the mathematical and physical background necessary to describe the noisy evolution of a system induced by teleporting through an imperfectly prepared entangled state.

1.1 Quantum Information

1.1.1 Quantum States and Evolution

In order to discuss the evolution of a quantum system under the teleportation protocol, a complete mathematical description of environments and unknown states is needed. In this work, we consider only integral numbers of qubits and may therefore restrict ourselves to finite dimensional systems throughout the entirety of this paper. Let $\mathcal{H}^{2^n}$ be a Hilbert space of dimension 2^n; i.e., the state space of an n-qubit system.

Definition 1 : A quantum state is described by a positive semi-definite, trace one, linear operator $\rho : \mathcal{H}^{2^n} \to \mathcal{H}^{2^n}$ called the *density operator*.

If ρ is a *pure state*, there exists a unit vector $|\psi\rangle \in \mathcal{H}^{2^n}$ such that $\rho = |\psi\rangle\langle\psi|$. Otherwise, ρ is a statistical ensemble of pure states and is called a *mixed state*. From the spectral theorem, there are probabilities p_i and unit vectors $|\psi_i\rangle \in \mathcal{H}^{2^n}$ such that $\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|$ partioning the sets of mixed states and pure states. When considering the pure state $\rho = |\psi\rangle\langle\psi|$, the state vector $|\psi\rangle$ completely characterizes the quantum system and the two will be used interchangeably.

The set of entangled states are particularly useful subset of pure states. For example, entanglement is necessary for many complex quantum protocols spanning sensing, computing, and networking; however, generating high-quality entangled states on demand is still a challenge area in experimental quantum information science.

Definition 2 : For a pure state $\rho = |\psi\rangle\langle\psi|$, if there exist $|\eta\rangle \in H^{2^n}$ and $|\nu\rangle \in H^{2^m}$ such that $\rho = |\eta\rangle\langle\eta| \otimes |\nu\rangle\langle\nu|$, then ρ is said to be *separable*. If no such decomposition into tensor products of smaller systems exists, then ρ is said to be an *entangled state*.

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Expanding a density operator in terms of measurement bases, allows for an explicit calculation of the density operator in terms of expectation values. For the single qubit, the popular choice is the basis of Pauli spin operators:

$$\sigma_1 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1}$$

For an n-qubit system, we use the n-qubit spin operators [1]:

Definition 3: Define n-qubit spin operators to be

$$\Lambda^n = \left\{ \sigma_{i_1} \otimes \cdots \otimes \sigma_{i_n} | i_i \in \{1, 2, 3, 4\} \right\}; \tag{2}$$

give Λ^n the dictionary order and call the i^{th}element λ_i.

In this notation, λ_1 is an n-fold tensor product of the 2-dimensional Identity matrix σ_1, which we denote by I and its dimension will be clear from context. Representing the last $4^n - 1$ elements in vector form $\lambda = [\lambda_2 \ \lambda_3 \ \cdots \lambda_4^n]^t$, we have a convenient method to represent an n-qubit quantum state:

$$\rho = \frac{I + cr \cdot \lambda}{2^n}, \tag{3}$$

where $r \in \mathbb{R}^{4^n}$ is called the *Bloch* vector and c is a constant ensuring pure states have a Bloch vector with norm 1. For the single qubit, the collection of Bloch vectors forms the unit sphere, with the pure states on the surface. For larger systems, the set of Bloch vectors is a deformation retract of the unit sphere

Definition 4: An n-qubit quantum channel is a completely positive, trace-preserving, linear map $\Phi : \mathcal{H}^{2^n} \to \mathcal{H}^{2^n}$.

Many definitions of quantum channels [2] include *complete positivity* which requires the system and environment to be in a pure product state. Explicitly, an n-qubit quantum channel is completely positive if and only if $I_{2^n} \otimes \Phi$ is also a positive map [3]. However, in this paper we are considering state teleportation which necessitates initial entanglement between a state and its environment; i.e. the system and environment are not in a pure product state. Despite this fact, we will show that for n-qubit teleportation, the dynamics are in fact completely positive.

Similar to the Bloch representation of states, there also exists a Euclidean representation for quantum channels. Being described by a linear operator, each channel is completely determined by its action on a basis. That is, each quantum channel Φ induces a unique affine map $f(r) = Mr + b$, called the Bloch representation of the channel, on the Bloch vector in the following way:

$$\Phi(\rho) = \frac{I + f(r) \cdot \lambda}{2^n} \tag{4}$$

When the channel fixes the identity, it is called *unital* and its Bloch representation is linear.

We have introduced this Euclidean representation because it offers a concrete way to envision the dynamics, where all of the coefficients in the expansion arise as expectation values of a particular measurement scheme. Additionally, previous work has utilized this representation to develop simpler computational methods for analyzing the informatics of quantum systems. A complete discussion of the Euclidean representation of a quantum system would be outside the scope of this paper, but can be found in [1, 4].

1.1.2 Quantum Networks

As quantum technologies mature, we will need to remotely communicate and transfer information between them in order to leverage their full potential. For example, an n-qubit quantum state lies in a 2^n-dimensional Hilbert space, and consequently, there are $4^n - 1$ real parameters that are needed to fully characterize the state. Therefore, classically transmitting the amplitudes necessary to reconstruct a 40-qubit state would require a minimum of 9 trillion terabytes of classical information; even if we do not assume an arbitrary degree of precession needed for many of the most sophisticated protocols. However, if we were to transmit the information quantum mechanically, it would be theoretically possible to do so with a 40-photon state. This transfer can be done with a *quantum network*.

