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SIMPLIFIED ANALYSIS OF R-C AND R-L NETWORKS

By Charles F. White

- Report R-2668 -

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ABSTRACT

A method of analysis is developed by which the steady-state transfer characteristics of resistance-capacitance and resistance-inductance filter networks are readily obtained. The procedure depends upon an analysis of certain functions that are shown to be basic terms in the transfer characteristic equation for any such filter networks. The functions are as follows:

$$1 + p/\omega_0, \quad \text{and} \quad p/\omega_0$$

where p is defined as $\sqrt{-1}$ times 2π times frequency, and ω_0 is a constant. A set of design data charts is given for various combinations of resistance, capacitance, parallel resistance-capacitance, and series resistance-capacitance used in either the series or the shunt branches of an unsymmetrical "L" section network.

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INTRODUCTION

1. A method of analysis is developed in the present report by which the steady-state transfer characteristics of resistance-capacitance and of resistance-inductance filter networks may be readily obtained. The networks discussed all come within the class called passive, linear networks. The analysis procedure when applied to a particular network will demand only a simple algebraic manipulation followed by the taking of algebraic sums of tabulated quantities in order to yield an accurate determination of the attenuation and phase shift as a function of frequency.

2. To establish the analysis method, extensive use is made of various techniques employed in feedback amplifier design as developed by Bode*. The particular points of interest in the Bode approach as used here in the filter studies are: (1) the procedure of plotting loss in decibels on a logarithmic frequency scale, (2) the concept of asymptotic slopes used to define a reference frequency, (3) the usefulness of the quantity "decibels per octave", and, (4) the symmetrical character of phase shift when plotted on a logarithmic frequency scale. The points just enumerated will be explained as they recur throughout the report.

3. The method of analysis developed in the present report depends upon an understanding and application of certain functions that are shown to be basic terms in the transfer characteristic equation for any filter network containing only resistance and capacitance or only resistance and inductance. The functions are as follows:

$$1 + p/\omega_0, \quad \text{and} \quad p/\omega_0$$

where p is defined as $\sqrt{-1}$ times 2π times frequency, and ω_0 is a constant. The significance of the above functions is fully explained later. By way of leading the reader to an intuitive conclusions that only such functions of frequency may occur, the approach used is one of studying in detail the characteristics of the simple resistance-capacitance and resistance-inductance low- and high-pass filter sections. This conclusion is further substantiated by application of the analysis method to all possible combinations of resistance, capacitance, resistance-capacitance in parallel, and resistance-capacitance in series used in either the series or the shunt branches of an unsymmetrical "L" section network. A proof that only such functions occur is given as an appendix.

* H. W. Bode, "Relations between Attenuation and Phase in Feedback Amplifier Design," Bell System Tech. Jour., Vol. 19, p. 421, July, 1940.

4. Before the analysis of specific network configurations is undertaken, the general procedure used in the determination of the steady-state response of a network is outlined. First, it should be noted that the effect produced by the insertion of a network into an electrical circuit is not only a function of the character of the network itself, but also of the terminating impedances, i.e., the nature of the input and output circuits. The immediate interest here is in the individual effects of the network itself. The analysis is based upon the assumption that the input circuit impedance, referred to as the generator impedance Z_g, is negligibly small and that the output circuit impedance, referred to as the load impedance Z_L, is effectively infinite. It is not proposed in the present report to consider circuit applications of the networks analyzed but it may be stated in passing that the values of Z_g and Z_L present when the network is used in the grid circuit of a low-frequency amplifier closely approximate these conditions. The unsymmetrical "L" section of Fig. 1 is taken as the basic configuration.

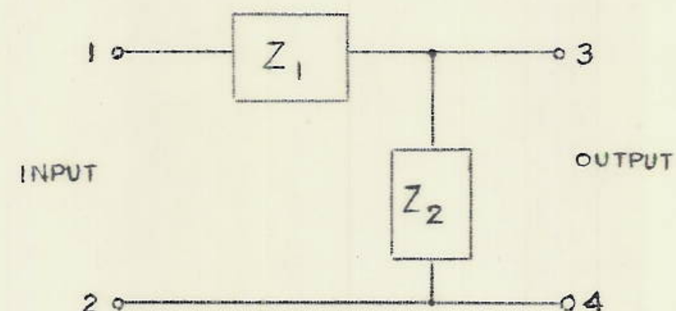

FIG. 1

5. The quantities Z_1 and Z_2 may contain resistance and reactance terms as indicated by equations in complex form as follows:

$$Z_1 = a_1 + j\,b_1 \quad \ldots\ldots\ldots\ldots \quad (1)$$

and

$$Z_2 = a_2 + j\,b_2 \quad \ldots\ldots\ldots\ldots \quad (2)$$

where a_1 and a_2 have values of zero or positive real numbers, and b_1 and b_2 have any real number values including zero with the limitation that b_1 and b_2 cannot have unlike signs. For resistance-inductance networks, b_1 and b_2 may be zero or positive. For resistance-capacitance networks, b_1 and b_2 may be zero or negative. The symbol j indicates the complex number, $\sqrt{-1}$.

6. Before obtaining the steady-state response equation, another term called the open-circuit driving-point impedance must be introduced. As used here, the impedance between terminals 1 and 2 of Fig. 1 measured with terminals 3 and 4 open-circuited is designated the input open-circuit driving-point impedance, Z_{12}. Similarly, the impedance between terminals 3 and 4 with terminals 1 and 2 open-circuited is designated the output open-circuit driving-point impedance, Z_{34}.

The steady-state transfer characteristic, α, may now be given by the general equation

$$\alpha = \frac{Z_{34}}{Z_{12}} = \frac{Z_2}{Z_1 + Z_2} = a + j\,b. \quad \ldots\ldots\ldots (3)$$

The magnitude of α is designated by the term attenuation, A, and is evaluated by

$$A = \sqrt{a^2 + b^2}, \quad \ldots\ldots\ldots\ldots (4)$$

whereas the phase of the output voltage relative to the input voltage is designated by the term phase shift, B, and is given by

$$B = \tan^{-1} \frac{b}{a}. \quad \ldots\ldots\ldots\ldots (5)$$

Since a is never negative, the sigh of b determines whether the output voltage leads or lags the input. For a positive value of b, a leading phase shift occurs; for a negative value of b, a lagging phase shift results. In all networks of the type under consideration the phase shift never exceeds 90 degrees.

SINGLE-SECTION LOW-PASS FILTER

7. The procedure outlined in the material on "L" section networks will be applied to the single-section low-pass filter of Fig. 2. As indicated in the introduction, a careful study of such basic networks leads to a generally applicable method of analysis by which all resistance-capacitance and all resistance-inductance networks may be analyzed. In this case, the series-branch impedance Z_1 becomes R and the shunt-branch impedance, Z_2 becomes $1/pC$ where p is defined as $j\omega$, $\omega = 2\pi f$, and f equals frequency in cycles per second.

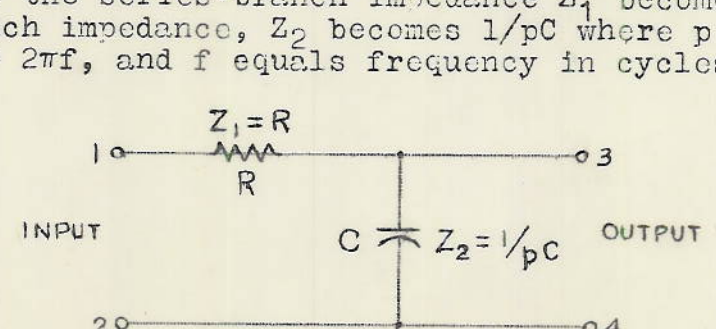

Fig. 2

The equation for the transfer characteristic becomes

$$\alpha = \frac{\frac{1}{pC}}{R + \frac{1}{pC}} = \frac{1}{1 + pRC} \quad . \ . \ . \ . \ . \ (6)$$

$$\alpha = \frac{1 - pRC}{(1 + pRC)(1 - pRC)} = \frac{1}{1 + \omega^2R^2C^2} +$$

$$j \frac{-\omega RC}{1 + \omega^2R^2C^2} . \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ (7)$$

The equation for attenuation is

$$A = (1 + \omega^2R^2C^2)^{-1/2}, \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ (8)$$

and the phase shift is

$$B = \tan^{-1} (-\omega RC). \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ (9)$$

8. As mentioned in the introduction, the plotting of both attenuation and phase shift on a logarithmic frequency scale has special merits in filter studies. If attenuation is expressed in terms of decibels, as determined by the formula*

$$A_{db} = 20 \log_{10} A, \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ (10)$$

the characteristic exhibits approximately straight line low and high frequency portions when plotted on a logarithmic frequency scale as indicated by Fig. 3. The lines approached by the attenuation characteristic are asymptotes of the curve and a graphical extension of these asymptotes results in an intersection at a frequency having a particularly important relationship to the parameters of the network being analyzed. Equation (8) substituted into (10)

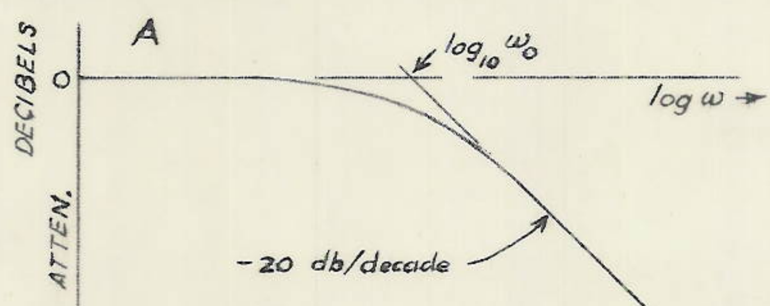

FIG. 3.

* Values of decibels determined on a voltage ratio basis may not be used in power level calculations unless the input and output impedances are equal in value and pure resistance.

gives the attenuation expressed in decibels,

$$A_{db} = 20 \log_{10} (1 + \omega^2R^2C^2)^{-1/2} \; . \; . \; . \; . \; .$$

$$= -10 \log_{10} (1 + \omega^2R^2C^2), \; . \; . \; . \; . \; . \; . \; (11)$$

from which it is possible to determine the asymptotes by retaining the portion within the parentheses having importance at extremely low frequencies and again at extremely high frequencies. For low frequencies the equation of the asymptote,

$$A_{db} = 0, \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (12)$$

is a zero-slope line at zero-attenuation level.

For very high frequencies the equation of the asymptote,

$$A_{db} = -20 \log_{10} \omega RC \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (13)$$

is the equation of a line crossing the zero decibel attenuation line at a frequency specified by the equation

$$\omega_0 = \frac{1}{RC} \; . \; \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (14)$$

To distinguish from the variable, ω, a sub-script zero has been used. The intersection of the asymptotes thus defines an angular frequency ω_0 which may be used as a reference frequency, f_0, specification ($\omega_0 = 2\pi f_0$). The significant nature of the value of ω specified by equation (14) is emphasized by the fact that the reciprocal of ω_0, namely RC, is the circuit time constant of transient analysis. Before discussing the slope of the high frequency asymptote it is desirable to introduce the term "decade". In conformity with a rising usage, the term decade serves to mean a frequency interval of ten to one, i.e., an interval of one cycle of graduations in a plot on common log paper. From a value of zero decibels for $\omega = \omega_0$, equation (13) gives a value of -20 decibels for $\omega = 10\omega_0$. Accordingly, the slope of the high frequency asymptote is -20 db/decade. In using the results of a filter analysis, the unit of slope, "decibels per decade", is highly satisfactory but an alternate unit that is probably more generally used should be mentioned. If the term "octave" is understood to mean a frequency interval of two to one, an alternate value of -6 db/octave results for the slope of the high frequency asymptote.

