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Report on

The Longitudinal Vibrations of a Projectile

During Armor Penetration

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Report on

The Longitudinal Vibrations of a Projectile During Armor Penetration

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NAVAL RESEARCH LABORATORY
WASHINGTON, D. C.
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NOTATION

t = time, t' = a particular instant, t_m = time at which $F(t)$ is maximum.

e = natural base = 2.71828···

β = constant

s = positive integer

$F = F(t)$ = force on the nose of a projectile during penetration as a function of time.

K = constant

$u = u(x,t)$ = displacement of an elementary cross-section of the projectile as a function of position in the projectile and time.

x = position in projectile of length ℓ, $0 \leq x \leq \ell$.

$c = \sqrt{\frac{E}{\rho}}$ = velocity of sound in the projectile.

E = modulus of elasticity of the projectile.

ρ = density of projectile.

V_0 = initial or striking velocity of projectile.

p = parameter in the Laplace Transformation greater than zero.

$\bar{u}(x,p) = \int_0^{\infty} e^{-pt} u(x,t)dt$ = Laplace Transform of $u(x,t)$.

A = cross-sectional area of projectile (considered a rod)

B_r = symbol denoting the Bromwich contour (See Plate II).

γ = positive constant large enough to bring the line $z = \gamma + i\eta$ to the right of all singularities of $\bar{u}(x,\lambda)$ and $\bar{F}(\lambda)$ in the complex λ-plane.

$\bar{F}(p) = \int_0^{\infty} e^{-pt} F(t)dt$ = Laplace Transform of $F(t)$.

[redacted]

NOTATION (Cont'd)

λ = complex variable = $\xi + i\eta$.

ξ = abscissa in complex λ-plane.

η = ordinate in complex λ-plane.

$M = AE\ell/c^2 = A\rho\ell$ = mass of the projectile.

$i = \sqrt{-1}$

$\tau = t/tm$

n = positive integer

H = total energy of the projectile.

$2b$ = generalized coefficient of viscosity (internal).

$2a$ = coefficient of velocity resistance.

σ_x = stress in projectile at x.

$\partial u/\partial x\big|_x$ = strain in projectile at x.

$$sOn = \frac{s(s-1)(s-2)\ldots(s-n)\ldots3\cdot2\cdot1}{n!(s-n)!} = \frac{s!}{n!(s-n)!}$$

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FORWORD

A few years ago when the first successful measurements of the force acting on a bullet during penetration were made at the Naval Research Laboratory, lack of precision in reading the photographic records justified using several approximations in analyzing the data. It was known that some important variations in the force might be overlooked, but any uneasiness due to this was diminished by enthusiasm for the new experiment and a desire to obtain preliminary results. After nearly thirty analyses were made and a report was written about force measurements of typical penetrations of STS and mild steel, it was realized that too much smoothing existed in the method and that all force curves were liable to look much alike. This required an improvement of the experimental apparatus and of the process of analyzing the data before desirable studies could be made such as comparing forces on a bullet from homogeneous armor to forces on a bullet from face hardened armor.

The apparatus was improved so as to give almost another order of magnitude to the precision of the measurements, and the first new photographic records were measured. But when the data were analyzed, persistent peculiarities were revealed which could be explained only by assuming that the bullet exhibited an elastic vibration superimposed upon its motion of penetration. This was disturbing,for,if the motion of the base of the projectile was showing the influence of bullet vibration to the extent that seemed to be indicated by the new photographic records, then the force resisting the penetration of the nose of the bullet could not be obtained

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any better than before unless a good approximate solution for the motion of the base could be obtained from the theory of elasticity.

This presented a problem to be solved which was: How does a short moving cylindrical rod become elastically deformed when it strikes a softer deformable material? The rod must be assumed to have an initial velocity, V_0 , and a form of the resisting force must be used that is consistent with smoothed analyses of the data. A careful search of technical literature failed to give a complete and satisfactory solution for this problem although, as mentioned in the report, two noteworthy attempts were made using the Fourier Integral Theorem. When a method making use of the Laplace-Transform was applied, a solution was obtained which is the subject of this report.

Like all mathematical treatments of physical phenomena, certain restrictive assumptions have been made in developing this solution. The justification of these assumptions will be shown in the report to arise from the excellent agreement found between the predicted results and the experimental observations. Thus, the solution predicts the existence of a longitudinal vibration arising from the impact that should affect the motion of the base of the bullet in a manner which agrees very well with the behavior observed experimentally. What is more important, the extent of the observed effect of vibration on the motion of the base of the bullet is in good agreement with that predicted by the solution when elastic properties of the bullet, its density, and a reasonable estimate of its effective length are introduced.

It was unexpectedly found to be possible, by means of the

[redacted]

solution, to prove several theorems that greatly simplified the task of analyzing the data from the photographic records. In fact, one of these theorems lets the force be deduced that comes into existence just as impact occurs, before an elastic wave of compression can reach the base to show a deceleration on the photographic record. This allows the initial forces of impact to be found and compared to initial forces measured in static punching experiments.

The method of solution used in this report ought to be of considerable interest to anyone who has to deal with some similar problem, because it gives explicit expressions as functions of time and initial and boundary conditions for such quantities as the strain at any point along the bullet and the positions of the center of mass. An an example of the possible uses of the theory, besides its application to measurement of the force on a bullet during penetration, the expression for the strain given in the report may be combined with Hooke's law and used to estimate the instantaneous average stress at any particular plane-section along the bullet. This would require, of course, a preliminary knowledge of the force resisting penetration, but the Naval Research Laboratory force measurements can give this.

In conclusion, it may be said that this report has been written not only to furnish a record of the advancement in the process of measuring the force on a bullet at the Naval Research Laboratory, but also because it is believed that there may be readers who will welcome this demonstration of the method of the Laplace Transform.

G. D. Kinzer

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On the Longitudinal Vibrations of a Projectile During Armor Penetration

ABSTRACT

This laboratory's more recent and improved time-displacement measurements for projectiles penetrating armor plate yield force-time curves which have distinct maxima and minima beyond probable error. The periodicity observed in the maxima and minima has led to the conclusion that, in measuring the deceleration of a projectile by observing the motion of the base across an illuminated slit, the results obtained are influenced by the longitudinal vibration of the projectile arising from its compressibility.

By considering the projectile to be a cylindrical rod as a first approximation, the one-dimensional wave-equation is solved operationally under appropriate boundary conditions by the method of the Laplace Transform and contour integration. A simple method is deduced for evaluating the force resisting penetration from data on the motion of the base: - multiply the mass of the projectile by the average acceleration over any time-interval equal to a fundamental period of vibration of the projectile, to give the force on the projectile at the midpoint of that interval.

An illustrative solution for the wave-equation is obtained in explicit form for force-time functions proportional to $t^s \cdot e^{-\beta t}$ where s is a positive integer and β a constant determined by $\underline{s}$ and the time at which the force reaches a maximum. It is shown empirically

[redacted]

ABSTRACT (Continued)

that for s = 2 such a force-function approximates that actually resisting armor penetration. Theoretical curves are presented showing the deceleration of the base of a projectile to be expected for the assumed force on the nose. The wave-equation with viscous and velocity damping is also solved by the same operational method.

[redacted]

INTRODUCTION

A. Authorization and References

1. This problem was authorized by Bureau letter, reference (a). Other references mentioned in this report are listed below, (b) to (j).

Reference: (a) BuOrd ltr. S13-1(4/173)(QB) of 13 December 1934.
(b) NRL Report No. 0-1591 of 6 February 1940.
(c) Frankford Arsenal Report No. 52 of October, 1940.
(d) NRL Report No. 0-2276 of 13 April 1944.
(e) Frankford Arsenal Report No. 52C of April, 1941.
(f) Private communication to NRL from N. Rosen, University of North Carolina.
(g) Bromwich, T.J. I'A, Proc. Lond. Math. Soc. (2), 15(1917), 401.
(h) Carslaw, H.S. and Jaeger, J.C., Operational Methods in Applied Mathematics, Chapt. IV, Oxford, 1941.
(i) Wagner, K.W., Archiv fur Electrotechnik, 4(1916),159.
(j) Lamb, H., The Dynamical Theory of Sound, 2nd Ed., E. Arnold & Co., Lond., 188.

B. Statement of Problem

2. The need for mathematical analysis of the behavior of elastically deformable projectiles while penetrating armor plate has arisen as a consequence of the experimental work, references (b), (c), and (d), purporting to measure the force opposing penetration. The experimental procedure consists essentially of obtaining a continuous photographic

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trace of the displacement of the base of a projectile during penetration as a function of time. An analysis of the time-displacement record will give the velocity and deceleration of the base of the projectile at short time intervals during the armor penetration.

3. It is clear that if the projectile suffers longitudinal strain during armor penetration the motion of the base will not be identical with that of the center of mass, and it is only a knowledge of the motion of the latter which can yield the true force on the projectile. This point has been investigated in a report by W. J. Kroeger and C. M. Hudson at the Frankford Arsenal, reference (e), and in a communication to this laboratory by N. Rosen, reference (f). These workers make use of a Fourier Integral representation of the hypothetical force encountered by the projectile. From this expression it is deduced that the motion of the center of mass can be obtained from that of the base by averaging over a time interval equal to the fundamental period of longitudinal vibration of the projectile.

C. Mathematical Method

4. This result is proven and extended here by an entirely different method depending on the Laplace Transformation. The advantages of the Laplace Transformation are that the force function is less restricted in character than those which are expressible by a Fourier Integral and that the method enables one to obtain a solution for the wave equation in explicit form - generally an infinite series whose convergence is sufficiently rapid for actual computation. In what follows we consider first the general solution for the wave-equation in one-dimension which is

[redacted]

applicable to a cylindrical projectile, under boundary conditions in which the force acting on the nose of the projectile during armor penetration is specified simply as a function of time, $F(t)$. Later, a particular functional form is taken which agrees empirically with those observed.

PART I

A. Formal Solution of the One-Dimensional Wave-Equation With Boundary Conditions

5. The problem of an ogival nosed bullet with small ogival radius which executes longitudinal vibrations while penetrating a piece of armor, Plate I, fig. 1a, can be simplified as a first step to the problem of an elastic rod which is moving with velocity V_o , fig. 1b, and which at time $t = o$ encounters a force, $F(t)$, acting on the leading face so as to oppose the motion. It is seen that $F(t)$ as a physically admissable force-function is in general continuous, differentiable, bounded, and absolutely integrable over the time: $\int_{-\infty}^{+\infty} \left| F(t) \right| dt < K$. In our problem we take $F(t) = o$ for $t < o$, so that $\int_{-\infty}^{+\infty} \left| F(t) \right| dt = \int_{o}^{\infty} \left| F(t) \right| dt$. We assume also that the longitudinal strain is small and that damping is negligible. The problem will be considered again when damping is not neglected.