A quantum network consists of various transmission media and protocols that allows for the transfer of entanglement between multiple nodes. While quantum networks possess many of the same engineering challenges of classical networks, they will need to overcome obstacles that are uniquely quantum mechanical in nature. For example, due to the no cloning theorem [2], the classical technique of signal amplification is not possible with quantum mechanical systems. Furthermore, because of a variety of factors [3], including fiber loss, quantum signals can only propagate for approximately 100 km before the signal becomes too lossy to extract useful information [13]. The current solution to this limitation of transmitting quantum states is the use of a *quantum repeater* . In this scenario, a quantum repeater is used to distribute a state while also being placed between Alice and Bob; i.e. minimizing the effective distance and therefore signal degradation. The basic quantum repeater works as follows: two shared entangled states are established, one between Alice and a repeater, R, and one between Bob and R. At this point the repeater performs a Bell measurement on the two states at R resulting in the repeater's initial entangled state with Alice being teleported to Bob. At the end, this process establishes entanglement between Alice and Bob's states. Since the Bell measurement destroys the information shared between Alice and the repeater and between the repeater and Bob, this process does not violate the no cloning theorem.

Quantum teleportation allows for a signal to be repeated, potentially an indefinite number of times [4], it also allows for information to be transferred without passing through physical space. In this case, Alice and Bob would not make use of a repeater necessarily, but instead use a similar protocol for Alice to transfer "information qubits" to Bob. Specifically, Alice would interact an information qubit with her half of the entangled pair. Through the use of local operations and nominal classical communication, Alice can transfer her information qubits to Bob without the qubits traversing physical space, and therefore preventing any interception by an eavesdropper.

In the case of a single qubit, the entangled *Bell state*

$$|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \tag{5}$$

is used to transfer the information qubit without error. However, there is no known method capable of generating perfect Bell states on demand. The natural question then arises, "How does imperfect preparation of an entangled state effect the transfer of information?". It is worth noting that this noise is not a result of environmental factors, but instead stems from faulty implementations of the protocol itself. This question was previous answered for single qubit teleportation in [14], and recently solved for the case of two-qubits in the first year of this NISE project. In this report we will characterize these effects for the teleportation of an arbitrary number of qubits. We generalize the results previously obtained and show that these effects are described by a completely positive quantum channel.

1.2 Single Qubit Teleportation

In this section we discuss each step involed single qubit teleportation. We begin by introducing an integral operation for single qubit teleportation, the Bell measurement.

The Bell Measurement

The Bell measurement is a joint measurement of two qubits that determines which of the four Bell states a system is in. Mathematically speaking, a Bell measurement is a projection onto the Bell state basis, and therefore, an entangling operation. That is, even if the system is not in a Bell state initially, it is after a Bell measurement is performed. On a computational note, it is significant that the Bell states form an orthonormal basis for the state space of a two qubit system. Explicitly, given an arbitrary two qubit state, $|\Phi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$, there exist $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \in \mathbb{C}$ such that

$$|\Phi\rangle = \tilde{a}|\Phi_1\rangle + \tilde{b}|\Phi_2\rangle + \tilde{c}|\Phi_3\rangle + \tilde{d}|\Phi_4\rangle, \tag{6}$$

where $|\Phi_i\rangle$ is the i^{th} Bell state and $|\tilde{a}|^2 + |\tilde{b}|^2 + |\tilde{c}|^2 + |\tilde{d}|^2 = 1$. Moreover, basic calculations show that

$$|\Phi_i\rangle = \text{CNOT H}|i\rangle, \tag{7}$$

where $|i\rangle$ is the i^{th} element of the standard computational basis for the state space of a two qubit system, H is the hadamard gate applied to the first qubit, and CNOT is the CNOT gate where the control is the first qubit and the target is the second qubit. It then follows that Alice's measurement may instead be viewed as the application of CNOT H followed by a measurement in the computational basis. While this might appear pedantic, it greatly simplifies calculations.

Single Qubit Teleportation Protocol

Single qubit teleportation allows a sender (Alice) to transmit a qubit $|\psi\rangle$ to a receiver (Bob) via the following steps:

1. Before the protocol begins, Alice and Bob each posses one qubit from the maximally entangled two qubit state
$$|\Phi_1\rangle = \frac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle\big). \tag{8}$$
We denote the total three qubit system by $|\Psi\rangle = |\psi\rangle \otimes |\Phi\rangle$, where Alice has the first 2 qubits and Bob holds the last.

2. Alice performs a Bell measurement on the two qubits she posses. Recall, this operation is equivalent to the application of CNOT H followed by a measurement in the computational basis. Through this method, Alice obtains one of the four possible resutls: $m = |00\rangle$, $m = |01\rangle$, $m = |10\rangle$, or $m = |11\rangle$.

3. Since the Bell measurement is an entangling operation, the state of Bob's qubit will be determined by the outcome of Alice's measurement in the previous step. Explicitly, Bob's state is

$$\begin{cases} \alpha|0\rangle + \beta|1\rangle & \text{if } m = |00\rangle \\ \alpha|1\rangle + \beta|0\rangle & \text{if } m = |01\rangle \\ \alpha|0\rangle - \beta|1\rangle & \text{if } m = |10\rangle \\ \alpha|1\rangle - \beta|0\rangle & \text{if } m = |11\rangle \end{cases} \tag{9}$$

4. Alice finally sends the classical bit string $m = ij$ associated with her measurement to Bob. He then applies the operator $\sigma_4^i \sigma_2^j$ to the qubit he holds, thereby completely recovering the original qubit $|\psi\rangle$. Note, the σ_4 and σ_2 operators act on a qubit as follows: $\sigma_4|0\rangle = |0\rangle$, $\sigma_4|1\rangle = -|1\rangle$, $\sigma_2|0\rangle = |1\rangle$, and $\sigma_2|1\rangle = |0\rangle$.