9. Equations (11) and (9) may now be written in another form as follows:

$$A_{db} = -10 \log_{10} \left[1 + \left(\frac{f}{f_o}\right)^2\right], \quad \ldots\ldots (15)$$

and

$$B = -\tan^{-1} \left(\frac{f}{f_o}\right), \quad \ldots\ldots\ldots\ldots (16)$$

where

$$f_o = \frac{1}{2\pi} \cdot \frac{1}{RC} . \quad \ldots\ldots\ldots\ldots (17)$$

The actual attenuation characteristic is a smooth curve deviating from the asymptotes by the greatest amount, -3 db, at the reference frequency. For a given number of decades either side of the reference frequency, the attenuation characteristic has the same deviation from the asymptotes. Accordingly, only a limited number of deviation values are needed in order to graph the characteristic. A study of the equation for phase shift, equation (16), reveals the fact that the phase characteristic has odd-function symmetry about the reference frequency. Equations (15), (16), and (17) are used to calculate the values tabulated in Table 1 and plotted on Plate 1. Attenuation and phase shift given as a function of frequency completely specify the steady-state transfer characteristic of a network. Both characteristics are implicitly expressed by one equation of the type of equation (6). Equation (6) may now be written in the form $(1 + p/\omega_o)^{-1}$ which is the reciprocal of one of the two basic functions upon which the simplified method of analysis depends.

SINGLE-SECTION HIGH-PASS FILTER

10. A somewhat different procedure will be used in the determination of the steady state attenuation and phase shift of the resistance-capacitance high-pass filter of Fig. 4.

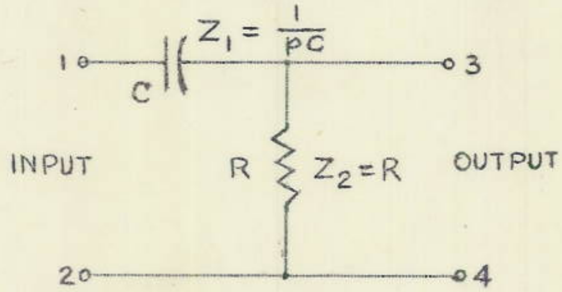

Fig. 4

The steady-state response equation is written using the procedure indicated by equation (3). Thus,

$$\alpha = \frac{R}{\frac{1}{pC} + R} = \frac{pRC}{1 + pRC} \; . \qquad \ldots \ldots \ldots (18)$$

By writing equation (18) in the following special form,

$$\alpha = (pRC)\left(\frac{1}{1 + pRC} \right), \qquad \ldots \ldots \ldots (19)$$

we have separated the expression for the response of the high-pass filter into two functions.

11. The second function of equation (19) has been discussed in the paragraphs on the low-pass filter, and its characteristics are fully known. The results of a similar analysis of the characteristics of the first function when properly combined with those already obtained will give the performance of the high-pass filter. Moreover, it will be shown that the function pRC, when expressed more generally, is the only other type of function involved in the transfer characteristics defining the steady-state response of any resistance-capacitance or resistance-inductance filter network. Accordingly, we proceed in the analysis of the first function of equation (19) by writing

$$\alpha = pRC, \qquad \ldots \ldots \ldots (20)$$

and

$$\alpha = 0 + j\omega RC. \qquad \ldots \ldots \ldots (21)$$

Equation (21) is in the form of a + jb. To obtain the attenuation in decibels we write

$$A_{db} = 20 \log_{10} (a^2 + b^2)^{-1/2}$$

$$= 10 \log_{10} (0 + \omega^2 R^2 C^2),$$

$$A_{db} = 20 \log_{10} \omega RC. \qquad \ldots \ldots \ldots (22)$$

Equation (22) shows that attenuation expressed in decibels has a straight-line characteristic when plotted on a log scale. An angular reference frequency ω_0 may be specified by setting equation (22) equal to zero, thus:

$$\omega_0 = \frac{1}{RC} \; . \qquad \ldots \ldots \ldots (23)$$

From ω_0 we have the reference frequency f_0 since $\omega_0 = 2\pi f_0$. Equation (22) may now be written in the form

$$A_{db} = 20 \log_{10} \left(\frac{f}{f_0}\right). \qquad \ldots \ldots \ldots (24)$$

12. The sign of the decibel value given by equation (24) depends upon whether the ratio f/f_0 is less than or greater than unity. At any frequency lower than f_0, a negative decibel value or attenuation results. For any frequency higher than f_0, a positive decibel value or amplification is indicated. The slope of the characteristic is a constant and equal to plus 20 decibels per decade (six decibels per octave) for all values of f. The phase shift is given by

$$B = \tan^{-1} \frac{b}{a} = \tan^{-1} \left[+ \frac{\omega RC}{0} \right] = 90^\circ \text{ lead} \quad \ldots\ldots\ldots\ldots (25)$$

and is seen to be the same for all values of frequency f. Plate 2 summarizes the above results for the function $\alpha = p/\omega_0$.

13. Each of the two terms in equation (19) are known as complex functions of the real variable ω, since $p = j\omega$. Using the subscripts 1 and 2 to refer to the first and the second function respectively, the functions may be indicated in the equivalent rectangular and polar forms

$$\alpha_1 = a_1 + jb_1 = A_1 \angle B_1 , \quad \ldots\ldots\ldots\ldots (26)$$

and

$$\alpha_2 = a_2 + jb_2 = A_2 \angle B_2 , \quad \ldots\ldots\ldots\ldots (27)$$

where the symbol $\angle B$ means at angle B. The product of two such functions is given by*

$$\alpha_1 \cdot \alpha_2 = (A_1 \angle B_1)(A_2 \angle B_2)$$

$$= A_1 \cdot A_2 \angle B_1 + B_2 . \quad \ldots\ldots (28)$$

Finally, since attenuation is expressed in decibels, the method of interpreting the product of two functions is given by

$$A_{db} = (A_1 \cdot A_2)_{db} = A_{1_{db}} + A_{2_{db}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (29)$$

and

$$B = B_1 + B_2 . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (30)$$

We are now able to combine the results given for the function (p/ω_0) by equations (24) and (25) with the previously obtained results for the function $(1 + p/\omega_0)^{-1}$, equations (15) and (16), to exhibit the steady-state attenuation and phase shift for the high-pass filter of Fig. 4. The equations are as follows:

*For a detailed discussion see a text on circuit theory; e.g., "Electric Circuits," M.I.T.Staff, 1943, John Wiley & Sons, Inc.

$$A_{db} = -10 \log_{10} [1 + (f_o/f)^2], \qquad (31)$$

and

$$B = \tan^{-1} f_o/f. \qquad (32)$$

However, the purpose of the foregoing study is not that of deriving equations but one of indicating an approach for obtaining the overall characteristics of the network by graphical means. The procedure indicated in mathematical terms by equations (29) and (30) is applied graphically by combining the data of Plates 1 and 2 as shown by Fig. 5. The method does not require that the reference frequencies of the functions involved be the same, as is the case here.

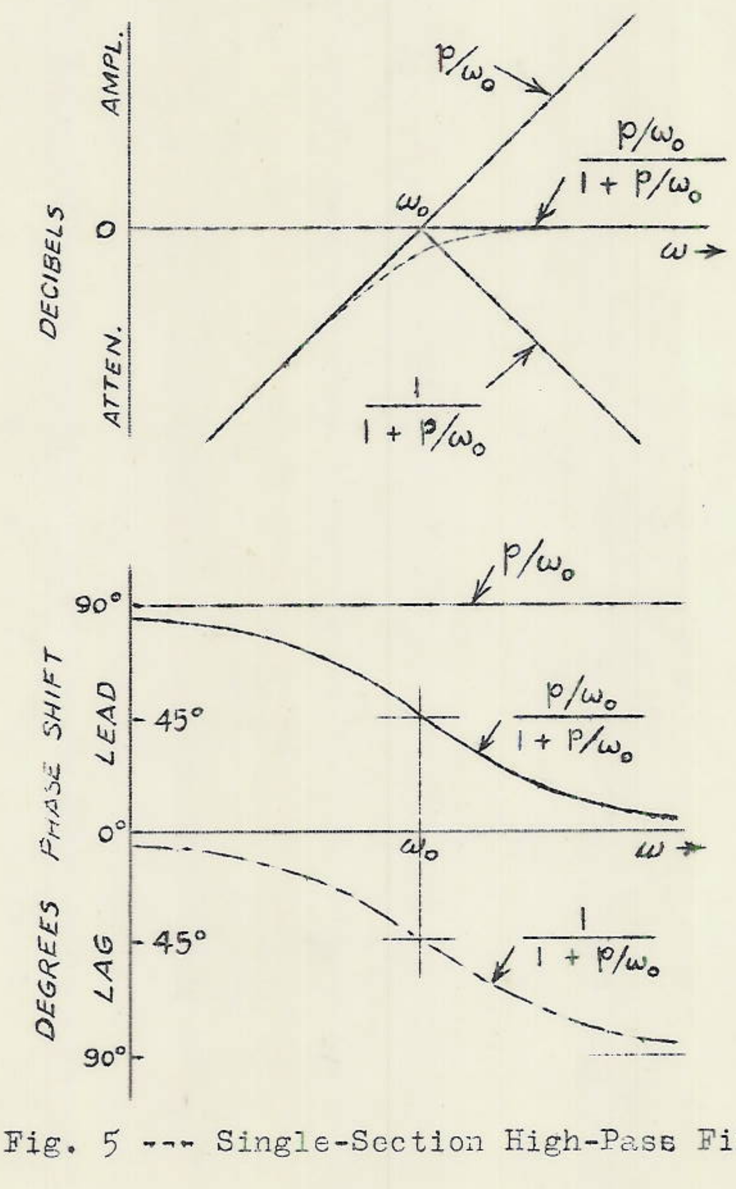

Fig. 5 --- Single-Section High-Pass Filter

THE FUNCTIONS $(1 + p/\omega_0)^{-1}$, $(1 + p/\omega_0)$, AND (p/ω_0).

14. In the introduction it was stated that functions of only two basic types are involved in the expressions for the steady-state response of resistance-capacitance and of resistance-inductance filter networks. It is shown in the appendix that one of them, p/ω_0, occurs only in the numerator but that the other, namely $(1 \mp p/\omega_0)$, may occur in either or both the numerator and denominator of the transfer characteristic equation. In the preceding discussion the only alternative not studied was the function $(1 + p/\omega_0)$ as a term in the numerator. Here, the results for the low-pass filter may be used directly by changing the sign of both the phase shift angle and attenuation. Phase shift starts from zero and rises to 90° lead instead of dropping to 90° lag. The "attenuation" becomes negative attenuation or amplification. The foregoing is summarized and shown graphically by Figures 6, 7, and 8; accurate plots of the same information are given on Plates 1, 3, and 2 respectively.

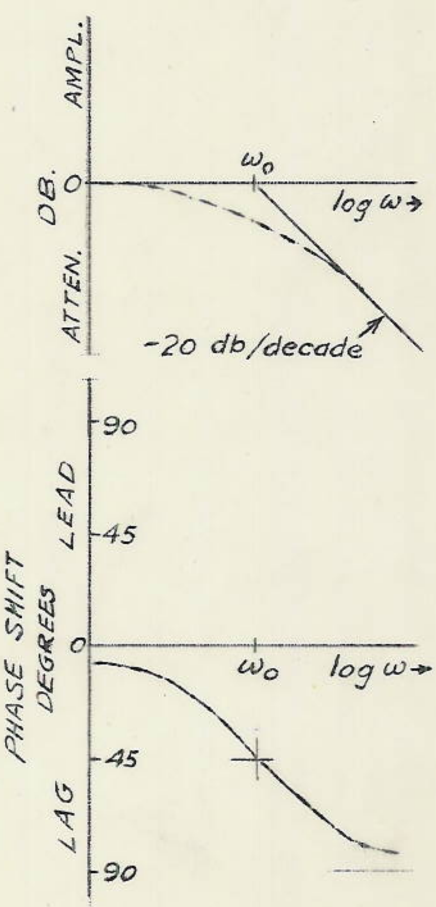

Fig. 6-- $\left(\frac{1}{1 + p/\omega_0}\right)$.

