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Time-Dependent Generalized Normalized Radar Cross Section for Improved SAR Performance

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14. ABSTRACT

The time-dependent Generalized Normalized Radar Cross Section (G-NRCS) is introduced using its time-harmonic version in conjunction with the frozen-surface assumption. The linear ocean-surface model is utilized along with the associated statistical relationships to yield an expression for a fully developed sea. The ratio of the surface displacement and the EM wavelength of the incident field is assumed to be very small compared to unity. Based on this property, the Small Amplitude Approximation (SAA) is employed to describe the scattering behavior of the sea surface and, thus, the Scattering Matrix for a terrain chosen with its necessary components defined in detail. These expressions are employed here to yield the time-dependent G-NRCS under SAA. This quantity is should be well suited for Synthetic Aperture Radar (SAR) applications where the sea-surface reflectivity is required at every point within the antenna footprint corresponding to a given pixel in a SAR image of the ocean surface. The application of the expressions derived here holds the promise of improved SAR imaging performance for the detection of ships in sea-clutter environments.

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EXECUTIVE SUMMARY

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TIME-DEPENDENT GENERALIZED NORMALIZED RADAR CROSS SECTION FOR IMPROVED SAR PERFORMANCE

INTRODUCTION

The purpose of this memorandum is to develop an expression for the time-dependent Generalized Normalized Radar Cross Section (G-NRCS) that can be used to evaluate the reflectivity of every point on the ocean surface inscribed within an antenna footprint. This expression could help to improve the performance of Synthetic Aperture Radar (SAR) in imaging the sea surface or in the detection of ships in sea-clutter for a given sea state by providing an estimate of the ambient sea clutter reflectivity. The G-NRCS is a quantity that relates the scattering, kinematic, and stochastic properties of the ocean surface in terms of the time-averaged power as measured at the receive antenna port. The quantity has been shown to characterize the spatial coherence of the time-harmonic rough-surface backscattered field for the case when the incident or radiated Electromagnetic (EM) Field is not a plane wave. The conventional means of modeling variations of the local NRCS is based on the linearized sea-surface representation (which is utilized herein). In addition, the linearized sea-surface displacement is related to the variations of the NRCS via the Real Aperture Radar (RAR) modulation transfer function [5]. This quantity is EM polarization, hydrodynamically, and geometrically dependent, which are essential quantities for evaluating NRCS. However, the incident field used in the conventional method is a plane wave and is therefore lacking. In this memorandum report, the expression for the time-dependent G-NRCS will be cast in the conventional framework, but the basis of the derivation will be generated from [2], which employs the antenna pattern function and not a single plane wave as the incident field. In [1-4], the small-slope approximation was examined for the time-harmonic G-NRCS. Here, the small-amplitude approximation in conjunction with the formulism used in [2] will be employed to derive the time-dependent G-NRCS. In addition, the modified form of the RAR modulation transfer function is presented wherein polarization, geometricaland kinematic dependent components will be referenced to the sea surface illuminated within the antenna footprint.

The G-NRCS is derived here using its time-harmonic version in conjunction with the frozen-surface assumption. The linear ocean-surface model is introduced along with associated statistical relationships to yield an expression for a fully developed sea. The ratio of the surface displacement and the EM wavelength of the incident field is assumed to be very small compared to unity. Based on this property, the Small Amplitude Approximation (SAA) is chosen to describe the scattering behavior of the sea surface and, thus, the scattering matrix for a terrain chosen with its necessary components defined in detail. Finally, all of these expressions are employed to yield the time-dependent G-NRCS under the SAA. This quantity is best suited for Synthetic Aperture Radar (SAR) applications where the sea-surface reflectivity is required at every point within the antenna footprint corresponding to a given pixel in a SAR image of the ocean surface.

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ANTENNA RESPONSE FROM TIME-VARYING ROUGH SURFACES AND G-NRCS

In order to simulate the reflectivity of radar returns from terrain or rough surfaces, an expression for the time-dependent G-NRCS is introduced in this memorandum. The formulism and definitions adopted in the previous literature will be utilized here (see [1]). Thus, where appropriate, one should seek the explanations discussed in [1- 8] for further details regarding the derivation of the expressions in the following. However, for the sake of consistency, the scattering geometry depicted in Fig. 1 will briefly be described. An antenna at height H above the mean displacement of a (statistically) rough surface $\varsigma(x,y)$ (or an ocean surface as in this application), whose Cartesian coordinate system serves as a reference for the spherical coordinates θ and ϕ as indicated by the axes shown in red, is illustrated in Fig. 1.