Full recovery of the qubit $|\psi\rangle$ relies on the assumption that one can generate the maximally entangled state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ on demand. As there is currently no experimental scheme that can achieve this, Lanzagorta and Martin explored the errors that arise when teleporting through the imperfectly prepared state [14]:

$$|\Phi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle, \tag{10}$$

where $a, b, c, d \in \mathbb{C}$ and $|a|^2 + |b|^2 + |c|^2 + |d|^2 = 1$. In [14] it was shown that the set of diagonal qubit channels in the Bloch representation is exactly the set of 3 by 3 matrices that describe the errors associated with teleportation through imperfectly prepared states. To this end, we refer to these matrices as the *qubit teleportation channels*.

1.3 Arbitrary n-Qubit Teleportation

Single qubit teleportation is well-established [5], and the noise model well-studied [6], the details of this protocol are briefly outlined in the previous section. In this section, we generalize a similar outline for single qubit teleportation to describe the teleportation of any arbitrary number of qubits. Note, this has been done in [7] for two-qubit teleporation and outlined for n-qubit teleportation using certain maximally entangled states; here we explicitly extend this analysis to n-qubit systems, allowing for any state to be used for teleportation.

The process of choosing states suitable for teleportation is not as straight forward as picking an arbitrary entangled state. For instance, there exist multiple interpretations for the degree of entanglement in systems larger than two qubits; and such systems are necessary for the teleportation of multiple qubits. We will begin by establishing appropriate entangled states for teleportation via a natural extension to Bell states. We call these states the "*teleportation states*" and denote them by $|g_{ij}\rangle$.

The Teleportation States

Consider the $2n$-qubit state $|i\rangle|j\rangle$, where each ket describes the state of an n-qubit system. In order to construct the 4^n teleportation states, we apply a Hadamard gate to each qubit in $|i\rangle$, followed by n CNOT gates, where the control is the k^{th} qubit and the target is the the $n+k^{\text{th}}$, respectively, and $k \in \{1, 2, ..., n\}$. Explicitly, we generate the $2n$-qubit teleportation state $|g_{ij}\rangle$ in the following way:

$$|g_{ij}\rangle = \left(\prod_{k=1}^{n} \text{CNOT}_k \prod_{k=1}^{n} \text{H}_k\right)|i\rangle|j\rangle. \tag{11}$$

We may simplify Eq. (11) by noting that each Hadamard gate effects only a single distinct qubit; the same is true for CNOT gates. Therefore, we may combine the two indexed products in Eq. (11) into one, leaving

$$|g_{ij}\rangle = \left(\prod_{k=1}^{n} \text{CNOT}_k \text{H}_k\right)|i\rangle|j\rangle. \tag{12}$$

Note, for the case of $n = 1$, we simply obtain the Bell states. That is, Eq. (12) is a direct generalization of the process used to construct the well-known Bell states. As shown in [7], Eq. (12) provides a systemic approach to constructing the teleportation states for 2 qubits. With our teleportation states defined, we can now formalize the protocol for n-qubit teleportation.

The n-Qubit Teleportation Protocol

Let us take a moment to introduce some notation which is both necessary for defining the teleportation protocol and will prove computationally beneficial throughout this report. Consider an n-qubit state written as a tensor product of each individual qubit system. That is,

$$|i\rangle = \bigotimes_{k=1}^{n} |i_k\rangle, \tag{13}$$

where $|i_k\rangle$ represents the state of the k^{th} qubit and is the k^{th} digit in the binary representation of $|i\rangle$. Under ideal assumptions, i.e., the ability to precisely genterate one of the pure states $|g_{ij}\rangle$, the process for teleporting an n-qubit state $|\psi\rangle$ is as follows:

1. Alice and Bob begin by equally sharing the $2n$-qubit teleportation state $|g_{11}\rangle$. The total $3n$-qubit system by $|\Psi\rangle = |\psi\rangle \otimes |g_{11}\rangle$, where Alice has the first $2n$ qubits and Bob holds the remaining n.

2. Alice performs a measurement on the $2n$ qubits she posses. This operation is equivalent to the application of $\prod \text{H}_k\text{CNOT}_k$ followed by a measurement in the computational basis. She then obtains one of the 4^n possible computational basis elements.

3. Alice encodes the outcome of her measurement in the classical $2n$ bit string $m = i_1 i_2 ... i_n j_1 j_2 ... j_n$, and sends it to Bob via a classical channel. Bob applies the operator $\prod_{k=1}^{n}(\sigma_4^{i_k})_k(\sigma_2^{j_k})_k$ to his qubits, where $(\sigma_2)_k$ and $(\sigma_4)_k$ are the spin x and z gates acting on the k^{th}qubit, respectively. This operation results in Bob completely recovering the original qubit $|\psi\rangle$.

Similar to the single qubit case [6], this protocol hinges on the ability to reliably generate the state $|g_{11}\rangle$, which is not currently achievable in general. To this end, we will characterize the effects of teleporting an n-qubit state through the state

$$|\phi\rangle = \sum_{i,j=1}^{2^n} a_{ij}|i\rangle|j\rangle. \tag{14}$$

2. N-QUBIT TELEPORTATION

In this section we calculate in a step-by-step fashion, as described in the outline for n-qubit teleportation above, the relationship between the density operators for the states of the information qubits Alice wishes to send and that of the n qubits held by Bob at the end of the protocol.