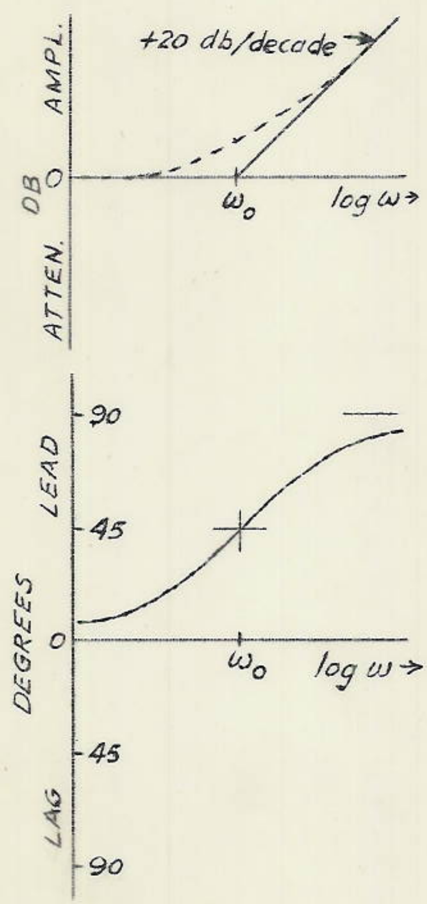

Fig. 7--$(1 + p/\omega_0)$.

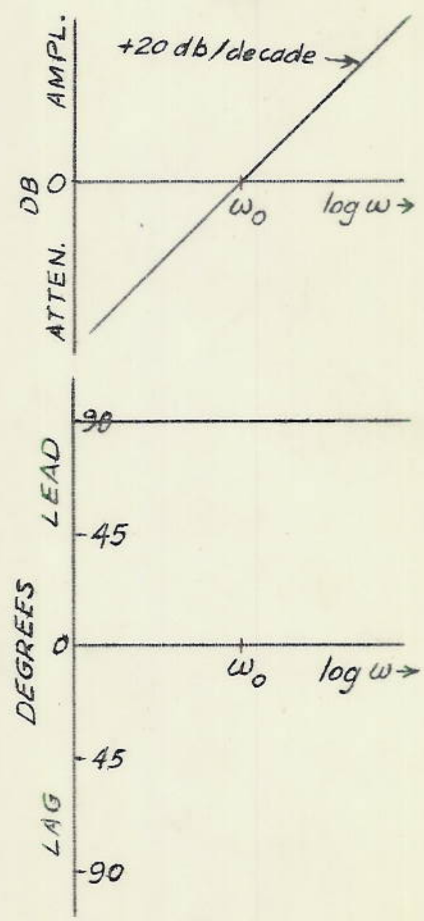

Fig. 8-- (p/ω_0).

RESISTANCE-INDUCTANCE NETWORK SECTIONS

15. The steady-state transfer characteristics of resistance-inductance filter networks may be obtained in an exactly analogous manner to that used in studying the resistance-capacitance networks. The same two basic forms of functions appear as is shown by the equations written for the networks of Figs. 9 and 10.

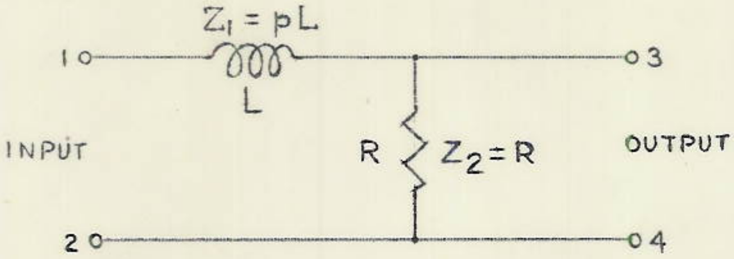

Fig. 9--- Low-Pass R-L Network.

The transfer characteristic equation is

$$\alpha = \frac{R}{pL + R} = \left(\frac{1}{1 + p/(R/L)}\right). \qquad \ldots\ldots(33)$$

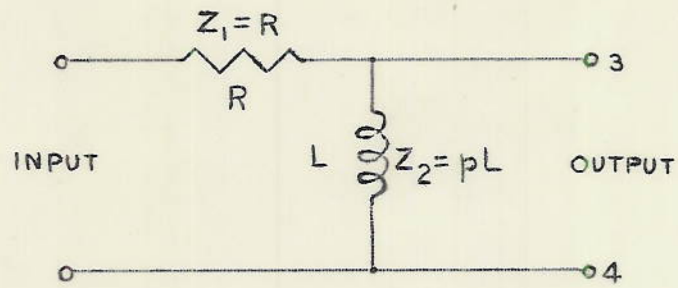

Fig. 10--- High-Pass R-L Network.

The transfer characteristic equation may be written in the form

$$\alpha = \frac{pL}{R + pL} = \frac{p/(R/L)}{1 + p/(R/L)}, \qquad \ldots\ldots\ldots(34)$$

or in the form

$$\alpha = \left[p/(R/L) \right] \cdot \left[\frac{1}{1 + p/(R/L)} \right]. \qquad \ldots\ldots(35)$$

By analogy with equations for the resistance-capacitance networks, reference frequency specifications of $\omega_o = (R/L)$ are noted throughout the above equations. All other details are the same for both types of networks.

APPLICATION OF THE SIMPLIFIED ANALYSIS PROCEDURE

16. The study of the basic functions upon which the method of analysis of the steady-state performance of resistance-capacitance and of resistance-inductance filter networks depends has been completed. The details of the application of the method are best shown by analyzing a more complicated configuration, namely, the circuit of Fig. 11.

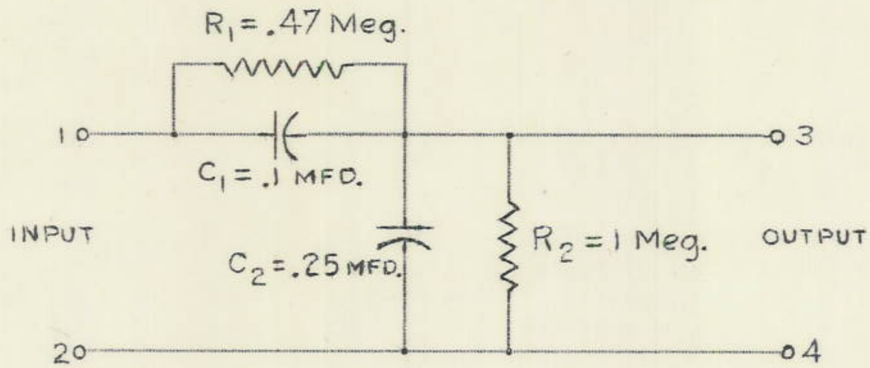

Fig. 11.

The procedure may be separated into six steps: (1) Write equation for transfer characteristic. (2) Note the functions involved. (3) Draw individual asymptotes and combined overall asymptotic attenuation characteristics. (4) Draw individual phase shifts and combined overall phase shift curves. (5) Compute values using specific circuit values. (6) Prepare accurate graphs using values assembled in (5). These steps are applied to the circuit of Fig. 11 as follows:

(1) Write equation for transfer characteristic.

$$\alpha = \frac{\dfrac{R_2}{1 + pR_2C_2}}{\dfrac{R_1}{1 + pR_1C_1} + \dfrac{R_2}{1 + pR_2C_2}}$$

$$\alpha = \frac{R_2}{R_1 + R_2} \cdot \frac{1 + pR_1C_1}{1 + p\left(\dfrac{R_1R_2}{R_1 + R_2}\right)(C_1 + C_2)} . \quad (36)$$

(2) Note the functions involved.

(a) $\dfrac{R_2}{R_1+R_2}$; a constant attenuation, no phase shift.

(b) $\dfrac{1}{1 + p\left(\dfrac{R_1 R_2}{R_1 + R_2}\right)(C_1 + C_2)}$; a low pass filter characteristic with an angular reference frequency ω_{o1} equal to the reciprocal of the coefficient of p.

(c) $1 + pR_1C_1$; the reciprocal of a low-pass filter with a reference frequency determined by $\omega_{o2} = 1/R_1C_1$.

(3) Draw individual asymptotes and combined attenuation characteristic.
See Fig. 12*.

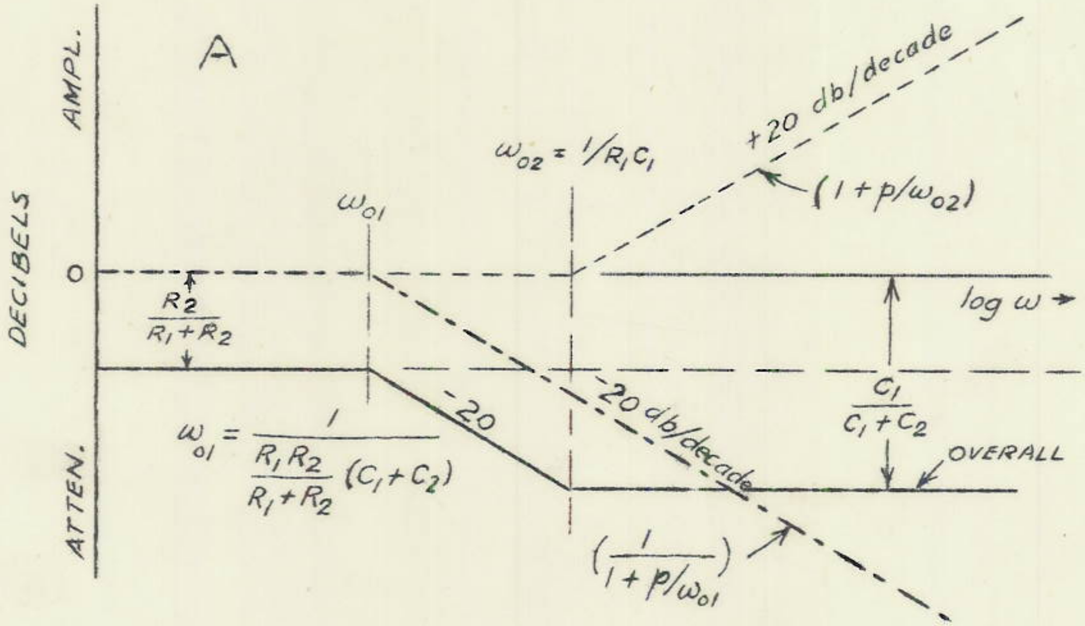

Fig. 12.

* Note that the -20 db/decade portion of the overall characteristic is not an asymptote in the mathematical sense but such usage is convenient.

(4) Draw individual phase shifts and combined overall phase shift curves. See Fig. 13.

