

REPORT DOCUMENTATION PAGE			Form Approved OMB NO. 0704-0188		
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1. REPORT DATE (DD-MM-YYYY) 15-04-2022		2. REPORT TYPE Final Report		3. DATES COVERED (From - To) 19-Dec-2018 - 18-Dec-2021	
4. TITLE AND SUBTITLE Final Report: Determination of electrical and geometrical characteristics of mine-like and IED-like targets using new globally convergent inverse algorithms			5a. CONTRACT NUMBER W911NF-19-1-0044		
			5b. GRANT NUMBER		
			5c. PROGRAM ELEMENT NUMBER 611102		
6. AUTHORS			5d. PROJECT NUMBER		
			5e. TASK NUMBER		
			5f. WORK UNIT NUMBER		
7. PERFORMING ORGANIZATION NAMES AND ADDRESSES University of North Carolina - Charlotte Contracts and Grants Office 9201 University City Blvd. Charlotte, NC 28223 -0001			8. PERFORMING ORGANIZATION REPORT NUMBER		
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS (ES) U.S. Army Research Office P.O. Box 12211 Research Triangle Park, NC 27709-2211			10. SPONSOR/MONITOR'S ACRONYM(S) ARO		
			11. SPONSOR/MONITOR'S REPORT NUMBER(S) 72975-MA-H.103		
12. DISTRIBUTION AVAILABILITY STATEMENT Approved for public release; distribution is unlimited.					
13. SUPPLEMENTARY NOTES The views, opinions and/or findings contained in this report are those of the author(s) and should not be construed as an official Department of the Army position, policy or decision, unless so designated by other documentation.					
14. ABSTRACT					
15. SUBJECT TERMS					
16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT UU	15. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON Michael Klivanov
a. REPORT UU	b. ABSTRACT UU	c. THIS PAGE UU			19b. TELEPHONE NUMBER 704-687-0620

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as of 18-Apr-2022

Agency Code: 21XD

Proposal Number: 72975MAH

Agreement Number: W911NF-19-1-0044

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EIN: 560791228

Report Date: 18-Mar-2022

Date Received: 15-Apr-2022

Final Report for Period Beginning 19-Dec-2018 and Ending 18-Dec-2021

Title: Determination of electrical and geometrical characteristics of mine-like and IED-like targets using new globally convergent inverse algorithms

Begin Performance Period: 19-Dec-2018

End Performance Period: 18-Dec-2021

Report Term: 0-Other

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Distribution Statement: 1-Approved for public release; distribution is unlimited.

STEM Degrees: 0

STEM Participants:

Major Goals: 1. MAJOR GOAL 1. The MAIN GOAL of this project was an intensive collaboration of the PI and his team with engineers of the US Army Research Laboratory (ARL) Drs. Anders Sullivan and Lam Nguyen.

2. MAJOR GOAL 2. Analytical and computational developments of various versions of the TRULY REVOLUTIONARY convexification inversion numerical method of the PI to solve various Coefficient Inverse Problems (CIPs) of Army needs.

The convexification is the FOUNDATIONAL numerical method of this project. CIPs are both highly nonlinear and highly ill-posed. Therefore, the problem of constructions of numerical methods for them is one of the MOST CHALLENGING PROBLEMS of Computational Mathematics.

The PI is proud to state that his convexification method addresses this problem for a very broad class of Coefficient Inverse Problems. This is clear from the PI's annual report on this grant.

Furthermore, the convexification is currently the SINGLE known numerical method for CIPs, which addresses this problem.

ALL versions of the convexification are supplied by both: rigorous analytical convergence analysis and numerical studies. The convergence analysis is based on a very powerful tool of Carleman estimates, which was introduced in the field of Inverse Problems by the PI. In all versions of the convexification rigorous error estimates are provided. These estimates depend on the noise level in the data.

Conventional numerical methods for CIPs are based on the minimization of least squares cost functionals. Since these functionals are typically non convex, then they are plagued by the phenomenon of multiple local minima and ravines. This makes minimization methods for those functionals to be unreliable and locally convergent ones, i.e. to ensure the convergence, one needs a good first guess about the solution. However, this guess is rarely available.

On the other hand, convexification is the single known numerical method in the field of Coefficient Inverse Problems, which avoids the phenomenon of multiple local minima and ravines.

3. MAJOR GOAL 3. This one is extra since the discovery described here was a VERY surprising one.

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Goal 3 consists in the developments of a convexification-based Nonlinear SAR imaging (CONSAR): a surprising REVOLUTIONARY joint discovery with Drs. Anders Sullivan and Lam Nguyen. The conventional SAR imaging is based on the linear Born approximation. This is the main reason why the conventional approach allows one to image only locations and shapes of targets of Army interest. But it does not allow one to compute dielectric constants of those targets. Unlike the conventional case, CONSAR solves a series of NONLINEAR Coefficient Inverse Problems using the convexification method of the PI. As a result, CONSAR accurately computes those dielectric constants. Since the dielectric constants characterize constitutive materials of targets, then CONSAR has a great potential to radically improve current SAR capabilities. Since the dielectric constants characterize constitutive materials of targets, then CONSAR has a great potential to radically improve current SAR capabilities via incorporating CONSAR in the currently available SAR software.

4. MAJOR GOAL 4. Analytical and numerical development of convexification-based numerical methods, which would be directly applicable to standoff detection and identification of antipersonnel land mines and improvised explosive devices. These methods should be tested on both computationally simulated experimental data. Two types of experimental data should be used:

4.1 The data collected by ARL.

4.2. The data collected by the Microwave Laboratory located in the campus of The University of North Carolina at Charlotte where the PI works. This laboratory has been funded by this ARO grant and the PI runs this laboratory.

5. MAJOR GOAL 5. This goal is an extra one since the idea for the problem of this goal occurred during the work on this project.

This goal is to solve a highly nonlinear 3D Coefficient Inverse Problem for a wave-like Partial Differential Equation via the solution of a linear integral equation of the first kind.

6. MAJOR GOAL 6. This is an extra goal since the idea for the problem of this goal occurred during the work on this project.

This goal is to solve via the convexification a well known 114 years old coefficient inverse problem. This is the so-called "Travel Time Tomography Problem" (TTTP). TTTP has an immediate application to the Army needs for standoff inspections of buildings in the case when the transmitted rather than the backscattering electric wave is measured. Another well known application is in seismology.

It is well known that TTTP is the single MOST CHALLENGING one among all other Coefficient Inverse Problems.

TTTP was first posed and solved for the 1D case by a German scientist Herglotz in 1905. Many attempts of many experts in Inverse Problems were made since then to solve the TTTP in 2D and 3D cases. Even though some uniqueness results were obtained, their common drawback was that the data were complete in 2D and 3D cases and, in addition, the data were redundant in 3D (i.e. overdetermined). Another significant drawback was the absence of globally convergent numerical methods.

7. MAJOR GOAL 7. This is also an extra goal since ideas for problems of this goal occurred during the work on this project. To solve via the convexification an enormously popular Coefficient Inverse Problem of Electrical Impedance Tomography (EIT). Again, even though many experts in Inverse Problems have worked on EIT since about 1987, globally convergent numerical methods were not developed for this problem prior to the work of the PI in 2019. EIT has many applications in medicine.

8. MAJOR GOAL 8. This is also an extra one since ideas for problems of this goal occurred during the work on this project. To demonstrate a truly UNIVERSAL nature of the convexification concept. The PI wanted to demonstrate that the convexification is applicable not only to inverse problems but to forward problems for CLASSIC nonlinear Partial Differential Equations as well.

9. MAJOR GOAL 9. This is also an extra goal.

This goal is to publish a research book about Carleman estimates and convexification with the acknowledgments of the support of this grant:

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10. MAJOR GOAL 10. This is also an extra goal.

This goal is to publish a textbook on Partial Differential Equations with the acknowledgments of the support of this grant.

Accomplishments: 1. MAJOR GOAL 1.

This goal was successfully achieved with seven (7) joint resulting publications during the reportage period. All seven acknowledge the support of this grant. In fact, this is a TEN (10) YEARS OLD LONG TERM collaboration since the first joint publication in 2012. The total number of joint publications with ARL in the ten (10) years period of 2012-2022 is thirteen (13). In all seven (7) papers published during the reportage period, experimental data are treated by various versions of the convexification method of the PI. Experimental data were collected either by Drs. Anders Sullivan and Lam Nguyen (ARL) or by the PI and his team. The latter was done in a Microwave Laboratory located in the campus of The University of North Carolina at Charlotte where the PI works, The PI is in charge of this laboratory and it was funded by this grant. Two types of applications were successfully addressed using these experimental data:

A. Standoff detection and identification of both buried and non-buried antipersonnel land mines and improvised explosive devices.

B. Standoff inspections of buildings.

2. MAJOR GOAL 2.

Various versions of the convexification globally convergent numerical method of the PI have been developed both analytically and numerically for many Coefficient Inverse Problems of this project. Convexification is currently the SINGLE numerical method for Coefficient Inverse Problems with rigorously guaranteed GLOBAL CONVERGENCE property. In other words, one does not need to know a good first guess for the solution. This is unlike all other currently known numerical method for Coefficient Inverse Problems. All global convergence theorems of this project are accompanied with a priori error estimates for the solutions depending on the noise in the data. A very good numerical performance of the convexification is well demonstrated for a very broad class of Coefficient Inverse Problems for both computationally simulated and experimental data.

3. MAJOR GOAL 3.

A RADICALLY NEW convexification-based Nonlinear SAR Imaging (CONSAR) technique was, quite surprisingly invented jointly by the PI and Drs. Anders Sullivan and Lam Nguyen, engineers of ARL. This method was successfully tested on two types of experimental data:

A. SAR data collected by ARL for the problem of standoff inspections of buildings. Accurate values of dielectric constants of targets were obtained. Furthermore, comparison of our results with the regular SAR images has demonstrated a clear superiority of CONSAR!

B. SAR data collected in the above mentioned Microwave Laboratory run by the PI. These data were collected for a target buried in a sandbox. The target closely mimicked 'Glassmine43'. a plastic antipersonnel landmine used by Germany in WWII era. The dielectric constant of this target was accurately computed by CONSAR.

4. MAJOR GOAL 4:

4.1. Three convexification-based globally convergent numerical methods for a 1D Coefficient Inverse Problem for a wave-like Partial Differential Equation in time domain were developed both analytically and numerically. Surprisingly, one of them, which is presented in section 8 of the pdf file, is based on a synthesis of the classic contraction mapping principle with the convexification. Actually, this is the SECOND GENERATION of the convexification. The connection with the contraction mapping allows the PI to make computations of experimental data in REAL TIME, which is obviously crucial for Army applications. All three methods were applied to experimental data collected by the Forward Looking Radar of ARL. Dielectric constants of real shallow mine-like targets were computed. Since dielectric constants were not directly measured during the data collection, then we have compared our results with tabulated published values of dielectric constants.

Our results coincide well with tabulated values. Drs. Anders Sullivan and Lam Nguyen, engineers of ARL, are

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happy with these results.

4.2. Two convexification-based globally convergent numerical methods were developed both analytically and numerically for 3D Coefficient Inverse Problems for the Helmholtz equation. This equation governs propagation of electric waves in the frequency domain. One of these methods were tested on both computationally simulated and experimental data. The second one was tested only on computationally simulated data. An important and quite rare feature of the first method is that it computes TWO coefficients of the Helmholtz equation AT ONCE: the dielectric constant and the electric conductivity of targets.

Experimental data were collected in the above mentioned Microwave Laboratory run by the PI. All five (5) targets were buried in a sandbox, which obviously made the problem of their imaging a VERY COMPLICATED one. All these targets were mimicking landmines and improvised explosive devices. One target closely modeled the above mentioned 'Glassmine43', which was used by Germany in World War II era. Other targets had quite sophisticated non-convex shapes. It is well known that targets with such shapes are hard to image.

Nevertheless, three (3) most important components of all five targets were accurately computed:

- A. Locations.
- B. Shapes
- C. Dielectric constants.

5. MAJOR GOAL 5.

A 3D Coefficient Inverse Problem with slightly overdetermined data was reduced to the solution of a linear integral equation of the first kind. A coupled system of elliptic Partial Differential Equations was derived from that integral equation. Boundary conditions on this system were obtained. Then the resulting boundary value problem was solved by the Quasi-Reversibility Method, which is a linear analog of the convexification method. Convergence analysis was provided.. This analysis is based on a Carleman estimate, so as any that analysis for any version of the convexification method. High quality numerical results were obtained.

6. MAJOR GOAL 6.

In two of his works of 2019 the PI has solved that 114 years old Travel Time Tomography Problem (TTTP). He has done this for the most challenging case when the data are both non overdetermined and incomplete. More precisely, he has developed a REVOLUTIONARY convexification-based globally convergent numerical method for TTTP. In the follow up work of 2020 of the PI with coauthors PIONEERING numerical studies are presented for the linearized case of TTTP by another version of the convexification. Numerical results demonstrate a REMARKABLE robustness of the algorithm of the PI. Indeed, good quality images of sophisticated targets were obtained for very high noise levels varying between 80% and 170%.

7. MAJOR GOAL 7.

A GLOBALLY convergent numerical method for the Coefficient Inverse Problem of Electrical Impedance Tomography in the n-D case, $n=2,3$ is developed. This is one of versions of the convexification concept. All other methods for EIT use over determined data in the 3D case, i.e. the data are redundant. Unlike this, the convexification works with non redundant data. Numerical testing demonstrates a very good performance. This is another BREAKTHROUGH result.

8. MAJOR GOAL 8.

Numerical methods for finding "viscosity" solutions of the classic Hamilton-Jacobi nonlinear Partial Differential Equation work only under restrictive conditions imposed on the underlying differential operator. Unlike this, the PI with coauthors has developed a version of the globally convergent convexification method for this problem. This method works for a broad class operators.

CONCLUSION: the convexification is a UNIVERSAL NUMERICAL METHOD.

The support of this grant is acknowledged in:

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9. MAJOR GOAL 9.

A research book is published: M.V. Klibanov and J. Li, Inverse Problems and Carleman Estimates: Global Uniqueness, Global Convergence and Experimental Data, De Gruyter, Germany, 2021.

10. MAJOR GOAL 10.

A textbook is published: M.V. Klibanov and J. Li, Partial Differential Equations, Nova Publishers, New York, USA, 2022.

Training Opportunities: The following six (6) people were trained:

1. Mr. Vo Khoa, Ph.D., a postdoctoral research associate. Dr. Khoa is a coauthor of five (5) joint papers with the acknowledgment of the support of this grant. Two (2) of them are about the REVOLUTIONARY convexification-based nonlinear SAR imaging, see Goal 3 and Accomplishments for Goal 3. Three (3) more papers are about two versions of the convexification method for a Coefficient Inverse Problem for the Helmholtz equation, see Goal 4.2 and Accomplishments for this goal. Four (4) out of five (5) publications work with experimentally collected data. Drs. Anders Sullivan and Lam Nguyen, engineers of ARL, are coauthors of these four.

These publications pave a way for Dr. Khoa to get a tenure track position at Florida A&M University. A peculiar point in the latter is that he got this position in 2021 during the pandemic when the number of such positions has severely declined.

2. Mrs. Thuy Le, a graduate student of the Department of Mathematics and Statistics of The University of North Carolina at Charlotte where the PI works. The PI is a Thesis Advisor of Le. Mrs. Le is a coauthor of seven (7) papers with the acknowledgment of the support of this grant. Two of these are about a 1D Coefficient Inverse Problem for a wave-like Partial Differential Equation. They work with experimental data collected by the Forward Looking Radar of ARL. Drs. Anders Sullivan and Lam Nguyen, engineers of ARL, are coauthors of these two. The PI is a coauthor of three out of those seven (7) papers.

3. Mr. Alexey Smirnov, a graduate student of the Department of Mathematics and Statistics of The University of North Carolina at Charlotte where the PI works. Mr. Smirnov has coauthored with the PI five (5) publications with the acknowledgment of the support of this grant. Two (2) of these five work treat SAR experimental data via CONSAR. One paper treats experimental data collected by the Forward Looking Radar of ARL. Drs. Anders Sullivan and Lam Nguyen, engineers of ARL, are coauthors of these three (3) papers with experimental data.

4. Mr. Ray Abney, a graduate student of the Department of Mathematics and Statistics of The University of North Carolina at Charlotte where the PI works.

5. Mr. William Powell, a graduate student of the Department of Mathematics and Statistics of The University of North Carolina at Charlotte where the PI works. Mr. Powell has coauthored one publication with the acknowledgment of the support of this grant.

6. Mr. Grant Bidney, a graduate student of the Department of Physics of The University of North Carolina at Charlotte. Mr Bidney has collected the experimental data in the Microwave Laboratory run by the PI. This laboratory has been funded by this grant. Mr. Bidney has coauthored with the PI three (3) publications with the acknowledgment of the support of this grant. Drs. Anders Sullivan and Lam Nguyen, engineers of ARL, are coauthors of these three (3) papers.

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Results Dissemination: 1. PRESENTATIONS:

The PI and his research team have presented results of this project in:

1. Two talks in North Carolina State University (2020).
2. Two talks on the 5th annual Meeting of SIAM Central States Section at Iowa State University (2020).
3. Three talks on an annual meeting of American Mathematical Society (2021).

Although a number of other presentations was planned, but all of them were cancelled due to the pandemic.

2. PUBLICATIONS

Total thirty three (33) publications with the acknowledgment of the support of this grant. This means $33/3=11$ publications annually as the average.

Among them:

A. Two books, see [1] and [2] below.

B. Seven (7) joint publications with Drs. Anders Sullivan and Lam Nguyen, engineers of ARL, see [3]-[9] below.

C. The total number of joint publications with Sullivan and Nguyen during long term 10 years old collaboration of the PI with them in 2012-2022 is thirteen (13).

LIST OF PUBLICATIONS WITH THE ACKNOWLEDGMENT OF THE SUPPORT OF THIS GRANT.

[1] M.V. Klibanov and J. Li, Partial Differential Equations: Theory, Numerical Methods and Ill-Posed Problems, Nova Publishing, 2022.

[2] M.V. Klibanov and J. Li, Inverse Problems and Carleman Estimates: Global Uniqueness, Global Convergence and Experimental Data, De Gruyter, a German publisher, 2021.

[3] M.V. Klibanov, A.V. Smirnov, K.A. Vo, A.J. Sullivan and Lam H. Nguyen, Through-the-wall nonlinear SAR imaging, IEEE Transactions on Geoscience and Remote Sensing, 59, 7475-7486, 2021.

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[16] M.V. Klibanov, T.T. Le and L.H. Nguyen, Convergent numerical method for a linearized travel time tomography problem with incomplete data, *SIAM J. Scientific Computing*, 45, B1173-B1192, 2020.

[17] M.V. Klibanov and A.G. Yagola, Convergent numerical methods for parabolic equations with reversed time via a new Carleman estimate, *Inverse Problems*, 35, 115012, 2019.

[18] M.V. Klibanov, J. Li and W. Zhang, Linear Lavrent'ev integral equation for the numerical solution of a nonlinear coefficient inverse problem, *SIAM J. Applied Mathematics*, 81, 1954-1978, 2021.

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Honors and Awards: The PI got two honors during his work on this grant:

1. THE FIRST HONOR:

A laudation paper published in a reputable Journal of Inverse and Ill-Posed Problems, 30, 1-4, 2022. This paper is written on the occasion of the PI's 70st anniversary. It is signed by eleven (11) members of the editorial board. The title of this paper is: "Research biography of a distinguished expert in the field of inverse problems: Professor Michael Victor Klibanov". This paper is published in a special issue dedicated to the PI's 70st anniversary, <https://www.degruyter.com/journal/key/jiip/30/1/html>

2. THE SECOND HONOR.:

A special minisimposium dedicated to the PI's 70st anniversary will be held on the conference "Inverse Problems: Modeling and Simulation" to be held in Malta in May 2022 <https://www.ipms-conference.org/ipms2022/index.php>, see minisimposium M9.

Protocol Activity Status:

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Technology Transfer: The MAIN GOAL of this project was an intensive collaboration of the PI with Drs. Anders Sullivan and Lam Nguyen, engineers of the US Army Research Laboratory. As a result of this collaboration, seven (7) joint papers of the PI with Sullivan and Nguyen were published, see references [1]-[7] below. All of them acknowledge the support of this grant. In fact, the collaboration of the PI with ARL is ongoing for the last ten (10) years with resulting thirteen (13) joint publications covering 2012-2022, see [1]-[13] below.

For a better explanation of the material below in this section, we recall that the key mathematical tool of this grant has been the REVOLUTIONARY convexification concept of the PI, see "Major Goals" and "Accomplished". The convexification is currently the SINGLE known numerical concept for Coefficient Inverse Problems with a rigorously guaranteed GLOBAL convergence property.

During this reportage period, the most intriguing result of this collaboration was a surprising joint discovery of the Convexification-based Nonlinear SAR Imaging (CONSAR). The PI refers here to publications [1,2] (below) as well as to Major Goal 3 and Accomplishments for Major Goal 3. CONSAR has very promising and FAR GOING future applications to the Army needs. In fact, all currently commonly used software for SAR imaging would be updated to incorporate CONSAR in it. This is why the discovery of CONSAR is a REVOLUTIONARY one!

The conventional SAR imaging is based on the linear Born approximation. This is the main reason why the conventional approach allows one to image only locations and shapes of targets of Army interest. But it does not allow one to compute dielectric constants of those targets. Unlike the conventional case, CONSAR solves a series of NONLINEAR Coefficient Inverse Problems using the convexification method of the PI. As a result, CONSAR accurately computes those dielectric constants. Since the dielectric constants characterize constitutive materials of targets, then CONSAR has a great potential to radically improve current SAR capabilities.

The performance of CONSAR was tested on two sets of experimentally collected SAR data:

1. The SAR data collected by ARL for the problem on standoff inspections of buildings [1]. It was demonstrated that CONSAR computes with a good accuracy dielectric constants of targets hidden inside the building. Furthermore, comparison with the regular SAR images demonstrates a clear superiority of CONSAR when computing dielectric constants of targets.
2. The SAR data collected in the Microwave Laboratory located in The University of North Carolina at Charlotte, where the PI works [2]. The PI is in charge of this laboratory and it has been funded by this grant. These data were collected for a target buried in a sandbox. The target closely mimicked 'Glassmine43', a plastic antipersonnel landmine used by Germany in WWII era. The dielectric constant of this target was accurately computed by CONSAR in [2].

In addition to CONSAR, the PI has collaborated during the reportage period with Drs. Anders Sullivan and Lam Nguyen from ARL on the problem of standoff detection and identification of antipersonnel land mines and improvised explosive devices.

More precisely, this collaboration was focused on the following two topics:

TOPIC A.

Application of three versions of the convexification method to experimental data collected by the Forward Looking Radar of ARL. The goal is to Dielectric constants of real shallow mine-like targets were computed. Since dielectric constants were not directly measured during the data collection, then we have compared our results with tabulated published values of dielectric constants. Our results coincide well with tabulated values. Drs. Anders Sullivan and Lam Nguyen, engineers of ARL, are happy with these results.

TOPIC B.