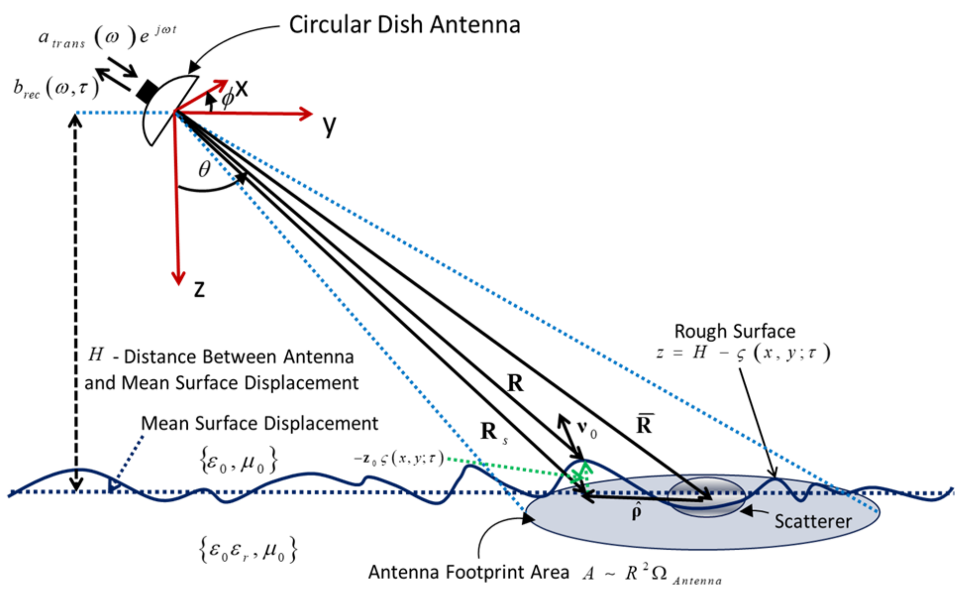

Fig. 1 — Description of the scattering geometry

The antenna can be steered to $\theta = \bar{\theta}$ in elevation and $\phi = \bar{\phi}$ in azimuth. The antenna gain will have its maximum at $\theta = \bar{\theta}$, $\phi = \bar{\phi}$ corresponding to a position-vector $\bar{\mathbf{R}}$. A second position vector $\mathbf{R}$ is directed from the phase center of the antenna to a given point on the rough surface with magnitude $R = \sqrt{x^2 + y^2 + (H - \varsigma(x,y))^2}$ and $R = R_s \equiv \sqrt{x^2 + y^2 + H^2}$ for $\varsigma(x,y) = 0$. The outward-surface normal vector is defined as $\mathbf{v}_0 = -(\mathbf{z}_0 + \nabla_t\varsigma)\Big/\sqrt{\left(1 + |\nabla_t\varsigma|^2\right)}$, where the transverse gradient is defined as $\nabla_t = \{\partial/\partial x, \partial/\partial y\}$. The area of the antenna footprint is defined as $A \sim R^2\Omega_{Antenna}$ with $\Omega_{Antenna}$ being the solid angle that subtends the radiated field on the surface.

THE GENERALIZED ANTENNA RECIPROCITY RELATIONSHIP (GARR) FOR TERRAIN

In order to correctly represent the reception of the rough-surface backscattered field by an antenna, the Generalized Antenna Reciprocity Relationship (GARR) [4], that predicts the response due to an incident arbitrary time-harmonic EM field in free space, must first be considered. From [6, 8], the GARR for a single angular frequency ω is given as:

$$a_{trans} b_{rec} = \frac{-j}{2\pi k_0 \sqrt{Z_0}} \int_{\left|\mathbf{k}_t'\right|^2 \le k_0^2} \mathbf{F}_{rad}\left(\mathbf{k}_t'\right) \cdot \widetilde{\mathbf{E}}_{scat}\left(\mathbf{k}_t'\right) d^2\mathbf{k}_t'. \tag{1}$$

In (1), the local antenna-port scattering parameters, a_{trans} and b_{rec}, are the transmitter output and antenna response to the backscattered field, respectively. The non-local EM-field quantities $\mathbf{F}_{rad}\left(\mathbf{k}_t'\right)$ and $\widetilde{\mathbf{E}}_{scat}\left(\mathbf{k}_t'\right)$ represent the far-field vector radiation amplitude of the antenna in transmission and the plane-wave spectrum of the incoming (rough-surface scattered) field, where the transverse wave-number vector is $\mathbf{k}_t' = \mathbf{x}_0 k_x' + \mathbf{y}_0 k_y'$. Also, $k_0 = \omega\sqrt{\mu_0 \varepsilon_0}$ is the free-space wave number and $Z_0 = \sqrt{\mu_0 / \varepsilon_0}$ is the free-space characteristic impedance. The single integral, which is to be carried out over the visible portion of the radiated-field spectrum $\left|\mathbf{k}_t'\right|^2 \le k_0^2$, is shorthand for a double-integral operator and $d^2\mathbf{k}_t' = dk_x' dk_y'$ is the associated differential variance. Note that $\widetilde{\mathbf{E}}_{scat}\left(\mathbf{k}_t'\right)$ is spatially invariant and not, in general, a far-field dependent quantity. Thus, (1) can be computed for a given incident field from any region with respect to the proximity of the antenna, provided that the scattered field from the antenna can be neglected. Following the formulation derived in [6-8] that utilizes (1), the GARR that represents the response due to a rough-surface EM backscattered field can be represented as:

$$a_{trans} b_{rec} = \frac{-2\pi j \left|a_{trans}\right|^2}{k_0} \int\int_A \frac{G(\theta,\phi)}{4\pi R_s^2} \mathbf{p}_\mathbf{r} \cdot \mathbf{S}_{distr}(\theta,\phi) \cdot \mathbf{p}_\mathbf{r} e^{-j2k_0 R} d^2\mathbf{R}, \tag{2}$$

where $*$ denotes the complex conjugate, $\mathbf{S}_{distr}(\theta,\phi)$ is the 2×2 complex rough-surface scattering matrix that describes the constitutive and kinematic properties of the surface, $G(\theta,\phi)$ is the available-gain function of the antenna, $\mathbf{p}_\mathbf{r}$ is the transmitter polarization vector associated with $\mathbf{F}_{rad}(\theta,\phi)$, and

$$\begin{aligned} d^2\mathbf{R} &= s(\theta,\phi)\,dxdy = s(\theta,\phi)\,dS \\ &\equiv \sqrt{1+\left|\nabla_t \varsigma(x,y)\right|^2}\,dS. \end{aligned}$$