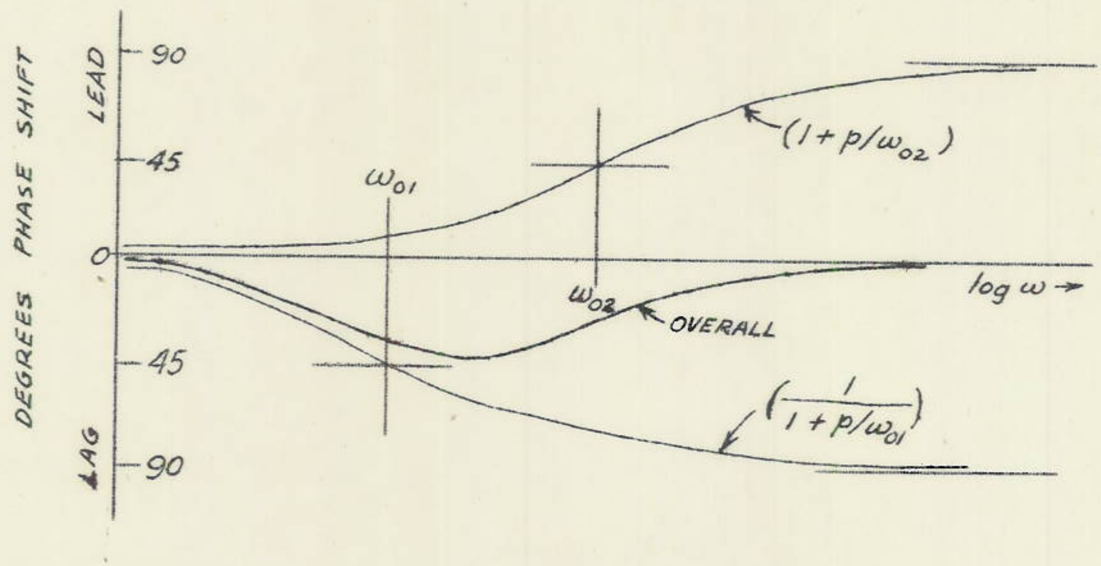

Fig. 13.

(5) Compute values using specific circuit values.

$$\frac{R_2}{R_1 + R_2} = \frac{1}{1 + .47} = .68; \quad \underline{-\ 3.35\ \text{db}}.$$

$$\frac{1}{\left(\frac{R_1\ R_2}{R_1 + R_2}\right)(C_1 + C_2)} = \frac{1}{\frac{.47}{1.47}(.1 + .25)} = 8.94; \quad \underline{f_{01} = 1.42\ \text{cps}}$$

$$1/R_1C_1 = 1/(.47 \cdot .1) = 21.17; \quad \underline{f_{02} = 3.38\ \text{cps}}$$

For convenience in graphing, prepare a table of attenuation and phase shift values with corresponding frequency values. See Fig. 14.

f/f_o	Frequency values, cps		$\pm A_{db}$	$B(\pm)$
	f_{o1} series	f_{o2} series		
1/16		.21	.02	3.6
1/8	.18	.42	.07	7.1
1/4	.36	.84	.33	14.0
1/2	.71	1.69	.97	26.6
1	1.42	3.38	3.0	45
2	2.84	6.76	7.0	63.4
4	5.68	13.52	12.3	76.0
8	11.36	27.14	18.1	82.9
16	22.72		24.1	86.4

Fig. 14

(6) Prepare accurate graphs using values assembled in (5).

See Plate 4. A variation of the magnitudes involved in the particular network studied can cause a somewhat wide variation in the overall results. See Plate 10.

AIDS IN FILTER DESIGN

17. In equalizer and filter design work, the general character of the attenuation and phase shift obtainable with a given network serves as a guide in selecting a particular circuit configuration. Adjustment of the characteristic to meet particular requirements is often a rather tedious cut-and-try process. This is especially true when the circuit is somewhat complicated and variation of the magnitude of a component in one branch causes interaction in other branches. When the network response has been found by the simplified method of analysis presented here, the effects of all elements in the circuit are clearly revealed with the result that suitable design values may be readily assigned. Thus, the designer has at his command a means whereby he can make minor adjustments in the characteristics to meet particular needs.

18. Design data for "L" section networks composed of resistance, capacitance, resistance-capacitance in parallel, and resistance-capacitance in series used in either the series or the shunt branches are presented on Plates 5 through 12. The sixteen possible combinations are shown. Each data chart is divided into five parts: (1) Circuit schematic of the network. (2) Steady-state transfer equation. (3) Attenuation asymptotes. (4) Attenuation characteristics. (5) Phase shift curves. Plate 13 provides an index to the material of Plates 1 - 12. The illustrations are meant to be general and the relative location on the frequency scale is not restricted to the one-octave interval between reference frequencies shown on many of the plates. On Plate 10 the circuit on the left half is capable of producing an asymptotic characteristic with a high-frequency portion other than zero. The characteristic shown in the illustration is a special case in which f_{01} is exactly one octave higher than f_0. In the right half of Plate 10, the phase shift shown should approach 90 degrees lag but fails to do so because of the particular choice of the attenuation reference frequencies used in the illustration.

CONCLUSIONS

19. The equations for the steady-state performance of resistance-capacitance and resistance-inductance filter networks are shown to have factors of the following basic forms:

$$1 + p/\omega_0, \qquad \text{and} \qquad p/\omega_0.$$

A simplified method of analysis is developed and applied to various configurations containing resistance and capacitance.

ACKNOWLEDGMENTS

20. The writer is pleased to acknowledge the interest and generous assistance of Lloyd O. Brown, Jr. and R. Latter, both of the Fire Control Division, Naval Research Laboratory.

APPENDIX 1 -- Types of Functions in Transfer Characteristic Equations

21. In paragraph 3 of the introduction, the statement was made that the transfer characteristic equations written for filter networks containing only resistance and capacitance or only resistance and inductance contain two types of functions. The purpose of the following is to prove that only these two functions occur in the basic "L" section of Fig. 1, repeated below:

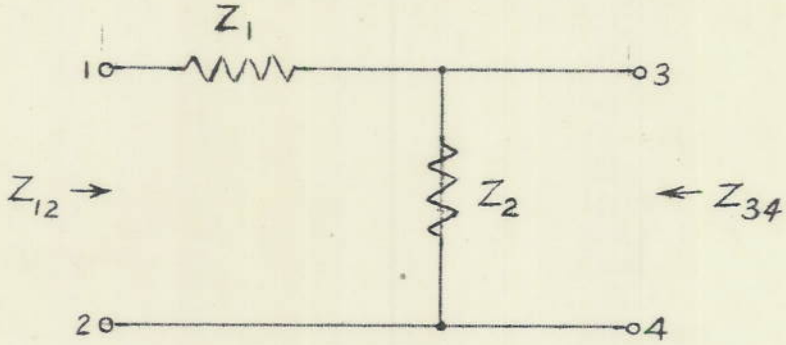

Fig. 1

22. In the circuit of Fig. 1, Z_1 and Z_2 are driving point impedances of two resistance-capacitance or resistance-inductance circuits. The characteristics of driving point impedances are discussed thoroughly in numerous reference books on filter theory. The treatment of E. A. Guillemin, "Communication Networks," Vol. II, John Wiley & Sons, Inc., is followed here with a change in the notation. In particular, see pages 208 and 212 of the above reference. For resistance-inductance circuits the general equation for driving point impedances is

$$Z(p) = H \frac{(p - p_1)(p - p_3)......(p - p_{2n-1})}{(p - p_2)(p - p_4)......(p - p_{2n-2})} \quad . . (37)$$

$$-\infty \leq p_{2n-1} < p_{2n-2} < .... < p_2 < p_1 \leq 0 \quad . . (38)$$

Note that the roots and poles of $Z(p)$ are $p = p_i \leq 0$. For circuits containing resistance and capacitance the general equation for driving point impedance is

$$Z(p) = H \frac{(p - p_1)(p - p_3).......(p - p_{2n-1})}{p(p - p_2)(p - p_4).......(p - p_{2n-2})} \quad . . (39)$$

$$-\infty \leq p_{2n-1} < p_{2n-2} < ..... < p_2 < p_1 \leq 0 \quad . . . (40)$$

The transfer characteristic for "L" sections networks is given by

$$\alpha = \frac{Z_2}{Z_1 + Z_2} \quad . . . . (3)$$

Since Z_2 is a driving point impedance and $Z_1 + Z_2$ is a driving point impedance, the function obtained by the quotient is of the form of either equation (37) or (39). Thus, the equation for α has only the two types of functions discussed in paragraph 3. As an additional consequence of the above proof, the transfer characteristic for "L" section filters has been shown to contain roots and poles with negative real parts only.

TABLE 1

Attenuation and Phase Shift for the function $\left(\frac{1}{1 + p/\omega_o} \right)$.

The formulas used in evaluating the function are:

$$A_{db} = - 10 \log_{10} \left(1 + \left(\frac{f}{f_o}\right)^2 \right) \quad . . . . (15)$$

$$B = - \tan^{-1} \left(\frac{f}{f_o}\right) \quad . . . . (16)$$

f/f_o	A_{db}	B
1/32	0	- 1.8
1/16	-.02	- 3.6
1/8	-.07	- 7.1
1/4	-.33	-14.0
1/2	-.97	-26.6
1	-3.0	-45
2	-7.0	-63.4
4	-12.3	-76.0
8	-18.1	-82.9
16	-24.1	-86.4
32	-30.1	-88.2