Application of a version of the convexification method for a 3D Coefficient Inverse Problem for the Helmholtz equation to the experimental data collected in the above mentioned Microwave Laboratory run by the PI. Five (5) targets were mimicking antipersonnel landmines and improvised explosive devices. In particular, one of targets was closely mimicking 'Glassmine43', a plastic mine used by Germany in the World War II era. Other targets had quite sophisticated non-convex shapes. It is well known that targets with such shapes are hard to image. .

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Nevertheless, three (3) most important components of all five targets were accurately computed:

A. Locations.

B. Shapes.

C. Dielectric constants.

The list joint publications with ARL covering the period 2012-2022 is below. Papers [1]-[7] were published during the reportage period and they acknowledge the support of this grant. Papers [8]-[13] were published prior this award.

[1] M.V. Klibanov, A.V. Smirnov, K.A. Vo, A.J. Sullivan and Lam H. Nguyen, Through-the-wall nonlinear SAR imaging, IEEE Transactions on Geoscience and Remote Sensing, 59, 7475-7486, 2021.

[2] M. V. Klibanov, V.A. Khoa, A.V. Smirnov, Loc H. Nguyen, G.W. Bidney, Lam H. Nguyen, A.V. Sullivan and V.N. Astratov, Convexification and inversion method for nonlinear SAR imaging with experimentally collected data, J. Applied and Industrial Mathematics, 15, 413-436, 2021.

[3] V. A. Khoa, G. W. Bidney, M. V. Klibanov, L. H. Nguyen, A. J. Sullivan, Lam Nguyen and V. N. Astratov, Convexification and experimental data for a 3D inverse scattering problem with the moving point source, Inverse Problems, 36, 085007, 2020.

[4] V. A. Khoa, G. W. Bidney, M. V. Klibanov, L. H. Nguyen, A. J. Sullivan, Lam Nguyen and V. N. Astratov, An inverse problem of a simultaneous reconstruction of the dielectric constant and conductivity from experimental backscattering data, Inverse Problems in Science and Engineering, 29, 712-735, 2021.

[5] M. V. Klibanov, T.T. Le, Loc Nguyen, A. Sullivan and Lam Nguyen, Convexification-based globally convergent numerical method for a 1D coefficient inverse problem with experimental data, preprint, Inverse Problems and Imaging, published online, doi:10.3934/ipi.2021068, 2021.

[6] T. T. Le, M.V. Klibanov, Loc H. Nguyen, A. Sullivan and Lam Nguyen, Carleman contraction mapping for a 1D inverse scattering problem with experimental time-dependent data, Inverse Problems, 38, 045002, 2022.

[7] A.V. Smirnov, M.V. Klibanov, A.J. Sullivan and L.H. Nguyen, Convexification for an inverse problem for a 1D wave equation with experimental data, Inverse Problems, 36, 095008, 2020.

Joint papers with ARL published prior this award:

[8] M.V. Klibanov, A. E. Kolesov, Lam Nguyen and A. Sullivan, A new version of the convexification method for a 1-D coefficient inverse problem with experimental data, Inverse Problems, 34, 115014, 2018.

[9] M.V. Klibanov, A. E. Kolesov, Lam Nguyen and A. Sullivan, Globally strictly convex cost functional for a 1-D inverse medium scattering problem with experimental data, SIAM J. Applied Mathematics, 77, 1733-1755, 2017.

[10] M.V. Klibanov, L.H. Nguyen, A. Sullivan and L. Nguyen, A globally convergent numerical method for a 1-D inverse problem with experimental data, Inverse Problems and Imaging, 10, 1057-1085, 2016.

[11] A. L. Karchevsky, M. V. Klibanov, N. Pantong, L. Nguyen and A. Sullivan, The Krein method and the globally convergent method for experimental data, Applied Numerical Mathematics, 74, 111-127, 2013.

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[12] A. V. Kuzhuget, L. Beilina, M. V. Klibanov, A. Sullivan, L. Nguyen and M. A. Fiddy, Quantitative image recovery from measured blind backscattered data using a globally convergent inverse method, IEEE Transaction for Geoscience and Remote Sensing, 51, 2937-2948, 2013.

[13] A. V. Kuzhuget, L. Beilina, M. V. Klibanov, A. Sullivan, L. Nguyen and M. A. Fiddy, Blind backscattering experimental data collected in the eld and an approximately globally convergent inverse algorithm, Inverse Problems, 28, 095007, 2012.

PARTICIPANTS:

Participant Type: PD/PI

Participant: Michael Victor Klibanov

Person Months Worked: 6.00

Project Contribution:

National Academy Member: N

Funding Support:

Participant Type: Co PD/PI

Participant: Astratov N Vasili

Person Months Worked: 3.00

Project Contribution:

National Academy Member: N

Funding Support:

Participant Type: Faculty

Participant: Loc Hoang Nguyen

Person Months Worked: 3.00

Project Contribution:

National Academy Member: N

Funding Support:

Participant Type: Postdoctoral (scholar, fellow or other postdoctoral position)

Participant: Vo Khoa

Person Months Worked: 15.00

Project Contribution:

National Academy Member: N

Funding Support:

Participant Type: Consultant

Participant: Steven Kitichin

Person Months Worked: 1.00

Project Contribution:

National Academy Member: N

Funding Support:

Participant Type: Graduate Student (research assistant)

Participant: Ray Abney

Person Months Worked: 3.00

Project Contribution:

National Academy Member: N

Funding Support:

RPPR Final Report as of 18-Apr-2022

Participant Type: Graduate Student (research assistant)
Participant: Grant Bidney
Person Months Worked: 15.00
Funding Support:
Project Contribution:
National Academy Member: N

Participant Type: Graduate Student (research assistant)
Participant: Thuy T. Le
Person Months Worked: 5.00
Funding Support:
Project Contribution:
National Academy Member: N

Participant Type: Graduate Student (research assistant)
Participant: Alexey V Smirnov
Person Months Worked: 12.00
Funding Support:
Project Contribution:
National Academy Member: N

ARTICLES:

Publication Type: Journal Article Peer Reviewed: Y **Publication Status:** 1-Published
Journal: Journal of Computational Physics
Publication Identifier Type: DOI **Publication Identifier:** <http://dx.doi.org/10.1016/j.apnum.2017.05.00>
Volume: 345 **Issue:** 1 **First Page #:** 17
Date Submitted: 10/27/18 12:00AM **Date Published:** 1/28/17 6:24AM
Publication Location:
Article Title: Numerical solution of a coefficient inverse problem with multi-frequency experimental raw data by a globally convergent algorithm
Authors: Dinh-Liem Nguyen, Michael V. Klibanov, Loc H. Nguyen, Aleksandr Kolesov, Michael A. Fiddy, Hui Liu
Keywords: experimental multi-frequency data, backscatter data, inverse scattering, globally convergent methods, coefficient identification, microwave imaging
Abstract: We analyze in this paper the performance of a newly developed globally convergent numerical method for a coefficient inverse problem for the case of multi-frequency experimental backscatter data associated to a single incident wave. These data were collected using a microwave scattering facility at the University of North Carolina at Charlotte. The challenges for the inverse problem under the consideration are not only from its high nonlinearity and severe ill-posedness but also from the facts that the amount of the measured data is minimal and that these raw data are contaminated by a significant amount of noise, due to a non-ideal experimental setup. This setup is motivated by our target application in detecting and identifying explosives. We show in this paper how the raw data can be preprocessed and successfully inverted using our inversion method. More precisely, we are able to reconstruct the dielectric constants and the locations of the scattering objects with a good accuracy.
Distribution Statement: 3-Distribution authorized to U.S. Government Agencies and their contractors
Acknowledged Federal Support: Y

RPPR Final Report as of 18-Apr-2022

Publication Type: Journal Article Peer Reviewed: Y **Publication Status:** 1-Published
Journal: Applied Numerical Mathematics
Publication Identifier Type: DOI **Publication Identifier:** <http://dx.doi.org/10.1016/j.apnum.2017.05.00>
Volume: 120 **Issue:** 1 **First Page #:** 176
Date Submitted: 10/27/18 12:00AM **Date Published:** 10/25/16 12:22AM
Publication Location:
Article Title: Single measurement experimental data for an inverse medium problem inverted by a multi-frequency globally convergent numerical method
Authors: Aleksandr Kolesov, Michael V. Klibanov, Loc H. Nguyen, Dinh-Liem Nguyen and Nguyen T. Thanh
Keywords: experimental time dependent data, multi-frequency data, global convergence, coefficient inverse problems, inverse scattering problems
Abstract: The recently developed globally convergent numerical method for an inverse medium problem for the Helmholtz equation proposed in [25] is tested on experimental data. The data were originally collected in the time domain, whereas the method works in the frequency domain with the multi-frequency data. Due to a huge discrepancy between the collected and computationally simulated data, the straightforward Fourier transform of the experimental data does not work. Hence, it is necessary to develop a heuristic data preprocessing procedure. This procedure is described. The preprocessed data are used as the input for the inversion algorithm. Numerical results demonstrate good accuracy in the reconstruction of both refractive indices and locations of targets. Furthermore, the reconstruction errors for refractive indices of dielectric targets are significantly less than errors of a posteriori direct measurements.
Distribution Statement: 3-Distribution authorized to U.S. Government Agencies and their contractors
Acknowledged Federal Support: Y

Publication Type: Journal Article Peer Reviewed: Y **Publication Status:** 1-Published
Journal: Journal of Inverse and Ill-Posed Problems
Publication Identifier Type: DOI **Publication Identifier:** 10.1515/jiip-2016-0039
Volume: 24 **Issue:** 6 **First Page #:** 761
Date Submitted: 10/27/18 12:00AM **Date Published:** 7/27/16 8:40PM
Publication Location: Germany
Article Title: Numerical solution of an ill-posed Cauchy problem for a quasilinear parabolic equation using a Carleman weight function
Authors: Michael V. Klibanov, Nikolaj A. Koshev, Jingzhi Li and Anatoly G. Yagola
Keywords: Ill-posed Cauchy problem, quasilinear parabolic PDE, numerical solution, Carleman weight function
Abstract: We solve numerically the side Cauchy problem for a 1-D parabolic equation. The initial condition is unknown. This is an ill-posed problem. The main difference with previous results is that our equation is quasilinear, whereas known publications on this topic work only with linear PDEs. The key idea is to minimize a weighted Tikhonov functional with the Carleman Weight Function (CWF) in it. Roughly, given a reasonable bounded set of any size in a reasonable Hilbert space, one can choose the parameter of the CWF in such a way that this functional becomes strictly convex on that set.
Distribution Statement: 3-Distribution authorized to U.S. Government Agencies and their contractors
Acknowledged Federal Support: Y

RPPR Final Report as of 18-Apr-2022

Publication Type: Journal Article Peer Reviewed: Y **Publication Status:** 1-Published

Journal: SIAM Journal on Imaging Sciences

Publication Identifier Type: DOI

Publication Identifier: 10.1137/18M1179560

Volume: 11

Issue: 4

First Page #: 2339

Date Submitted: 10/27/18 12:00AM

Date Published: 10/27/18 7:52PM

Publication Location: Philadelphia

Article Title: A numerical method to solve a phaseless coefficient inverse problem from a single measurement of experimental data

Authors: Michael V. Klibanov, Nikolaj A. Koshev, Dinh-Liem Nguyen, Loc H. Nguyen, Aaron Brettin, Vasilii N. Ast

Keywords: phaseless coefficient inverse problem, phased coefficient inverse problem, optical experimental data, single measurement, new numerical method, numerical reconstructions

Abstract: The goal of this paper is to develop a globally convergent numerical method for the inverse problem which would work with the optical experimental data collected by this research group. Only the intensity (modulus square) of the total complex valued wave field is measured at an array of light detectors located on a measurement plane. The phase is not measured. We solve numerically the phaseless coefficient inverse problem to reconstruct locations and refractive indices of unknown scatterers from those measurements. Our method consists of two stages. In the first stage, we obtain an upper estimate for the modulus of the scattered wave field. This estimate allows us to approximately reconstruct the wave field at the measurement plane using a resulting inversion formula. This reduces the phaseless inverse scattering problem to the phased one. In the second stage, we apply a recently developed globally convergent numerical method to reconstruct the desired refractive index.

Distribution Statement: 3-Distribution authorized to U.S. Government Agencies and their contractors

Acknowledged Federal Support: Y

Publication Type: Journal Article

Peer Reviewed: Y

Publication Status: 1-Published

Journal: Inverse Problems

Publication Identifier Type: DOI

Publication Identifier: <https://doi.org/10.1088/1361-6420/aadbc6>

Volume: 34

Issue: 9

First Page #:

Date Submitted: 10/27/18 12:00AM

Date Published: 9/21/18 8:00PM

Publication Location: United Kingdom

Article Title: A new version of the convexification method for a 1D coefficient inverse problem with experimental data

Authors: Michael V. Klibanov, Aleksandr E. Kolesov, Anders Sullivan and Lam Nguyen

Keywords: experimental data, coefficient inverse problem, convexification, Carleman weight function, numerical method, global convergence

Abstract: A new version of the convexification method is developed analytically and tested numerically for a 1D coefficient inverse problem in the frequency domain. Unlike the previous version, this one does not use the so-called 'tail function', which is a complement of a certain truncated integral with respect to the wave number. Globally strictly convex cost functional is constructed with the Carleman weight function. Global convergence of the gradient projection method to the correct solution is proved. Numerical tests are conducted for both computationally simulated and experimental data.

Distribution Statement: 3-Distribution authorized to U.S. Government Agencies and their contractors

Acknowledged Federal Support: Y

RPPR Final Report
as of 18-Apr-2022

Publication Type: Journal Article Peer Reviewed: Y **Publication Status:** 1-Published

Journal: SIAM Journal on Applied Mathematics

Publication Identifier Type: DOI

Publication Identifier: 10.1137/17M1122487

Volume: 77

Issue: 5

First Page #: 1733

Date Submitted: 10/27/18 12:00AM

Date Published: 10/4/17 8:12PM

Publication Location: Philadelphia

Article Title: Globally strictly convex cost functional for a 1-D inverse medium scattering problem with experimental data

Authors: Michael V. Klibanov, Alexandr E. Kolesov, Lam Nguyena and Anders Sullivan

Keywords: global convergence, coefficient inverse problem, multifrequency data, Carleman weight function

Abstract: A new numerical method is proposed for a one-dimensional inverse medium scattering problem with multifrequency data. This method is based on the construction of a weighted cost functional. The weight is a Carleman weight function (CWF). In other words, this is the function which is present in the Carleman estimate for the underlying differential operator. The presence of the CWF makes this functional strictly convex on any a priori chosen ball with the center at f_0g in an appropriate Hilbert space. Convergence of the gradient minimization method to the exact solution starting from any point of that ball is proved. Computational results for both computationally simulated and experimental data show a good accuracy of this method.

Distribution Statement: 3-Distribution authorized to U.S. Government Agencies and their contractors

Acknowledged Federal Support: Y

Publication Type: Journal Article Peer Reviewed: Y **Publication Status:** 1-Published

Journal: Journal of Inverse and Ill-Posed Problems

Publication Identifier Type: DOI

Publication Identifier: 10.1515/jiip-2017-0067

Volume: 25

Issue: 5

First Page #: 669

Date Submitted: 10/27/18 12:00AM

Date Published: 10/27/17 9:13PM

Publication Location: Germany

Article Title: Convexification of Restricted Dirichlet-to-Neumann Map

Authors: Michael V. Klibanov

Keywords: restricted Dirichlet-to-Neumann data, convexification, global strict convexity, Carleman Weight Functions

Abstract: By our definition, "restricted Dirichlet-to-Neumann map" (DN) means that the Dirichlet and Neumann boundary data for a Coefficient Inverse Problem (CIP) are generated by a point source running along an interval of a straight line. On the other hand, the conventional DN data can be generated, at least sometimes, by a point source running along a hypersurface. CIPs with the restricted DN data are non-overdetermined in the n -D case with $n \geq 2$. We develop, in a unified way, a general and a radically new numerical concept for CIPs with restricted DN data for a broad class of PDEs of the second order, such as, e.g. elliptic, parabolic and hyperbolic ones. Namely, using Carleman Weight Functions, we construct globally convergent numerical methods. Hölder stability and uniqueness are also proved. The price we pay for these features is a well acceptable one in the Numerical Analysis: we truncate a certain Fourier-like series with respect to some functions depending on the position of that source.

Distribution Statement: 3-Distribution authorized to U.S. Government Agencies and their contractors

Acknowledged Federal Support: Y

RPPR Final Report

as of 18-Apr-2022

Publication Type: Journal Article Peer Reviewed: Y **Publication Status:** 1-Published

Journal: SIAM Journal on Imaging Sciences

Publication Identifier Type: DOI

Publication Identifier: DOI. 10.1137/18M1191658

Volume: 12

Issue: 1

First Page #: 576

Date Submitted: 4/21/19 12:00AM

Date Published: 1/20/19 2:30AM

Publication Location:

Article Title: Convexification method for a coefficient inverse problem and its performance for experimental backscatter data for buried targets

Authors: Michael V. Klibanov, Aleksandr E. Kolesov and Dinh-Liem Nguyen

Keywords: Carleman weight function, Carleman estimates, reconstruction method, convexification, global convergence, coefficient inverse problem, experimental data

Abstract: We present in this paper a novel numerical reconstruction method for solving a 3D coefficient inverse problem with scattering data generated by a single direction of the incident plane wave. This inverse problem is well-known to be a highly nonlinear and ill-posed problem. Therefore, optimization-based reconstruction methods for solving this problem would typically suffer from the local-minima trapping and require strong a priori information of the solution. To avoid these problems, in our numerical method, we aim to construct a cost functional with a globally strictly convex property, whose minimizer can provide a good approximation for the exact solution of the inverse problem. The key ingredients for the construction of such functional are an integro-differential formulation of the inverse problem and a Carleman weight function. Under a (partial) finite difference approximation, the global strict convexity is proven using the tool of Carleman estimates.

Distribution Statement: 3-Distribution authorized to U.S. Government Agencies and their contractors

Acknowledged Federal Support: Y

Publication Type: Journal Article

Peer Reviewed: Y

Publication Status: 1-Published

Journal: Inverse Problems and Imaging

Publication Identifier Type: DOI

Publication Identifier: 10.3934/ipi.2018021

Volume: 12

Issue: 2

First Page #: 493

Date Submitted: 10/27/18 12:00AM

Date Published:

Publication Location:

Article Title: Globally convergent numerical method for a 3D coefficient inverse problem with a single measurement of multi-frequency data

Authors: Michael V. Klibanov, Dinh-Liem Nguyen, Loc H. Nguyen, Hui Liu

Keywords: global convergence, coefficient inverse problem, inverse medium problem, coefficient identification, multi-frequency data, experimental data, single measurement

Abstract: The goal of this paper is to reconstruct spatially distributed dielectric constants from complex-valued scattered wave field by solving a 3D coefficient inverse problem for the Helmholtz equation at multi-frequencies. The data are generated by only a single direction of the incident plane wave. To solve this inverse problem, a globally convergent algorithm is analytically developed. We prove that this algorithm provides a good approximation for the exact coefficient without any a priori knowledge of any point in a small neighborhood of that coefficient. This is the main advantage of our method, compared with classical approaches using optimization schemes. Numerical results are presented for both computationally simulated data and experimental data. Potential applications of this problem are in detection and identification of explosive-like targets.

Distribution Statement: 3-Distribution authorized to U.S. Government Agencies and their contractors

Acknowledged Federal Support: Y

RPPR Final Report as of 18-Apr-2022

Publication Type: Journal Article Peer Reviewed: Y **Publication Status:** 1-Published
Journal: Inverse Problems
Publication Identifier Type: DOI Publication Identifier: <https://doi.org/10.1088/1361-6420/aa7a18>
Volume: 32 Issue: 6 First Page #:
Date Submitted: 10/27/18 12:00AM Date Published: 1/2/17 11:10PM
Publication Location: United Kingdom
Article Title: Uniqueness of a 3-D coefficient inverse scattering problem without the phase information
Authors: Michael V. Klibanov and Vladimir G. Romanov
Keywords: phaseless data, inverse scattering problem, uniqueness theorem
Abstract: We use a new method to prove the uniqueness theorem for a coefficient inverse scattering problem without the phase information for the 3-D Helmholtz equation. We consider the case when only the modulus of the scattered wave field is measured and the phase is not measured. The spatially distributed refractive index is the subject of interest in this problem. Applications of this problem are in imaging of nanostructures and biological cells.
Distribution Statement: 3-Distribution authorized to U.S. Government Agencies and their contractors
Acknowledged Federal Support: Y

Publication Type: Journal Article Peer Reviewed: Y **Publication Status:** 1-Published
Journal: Inverse Problems and Imaging
Publication Identifier Type: DOI Publication Identifier: [doi:10.3934/ipi.2017013](https://doi.org/10.3934/ipi.2017013)
Volume: 11 Issue: 2 First Page #: 263
Date Submitted: 10/27/18 12:00AM Date Published: 3/2/17 11:21PM
Publication Location:
Article Title: A phaseless inverse scattering problem for the 3-D Helmholtz equation
Authors: Michael V. Klibanov
Keywords: Lensless quality control of fabricated nanostructures, high frequencies, phaseless inverse scattering, uniqueness
Abstract: An inverse scattering problem for the 3-D Helmholtz equation is considered. Only the modulus of the complex valued scattered wave field is assumed to be measured and the phase is not measured. This problem naturally arises in the lensless quality control of fabricated nanostructures. Uniqueness theorem is proved.
Distribution Statement: 3-Distribution authorized to U.S. Government Agencies and their contractors
Acknowledged Federal Support: Y

Publication Type: Journal Article Peer Reviewed: Y **Publication Status:** 1-Published
Journal: Inverse Problems
Publication Identifier Type: DOI Publication Identifier: <https://doi.org/10.1088/1361-6420/ab0133>
Volume: 35 Issue: 3 First Page #:
Date Submitted: 4/21/19 12:00AM Date Published: 3/22/19 2:31AM
Publication Location: United Kingdom
Article Title: PDE-based numerical method for a limited angle X-ray tomography
Authors: Michael V. Klibanov, Loc H. Nguyen
Keywords: tomographic inverse problem, X-ray transform, incomplete data, Carleman estimate
Abstract: A new numerical method for X-ray tomography for a specific case of incomplete Radon data is proposed. Potential applications are in checking out bulky luggage in airports. This method is based on the analysis of the transport PDE governing the X-ray tomography rather than on the conventional integral formulation. The quasi-reversibility method is applied. Convergence analysis is performed using a new Carleman estimate. Numerical results are presented and compared with the inversion of the Radon transform using the well-known filtered back projection algorithm. In addition, it is shown how to use our method to study the inversion of the attenuated X-ray transform for the same case of incomplete data.
Distribution Statement: 3-Distribution authorized to U.S. Government Agencies and their contractors
Acknowledged Federal Support: Y