6. The equation to be solved is

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2} \quad , \quad c^2 = \frac{E}{\rho} \qquad (1)$$

[redacted]

subject to the boundary conditions

(1a) $t = o : \quad u = o , \quad \partial u/\partial t = V_o$

(1b) $x = o : \; \partial u/\partial x\Big|_{x=o} = o , \quad x = \ell : \; \partial u/\partial x\Big|_{x=\ell} = F(t)/AE.$

7. The application of the Laplace Transformation to Eq. (1) yields formally the relation

$$\frac{d^2}{dx^2}\int_o^\infty e^{-pt}\,u(x,t)\,dt = \frac{1}{c^2}\int_o^\infty e^{-pt}\,\frac{\partial^2 u(x,t)}{\partial t^2}\,dt.$$

We write $\bar{u}(x,p) = \int_o^\infty e^{-pt}u(x,t)dt$ and integrate the right hand integral twice by parts to get

$$\frac{d^2\bar{u}}{dx^2} = \frac{1}{c^2}\Bigg[\Bigg(e^{-pt}\,\frac{\partial u}{\partial t}\,\Bigg|_o^\infty + p\,\Bigg\{\Bigg(e^{-pt}u\,\Bigg|_o^\infty + p\int_o^\infty e^{-pt}u\,dt\Bigg\}\Bigg] .$$

For physically acceptable solutions it is clear that

$$\lim_{t\to\infty}\left[e^{-pt}\,\frac{\partial u}{\partial t}\right] = \lim_{t\to\infty}\left[e^{-pt}\,u\right] = o , \quad \text{and by the}$$

conditions (1a) it follows that

$$\lim_{t\to o}\left[e^{-pt}\,\frac{\partial u}{\partial t}\right] = V_o \quad \text{and} \quad \lim_{t\to o}\left[e^{-pt}\,u\right] = o,$$

so that the Laplace transform of Eq. (1) is given by

(2) $$\frac{d^2\bar{u}(x,p)}{dx^2} - \frac{p^2}{c^2}\,\bar{u}(x,p) = -\frac{V_o}{c^2}$$

[redacted]

subject to the conditions

$$(2b)\qquad \left.\frac{d\bar{u}}{dx}\right|_{x=0} = 0 \quad \text{and} \quad \left.\frac{d\bar{u}}{dx}\right|_{x=\ell} = \frac{1}{AE}\int_0^{\infty} e^{-pt}F(t)dt = \frac{\bar{F}(p)}{AE}$$

8. The problem of obtaining a solution for the wave-equation for the propagation of displacement u, has thus been reduced to obtaining a solution for the ordinary differential equation, Eq. (2), involving the Laplace Transform of u. The solution for Eq. (2) under conditions (2b) is

$$(3)\qquad \bar{u}(x,p) = \frac{V_c}{p^2} + \frac{C}{AE} \cdot \bar{F}(p) \cdot \frac{\cosh\frac{px}{c}}{p \cdot \sinh\frac{p\ell}{c}}$$

as may be verified directly.

9. We desire a function $u(x,t)$ such that

$$\int_0^{\infty} e^{-pt}u(x,t)dt = \bar{u}(x,p) = \frac{V_c}{p^2} + \frac{c}{AE} \cdot \bar{F}(p) \cdot \frac{\cosh\frac{px}{c}}{p \cdot \sinh\frac{p\ell}{c}}$$

The function $u(x,t)$ which is the solution for Eq. (1) and which has the Laplace Transform given by (3) is found as the sum of the residues of the following contour integral, references (g) and (h)

$$(4)\qquad u(x,t) = \frac{1}{2\pi i}\int_{Br} e^{\lambda t}\left[\frac{V_o}{\lambda^2} + \frac{c}{AE} \cdot \bar{F}(\lambda) \cdot \frac{\cosh\frac{\lambda x}{c}}{\lambda \sinh\frac{\lambda\ell}{c}}\right] d\lambda$$

in which the symbol Br represents the contour indicated by the

[redacted]

arrows in Plate II and was first given by Bromwich, reference g, and Wagner, reference i, independently. The contour may have to be deformed if the transform $\bar{F}(\lambda)$ introduces branch points. If $\bar{F}(\lambda)$ introduces functional forms* which cause the integral to be non-vanishing over the arc as the radius is allowed to increase indefinitely, the method will fail. These complications will not be encountered in the problem at hand because first, branch points are not introduced except possibly when the partial derivatives with respect to time and position in the original partial differential equation are not of equal order and, second, physically admissable force-functions $F(t)$ will have transforms whose absolute values as functions of a complex variable, λ, tend to zero as the argument tends to infinity. (We cannot consider force functions which do not possess Laplace transforms.) We are concerned hereafter with the formal solution for Eq. (1) expressed in Eq. (4) and there is left to show that the solution does in fact satisfy the original equation and the boundary conditions specified.

10. That the solution in (4) satisfies Eq. (1) is easily seen by direct differention under the sign of integration and by noting that

$$\int_{B_r} v_0 e^{\lambda t} d\lambda = 0 \qquad \text{and} \qquad \frac{\partial}{\partial x}\left(v_0 \frac{e^{\lambda t}}{\lambda^2}\right) = 0$$

---

*Such forms are introduced when $F(t)$ is e.g.: e^{kt^2}, t^{-n}, where n is a positive integer.

[redacted]

Further,

$$|u(x,o)| \leq \frac{1}{2\pi}\int_{B_r} \left| \frac{V_o}{\lambda^2} + \frac{c}{AE}\,\frac{\bar{F}(\lambda)}{\lambda}\,\frac{\cosh\frac{\lambda x}{c}}{\sinh\frac{\lambda \ell}{c}} \right| d\lambda$$

$$= \frac{1}{2\pi}\int_{B_r} \left| \frac{c}{AE}\,\frac{\bar{F}(\lambda)}{\lambda}\,\frac{\cosh\frac{\lambda x}{c}}{\sinh\frac{\lambda \ell}{c}} \right| d\lambda$$

$$= \frac{1}{2\pi}\lim_{|\lambda|\to\infty}\oint \frac{c}{AE}\left|\frac{\bar{F}(\lambda)}{\lambda}\,e^{\frac{\lambda(x-\ell)}{c}}\right| d\lambda \leq \frac{1}{2\pi}\lim_{|\lambda|\to\infty}\oint\frac{c}{AE}\left|\frac{\bar{F}(\lambda)}{\lambda}\right| d\lambda = 0$$

Similarly, we can show

$$\left| \left.\frac{\partial u(x,t)}{\partial t}\right|_{t=o} \right| = \frac{1}{2\pi}\left| \int_{B_r} \left[\frac{V_o}{\lambda} + \frac{c}{AE}\,\frac{\bar{F}(\lambda)}{\lambda}\,\frac{\cosh\frac{\lambda x}{c}}{\sinh\frac{\lambda \ell}{c}} \right] d\lambda \right| = V_o.$$

Here we have had recourse to the assumption that the force function $F(t)$ is such that

$$\lim_{|\lambda|\to\infty}\left|\bar{F}(\lambda)\right| = \lim_{|\lambda|\to\infty}\left|\int_o^\infty e^{-\lambda t}\,F(t)\,dt\right| = 0.$$

We have thus shown that the initial conditions are satisfied with respect to time. For the end conditions on the projectile we have at once that

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=o} = \frac{1}{2\pi i}\int_{B_r} \left.\frac{\partial}{\partial x}\left\{ e^{\lambda t}\left[\frac{V_o}{\lambda^2} + \frac{c}{AE}\,\frac{\bar{F}(\lambda)}{\lambda}\cdot\frac{\cosh\frac{\lambda x}{c}}{\sinh\frac{\lambda \ell}{c}} \right]\right\}\right|_{x=o} d\lambda = 0$$

[redacted]

and

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=\ell} = \frac{1}{2\pi i}\int_{B_r} \frac{1}{AE}\, e^{\lambda t}\, \bar{F}(\lambda)\, d\lambda = \frac{F(t)}{AE}.$$

This completes the formal solution.

B. Motion of the Base of the Projectile

11. The displacement, velocity, and acceleration of the base is obtained at once by setting $x = o$ in Eq. (4) and these quantities are given respectively by

$$(5) \qquad u(o,t) = \frac{1}{2\pi i}\int_{B_r} e^{\lambda t}\left[\frac{V_o}{\lambda^2} + \frac{c}{AE}\,\frac{\bar{F}(\lambda)}{\lambda \sinh \frac{\lambda \ell}{c}}\right] d\lambda,$$

$$(6) \qquad \left.\frac{\partial u(x,t)}{\partial t}\right|_{x=o} = \frac{1}{2\pi i}\int_{B_r} e^{\lambda t}\left[\frac{V_o}{\lambda} + \frac{c}{AE}\,\frac{\bar{F}(\lambda)}{\sinh \frac{\lambda \ell}{c}}\right] d\lambda,$$

$$(7) \qquad \left.\frac{\partial^2 u(x,t)}{\partial t^2}\right|_{x=o} = \frac{1}{2\pi i}\int_{B_r} e^{\lambda t}\left[V_o + \frac{c}{AE}\,\frac{\lambda \bar{F}(\lambda)}{\sinh \frac{\lambda \ell}{c}}\right] d\lambda.$$

$$= \frac{1}{2\pi i}\int_{B_r} \frac{c}{AE}\cdot\frac{\lambda\, e^{\lambda t}\, \bar{F}(\lambda)}{\sinh \frac{\lambda \ell}{c}}\, d\lambda$$

[redacted]

In Eq. (7) we note that $\int_{Br} V_o\, e^{\lambda t}\, d\lambda = o$

C. Motion of the Center of Mass of the Projectile

12. In this section we state and prove some theorems which are useful in obtaining the motion of the center of mass from that of the base. Since it is the motion of the base that is observed, the following theorems constitute the main results of the report.

Theorem I:

13. The average value of the displacement of the base over any time-interval, $(t' - \frac{\ell}{c},\ t' + \frac{\ell}{c})$, is equal to the displacement of the center of mass at the midpoint, t', of that interval.

Proof: We have for the average by Eq. (5)

$$\frac{c}{2\ell}\int_{t'-\frac{\ell}{c}}^{t'+\frac{\ell}{c}} u(o,t)dt = \frac{c}{2\ell}\cdot\frac{1}{2\pi i}\int_{Br}\left[\int_{t'-\frac{\ell}{c}}^{t'+\frac{\ell}{c}} e^{\lambda t}dt\right]\left[\frac{V_o}{\lambda^2} + \frac{c}{AE}\,\frac{\bar{F}(\lambda)}{\lambda\sinh\frac{\lambda\ell}{c}}\right]d\lambda$$

$$= \frac{c}{\ell}\cdot\frac{1}{2\pi i}\int_{Br}\left[e^{\lambda t'}\cdot\frac{\sinh\frac{\lambda\ell}{c}}{\lambda^3}\cdot V + \frac{c}{AE}\cdot e^{\lambda t'}\cdot\frac{\bar{F}(\lambda)}{\lambda^2}\right]d\lambda$$

$$= V_o t' + \frac{1}{M}\cdot\frac{1}{2\pi i}\int_{Br}\left[\int_{-\infty}^{t'}\left[\int_{-\infty}^{t'} e^{\lambda t}\,dt\right]dt\right]\bar{F}(\lambda)d\lambda$$

$$= V_o t' + \frac{1}{M}\cdot\frac{1}{2\pi i}\int_{-\infty}^{t'}\left[\int_{-\infty}^{t'}\left[\int_{Br} e^{\lambda t}\,\bar{F}(\lambda)d\lambda\right]dt\right]dt$$

$$= V_o t' + \frac{1}{M}\cdot\int_o^{t'}\cdot\int_o^{t'} F(t)dt\cdot dt.$$

[redacted]

and this is precisely the displacement of a point-mass, $M\ (= AE\ell/c^2)$, which has an initial velocity, V_o, and is acted upon by a force $F(t)$ for the time t'. The lower limit can be changed from $-\infty$ to zero in the last expression because $F(t)$ is zero for $t < o$.

Theorem II:

14. The average value of the velocity of the base over any time-interval $(t' - \frac{\ell}{c}, t' + \frac{\ell}{c})$ is equal to the velocity of the center of mass at the midpoint, t', of that interval.