Note that aside from the fact that the surface is in the far field of the antenna or that $\left|\varsigma(x,y)\right| / R_s \ll 1$, it should be understood that in (2) $\mathbf{S}_{distr}(\theta,\phi)$ is given without any scattering assumptions [1-3]. Here, $\mathbf{S}_{distr}(\theta,\phi)$ is directly related to the equivalent surface-current distribution induced by the incident field. Stated more generally, the source of each ray of the plane-wave spectrum of the scattered field $\widetilde{\mathbf{E}}_{scat}\left(\mathbf{k}_t'\right)$ received by the antenna can be associated with a localized distribution of equivalent electric and magnetic currents on the surface within the antenna footprint. This property makes (2) unique in that the NRCS or more appropriately, the G-NRCS, can be computed at every point within the antenna footprint. Note that (2) is used to compute the response b_{rec} for a given frequency ω. If the radar system is excited with $a_{trans}(\omega)e^{j\omega t}$ and the surface is allowed to vary as a function of (slow) time τ, then the surface equation can be written as $z = H - \varsigma(x, y; \tau)$ and (2) can be expressed as:

$$a_{trans}(\omega)b_{rec}(\omega,\tau)e^{j\omega t} \approx \frac{-2\pi j\left|a_{trans}(\omega)\right|^2}{k_0}\iint \frac{G(\theta,\phi;\omega)}{4\pi R_s^2}Q(\theta,\phi;\omega,\tau)e^{j\omega t}dS;$$
$$Q(\theta,\phi;\omega,\tau) = s(\theta,\phi;\tau)\mathbf{p}_\mathbf{r}(\theta,\phi;\omega)\cdot\mathbf{S}_{distr}(\theta,\phi;\omega,\tau)\cdot\mathbf{p}_\mathbf{r}(\theta,\phi;\omega)e^{-j2k_0R(\theta,\phi;\tau)}, \tag{3}$$

where the scattering variable $Q(\theta,\phi;\omega,\tau)$ [2] is further described later in this section. The expression above is valid given that the surface $\varsigma(x, y;\tau)$ is relatively motionless compared to the speed of the impinging EM field. In (3), the available gain-function defined as:

$$G(\theta,\phi;\omega) = 4\pi\left|\frac{\mathbf{F}_{rad}(\theta,\phi;\omega)}{a_{trans}(\omega)}\right|^2, \tag{4}$$

and the polarization vector defined as:

$$\mathbf{p}_\mathbf{r}(\theta,\phi;\omega) = \frac{\mathbf{F}_{rad}(\theta,\phi;\omega)}{\left|\mathbf{F}_{rad}(\theta,\phi;\omega)\right|}, \tag{5}$$

now explicitly depend on ω. It should be noted that now $d^2\mathbf{R} = \sqrt{1+\left|\nabla_t\varsigma(x,y;\tau)\right|^2}dS$ and $R(\theta,\phi;\tau) = \sqrt{x^2+y^2+(H-\varsigma(x,y;\tau))^2}$.

DERIVIATION OF THE TIME-DEPENDENT G-NRCS $\sigma(S;\omega,\tau,\tau')$

The antenna response, $b_{rec}(\omega,\tau)$, for a time-varying surface whose scattering properties are represented by $\mathbf{S}_{distr}(\theta,\phi;\omega,\tau)$ when stimulated by the frequency spectrum of the transmit waveform, $a_{trans}(\omega)e^{j\omega t}$, is described in (3). This particular version of the GARR relates the constitutive EM-field and kinematic properties of $\varsigma(x, y;\tau)$ to a single frequency component of $b_{rec}(\omega,\tau)$ via a surface integral within the antenna footprint. That is, each spectral component of $b_{rec}(\omega,\tau)$ represents a sampling of the equivalent surface-current distribution and surface dynamics. In essence, (3) represents a single component of the total output frequency response of a linear time-varying system or a two-port microwave network (monostatic radar) for a rough surface [1, 7]. This property is not the current focus of this memorandum report. However, (3) will be further explored to derive the time-dependent G-NRCS.