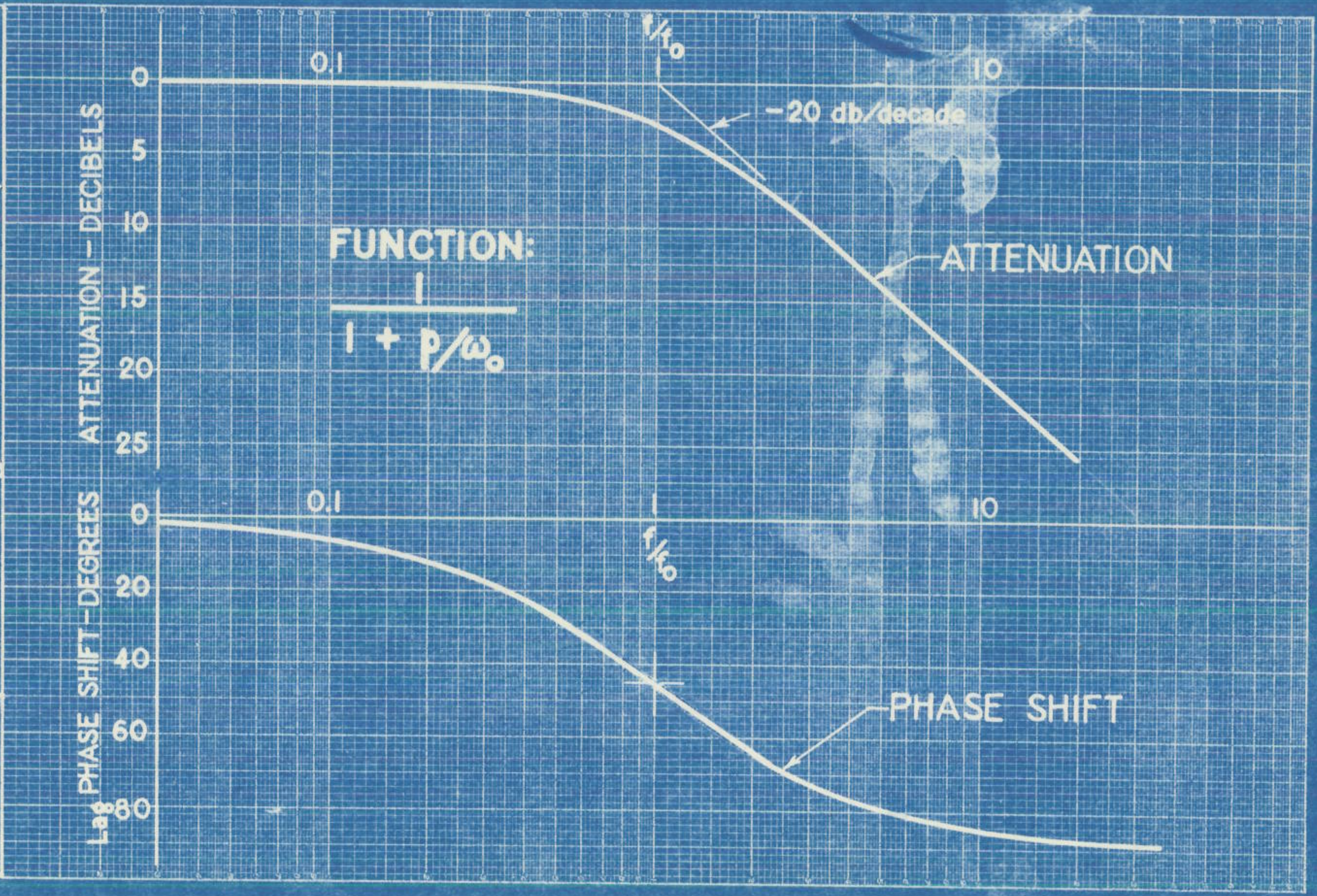

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PLATE I

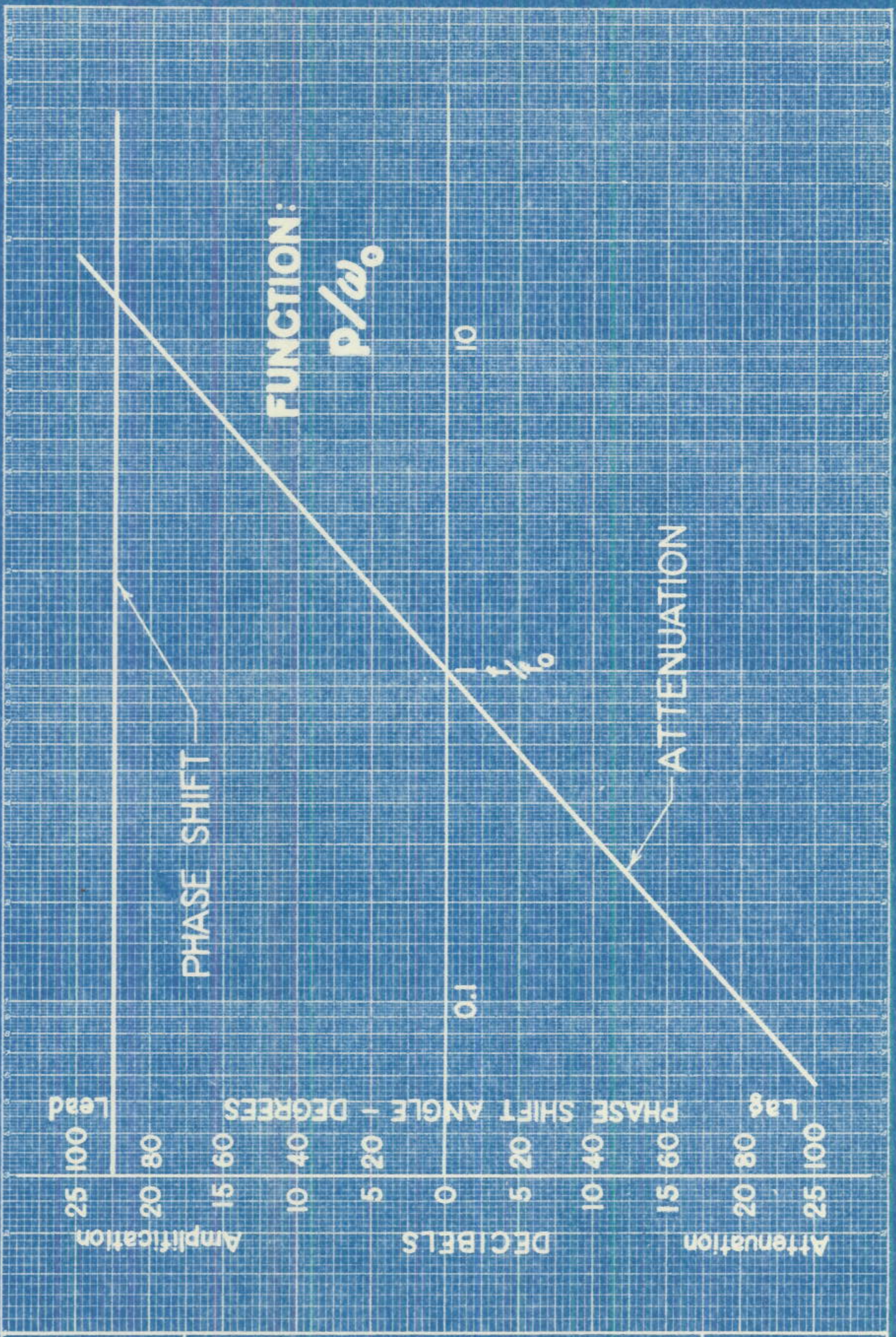

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PLATE 2

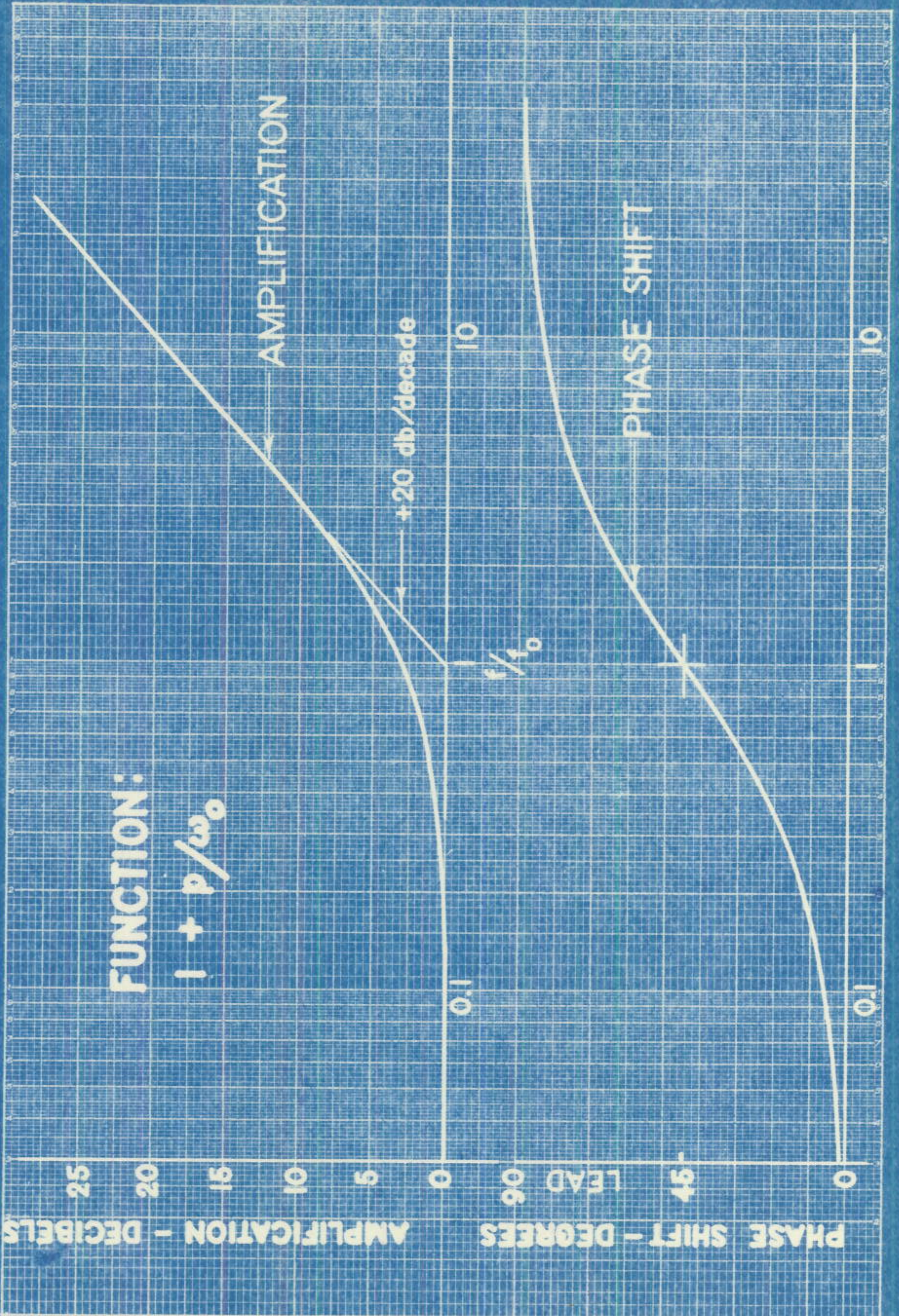

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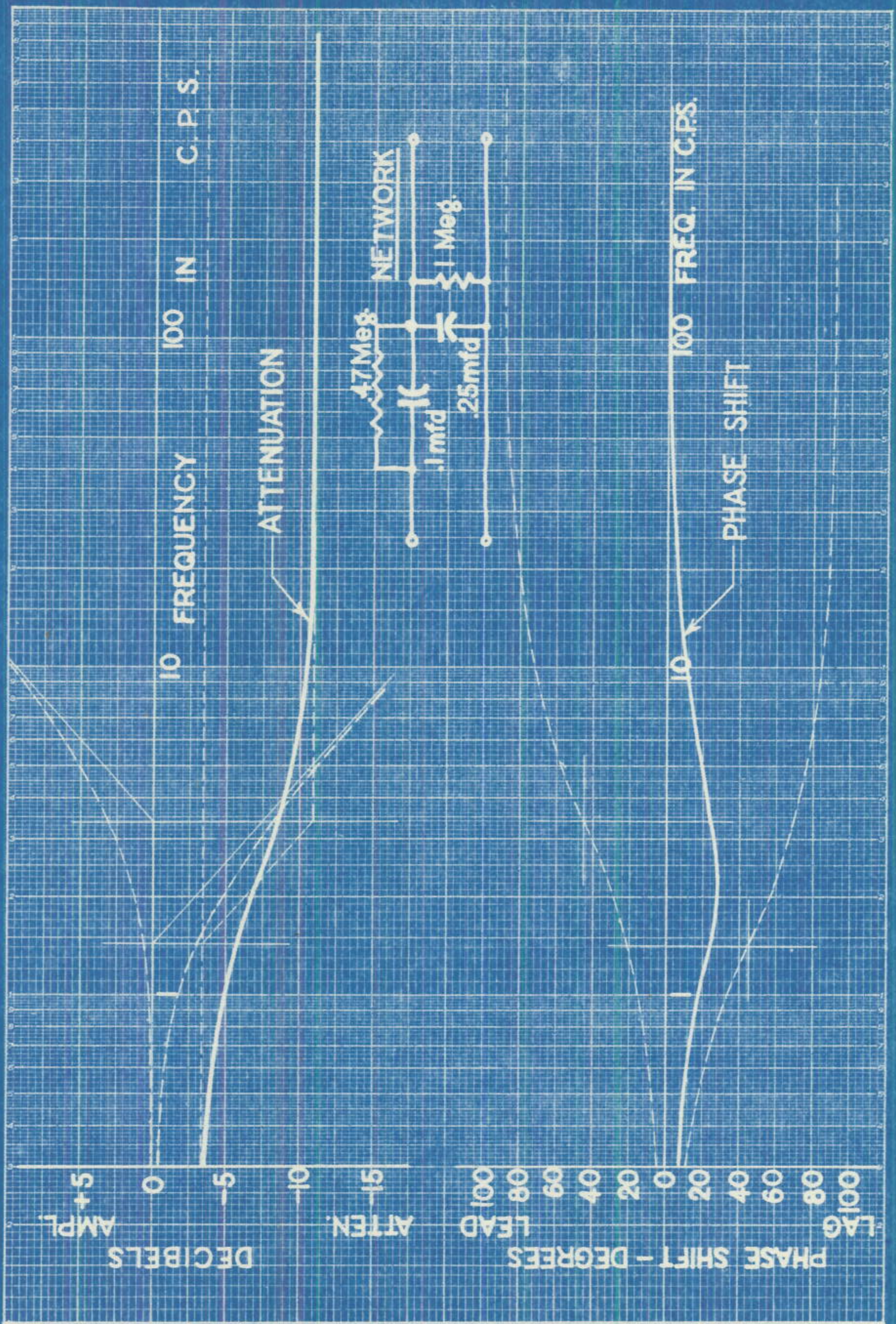
PLATE 3