Proof: We have for the average by Eq. (6)

$$\frac{c}{2\ell}\int_{t'-\frac{\ell}{c}}^{t'+\frac{\ell}{c}} \left.\frac{\partial u(x,t)}{\partial t}\right|_{x=o} dt = \frac{c}{2\ell}\ \frac{1}{2\pi i}\int_{B_r}\left[\int_{t'-\frac{\ell}{c}}^{t'+\frac{\ell}{c}} e^{\lambda t}dt\right]\left[\frac{V_o}{\lambda} + \frac{c}{AE}\ \frac{\bar{F}(\lambda)}{\sinh\frac{\lambda\ell}{c}}\right] d\lambda$$

$$= \frac{c}{\ell}\cdot\frac{1}{2\pi i}\int_{B_r}\left[V_o \cdot \frac{e^{\lambda t'}\sinh\frac{\lambda\ell}{c}}{\lambda^2} + \frac{c}{AE}\ \frac{e^{\lambda t'}\bar{F}(\lambda)}{\lambda}\right] d\lambda$$

$$= V_o + \frac{1}{M}\cdot\frac{1}{2\pi i}\int_{B_r}\left[\int_{-\infty}^{t'} e^{\lambda t}dt\right]\bar{F}(\lambda)d\lambda$$

$$= V_o + \frac{1}{M}\int_o^{t'} F(t)dt,$$

and this is the velocity a point-mass M will attain which has

[redacted]

initial velocity V_o and is acted on by a force $F(t)$ for the time t' .

Corollary:

15. The velocity of the center of mass at the midpoint, t' , of any time-interval $(t'-\frac{\ell}{c}, t'+\frac{\ell}{c})$ is equal to the difference between the displacements of the base at the end points of that time-interval divided by the magnitude of the interval (the fundamental period of longitudinal vibration of the projectile, $\frac{2\ell}{c}$.)

Proof:

$$\frac{c}{2\ell}\int_{t'-\frac{\ell}{c}}^{t'+\frac{\ell}{c}} \left.\frac{\partial u(x,t)}{\partial t}\right|_{x=o} dt = \frac{c}{2\ell}\left[u(o,t'+\tfrac{\ell}{c}) - u(o,t'-\tfrac{\ell}{c})\right] \tag{8}$$

Theorem III:

16. The average value of the acceleration of the base over any time-interval $(t'-\frac{\ell}{c}, t'+\frac{\ell}{c})$ is equal to the acceleration of the center of mass at the midpoint, t' , of that interval.

Proof: We have for the average by Eq. (7)

$$\frac{c}{2\ell}\int_{t'-\frac{\ell}{c}}^{t'+\frac{\ell}{c}} \left.\frac{\partial^2 u(x,t)}{\partial t^2}\right|_{x=o} dt = \frac{c}{2\ell}\cdot\frac{1}{2\pi i}\int_{B_r}\left[\int_{t'-\frac{\ell}{c}}^{t'+\frac{\ell}{c}} \lambda e^{\lambda t}dt\right]\cdot\frac{c}{AE}\cdot\frac{\overline{F}(\lambda)}{\sinh\frac{\lambda\ell}{c}}\,d\lambda$$

$$= \frac{1}{2\pi i}\cdot\frac{1}{M}\int_{B_r} e^{\lambda t'}\cdot\left[\frac{e^{\lambda\ell/c} - e^{-\lambda\ell/c}}{2}\right]\cdot\frac{\overline{F}(\lambda)}{\sinh\frac{\lambda\ell}{c}}\,d\lambda$$

$$= \frac{1}{2\pi i}\cdot\frac{1}{M}\int_{B_r} e^{\lambda t'}\overline{F}(\lambda)d\lambda = \frac{1}{M}F(t') ,$$

[redacted]

which is clearly the acceleration of a point-mass M at the instant t' for the force function $F(t)$.

Corollary:

17. The acceleration of the center of mass at the midpoint t' of any time-interval $\left(t'-\frac{\ell}{c},\ t'+\frac{\ell}{c}\right)$ is equal to the difference between the velocities of the base at the endpoints of that time-interval divided by the magnitude of the interval.

Proof:

$$(a)\quad \frac{c}{2\ell}\int_{t'-\frac{\ell}{c}}^{t'+\frac{\ell}{c}} \left.\frac{\partial^2 u(x,t)}{\partial t^2}\right|_{x=0} dt = \frac{c}{2\ell}\left[\left.\frac{\partial u(x,t)}{\partial t}\right|_{\substack{x=0\\ t=t'+\frac{\ell}{c}}} - \left.\frac{\partial u(x,t)}{\partial t}\right|_{\substack{x=0\\ t=t'-\frac{\ell}{c}}}\right].$$

18. In each of the proofs for the above theorems and the associated corollaries, the effect of the averaging process has been to cancel the factor $\sinh\frac{\lambda\ell}{c}$ from the denominator in the integrand. The consequence of this mathematically is to eliminate the spectrum of poles distributed in pairs along the entire imaginary axis. This is equivalent to the elimination from the solution of oscillatory terms which are always introduced by poles located anywhere in the complex-plane except on the real-axis. We now state some theorems which are of no practical importance for the experimental work but which complete the mathematical treatment.

Theorems IV, V, and VI:

19. The displacement, velocity, and acceleration of the center of mass at any instant are respectively equal to the average of the values of the

[redacted]

displacement, velocity and acceleration <u>distributed throughout the projectile at that instant</u>.

Proof:

20. These theorems differ from the preceeding three in that the average is taken over the space variable instead of the time. In this connection it is to be noted that the initial conditions (1a) signify that the displacement of any cross-section in the projectile is measured from <u>its original</u> position so that the displacement values are "absolute." The proof need only be given for the case of displacement. We evaluate

$$\frac{1}{\ell}\int_0^\ell u(x,t)dx = \frac{1}{\ell}\cdot\frac{1}{2\pi i}\int_{B_r} d\lambda \int_0^\ell e^{\lambda t}\left[\frac{V_o}{\lambda^2} + \frac{c}{AE}\,\frac{\overline{F}(\lambda)\cosh\frac{\lambda x}{c}}{\lambda\sinh\frac{\lambda\ell}{c}}\right]dx$$

$$= \frac{1}{2\pi i}\int_{B_r} e^{\lambda t}\left[\frac{V_o}{\lambda^2} + \frac{1}{M}\,\frac{\overline{F}(\lambda)}{\lambda^2}\right]d\lambda = V_o t + \frac{1}{M}\int_{B_r} d\lambda \int_{-\infty}^{t} dt \int_{-\infty}^{t} e^{\lambda t}\overline{F}(\lambda)dt$$

$$= V_o t + \frac{1}{M}\int_{-\infty}^{t} dt \int_{-\infty}^{t} dt \int_{B_r} e^{\lambda t}\overline{F}(\lambda)d\lambda = V_o t + \frac{1}{M}\int_0^t dt \int_0^t F(t)dt$$

which is identical with the result in Theorem I for the upper limit $t=t'$.
The proofs for velocity and acceleration follow at once.

[redacted]

PART II

A. Application of Formal Solution

21. In this part of the report we present an application of the general theory given in Part I by determining the explicit form of the solution for an empirically determined force function. In Plates III and IV there are presented a set of force-time curves for the base obtained from near limit velocity shots at various thicknesses of mild steel and STS armor plates with .27 Cal. darts in each case. The smooth curves are hypothetical force-functions plotted from the assumed form $F(t) = F_o t^2 e^{-\beta t}$ where F_o and β have been adjusted to give reasonable approximations to smoothened curves. While a close agreement is not obtained over the entire penetration cycle, the functional form $t^2 e^{-\beta t}$ exhibits the essential features of the type of force actually opposing armor penetration.

22. Let it be assumed that the maximum force, F_m , is reached at a time, t_m . Our force-function will then be given by $F(t) = -F_m(\frac{e}{t_m})^s t^s e^{-st/t_m}$ where we write <u>s</u> instead of <u>2</u> for generality, $F_o = -F_m(\frac{e}{t_m})^s$, and $\beta = \frac{s}{t_m}$. In order to evaluate the contour integral in the general solution of Eq. (4), we substitute

$$(10) \qquad \bar{F}(\lambda) = \int_0^\infty F_o t^s e^{-\beta t} e^{-\lambda t} dt = F_o \frac{s!}{(\lambda+\beta)^{s+1}}$$

to give the contour integral

$$(11) \quad u(x,t) = \frac{1}{2\pi i} \int_{B_r} e^{\lambda t} \left[\frac{V}{\lambda^2} + F_o \frac{c \cdot s!}{AE} \cdot \frac{\cosh \frac{\lambda x}{c}}{\lambda(\lambda+\beta)^{s+1} \sinh \frac{\lambda \ell}{c}} \right] d\lambda$$

[redacted]

23. The displacement function, $u(x,t)$, is then the sum of the residues at the poles

$$\lambda = 0,\ -\beta\ (s+1\text{-tuple}),\ \pm \frac{n\pi c}{\ell} i\ ,\ \text{where } n = 1, 2, 3, \ldots$$

Calculation of the Residues

24. In giving the residues at the various poles, we omit the factor $2\pi i$ since it is automatically cancelled.

$\underline{\lambda = 0}$:

25. The residue contributed by $V_o \frac{e^{\lambda t}}{\lambda^2}$ is $\underline{V_o t}$. The residue contributed by

$$F_o \frac{c \cdot s!}{AE} \cdot \frac{e^{\lambda t} \cdot \cosh \frac{\lambda x}{c}}{\lambda(\lambda+\beta)^{s+1} \sinh \frac{\lambda \ell}{c}}$$

is

$$F_o \frac{c^2 . s!}{AE \ell \beta^{s+1}} \left(t - \frac{s+1}{\beta} \right)$$

$\underline{\lambda = -\beta}$:

26. Expressed differentially the residue is

$$F_o \frac{c}{AE} \frac{d^s}{d\lambda^s} \left[\frac{e^{\lambda t} \cosh \frac{\lambda x}{c}}{\lambda \sinh \frac{\lambda \ell}{c}} \right|_{\lambda = -\beta}$$

$\lambda = \pm \frac{n\pi c}{\ell} i$ ::

27. By expansion we find the residues to be

$$F_o \cdot \frac{c \cdot s!}{AE} \frac{(-1)^n (\mp i) \left(\beta \mp \frac{n\pi c}{\ell} i\right)^{s+1}}{n\pi \left\{\beta^2 + \left(\frac{n\pi c}{\ell}\right)^2\right\}^{s+1}} \cdot e^{\pm i \cdot \frac{n\pi ct}{\ell}} \cdot \cos \frac{n\pi x}{\ell} \quad .$$

Now by writing

$$e^{\pm i \frac{n\pi ct}{\ell}} = \cos \frac{n\pi ct}{\ell} \pm i \sin \frac{n\pi ct}{\ell} \quad ,$$

and expanding the expressions

$$\mp i \left(\beta \mp i \frac{n\pi c}{\ell}\right)^{s+1} \quad ,$$

these residues become

$$F_o \cdot \frac{c \cdot s!}{AE} \cdot \frac{(-1)^n}{n\pi \left\{\beta^2 + \left(\frac{n\pi c}{\ell}\right)^2\right\}^{s+1}} \left[\sum_{p=0}^{s+1} (\mp i)^{p+1} \frac{(s+1)!}{p!\,(s-p+1)!} \beta^{s-p+1} \left(\frac{n\pi c}{\ell}\right)^p \right] x$$

$$\left[\cos \frac{n\pi ct}{\ell} \pm i \sin \frac{n\pi ct}{\ell}\right] \cdot \cos \frac{n\pi x}{\ell} \quad .$$

Hence, on adding all the residues, those on the imaginary axis being

[redacted]

taken in pairs at poles of equal magnitude and opposite sign, we obtain as the solution for the displacement throughout the projectile

$$(12a)\quad u(x,t) = V_0 t + F_0 \frac{c^2 \cdot s!}{AE\,\ell\,\beta^{s+1}}\left(t - \frac{s+1}{\beta}\right) + F_0 \frac{c}{AE}\frac{d^s}{d\lambda^s}\left[\frac{e^{\lambda t}\cosh\frac{\lambda x}{c}}{\lambda \sinh\frac{\lambda \ell}{c}}\right]_{\lambda=-\beta}$$

$$+F_0 \frac{2 \cdot c \cdot s!}{AE}\sum_{n=1}^{\infty}\left[\frac{(-1)^n}{n\pi\left\{\beta^2 + \left(\frac{n\pi c}{\ell}\right)^2\right\}^{s+1}}\left\{(P)_n \sin\frac{n\pi ct}{\ell} + (Q)_n \cos\frac{n\pi ct}{\ell}\right\}\cos\frac{n\pi x}{\ell}\right]$$

where

$$(P)_n = \beta^{s+1} - \frac{(s+1)\cdot s}{2!}\beta^{s-1}\left(\frac{n\pi c}{\ell}\right)^2 + \frac{(s+1)\cdot s\cdot(s-1)(s-2)}{4!}\beta^{s-3}\left(\frac{n\pi c}{\ell}\right)^4 - \cdots$$

$$(Q)_n = -(s+1)\beta^{s}\left(\frac{n\pi c}{\ell}\right) + \frac{(s+1)\cdot s\cdot(s-1)}{3!}\beta^{s-2}\left(\frac{n\pi c}{\ell}\right)^3 - \cdots ,$$

and $t \geq 0$.