The focus is now redirected at (3) to represent it in a more familiar form. First, (3) is cross-divided through by $a_{trans}(\omega)$ and the fast-time dependence ($e^{j\omega t}$) is suppressed to obtain,

$$\frac{b_{rec}(\omega,\tau)}{a_{trans}(\omega)} \approx \frac{a_{trans}^{*}(\omega)}{a_{trans}(\omega)} \frac{-2\pi j}{k_0} \iint \frac{G(\theta,\phi;\omega)}{4\pi R_s^2} Q(\theta,\phi;\omega,\tau)\, dS$$
$$= \frac{a_{trans}^{*}(\omega)}{a_{trans}(\omega)} \frac{-2\pi j}{k_0} \iint \frac{G(\theta,\phi;\omega)}{4\pi R_s^2} s(\theta,\phi;\tau)\, \mathbf{p_r}(\theta,\phi;\omega)\cdot \mathbf{S}_{distr}(\theta,\phi;\omega,\tau)\cdot \mathbf{p_r}(\theta,\phi;\omega)\, e^{-j2k_0R(\theta,\phi;\tau)} dS. \tag{6}$$

Then, the ratio of the frequency response to the stimulus $b_{rec}(\omega,\tau)/a_{trans}(\omega)$ is redefined as,

$$\widehat{b}_{rec}(\omega,\tau) = \frac{b_{rec}(\omega,\tau)}{a_{trans}(\omega)} . \tag{7}$$

Next, the ensemble average of $\left\langle \widehat{b}_{rec}(\omega,\tau)\widehat{b}_{rec}^{*}(\omega,\tau')\right\rangle$ is computed to yield:

$$\left\langle \widehat{b}_{rec}(\omega,\tau)\widehat{b}_{rec}^{*}(\omega,\tau')\right\rangle = \lambda^2 \iint \frac{G(\theta,\phi;\omega)}{4\pi R_s^2} \iint \frac{G(\theta',\phi';\omega)}{4\pi R_s'^2} \left\langle Q(\theta,\phi;\omega,\tau) Q^{*}(\theta',\phi';\omega,\tau')\right\rangle dS\, dS' . \tag{8}$$

With further algebraic manipulations of (8), one obtains

$$\left\langle \widehat{b}_{rec}(\omega,\tau)\widehat{b}_{rec}^{*}(\omega,\tau')\right\rangle =$$
$$\frac{\lambda^2}{(4\pi)^3} \iint \frac{G^2(\theta,\phi;\omega)}{R_s^4} \left\{ \frac{4\pi R_s^2}{G(\theta,\phi;\omega)} \iint \frac{G(\theta',\phi';\omega)}{R_s'^2} \left\langle Q(\theta,\phi;\omega,\tau) Q^{*}(\theta',\phi';\omega,\tau')\right\rangle dS' \right\} dS . \tag{9}$$

The final expression can then be written as,

$$\left\langle \widehat{b}_{rec}(\omega,\tau)\widehat{b}_{rec}^{*}(\omega,\tau')\right\rangle = \frac{\lambda^2}{(4\pi)^3} \iint \frac{G^2(\theta,\phi;\omega)\,\sigma(S;\omega,\tau,\tau')}{R_s^4} dS; \tag{10}$$

and the time-dependent G-NRCS is now defined as,

$$\sigma(S;\omega,\tau,\tau') = \frac{4\pi R_s^2}{G(\theta,\phi;\omega)} \iint \frac{G(\theta',\phi';\omega)}{R_s'^2} \left\langle Q(\theta,\phi;\omega,\tau) Q^{*}(\theta',\phi';\omega,\tau')\right\rangle dS'; \tag{11}$$

with the scattering variable expressed as

$$Q(\theta,\phi;\omega,\tau) = s(\theta,\phi;\tau)\mathbf{p_r}(\theta,\phi;\omega)\cdot\mathbf{S}_{distr}(\theta,\phi;\omega,\tau)\cdot\mathbf{p_r}(\theta,\phi;\omega)e^{-j2k_0R(\theta,\phi;\tau)}. \tag{12}$$

Equation (10) resembles the standard radar equation seen in [1-4], except in the case wherein the average relative received power is a function of the referenced and observed (slow) times. Note that (11) is a measure of the backscattered power from the illuminated surface and relates the statistical and physical properties of the surface, the kinematic description of the surface, and the radiation properties of the antenna (via the gain and polarization patterns) as a function of space, frequency, and slow-time [1-2]. The scattering variable $Q(\theta,\phi;\omega,\tau)$ can be seen as a measure of spatial-temporal coherence, because it has $s(\theta,\phi;\tau)$, $\mathbf{S}_{distr}(\theta,\phi;\omega,\tau)$, and $e^{-j2k_0R(\theta,\phi;\tau)}$ as its randomly-dependent components, which are attributed to the spatial-coherence of the backscattered field. Note that (11) is a complex function because of the loss of Hermitian symmetry due to the assembly of the terms used in the ensemble average and the fact that the surface integration is only over the primed coordinates. This is also due to (11) containing terms that are completely deterministic, $\{G(\theta,\phi;\omega), R_s^2, \mathbf{p_r}(\theta,\phi;\omega)\}$, and terms that are stochastic, $\{\varsigma(x,y;\tau), \nabla_t\varsigma(x,y;\tau)\}$. However, (10) is and always will be a real quantity because the imaginary terms are cancelled out [1, 3].

As it stands, $Q(\theta,\phi;\omega,\tau)$ is general in that it involves no implicit assumptions of the nature of the surface, and as a consequence, nor does it implicitly describe the scattering mechanism of the surface in its formulation. Thus, the surface's nature and scattering properties must be defined empirically or analytically.