NRL - FIRE CONTROL DIVISION	PLATE 4
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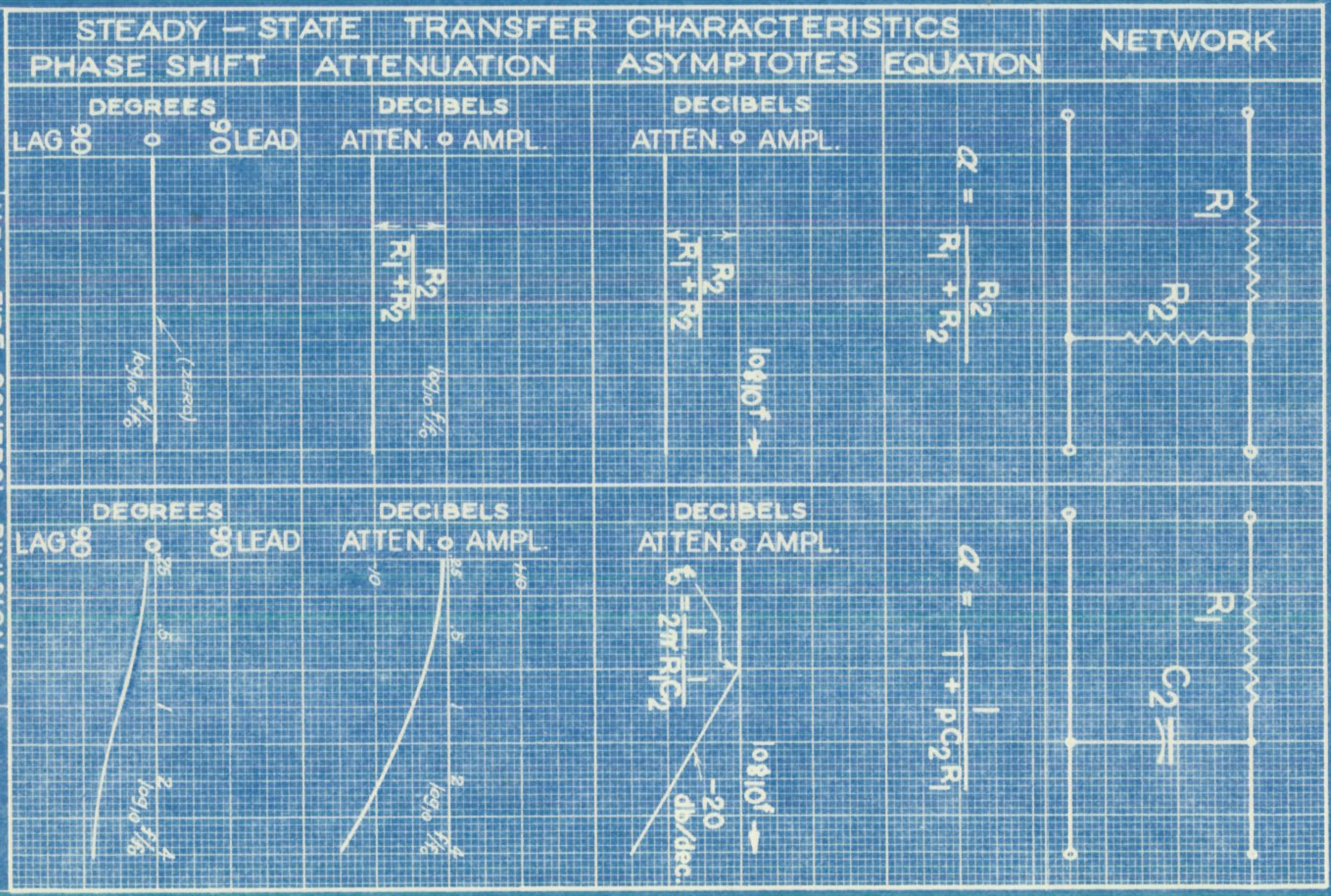

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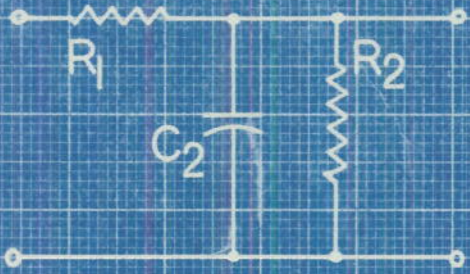
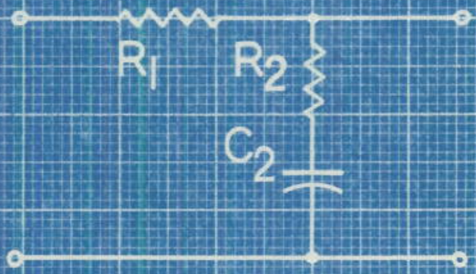
PLATE 5

STEADY – STATE TRANSFER CHARACTERISTICS

NETWORK





EQUATION

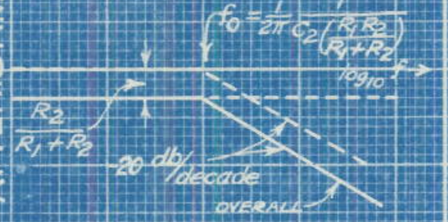
$$\alpha = \left(\frac{R_2}{R_1 + R_2}\right) \frac{1}{1 + pC_2\left(\frac{R_1 R_2}{R_1 + R_2}\right)}$$

$$\alpha = \frac{1 + pR_2C_2}{1 + p(R_1+R_2)C_2}$$

ASYMPTOTES

DECIBELS

ATTEN. 0 AMPL.



DECIBELS

ATTEN. 0 AMPL.

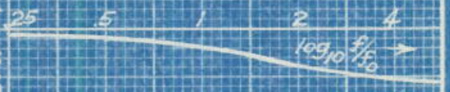
ATTENUATION

DECIBELS

ATTEN. 0 AMPL.

DECIBELS

ATTEN. 0 AMPL.



PHASE SHIFT

DEGREES

LAG 90 0 90 LEAD

DEGREES

LAG 90 0 90 LEAD

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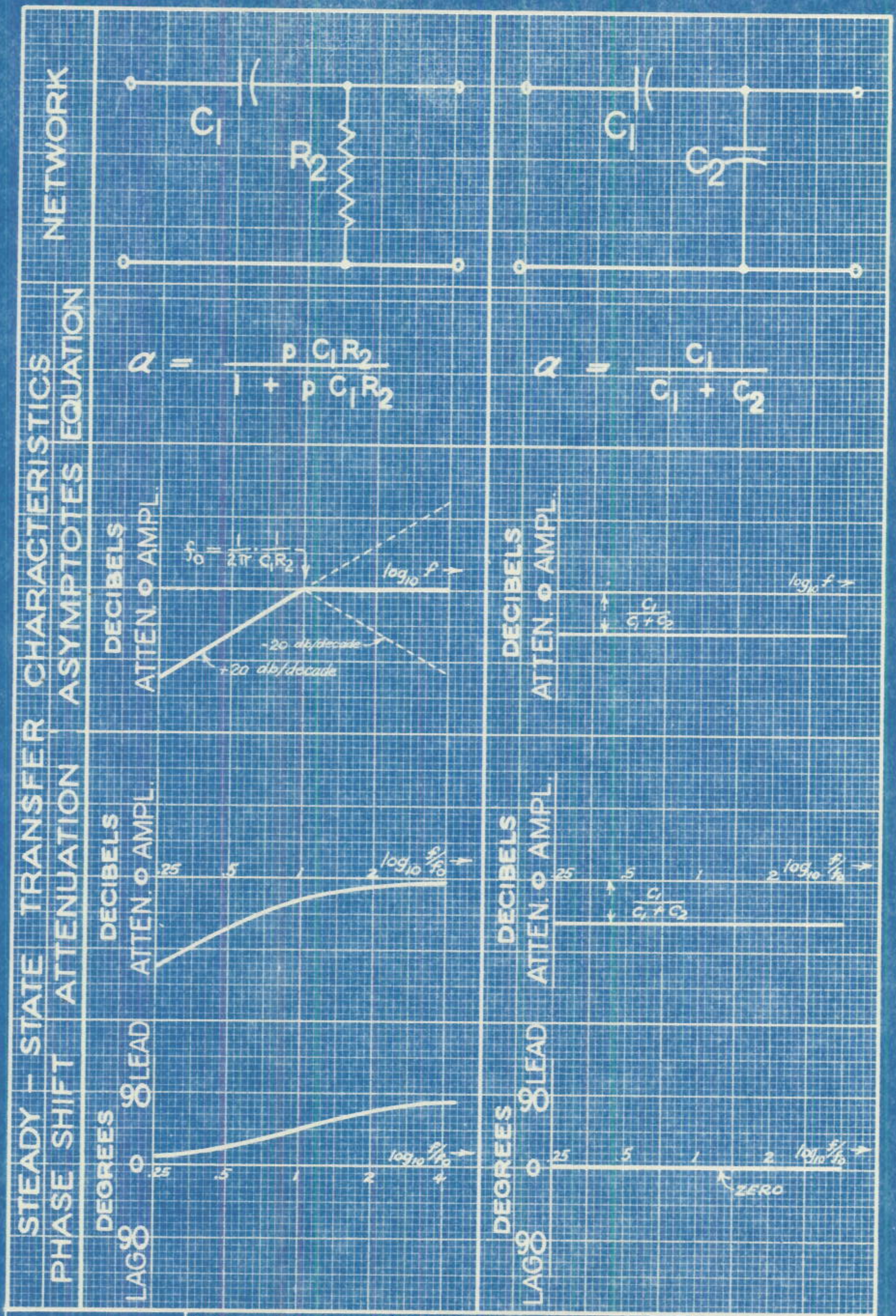