RPPR Final Report
as of 18-Apr-2022

Publication Type: Journal Article Peer Reviewed: Y **Publication Status:** 1-Published
Journal: Inverse Problems
Publication Identifier Type: DOI **Publication Identifier:** <https://doi.org/10.1088/1361-6420/aafecd>
Volume: 35 **Issue:** 2 **First Page #:**
Date Submitted: 4/21/19 12:00AM **Date Published:** 2/4/19 7:55PM
Publication Location: United Kingdom
Article Title: ELECTRICAL IMPEDANCE TOMOGRAPHY WITH RESTRICTED DIRICHLET-TO-NEUMANN MAP DATA
Authors: Michael V. Klibanov, Jingzhi Li, Wenlong Zhang
Keywords: inverse problem, Carleman Weight Function, global strict convexity, global convergence
Abstract: We propose a new numerical method to reconstruct the isotropic electrical conductivity from measured restricted Dirichlet-to-Neumann map data in electrical impedance tomography (EIT) model. "Restricted Dirichlet-to-Neumann (DtN) map data" means that the Dirichlet and Neumann boundary data for EIT are generated by a point source running either along an interval of a straight line or along a curve located outside of the domain of interest. We "convexify" the problem via constructing a globally strictly convex Tikhonov-like functional using a Carleman Weight Function. In particular, two new Carleman estimates are established. Global convergence to the correct solution of the gradient projection method for this functional is proven. Numerical examples demonstrate a good performance of this numerical procedure.
Distribution Statement: 3-Distribution authorized to U.S. Government Agencies and their contractors
Acknowledged Federal Support: Y

Publication Type: Journal Article Peer Reviewed: Y **Publication Status:** 1-Published
Journal: SIAM Journal on Applied Mathematics
Publication Identifier Type: DOI **Publication Identifier:** DOI: 10.1137/18M1236034
Volume: 79 **Issue:** 5 **First Page #:** 1722
Date Submitted: 12/3/19 12:00AM **Date Published:** 11/12/19 4:36AM
Publication Location:
Article Title: CONVEXIFICATION FOR THE INVERSION OF A TIME DEPENDENT WAVE FRONT IN A HETEROGENEOUS MEDIUM
Authors: Michael V. Klibanov, Jingzhi Li, Wenlong Zhang
Keywords: Coefficient Inverse Problem, convexification, Carleman estimate, global strict convexity, numerical examples
Abstract: An inverse scattering problem for the 3D acoustic equation in time domain is considered. The unknown spatially distributed speed of sound is the subject of the solution of this problem. A single location of the point source is used. Using a Carleman Weight Function, a globally strictly convex cost functional is constructed. A new Carleman estimate is proven. Global convergence of the gradient projection method is proven. Numerical experiments are conducted.
Distribution Statement: 3-Distribution authorized to U.S. Government Agencies and their contractors
Acknowledged Federal Support: Y

RPPR Final Report

as of 18-Apr-2022

Publication Type: Journal Article Peer Reviewed: Y **Publication Status:** 1-Published
Journal: Computers and Mathematics with Applications
Publication Identifier Type: DOI **Publication Identifier:** <https://doi.org/10.1016/j.camwa.2018.03.016>
Volume: 77 **Issue:** **First Page #:** 1681
Date Submitted: 4/21/19 12:00AM **Date Published:** 3/15/17 7:18PM
Publication Location:
Article Title: Convexification of a 3-D coefficient inverse scattering problem
Authors: Michael V. Klibanov, Alexandr E. Kolesov
Keywords: Coefficient inverse scattering problem, Carleman weight function, Globally convergent numerical method
Abstract: A version of the so-called "convexification" numerical method for a coefficient inverse scattering problem for the 3D Helmholtz equation is developed analytically and tested numerically. Backscattering data are used, which result from a single direction of the propagation of the incident plane wave on an interval of frequencies. The method converges globally. The idea is to construct a weighted Tikhonov-like functional. The key element of this functional is the presence of the so-called Carleman Weight Function (CWF). This is the function which is involved in the Carleman estimate for the Laplace operator. This functional is strictly convex on any appropriate ball in a Hilbert space for an appropriate choice of the parameters of the CWF. Thus, both the absence of local minima and convergence of minimizers to the exact solution are guaranteed. Numerical tests demonstrate a good performance of the resulting algorithm.
Distribution Statement: 3-Distribution authorized to U.S. Government Agencies and their contractors
Acknowledged Federal Support: Y

Publication Type: Journal Article Peer Reviewed: Y **Publication Status:** 1-Published
Journal: SIAM J. Applied Mathematics
Publication Identifier Type: DOI **Publication Identifier:** \bFD\bFO\bfl . 10.1137/18M1168303
Volume: 79 **Issue:** 1 **First Page #:** 1
Date Submitted: 4/21/19 12:00AM **Date Published:** 1/10/19 8:28PM
Publication Location:
Article Title: A COEFFICIENT INVERSE PROBLEM WITH A SINGLE MEASUREMENT OF PHASELESS SCATTERING DATA
Authors: Michael V. Klibanov, Dinh-Liem Nguyen, Loc H. Nguyen
Keywords: phaseless inverse scattering, coefficient inverse problem, one measurement, uniqueness theorems, reconstruction method
Abstract: We present a reconstruction method for solving a 3D coefficient inverse problem with a single measurement of phaseless scattering data. These are multifrequency data generated by a single direction of the incident plane wave. Our method consists of two stages, each of which is challenging in its own right. The first stage is the problem of the reconstruction of the wave field itself on the measurement plane from the measured intensity. In this stage, we prove a uniqueness result and study a numerical method for this reconstruction. After obtaining the approximate scattered field on the measurement plane, in the second stage, we exploit our newly developed globally convergent numerical method to solve the coefficient inverse problem with the phased scattering data. Our two stage method does not require any advanced information about the true solution of the phaseless coefficient inverse problem. Numerical examples are presented to demonstrate the performance of the method.
Distribution Statement: 3-Distribution authorized to U.S. Government Agencies and their contractors
Acknowledged Federal Support: Y

RPPR Final Report

as of 18-Apr-2022

Publication Type: Journal Article Peer Reviewed: Y **Publication Status:** 1-Published

Journal: SIAM Journal on Scientific Computing

Publication Identifier Type: DOI

Publication Identifier: doi:10.3934/ipi.2019048

Volume: 13

Issue: 5

First Page #: 1067

Date Submitted: 12/3/19 12:00AM

Date Published: 10/16/19 6:04AM

Publication Location:

Article Title: ON AN INVERSE SOURCE PROBLEM FOR THE FULL RADIATIVE TRANSFER EQUATION WITH INCOMPLETE DATA

Authors: Alexey Smirnov, Michael Klibanov, Loc Nguyen

Keywords: radiative transfer equation, absorption term, scattering term, inverse source problem, discrete Carleman estimate, quasi-reversibility method

Abstract: A new numerical method to solve an inverse source problem for the radiative transfer equation involving the absorption and scattering terms, with incomplete data, is proposed. No restrictive assumption on those absorption and scattering coefficients is imposed. The original inverse source problem is reduced to boundary value problem for a system of coupled partial differential equations of the first order. The unknown source function is not a part of this system. Next, we write this system in the fully discrete form of finite differences. That discrete problem is solved via the quasi-reversibility method. We prove the existence and uniqueness of the regularized solution. Especially, we prove the convergence of regularized solutions to the exact one as the noise level in the data tends to zero via a new discrete Carleman estimate. Numerical simulations demonstrate good performance of this method even when the data is highly noisy.

Distribution Statement: 3-Distribution authorized to U.S. Government Agencies and their contractors

Acknowledged Federal Support: Y

Publication Type: Journal Article

Peer Reviewed: Y

Publication Status: 1-Published

Journal: arXiv: posted online

Publication Identifier Type: DOI

Publication Identifier: doi:10.3934/ipi.2019048

Volume: 13

Issue: 5

First Page #: 1067

Date Submitted: 12/3/19 12:00AM

Date Published: 10/22/19 12:46AM

Publication Location:

Article Title: A convergent numerical method for a multi-frequency inverse source problem in inhomogenous media

Authors: Loc H. Nguyen, Qitong Li, Michael V. Klibanov

Keywords: inverse source problem, truncated Fourier series, approximation, Carleman estimate, convergence

Abstract: A new numerical method to solve an inverse source problem for the Helmholtz equation in inhomogenous media is proposed. This method reduces the original inverse problem to a boundary value problem for a coupled system of elliptic PDEs, in which the unknown source function is not involved. The Dirichlet boundary condition is given on the entire boundary of the domain of interest and the Neumann boundary condition is given on a part of this boundary. To solve this problem, the quasi-reversibility method is applied. Uniqueness and existence of the minimizer are proven. A new Carleman estimate is established. Next, the convergence of those minimizers to the exact solution is proven using that Carleman estimate. Results of numerical tests are presented.

Distribution Statement: 3-Distribution authorized to U.S. Government Agencies and their contractors

Acknowledged Federal Support: Y

RPPR Final Report as of 18-Apr-2022

Publication Type: Journal Article Peer Reviewed: Y **Publication Status:** 1-Published

Journal: arXiv: posted online

Publication Identifier Type: DOI

Publication Identifier: doi:10.3934/ipi.2019060

Volume: 13

Issue: 6

First Page #: 1365

Date Submitted: 12/3/19 12:00AM

Date Published: 10/4/19 2:16AM

Publication Location:

Article Title: Travel time tomography with formally determined incomplete data in 3D

Authors: Michael V. Klibanov

Keywords: inverse kinematic problem, Carleman-like estimate for the Volterra operator, Lipschitz stability, convexification, globally convergent numerical method

Abstract: This is the FIRST SOLUTION OF A WELL KNOWN AND LONG STANDING PROBLEM. The second name for this problem is "inverse kinematic problem". A number of radically new ideas is presented. The problem was originally posed by German mathematicians Herglotz in 1905 and Wiehert and Zoeppritz in 1907. The problem is about the determination of the spatial distribution of the sound speed using measurements of first times of arrival of sound waves on the surface of an unknown medium. Direct applications in Geophysics are well known. In addition, the author foresees quite important applications to Army needs as well. One of them is to "see behind the walls" using acoustical signals. For the first time, a globally convergent numerical method is developed for the challenging problem of travel time tomography in 3D for formally determined incomplete data. All other works use over determined data. Carleman Weight Functions are applied to this problem for the first time.

Distribution Statement: 3-Distribution authorized to U.S. Government Agencies and their contractors

Acknowledged Federal Support: Y

Publication Type: Journal Article

Peer Reviewed: Y

Publication Status: 1-Published

Journal: Inverse Problems

Publication Identifier Type: DOI

Publication Identifier: <https://doi.org/10.1088/1361-6420/ab2777>

Volume: 35

Issue: 10

First Page #: 1

Date Submitted: 12/3/19 12:00AM

Date Published: 10/2/19 1:15AM

Publication Location: England

Article Title: Convergent numerical methods for parabolic equations with reversed time via a new Carleman estimate

Authors: Michael V. Klibanov, Anatoly G. Yagola

Keywords: linear and quasilinear parabolic equations, reversed time, Carleman estimate, stability estimate, convergent quasi-reversibility method for the linear case, globally convergent numerical method for the quasilinear case

Abstract: The key tool of this paper is a new Carleman estimate for an arbitrary parabolic operator of the second order for the case of reversed time data. This estimate works on an arbitrary time interval. On the other hand, the previously known Carleman estimate for the reversed time case works only on a sufficiently small time interval. First, a stability estimate is proven. Next, the quasi-reversibility numerical method is proposed for an arbitrary time interval for the linear case. This is unlike a sufficiently small time interval in the previous work. The convergence rate for the quasi-reversibility method is established. Finally, the quasilinear parabolic equation with reversed time is considered. A weighted globally strictly convex Tikhonov-like functional is constructed. The weight is the Carleman Weight Function which is involved in that Carleman estimate. The global convergence of the gradient projection method to the exact solution is proved for this functional.

Distribution Statement: 3-Distribution authorized to U.S. Government Agencies and their contractors

Acknowledged Federal Support: Y

RPPR Final Report as of 18-Apr-2022

Publication Type: Journal Article Peer Reviewed: Y **Publication Status:** 5-Submitted

Journal: posted on arxiv

Publication Identifier Type:

Publication Identifier: arxiv: 1911.04581

Volume: n/a

Issue: n/a

First Page #: n/a

Date Submitted: 12/3/19 12:00AM

Date Published: 11/3/19 1:35AM

Publication Location: www.arxiv.org

Article Title: CONVERGENT NUMERICAL METHOD FOR A LINEARIZED TRAVEL TIME TOMOGRAPHY PROBLEM WITH INCOMPLETE DATA

Authors: Michael Klibanov, Thuy Le, Loc Nguyen

Keywords: linearization, inverse kinematic problem, travel time tomography, numerical solution, convergence

Abstract: We propose a new numerical method to solve the linearized problem of travel time tomography with incomplete data. Our method is based on the technique of the truncation of the Fourier series with respect to a special basis of L_2 . This way we derive a boundary value problem for a system of coupled partial differential equations (PDEs) of the first order. This system is solved by the quasi-reversibility method. Hence, the spatially dependent Fourier coefficients of the solution to the linearized Eikonal equation are obtained. The convergence of this method is established. Numerical results for highly noisy data are presented.

Distribution Statement: 3-Distribution authorized to U.S. Government Agencies and their contractors

Acknowledged Federal Support: Y

Publication Type: Journal Article Peer Reviewed: Y **Publication Status:** 1-Published

Journal: SIAM Journal on Imaging Sciences

Publication Identifier Type: DOI

Publication Identifier: 10.1137/19M1303101

Volume: 13

Issue: 2

First Page #: 871

Date Submitted: 6/19/20 12:00AM

Date Published: 4/24/20 6:52AM

Publication Location: United States

Article Title: Convexification for a Three-Dimensional Inverse Scattering Problem with the Moving Point Source

Authors: Vo Khoa, Michael Klibanov, Loc Nguyen

Keywords: Coefficient inverse scattering problem, point sources, Carleman weight function, globally convergent numerical method, data completion, Fourier truncation

Abstract: For the first time, we develop in this paper the globally convergent convexification numerical method for a coefficient inverse problem for the three-dimensional Helmholtz equation for the case when the backscattering data are generated by a point source running along an interval of a straight line and the wavenumber is fixed. Thus, by varying the wavenumber, one can reconstruct the dielectric constant depending not only on spatial variables but on the wavenumber (i.e., frequency) as well. Our approach relies on a new derivation of a boundary value problem for a system of coupled quasilinear elliptic partial differential equations. This is done via an application of a special truncated Fourier-like method. First, we prove the Lipschitz stability estimate for this problem. Next, using the Carleman weight function, we construct a globally strictly convex cost functional and prove the global convergence to the exact solution of a gradient-like method. Numerical results are presented.

Distribution Statement: 2-Distribution Limited to U.S. Government agencies only; report contains proprietary info

Acknowledged Federal Support: Y

RPPR Final Report as of 18-Apr-2022

Publication Type: Journal Article

Peer Reviewed: Y

Publication Status: 1-Published

Journal: Inverse Problems

Publication Identifier Type: DOI

Publication Identifier: doi.org/10.1088/1361-6420/ab95aa

Volume: 36

Issue: n/a

First Page #: 1

Date Submitted: 3/20/21 12:00AM

Date Published: 6/10/20 4:31AM

Publication Location: United Kingdom

Article Title: Convexification and experimental data for a 3D inverse scattering problem with the moving point source

Authors: Vo Khoa, Grant Bidney, Michael Klibanov, Loc Nguyen, Lam Nguyen, Anders Sullivan, Vasily Astratov

Keywords: Coefficient inverse problem, multiple point sources, experimental data, Carleman weight, global convergence, Fourier series

Abstract: Inverse scattering problems of the reconstructions of physical properties of a medium from boundary measurements are substantially challenging ones. This work aims to verify the performance on experimental data of a newly developed convexification method for a 3D coefficient inverse problem for the case of objects buried in a sandbox a fixed frequency and the point source moving along an interval of a straight line. Using a special Fourier basis, the method of this work strongly relies on a new derivation of a boundary value problem for a system of coupled quasilinear elliptic equations. This problem, in turn is solved via the minimization of a Tikhonov-like functional weighted by a Carleman Weight Function. The global convergence of the numerical procedure is established analytically. The numerical verification is performed using experimental data, which are raw backscatter data of the electric field.

Distribution Statement: 2-Distribution Limited to U.S. Government agencies only; report contains proprietary info
Acknowledged Federal Support: Y

Publication Type: Journal Article

Peer Reviewed: Y

Publication Status: 1-Published

Journal: Inverse Problems in Science and Engineering

Publication Identifier Type: DOI

Publication Identifier: https://doi.org/10.1080/17415977.2020.1802

Volume: 29

Issue: 5

First Page #: 712

Date Submitted: 7/5/21 12:00AM

Date Published: 6/20/21 5:47AM

Publication Location: England

Article Title: An inverse problem of a simultaneous reconstruction of the dielectric constant and conductivity from experimental backscattering data

Authors: Vo Khoa, Grant Bidney, Michael Klibanov, Loc Nguyen, Lam Nguyen, Anders Sullivan, Vasily Astratov

Keywords: Coefficient inverse problem, multiple point sources, experimental data, Carleman weight, global convergence, Fourier series

Abstract: This report extends our recent progress in tackling a challenging 3D inverse scattering problem governed by the Helmholtz equation. Our target application is to reconstruct dielectric constants, electric conductivities and shapes of front surfaces of objects buried very closely under the ground. These objects mimic explosives, like, e.g., antipersonnel land mines and improvised explosive devices. We solve a coefficient inverse problem with the backscattering data generated by a moving source at a fixed frequency. In our previous work only the unknown dielectric constants of objects and shapes of their front surfaces were calculated. Unlike this, in the current work performance of our numerical convexification algorithm is verified for the case when the dielectric constants, the electric conductivities and shapes of targets are unknown. Our method successfully images all these three parameters. The global convergence theory is presented.

Distribution Statement: 2-Distribution Limited to U.S. Government agencies only; report contains proprietary info
Acknowledged Federal Support: Y

RPPR Final Report as of 18-Apr-2022

Publication Type: Journal Article

Peer Reviewed: Y

Publication Status: 1-Published

Journal: Inverse Problems

Publication Identifier Type: Other

Publication Identifier: <https://doi.org/10.1088/1361-6420/abac9a>

Volume: 36

Issue: 9

First Page #: n/a

Date Submitted: 3/2/21 12:00AM

Date Published: 4/10/20 4:12AM

Publication Location: United Kingdom

Article Title: CONVEXIFICATION FOR AN INVERSE PROBLEM FOR A 1D WAVE EQUATION WITH EXPERIMENTAL DATA

Authors: Alexey Smirnov, Michael Klibanov, Anders Sullivan, Lam Nguyen

Keywords: experimental data, 1D hyperbolic equation with a variable coefficient, coefficient inverse problem, convexification, globally convergent numerical method, Carleman estimate, numerical results

Abstract: The forward problem here is the Cauchy problem for a 1D hyperbolic PDE with a variable coefficient in the principal part of the operator. That coefficient is the spatially distributed dielectric constant. The inverse problem consists of the recovery of that dielectric constant from backscattering boundary measurements. The data depend on one variable, which is time. To address this problem, a new version of the convexification method is analytically developed. The theory guarantees the global convergence of this method. Numerical testing is conducted for both computationally simulated and experimental data. Experimental data, which are collected in the field, mimic the problem of the recovery of the spatially distributed dielectric constants of antipersonnel land mines and improvised explosive devices.

Distribution Statement: 2-Distribution Limited to U.S. Government agencies only; report contains proprietary info
Acknowledged Federal Support: Y

Publication Type: Journal Article

Peer Reviewed: Y

Publication Status: 1-Published

Journal: Inverse Problems and Imaging

Publication Identifier Type: DOI

Publication Identifier: doi:10.3934/ipi.2020042

Volume: 14

Issue: 5

First Page #: 913

Date Submitted: 3/2/21 12:00AM

Date Published: 2/5/20 11:18AM

Publication Location: United States

Article Title: CONVEXIFICATION FOR A 1D HYPERBOLIC COEFFICIENT INVERSE PROBLEM WITH SINGLE MEASUREMENT DATA

Authors: Alexey Smirnov, Michael Klibanov, Loc Nguyen

Keywords: 1D hyperbolic equation, coefficient inverse problem, globally convergent method, convexification, Carleman estimate

Abstract: A version of the convexification numerical method for a Coefficient Inverse Problem for a 1D hyperbolic PDE is presented. The data for this problem are generated by a single measurement event. This method converges globally. The most important element of the construction is the presence of the Carleman Weight Function in a weighted Tikhonov-like functional. This functional is strictly convex on a certain bounded set in a Hilbert space, and the diameter of this set is an arbitrary positive number. The global convergence of the gradient projection method is established. Computational results demonstrate a good performance of the numerical method for noisy data

Distribution Statement: 2-Distribution Limited to U.S. Government agencies only; report contains proprietary info
Acknowledged Federal Support: Y

RPPR Final Report as of 18-Apr-2022

Publication Type: Journal Article

Peer Reviewed: Y

Publication Status: 1-Published

Journal: Inverse Problems

Publication Identifier Type: DOI

Publication Identifier: <https://doi.org/10.1088/1361-6420/ab9893>

Volume: 36

Issue: 6

First Page #: n/a

Date Submitted: 3/2/21 12:00AM

Date Published: 6/6/18 5:42AM

Publication Location: United Kingdom

Article Title: CONVEXIFICATION FOR AN INVERSE PARABOLIC PROBLEM

Authors: Michael Klibanov, J. Li, W. Zhang

Keywords: parabolic equation, coefficient inverse problem, globally convergent numerical method, convexification, Carleman estimate, numerical studies

Abstract: A convexification-based numerical method for a Coefficient Inverse Problem for a parabolic PDE is presented. The key element of this method is the presence of the so-called Carleman Weight Function in the numerical scheme. Convergence analysis ensures the global convergence of this method, as opposed to the local convergence of the conventional least squares minimization techniques. Numerical results demonstrate a good performance.

Distribution Statement: 2-Distribution Limited to U.S. Government agencies only; report contains proprietary info
Acknowledged Federal Support: Y

Publication Type: Journal Article

Peer Reviewed: Y

Publication Status: 1-Published

Journal: Inverse Problems in Science and Engineering

Publication Identifier Type: DOI

Publication Identifier: DOI: 10.1080/17415977.2021.1943384

Volume: n/a

Issue: n/a

First Page #: n/a

Date Submitted: 10/18/21 12:00AM

Date Published: 6/25/21 5:41AM

Publication Location: United Kingdom

Article Title: Convexification numerical algorithm for a 2D inverse scattering problem with backscatter data

Authors: Trung Truong, Dinh-Liem Nguyen, Michael Klibanov

Keywords: inverse scattering, numerical reconstruction, convexification, backscatter data, Carleman weight function, coefficient identification

Abstract: This paper is concerned with the inverse scattering problem which aims to determine the spatially distributed dielectric constant coefficient of the 2D Helmholtz equation from multifrequency backscatter data associated with a single direction of the incident plane wave. We propose a globally convergent convexification numerical algorithm to solve this nonlinear and ill-posed inverse problem. The key advantage of our method over conventional optimization approaches is that it does not require a good first guess about the solution. First, we eliminate the coefficient from the Helmholtz equation using a change of variables. Using a truncated expansion with respect to a special Fourier basis, we reformulate the inverse problem as a system of quasilinear elliptic PDEs. The cost functional for the weighted quasi-reversibility method is constructed as a Tikhonov-like functional that involves a Carleman Weight Function. Numerical results are presented.