THE LINEARIZED SEA-SURFACE MODEL – $\varsigma(\boldsymbol{\rho};\tau)$ AS A RANDOM PROCESS

The behaviour (or motion and displacement) of the scattering properties and the EM constitutive properties of the media bounded by the surface, are all contained in $s(\theta,\phi;\tau)$, $\mathbf{S}_{distr}(\theta,\phi;\omega,\tau)$, and $e^{-j2k_0R(\theta,\phi;\tau)}$. The required properties of the three aforementioned quantities will be chosen later in this section. At this point, $\varsigma(x,y;\tau)$ is described as a zero-mean random process that is temporally stationary and spatially homogenous. To that effect, the linear propagation model for ocean surfaces will be employed in which $\varsigma(x,y;\tau)$ is defined as [2, 9]

$$\varsigma(x,y;\tau) = \int\left(\mathcal{A}^{+}(\boldsymbol{\kappa})e^{j\Omega_w\tau} + \mathcal{A}^{-}(\boldsymbol{\kappa})e^{-j\Omega_w\tau}\right)e^{-j\boldsymbol{\kappa}\cdot\boldsymbol{\rho}}d^2\boldsymbol{\kappa}. \tag{13}$$

Equation (13) represents the solution to the linearized Momentum Equation for an incompressible, irrotational fluid flow-field (the ocean) expressed in terms of a Fourier Transform over the spatial-spectrum wave-vector $\boldsymbol{\kappa} = \{\kappa_x, \kappa_y\}$ and $\boldsymbol{\rho} = \{x, y\}$. The spatial-spectral quantities $\{\mathcal{A}^{+}(\boldsymbol{\kappa}), \mathcal{A}^{-}(\boldsymbol{\kappa})\}$ are based on the spatial cross-temporal spectral density of the vertical displacement $\mathcal{S}(\boldsymbol{\kappa},\bar{\tau})$ and are yet to be defined. The water-wave dispersion relationship is defined as $\Omega_w = \sqrt{\kappa g + \mathcal{T}\kappa^3/\rho_v}$ and g, ρ_v, and $\mathcal{T}$ are the gravitational acceleration, density, and surface tension of the water, respectively. Earlier it was mentioned that $\varsigma(\boldsymbol{\rho};\tau)$ is a zero-mean random process that is spatially homogeneous and temporally stationary. Hence, its autocorrelation function is defined as:

$$\mathcal{R}(\boldsymbol{\rho}-\boldsymbol{\rho}';\tau-\tau') = \mathcal{R}(\hat{\boldsymbol{\rho}};\hat{\tau}) = \langle \varsigma(\boldsymbol{\rho};\tau)\varsigma(\boldsymbol{\rho}';\tau')\rangle. \tag{14}$$

The surface $\varsigma(\boldsymbol{\rho};\tau)$ is real and thus $\mathcal{R}(\hat{\boldsymbol{\rho}};\hat{\tau})$ is real and symmetric. Hence, the cross-temporal spectral density $\mathcal{S}(\boldsymbol{\kappa},\hat{\tau})$ is uncorrelated, and is related to $\mathcal{R}(\hat{\boldsymbol{\rho}};\hat{\tau})$ via

$$\mathcal{R}(\hat{\boldsymbol{\rho}};\hat{\tau}) = \int \mathcal{S}(\boldsymbol{\kappa},\hat{\tau}) e^{-j\boldsymbol{\kappa}\cdot\hat{\boldsymbol{\rho}}} d^2\boldsymbol{\kappa}\,. \tag{15}$$

Note that it can be shown that

$$\begin{aligned}\left\langle \mathcal{A}^{+}(\boldsymbol{\kappa})\mathcal{A}^{+*}(\boldsymbol{\kappa}')\right\rangle &= \frac{1}{2}\psi(\boldsymbol{\kappa})\delta(\boldsymbol{\kappa}-\boldsymbol{\kappa}');\\ \left\langle \mathcal{A}^{-}(\boldsymbol{\kappa})\mathcal{A}^{-*}(\boldsymbol{\kappa}')\right\rangle &= \frac{1}{2}\psi(-\boldsymbol{\kappa})\delta(\boldsymbol{\kappa}-\boldsymbol{\kappa}').\end{aligned} \tag{16}$$

SINGLE-SIDED WAVE-NUMBER SPECTRUM

By applying (16) in (13) and (14), the cross-temporal spectral density can now be defined as

$$\mathcal{S}(\boldsymbol{\kappa},\hat{\tau}) = \frac{1}{2}\left[\psi(\boldsymbol{\kappa})e^{j\Omega_w\hat{\tau}} + \psi(-\boldsymbol{\kappa})e^{-j\Omega_w\hat{\tau}}\right] \tag{17}$$

and the autocorrelation function can be rewritten as,

$$\mathcal{R}(\hat{\boldsymbol{\rho}};\hat{\tau}) = \int \frac{1}{2}\left[\psi(\boldsymbol{\kappa})e^{j\Omega_w\hat{\tau}} + \psi(-\boldsymbol{\kappa})e^{-j\Omega_w\hat{\tau}}\right] e^{-j\boldsymbol{\kappa}\cdot\hat{\boldsymbol{\rho}}} d^2\boldsymbol{\kappa}. \tag{18}$$

Here, $\psi(\boldsymbol{\kappa}) = \psi(\kappa_x,\kappa_y)$ is formally known as the Single-Sided Wave-Number Spectrum of $\varsigma(\boldsymbol{\rho};\tau)$. Generally, $\psi(\boldsymbol{\kappa})$ can be any real function provided that the necessary hydrodynamic conditions are upheld. This would mean that $\psi(\boldsymbol{\kappa})$ could be determined either analytically or empirically. In essence, by virtue of (13) and (18) the characteristics of $s(\theta,\phi;\tau)$ and $e^{-j2k_0R(\theta,\phi;\tau)}$ have been fully described, and thus, the focus is now redirected towards the scattering matrix $\mathbf{S}_{distr}(\theta,\phi;\omega,\tau)$.