For $s = 2$, these become

$$(P)_n = \beta^3 - 3\,\beta\left(\frac{n\pi c}{\ell}\right)^2$$

$$(Q)_n = -3\,\beta^2\left(\frac{n\pi c}{\ell}\right) + \left(\frac{n\pi c}{\ell}\right)^3$$

[redacted]

28. The infinite series portion of the solution converges because the terms decrease to zero and alternate in sign. From physical considerations it should be possible to differentiate the solution for the displacement partially with respect to time to obtain velocity and acceleration expressions. We note, however, that for large values of $\underline{n}$, the coefficients for the sine and cosine functions in position are of the order of $(1/n)^{s+2}$ at worst, so that in fact the solution and the first $\underline{s}$ derivatives are all absolutely and uniformly convergent for all $0 \leq x \leq \ell$, $0 \leq t < \infty$ and the differentiation is rigorously justified for $s \geq 2$. Since the solution satisfies the wave-equation and boundary conditions, it is complete.

29. From the solution (12) for the displacement, the expressions for velocity and acceleration are obtained straightforwardly as

$$(12b) \quad \frac{\partial u}{\partial t} = V_o + F_o \frac{c^2 \cdot s!}{AE \ell \beta^{s+1}} + F_o \cdot \frac{c}{AE} \frac{d^s}{d\lambda^s} \left[\frac{e^{\lambda t} \cosh \frac{\lambda x}{c}}{\sinh \frac{\lambda \ell}{c}} \right|_{\lambda = -\beta}$$

$$+ F_o \cdot \frac{2c \cdot s!}{AE} \sum_{n=1}^{\infty} \left[\frac{(-1)^n \frac{n\pi c}{\ell}}{n\pi \left\{ \beta^2 + \left(\frac{n\pi c}{\ell} \right)^2 \right\}^{s+1}} \left\{ (P)_n \cos \frac{n\pi ct}{\ell} - (Q)_n \sin \frac{n\pi ct}{\ell} \right\} \cos \frac{n\pi x}{\ell} \right]$$

and

$$(12c) \quad \frac{\partial^2 u}{\partial t^2} = F_o \frac{c}{AE} \frac{d^s}{d\lambda^s} \left[\frac{\lambda e^{\lambda t} \cosh \frac{\lambda x}{c}}{\sinh \frac{\lambda \ell}{c}} \right|_{\lambda = -\beta}$$

$$- F_o \frac{2c \cdot s!}{AE} \sum_{n=1}^{\infty} \left[\frac{(-1)^n \left(\frac{n\pi c}{\ell} \right)^2}{n\pi \left\{ \beta^2 + \left(\frac{n\pi c}{\ell} \right)^2 \right\}^{s+1}} \left\{ (P)_n \sin \frac{n\pi ct}{\ell} + (Q)_n \cos \frac{n\pi ct}{\ell} \right\} \cos \frac{n\pi x}{\ell} \right].$$

[redacted]

30. The displacement, velocity, and acceleration of the base is then obtained by setting $x = 0$ in (12a), (12b), and (12c) respectively. These expressions are in a form suitable for computation. In Plate V curves are presented which show the acceleration of the base to be expected from the assumed force on the nose. Curve I is for a projectile 1" long, $c = 16{,}600$ ft/sec, while curve II is for a projectile 3/4" long, $c = 16{,}600$ ft/sec. The force-function and the acceleration curves for the base are plotted as fractional values with respect to the maximum force on the nose, and the argument τ is the fractional value with respect to the time at which the maximum force occurs on the nose. It is seen that no acceleration of the base is "observed" until a half-period after impact $(\tau = 0)$; this is to be expected since that is the time necessary for the strain- or stress-wave to reach the trailing face from the nose. In other words, Eq. (12c) with $x = 0$ is a combined exponential and trigonometric series which is equal to zero for values of t between zero and ℓ/c inclusive and at the latter value rises smoothly to give the indicated curves for the two values of ℓ chosen. The time at which $F(t) = F_o t^2 e^{-\beta t}$ reached its maximum was taken to be 20×10^{-6} sec. for both bullet lengths. This made the calculations convenient for $\ell = 1"$ since there the fundamental period would be 10^{-5} sec.

31. As a result of these calculations which show the maxima and minima at equal time intervals, it is calculated from the period observed experimentally, reference (d), that the effective length of the projectiles used is approximately 7/8". They were actually 1" overall with an ogival height of 0.40" and an ogival

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radius of 2.5 times the calibre (.27), so that the result is favorable to the assumption that the projectiles behave like uniform rods.

D. Residual Velocity, Strain, and Energy of a Projectile After Impact.

32. For even moderately large values of t, an inspection of the Eqs. (12a,b,c) shows that the residues contributed by the pole at $\lambda = -\beta$ become vanishingly small on account of the factor $e^{-\beta t}$. This is merely a consequence of the fact that $F(t) = F_o t^s e^{-\beta t}$ becomes vanishingly small with time. By neglecting the residue at $\lambda = -\beta$, we are considering the solution, in effect, for times when the "impact" can be considered as over. The expression for velocity in Eq. (12b) therefore expresses residual velocity if we write

$$(13)\qquad \left.\frac{\partial u}{\partial t}\right|_{t \gg tm} \simeq V_o + F_o \frac{c^2 \cdot s!}{AE \ell \beta^{s+1}}$$

$$+ F_o \cdot \frac{2c^2 \cdot s!}{AE \ell} \sum_{n=1}^{\infty} \left[\frac{(-1)^n}{\left\{\beta^2 + \left(\frac{n\pi c}{\ell}\right)^2\right\}^{s+1}} \left\{ (P)_n \cos\frac{n\pi ct}{\ell} - (Q)_n \sin\frac{n\pi ct}{\ell} \right\} \cos\frac{n\pi x}{\ell} \right]$$

33. Hence, the projectile emerges from impact with a constant velocity and in a state of vibration which can best be described as giving a velocity distribution throughout the projectile which is varying with time in a manner described by the infinite compound trigonometric series. Similarly, there exists in the projectile a

[redacted]

residual dynamic state of strain which is expressed by the distribution

$$(14)\quad \left.\frac{\partial u}{\partial x}\right|_{t \gg t_m} \simeq -F_o \frac{2c\cdot s!}{AE\ell} \sum_{n=1}^{\infty} \left[\frac{(-1)^n}{\left\{\beta^2+\left(\frac{n\pi c}{\ell}\right)^2\right\}^{s+1}} \left\{ (P)_n \sin\frac{n\pi ct}{\ell} + (Q)_n \cos\frac{n\pi ct}{\ell} \right\} \sin\frac{n\pi x}{\ell} \right]$$

Note that $\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right)$. This corresponds with a difference in phase of 1/4 wave-length for both the time dependent and position dependent functions in the series.

34. With the expressions for velocity and strain distributions known, we are in a position to evaluate the residual energy of the projectile, particularly with the object of evaluating the energy of vibration. Because the total residual energy is distributed throughout the projectile, its value is determined by

$$(15)\quad H = \frac{1}{2} A\rho \int_0^{\ell} \left(\frac{\partial u}{\partial t}\right)^2 dx + \frac{1}{2} AE \int_0^{\ell} \left(\frac{\partial u}{\partial x}\right)^2 dx$$

in which the first integral gives the sum of the kinetic energies of translation and vibration and the second integral gives the potential energy of strain. We must find, of course, that H is a constant. From our approximation we have

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$$\frac{1}{2} A\rho \int_0^{\ell} \left(\frac{\partial u}{\partial t}\right)^2 dx \simeq \frac{1}{2} A \rho \int_0^{\ell} \left\{ \left(V_0 + F_0 \frac{c^2 \cdot s!}{AE \ell \beta^{s+1}}\right)^2 \right.$$

$$+ 2\left(V_0 + F_0 \frac{c^2 \cdot s!}{AE \ell \beta^{s+1}}\right) \left(F_0 \cdot \frac{2c^2 \cdot s!}{AE \ell} \sum_{n=1}^{\infty} \left[\frac{(-1)^n}{\left\{\beta^2 + \left(\frac{n\pi c}{\ell}\right)^2\right\}^{s+1}} \alpha_n(t) \cdot \cos \frac{n\pi x}{\ell} \right]\right)$$

$$+ \left(F_0 \cdot \frac{2c^2 \cdot s!}{AE \ell}\right)^2 \cdot \sum_{n=1}^{\infty} \left[\frac{\alpha_n^2(t)}{\left\{\beta^2 + \left(\frac{n\pi c}{\ell}\right)^2\right\}^{2s+2}} \cos^2 \frac{n\pi x}{\ell} \right]$$

$$\left. + \left(F_0 \cdot \frac{2c^2 \cdot s!}{AE \ell}\right)^2 \sum_{n=1}^{\infty} \sum_{\substack{m=1 \\ n \neq m}}^{\infty} \left[\frac{2\alpha_n(t) \cdot \alpha_m(t)}{\left\{\beta^2 + \left(\frac{n\pi c}{\ell}\right)^2\right\}^{s+1} \left\{\beta^2 + \left(\frac{m\pi c}{\ell}\right)^2\right\}^{s+1}} \cos \frac{n\pi x}{\ell} \cos \frac{m\pi x}{\ell} \right] \right\} dx$$

where $\alpha_n(t) = (P)_n \cos \frac{n\pi ct}{\ell} - (Q)_n \sin \frac{n\pi ct}{\ell}$,

and

$$\frac{1}{2} AE \int_0^{\ell} \left(\frac{\partial u}{\partial x}\right)^2 dx \simeq \frac{1}{2} AE \int_0^{\ell} \left(F_0 \cdot \frac{2c \cdot s!}{AE \ell}\right)^2 \sum_{n=1}^{\infty} \left[\frac{\beta_n^2(t)}{\left\{\beta^2 + \left(\frac{n\pi c}{\ell}\right)^2\right\}^{2s+2}} \sin^2 \frac{n\pi x}{\ell} \right] dx$$

$$+ \frac{1}{2} AE \int_0^{\ell} \left(F_0 \cdot \frac{2c \cdot s!}{AE \ell}\right)^2 \sum_{n=1}^{\infty} \sum_{\substack{m=1 \\ n \neq m}}^{\infty} \left[\frac{2\, \beta_n(t) \cdot \beta_m(t)}{\left\{\beta^2 + \left(\frac{n\pi c}{\ell}\right)^2\right\}^{s+1} \left\{\beta^2 + \left(\frac{m\pi c}{\ell}\right)^2\right\}^{s+1}} \sin \frac{n\pi x}{\ell} \sin \frac{m\pi x}{\ell} \right] dx ,$$

[redacted]

where $\beta_n(t) = (P)_n \sin \frac{n\pi ct}{\ell} + (Q)_n \cos \frac{n\pi ct}{\ell}$.