Distribution Statement: 2-Distribution Limited to U.S. Government agencies only; report contains proprietary info
Acknowledged Federal Support: Y

RPPR Final Report as of 18-Apr-2022

Publication Type: Journal Article Peer Reviewed: Y **Publication Status:** 4-Under Review

Journal: Journal of Inverse and Ill-Posed Problems

Publication Identifier Type: Other

Publication Identifier: arxiv: 2004.05660

Volume: n/a

Issue: n/a

First Page #: n/a

Date Submitted: 6/19/20 12:00AM

Date Published: 4/13/20 1:02AM

Publication Location: United States

Article Title: Convergence of a series associated with the convexification method for coefficient inverse problems

Authors: Michael Klibanov, Dinh-Liem Nguyen

Keywords: Convexification method, global convergence, coefficient inverse problem, truncated Fourier series, numerical reconstruction

Abstract: This paper is concerned with the convergence of a series associated with a certain version of the convexification method. That version has been recently developed by the research group of the first author for solving coefficient inverse problems. The convexification method aims to construct a globally convex Tikhonov-like functional with a Carleman Weight Function in it. In the previous works the construction of the strictly convex weighted Tikhonov-like functional assumes a truncated Fourier series (i.e. a finite series instead of an infinite one) for a function generated by the total wave field. In this paper we prove a convergence property for this truncated Fourier series approximation. More precisely, we show that the residual of the approximate PDE obtained by using the truncated Fourier series tends to zero in L_2 as the truncation index in the truncated Fourier series tends to infinity.

Distribution Statement: 2-Distribution Limited to U.S. Government agencies only; report contains proprietary info
Acknowledged Federal Support: Y

Publication Type: Journal Article Peer Reviewed: Y **Publication Status:** 1-Published

Journal: Journal of Inverse and Ill-Posed Problems

Publication Identifier Type: DOI

Publication Identifier: <https://doi.org/10.1515/jiip-2020-0028>

Volume: n/a

Issue: n/a

First Page #: n/a

Date Submitted: 3/20/21 12:00AM

Date Published: 5/20/20 9:23AM

Publication Location: Germany

Article Title: A convergent numerical method to recover the initial condition of nonlinear parabolic equations from lateral Cauchy data

Authors: Thuy Le, Loc Nguyen

Keywords: Nonlinear parabolic equations, initial condition, convergent numerical method, Carleman estimate, iteration

Abstract: We propose a robust numerical method to find the coefficient of the creation or depletion term of parabolic equations from the measurement of the lateral Cauchy information of their solutions. Most papers in the field study this nonlinear and severely ill-posed problem using optimal control. The main drawback of this widely used approach is the need of some advanced knowledge of the true solution. In this paper, we propose a new method that opens a door to solve nonlinear inverse problems for parabolic equations without any initial guess of the true coefficient. This claim is confirmed numerically. The key point of the method is to derive a system of nonlinear elliptic equations for the Fourier coefficients of the solution to the governing equation with respect to a special basis of L_2 . We then solve this system by a predictor-corrector process to obtain the first and second predictors. The desired solution to the inverse problem under consideration follows.

Distribution Statement: 2-Distribution Limited to U.S. Government agencies only; report contains proprietary info
Acknowledged Federal Support: Y

RPPR Final Report as of 18-Apr-2022

Publication Type: Journal Article Peer Reviewed: Y **Publication Status:** 1-Published

Journal: Journal of Inverse and Ill-Posed Problems

Publication Identifier Type: DOI

Publication Identifier: doi.org/10.1515/jiip-2019-0026

Volume: 28

Issue: n/a

First Page #: 323

Date Submitted: 3/20/21 12:00AM

Date Published: 12/10/20 10:08PM

Publication Location: Germany

Article Title: A numerical method for an inverse source problem for parabolic equations and its application to a coefficient inverse problem

Authors: Phuong Nguyen, Loc Nguyen

Keywords: Parabolic equation, inverse source problem, coefficient inverse problem, numerical method, quasi-reversibility method

Abstract: Two main aims of this paper are to develop a numerical method to solve an inverse source problem for parabolic equations and apply it to solve a nonlinear coefficient inverse problem. The inverse source problem in this paper is the problem to reconstruct a source term from external observations. Our method to solve this inverse source problem consists of two stages. We first establish an equation of the derivative of the solution to the parabolic equation with respect to the time variable. Then, in the second stage, we solve this equation by the quasi-reversibility method. The inverse source problem considered in this paper is the linearization of a nonlinear coefficient inverse problem. Hence, iteratively solving the inverse source problem provides the numerical solution to that coefficient inverse problem. Numerical results for the inverse source problem under consideration and the corresponding nonlinear coefficient inverse problem are presented.

Distribution Statement: 2-Distribution Limited to U.S. Government agencies only; report contains proprietary info
Acknowledged Federal Support: Y

Publication Type: Journal Article Peer Reviewed: Y **Publication Status:** 1-Published

Journal: Computers and Mathematics with Applications

Publication Identifier Type: DOI

Publication Identifier: doi.org/10.1016/j.camwa.2020.09.010

Volume: 80

Issue: 4

First Page #: 2135

Date Submitted: 3/20/21 12:00AM

Date Published: 6/6/20 5:39AM

Publication Location: United States

Article Title: A new algorithm to determine the creation or depletion term of parabolic equations from boundary measurements

Authors: Loc Nguyen

Keywords: coefficient inverse problem parabolic equations approximation Fourier coefficients

Abstract: We propose a robust numerical method to find the coefficient of the creation or depletion term of parabolic equations from the measurement of the lateral Cauchy information of their solutions. Most papers in the field study this nonlinear and severely ill-posed problem using optimal control. The main drawback of this widely used approach is the need of some advanced knowledge of the true solution. In this paper, we propose a new method that opens a door to solve nonlinear inverse problems for parabolic equations without any initial guess of the true coefficient. This claim is confirmed numerically. The key point of the method is to derive a system of nonlinear elliptic equations for the Fourier coefficients of the solution to the governing equation with respect to a special basis of L_2 . We then solve this system by a predictor-corrector process. The desired solution to the inverse problem follows. Numerical results are presented.

Distribution Statement: 2-Distribution Limited to U.S. Government agencies only; report contains proprietary info
Acknowledged Federal Support: Y

RPPR Final Report as of 18-Apr-2022

Publication Type: Journal Article Peer Reviewed: Y **Publication Status:** 1-Published

Journal: SIAM J. on Scientific Computing

Publication Identifier Type: Other

Publication Identifier: <https://doi.org/10.1137/19M1299487>

Volume: 42

Issue: 5

First Page #: B1173

Date Submitted: 3/2/21 12:00AM

Date Published:

Publication Location:

Article Title: NUMERICAL SOLUTION OF A LINEARIZED TRAVEL TIME TOMOGRAPHY PROBLEM WITH INCOMPLETE DATA

Authors: Michael V. Klibanov, Thuy T. Le and Loc H. Nguyen

Keywords: linearization, inverse kinematic problem, travel time tomography, numerical solution, convergence

Abstract: We propose a new numerical method to solve the linearized problem of the travel time tomography with incomplete data. Our method is based on the technique of the truncation of the Fourier series with respect to a special basis of L_2 . This way we derive a boundary value problem for a system of coupled PDEs of the first order. This problem is solved by the quasi-reversibility method. The spatially dependent Fourier coefficients of the solution to the linearized eikonal equation are obtained this way. Numerical results for highly noisy data are presented.

Distribution Statement: 2-Distribution Limited to U.S. Government agencies only; report contains proprietary info
Acknowledged Federal Support: Y

Publication Type: Journal Article Peer Reviewed: Y **Publication Status:** 5-Submitted

Journal: arxiv

Publication Identifier Type: Other

Publication Identifier: arxiv: 2010.1414

Volume: n/a

Issue: n/a

First Page #: n/a

Date Submitted: 12/27/20 12:00AM

Date Published: 12/27/20 11:33PM

Publication Location:

Article Title: LINEAR LAVRENT'EV INTEGRAL EQUATION FOR THE NUMERICAL SOLUTION OF A NONLINEAR COEFFICIENT INVERSE PROBLEM

Authors: Michael V. Klibanov, Jingzhi Li and Wenlong Zhang

Keywords: coefficient inverse problem, Lavrent'ev equation, non overdetermined data, Carleman estimate, quasi-reversibility method, convergence rate

Abstract: For the first time, we develop a convergent numerical method for the linear integral equation derived by M.M. Lavrent'ev in 1964 with the goal to solve a coefficient inverse problem for a wave-like equation in 3D. The data are non overdetermined. Convergence analysis is presented along with the numerical results. An intriguing feature of the Lavrent'ev equation is that, without any linearization, it reduces a highly nonlinear coefficient inverse problem to a linear integral equation of the first kind. Nevertheless, numerical results for that equation, which use the data generated for that coefficient inverse problem, show a good reconstruction accuracy. This is similar with the classical Gelfand-Levitan equation derived in 1951, which is valid in the 1D case.

Distribution Statement: 2-Distribution Limited to U.S. Government agencies only; report contains proprietary info
Acknowledged Federal Support: Y

RPPR Final Report as of 18-Apr-2022

Publication Type: Journal Article Peer Reviewed: Y **Publication Status:** 1-Published

Journal: Inverse Problems

Publication Identifier Type: DOI

Publication Identifier: doi.org/10.1088/1361-6420/ab9893

Volume: 36

Issue: n/a

First Page #: n/a

Date Submitted: 12/27/20 12:00AM

Date Published: 6/9/20 4:00AM

Publication Location: United Kingdom

Article Title: Convexification for an inverse parabolic problem

Authors: Michael V. Klibanov, Jingzhi Li and Wenlong Zhang

Keywords: parabolic equation, coefficient inverse problem, globally convergent numerical method, convexification, Carleman estimate, numerical studies

Abstract: For the first time, we develop a convergent numerical method for the linear integral equation derived by M.M. Lavrent'ev in 1964 with the goal to solve a coefficient inverse problem for a wave-like equation in 3D. The data are non overdetermined. Convergence analysis is presented along with the numerical results. An intriguing feature of the Lavrent'ev equation is that, without any linearization, it reduces a highly nonlinear coefficient inverse problem to a linear integral equation of the first kind. Nevertheless, numerical results for that equation, which use the data generated for that coefficient inverse problem, show a good reconstruction accuracy. This is similar with the classical Gelfand-Levitan equation derived in 1951, which is valid in the 1D case.

Distribution Statement: 2-Distribution Limited to U.S. Government agencies only; report contains proprietary info
Acknowledged Federal Support: Y

Publication Type: Journal Article Peer Reviewed: Y **Publication Status:** 1-Published

Journal: IEEE TRANSACTIONS ON GEOSCIENCE AND REMOTE SENSING

Publication Identifier Type: DOI

Publication Identifier: 10.1109/TGRS.2021.3055805

Volume: n/a

Issue: n/a

First Page #: n/a

Date Submitted: 3/2/21 12:00AM

Date Published: 3/3/21 12:18AM

Publication Location: United Kingdom

Article Title: Through-the-Wall Nonlinear SAR Imaging

Authors: Michael V. Klibanov, Alexey V. Smirnov, Khoa Anh Vo, Anders J. Sullivan, and Lam H. Nguyen

Keywords: inverse scattering problem, through-wall imaging, synthetic aperture radar; SAR; convexification, experimental data.

Abstract: An inverse scattering problem for SAR data in application to through-the-wall imaging is addressed. In contrast with the conventional algorithms of SAR imaging, that work with the linearized mathematical model based on the Born approximation, the fully nonlinear case is considered here. To avoid the local minima problem, the so-called "convexification" globally convergent inversion scheme is applied to approximate the distribution of the slant range (SR) dielectric constant in the 3-D domain. The benchmark scene of this paper comprises a homogeneous dielectric wall and different dielectric targets hidden behind it. The results comprise two dimensional images of the SR dielectric constant of the scene of interest. Numerical results are obtained by the proposed inversion method for both the computationally simulated and experimental data. Our results show that the values, cross-range sizes and locations of SR dielectric constants for targets hidden behind the wall are close to real ones.

Distribution Statement: 2-Distribution Limited to U.S. Government agencies only; report contains proprietary info
Acknowledged Federal Support: Y

RPPR Final Report as of 18-Apr-2022

Publication Type: Journal Article

Peer Reviewed: Y

Publication Status: 1-Published

Journal: arxiv

Publication Identifier Type: DOI

Publication Identifier: 10.1109/TGRS.2021.3055805

Volume: n/a

Issue: n/a

First Page #: n/a

Date Submitted: 3/2/21 12:00AM

Date Published: 9/18/20 12:00PM

Publication Location: United Kingdom

Article Title: Through-the-Wall Nonlinear SAR Imaging

Authors: Michael V. Klibanov, Alexey V. Smirnov, Khoa Anh Vo, Anders J. Sullivan, and Lam H. Nguyen

Keywords: inverse scattering problem, through-wall imaging (TWI(; synthetic aperture radar (SAR(; convexification, experimental data.

Abstract: An inverse scattering problem for SAR data in application to through-the-wall imaging is addressed. In contrast with the conventional algorithms of SAR imaging, that work with the linearized mathematical model based on the Born approximation, the fully nonlinear case is considered here. To avoid the local minima problem, the so-called "convexification" globally convergent inversion scheme is applied to approximate the distribution of the slant range (SR) dielectric constant in the 3-D domain. The benchmark scene of this paper comprises a homogeneous dielectric wall and different dielectric targets hidden behind it. The results comprise two dimensional images of the SR dielectric constant of the scene of interest. Numerical results are obtained by the proposed inversion method for both the computationally simulated and experimental data. Our results show that the values, cross-range sizes and locations of SR dielectric constants for targets hidden behind the wall are close to real ones.

Distribution Statement: 2-Distribution Limited to U.S. Government agencies only; report contains proprietary info
Acknowledged Federal Support: Y

Publication Type: Journal Article

Peer Reviewed: Y

Publication Status: 1-Published

Journal: SIAM J. on Scientific Computing

Publication Identifier Type: DOI

Publication Identifier: DOI. 10.1137/19M1253605

Volume: 41

Issue: 5

First Page #: B929

Date Submitted: 12/27/20 12:00AM

Date Published: 12/18/19 5:00AM

Publication Location:

Article Title: ON AN INVERSE SOURCE PROBLEM FOR THE FULL RADIATIVE TRANSFER EQUATION WITH INCOMPLETE DATA

Authors: Alexey Smirnov, Michael V. Klibanov and Loc Nguyen

Keywords: radiative transfer equation, absorption term, scattering term, inverse source problem, discrete Carleman estimate, quasi-reversibility method

Abstract: A new numerical method to solve an inverse source problem for the radiative transfer equation involving the absorption and scattering terms, with incomplete data, is proposed. No restrictive assumption on those absorption and scattering coefficients is imposed. The original inverse source problem is reduced to boundary value problem for a system of coupled partial differential equations of the first order. The unknown source function is not a part of this system. Next, we write this system in the fully discrete form of finite differences. That discrete problem is solved via the quasi-reversibility method. We prove the existence and uniqueness of the regularized solution. Especially, we prove the convergence of regularized solutions to the exact one as the noise level in the data tends to zero via a new discrete Carleman estimate. Numerical simulations demonstrate good performance of this method even when the data are highly noisy.

Distribution Statement: 2-Distribution Limited to U.S. Government agencies only; report contains proprietary info
Acknowledged Federal Support: Y

RPPR Final Report as of 18-Apr-2022

Publication Type: Journal Article Peer Reviewed: Y **Publication Status:** 1-Published

Journal: Journal of Inverse and Ill-Posed Problems

Publication Identifier Type: DOI

Publication Identifier: doi.org/10.1515/jiip-2019-0026

Volume: 28

Issue: 3

First Page #: 323

Date Submitted: 3/20/21 12:00AM

Date Published: 3/21/20 4:58AM

Publication Location: Germany

Article Title: A numerical method for an inverse source problem for parabolic equations and its application to a coefficient inverse problem

Authors: Phuong Mai Nguyen, Loc Hoang Nguyen

Keywords: Parabolic equation, inverse source problem, coefficient inverse problem, numerical method, quasi-reversibility method

Abstract: We propose a robust numerical method to find the coefficient of the creation or depletion term of parabolic equations from the measurement of the lateral Cauchy information of their solutions. Most papers in the field study this nonlinear and severely ill-posed problem using optimal control. The main drawback of this widely used approach is the need of some advanced knowledge of the true solution. In this paper, we propose a new method that opens a door to solve nonlinear inverse problems for parabolic equations without any initial guess of the true coefficient. This claim is confirmed numerically. The key point of the method is to derive a system of nonlinear elliptic equations for the Fourier coefficients of the solution to the governing equation with respect to a special basis of L^2 . We then solve this system by a predictor-corrector process to obtain the first and second predictors. The desired solution to the inverse problem under consideration follows.

Distribution Statement: 2-Distribution Limited to U.S. Government agencies only; report contains proprietary info
Acknowledged Federal Support: Y

Publication Type: Journal Article

Peer Reviewed: Y

Publication Status: 5-Submitted

Journal: preprint

Publication Identifier Type: Other

Publication Identifier: arxiv: 2103.04159

Volume: n/a

Issue: n/a

First Page #: n/a

Date Submitted: 3/24/21 12:00AM

Date Published: 3/23/21 3:28AM

Publication Location:

Article Title: The gradient descent method for the convexification to solve boundary value problems of quasi-linear PDEs and a coefficient inverse problem

Authors: Thuy T. Le, Loc H. Nguyen

Keywords: boundary value problem, quasi-linear, convexification, gradient descent method, coefficient inverse problem.

Abstract: We study the global convergence of the gradient descent method of the minimization of strictly convex functionals on an open and bounded set of a Hilbert space. Such results are unknown for this type of sets, unlike the case of the entire Hilbert space. The proof of this convergence is based on the classical contraction principle. Then, we use our result to establish a general framework to numerically solve boundary value problems for quasi-linear partial differential equations (PDEs) with noisy Cauchy data. The procedure involves the use of Carleman weight functions to convexify a cost functional arising from the given boundary value problem and thus to ensure the convergence of the gradient descent method above. We prove the global convergence of the method as the noise tends to 0. The convergence rate is Lipschitz. Next, we apply this method to solve a highly nonlinear and severely ill-posed coefficient inverse problem, which is the so-called back scattering inverse problem.

Distribution Statement: 2-Distribution Limited to U.S. Government agencies only; report contains proprietary info
Acknowledged Federal Support: Y

RPPR Final Report

as of 18-Apr-2022

Publication Type: Journal Article

Peer Reviewed: Y

Publication Status: 1-Published

Journal: preprint

Publication Identifier Type: Other

Publication Identifier: arxiv:2013.10431

Volume: n/a

Issue: n/a

First Page #: n/a

Date Submitted: 3/31/21 12:00AM

Date Published: 3/17/21 4:00AM

Publication Location:

Article Title: Convexification inversion method for nonlinear SAR imaging with experimentally collected data

Authors: M. V. Klibanov, V. A. Khoa, A. V. Smirnov, L. H. Nguyen, G. W. Bidney, L. H. Nguyen, A. J. Sullivan, V.

Keywords: SAR imaging, gradient descent method, convexification, global convergence, experimental data, data propagation

Abstract: This paper is concerned with the study of a version of the globally convergent convexification method with direct application to synthetic aperture radar (SAR) imaging. Results of numerical testing are presented for experimentally collected data for a fake landmine. The SAR imaging technique is a common tool used to create maps of parts of the surface of the Earth or other planets. Recently, it has been applied in the context of non-invasive inspections of buildings in military and civilian services. Nowadays, any SAR imaging software is based on the Born approximation, which is a linearization of the original wave-like partial differential equation. One of the essential assumptions this linearization procedure needs is that only those dielectric constants are imaged whose values are close to the constant background. In this work, we propose a radically new idea: to work without any linearization while still using the same data as the conventional SAR imaging technique uses. We cons

Distribution Statement: 2-Distribution Limited to U.S. Government agencies only; report contains proprietary info
Acknowledged Federal Support: Y

Publication Type: Journal Article

Peer Reviewed: Y

Publication Status: 1-Published

Journal: Journal of Computational Physics

Publication Identifier Type: DOI

Publication Identifier: <https://doi.org/10.1016/j.jcp.2021.110828>

Volume: n/a

Issue: n/a

First Page #: n/a

Date Submitted: 11/22/21 12:00AM

Date Published: 11/21/21 7:34PM

Publication Location: United States

Article Title: Numerical viscosity solutions to Hamilton-Jacobi equations via a Carleman estimate and the convexification method

Authors: Michael Klibanov, Loc H. Nguyen, Hung V. Tran

Keywords: numerical methods; convexification method; gradient descent method; viscosity solutions; Hamilton-Jacobi equations; vanishing viscosity process; boundary value problems

Abstract: We propose a globally convergent numerical method, called the convexification, to numerically compute the viscosity solution to first-order Hamilton-Jacobi equations through the vanishing viscosity process where the viscosity parameter is a fixed small number. By convexification, we mean that we employ a suitable Carleman weight function to convexify the cost functional defined directly from the form of the Hamilton-Jacobi equation under consideration. The strict convexity of this functional is rigorously proved using a new Carleman estimate. We also prove that the unique minimizer of the this strictly convex functional can be reached by the gradient descent method. Moreover, we show that the minimizer well approximates the viscosity solution of the Hamilton-Jacobi equation as the noise contained in the boundary data tends to zero. Some interesting numerical illustrations are presented.

Distribution Statement: 2-Distribution Limited to U.S. Government agencies only; report contains proprietary info
Acknowledged Federal Support: Y

RPPR Final Report as of 18-Apr-2022

Publication Type: Journal Article Peer Reviewed: Y **Publication Status:** 1-Published

Journal: Inverse Problems and Imaging

Publication Identifier Type: DOI

Publication Identifier: doi:10.3934/ipi.2021068

Volume: n/a

Issue: n/a

First Page #: n/a

Date Submitted: 11/3/21 12:00AM

Date Published: 11/1/21 8:00AM

Publication Location: United States

Article Title: Convexification-based globally convergent numerical method for a 1D coefficient inverse problem with experimental data

Authors: Michael Klibanov, Thuy Le, Loc Nguyen, Anders Sullivan, Lam Nguyen

Keywords: experimental data, convexification, 1D hyperbolic equation, coefficient inverse problem, globally convergent numerical method, Carleman estimate, numerical results

Abstract: To compute the spatially distributed dielectric constant from the backscattering data, we study a coefficient inverse problem for a 1D hyperbolic equation. To solve the inverse problem, we establish a new version of Carleman estimate and then employ this estimate to construct a cost functional which is strictly convex on a convex bounded set with an arbitrary diameter in a Hilbert space. The strict convexity property is rigorously proved. This result is called the convexification theorem and is considered as the central analytical result of this paper. Minimizing this convex functional by the gradient descent method, we obtain the desired numerical solution to the coefficient inverse problems. We prove that the gradient descent method generates a sequence converging to the minimizer and we also establish a theorem which guarantees that the minimizer converges to the true solution as the noise in the measured data and the regularization parameter tend to zero.