THE SCATTERING MATRIX $\mathbf{S}_{distr}(\theta,\phi;\omega,\tau)$ UNDER THE SMALL AMPLITUDE APPROXIMATION

The scattering nature of an object or terrain depends (in most cases) on the frequency (wavelength) and direction of the impinging EM field, the physical and material properties of the surface, and the kinematics of the scatterer. By virtue of examining the spatial and temporal coherence of ocean surface backscatter, what is left to be defined are surface properties in relation to the frequency of the radiated field. In this case, it is assumed that the wavelength λ of the incident field is larger than the fluctuations of a given roughness scale of the sea surface $\varsigma(\boldsymbol{\rho};\tau)$ or, stated plainly, $\left|k_0\varsigma(\boldsymbol{\rho};\tau)\right| \ll 1$ [4]. Thus, due to the case of small surface perturbations relative to λ, the Small Amplitude Approximation (SAA) is chosen to describe the scattering nature of the surface. Correspondingly, the time-dependent scattering matrix under the SAA has the following expression [3-4],

$$\mathbf{S}_{distr}(\theta,\phi;\omega,\tau) = \frac{k_0^2\cos^2\theta\varsigma(\boldsymbol{\rho};\tau)}{\pi\sqrt{1+\left|\nabla_t\varsigma(\boldsymbol{\rho};\tau)\right|^2}}\begin{bmatrix}\varpi(\theta;\omega)\Gamma'(\theta;\omega) & 0 \\ 0 & -\Gamma''(\theta;\omega)\end{bmatrix}. \tag{19}$$

Equation (19) has an explicit dependence on $\varsigma(x,y;\tau)$ where $\Gamma'(\theta;\omega)$ and $\Gamma''(\theta;\omega)$ represent the Fresnel Reflection Coefficients for parallel and perpendicular polarization, respectively. The quantity $\varpi(\theta;\omega)$ is a multiplicative factor related to the slopes of the surface. Note that $\varpi(\theta;\omega)\sim 1$ when $\left|\nabla_t\varsigma(\boldsymbol{\rho};\tau)\right| \ll 1$. The frequency-dependent Fresnel Reflection Coefficients and the slope factor are respectively defined as follows,

$$\Gamma'(\theta;\omega) = \frac{\sqrt{\varepsilon_r(\omega)-\sin^2\theta}-\varepsilon_r(\omega)\cos\theta}{\sqrt{\varepsilon_r(\omega)-\sin^2\theta}+\varepsilon_r(\omega)\cos\theta}; \tag{20}$$

$$\Gamma''(\theta;\omega) = \frac{\cos\theta-\sqrt{\varepsilon_r(\omega)-\sin^2\theta}}{\cos\theta+\sqrt{\varepsilon_r(\omega)-\sin^2\theta}}; \tag{21}$$

and

$$\varpi(\theta;\omega) = \frac{\varepsilon_r(\omega)-(1-\varepsilon_r(\omega))\sin^2\theta}{\varepsilon_r(\omega)+(1-\varepsilon_r(\omega))\sin^2\theta}. \tag{22}$$

DERIVIATION OF TIME-DEPENDENT G-NRCS $\sigma(S;\omega,\tau,\tau')$ UNDER THE SAA

The characteristics of $\mathbf{S}_{distr}(\theta,\phi;\omega,\tau)$, $s(\theta,\phi;\tau)$, and $e^{-j2k_0R(\theta,\phi;\tau)}$ have now been fully described and defined. Using the expressions detailed in (14) and (18), and (19) through (22), and following the developments in [4], the scattering variable as defined in (12) is now written as,

$$\begin{aligned} Q(\theta,\phi;\omega,\tau) &= \frac{k_0^2}{\pi}\varsigma(\boldsymbol{\rho};\tau)f_s(\theta,\phi;\omega)e^{-j2k_0R(\theta,\phi;\tau)}; \\ f_s(\theta,\phi;\omega) &= \cos^2\theta\left[\varpi(\theta,\phi;\omega)p_{r\theta}^2(\theta,\phi;\omega)\Gamma'(\theta,\phi;\omega)-p_{r\phi}^2(\theta,\phi;\omega)\Gamma''(\theta,\phi;\omega)\right]. \end{aligned} \tag{23}$$