34. Now on account of the orthogonality of the functions $\sin \frac{n\pi x}{\ell}$ and $\cos \frac{n\pi x}{\ell}$ over the ground region $o \leqq x \leqq \ell$, it is seen that

$$\int_o^{\ell} \sin^2 \frac{n\pi x}{\ell}\, dx = \int_o^{\ell} \cos^2 \frac{n\pi x}{\ell}\, dx = \frac{1}{2}\ell$$

and

$$\int_o^{\ell} \cos \frac{n\pi x}{\ell}\, dx = \int_o^{\ell} \cos \frac{n\pi x}{\ell} \cos \frac{m\pi x}{\ell}\, dx = \int_o^{\ell} \sin \frac{n\pi x}{\ell} \sin \frac{m\pi x}{\ell}\, dx = o$$

Consequently,

$$\frac{1}{2} A\rho \int_o^{\ell} \left(\frac{\partial u}{\partial t}\right)^2 dx \simeq \frac{1}{2} A\rho \ell \left(V_o + F_o \frac{c^2 \cdot s!}{AE\ell \beta^{s+1}}\right)^2$$

$$+ \frac{1}{4} A\rho \ell \left(F_o \cdot \frac{2c^2 \cdot s!}{AE\ell}\right)^2 \sum_{n=1}^{\infty} \frac{\alpha_n^2(t)}{\left\{\beta^2 + \left(\frac{n\pi c}{\ell}\right)^2\right\}^{2s+2}}$$

and

$$\frac{1}{2} AE \int_o^{\ell} \left(\frac{\partial u}{\partial x}\right)^2 dx \simeq \frac{1}{4} AE \ell \left(F_o \cdot \frac{2c \cdot s!}{AE\ell}\right)^2 \sum_{n=1}^{\infty} \frac{\beta_n^2(t)}{\left\{\beta^2 + \left(\frac{n\pi c}{\ell}\right)^2\right\}^{2s+2}}$$

However,

$$A\rho\ell \left(F_o \cdot \frac{2c^2 \cdot s!}{AE\ell}\right)^2 = AE\ell \left(F_o \cdot \frac{2c \cdot s!}{AE\ell}\right)^2 \quad , \quad A\rho\ell = M ,$$

$E = \rho c^2$, and

$$\alpha_n^2(t) + \beta_n^2(t) = (P)_n^2 + (Q)_n^2$$

[redacted]

so that finally

$$(16) \qquad H \cong \frac{1}{2}M\left(V_o + F_o\frac{s!}{M\beta^{s+1}}\right)^2$$

$$+ F_o^2 \cdot \frac{(s!)^2}{M} \cdot \sum_{n=1}^{\infty} \frac{(P)_n^2 + (Q)_n^2}{\left\{\beta^2 + \left(\frac{n\pi c}{\ell}\right)^2\right\}^{2s+2}} \quad ,$$

which is constant and gives separately the energies of translation and vibration after "impact." The rate at which the vibrational terms are dissipated will be considered under the effect of damping.

C. Effect of Damping

35. It is generally observed that the amplitude of the oscillations in the force-time curves for the base diminish very rapidly, reference (d), so that not more than four periods can be distinguished during any one penetration. This can be accounted for almost entirely by loss of vibrational energy to the armor while it is being plastically deformed. The elastic-plastic interface presents a low reflection boundary at the nose of the projectile. An attempt to evaluate losses at this boundary is beyond the scope of this report. Our interest lies in evaluating the motion of the center of mass from that of the base of the projectile. Whatever stress impulse may be impressed on the nose, it cannot suffer loss in value by reflection at the nose until a time equal to a full vibration period later. The process of averaging over a full vibration

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period, therefore, will not be affected in principle by losses arising from reflections.

36. On the other hand, the damping of the amplitude of vibrations which arises from velocity resistance and internal viscosity will alter the vibration period as is well known. In extending the classical equation given by Lamb, reference (j), to include surface friction, care must be taken so that the damping will apply only to the <u>oscillatory</u> terms of the solution because if the solution is to represent the motion of the projectile during passage through the armor, it is only the oscillatory motion which suffers internal damping and velocity damping. (The diminution of the linear velocity is effected by the force-function.)

37. Under these conditions of damping, we let $2b$ be the generalized coefficient of viscosity and $2a$ the coefficient of velocity resistance, then the displacement function, $u(x,t)$, must now satisfy the equation

$$(17) \qquad \frac{\partial^2}{\partial x^2}\left[\frac{\partial u}{\partial t} + \frac{2b}{E}\frac{\partial^2 u}{\partial t^2}\right] = \frac{1}{c^2}\frac{\partial^3 u}{\partial t^3} + \frac{2a}{E}\frac{\partial^2 u}{\partial t^2} - \frac{2a}{EM}F(t)$$

subject to the boundary conditions

$$(18a) \qquad t = o: \quad u = o, \quad \partial u/\partial t = V_o, \quad \partial^2 u/\partial t^2 = o;$$

$$(18b) \qquad x = o: \quad \partial u/\partial x \Big|_{x=o} = o: \quad x = \ell: \quad \frac{\partial u}{\partial x}\Big|_{x=\ell} = F(t)/AE$$

[redacted]

38. Eq. (17) is derived and solved in the Appendix with the force-function $F(t) = F_o t^s e^{-\beta t}$. The formal solution is found to be (the sum of the residues of) the contour integral

$$(19)\quad u(x,t) = \frac{1}{2\pi i}\int_{B_r} e^{\lambda t}\left[\frac{V_o}{\lambda^2} + \frac{2a}{EM}\,\frac{\bar{F}(\lambda)}{\lambda\left(\frac{\lambda^2}{c^2}+\frac{2a\lambda}{E}\right)} + \frac{1}{AE}\,\frac{\bar{F}(\lambda)\cosh x\cdot\sqrt{\frac{\frac{\lambda^2}{c^2}+\frac{2a\lambda}{E}}{1+\frac{2b}{E}\lambda}}}{\sqrt{\frac{\frac{\lambda^2}{c^2}+\frac{2a\lambda}{E}}{1+\frac{2b}{E}\lambda}}\sinh \ell\sqrt{\frac{\frac{\lambda^2}{c^2}+\frac{2a\lambda}{E}}{1+\frac{2b}{E}\lambda}}}\right]d\lambda$$

where, for $\bar{F}(\lambda) = F_o\,\frac{s!}{(\lambda+\beta)^{s+1}}$, the explicit solution is

$$(20)\ u(x,t;a,b) = V_o t + F_o\cdot\frac{s!}{M\beta^{s+1}}\left(t-\frac{s+1}{\beta}\right) + F_o\cdot\frac{2a}{EM}\,\frac{d^s}{d\lambda^s}\left[\frac{e^{\lambda t}}{\lambda\left(\frac{\lambda^2}{c^2}+\frac{2a\lambda}{E}\right)}\right]_{\lambda=-\beta}$$

$$+ F_o\,\frac{1}{AE}\,\frac{d^s}{d\lambda^s}\left[\frac{e^{\lambda t}\cosh\ x\cdot\sqrt{\frac{\frac{\lambda^2}{c^2}+\frac{2a\lambda}{E}}{1+\frac{2b}{E}\lambda}}}{\sqrt{\frac{\frac{\lambda^2}{c^2}+\frac{2a\lambda}{E}}{1+\frac{2b}{E}\lambda}}\left\{\sinh \ell\cdot\sqrt{\frac{\frac{\lambda^2}{c^2}+\frac{2a\lambda}{E}}{1+\frac{2b}{E}\lambda}}\right\}}\right]_{\lambda=-\beta}$$

$$+ F_o\,\frac{2c^2\cdot s!}{AE\ell}\,e^{-\frac{a}{\rho}t}\sum_{n=1}^{\infty}\left[\frac{(-1)^n\ e^{-\frac{n^2\pi^2}{\ell^2}\frac{b}{\rho}t}}{\psi\left\{(\beta-\phi)^2+\psi^2\right\}^{s+1}}\left\{(P)'_n\sin\psi t+(Q)'_n\cos\psi t\right\}\cos\frac{n\pi x}{\ell}\right]$$

$$+ \frac{2b}{E}F_o\cdot\frac{2c^2\cdot s!}{AE\ell}\,e^{-\frac{a}{\rho}t}\sum_{n=1}^{\infty}\left[\frac{(-1)^n\ e^{\frac{n^2\pi^2}{\ell^2}\frac{b}{\rho}t}}{\psi\left\{(\beta-\phi)^2+\psi^2\right\}^{s+1}}\left\{(P)'_n\sin(\alpha-\psi t)+(Q)'_n\cos(\alpha-\psi t)\right\}\cos\frac{n\pi x}{\ell}\right]$$

[redacted]

where

$$\alpha = \arctan\left(\frac{\pm\psi}{\phi}\right) ,$$

$$\phi = \frac{1}{\rho}\left(a + \frac{n^2\pi^2}{\ell^2} b\right) ,$$

$$\psi = \sqrt{\left(\frac{n\pi c}{\ell}\right)^2 - \left(\frac{1}{\rho}\right)^2\left(a + \frac{n^2\pi^2}{\ell^2} K\right)^2} ,$$

$$(P)_n' = (\beta-\phi)^{s+1} - \frac{(s+1)\cdot s}{2!}(\beta-\phi)^{s-1}\psi^2 + \frac{(s+1)\cdot s.(s-1)(s-2)}{4!}(\beta-\phi)^{s-3}\psi^4 \ldots$$

and $(O)_n' = -(s+1)(\beta-\phi)^s\cdot\psi + \frac{(s+1)\cdot s\cdot(s-1)}{3!}(\beta-\phi)^{s-2}\psi^3 - \ldots ,$

each of $(P)_n'$ and $(O)_n'$ terminating when the exponent of $(\beta-\phi)$ is unity or zero.

39. The solution (20) reduces to (12) when $a = b = o$, as it must. We note the period has been increased by the factor $\left\{1 - \left(\frac{\frac{a\ell}{n\pi} + b}{\rho c}\right)^2\right\}^{-1/2}$ because of the damping. This is necessarily small enough to have little effect on the natural period. The most interesting observation to be drawn from the form of the solution is that while the damping factor $e^{-\frac{a}{\rho}t}$ applies equally to all frequencies, the damping factor arising from viscosity is increasingly effective on the higher frequencies since in the factor $e^{-\frac{b}{\rho}\frac{n^2\pi^2}{\ell^2}t}$ we see that $n \rightarrow \infty$ through all integers.

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APPENDIX

The Partial Differential Equation for Viscous and Velocity Damping of the Longitudinal Vibrations of a Projectile During Impact.

In this discussion we let <u>2b</u> denote a generalized viscosity coefficient such that if the strain at position <u>x</u> in a rod is $\partial u/\partial x$, then the stress across the face is given by

$$\sigma_x = E \cdot \partial u/\partial x + 2\,b \cdot \partial/\partial t\,(\partial u/\partial x)$$

where the term $2b \dfrac{\partial}{\partial t}\left(\dfrac{\partial u}{\partial x}\right) = 2b \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial t}\right)$ is the familiar stress arising from strain-rate. The stress across the face at <u>x + dx</u> is $\sigma_x + (\partial \sigma_x/\partial x)\,dx$, so that the net force on the infinitesimal cross-section is

$$\text{(A1)} \qquad A \cdot \frac{\partial}{\partial x}\left\{E\,\frac{\partial u}{\partial x} + 2b\,\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right)\right\} dx$$

This net force will be equivalent to the kinetic reaction plus frictional resistance. The latter is supposed to arise only from the relative displacement of the infinitesimal cross-section with respect to its <u>center of oscillation</u>. We let <u>2a</u> be the coefficient of this oscillatory velocity resistance per unit volume. Now from our definition of <u>u</u> as a displacement from original position, i.e.: $u(x,t)\big|_{t=o} = o$, it follows that $u(x,t) - \left(V_o t + \dfrac{1}{M}\displaystyle\int_o^t dt \int_o^t F(t)dt\right)$ is the relative

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displacement at the time t after impact (see Theorem I). The frictional resistance to the oscillatory motion is then

$$2a \frac{\partial}{\partial t} \left(u - V_o t - \frac{1}{M} \int_o^t dt \int_o^t F(t)\, dt \right) A dx.$$

The total reaction of the elementary cross-section $\underline{Adx}$ is

$$(A2) \qquad A\rho \frac{\partial^2 u}{\partial t^2} dx + 2a \frac{\partial}{\partial t} \left(u - V_o t - \frac{1}{M} \int_o^t dt \int_o^t F(t) dt \right) A dx.$$

Equating (A1) and (A2) and differentiating with respect to time, we obtain the equation

$$(A3) \qquad \frac{\partial^2}{\partial x^2} \left(\frac{\partial u}{\partial t} + \frac{2b}{E} \frac{\partial^2 u}{\partial t^2} \right) = \frac{1}{c^2} \frac{\partial^3 u}{\partial t^3} + \frac{2a}{E} \frac{\partial^2 u}{\partial t^2} - \frac{2a}{EM} F(t)$$

with the boundary conditions

$$t = o: \quad u(x,t)\Big|_{t=o} = o, \quad \frac{\partial u}{\partial t}\Big|_{t=o} = V_o , \quad \frac{\partial^2 u}{\partial t^2}\Big|_{t=o} = o ,$$

$$x = o: \quad \frac{\partial u}{\partial x}\Big|_{x=o} = o ; \quad x = \ell : \quad \frac{\partial u}{\partial x}\Big|_{x=\ell} = F(t)/AE$$

We have an initial condition on acceleration because (A3) is of the third order in time and we will assume $F(o) = o$.