Distribution Statement: 2-Distribution Limited to U.S. Government agencies only; report contains proprietary info
Acknowledged Federal Support: Y

Publication Type: Journal Article Peer Reviewed: Y **Publication Status:** 1-Published

Journal: Journal of Scientific Computing

Publication Identifier Type: DOI

Publication Identifier: <https://doi.org/10.1007/s10915-021-01501-3>

Volume:

Issue: n/a

First Page #: n/a

Date Submitted: 5/8/21 12:00AM

Date Published: 5/7/21 3:28AM

Publication Location:

Article Title: The Quasi-reversibility Method to Numerically Solve an Inverse Source Problem for Hyperbolic Equations

Authors: Thuy Le, · Loc Nguyen, Thi-Phong Nguyen, William Powell

Keywords: Inverse source problem, hyperbolic equation, quasi-reversibility method, orthonormal basis

Abstract: We propose a numerical method to solve an inverse source problem of computing the initial condition of hyperbolic equations from the measurements of Cauchy data. This problem arises in thermo- and photo-acoustic tomography in a bounded cavity, in which the reflection of the wave makes the widely-used approaches, such as the time reversal method, not applicable. In order to solve this inverse source problem, we approximate the solution to the hyperbolic equation by its Fourier series with respect to a special orthonormal basis of L^2 . Then, we derive a coupled system of elliptic equations for the corresponding Fourier coefficients. We solve it by the quasi-reversibility method. The desired initial condition follows. We rigorously prove the convergence of the quasi-reversibility method as the noise level tends to 0. Some numerical examples are provided. In addition, we numerically prove that the use of the special basis above is significant.

Distribution Statement: 2-Distribution Limited to U.S. Government agencies only; report contains proprietary info
Acknowledged Federal Support: Y

RPPR Final Report as of 18-Apr-2022

Publication Type: Journal Article Peer Reviewed: Y **Publication Status:** 1-Published
Journal: Inverse Problems in Science and Engineering
Publication Identifier Type: DOI **Publication Identifier:** <https://doi.org/10.1080/17415977.2021.1943>
Volume: n/a **Issue:** n/a **First Page #:** n/a
Date Submitted: 7/5/21 12:00AM **Date Published:** 7/5/21 11:43PM
Publication Location: England
Article Title: Convexification numerical algorithm for a 2D inverse scattering problem with backscatter data
Authors: T. Truong, D.-L. Nguyen, M. V. Klibanov
Keywords: Inverse scattering; numerical reconstruction; convexification; backscatter data; carleman weight function; coefficient identification
Abstract: This paper is concerned with the inverse scattering problem which aims to determine the spatially distributed dielectric constant coefficient of the 2D Helmholtz equation from multifrequency backscatter data associated with a single direction of the incident plane wave. We propose a globally convergent convexification numerical algorithm to solve this nonlinear and ill-posed inverse problem. The key advantage of our method over conventional optimization approaches is that it does not require a good first guess about the solution. First, we eliminate the coefficient from the Helmholtz equation using a change of variables. Next, we approximately reformulate the inverse problem as a system of quasilinear elliptic PDEs, which can be numerically solved by a weighted quasi-reversibility approach. The cost functional for the weighted quasi-reversibility method is constructed as a Tikhonov-like functional that involves a Carleman Weight Function. Numerical examples are presented.
Distribution Statement: 2-Distribution Limited to U.S. Government agencies only; report contains proprietary info
Acknowledged Federal Support: Y

Publication Type: Journal Article Peer Reviewed: Y **Publication Status:** 1-Published
Journal: Journal of Inverse and Ill-Posed Problems
Publication Identifier Type: DOI **Publication Identifier:** <https://doi.org/10.1515/jiip-2020-0039>
Volume: n/a **Issue:** n/a **First Page #:** n/a
Date Submitted: 7/5/21 12:00AM **Date Published:** 7/5/21 11:51PM
Publication Location: Germany
Article Title: Solving a 1-D inverse medium scattering problem using a new multi-frequency globally strictly convex objective functional
Authors: N. T. Thanh, M.V. Klibanov
Keywords: Inverse scattering, multi-frequency measurement, global convergence, error estimates, Carleman estimates
Abstract: We propose a new approach to constructing globally strictly convex objective functional in a 1-D inverse medium scattering problem using multi-frequency backscattering data. The global convexity of the proposed objective functional is proved. We also prove the global convergence of the gradient projection algorithm and derive an error estimate. Numerical examples are presented to illustrate the performance of the proposed algorithm.
Distribution Statement: 2-Distribution Limited to U.S. Government agencies only; report contains proprietary info
Acknowledged Federal Support: Y

RPPR Final Report as of 18-Apr-2022

Publication Type: Journal Article Peer Reviewed: Y **Publication Status:** 5-Submitted

Journal: SIAM Journal on Scientific Computing, submitted

Publication Identifier Type: Other

Publication Identifier: arxiv: 2108.03500v3

Volume:

Issue:

First Page #:

Date Submitted: 8/27/21 12:00AM

Date Published: 8/26/21 5:27AM

Publication Location:

Article Title: Carleman estimates and the contraction principle for an inverse source problem for nonlinear hyperbolic equations

Authors: Loc Nguyen, Michael Klibanov

Keywords: numerical methods; Carleman estimate; contraction principle; globally convergent numerical method, nonlinear hyperbolic equations

Abstract: The main aim of this paper is to solve an inverse source problem for a general nonlinear hyperbolic equation. Combining the quasi-reversibility method and a suitable Carleman weight function, we define a map of which fixed point is the solution to the inverse problem. To find this fixed point, we define a recursive sequence with an arbitrary initial term by the same manner as in the classical proof of the contraction principle. Applying a Carleman estimate, we show that the sequence above converges to the desired solution with the exponential rate. Therefore, our new method can be considered as an analog of the contraction principle. We rigorously study the stability of our method with respect to noise. Numerical examples are presented.

Distribution Statement: 2-Distribution Limited to U.S. Government agencies only; report contains proprietary information
Acknowledged Federal Support: Y

Publication Type: Journal Article

Peer Reviewed: Y

Publication Status: 1-Published

Journal: Inverse Problems

Publication Identifier Type: DOI

Publication Identifier: 2108.03500v3

Volume: 38

Issue: n/a

First Page #: 035009

Date Submitted: 8/27/21 12:00AM

Date Published: 3/7/22 4:09PM

Publication Location: United States

Article Title: Carleman estimates and the contraction principle for an inverse source problem for nonlinear hyperbolic equations

Authors: Loc Nguyen and Michael Klibanov

Keywords: numerical methods; Carleman estimate; contraction principle; globally convergent numerical method, nonlinear hyperbolic equations

Abstract: We propose a new iterative scheme to compute the numerical solution to an over-determined boundary value problem for a general quasilinear elliptic PDE. The main idea is to repeatedly solve its linearization by using the quasi-reversibility method with a suitable Carleman weight function. The presence of the Carleman weight function allows us to employ a Carleman estimate to prove the convergence of the sequence generated by the iterative scheme above to the desired solution. The convergence of the iteration is fast at an exponential rate without the need of an initial good guess. We apply this method to compute solutions to some general quasilinear elliptic equations and a large class of first-order Hamilton-Jacobi equations. Numerical results are presented.

Distribution Statement: 3-Distribution authorized to U.S. Government Agencies and their contractors

Acknowledged Federal Support: Y

RPPR Final Report as of 18-Apr-2022

Publication Type: Journal Article Peer Reviewed: Y **Publication Status:** 5-Submitted

Journal: Journal of Scientific Computing

Publication Identifier Type: Other

Publication Identifier: 2108.07914v1

Volume: n/a

Issue: n/a

First Page #: n/a

Date Submitted: 8/27/21 12:00AM

Date Published: 8/18/21 4:00AM

Publication Location: United States

Article Title: A Carleman-based numerical method for quasilinear elliptic equations with over-determined boundary data and applications

Authors: Thuy Le, Loc Nguyen, Hung Tran

Keywords: numerical methods; Carleman estimate; linearization; boundary value problems; quasi-linear elliptic equations; Hamilton-Jacobi equations; viscosity solutions; vanishing viscosity process.

Abstract: The main aim of this paper is to solve an inverse source problem for a general nonlinear hyperbolic equation. Combining the quasi-reversibility method and a suitable Carleman weight function, we define a map of which fixed point is the solution to the inverse problem. To find this fixed point, we define a recursive sequence with an arbitrary initial term by the same manner as in the classical proof of the contraction principle. Applying a Carleman estimate, we show that the sequence above converges to the desired solution with the exponential rate. Therefore, our new method can be considered as an analog of the contraction principle. We rigorously study the stability of our method with respect to noise. Numerical examples are presented.

Distribution Statement: 2-Distribution Limited to U.S. Government agencies only; report contains proprietary info
Acknowledged Federal Support: Y

Publication Type: Journal Article

Peer Reviewed: Y

Publication Status: 1-Published

Journal: Inverse Problems, submitted

Publication Identifier Type: Other

Publication Identifier: arxiv: 2109.11098

Volume:

Issue:

First Page #:

Date Submitted: 10/18/21 12:00AM

Date Published: 9/24/21 1:54AM

Publication Location:

Article Title: Carleman contraction mapping for a 1D inverse scattering problem with experimental time-dependent data

Authors: Thuy Le, Michael Klibanov, Loc Nguyen, Anders Sullivan, Lam Nguyen

Keywords: one-dimensional wave equation, Carleman estimate, iterative method, contraction principle, numerical solution, experimental data, global convergence

Abstract: It is shown that the contraction mapping principle with the involvement of a Carleman Weight Function works for a Coefficient Inverse Problem for a 1D hyperbolic equation. Using a Carleman estimate, the global convergence of the corresponding numerical method is established. Numerical studies for both computationally simulated and experimentally collected data are presented. The experimental part is concerned with the problem of computing dielectric constants of explosive-like targets in the standoff mode using severely underdetermined data.

Distribution Statement: 2-Distribution Limited to U.S. Government agencies only; report contains proprietary info
Acknowledged Federal Support: Y

RPPR Final Report as of 18-Apr-2022

Publication Type: Journal Article Peer Reviewed: Y **Publication Status:** 1-Published

Journal: SIAM Journal on Applied Mathematics

Publication Identifier Type: DOI

Publication Identifier: DOI. 10.1137/20M1376558

Volume: 81

Issue: 5

First Page #: 1954

Date Submitted: 10/18/21 12:00AM

Date Published: 10/2/21 6:02AM

Publication Location:

Article Title: LINEAR LAVRENT'EV INTEGRAL EQUATION FOR THE NUMERICAL SOLUTION OF A NONLINEAR COEFFICIENT INVERSE PROBLEM

Authors: Michael Klibanov, Jingzhi Li, Wenlong Zhang

Keywords: coefficient inverse problem, Lavrent'ev equation, nonoverdetermined data, quasireversibility method, convergence rate

Abstract: For the first time, we develop a convergent numerical method for the linear integral equation derived by M.M. Lavrent'ev in 1964 with the goal of solving a coefficient inverse problem for a wavelike equation in 3 dimensions. The data are non overdetermined. Convergence analysis is presented along with the numerical results. An intriguing feature of the Lavrent'ev equation is that, without any linearization, it reduces a highly nonlinear coefficient inverse problem to a linear integral equation of the first kind. Nevertheless, numerical results for that equation, which use the data generated for that coefficient inverse problem, show a good reconstruction accuracy. This is similar with the classical Gelfand–Levitan equation derived in 1951, which is valid in the one-dimensional case.

Distribution Statement: 2-Distribution Limited to U.S. Government agencies only; report contains proprietary info
Acknowledged Federal Support: Y

Publication Type: Journal Article Peer Reviewed: Y **Publication Status:** 1-Published

Journal: Journal of Applied and Industrial Mathematics

Publication Identifier Type: Other

Publication Identifier:

Volume: 15

Issue: 3

First Page #: 413

Date Submitted: 1/24/22 12:00AM

Date Published: 12/28/21 5:00AM

Publication Location: Germany

Article Title: Convexification Inversion Method for Nonlinear SAR Imaging with Experimentally Collected Data

Authors: Michael Klibanov, Vo Khoa, Alexey Smirnov, Loc Nguyen, Grant Bidney, Lam Nguyen, Anders Sullivan,

Keywords: SAR imaging, gradient descent method, convexification, global convergence, experimental data, data propagation

Abstract: This paper is concerned with the study of a version of the globally convergent convexification method with direct application to synthetic aperture radar (SAR) imaging. Results of numerical testing are presented for experimentally collected data for a fake landmine. The SAR imaging technique is a common tool used to create maps of parts of the surface of the Earth or other planets. Recently, it has been applied in the context of noninvasive inspections of buildings in military and civilian services. Nowadays, any SAR imaging software is based on the Born approximation which is a linearization of the original wave-like partial differential equation. One of the essential assumptions this linearization procedure needs is that only those dielectric constants are imaged whose values are close to the constant background. In this work, we propose a radically new idea: to work without any linearization while still using the same data as the conventional SAR imaging technique uses.

Distribution Statement: 1-Approved for public release; distribution is unlimited.

Acknowledged Federal Support: Y

RPPR Final Report as of 18-Apr-2022

Publication Type: Journal Article Peer Reviewed: Y **Publication Status:** 1-Published
Journal: Inverse Problems
Publication Identifier Type: DOI Publication Identifier: <https://doi.org/10.1088/1361-6420/ac50b8>
Volume: 38 Issue: 4 First Page #: 1
Date Submitted: 3/10/22 12:00AM Date Published: 3/10/22 6:07PM
Publication Location: United Kingdom
Article Title: Carleman contraction mapping for a 1D inverse scattering problem with experimental time-dependent data
Authors: Thuy Le, Michael Klibanov1, Loc Nguyen1, Anders Sullivan, Lam Nguyen
Keywords: one-dimensional wave equation, Carleman estimate, iterative method, contraction principle, numerical solution, experimental data, global convergence
Abstract: It is demonstrated that the contraction mapping principle with the involvement of a Carleman weight function works for a coefficient inverse problem for a 1D hyperbolic equation. Using a Carleman estimate, the global convergence of the corresponding numerical method is established. Numerical studies for both computationally simulated and experimentally collected data are presented. The experimental part is concerned with the problem of computing dielectric constants of explosive-like targets in the standoff mode using severely underdetermined data.
Distribution Statement: 1-Approved for public release; distribution is unlimited.
Acknowledged Federal Support: Y

Publication Type: Journal Article Peer Reviewed: Y **Publication Status:** 4-Under Review
Journal: preprint
Publication Identifier Type: Other Publication Identifier: arXiv:2203.12694
Volume: Issue: First Page #:
Date Submitted: 3/30/22 12:00AM Date Published: 3/22/22 4:00AM
Publication Location:
Article Title: The Carleman-contraction method to solve quasi-linear elliptic equations
Authors: Loc Hoang Nguyen
Keywords: Carleman estimate, the contraction principle, quasi-linear PDEs, Cauchy data
Abstract: We propose a globally convergent numerical method to compute solutions to a general class of quasi-linear PDEs. Combining the quasi-reversibility method and a suitable Carleman weight function, we define a map of which fixed point is the solution to the PDE under consideration. To find this fixed point, we define a recursive sequence with an arbitrary initial term using the same manner as in the proof of the contraction principle. Applying a Carleman estimate, we show that the sequence above converges to the desired solution. On the other hand, we also show that our method delivers reliable solutions even when the given data are noisy. Numerical examples are presented.
Distribution Statement: 2-Distribution Limited to U.S. Government agencies only; report contains proprietary info
Acknowledged Federal Support: Y

BOOKS:

Publication Type: Book Peer Reviewed: Y **Publication Status:** 1-Published
Publication Identifier Type: DOI Publication Identifier: <https://doi.org/10.1515/9783110745481-202>
Book Edition: n/a Volume: n/a Publication Year: 2021 Date Received: 22-Oct-2021
Publication Location: Germany
Publisher: De Gruyter
Book Title: Inverse Problems and Carleman Estimates. Global Uniqueness, Global Convergence and Experimental Data
Authors: Michael Klibanov, Jingzhi Li
Editor:
Acknowledged Federal Support: Y

Publication Type: Book Peer Reviewed: Y **Publication Status:** 1-Published
Publication Identifier Type: DOI Publication Identifier: <https://doi.org/10.1080/17415977.2021.1943>
Book Edition: n/a Volume: n/a Publication Year: 2021 Date Received: 22-Oct-2021

RPPR Final Report

as of 18-Apr-2022

Publication Location: Germany

Publisher: De Gruyter

Book Title: Inverse Problems and Carleman Estimates. Global Uniqueness, Global Convergence and Experimental Data

Authors: Michael Klibanov, Jingzhi Li

Editor:

Acknowledged Federal Support: Y

Partners

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I certify that the information in the report is complete and accurate:

Signature: Michael V. Klibanov

Signature Date: 4/15/22 9:00PM

FINAL REPORT

US Army Research Laboratory and Office of Army Research grant W911NF-19-1-0044

Grant Title: Determination of electrical and geometrical characteristics of mine-like and IED-like targets using new globally convergent inverse algorithms

Reportage Period: December 19, 2018–December 18, 2021

Principal Investigator: Michael V. Klibanov, Professor, Department of Mathematics and Statistics, University of North Carolina at Charlotte, Charlotte, NC 28223, email: mklibanv@uncc.edu

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ABSTRACT

The **MAIN GOAL** of this project has been an intensive collaboration of the PI and his team with engineers of the US Army Research Laboratory (ARL) Drs. Anders Sullivan and Lam Nguyen. This goal was successfully achieved with seven (7) joint resulting publications during the reportage period [4]-[10]. In fact, this is a **TEN (10) YEARS OLD LONG TERM** collaboration since the first joint publication in 2012 [40]. The total number of joint publications in the ten (10) years period of 2012-2022 is thirteen (13), see [4]-[10], [35]-[40]. In this project, the **TRULY REVOLUTIONARY CONVEXIFICATION** inversion method of the PI has been applied to numerically solve Coefficient Inverse Problems (CIPs) for both experimentally collected microwave data and the computationally simulated data. The experimental data were collected by both ARL and the microwave laboratory of the campus where the PI works. The PI leads this laboratory and its work has been sponsored by this ARO grant.

The total number of publications with the acknowledgment of the support of this grant is thirty four (33), i.e. eleven (11) publications annually [2]-[34]. Among those are two books [2, 3].

The following eight (8) most prominent results were obtained in this project:

1. Direct applications to standoff detection and identification of antipersonnel land mines and improvised explosive devices were addressed for both computationally simulated and the above mentioned experimental data [3], [5]-[11], [23].
2. Direct applications to the problem of standoff inspections of buildings was addressed for experimental data of ARL [4].
3. The collaboration with ARL has resulted in a **SURPRISING BREAKTHROUGH JOINT DISCOVERY** with far going applications to the Army needs. This is the Convexification Based Nonlinear SAR Imaging (CONSAR) [4, 5]. CONSAR has demonstrated a very good performance on experimental data, in particular in the problem of standoff inspections of buildings.
4. In his **BREAKTHROUGH** publications in 2019 [15, 16] and then in 2020 [17], and then in the book [3, Chapters 11,12] (2021), the PI has solved, **for the first time**, a 114 years old the so-called Travel Time Tomography Problem (TTTP), which was first posed in 1905 by a German scientist Herglotz [70]. Furthermore, numerical results of [17] demonstrate a **REMARKABLE ROBUSTNESS** of the PI's convexification-based technique. Indeed, good quality images were obtained for large noise levels varying between 80% and 170%. TTTP is **THE SINGLE MOST CHALLENGING** Coefficient Inverse Problem.
5. **For the first time**, a globally convergent numerical method was developed for the famous and enormously popular Coefficient Inverse Problem of Electrical Impedance Tomography [3, Chapter 7], [12].
6. **The first honor of the PI:** A laudation paper [76] published in a reputable *Journal of Inverse and Ill-Posed Problems*, 30, 1-4, 2022. This paper is written on the occasion of the PI's 70st anniversary. It is signed by eleven (11) members of the editorial board. **The title of this paper is:** "*Research biography of a distinguished expert in the field of inverse problems: Professor Michael Victor Klivanov*".

7. **The second honor of the PI:** A special minisymposium dedicated to the PI's 70st anniversary on the conference "Inverse Problems: Modeling and Simulation" to be held in Malta in May 2022 <https://www.ipms-conference.org/ipms2022/index.php>.
8. Two books of the PI were published [2, 3]. Both acknowledge the support of this grant.

The **key mathematical tool** of this project, including the above mentioned results of items 1-4, is the convexification method originally invented by the PI [80, 81]. While those initial publications were only theoretical ones, the PI and his research team have started active numerical studies of the convexification only much later, after publishing the paper [43] of the PI with coauthors. This paper has removed a number of important obstacles which were standing on the way of numerical studies. The vast majority of the above mentioned thirty three (33) publications [2]-[34] acknowledging the support of this grant combine both analytical and numerical studies of the convexification.

The convexification has demonstrated a **REMARKABLE** robustness. Indeed, in the case of computationally simulated data acceptable quality images are obtained in section 14 even for very high levels of noise between 80% and 170%, see section 14 as well as [3, Chapter 12], [17]. In the case of experimental data, the convexification excellently handles the natural accompanying noise in these data [3, Chapter 10], [4]-[10].

The convexification is currently the **SINGLE** known numerical method for CIPs, which has three (3) most prominent properties:

1. Rigorously guaranteed global convergence, i.e. a good first guess for the solution is not required. Whereas all other numerical method for CIPs converge only locally, i.e. they require a good first guess for the solution. However, such a guess is rarely available in Army applications.
2. Rigorous estimates of accuracies of the solutions depending on the noise level in the data.
3. This method is applicable to a very broad class of CIPs, as it is clear from this report.

1 Introduction

The **MAIN GOAL** of this project has been an intensive collaboration of the PI and his team with engineers of the US Army Research Laboratory (ARL) Drs. Anders Sullivan and Lam Nguyen.

The following six (6) most prominent results were obtained in this project:

1. Direct applications to standoff detection and identification of antipersonnel land mines and improvised explosive devices were addressed for both computationally simulated and the experimental data [3], [5]-[11], [23]. Experimental data were collected by both ARL and the microwave laboratory located in the campus where the PI works. The PI leads this laboratory, and it has been funded by this grant.
2. Direct applications to the problem of standoff inspections of buildings was addressed for experimental data of ARL [4].
3. The collaboration with ARL has resulted in a **SURPRISING BREAKTHROUGH JOINT DISCOVERY** with far going applications to the Army needs. This is the Convexification Based Nonlinear SAR Imaging (CONSAR) [4, 5]. CONSAR has demonstrated a very good performance on experimental data, in particular in the problem of standoff inspections of buildings, see sub-subsections 2.1.1, 2.3.1, section 7 and publications [4, 5].
4. In his **BREAKTHROUGH** publications in 2019 [15, 16] and then in 2020 [17], and then in the book [3, Chapters 11,12] (2021), the PI has solved, **for the first time**, a 114 years old the so-called Travel Time Tomography Problem (TTTP), which was first posed in 1905 by a German scientist Herglotz [70]. Furthermore, numerical results of [17] demonstrate a **REMARKABLE ROBUSTNESS** of the PI's convexification-based technique. Indeed, good quality images were obtained for large noise levels varying between 80% and 170%. TTTP is **THE SINGLE MOST CHALLENGING** Coefficient Inverse Problem.
5. **For the first time**, a globally convergent numerical method was developed for the famous and enormously popular Coefficient Inverse Problem of Electrical Impedance Tomography [3, Chapter 7], [12].
6. **The first honor of the PI:** A laudation paper [76] published in a reputable Journal of Inverse and Ill-Posed Problems, 30, 1-4, 2022. This paper is written on the occasion of the PI's 70st anniversary. It is signed by eleven (11) members of the editorial board.
7. **The second honor of the PI:** A special minisymposium dedicated to the PI's 70st anniversary on the conference "Inverse Problems: Modeling and Simulation" to be held in Malta in May 2022 <https://www.ipms-conference.org/ipms2022/index.php>.
8. Two books of the PI were published [2, 3]. Both acknowledge the support of this grant.