2.0.1 Step 1: The Composite State

The overall state of the n information qubits can be written as

$$|\psi\rangle = \sum_{a=1}^{2^n} \psi_a|a\rangle = \sum_{a=1}^{2^n} \psi_a \bigotimes_{k=1}^{n} |a_k\rangle, \tag{15}$$

and the imperfectly prepared teleportation state is given by

$$|\phi\rangle = \sum_{b,c=1}^{2^n} \alpha_{bc} \left(\prod_{k=1}^{n} \mathrm{CNOT}_k H_k\right) |b\rangle|c\rangle, \tag{16}$$

where the α_{bc}'s are the coefficients for $|\phi\rangle$ when it is written in the basis of the teleportation states. Note, while we have written $|\phi\rangle$ in the computational basis, we have done so by first decomposing our state in the teleportation basis and then replaced each $|g_{bc}\rangle$ with the expression given in Eq. (12). We have chosen to do this as it greatly simplifies future calculations.

2.0.2 Step 2: The Measurement

The teleportation protocol again begins with Alice and Bob sharing half of this state and then making a measurement in the teleportation basis on $|\psi\rangle$ and her remaining half of $|\phi\rangle$. The effects of this measurement on the overall $3n$-qubit state are given by $\langle g_{ij}||\psi\rangle|\phi\rangle$. With this in mind, we define the following three-qubit operator:

$$\begin{aligned} \hat{M} &\equiv (H \otimes I \otimes I)\,(\mathrm{CNOT} \otimes I)\,(I \otimes \mathrm{CNOT})\,(I \otimes H \otimes I) \\ &= \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 & 0 & -1 & 0 & -1 \end{pmatrix}. \end{aligned} \tag{17}$$

In particular, this operator accounts for Alice's Bell measurement in the single qubit teleportation protocol when $\langle g_{ij}||\psi\rangle|\phi\rangle$ is expressed in the computational basis.

Similar to that seen in Eq. (12), we use a single subscript to denote which qubits this operator affects. In this case, $\hat{M}_k$ denotes the operator whose actions applies Hadamard gates to qubits k and $n+k$, and CNOT gates to qubits $n+k$ and $2n+k$ with control qubits k and $n+k$, respectively. Using this notation, $\langle g_{ij}||\psi\rangle|\phi\rangle$ then becomes

$$\langle g_{ij}||\psi\rangle|\phi\rangle = \langle j|\langle i|\prod_{k=1}^{n}\hat{M}_k\sum_{abc}\psi_a\alpha_{bc}|a\rangle|b\rangle|c\rangle, \tag{18}$$

where the sums run from 1 to 2^n. At this point, Eq. (18) describes the n-qubit state Bob is holding after Alice makes her measurement.

The action of $\hat{M}_k$ is determined by a 8^n square matrix, and in order to simplify these calculations, we apply a projector $|l\rangle\langle l|$ on the left of Eq. (18). Noting that $\hat{M}_k$ acts on only three-qubit at a time, the states $|a_k\rangle$, $|b_k\rangle$, and $|c_k\rangle$ are coupled to $\langle i_k|$, $\langle j_k|$, and $\langle l_k|$, and we can rewrite the equation above in terms of single-qubit states. That is,

$$\begin{aligned}\langle g_{ij}||\psi\rangle|\phi\rangle &= \sum_{l}|l\rangle\langle l|\langle g_{ij}|\psi\rangle|\phi\rangle \\ &= \sum_{labc}\psi_a\alpha_{bc}\bigotimes_{k=1}^{n}|l_k\rangle\langle l_k|\langle j_k|\langle i_k|\hat{M}|a_k\rangle|b_k\rangle|c_k\rangle,\end{aligned} \tag{19}$$

where $\hat{M}$ is now the the 8×8 matrix from Eq. (17), which acts on the 8-vector $|a_k\rangle|b_k\rangle|c_k\rangle$.

2.0.3 Step 3: Bob's Interaction

As described by the protocol in Section 3 C, Alice then sends the outcome of her measurement to Bob via a classical bit string, i.e., $m = i_1 i_2 ... i_n j_1 j_2 ... j_n$. With this information, Bob applies a series of σ_2 and σ_3 gates to each qubit in the following way: If j_k is 1, Bob applies σ_2 to qubit k. If j_k is 0, he does nothing. Next, if i_k is 1, Bob applies σ_4 to qubit k. If i_k is 0, he does nothing. This process is repeated for all n qubits in Bob's possession.

This sequence of operations can be written more compactly as

$$|\psi_B\rangle = \sum_{labc}\psi_a\alpha_{bc}\bigotimes_{k=1}^{n}\sigma_4^{i_k}\sigma_2^{j_k}|l_k\rangle\langle l_k|\langle j_k|\langle i_k|\hat{M}|a_k\rangle|b_k\rangle|c_k\rangle. \tag{20}$$

From Definition 1 and Eq. (20), we can then construct the density operator associated with the quantum state Bob holds at the end of the teleportation protocol,

$$\begin{aligned}\rho_{\mathrm{B}} = |\psi_B\rangle\langle\psi_B| = \sum_{\substack{abc \\ a'b'c'}}\alpha_{bc}\alpha^*_{b'c'}\psi_a\psi^*_{a'}\bigotimes_{k=1}^{n}\sum_{\substack{i_k,j_k, \\ l_k,l'_k=0}}^{1}\sigma_4^{i_k}\sigma_2^{j_k}|l_k\rangle\Big[\langle l_k|\langle j_k|\langle i_k|\hat{M}|a_k\rangle|b_k\rangle|c_k\rangle\Big] \\ \Big[\langle c'_k|\langle b'_k|\langle a'_k|\hat{M}^\dagger|i_k\rangle|j_k\rangle|l'_k\rangle\Big]\Big[\langle l'_k|\sigma_2^{j_k}\sigma_4^{i_k}\Big]\end{aligned} \tag{21}$$

Projecting ρ_{B} onto each n-qubit spin operator λ_i, we obtain the i^{th}component of the Bloch vector associated with our state.