[redacted]

The Laplace transform of (A3) subject to the initial conditions in time is found to be

$$(A4)\quad \frac{d^2\bar{u}}{dx^2} - \left\{ \frac{\frac{\lambda^2}{c^2} + \frac{2a\lambda}{E}}{1 + \frac{2b}{E}\lambda} \right\} \bar{u} = -V_o \left\{ \frac{\frac{1}{c^2} + \frac{2a}{E}\frac{1}{\lambda}}{1 + \frac{2b}{E}\lambda} \right\} - \frac{2a}{EM} \frac{\bar{F}(\lambda)}{\lambda\left(1 + \frac{2b}{E}\lambda\right)}$$

With the end conditions

$$x = o:\ \left.\frac{d\bar{u}}{dx}\right|_{x=o} = o \quad ; \quad x = \ell :\ \left.\frac{d\bar{u}}{dx}\right|_{x=\ell} = \bar{F}(\lambda)/AE$$

The solution for (A4) is verified readily to be

$$(A5)\quad \bar{u}(x,\lambda) = \frac{V_o}{\lambda^2} + \frac{2a}{EM} \frac{\bar{F}(\lambda)}{\lambda\left(\frac{\lambda^2}{c^2} + \frac{2a\lambda}{E}\right)} + \frac{1}{AE} \frac{\bar{F}(\lambda)\cdot\cosh\left\{x\cdot\sqrt{\frac{\frac{\lambda^2}{c^2} + \frac{2a\lambda}{E}}{1 + \frac{2b}{E}\lambda}}\right\}}{\sqrt{\frac{\frac{\lambda^2}{c^2} + \frac{2a\lambda}{E}}{1 + \frac{2b}{E}\lambda}}\cdot\sinh\left\{\ell\cdot\sqrt{\frac{\frac{\lambda^2}{c^2} + \frac{2a\lambda}{E}}{1 + \frac{2b}{E}\lambda}}\right\}}$$

which is the Laplace transform of the displacement function with respect to time. The function $u(x,t)$ is then determined as the sum of the residues of the contour integral

$$(A6)\quad u(x,t;a,b) = \frac{1}{2\pi i}\int_{B_r} e^{\lambda t}\,\bar{u}(x,\lambda)\,d\lambda$$

[redacted]

in which $\bar{u}(x,\lambda)$ is given by (A5) and the contour is the same as that given in Plate 2.

The poles of the integrand are found at $\lambda = 0$, $-2a/\rho$, the singularities of $\bar{F}(\lambda)$, and $-\phi \pm i\psi$, where $\phi = \frac{1}{\rho}\left(a + \frac{n^2\pi^2}{\ell^2}b\right)$ and $\psi = \sqrt{\frac{n^2\pi^2c^2}{\ell^2} - \frac{1}{\rho^2}\left(a + \frac{n^2\pi^2}{\ell^2}b\right)^2}$.

For the assumed force-function $F(t) = F_0 t^s e^{-\beta t}$, we have as before: $\bar{F}(\lambda) = F_0 \cdot s!/(\lambda+\beta)^{s+1}$. The pole contributed by $\bar{F}(\lambda)$ is then $\lambda = -\beta$ and is multiple to the order <u>s+1</u> .

<u>Calculation of the Residues:</u>

The residues are found by the usual methods of expansion or differentiation. We state only the results in most cases but give some details in the case of the poles distributed all along the upper and lower branches of the function in the complex λ-plane whose parametric representation in terms of <u>n</u> is

$$-\xi \pm i\eta : \begin{cases} \xi = \frac{1}{\rho}\left(a + \frac{n^2\pi^2}{\ell^2}b\right) \\ \eta = \sqrt{\frac{n^2\pi^2c^2}{\ell^2} - \frac{1}{\rho^2}\left(a + \frac{n^2\pi^2}{\ell^2}b\right)^2} \end{cases}$$

——— o ———

[redacted]

$\underline{\lambda = 0}$:

$$V_0 t + F_0 \cdot \frac{s!}{M\beta^{s+1}} \left(t - \frac{s+1}{\beta} - \frac{\rho}{2a} \right) + F_0 \frac{s!}{M\beta^{s+1}} \cdot \frac{\rho}{2a}$$

$$= V_0 t + F_0 \frac{s!}{M\beta^{s+1}} \left(t - \frac{s+1}{\beta} \right)$$

$\underline{\lambda = -\beta}$:

$$F_0 \cdot \frac{2a}{EM} \cdot \frac{d^s}{d\lambda^s} \left[\frac{e^{\lambda t}}{\lambda \left(\frac{\lambda^2}{c^2} + \frac{2a\lambda}{E} \right)} \right|_{\lambda = -\beta}$$

$$+ F_0 \cdot \frac{1}{A^2} \frac{d^s}{d\lambda^s} \left[\frac{e^{\lambda t} \cdot \cosh \left\{ x \cdot \sqrt{\frac{\frac{\lambda^2}{c^2} + \frac{2a\lambda}{E}}{1 + \frac{2b}{E}\lambda}} \right\}}{\sqrt{\frac{\frac{\lambda^2}{c^2} + \frac{2a\lambda}{E}}{1 + \frac{2b}{E}\lambda}} \cdot \sinh \left\{ x \cdot \sqrt{\frac{\frac{\lambda^2}{c^2} + \frac{2a\lambda}{E}}{1 + \frac{2b}{E}\lambda}} \right\}} \right|_{\lambda = -\beta}$$

$\underline{\lambda = -2a/\rho}$:

Residues add to zero.

$\underline{\lambda = -\theta \pm i\omega}$:

The residues arising from the complex conjugate poles will be treated together. Now the complex poles are only contributed by the term in which the hyperbolic sine appears in the denominator which is

[redacted]

zero for values of its argument equal to $\pm n\pi i$, $n = 1, 2, 3, \cdots$ (The pole arising at $\lambda = o$ includes the case $n = o$.) With this in mind, we find the residues at $-\phi \pm i\psi$ to be expressed differentially as

$$F_o \cdot \frac{s!}{AE} \; \frac{e^{-\phi t} \cdot \left\{\beta - \phi \mp i\psi\right\}^{s+1} \cdot e^{\pm i\psi t} \cdot \cos\frac{n\pi x}{\ell}}{\pm \frac{n\pi i}{\ell}\left\{(\beta - \phi)^2 + \psi^2\right\}^{s+1} \; \frac{d}{d\lambda}\left\{\sinh \ell \cdot \sqrt{\frac{\frac{\lambda^2}{c^2} + \frac{2a\lambda}{E}}{1 + \frac{2b}{E}\lambda}}\right\}_{\lambda = -\phi \pm i\psi}}$$

Differentiating the hyperbolic sine function, we get:

$$(-1)^n \left[\frac{\ell}{2}\left(\frac{\frac{\lambda^2}{c^2} + \frac{2a\lambda}{E}}{1 + \frac{2b}{E}\lambda}\right)^{-1/2} \left\{\frac{2\left(\frac{\lambda}{c^2} + \frac{a}{E}\right)}{1 + \frac{2b}{E}\lambda} - \frac{\frac{2b}{E}\left(\frac{\lambda^2}{c^2} + \frac{2a\lambda}{E}\right)}{\left(1 + \frac{2b}{E}\lambda\right)^2}\right\} \right]_{\sqrt{\frac{\frac{\lambda^2}{c^2} + \frac{2a\lambda}{E}}{1 + \frac{2b}{E}\lambda}} = \pm\frac{n\pi i}{\ell}, \; \lambda = -\phi \pm i\psi}$$

$$= (-1)^n \frac{1}{\pm n\pi i} \cdot \frac{\ell^2}{1 + \frac{2b}{E}(-\phi \pm i\psi)} \left[\frac{-\phi \pm i\psi}{c^2} + \frac{a}{E} + \frac{b}{E} \cdot \frac{n^2\pi^2}{\ell^2}\right]$$

$$= (-1)^n \cdot \frac{1}{\pm n\pi i} \cdot \frac{\ell^2}{1 + \frac{2b}{E}(-\phi \pm i\psi)} \cdot \frac{\pm i\psi}{c^2} .$$

[redacted]

Substituting this in the residues above and making the necessary cancellations, we obtain

$$F_o \cdot \frac{c^2 \cdot s!}{AE} \cdot \frac{\dfrac{e^{-\phi t} \cdot (-1)^n \cdot (\mp i) \{\beta - \phi \mp i\psi\}^{s+1} \cdot e^{\pm i\psi t}}{\ell}}{1 + \dfrac{2b}{E}(-\phi \pm i\psi) \cdot \psi \cdot \left\{(\beta - \phi)^2 + \psi^2\right\}^{s+1}} \cdot \cos\frac{n\pi x}{\ell}$$

$$= F_o \cdot \frac{c^2 \cdot s!}{AE\,\ell} \cdot \frac{(-1)^n (\mp i) \cdot e^{-\phi t} \cdot \{\beta - \phi \mp i\psi\}^{s+1} \cdot e^{\pm i\psi t}}{\psi \left\{(\beta - \phi)^2 + \psi^2\right\}^{s+1}} \cdot \cos\frac{n\pi x}{\ell}$$

$$- \frac{2b}{E} \cdot F_o \cdot \frac{c^2 \cdot s!}{AE\,\ell} \cdot \frac{(-1)^n (\mp i)(\phi \mp i\psi)(\beta - \phi \mp i\psi)^{s+1} \cdot e^{\pm i\psi t}\, e^{-\phi t}}{\psi \left\{(\beta - \phi)^2 + \psi^2\right\}^{s+1}} \cdot \cos\frac{n\pi x}{\ell}$$

We can write $(\phi \mp i\psi)\, e^{\pm i\psi t} = e^{\pm i(\psi t - \alpha)}$, where $\phi = \cos\alpha$, $\psi = \sin\alpha$, $\alpha = \arctan \psi/\phi$. The residues become

$$F_o \cdot \frac{c^2 \cdot s!}{AE\,\ell} \cdot \frac{(-1)^n e^{-\phi t} (\mp i)\, (\beta - \phi \mp i\psi)^{s+1} \cdot e^{\pm i\psi t}}{\psi \left\{(\beta - \phi)^2 + \psi^2\right\}^{s+1}} \cdot \cos\frac{n\pi x}{\ell}$$

$$+ \frac{2b}{E} \cdot F_o \cdot \frac{c^2 \cdot s!}{AE\,\ell} \cdot \frac{(-1)^n \cdot e^{-\phi t} \cdot (\mp i)(\beta - \phi \mp i\psi)^{s+1}\, e^{\pm i(\psi t - \alpha + \pi)}}{\psi \left\{(\beta - \phi)^2 + \psi^2\right\}^{s+1}} \cdot \cos\frac{n\pi x}{\ell} .$$