Here, $f_s(\theta,\phi;\omega)$ is the polarization factor and represents the constitutive properties of the dielectric medium bounded by $\varsigma(\boldsymbol{\rho};\tau)$, where

$$\mathbf{p_r}(\theta,\phi;\omega) = p_{r\theta}(\theta,\phi;\omega)\boldsymbol{\theta}_0 + p_{r\phi}(\theta,\phi;\omega)\boldsymbol{\phi}_0,$$

are the components of the polarization vector of the radiated field. Note that $Q(\theta,\phi;\omega,\tau)$ is becomes linearly dependent on $\varsigma(\boldsymbol{\rho};\tau)$ when $R(\theta,\phi;\tau)=\sqrt{x^2+y^2+(H-\varsigma(x,y;\tau))^2}$ is combined with the constraint $\left|\varsigma(x,y)\right|/R_s \ll 1$ in (23) to yield

$$R(\theta,\phi;\tau) \sim R_s(\theta,\phi) - \varsigma(\boldsymbol{\rho};\tau)\cos\theta. \tag{24}$$

Substituting (24) into (23) produces

$$Q(\theta,\phi;\omega,\tau) = \frac{k_0^2}{\pi}\varsigma(\boldsymbol{\rho};\tau) f_s(\theta,\phi;\omega) e^{-j2k_0 R_s(\theta,\phi)}. \tag{25}$$

The exponential term $e^{j2k_0\varsigma(\boldsymbol{\rho};\tau)\cos\theta}$ is set to unity due to the small amplitude constraint $\left|k_0\varsigma(\boldsymbol{\rho};\tau)\right| \ll 1$ for a first order approximation. Now, by substituting (25) into $\left\langle Q(\theta,\phi;\omega,\tau)Q^*(\theta',\phi';\omega,\tau')\right\rangle$, the coherence measure is written as

$$\left\langle Q(\theta,\phi;\omega,\tau)Q^*(\theta',\phi';\omega,\tau')\right\rangle = \frac{k_0^4}{\pi^2} f_s(\theta,\phi;\omega) f_s^*(\theta',\phi';\omega) e^{-j2k_0 R_s(\theta,\phi)} \mathscr{R}(\widehat{\boldsymbol{\rho}};\widehat{\tau}) e^{j2k_0 R_s(\theta',\phi')}. \tag{26}$$

Finally, by employing (26) and (18) into (11) the time-dependent G-NRCS is obtained and is expressed as,

$$\sigma(S;\omega,\widehat{\tau}) = \frac{4\pi k_0^4 f_s(\theta,\phi;\omega) e^{-j2k_0 R_s(\theta,\phi)} R_s^2}{\pi^2 G(\theta,\phi;\omega)} \iint \frac{f_s^*(\theta',\phi';\omega) G(\theta',\phi';\omega) e^{j2k_0 R_s(\theta',\phi')}}{R_s'^2} \mathscr{R}(\widehat{\boldsymbol{\rho}};\widehat{\tau})\, dS'; \tag{27}$$

or expressed as,

$$\sigma(S;\omega,\widehat{\tau}) = \frac{4\pi k_0^4 f_s(\theta,\phi;\omega) e^{-j2k_0 R_s(\theta,\phi)} R_s^2}{G(\theta,\phi;\omega)} \cdot \left\{ \frac{2}{(2\pi)^2} \int \left[H(\boldsymbol{\kappa};\omega)\psi(\boldsymbol{\kappa}) e^{j\Omega_w \widehat{\tau}} + H(\boldsymbol{\kappa};\omega)\psi(-\boldsymbol{\kappa}) e^{-j\Omega_w \widehat{\tau}} \right] e^{-j\boldsymbol{\kappa}\cdot\boldsymbol{\rho}} d^2\boldsymbol{\kappa} \right\}; \tag{28}$$

with

$$H(\boldsymbol{\kappa};\omega) = \iint e^{j\boldsymbol{\kappa}\cdot\boldsymbol{\rho}'} \frac{f_s^*(\theta',\phi';\omega) G(\theta',\phi';\omega) e^{j2k_0 R_s(\theta',\phi')}}{R_s'^2} dS'.$$

(29)

A special case arises when the antenna beamwidth $\Omega_{Antenna}$ is small. That is, the ratios of the antenna footprint area to the square of the slant range (at beam center) and the antenna footprint dimensions to slant range are both small, or respectively, $A/\overline{R}^2 \ll 1$, $|\hat{x}|/\overline{R} \ll 1$, and $|\hat{y}|/\overline{R} \ll 1$. Together these constraints are known as the narrowbeam approximation, and the application of this approximation to (28) and (29) yields:

$$\sigma(S;\omega,\hat{\tau}) = \frac{4\pi k_0^4 f_s(\theta,\phi;\omega)}{G(\theta,\phi;\omega)} \cdot \left\{ \frac{2}{(2\pi)^2} \int \begin{bmatrix} H(\hat{\boldsymbol{\kappa}};\omega)\psi(\hat{\boldsymbol{\kappa}} - 2k_0 \sin\overline{\theta}\overline{\boldsymbol{\rho}}_0) e^{j\Omega_w(\hat{\boldsymbol{\kappa}}-2k_0\sin\overline{\theta}\overline{\boldsymbol{\rho}}_0)\hat{\tau}} + \cdots \\ H(\hat{\boldsymbol{\kappa}};\omega)\psi(-\hat{\boldsymbol{\kappa}} + 2k_0 \sin\overline{\theta}\overline{\boldsymbol{\rho}}_0) e^{-j\Omega_w(\hat{\boldsymbol{\kappa}}-2k_0\sin\overline{\theta}\overline{\boldsymbol{\rho}}_0)\hat{\tau}} \end{bmatrix} e^{-j\hat{\boldsymbol{\kappa}}\cdot\hat{\boldsymbol{\rho}}} d^2\hat{\boldsymbol{\kappa}}; \right\} \tag{30}$$