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PLATE 7

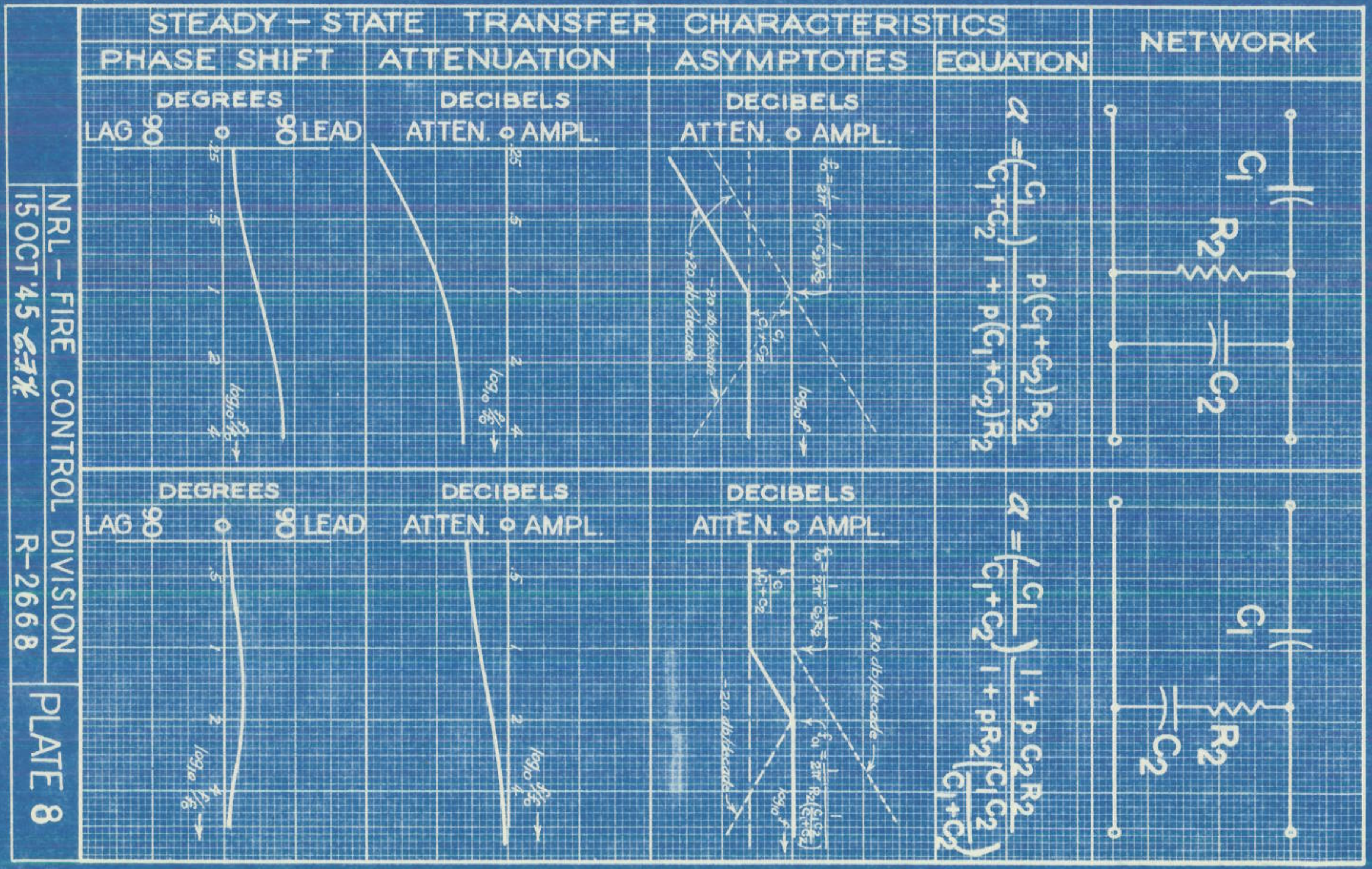
STEADY - STATE TRANSFER CHARACTERISTICS
PHASE SHIFT
ATTENUATION
ASYMPTOTES
EQUATION
NETWORK
DEGREES
LAG 90 0 90 LEAD
DECIBELS
ATTEN. 0 AMPL.
C1
R2
C2
α = (C1 / (C1+C2)) · p(C1+C2)R2 / (1 + p(C1+C2)R2)
α = (C1 / (C1+C2)) · (1 + p C2R2) / (1 + pR2(C1C2 / (C1+C2)))
+20 db/decade
-20 db/decade
.25
.5
1
2
4
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PLATE 8