[redacted]

The addition of all residues with those on the imaginary part of the plane added in complex conjugate pairs gives the solution

(A7) $u(x,t;a,b) =$

$$V_o t + F_o \cdot \frac{s!}{M\beta^{s+1}}\left(t - \frac{s+1}{\beta}\right) + F_o \cdot \frac{2a}{EM}\frac{d^s}{d\lambda^s}\left[\frac{e^{\lambda t}}{\lambda\left(\frac{\lambda^2}{c^2} + \frac{2a\lambda}{E}\right)}\right]_{\lambda = -\beta}$$

$$+ F_o \cdot \frac{1}{AE} \cdot \frac{d^s}{d\lambda^s}\left[\frac{e^{\lambda t} \cdot \cosh\left\{x \cdot \sqrt{\dfrac{\frac{\lambda^2}{c^2} + \frac{2a\lambda}{E}}{1 + \frac{2b}{E}\lambda}}\right\}}{\left\{\sqrt{\dfrac{\frac{\lambda^2}{c^2} + \frac{2a\lambda}{E}}{1 + \frac{2b}{E}\lambda}}\right\} \sinh\left\{\ell \cdot \sqrt{\dfrac{\frac{\lambda^2}{c^2} + \frac{2a\lambda}{E}}{1 + \frac{2b}{E}\lambda}}\right\}}\right]_{\lambda = -\beta}$$

$$+ F_o \cdot \frac{2(s!)}{M} \cdot e^{-\frac{a}{\beta}t} \cdot \sum_{n=1}^{\infty}\left[\frac{(-1)^n e^{-\frac{n^2\pi^2}{\ell^2}\frac{b}{\beta}t}}{\psi\left\{(\beta-\phi)^2 + \psi^2\right\}^{s+1}}\left\{(P)_n' \sin\psi t + (Q)_n' \cos\psi t\right\} \cos\frac{n\pi x}{\ell}\right]$$

$$+ F_o \cdot \frac{2b}{E} \cdot \frac{2(s!)}{M} e^{-\frac{a}{\beta}t} \sum_{n=1}^{\infty}\left[\frac{(-1)^n e^{-\frac{n^2\pi^2}{\ell^2}\frac{b}{\beta}t}}{\psi\left\{(\beta-\phi)^2 + \psi^2\right\}^{s+1}}\left\{(P)_n' \sin(\psi t - \alpha + \pi) + (Q)_n' \cos(\psi t - \alpha + \pi)\right\} \cos\frac{n\pi x}{\ell}\right]$$

where

$$(P)_n' = (\beta-\phi)^{s+1} - {}^{s+1}c_2 \cdot (\beta-\phi)^{s-1} \cdot \psi^2 + {}^{s+1}c_4 \cdot (\beta-\phi)^{s-3} \cdot \psi^4 - \cdots,$$

$$(Q)_n' = -(s+1) \cdot (\beta-\phi)^s \cdot \psi + {}^{s+1}c_3 \cdot (\beta-\phi)^{s-2} \cdot \psi^3 - {}^{s+1}c_5 \cdot (\beta-\phi)^{s-4}\psi^5 + \cdots$$

terminating when the exponent of $(\beta-\phi)$ is unity or zero.

[redacted]

Analytic Representation of a Projectile Piercing Armor

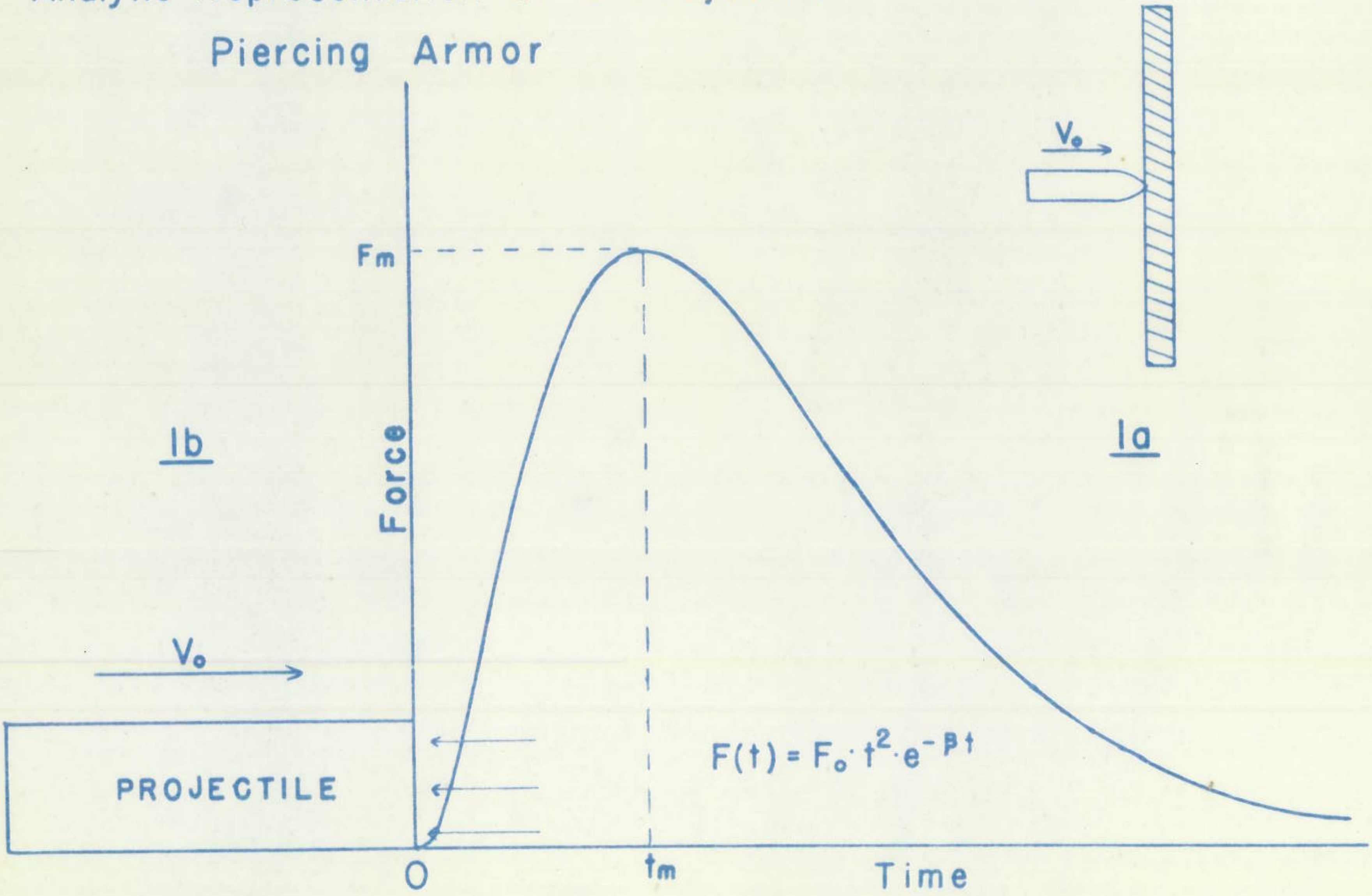

PLATE I

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Contour in the Complex Plane