$$H(\hat{\boldsymbol{\kappa}};\omega) = \iint e^{j\hat{\boldsymbol{\kappa}}\cdot\hat{\boldsymbol{\rho}}'} f_s^*(\theta',\phi';\omega) G(\theta',\phi';\omega)\, dS'; \tag{31}$$

where $\hat{\boldsymbol{\rho}} = \boldsymbol{\rho} - \overline{\boldsymbol{\rho}}$, $R_s = \overline{R} + \overline{\boldsymbol{\rho}}_0 \cdot \hat{\boldsymbol{\rho}} \sin\overline{\theta}$, $\hat{\boldsymbol{\kappa}} = \boldsymbol{\kappa} + 2k_0 \sin\overline{\theta}\overline{\boldsymbol{\rho}}_0$, and

$$\Omega_w(\hat{\boldsymbol{\kappa}} - 2k_0 \sin\overline{\theta}\overline{\boldsymbol{\rho}}_0) = \sqrt{|\hat{\boldsymbol{\kappa}} - 2k_0 \sin\overline{\theta}\overline{\boldsymbol{\rho}}_0| g + \mathcal{T} |\hat{\boldsymbol{\kappa}} - 2k_0 \sin\overline{\theta}\overline{\boldsymbol{\rho}}_0|^3 / \rho_v}.$$

Examining $\sigma(S;\omega,\hat{\tau})$ for when $\overline{\theta} = 0$ yields:

$$\sigma(S;\omega,\hat{\tau}) \approx \frac{4\pi k_0^4 f_s(\theta,\phi;\omega)}{G(\theta,\phi;\omega)} \cdot \left\{ \frac{2}{(2\pi)^2} \int \left[H(\hat{\boldsymbol{\kappa}};\omega)\psi(\hat{\boldsymbol{\kappa}}) e^{j\Omega_w\hat{\tau}} + H(\hat{\boldsymbol{\kappa}};\omega)\psi(-\hat{\boldsymbol{\kappa}}) e^{-j\Omega_w\hat{\tau}} \right] e^{-j\boldsymbol{\kappa}\cdot\boldsymbol{\rho}} d^2\hat{\boldsymbol{\kappa}} \right\}. \tag{32}$$

Equations (30), (31) , and (32) are independent of the slant range R_s, which produces a computational limitation due to the large number of spatial samples needed to track the oscillations of $e^{j2k_0R_s}$ for $k_0R_s \gg 1$. Thus, making these expressions well-suited for high-altitude airborne ocean-surface-imaging SAR applications.

Generally speaking, the quantity $H(\boldsymbol{\kappa};\omega)$ (or $H(\hat{\boldsymbol{\kappa}};\omega)$) can be understood as the Real Aperture Modulation Transfer Function, which essentially filters the spatial spectrum $\psi(\boldsymbol{\kappa})$ within the spectral band that contains the product the monostatic radar quantities $G(\theta',\phi';\omega)$, $f_s^*(\theta',\phi';\omega)$, and $e^{j2k_0R_s(\theta',\phi')}$. That is, the range of values of $\boldsymbol{\kappa}$ for $\psi(\boldsymbol{\kappa})$ overlap those values of $\boldsymbol{\kappa}$ for $H(\boldsymbol{\kappa};\omega)$ [3, 4]. Thus, in (28) is the mathematical representation of the Resonance Scattering phenomena for rough surfaces, which is a general case of Bragg Scattering. However, in this case the condition for constant gain and polarization factor $\left\{G\left(\theta,\phi;\omega\right), f_s\left(\theta,\phi;\omega\right)\right\}$ within the antenna footprint is not imposed. Note that because (28) and (32) is represented in terms of direct Fourier transform pairs $\sigma\left(S;\omega,\hat{\tau}\right)$ can be computed efficiently via the Fast Fourier Transform (FFT). This benefit is needed especially for low grazing-angle RAR and SAR applications when the antenna footprint dimension are fairly large compared to the slant range R_s, which is typically the case for small aperture antennas.

SUMMARY AND CONCLUSIONS

The G-NRCS was derived here using its time-harmonic version in conjunction with the frozen-surface assumption. The linear ocean-surface model was introduced along with associated statistical relationships to yield an expression for a fully developed sea. The ratio of the surface displacement and the EM wavelength of the incident field was assumed to be very small compared to unity. Based on this property, the Small Amplitude Approximation (SAA) was chosen to describe the scattering behaviour of the sea surface and, thus, the Scattering Matrix for a terrain was chosen with its necessary components defined in detail. These expressions were employed here to yield the time-dependent G-NRCS under SAA. This quantity should be suited for Synthetic Aperture Radar (SAR) applications where the sea-surface reflectivity is required at every point within the antenna footprint corresponding to a given pixel in a SAR image of the ocean surface. The application of the expressions derived here holds the promise of improved SAR imaging performance for the detection of ships in sea-clutter environments.

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