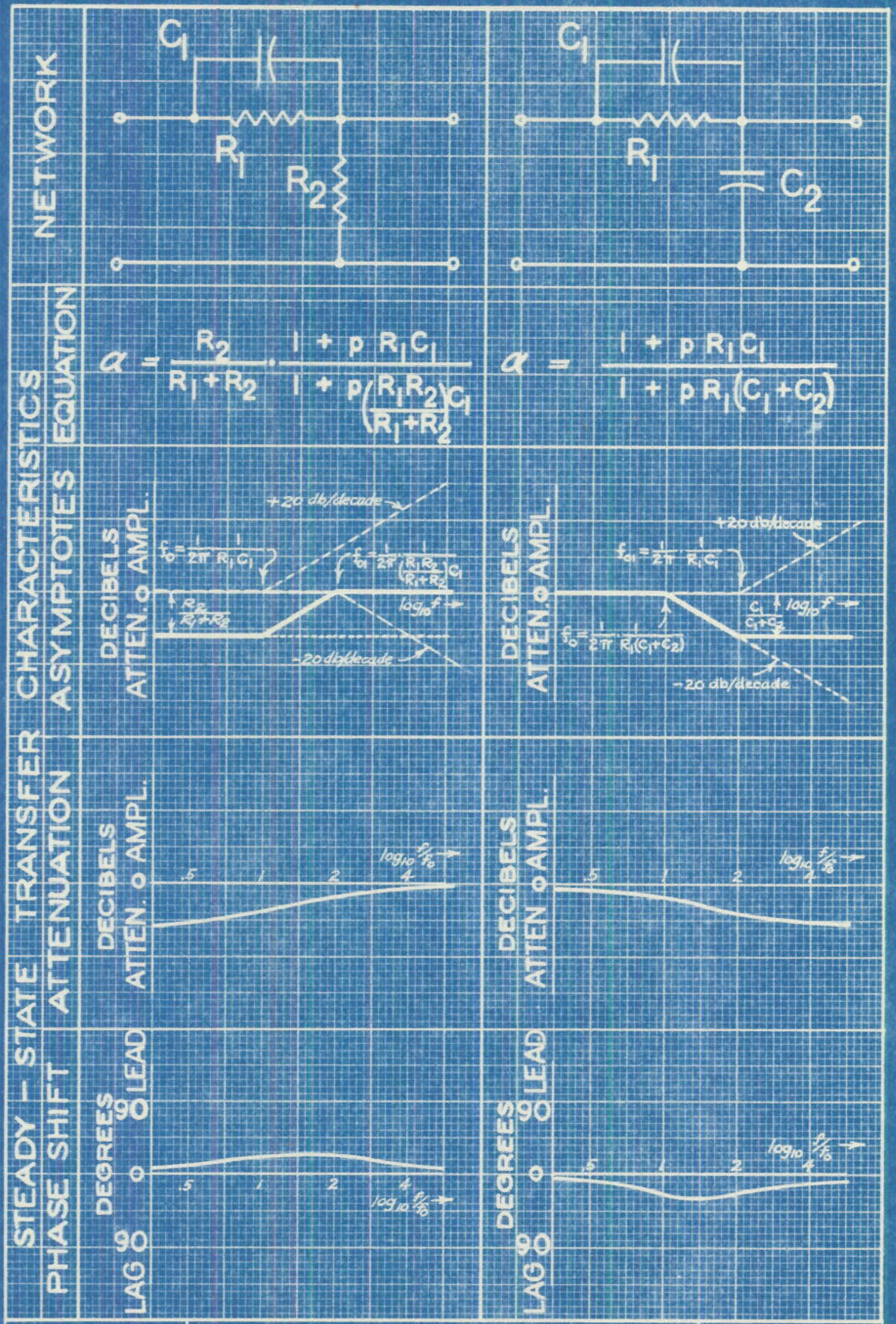

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PLATE 9

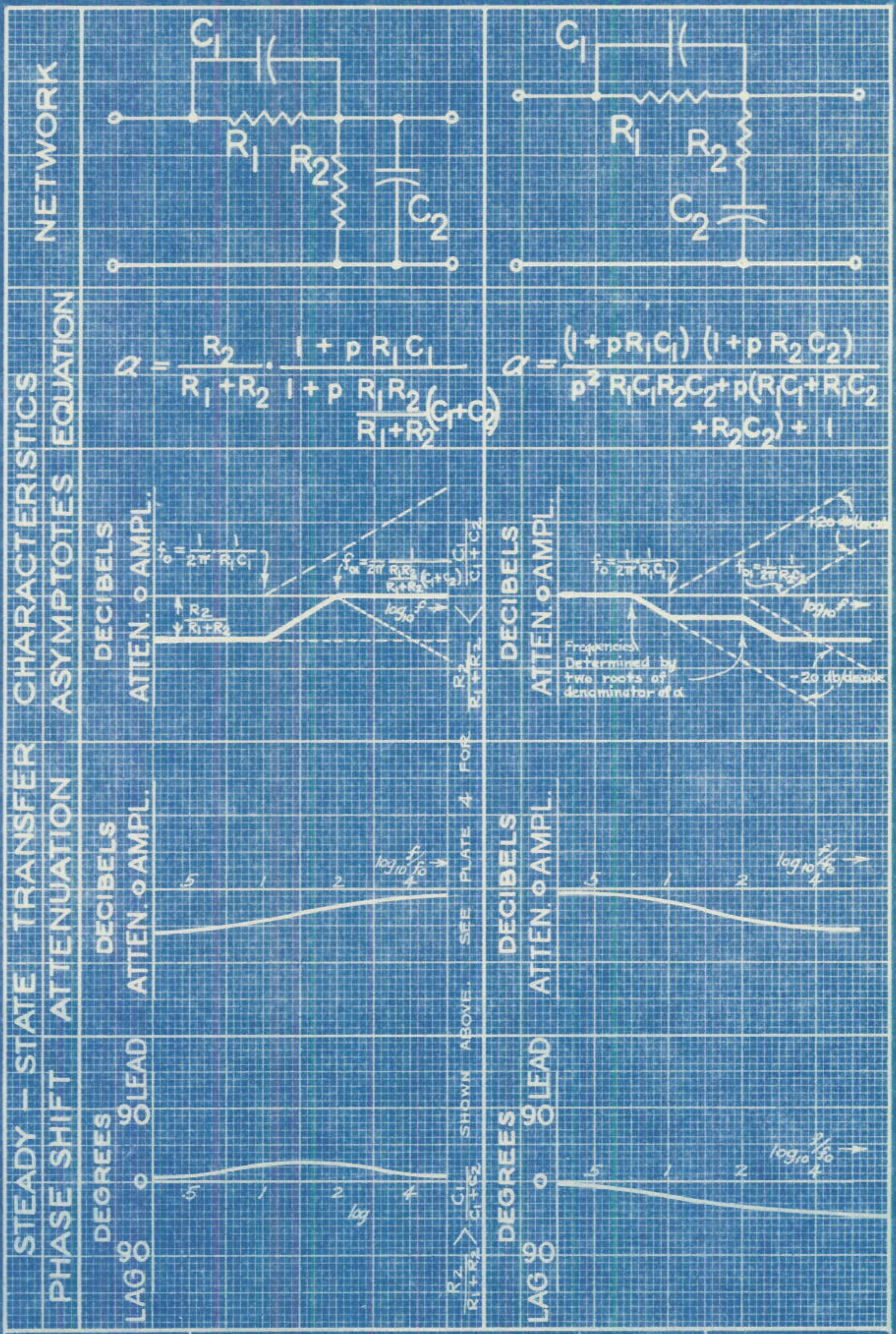

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PLATE 10

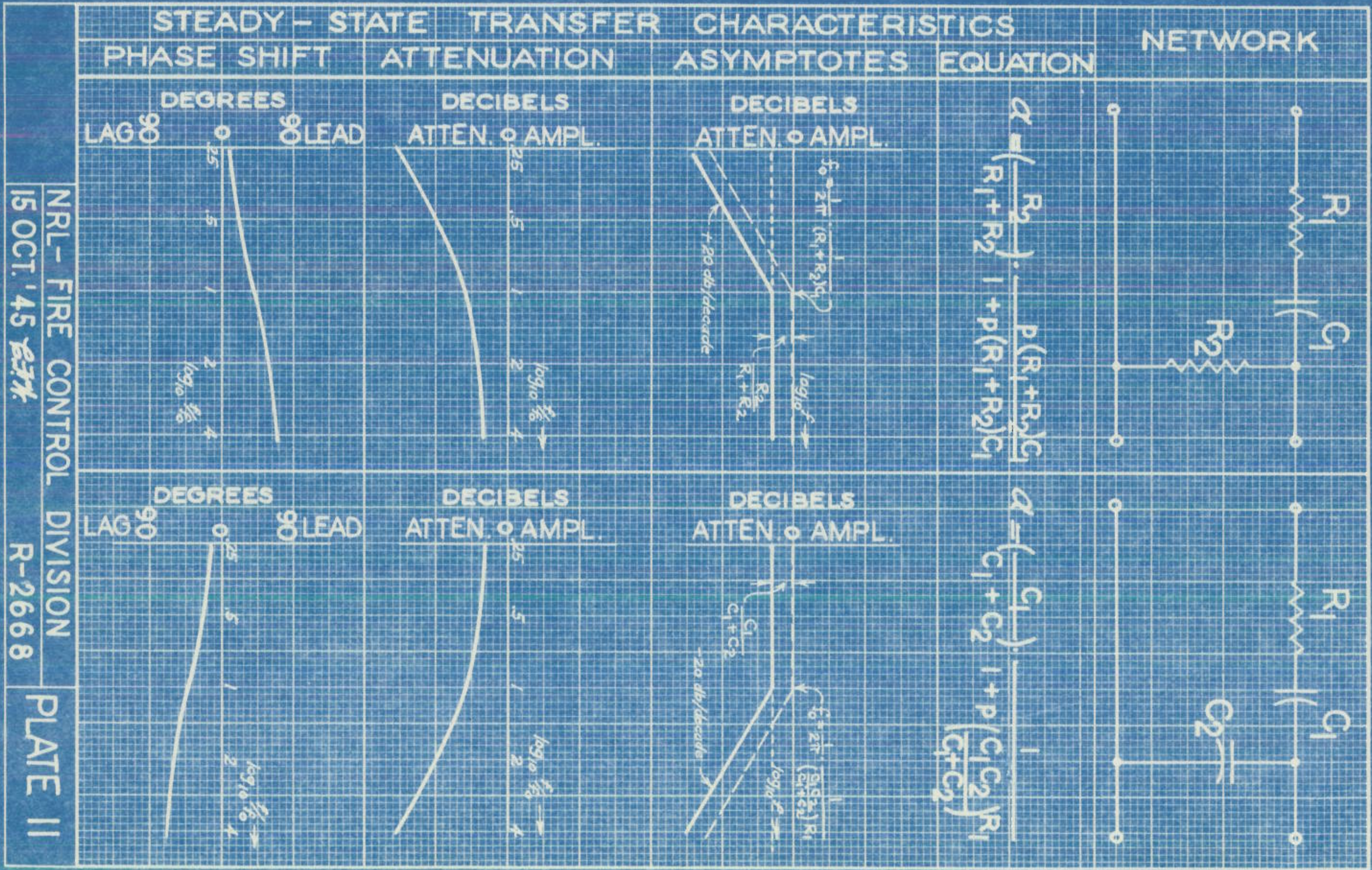
STEADY - STATE TRANSFER CHARACTERISTICS
PHASE SHIFT
ATTENUATION
ASYMPTOTES
EQUATION
NETWORK
DEGREES
LAG 90 0 90 LEAD
DECIBELS
ATTEN. 0 AMPL.
.25 .5 1 2 4
$\log_{10} f/f_0$
$f_0 = \frac{1}{2\pi}\cdot\frac{1}{(R_1+R_2)C_1}$
+20 db/decade
$\frac{R_2}{R_1+R_2}$
$\log_{10} f$
$\alpha = \left(\frac{R_2}{R_1+R_2}\right)\cdot\frac{p(R_1+R_2)C_1}{1+p(R_1+R_2)C_1}$
R1 C1 R2
$f_0 = \frac{1}{2\pi}\cdot\frac{1}{\left(\frac{C_1C_2}{C_1+C_2}\right)R_1}$
-20 db/decade
$\frac{C_1}{C_1+C_2}$
$\alpha = \left(\frac{C_1}{C_1+C_2}\right)\cdot\frac{1}{1+p\left(\frac{C_1C_2}{C_1+C_2}\right)R_1}$
R1 C1 C2
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PLATE II

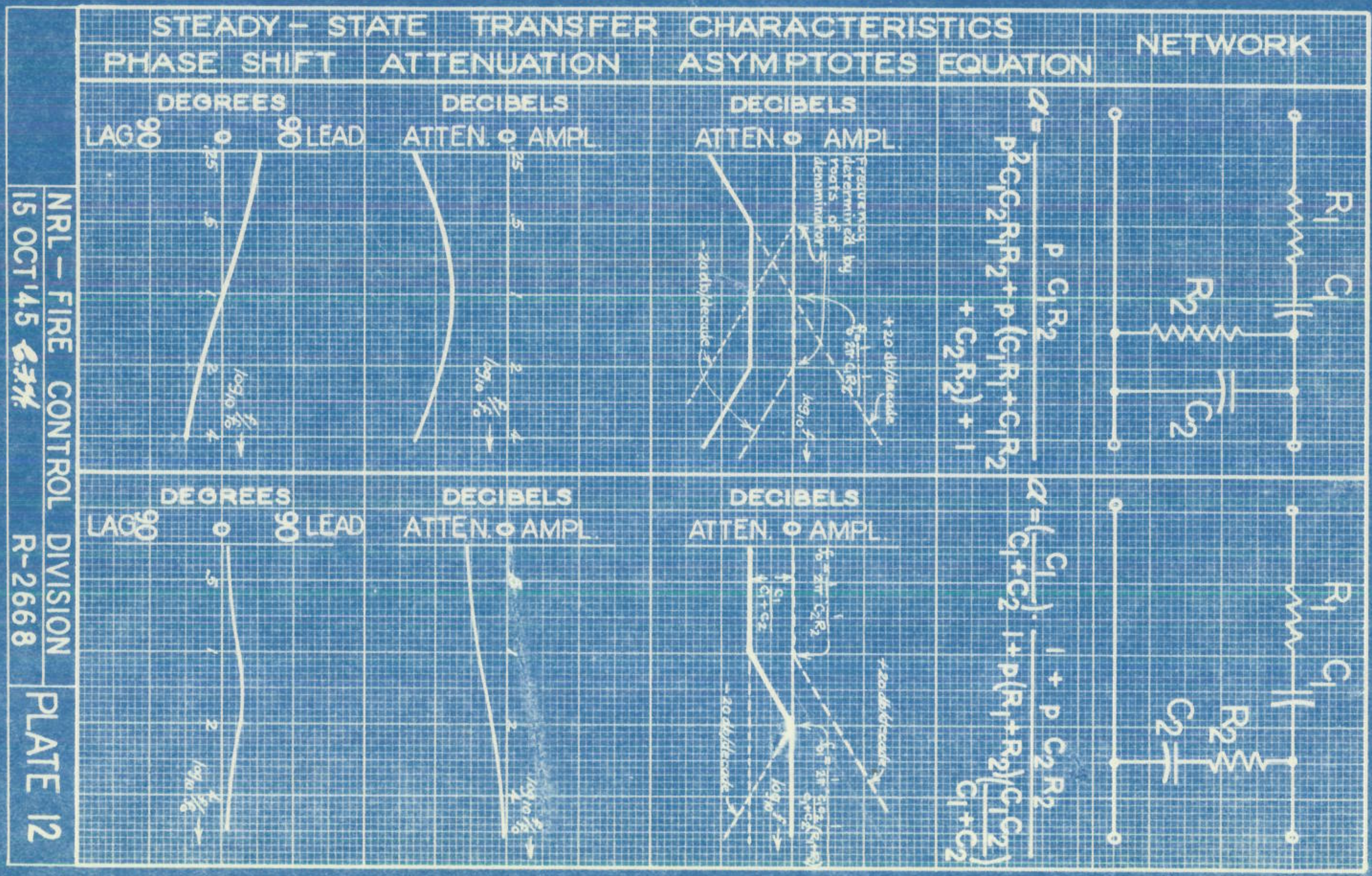
STEADY - STATE TRANSFER CHARACTERISTICS
PHASE SHIFT
ATTENUATION
ASYMPTOTES
EQUATION
NETWORK
DEGREES
LAG 90 0 90 LEAD
DECIBELS
ATTEN. 0 AMPL.
$\alpha = \frac{p\,C_1R_2}{p^2C_1C_2R_1R_2 + p(C_1R_1 + C_1R_2 + C_2R_2) + 1}$
$\alpha = \left(\frac{C_1}{C_1+C_2}\right)\cdot\frac{1 + p\,C_2R_2}{1 + p(R_1+R_2)\left(\frac{C_1C_2}{C_1+C_2}\right)}$
R_1
C_1
R_2
C_2
Frequency determined by roots of denominator
+20 db/decade
-20 db/decade
$\log_{10} f$
$\log_{10} f/f_0$
.25
.5
1
2
4
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PLATE 12

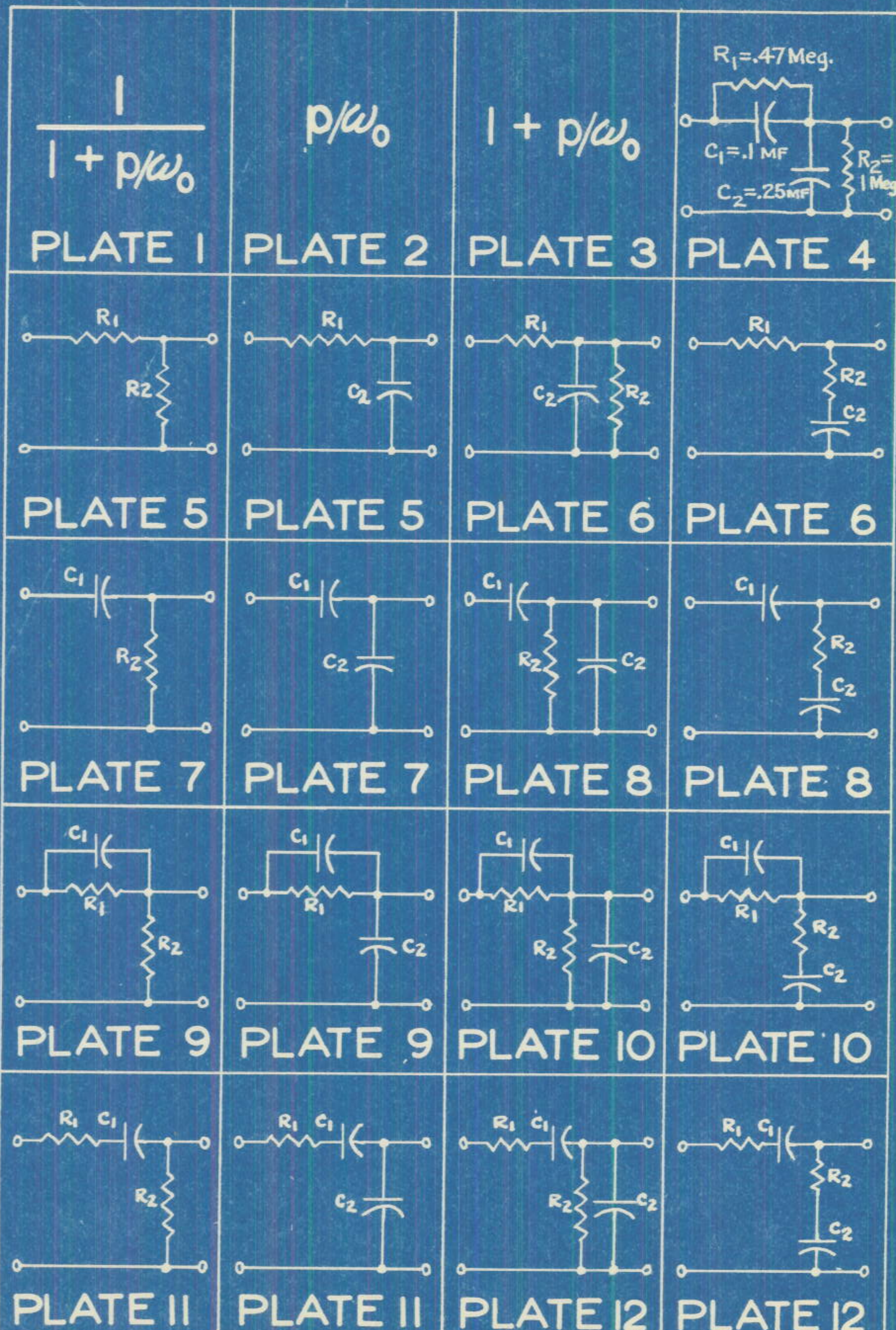

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PLATE 13