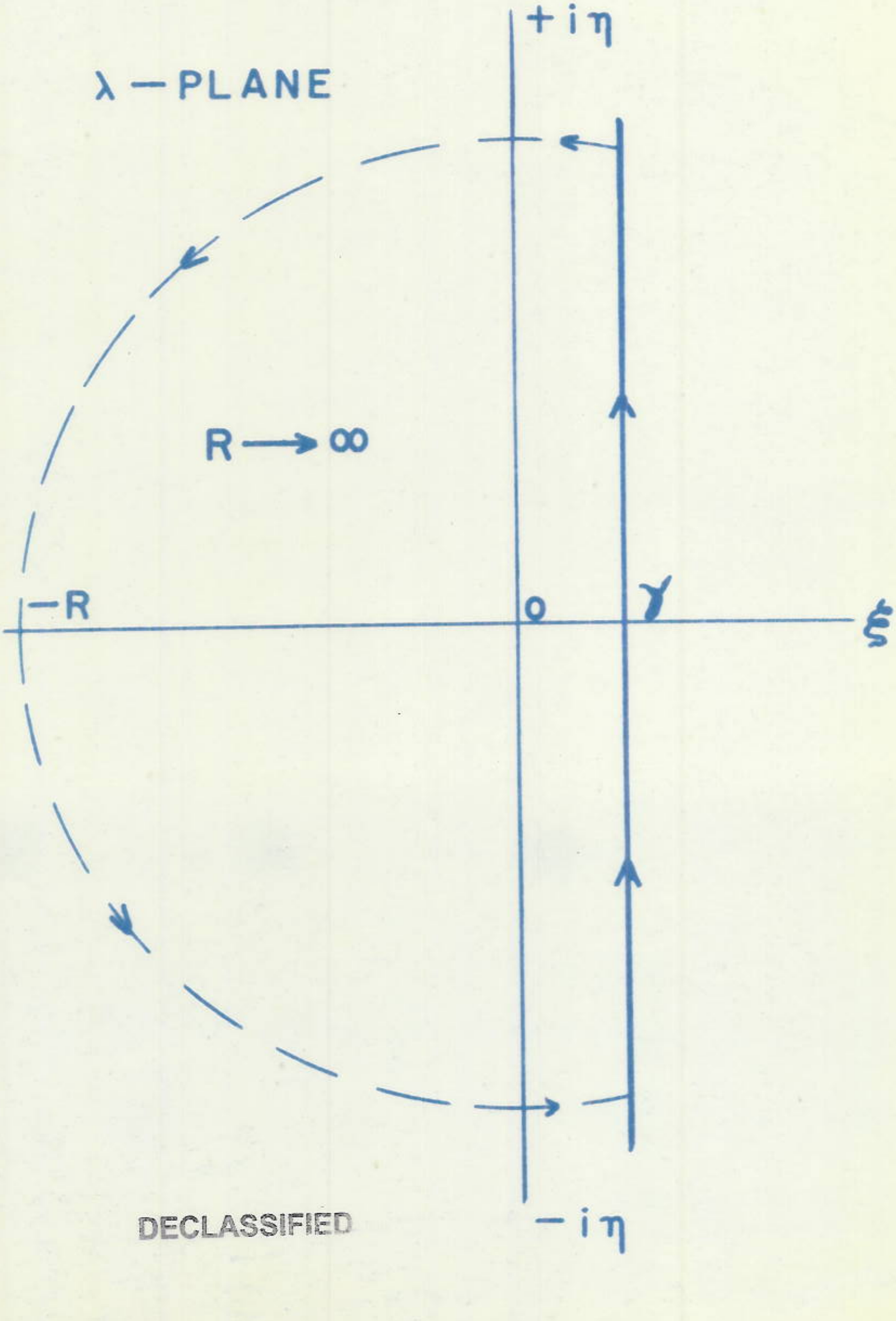

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PLATE II

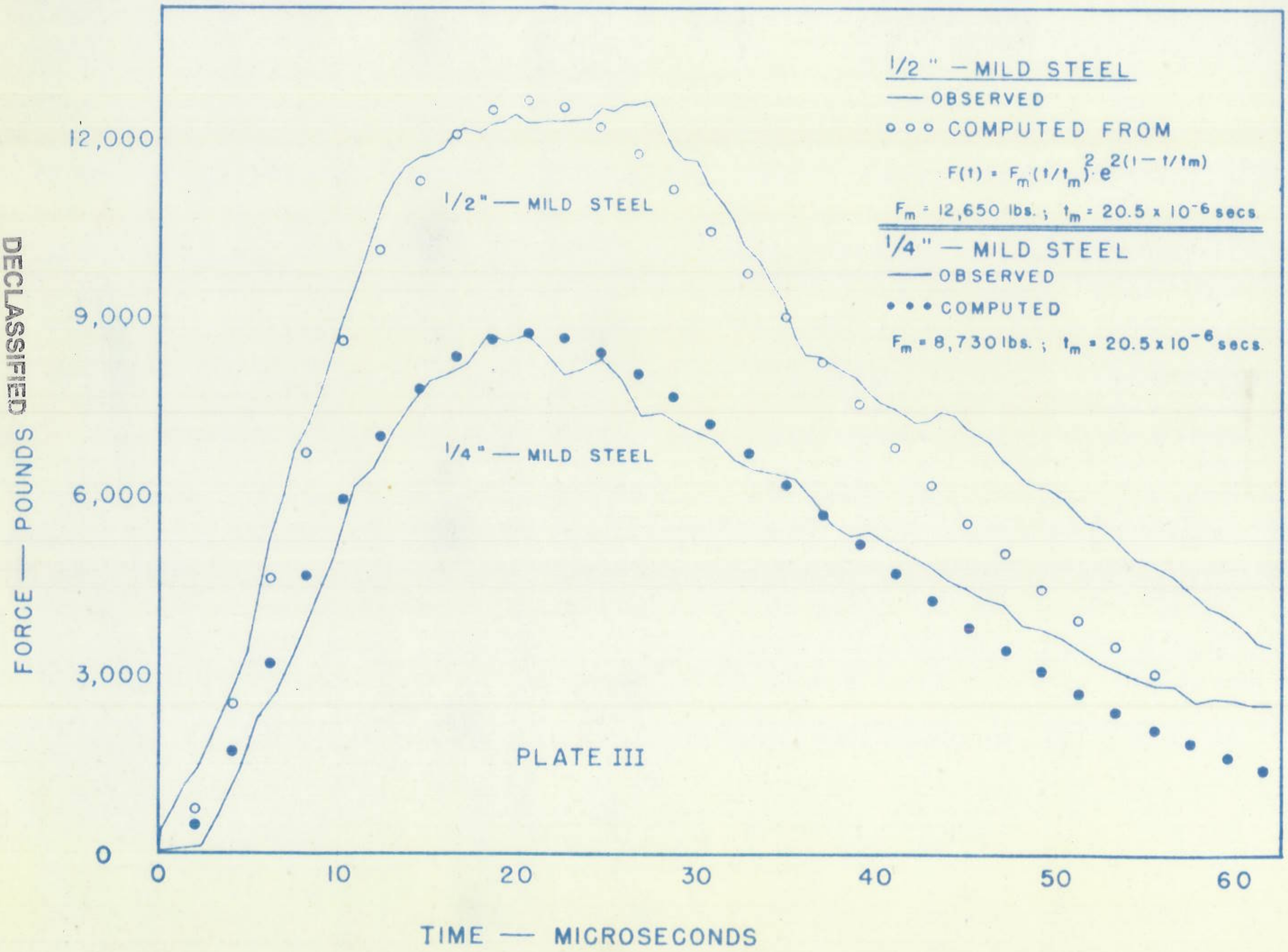
1/2" — MILD STEEL
— OBSERVED
○○○ COMPUTED FROM
$F(t) = F_m (t/t_m)^2 \cdot e^{2(1-t/tm)}$
F_m = 12,650 lbs. ; t_m = 20.5 x 10^{-6} secs.
1/4" — MILD STEEL
— OBSERVED
●●● COMPUTED
F_m = 8,730 lbs. ; t_m = 20.5 x 10^{-6} secs.
1/2" — MILD STEEL
1/4" — MILD STEEL
FORCE — POUNDS
0
3,000
6,000
9,000
12,000
TIME — MICROSECONDS
0
10
20
30
40
50
60

PLATE III

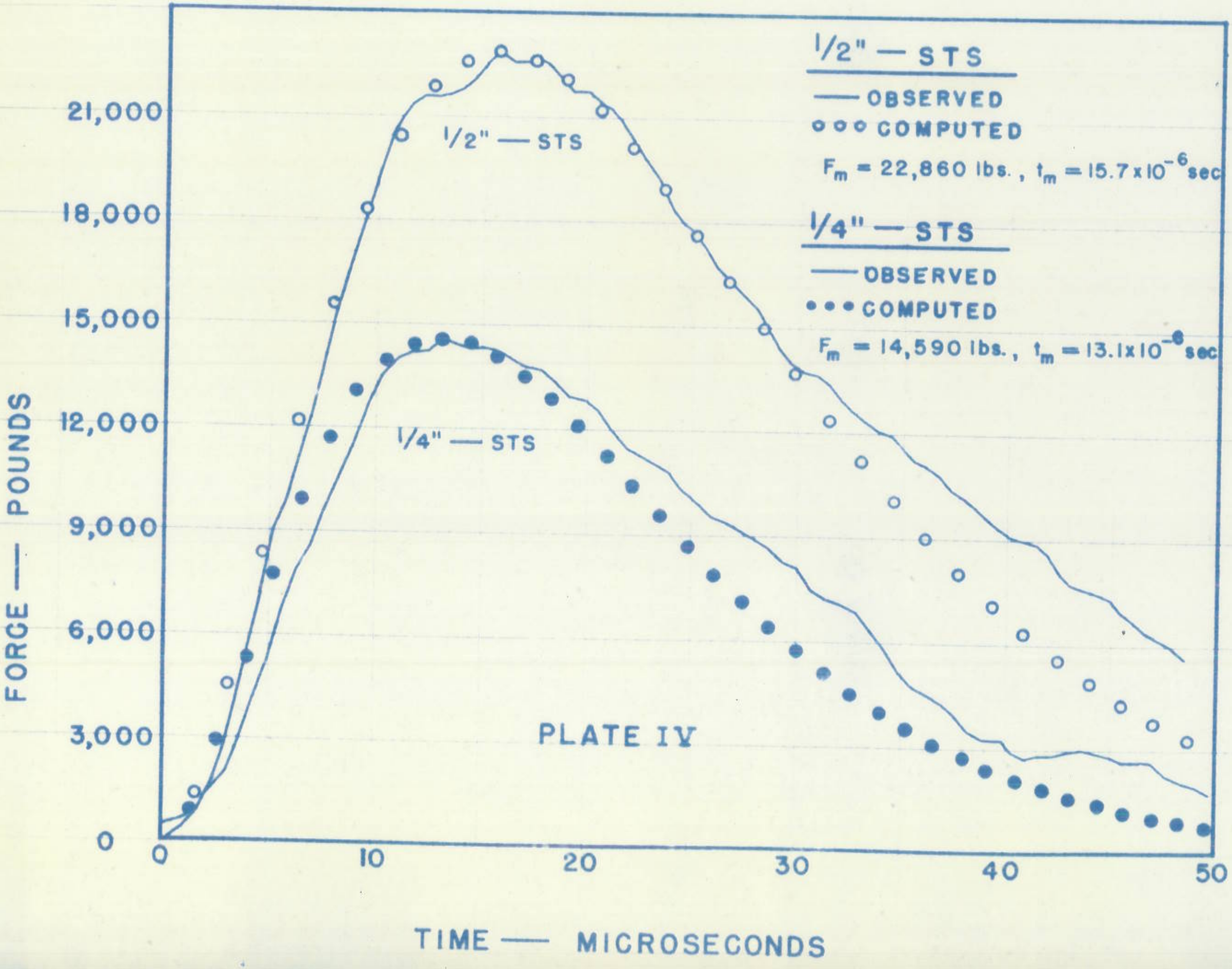
1/2" — STS
— OBSERVED
○○○ COMPUTED
$F_m = 22{,}860$ lbs., $t_m = 15.7 \times 10^{-6}$ sec
1/4" — STS
— OBSERVED
●●● COMPUTED
$F_m = 14{,}590$ lbs., $t_m = 13.1 \times 10^{-6}$ sec
1/2" — STS
1/4" — STS
21,000
18,000
15,000
12,000
9,000
6,000
3,000
0
FORCE — POUNDS
0
10
20
30
40
50
TIME — MICROSECONDS

PLATE IV

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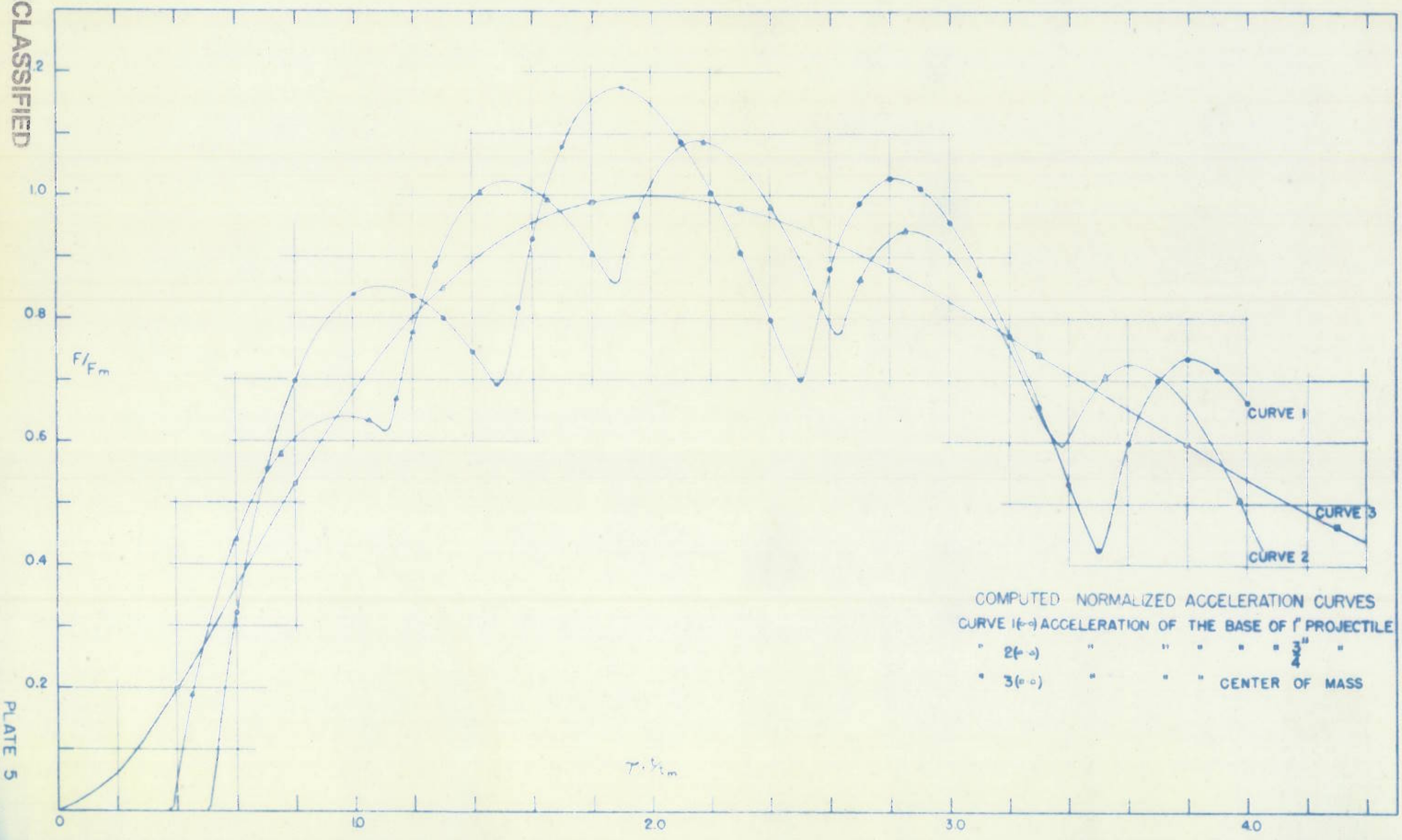

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PLATE 5