PUBLICATIONS:

1. The support of this grant was acknowledged in thirty three (33) publications [2]-[34]. This amounts to $33/3 = 11$ publications annually as the average.
2. In particular, two books were published with the acknowledgment of the support of this grant [2, 3].
3. Seven (7) joint papers of the PI's team with ARL with the acknowledgments of the support of this grant were published during the reportage period of [4]-[10]. The total number of joint publications of the PI's team with ARL reflecting their long term ten (10) years collaboration is thirteen (13) during the period of 2012-2022 [4]-[10], [35]-[40].

Remark 1.1. *One comment about the content of this report. Since the results of this project were discussed in thirty four (34) publications [2]-[34], then, for brevity, the PI presents below only the most important results. Although other results are also interesting and new.*

This report is structured as follows. In section 2, which the PI labels as “Summary of Research Achievements”, main results of this project are briefly outlined. In the cases of experimental data images of computed dielectric targets and certain tables reflecting their values are also included in section 2. Those computed values are compared with the true ones in corresponding tables.

On the other hand, in follow up sections the mathematical theory of several versions of the **PI's TRULY REVOLUTIONARY CONVEXIFICATION INVERSION METHOD** is presented. It is this theory, on which all those computational results for experimentally collected data are based. In addition, some related numerical results for computationally simulated data are presented in those follow up sections.

The PI points out that the convexification is based on an earlier **GROUND BREAKING** idea of the PI published in 1981 [53]. In this work the powerful tool of Carleman estimates was introduced in the field of Coefficient Inverse Problems (CIPs) **FOR THE FIRST TIME**. However, initially this idea was applied only for proofs of uniqueness and stability theorems for CIPs, see, e.g. follow up works [45, 77, 78, 82, 83] and it is still applied this way by many authors, see, e.g. [48, 120] and references cited therein. The PI is the **ONLY** one who has figured out how to apply Carleman estimates to get a globally convergent numerical method for CIPs, see some details in sub-subsection 1.2.2 and then more detailed descriptions in follow up sections. The PI calls this method “convexification”.

The convexification has demonstrated a **REMARKABLE** robustness. Indeed, in the case of computationally simulated data acceptable quality images are obtained in section 14 even for very high levels of noise between 80% and 170%, see section 14 as well as [3, Chapter 12], [17]. In the case of experimental data, the convexification excellently handles the natural accompanying noise in these data [3, Chapter 10], [4]-[10].

Definition 1.1. *A numerical method for a CIP is called globally convergent if there is a theorem which guarantees that this method provides a good approximation for the exact solution of that CIP without a priori knowledge of a small neighborhood of this solution.*

CIPs are both highly nonlinear and highly ill-posed. Therefore, the problem of constructions of numerical methods for them is one of the **MOST CHALLENGING PROBLEMS** of Computational Mathematics. Conventional numerical methods for CIPs are based on the minimization of least squares cost functionals, see, e.g. the book [54].

Since these functionals are typically non convex, then they are plagued by the phenomenon of multiple local minima and ravines. This makes minimization methods for those functionals to be unreliable and locally convergent ones, i.e. to ensure the convergence, one needs a good first guess about the solution. And this guess is rarely available. Figure 1 illustrates this point well. On the other hand, the convexification is currently the **SINGLE** known numerical method for CIPs, which avoids the above mentioned phenomenon of multiple local minima and ravines.

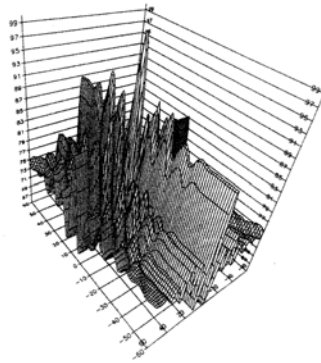


Figure 1: *An example of the level surface of the conventional least squares cost functional for a 1D CIP [110]. Minima are depicted as maxima for a better presentation. The phenomenon of multiple local minima, which plagues the minimization procedure, is evident.*

The convexification features three (3) most prominent properties:

1. Rigorously guaranteed global convergence, i.e. a good first guess for the solution is not required. Whereas all other numerical method for CIPs converge only locally, i.e. they require a good first guess for the solution. However, such a guess is rarely available in Army applications.
2. Rigorous estimates of accuracies of the solutions depending on the noise level in the data.
3. This method is applicable to a very broad class of CIPs, as it is clear from this report.

The PI got TWO HONORS during the reportage period, see section 3:

1. Journal of Ill-Posed and Inverse Problems in issue 1 of volume 30, 2022, <https://doi.org/10.1515/2022-2079> has published an Editorial signed by twelve (12) members of the Editorial Board [76]. **The title of this paper is:** “*Research biography of a distinguished expert in the field of inverse problems: Professor Michael Victor Klibanov*”.
2. The International Conference “Inverse Problems: Modeling and Simulation” to be held in Malta in May 2022 <https://www.ipms-conference.org/ipms2022/index.php> will have a special minisymposium dedicated to “the 70th anniversary of a distinguished expert in the field of inverse problems Professor Michael V. Klibanov”, <https://www.ipms-conference.org/ipms2022/m9>.

2 Summary of Research Achievements

2.1 The main goal of the project and direct applications to the interests of the US Army

2.1.1 The main goal

The **MAIN GOAL** of this project has been an intensive collaboration of the PI and his team with engineers of the US Army Research Laboratory Drs. Anders Sullivan and Lam Nguyen. This goal was successfully achieved with seven (7) joint resulting publications during the reportage period [4]-[10]. In fact, this is a **ten (10) years old long term** collaboration since the first joint publication in 2012 [40]. The total number of joint publications in the ten (10) years period of 2012-2022 is thirteen (13) [4]-[10], [35]-[40]. In this collaboration the **revolutionary convexification** inversion method of the PI has been applied to solve CIPs numerically for both experimentally collected microwave data and the computationally simulated data. The convexification is the **key mathematical tool** of this project. The experimental data were collected by both ARL and the team of the PI. The PI leads a microwave laboratory in the campus of the University of North Carolina at Charlotte. The laboratory has been funded by this ARO grant.

The PI points out that, it was numerically demonstrated in the [4] that, unlike the convexification method, the conventional least squares minimization method is not capable to work for the **BREAKTHROUGH JOINT DISCOVERY** of the ARL and PI, the Convexification Based Nonlinear SAR Imaging (CONSAR). This is a clear manifestation of a truly **CRUCIAL** advantage of the convexification over the conventional inversion technique for the Army needs .

2.1.2 Applications to the Army needs

Applications to the following interests of the US Army were studied in this project:

1. Standoff imaging and identification of buried antipersonnel land mines via calculations of their dielectric constants, shapes and locations.
2. Distant inspections of buildings.

Both computationally simulated and experimentally collected data were used for these purposes.

2.2 Preliminaries: publications and the convexification

2.2.1 Publications with the acknowledgments of the support of this grant

As it was stated in section 1, thirty three (33) publications [2]-[34] acknowledge the support of this grant, i.e. $33/3 = 11$ publications annually as the average. Two books [3, 2] are among them. Both are published in less than one (1) year time interval between publications. Total seven joint papers [4]-[10] with engineers of ARL Drs. Anders Sullivan and Lam Nguyen were published during the reportage period with the acknowledgment of the support of this grant. Total thirteen (13) joint papers of the PI with these people were published in the ten (10) years period of our collaboration 2012-2022 [4]-[10], [35]-[40].

2.2.2 The convexification: a REVOLUTIONARY mathematical tool

The convexification concept of the PI is currently the **ONLY** numerical method for CIPs with the rigorously guaranteed global convergence property. The convexification is based on an elegant and sophisticated mathematical tool of Carleman estimates, mentioned in section 1. The idea of the convexification is to construct a globally strictly convex cost functional for a considered CIP, thus avoiding that phenomenon. The key element of this functional is the presence of the Carleman Weight Function in it, i.e. the weight function, which is involved in the Carleman estimate for the corresponding partial differential operator. A **CRUCIAL** feature of the convexification is that it provides a **RIGOROUS GLOBAL CONVERGENCE ANALYSIS** with error estimates depending on the level of noise in the data, see Definition 1.1 in section 1 for the notion of global convergence.

The convexification has been the main mathematical tool of this project. The originating works of the PI were published in [80, 81] and later in [47, 85]. While these initial papers were purely theoretical ones, an active numerical work of the PI and his research team has started from the publication [43], in which a number of obstacles for numerical studies was removed. A good numerical performance of the convexification was confirmed in many recent publications of the PI and his research team for both computationally simulated and experimentally collected data [3]-[36].

2.3 Intensive collaboration with the US Army Research Laboratory

Below SAR means synthetic aperture radar imaging.

The collaboration with ARL has been going on in three directions. All three are described below in sub-subsections 2.3.1-2.3.3. These directions are:

1. Nonlinear SAR imaging via the convexification [4, 5]. In [4] experimentally collected SAR data collected by ARL [101] were used. On the other hand, in [5] SAR data were collected for a buried target in the microwave laboratory in the laboratory led by the PI. The PI labels this new idea as CONSAR, i.e. convexification based nonlinear SAR imaging. This is a **truly joint BREAKTHROUGH discovery** of the PI and engineers of ARL Drs. Anders Sullivan and Lam Nguyen.
2. Various versions of the convexification method for 1D CIPs [9, 8, 10, 32] to work on experimental data collected by the Forward Looking Radar of ARL [100]. These data were collected with the goal of detection and identification of antipersonnel land mines [100].
3. The work on microwave experimental data for buried targets collected by the above mentioned microwave laboratory of the PI [6, 7]. In this case the convexification was applied to solve 3D CIPs. Application in the standoff detection and identification of antipersonnel land mines.

2.3.1 CONSAR: Convexification based nonlinear SAR imaging: a surprising joint breakthrough discovery of ARL and the PI [4, 5]

The most recent **SURPRISING BREAKTHROUGH, JOINT WITH ARL**, is the nonlinear via the convexification [4, 5]. SAR imaging is widely used by the Army on aircraft, drones, inspections of buildings, etc. ARL routinely collects SAR data distant inspection of building, imaging of antipersonnel land mines, etc. However, the conventional SAR imaging software relies on the linearization of the corresponding CIP using the Born approximation [67]. As a result, while shapes and locations of targets are accurately imaged by this software, dielectric constants are not accurately computed [106].

Unlike the conventional case, CONSAR accurately computes dielectric constants of targets. Since dielectric constants characterize the constituent materials, then CONSAR brings in a fundamentally new and very important feature to the entire field of SAR imaging.

Namely, CONSAR is capable to provide:

1. Shapes and locations of targets, which is the case of the conventional SAR software.
2. **A RADICALLY NEW ITEM:** Characterizations of constituent materials of targets.
3. Therefore, CONSAR imaging has a potential to **REVOLUTIONIZE THE CONVENTIONAL SAR IMAGING SOFTWARE**, see, e.g. Figure 1.

Items 1-3 were confirmed on both computationally simulated and experimental data [4, 5]. Two types of experimental data were successfully treated by CONSAR: the data collected by ARL [4] and the microwave data collected in the microwave laboratory of the Department of Physics and Optical Science of the University of North Carolina at Charlotte [5]. This laboratory has been funded by this ARO grant and the PI is in charge of this laboratory.

It was shown in [4] that it is reasonable to assume that the dielectric constant of the wall was $\varepsilon_r(\text{wall}) = 2.5$. Let $\varepsilon_{r,\text{comp}}(\text{target})$ be the computed dielectric constant of a target, which is either metallic container, or a shower stall or a side wall. We define the relative dielectric constant of target computed by CONSAR as

$$\nu_{\text{comp}} = \frac{\varepsilon_{r,\text{comp}}(\text{target})}{\varepsilon_r(\text{wall})} = \frac{\varepsilon_{r,\text{comp}}(\text{target})}{2.5}. \quad (2.1)$$

Similarly we define the relative dielectric constant of target computed by conventional SAR imaging technique,

$$\nu_{\text{comp}}^{\text{SAR}} = \frac{\varepsilon_{r,\text{comp}}^{\text{SAR}}(\text{target})}{2.5}. \quad (2.2)$$

We also note that shower stalls were made of Phenol Formaldehyde Resin and/or Acrylic Resin, the dielectric constant varies greatly, depending on the type of a mixture used, see the table of dielectric constants [61]. The so-called "apparent dielectric constant" of the metallic container is in the interval [10, 30], see [40].

Table 1 [4]. Computed relative dielectric constants of targets of Figure 2c) by CONSAR (2.1) and by the regular SAR imaging (2.2). Tabulated values of true dielectric constants are given as intervals, just as in the table of dielectric constants [61] and in [40].

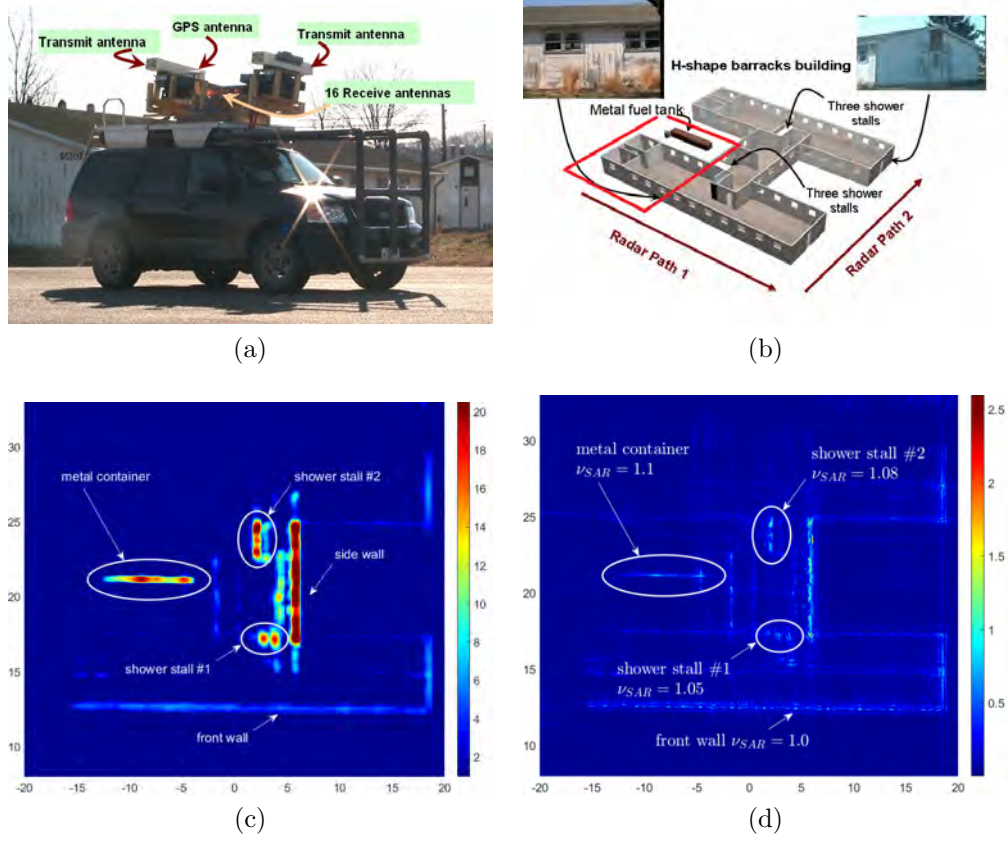


Figure 2: One of results of [4]. Performance of our breakthrough convexification based nonlinear SAR imaging technique for the SAR experimental data collected by ARL, see for these data [101]. a) The photograph of the vehicle used for by ARL for the SAR data collection [101]. b) A schematic diagram of the both the inspected building and measurements. c) Computed dielectric constants via CONSAR of targets inside of the inspected building. Table 1 demonstrates that the dielectric constants of the metallic container and two shower stalls were computed accurately. As to the overestimated value of the dielectric constants of the side wall, this is an artifact and the PI hopes to improve CONSAR in the future. d) Conventional SAR image from the same data if using the conventional method based on the Born approximation. The dielectric constant of the metallic container is underestimated by the factor of 6.5.

Conventional SAR imaging (column 5) underestimates the dielectric constant ν_{table} of the metallic container by the factor of 9.

Target	$\varepsilon_{r,\text{comp}}$	ν_{comp}	$\nu_{\text{comp}}^{\text{SAR}}$	ν_{table}
Shower stall #1	11.8	4.7	1.05	[4, 15] [61]
Shower stall #1	14.5	5.8	1.08	[4, 15] [61]
Metallic container	17.9	7.2	1.1	[10, 30] [40]

2.3.2 Various versions of the convexification method for 1D CIPs [9, 8, 10, 32]

To work on CONSAR as well as with the experimental data of the Forward Looking Radar of ARL [100], 1D CIP (2.3)-(2.7) for the wave-like Partial Differential Equation (2.5) was

studied. Various versions of the convexification method were developed in [9, 8, 10, 32] for this CIP. In this sub-subsection, we present the most recent result of [9] for the 1D CIP (2.3)-(2.7) for those experimental data. The reason of our preference of [9] is that this work represents the **SECOND GENERATION OF THE CONVEXIFICATION**. Unlike the first generation, the method of [9] has computed experimental ARL data of [100] in **real time**. The latter is definitely a **clear advantage** of the second generation over the first one.

The true reason of such a high speed of the second generation of the convexification in [9] is that, **quite surprisingly, the classic contraction principle is used there**. In our case, this principle works only if the Carleman Weight Function is involved in our numerical scheme. The rapid convergence of the contraction method is well known from the classic Real Analysis course of mathematical graduates. The idea of the second generation of the convexification method about the contraction principle was also used for another inverse problem in the paper of L.H. Nguyen and M.V. Klibanov [30] with the acknowledgment of the support of this grant.

Remark 2.1. *It seems to be at the first glance that the classical Gelfand-Levitan (GL) [95] method should work well for our 1D CIP. GL reduces the nonlinear 1D CIP to a linear integral equation of the second kind. However, it is unclear whether or not a solution of the GL equation is indeed the solution of the original CIP. In other words, uniqueness theorem for GL is not yet proven. Furthermore, it was demonstrated numerically in [38] that GL does not work for the same experimental data collected by the Forward Looking Radar of ARL [100] which we discuss both in [9, 8, 10, 32] and in this report.*

Let $b > 1$ be a known number, $x \in \mathbb{R}$ be the spatial variable and the function $c(x) \in C^3(\mathbb{R})$, represents the spatially distributed dielectric constant. We assume that

$$c(x) \in [1, b], x \in \mathbb{R}, \quad (2.3)$$

$$c(x) = 1 \text{ if } x \in (-\infty, 0] \cup [1, \infty). \quad (2.4)$$

Let T be a positive number. We consider the following Cauchy problem for a 1D hyperbolic PDE with a variable coefficient in the principal part of the operator:

$$c(x)u_{tt} = u_{xx}, \quad x \in \mathbb{R}, t \in (0, T), \quad (2.5)$$

$$u(x, 0) = 0, u_t(x, 0) = \delta(x). \quad (2.6)$$

The problem of finding the function $u(x, t)$ from conditions (2.5), (2.6) is our forward problem.

1D Coefficient Inverse Problem (1D CIP). *Suppose that the following two functions $g_0(t)$, $g_1(t)$ are known:*

$$u(0, t) = g_0(t), \quad u_x(0, t) = g_1(t), \quad t \in (0, T). \quad (2.7)$$

Determine the function $c(x)$ for $x \in (0, 1)$, assuming that the number $b > \varepsilon$ in (2.3) is known.

In the case of CONSAR imaging [4, 5], the version of [10] was used. In this version, a nonlinear change of variables is used to transform equation (2.5) in a simpler equation

$$v_{tt} = v_{xx} + q(x)v, \quad x \in \mathbb{R}, t \in (0, T) \quad (2.8)$$

with the unknown coefficient $q(x)$ which depends on $c(x)$. Equation (2.8) is simpler than equation (2.5) since the unknown coefficient $q(x)$ is not involve in the principal part of the differential operator in (2.8) where the second derivatives are involved.

A version of the convexification for simpler PDE (2.8) was first developed in [32]. Then it was extended in [10] to the more practically valuable case of equation (2.5) and then applied in CONSAR in [4, 5]. However, this version is inconvenient for computations due to the necessity of performing the inversion of the above mentioned change of variables [10].

Thus, in [8] a more advanced version of the convexification for the above CIP was developed. The version of [8] does not use that change of variables and works directly with equation (2.5) instead. Finally, in the most recent work [9] even more advanced version of the convexification was developed for CIP (2.3)-(2.7) without that change of variables.

The experimental data of the Forward Looking Radar of ARL [101] were treated by versions of [9, 8, 10] of the convexification.

We now present the results of [9] for the case of experimental data of the Forward Looking Radar of ARL. This radar collects time resolved backscattering data. Locations of targets were not of our interest since horizontal coordinates of targets were known from GPS very accurately. Targets were of two sorts: buried ones and ones located in air. Figures 2 display schematic diagrams of the data collection.

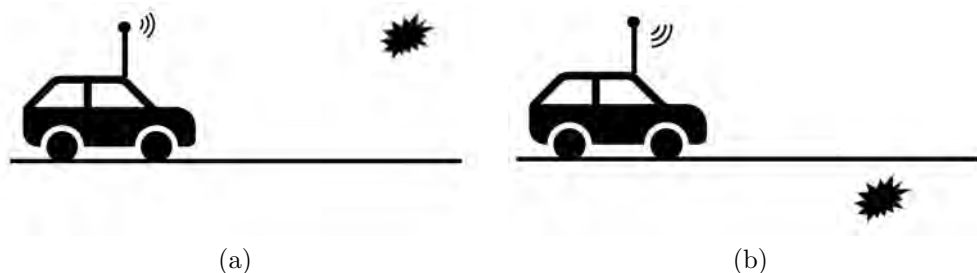


Figure 3: *Schematic diagrams of data collection by the Forward Looking Radar of ARL. a) The target located in air. b) The buried target.*

Dielectric constants of targets were not measured. Therefore, we had no choice but to compare our computed dielectric constants with tabulated ones. Tables [62] were used. As to the metallic targets, it was pointed out in sub-subsection 2.3.1 that it has an "apparent dielectric constant" whose values are in the interval $[10, 30]$, see [40]. The dielectric constants of the background were:

$$c_{\text{bckgr}} = \begin{cases} 1 & \text{for targets located in air,} \\ \in [3, 5] & \text{for targets buried in a dry sand [61].} \end{cases} \quad (2.9)$$

When solving the CIP, we have computed the relative spatially distributed dielectric constant $c_{\text{rel}}(x)$. Let $(a, b) \subset (0, 1)$ be the interval occupied by the target. The function $c_{\text{rel}}(x)$ is given by

$$c_{\text{rel}}(x) = \begin{cases} \frac{c_{\text{target}}(x)}{c_{\text{bckgr}}} & \text{if } x \in [a, b], \\ 1 & \text{otherwise.} \end{cases} \quad (2.10)$$

Next, we have taken the computed value of $c_{\text{rel}}(x)$ as

$$\text{comp}(c_{\text{rel}}(x)) = \begin{cases} \max c_{\text{rel}}(x) & \text{if this value was } > 1, \\ \min c_{\text{rel}}(x) & \text{if this value was } < 1. \end{cases} \quad (2.11)$$

Finally we set computed dielectric constant of the target as:

$$\text{comp}(c_{\text{target}}) = \text{comp}(c_{\text{rel}}(x)) c_{\text{bckgr}}, \quad (2.12)$$

where c_{bckgr} was taken from (2.9).