2.1 Simplifying with the Bloch Representation

Beginning with the Eq. (21), we calculate the s^{th}component of the Bloch representation for ρ_{B}, denoted by r_s^B, by projecting it onto the n-qubit spin operator, λ_s; which is defined in Definition 3. Recall that the k^{th} term of λ_s is denoted as σ_{s_k}, where $\sigma_{s_k} \in \{I, \sigma_2, \sigma_3, \sigma_4\}$, defined in Eq. (1). We then have that the Bloch representation of the density matrix ρ_{B}, is $\sum_{s=i}^{4^n} r_s \lambda_s$, where

$$r_s^B = \frac{1}{2^n}\mathrm{Tr}(\rho_B \lambda_s) = \frac{1}{2^n} \sum_{\substack{abc \\ a'b'c'}} \alpha_{bc}\alpha^*_{b'c'}\psi_a\psi^*_{a'} \prod_{k=1}^{n} \sum_{i_k j_k l_k l'_k=0}^{1} \langle c'_k|\langle b'_k|\langle a'_k|\hat{M}^\dagger|i_k\rangle|j_k\rangle|l'_k\rangle \times\langle l'_k|\sigma_2^{j_k}\sigma_4^{i_k}\hat{\sigma}_{s_k}\sigma_2^{j_k}\sigma_4^{i_k}|l_k\rangle\langle l_k|\langle j_k|\langle i_k|\hat{M}|a_k\rangle|b_k\rangle|c_k\rangle. \tag{22}$$

By utilizing some properties of Pauli matrices, we can simplify these equations. In particular, we can make the following substitution:

$$\sigma_4^{i_k}\sigma_2^{j_k}\sigma_{s_k}\sigma_2^{j_k}\sigma_4^{i_k} = (-1)^{\gamma_{i_k j_k s_k}}\sigma_{s_k}, \tag{23}$$

where

$$\gamma_{i_k j_k s_k} = \begin{cases} 1 & \text{if } s_k = 2 \text{ and } i_k = 1 \\ 1 & \text{if } s_k = 3 \text{ and } i_k + j_k = 1 \\ 1 & \text{if } s_k = 4 \text{ and } j_k = 1 \\ 0 & \text{otherwise} \end{cases} \tag{24}$$

The s^{th}component of the Bloch representation of ρ_{B} simplifies to

$$\begin{aligned} r_s^{\mathrm{B}} = &\frac{1}{2^n}\sum_{bc,b'c'}\alpha_{bc}\alpha^*_{b'c'}\sum_{aa'}\psi_a\psi^*_{a'} \\ &\times\prod_{k=1}^{n}\sum_{i_k j_k l_k l'_k=0}^{1}(-1)^{\gamma_{i_k j_k s_k}}\langle c'_k|\langle b'_k|\langle a'_k|\hat{M}^\dagger|i_k\rangle|j_k\rangle|l'_k\rangle \\ &\times\langle l'_k|\hat{\sigma}_{s_k}|l_k\rangle\langle l_k|\langle j_k|\langle i_k|\hat{M}|a_k\rangle|b_k\rangle|c_k\rangle. \end{aligned} \tag{25}$$

To explicitly perform the sum over i_k, j_k, l_k, and l'_k in the above expression. Consider the following expression found in the middle of Eq. (25):

$$(-1)^{\gamma_{i_k j_k s_k}}\hat{M}^\dagger|i_k\rangle|j_k\rangle|l'_k\rangle\langle l'_k|\hat{\sigma}_{s_k}|l_k\rangle\langle l_k|\langle j_k|\langle i_k|\hat{M}. \tag{26}$$

The sum of this expression over i_k, j_k, l_k, and l'_k can be formulated as a multiplication, where the factor of $(-1)^{\gamma_{i_k j_k s_k}}$ acts as a diagonal matrix sandwiched between $\hat{M}^\dagger$ and $\hat{M}$ on either side of $\langle l'_{s_k}|\sigma_{s_k}|l_k\rangle$. This diagonal matrix can be written as a tensor product of the identity acting on the third qubit with either the identity or σ_4 acting on the first two qubits, depending on whether s_k is 1, 2, 3, or 4. We will therefore write this matrix as $\hat{\sigma}_{1,s_k} \otimes \hat{\sigma}_{2,s_k} \otimes I$, where we define

$$\begin{aligned} \hat{\sigma}_{1,s_k} &= \begin{cases} \sigma_4 & \text{if } s_k = 2 \text{ or } 3 \\ I & \text{otherwise} \end{cases} \\ \hat{\sigma}_{2,s_k} &= \begin{cases} \sigma_4 & \text{if } s_k = 3 \text{ or } 4 \\ I & \text{otherwise} \end{cases} \end{aligned} \tag{27}$$

Similarly, the Pauli matrix σ_{s_k} acts only on the third qubit, and acts as the identity on the two qubits. We can therefore rewrite our expression from Eq. (26) as

$$\begin{aligned} &\hat{M}^\dagger \left(\hat{\sigma}_{1,s_k} \otimes \hat{\sigma}_{2,s_k} \otimes I\right) \left(I \otimes I \otimes \hat{\sigma}_{s_k}\right) \hat{M} \\ &\quad = \hat{M}^\dagger \left(\hat{\sigma}_{1,s_k} \otimes \hat{\sigma}_{2,s_k} \otimes \hat{\sigma}_{s_k}\right) \hat{M}. \end{aligned} \tag{28}$$