We had the following five targets:

Target 1. Bush. Its true dielectric constant is in the interval $[3, 20]$, see [56].

Target 2. A wood stake located in air. Its true dielectric constant was in the interval $[2, 6]$, see [61].

Target 3. Buried metallic box. Its apparent dielectric constant is in the interval $[10, 30]$, see [40].

Target 4. Buried metallic cylinder. Its apparent dielectric constant is in the interval $[10, 30]$, see [40].

Target 5. Buried plastic cylinder. Its true dielectric constant was in the interval $[1.6, 2.1]$, see [61].

Figures 4a),b) display samples of experimentally collected data for two targets and their smoothing by a data preprocessing procedure. Figures 4c),d) display spatial distributions $c_{\text{rel}}(x)$ of computed relative dielectric constants.

Table 2. Values of dielectric constants, computed from experimental data of the Forward Looking Radar of ARL by the second generation of the convexification method [9] for the 1D CIP (2.3)-(2.7). Formulas (2.9)-(2.12) were used. Here c_{table} means the dielectric constant taken from the above mentioned tables. Note that values of dielectric constants are given within certain intervals in those tables.

Object	$\text{comp}(c_{\text{target}})$	c_{bckgr}	c_{table}
Bush	7.62	1	$\in [3, 20]$
Wood stake	2.01	1	$\in [2, 6]$
Buried metallic box	$\in [12, 20]$	$\in [3, 5]$	$\in [10, 30]$
Buried metallic cylinder	$\in [12.3, 20.5]$	$\in [3, 5]$	$\in [10, 30]$
Buried plastic cylinder	$\in [1.6, 2.95]$	$\in [3, 5]$	$\in [1.1, 3.2]$

CONCLUSION

Since values of computed dielectric constants $\text{comp}(c_{\text{target}})$ of targets are within tabulated limits in Table 2, then we conclude that the second generation of the convexification method of [9] works well for these experimental data of ARL.

2.3.3 Joint work with ARL on microwave experimental data for buried targets [6, 7]

First, an analytical apparatus was developed for a version of the convexification for a 3D CIP for a wave-like Partial Differential Equation (PDE) in the frequency domain and computationally simulated data were used [11]. Next, a modified version of the method of

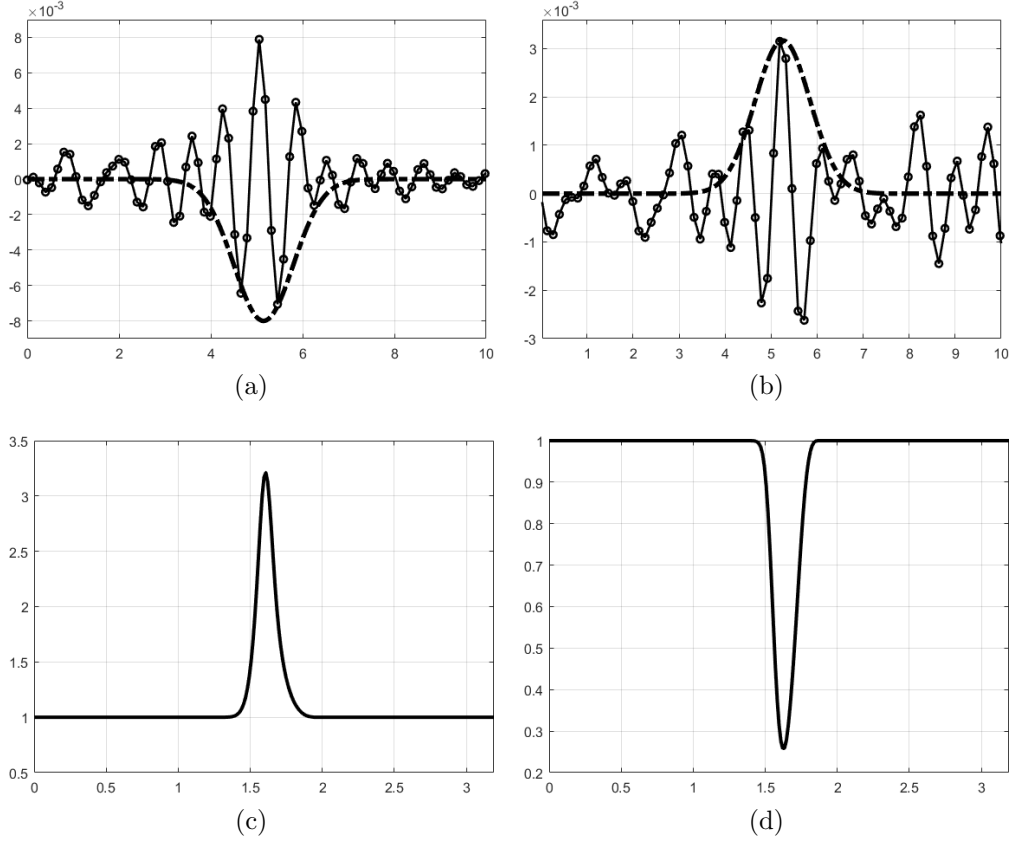


Figure 4: *Two samples of experimental data collected by the Forward Looking Radar of ARL for two targets and their computed images. a) Experimental data for Target 2, a wood stake in the air, and their smoothing by our data preprocessing procedure. b) Experimental data for Target 5, a buried plastic cylinder, and their smoothing by our data preprocessing procedure. c) Computed spatial distribution $c_{\text{rel}}(x)$ of the relative dielectric constant of a). d) Computed spatial distribution $c_{\text{rel}}(x)$ of the relative dielectric constant of b).*

[11] was applied to the experimental data collected by the above mentioned PI's microwave laboratory to image dielectric constants of mine-like targets buried in a box filled with the dry sand [6]. Next, a slightly modified version of the convexification method of was applied in to simultaneously image both dielectric constants and electrical conductivities of those buried targets [7].

Remark 2.2. *Note that the publication [7] has is a **very rare** case when two coefficients of a PDE are recovered simultaneously. Usually only one coefficient is recovered.*

Denote $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$ points of the space $\mathbb{R}^3$. Consider a rectangular prism

$$\Omega = (-R, R) \times (-R, R) \times (0, b) \text{ in } \mathbb{R}^3$$

for numbers $R, b > 0$. This prism is our computational domain of interest. Let $c = c(\mathbf{x})$ and $\sigma = \sigma(\mathbf{x})$ be respectively the spatially distributed dielectric constant and the electrical conductivity of the medium [7]. We assume that these functions are smooth and satisfy the following conditions:

$$\begin{cases} c(\mathbf{x}) \geq 1 & \text{in } \Omega, \\ c(\mathbf{x}) = 1 & \text{in } \mathbb{R}^3 \setminus \Omega, \end{cases} \quad \text{and} \quad \begin{cases} \sigma(\mathbf{x}) \geq 0 & \text{in } \Omega, \\ \sigma(\mathbf{x}) = 0 & \text{in } \mathbb{R}^3 \setminus \Omega. \end{cases} \quad (2.13)$$

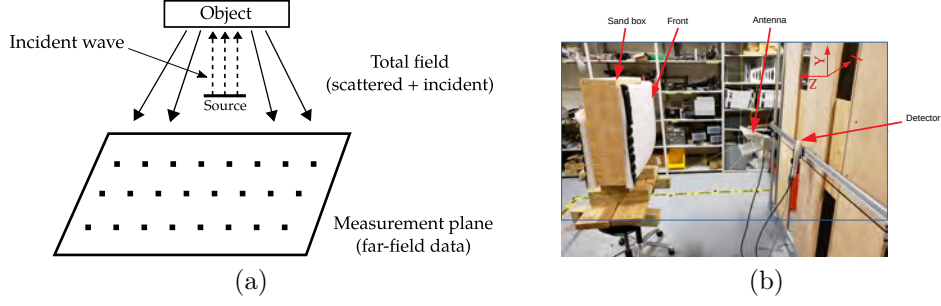


Figure 5: *Collection of microwave backscattering data. The data are collected from at a single frequency for many sources running along a straight line. a) Schematic diagram of data collection. b) A photograph of our experimental device.*

The second line of (2.13) means that we are assuming to have vacuum outside the domain of interest Ω . Figure 2a represents a schematic diagram of our measurements and Figure 2b is a photograph of our experimental device.

Let the number $d > 0$ and let $a_1 < a_2$. We consider the line L_{src} along which our sources run,

$$L_{\text{src}} = \{(\alpha, 0, -d) : a_1 \leq \alpha \leq a_2\}. \quad (2.14)$$

The distance from L_{src} to the xy -plane is d , and the length of the line of sources is $a = (a_2 - a_1)$. Using this setting, for each $\alpha \in [a_1, a_2]$ we arrange the point source $\mathbf{x}_\alpha := (\alpha, 0, -d)$ located on the straight line L_{src} . We also define the near-field measurement site as the lower side of the prism Ω ,

$$\Gamma := \{\mathbf{x} : |x|, |y| < R, z = 0\}.$$

We use notations of either α or $\mathbf{x}_\alpha$ to indicate the dependence of a functions from the source position $(\alpha, 0, -d)$. In practice, we measure backscattering data far from the domain Ω at a plane $P = \{\mathbf{x} : |x|, |y| < R, z = -s\}$ for a certain number $s > 0$. However, we use the so-called “data propagation procedure” described in [6] to “move” these far field to the part Γ of the near field plane $\{z = 0\}$.

We assume that the incident electric field is $E = (E_1, E_2, E_3) = (0, u_{\text{inc}}, 0)$. It was shown numerically in [46, 90] and analytically in [109] that the single Helmholtz-like PDE governs well the propagation of the second component $E_2(\mathbf{x}, \alpha) = u(\mathbf{x}, \alpha)$ in the medium. We also refer to the classical textbook of Born and Wolf [50, pages 695,696] for a similar conclusion, which is based on heuristic arguments.

The Helmholtz equation for the function $u(\mathbf{x}, k)$ with the radiation condition is:

$$\Delta u + (k^2 c(\mathbf{x}) - ik\sigma(\mathbf{x})) u = -\delta(\mathbf{x} - \mathbf{x}_\alpha) \quad \text{in } \mathbb{R}^3, \quad (2.15)$$

$$\lim_{r \rightarrow \infty} r(\partial_r u - iku) = 0 \quad \text{for } r = |\mathbf{x} - \mathbf{x}_\alpha|, \quad (2.16)$$

where k is the wave number. Here the incident wave is

$$u_{\text{inc}}(\mathbf{x}, \alpha) = \frac{\exp(ik|\mathbf{x} - \mathbf{x}_\alpha|)}{4\pi|\mathbf{x} - \mathbf{x}_\alpha|}. \quad (2.17)$$

Coefficient Inverse Problem. *Given a fixed value of the wave number $k > 0$, reconstruct both the dielectric constant $c(\mathbf{x})$ and the electric conductivity $\sigma(\mathbf{x})$ for $\mathbf{x} \in \Omega$ from*

the boundary measurements $F_0(\mathbf{x}, \mathbf{x}_\alpha)$, $F_1(\mathbf{x}, \mathbf{x}_\alpha)$ of the near field data,

$$F_0(\mathbf{x}, \mathbf{x}_\alpha) = u(\mathbf{x}, \alpha) \quad \text{for } \mathbf{x} \in \Gamma, \mathbf{x}_\alpha \in L_{\text{src}}, \quad (2.18)$$

$$F_1(\mathbf{x}, \mathbf{x}_\alpha) = u_z(\mathbf{x}, \alpha) \quad \text{for } \mathbf{x} \in \Gamma, \mathbf{x}_\alpha \in L_{\text{src}}, \quad (2.19)$$

where $u(\mathbf{x}, \alpha)$ is the total wave generated by the incident wave u_{inc} of (2.17).

It seems to be that we should know only the function $F_0(\mathbf{x}, \mathbf{x}_\alpha)$ since only the function $u(\mathbf{x}, \alpha)$ (voltage) can be measured, but not its z -derivative. However, as it was stated above, in this sub-subsection 2.3.3, in practice we have collected the far field data and then we have applied the data propagation procedure to “move” the data to the part Γ of the plane $\{z = 0\}$ in the near field. This procedure implies that we actually know both functions $F_0(\mathbf{x}, \mathbf{x}_\alpha)$ and $F_1(\mathbf{x}, \mathbf{x}_\alpha)$.

The version of the convexification method [7] has solved CIP (2.15)-(2.19) for our experimentally collected data. Our measurement device has the frequency range of $[1, 10]$ GHz. We have measured the data for the following five targets:

1. **Target 1.** Metallic cylinder, see Figure 6.
2. **Target 2.** Metallic letter ‘O’, see Figure 7.
3. **Target 3.** A piece of wood, see Figure 8.
4. **Target 4.** Metallic letter ‘A’, see Figure 9.
5. **Target 5.** Bottle of water, see Figure 10.

In computations by the version of the convexification method of section 9, we have chosen the following values of parameters listed in that section:

$$\lambda = 1.1, \gamma = 0, N = 6, \quad (2.20)$$

also, see (9.42) and Remark 9.3 in subsection 9.9.

Remarks 2.2:

1. *We have discovered that each target requires its own optimal value of the wave number k , see Table 3. This question needs a further study.*
2. *It was established in [40] that the so-called “apparent” dielectric constant of metals is in the interval $[10, 30]$, which is reflected in the values of c_{true} in Table 3.*
3. *The dielectric constant of water for this range of frequencies was directly measured in [115], and it was 23.8 there, see column 5 in Table 4.*
4. *Target 5 models well the ‘Glassmine 43’, which was used by Germany in the WWII era [60].*
5. *We have chosen non convex shapes of targets 2,3,4 since it is well known that it is very hard to image non convex targets when solving CIPs.*
6. *In Table 3 $c = \max(c_{\text{comp}}(\mathbf{x}))$ and $\sigma = \max(\sigma_{\text{comp}}(\mathbf{x}))$, where $c_{\text{comp}}(\mathbf{x})$ and $\sigma_{\text{comp}}(\mathbf{x})$ denote computed functions $c(\mathbf{x})$ and $\sigma(\mathbf{x})$ respectively and c_{true} is the true value of the dielectric constant.*
7. *Since values of σ are low in Table 4, then this question needs a further study.*

Table 3: Optimal wavenumbers and frequencies for Targets 1-5, see item 1 of Remarks 1.2.

Target	1	2	3	4	5
k	8.51	8.79	11.43	9.55	6.62
Frequency (GHz)	4.06	4.196	5.45	4.56	3.16

Table 4. Computed dielectric constants c and electrical conductivities σ for targets 1-5, see items 2-7 of Remarks 1.2.

Target	1	2	3	4	5
c	28.78	8.79	6.33	16.24	23.18
c_{true}	[10, 30]	[10, 30]	[2, 6] [62]	[10, 30]	23.8
σ	2.33	1.66	0.94	1.32	0.94

Remark 1.3. *It follows from Table 3 that we have accurately computed dielectric constants of all targets.*

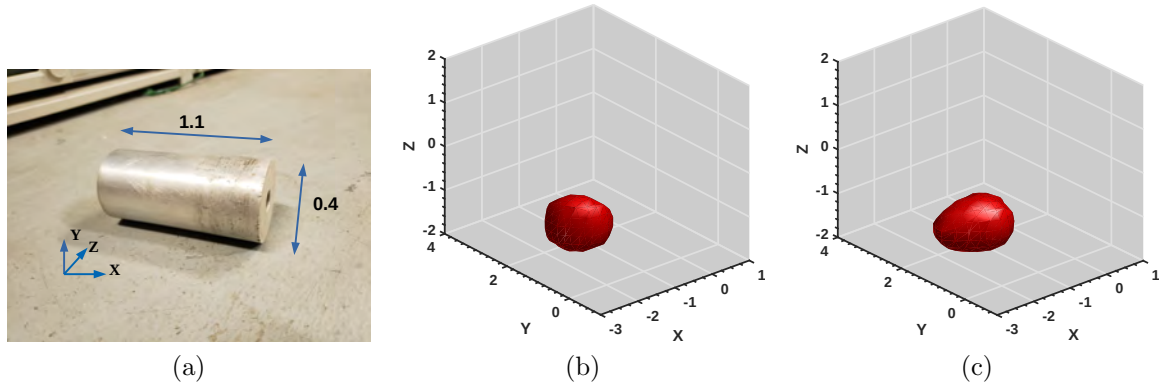


Figure 6: Images of the metallic cylinder obtained by the convexification method, see Tables 2,3 and Remark 1.3. a) A photograph of this target. b) The computed dielectric constant $c(\mathbf{x})$. c) The computed electric conductivity $\sigma(\mathbf{x})$.

2.4 Three versions of the convexification method for three 3D Coefficient Inverse Problems of wave propagation [3, 13, 19, 23]

We pose in this subsection 3.1 three 3D CIPs of wave propagation, to which three different versions of the convexification method are applied. These CIPs are different from the ones considered in 2.3.3. Therefore, three different versions of the convexification were developed for them during the reportage period.

2.4.1 3D CIP for the Helmholtz equation (2.15) with a single location of the point source [23]

A 3D CIP (2.15)-(2.19) was posed in sub-subsection 1.3.3. It was assumed in that CIP that the backscattering data are given at a single value of the wave number k , for all point

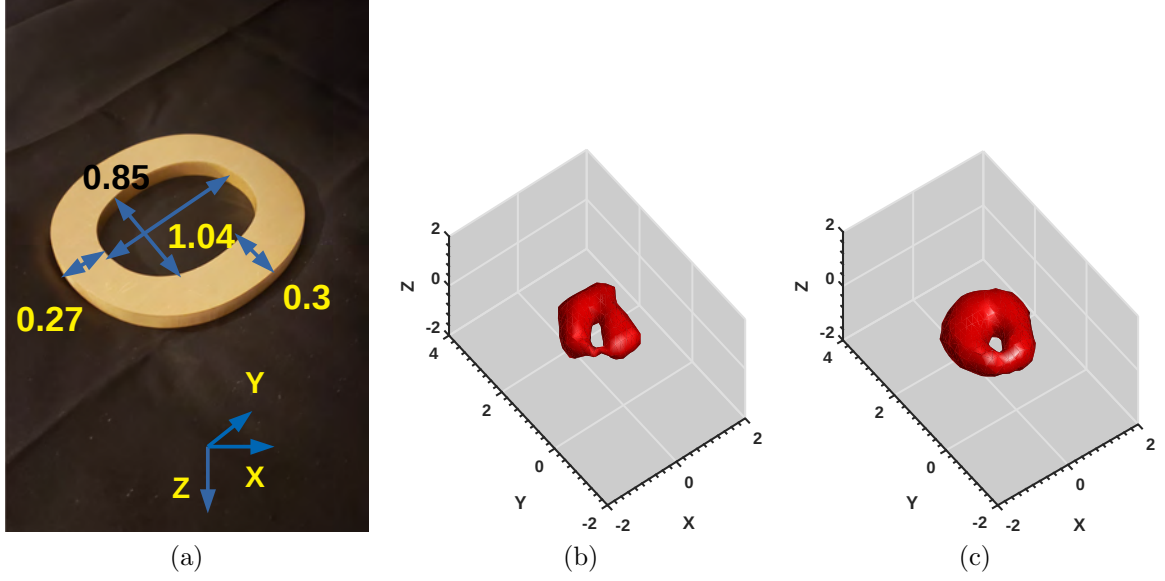


Figure 7: Images of the metallic letter ‘O’ obtained by the convexification method, see Tables 3,4 and Remark 1.3. a) A photograph of this target. b) The computed dielectric constant $c(\mathbf{x})$. c) The computed electric conductivity $\sigma(\mathbf{x})$.

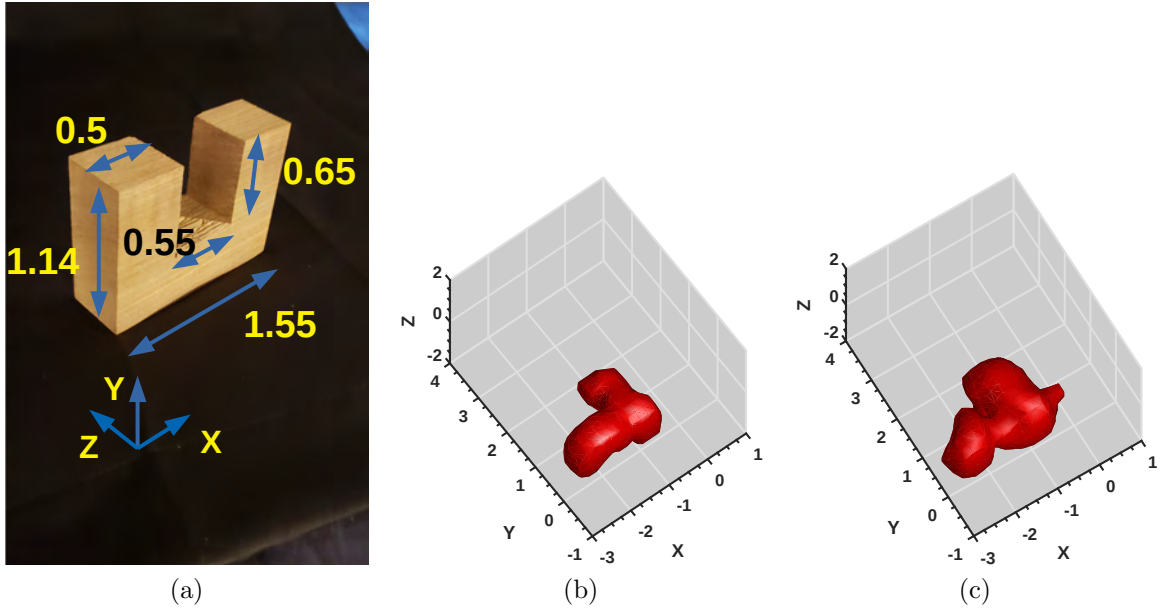


Figure 8: Images of a piece of wood having the shape of the letter ‘U’ obtained by the convexification method, see Tables 3,4 and Remark 1.3. a) A photograph of this target. b) The computed dielectric constant $c(\mathbf{x})$. c) The computed electric conductivity $\sigma(\mathbf{x})$.

sources running along the interval L_{src} (2.14) of a straight line and for all points $\mathbf{x} \in \Gamma$, where Γ is the backscattering side of the rectangular prism of interest Ω , see Figure 4a for a schematic diagram of measurements. We now change the statement of that CIP. First, we assume that $\Omega \subset \mathbb{R}^3$ is a cube and Γ is its lower side,

$$\Omega = (-R, R) \times (-R, R) \times (-R, R), \quad (2.21)$$

$$\Gamma = \{\mathbf{x} : -R < x, y < R, z = -R\}. \quad (2.22)$$

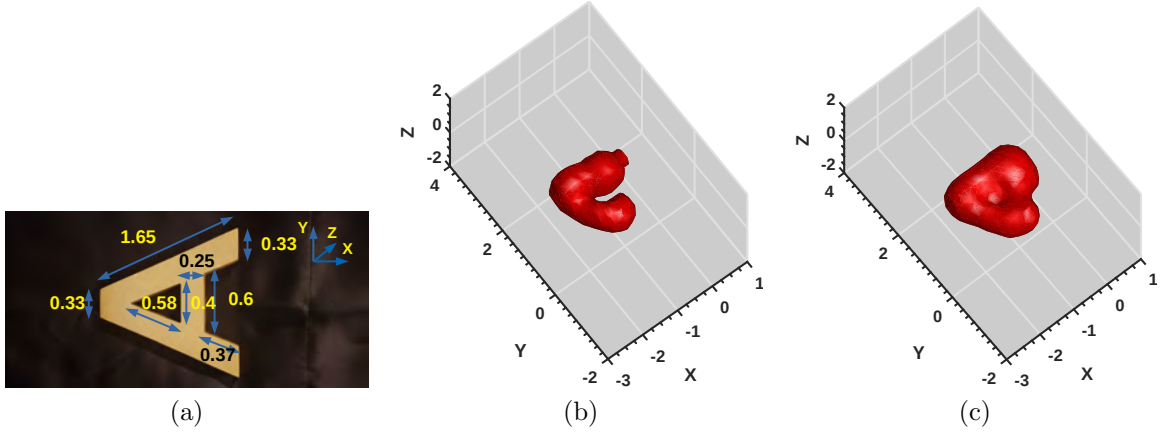


Figure 9: Images of the metallic letter ‘A’ obtained by the convexification method, see Tables 3,4 and Remark 1.3. a) A photograph of this target. b) The computed dielectric constant $c(\mathbf{x})$. c) The computed electric conductivity $\sigma(\mathbf{x})$.