Upon explicit calculation, we find that the righthand side of Eq. (28) reduces to one of the four possible matrices:

$$\text{Eq. (28)} = \begin{cases} I \otimes I \otimes I & \text{if } s_k = 1 \\ \sigma_2 \otimes \sigma_4 \otimes I & \text{if } s_k = 2 \\ \sigma_3 \otimes \sigma_4 \otimes \sigma_4 & \text{if } s_k = 3 \\ \sigma_4 \otimes I \otimes \sigma_4 & \text{if } s_k = 4 \end{cases} \tag{29}$$

Note that the first term in each tensor product is σ_{s_k}, and the second and third terms are $\hat{\sigma}_{1,s_k}$ and $\hat{\sigma}_{2,s_k}$ as defined in Eq. (27), respectively. Therefore, we can write that Eq. (28) is equal to

$$\sigma_{s_k} \otimes \hat{\sigma}_{1,s_k} \otimes \hat{\sigma}_{2,s_k}. \tag{30}$$

Due to the fact that the second and third terms in the tensor product are diagonal, the s^{th}component of the Bloch representation of $\rho_{\text{B}}$is reduced to

$$\begin{aligned} r_s^{\text{B}} = \frac{1}{2^n} \sum_{bc} |\alpha_{bc}|^2 \sum_{aa'} \psi_a \psi_{a'}^* \\ \times \prod_k \langle a'_k | \sigma_{s_k} | a_k \rangle \langle b_k | \hat{\sigma}_{1,s_k} | b_k \rangle \langle c_k | \hat{\sigma}_{2,s_k} | c_k \rangle \end{aligned} \tag{31}$$

Noting that the Bloch representation corresponding to the state $\rho = |\psi\rangle\langle\psi|$ that Alice wanted to send to Bob, is given by

$$r_s = \frac{1}{2^n} \text{Tr}\left(|\psi\rangle\langle\psi|\lambda_s\right) = \frac{1}{2^n} \sum_{aa'} \psi_a \psi_{a'}^* \prod_k \langle a'_k | \hat{\sigma}_{s_k} | a_k \rangle, \tag{32}$$

it then follows that we can factor ρ_s^{ψ} out of Eq. (31), and we have that

$$r_s^{\mathrm{B}} = \left(\sum_{bc} \alpha_{bc} \alpha_{bc}^* \prod_k \langle b_k | \hat{\sigma}_{1,s_k} | b_k \rangle \langle c_k | \hat{\sigma}_{2,s_k} | c_k \rangle \right) r_s. \tag{33}$$

Lastly, we can evaluate the matrix elements of $\hat{\sigma}_{1,s_k}$ and $\hat{\sigma}_{2,s_k}$ that appear in the above expression explicitly by remembering that b_k and c_k are either 0 or 1, meaning that

$$\begin{aligned} \langle b_k | \hat{\sigma}_{1,s_k} | b_k \rangle &= \begin{cases} (-1)^{b_k} & \text{if } s_k = 2 \text{ or } 3 \\ 1 & \text{otherwise} \end{cases} \\ \langle c_k | \hat{\sigma}_{2,s_k} | c_k \rangle &= \begin{cases} (-1)^{c_k} & \text{if } s_k = 3 \text{ or } 4 \\ 1 & \text{otherwise} \end{cases} \end{aligned} \tag{34}$$

Defining the function $S_{s_k}(b_k, c_k) \in \{-1, 1\}$, such that

$$S_{s_k}(b_k, c_k) = \begin{cases} 1 & \text{if } s_k = 1, \\ (-1)^{b_k} & \text{if } s_k = 2 \\ (-1)^{b_k}(-1)^{c_k} & \text{if } s_k = 3 \\ (-1)^{c_k} & \text{if } s_k = 4 \end{cases} \tag{35}$$

and our final expression for r_s^{B} becomes

$$r_s^{\mathrm{B}} = \left(\sum_{bc} \alpha_{bc} \alpha_{bc}^* \prod_{k=1}^{n} S_{s_k}^{b_k c_k} \right) r_s. \tag{36}$$

it can be shown that the components of the Bloch representation of ρ_{B} are given by

$$r_i^B = \frac{1}{2^n} \mathrm{Tr} \left(|\psi_B\rangle\langle\psi_B| \lambda_i \right) = \frac{1}{2^n} \sum_{aa'} \psi_a \psi_{a'}^* \prod_k \langle a_k' | \hat{\sigma}_{i_k} | a_k \rangle, \tag{37}$$

When compared to the Bloch vector of the original information qubit, i.e.

$$r_i = \frac{1}{2^n} \mathrm{Tr} \left(|\psi\rangle\langle\psi| \lambda_i \right), \tag{38}$$

we see that the components of these Bloch vectors are related by the following equation

$$r_i^B = \left(\sum_{bc} |\alpha_{bc}|^2 \prod_{k=1}^{n} S_{i_k}(b_k, c_k) \right) r_i, \tag{39}$$

where the value $i_k \in \{1,2,3,4\}$ corresponds to the k^{th}-spin operator in the tensor product that defines λ_i, and $S_{i_k}(b_k, c_k)$ denotes the i^{th}-diagonal entry of the matrix $S(b_k, c_k)$, defined to be

$$S(b_k, c_k) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & (-1)^{b_k} & 0 & 0 \\ 0 & 0 & (-1)^{b_k+c_k} & 0 \\ 0 & 0 & 0 & (-1)^{c_k} \end{pmatrix}. \tag{40}$$

Upon inspection of Eq. (39), we see that each i^{th}component of the Bloch vectors r_i^B and r_i are proportional. That is, there exists a 4^n-dimensional diagonal matrix that describes the relation between the 4^n-dimensional vectors r_i^B and r_i. Or explicitly, the mapping that describes the effects of imperfect n-qubit teleportation is given by the diagonal 4^n-dimensional matrix

$$f = \sum_{bc} |\alpha_{bc}|^2 \bigotimes_{k=1}^{n} S(b_k, c_k). \tag{41}$$

Note that for all values of b_k and c_k, the matrix $S(b_k, c_k)$ is either the idenity matrix or the Bloch representation a single qubit spin operator. It then follows that each tensor product in Eq. (41) is an n-qubit spin operator as defined by Definition 3; each of which defines a completely positive n-qubit channel. Furthermore, since the set of completely positive n-qubit channels is closed under convex sums, and because the sum of the non-negative $|\alpha_{bc}|^2$ terms is equal to 1, then f itself is a completely positive n-qubit channel [8].

2.2 The n-Qubit Teleportation Channel

When performing the teleporation with one of the teleportation states, $|g_{ab}\rangle$, then all but one of the $|\alpha_{bc}|^2$ terms would be equal to 0, with the remaining equal to 1. That is, the effects of n-qubit teleportation through any teleportation state is described by a diagonal n-qubit spin operator. Furthermore, this technique provides a reliable and efficient method for calculating the quantum channel associated with teleportation through an arbitrary state $|\phi\rangle$.

Theorem: The n-qubit teleportation channel induced by teleporting through the $2n$-qubit state

$$|\phi\rangle = \sum_{i,j=1}^{2^n} \alpha_{ij} |g_{ij}\rangle. \tag{42}$$

is unital with its diagonal Bloch representation given by

$$f = \sum_{i,j=1}^{2^n} |\alpha_{ij}|^2 f_{\lambda_{ij}}, \tag{43}$$

where $f_{\lambda_{ij}}$ is the Bloch representation of the n-qubit spin operator λ_{ij} that is generated by $|g_{ij}\rangle$.

Consider an n-qubit channel that has a diagonal Bloch representation, $f = \sum a_{ij}\lambda_{ij}$. From [1], this sum must be convex, and the coeffecients, non-negative and between 0 and 1. Then, there exists some state $|\phi\rangle$ which induces the channel f via n-qubit teleportation from the Theorem above. This identification between $2n$-qubit pure states and the diagonal n-qubit diagonal Bloch representations is bijective up to phases and has a natural convex extension to the set of $2n$-qubit mixed states.

3. OPEN QUESTIONS AND FUTURE RESEARCH

While the results of this work fully characterize the effects of teleportation of an arbitrary number of qubit states through an imperfectly prepared qubit state, there remain many important questions. For instance, how can we correct for these adverse effects? Is it beneficial to implement adaptive schemes, or simply repeat a given protocol when considering resource management? Is it advantageous or even possible to introduce a bias to the errors that we observe in imperfect teleportation? All of these questions are closely tied to informatic concepts and quantities that still need to be studied for non-ideal conditions in quantum teleportation. We plan to perform such an analysis for imperfect teleportation as well as other protocols and processes that are integral to quantum networks.

Fully implementing quantum technologies necessitates a method for distributing entanglement. While these systems, called quantum networks, possess many of the same obstacles as their classical counterparts, viable solutions are less readily obtained for quantum systems. In particular, current experimental capabilities may suppress, and in some cases even prohibit, ideal conditions for the performance of secure quantum communication protocols. The contents of this work lay out the mathematical and informatic foundations for such a case, i.e. when imperfect entangled states are used for quantum teleportation. Other examples of sources of potential adverse effects include: imperfect measurement procedures, non-fault tolerant gates for routing, and lossy fiber. A comprehensive understanding of how these potential handicaps impact the exchange of quantum information will have a direct affect on the design and security of networking protocols. Such knowledge holds the potential to provide techniques for mitigating errors, improving performance and security, or even taking advantage of quantum re- sources, such as additional degrees of freedom. To this end, such resource estimation is paramount for successful realization of quantum technologies.

REFERENCES

1. T. Crowder, "A Quantum Representation for Involution Groups," *Electronic Notes in Theoretical Computer Science* **276**, 145–158 (2011), ISSN 1571-0661, doi:https://doi.org/10.1016/j.entcs.2011.09.019. URL https://www.sciencedirect.com/science/article/pii/S1571066111001101, Twenty-seventh Conference on the Mathematical Foundations of Programming Semantics (MFPS XXVII).

2. M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information: 10th Anniversary Edition* (Cambridge University Press, 2011), ISBN 9781107002173.

3. M. D. Choi, "Completely positive linear maps on complex matrices," *Linear Algebra and its Applications* **10**(3), 285–290 (1975), ISSN 0024-3795, doi:https://doi.org/10.1016/0024-3795(75)90075-0. URL https://www.sciencedirect.com/science/article/pii/0024379575900750.

4. T. Crowder, *Representations of Quantum Channels*, PhD thesis, 2013).

5. C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, "Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels," *Phys. Rev. Lett.* **70**, 1895–1899 (Mar 1993), doi:10.1103/PhysRevLett.70.1895. URL https://link.aps.org/doi/10.1103/PhysRevLett.70.1895.

6. M. Lanzagorta and K. Martin, "Teleportation with an imperfect state," *Theor. Comput. Sci.* **430**, 117–125 (2012), ISSN 0304-3975, doi:https://doi.org/10.1016/j.tcs.2012.01.003. URL https://www.sciencedirect.com/science/article/pii/S0304397512000187, Mathematical Foundations of Programming Semantics (MFPS XXV).

7. G. Rigolin, "Quantum teleportation of an arbitrary two-qubit state and its relation to multipartite entanglement," *Phys. Rev. A* **71**, 032303 (Mar 2005), doi:10.1103/PhysRevA.71.032303. URL https://link.aps.org/doi/10.1103/PhysRevA.71.032303.

8. K. Martin, "The scope of a quantum channel," *Mathematical Structures in Computer Science, Cambridge(UK)* (2008).

9. H. Aschauer and H. J. Briegel, "Private Entanglement over Arbitrary Distances, Even Using Noisy Apparatus," *Phys. Rev. Lett.* **88**, 047902 (Jan 2002), doi:10.1103/PhysRevLett.88.047902. URL https://link.aps.org/doi/10.1103/PhysRevLett.88.047902.

10. R. Beals, S. Brierley, O. Gray, A. W. Harrow, S. Kutin, N. Linden, D. Shepherd, and M. Stather, "Efficient distributed quantum computing," *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* **469**(2153), 20120686 (2013), doi:10.1098/rspa.2012.0686. URL https://royalsocietypublishing.org/doi/abs/10.1098/rspa.2012.0686.

11. C. H. Bennett, H. J. Bernstein, S. Popescu, and B. Schumacher, "Concentrating partial entanglement by local operations," *Phys. Rev. A* **53**(4), 2046 (1996).

12. J. I. Cirac, A. K. Ekert, S. F. Huelga, and C. Macchiavello, "Distributed quantum computation over noisy channels," *Phys. Rev. A* **59**, 4249–4254 (Jun 1999), doi:10.1103/PhysRevA.59.4249. URL https://link.aps.org/doi/10.1103/PhysRevA.59.4249.

13. L. M. Duan, M. D. Lukin, J. I. Cirac, and P. Zoller, "Long-distance quantum communication with atomic ensembles and linear optics," *Nature* **414**(6862), 413–418 (2001), doi:10.1038/35106500. URL `https://doi.org/10.1038/35106500`.

14. Z. Eldredge, M. Foss-Feig, J. A. Gross, S. L. Rolston, and A. V. Gorshkov, "Optimal and secure measurement protocols for quantum sensor networks," *Phys. Rev. A* **97**, 042337 (Apr 2018), doi:10.1103/PhysRevA.97.042337. URL `https://link.aps.org/doi/10.1103/PhysRevA.97.042337`.

15. D. Gottesman and I. L. Chuang, "Demonstrating the viability of universal quantum computation using teleportation and single-qubit operations (1999).

16. D. G. Im, C. H. Lee, Y. Kim, H. Nha, M. S. Kim, S. W. Lee, and Y. H. Kim, "Optimal teleportation via noisy quantum channels without additional qubit resources," *npj Quantum Inf.* **7**(1), 86 (2021), doi:10.1038/s41534-021-00426-x. URL `https://doi.org/10.1038/s41534-021-00426-x`.

17. Y. S. Kim, J. C. Lee, O. Kwon, and Y. H. Kim, "Protecting entanglement from decoherence using weak measurement and quantum measurement reversal," *Nat. Phys.* **8**(2), 117–120 (2012).

18. E. Knill, R. Laflamme, and G. J. Milburn, "A scheme for efficient quantum computation with linear optics," *nature* **409**(6816), 46–52 (2001).

19. B. H. Liu, L. Li, Y. F. Huang, C. F. Li, G. C. Guo, E. M. Laine, H. P. Breuer, and J. Piilo, "Experimental control of the transition from Markovian to non-Markovian dynamics of open quantum systems," *Nat. Phys.* **7**(12), 931–934 (2011).

20. T. Liu, "The Applications and Challenges of Quantum Teleportation," *J. Phys. Conf. Ser.* **1634**(1), 012089 (sep 2020), doi:10.1088/1742-6596/1634/1/012089. URL `https://doi.org/10.1088/1742-6596/1634/1/012089`.

21. A. Orieux, A. d'Arrigo, G. Ferranti, R. L. Franco, G. Benenti, E. Paladino, G. Falci, F. Sciarrino, and P. Mataloni, "Experimental on-demand recovery of entanglement by local operations within non-Markovian dynamics," *Sci. Rep.* **5**(1), 1–8 (2015).

22. J. W. Pan, S. Gasparoni, R. Ursin, G. Weihs, and A. Zeilinger, "Experimental entanglement purification of arbitrary unknown states," *Nature* **423**(6938), 417–422 (2003).

23. N. H. Valencia, S. Goel, W. McCutcheon, H. Defienne, and M. Malik, "Unscrambling entanglement through a complex medium," *Nat. Phys.* **16**(11), 1112–1116 (2020).

24. M. Zwerger, A. Pirker, V. Dunjko, H. J. Briegel, and W. Dür, "Long-Range Big Quantum-Data Transmission," *Phys. Rev. Lett.* **120**, 030503 (Jan 2018), doi:10.1103/PhysRevLett.120.030503. URL `https://link.aps.org/doi/10.1103/PhysRevLett.120.030503`.

25. Z. Zhao, T. Yang, Y. A. Chen, A. N. Zhang, and J. W. Pan, "Experimental Realization of Entanglement Concentration and a Quantum Repeater," *Phys. Rev. Lett.* **90**, 207901 (May 2003), doi:10.1103/PhysRevLett.90.207901. URL `https://link.aps.org/doi/10.1103/PhysRevLett.90.207901`.