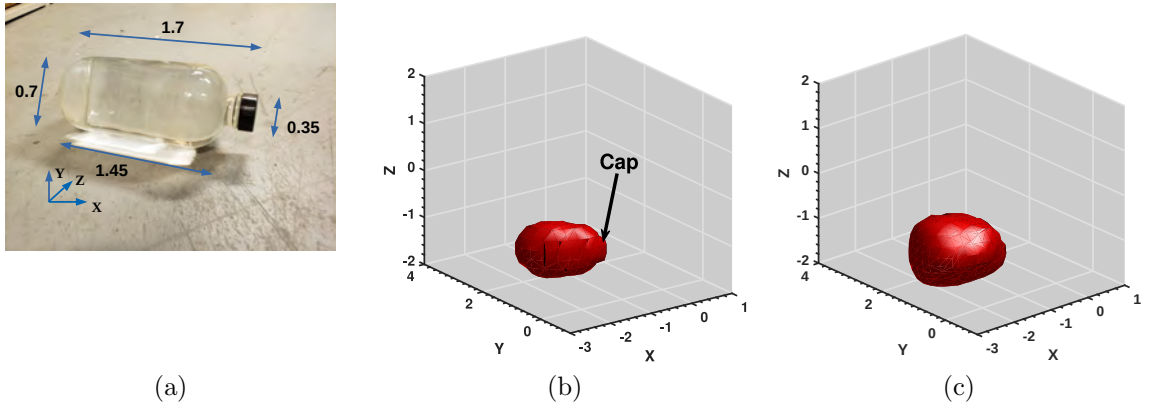


Figure 10: This target models well ‘Glassmine 43’, a non-metallic anti-personnel land mine largely with a glass body that Germany used to make detection harder in the World War II era [60]. Images of the bottle of water obtained by the convexification method, see Tables 3,4 and Remark 1.3. a) A photograph of this target. b) The computed dielectric constant $c(\mathbf{x})$. c) The computed electric conductivity $\sigma(\mathbf{x})$.

Second, and most importantly, we assume that the backscattering data at Γ are generated by a single location of the point source

$$\mathbf{x}_0 = (0, 0, -s), s = \text{const.} > R, \quad (2.23)$$

which is located below the cube Ω . The data are available for an interval of wave numbers $k \in (k_1, k_2) \subset (0, \infty)$, see Figure 11 for a schematic diagram of measurements.

We now specify the statement of our CIP. We ignore the electric conductivity coefficient $\sigma(\mathbf{x})$ in (2.15). For the fixed point source $\mathbf{x}_0$, let a component $u(\mathbf{x}, k)$ of the electric wave

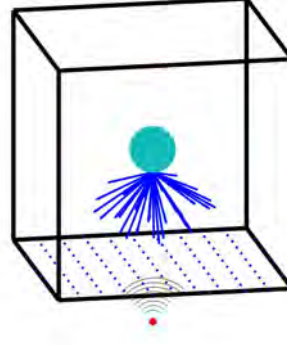


Figure 11: A schematic diagram of measurements for the case of a single location of the point source in CIP (2.24)-(2.26).

field be the solution of the Helmholtz equation:

$$\Delta u + k^2 c(\mathbf{x}) u = -\delta(\mathbf{x} - \mathbf{x}_0) \quad \text{in } \mathbb{R}^3, \quad (2.24)$$

$$\lim_{r \rightarrow \infty} r(\partial_r u - iku) = 0 \quad \text{for } r = |\mathbf{x} - \mathbf{x}_0|, \quad (2.25)$$

where the dielectric constant function $c(\mathbf{x})$ satisfies conditions (2.13) and is sufficiently smooth.

Coefficient Inverse Problem. Suppose that conditions (2.13), (2.24) and (2.25) hold. Determine the function $c(\mathbf{x})$ for $\mathbf{x} \in \Omega$, given functions $g_0(\mathbf{x}, k)$, $g_1(\mathbf{x}, k)$,

$$u(\mathbf{x}, k)|_{\Gamma} = g_0(\mathbf{x}, k), \quad u_z(\mathbf{x}, k)|_{\Gamma} = g_1(\mathbf{x}, k), \quad k \in (k_1, k_2). \quad (2.26)$$

It was explained in sub-subsection 2.3.3 that the data propagation procedure of [6] allows us to know both functions $g_0(\mathbf{x}, k)$ and $g_1(\mathbf{x}, k)$ rather than only $g_0(\mathbf{x}, k)$.

2.4.2 3D CIP for a wave-like equation in time domain [3, Chapter 8], [13]

We now pose a CIP which is known to be one of most *challenging* ones among Coefficient Inverse Problems. This is CIP of imaging of a heterogeneous medium using boundary measurements of a time dependent wave field. The governing PDE here is the time dependent analog of the Helmholtz equation. During the reportage period, we have solved this CIP only for the case of time dependent transmitted data. The case of backscattering data, which is quite applicable to the Army needs, is much more complicated. The PI plans consider this case in the future.

Let $A > 0$ be a number. Similarly with (2.21) we set the domain of interest $\Omega \subset \mathbb{R}^3$ is a cube

$$\Omega = \{\mathbf{x} = (x, y, z) : -A/2 < x, y < A/2, z \in (0, A)\}. \quad (2.27)$$

Let the number $a > 0$. We set the single point source we use as $\mathbf{x}_0 = (0, 0, -a)$. Hence, this source is located below the domain Ω . Let Γ_0 be the transmitted side of the boundary $\partial\Omega$ of the domain Ω and Γ_1 be the rest of this boundary,

$$\Gamma_0 = \{\mathbf{x} = (x, y, z) : -A/2 < x, y < A/2, z = A\}, \Gamma_1 = \partial\Omega \setminus \Gamma_0. \quad (2.28)$$

Let the number $T > 0$. Denote

$$S_T = \partial\Omega \times (0, T) \text{ and } \Gamma_{0,T} = \Gamma_0 \times (0, T). \quad (2.29)$$

Figure 12 depicts a schematic diagram of measurements.

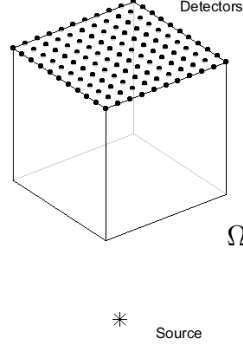


Figure 12: A schematic measurement diagram for CIP (2.30)-(2.32). Here a single location x_0 of the point source is used. Data on the transmitted side Γ_0 of the domain Ω are measured, see (2.30)-(2.32).

As the forward problem, we consider the following Cauchy problem for the equation which is connected with the Helmholtz equation via the Fourier transform:

$$c(\mathbf{x}) u_{tt} = \Delta u, \mathbf{x} \in \mathbb{R}^3, t > 0, \quad (2.30)$$

$$u(\mathbf{x}, 0) = 0, u_t(\mathbf{x}, 0) = \delta(\mathbf{x} - \mathbf{x}_0). \quad (2.31)$$

Since we work with only a single position $\mathbf{x}_0$ of the point source, we will omit below indications of dependencies on $\mathbf{x}_0$.

Coefficient Inverse Problem. Let formulas (2.26)-(2.29) be valid. Suppose that the following two functions are given:

$$u(\mathbf{x}, t) |_{(\mathbf{x}, t) \in S_T} = f_0(\mathbf{x}, t), \partial_z u(\mathbf{x}, t) |_{(\mathbf{x}, t) \in \Gamma_{0,T}} = f_1(\mathbf{x}, t), \quad (2.32)$$

Determine the function $c(\mathbf{x})$ for $\mathbf{x} \in \Omega$.

2.4.3 A linear integral equation for a highly nonlinear 3D CIP for the wave-like equation [19]

In this sub-subsection we present a **VERY INTRIGUING** idea of solving a highly nonlinear 3D CIP for equation (2.30) via a linear integral equation of the first kind, see formula (7.18) in the book [94] for this equation. The main reason why it is possible to derive a linear equation for that highly nonlinear 3-D CIP is that the data for that CIP are slightly overdetermined ones, i.e. redundant. On the other hand, in all other works on the convexification cited above as well as CIPs considered above the data are non overdetermined, i.e. non redundant.

We describe here a linear version of the convexification method, which solves the Lavrent'ev equation numerically [19]. Because of its linearity, this equation is similar with the classical Gelfand-Levitan integral equation (GL) in the 1-D case [?]. We point out, however, that an analog of the critique of GL presented in Remark 2.1 of subsection 2.3.2 does not work for the equation of our interest. Indeed, uniqueness of the solution of this equation follows in a certain sense from Theorem 12.3 in subsection 12.5. In addition, a full uniqueness theorem for the same case of data collection as in CIP7.1 (below) was proven in [91]. This means that the solution of that integral equation is indeed the solution of the originating CIP. On the other hand, a similar statement is not yet proven for GL since the publication of GL in 1950.

The first numerical study of that integral equation was published only in 2020 [44]. The numerical method of [44] is based on the standard regularization method. However, it turns out that this standard method is quite unstable: the technique of [44] works only for the case when the noise level is well too small, $\sigma < 10^{-5}$. Also, convergence estimates are not provided in [44].

On the other hand, our version of the convexification method of [19], which is outlined in section 12, turns the that equation in a problem for recovering the right hand side of an elliptic PDE. The resulting differential form of the problem to solve leads, in turn to a boundary value problem for a system of coupled elliptic PDEs which does not contain that unknown right hand side. Finally, this system is solved by the Quasi-Reversibility Method (QRM) [3, Chapter 4], [84] and a Carleman estimate leads to an estimate of the convergence rate of corresponding minimizers to the true solution depending on the noise level σ . Numerical study for $\sigma = 0.05$, i.e. 5% noise show a good performance of our technique. Note that $0.05/10^{-5} = 500$, i.e. our method can sustain noise whose level is 500 times larger than the one of [44]. The initial publication on the QRM was [93].

We consider now a 3-D CIP with multiple locations of the source and time dependent data measured on either the backscattering plane or on the forward scattering plane, see Figures 13 for schematic diagram of data collection.

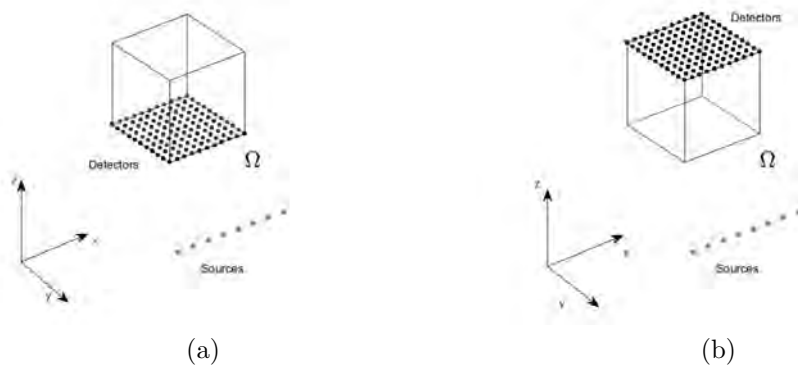


Figure 13: A schematic diagrams of data collection for the case of CIP (2.33)-(2.41). a) Backscattering data. b) Forward scattering (transmitted) data.

We choose the domain of interest Ω and the line of sources L_{src} as:

$$\Omega = (-R, R) \times (-R, R) \times (-R, R), \quad (2.33)$$

$$\Gamma = \{\mathbf{x} : -R < x, y < R, z = -R\}, \quad (2.34)$$

$$L_{\text{src}} := \{(\alpha, 0, -R - b) : a_1 \leq \alpha \leq a_2\}, \quad (2.35)$$

where $b = \text{const.} > 0$ and Γ is the backscattering side on which measurements are conducted.

Let the function $c(\mathbf{x})$ be sufficiently smooth in $\mathbb{R}^3$ and also

$$c(\mathbf{x}) \geq 1 \text{ in } \mathbb{R}^3, \quad (2.36)$$

$$c(\mathbf{x}) = 1 \text{ in } \mathbb{R}^3 \setminus \Omega. \quad (2.37)$$

Consider the following Cauchy problem

$$c(\mathbf{x}) u_{tt} = \Delta u, \quad \mathbf{x} \in \mathbb{R}^3, t > 0, \quad (2.38)$$

$$u(\mathbf{x}, 0) = 0, u_t(\mathbf{x}, 0) = \delta(\mathbf{x} - \mathbf{x}_0). \quad (2.39)$$

Coefficient Inverse Problem. *Let conditions (2.33)-(2.39) be satisfied. Find the function $c(\mathbf{x})$ for $\mathbf{x} \in \Omega$, given the functions $p_0(\mathbf{x}, \mathbf{x}_0, t)$ and $p_1(\mathbf{x}, \mathbf{x}_0, t)$,*

$$u(\mathbf{x}, \mathbf{x}_0, t) = p_0(\mathbf{x}, \mathbf{x}_0, t), \quad (\mathbf{x}, \mathbf{x}_0, t) \in \Gamma \times L_{\text{src}} \times (0, \infty), \quad (2.40)$$

$$\frac{\partial}{\partial n(\mathbf{x})} u(\mathbf{x}, \mathbf{x}_0, t) = p_1(\mathbf{x}, \mathbf{x}_0, t), \quad (\mathbf{x}, \mathbf{x}_0, t) \in \Gamma \times L_{\text{src}} \times (0, \infty), \quad (2.41)$$

where $n = n(\mathbf{x})$ is the outward unit normal vector to Γ at the point $\mathbf{x} \in \Gamma$.

It follows from results of [117] that, under some non restrictive conditions imposed on the unknown coefficient $c(\mathbf{x})$, one can apply the Fourier transform with respect to t to the function u , -

$$w(\mathbf{x}, \mathbf{x}_0, k) = \int_0^\infty u(\mathbf{x}, \mathbf{x}_0, t) e^{ikt} dt, \quad k \geq 0. \quad (2.42)$$

Furthermore, one can differentiate the function $w(\mathbf{x}, \mathbf{x}_0, k)$ at least twice with respect to $\mathbf{x}$ and infinitely many times with respect to k , and these derivatives can be calculated under the sign of the integral. Differentiate (2.42) twice with respect to k and then set $k = 0$. Denote

$$q(\mathbf{x}) = c(\mathbf{x}) - 1. \quad (2.43)$$

$$w^0(\mathbf{x}, \mathbf{x}_0) = \partial_k^2 w(\mathbf{x}, \mathbf{x}_0, k) |_{k=0}. \quad (2.44)$$

By (2.42)

$$w^0(\mathbf{x}, \mathbf{x}_0) = - \int_0^\infty t^2 u(\mathbf{x}, \mathbf{x}_0, t) dt. \quad (2.45)$$

Then it was derived from (2.42)-(2.45) in [19] that

$$\frac{1}{8\pi^2} \int_\Omega \frac{q(\boldsymbol{\xi})}{|\mathbf{x} - \boldsymbol{\xi}| |\boldsymbol{\xi} - \mathbf{x}_0|} d\boldsymbol{\xi} = w^0(\mathbf{x}, \mathbf{x}_0) + \frac{|\mathbf{x} - \mathbf{x}_0|}{4\pi}. \quad (2.46)$$

Linear equation (2.46) with respect to the function $q(\mathbf{x})$ is called “the Lavrent’ev equation”.

We now reformulate the CIP (2.40), (2.41) in terms of equation (2.46). Denote

$$g_0(\mathbf{x}, \mathbf{x}_0) = -2\pi \int_0^\infty p_0(\mathbf{x}, \mathbf{x}_0, t) t^2 dt + \frac{|\mathbf{x} - \mathbf{x}_0|}{2}, \text{ for } (\mathbf{x}, \mathbf{x}_0) \in \Gamma \times L_{\text{src}}, \quad (2.47)$$

$$g_1(\mathbf{x}, \mathbf{x}_0) = -2\pi \int_0^\infty p_1(\mathbf{x}, \mathbf{x}_0, t) t^2 dt + \frac{\partial}{\partial n(\mathbf{x})} \left(\frac{|\mathbf{x} - \mathbf{x}_0|}{4\pi} \right), \text{ for } (\mathbf{x}, \mathbf{x}_0) \in \Gamma \times L_{\text{src}}. \quad (2.48)$$

Thus, we have reduced CIP (2.40), (2.41) to the following inverse problem:

Inverse Problem. Find the function $q(\mathbf{x})$, $\mathbf{x} \in \Omega$ from functions $g_0(\mathbf{x}, \mathbf{x}_0)$ and $g_1(\mathbf{x}, \mathbf{x}_0)$ given, where

$$\frac{1}{4\pi} \int_{\Omega} \frac{q(\boldsymbol{\xi})}{|\mathbf{x} - \boldsymbol{\xi}| |\boldsymbol{\xi} - \mathbf{x}_0|} d\boldsymbol{\xi} = g_0(\mathbf{x}, \mathbf{x}_0), \quad (\mathbf{x}, \mathbf{x}_0) \in \Gamma \times L_{\text{src}}, \quad (2.49)$$

$$\frac{\partial}{\partial n(\mathbf{x})} \left(\frac{1}{4\pi} \int_{\Omega} \frac{q(\boldsymbol{\xi})}{|\mathbf{x} - \boldsymbol{\xi}| |\boldsymbol{\xi} - \mathbf{x}_0|} d\boldsymbol{\xi} \right) = g_1(\mathbf{x}, \mathbf{x}_0), \quad (\mathbf{x}, \mathbf{x}_0) \in \Gamma \times L_{\text{src}}, \quad (2.50)$$

where functions $g_0(\mathbf{x}, \mathbf{x}_0)$ and $g_1(\mathbf{x}, \mathbf{x}_0)$ are given in (2.47) and (2.48) respectively.

CONCLUSION:

We have reduced a highly nonlinear CIP (2.33)-(2.41) to a system (2.49)-(2.50) of two linear integral equations. Uniqueness of the solution of this system was proven in [91]. Therefore, if we find numerically the function $q(\mathbf{x})$, then the function $c(\mathbf{x}) = q(\mathbf{x}) + 1$ (see (2.43)) will be exactly our unknown target coefficient.

2.5 The first complete solution of a 114 years old Coefficient Inverse Problem [15, 16, 17], the single most challenging one among all CIPs

This subsection is about the so-called Travel Time Tomography Problem (TTTP).

The TTTP has immediate applications to the Army needs in the problem of seeing through the walls if using transmitted acoustic/electric signals. It is well known among experts in inverse problems that TTTP is the **SINGLE MOST CHALLENGING PROBLEM** among all CIPs.

2.5.1 Preliminaries

Let $\Phi \subset \mathbb{R}^3$ be a bounded domain. Let Γ_0 be a surface/curve and $\Gamma_1 \subseteq \partial\Phi$ be a part of the boundary of the domain Φ . Suppose that there are some point sources $\{x_0\} \subseteq \Gamma_0$ of the acoustic/electric signals. Suppose that at each point $x \in \Gamma_1$ one is measuring the first time $\tau(x, x_0)$ of arrival of those signals from points $x_0 \in \Gamma_0$. The TTTP is: Having

$\tau(x, x_0)$ for $x_0 \in \Gamma_0, x \in \Gamma$, find the spatially distributed speed of sound in the acoustic case and spatially distributed dielectric constants in the electric case.

Another important application of the TTTP is in Geophysics, in which case Φ is the Earth, $\Gamma_0 \subseteq \partial\Phi$ is the surface where explosions/earthquakes are and $\Gamma \subseteq \partial\Phi$ is the surface where seismic stations are located.

In 1905, 114 years prior publications [15, 16, 17] of the PI in 2019, a German scientist Herglotz and then, in 1907, two other German scientists Wiechert and Zoeppritz have posed and solved the TTTP in the 1D case [70, 119]. They kept in mind Geophysical applications. Since then many efforts of many mathematicians were devoted to the solution of the TTTP in the n -D case, $n = 2, 3$, see, e.g. [98, 99, 104, ?, 111, 113, 121]. However, **most importantly**, globally convergent numerical methods were not developed in those publications. In addition, all works in the 3D case have assumed that the data are over determined ones. In other words, the number m of free variables in the data exceeds the number n of free variables in the unknown coefficient, $m = 4 > n = 3$.

Also, all previous publications work with complete data. On the other hand, works of the PI [15, 16, 17] are for the case of the incomplete data. Complete data for the TTTP are generated by the point source running along the entire surface of the boundary of the domain of interest Φ . Unlike these, our point source runs along an interval of a straight line located outside of the domain of interest. This means that the data are incomplete.

Figure 14 displays the schematic diagram of the sources/detectors configuration in our case. Curves indicate geodesic lines along which the first arrival signals propagate. The purpose of Figures 15 is to illustrate the incompleteness of our data for the simplest case when the geodesic lines are straight lines, i.e. Radon transform data in 2D.

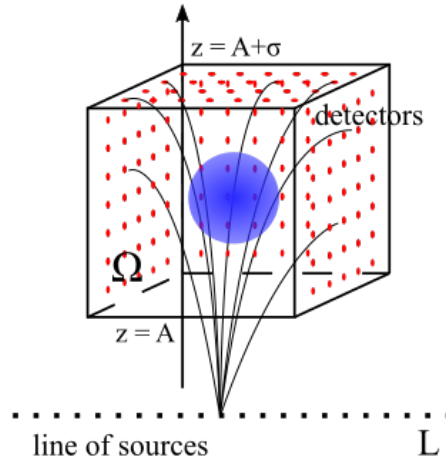


Figure 14: A schematic diagram of measurements in Travel Time Tomography Problem. Curves indicate geodesic lines.

114 years later three **GROUND BREAKING** publications [15, 16, 17] of the PI have completely solved TTTP for the first time in 2D and 3D cases for the case of incomplete non overdetermined data. Results of [15, 16, 17] are:

1. Two globally convergent numerical methods for the TTTP were developed.
2. The data are non overdetermined ones in the n -D case, $n = 2, 3$.

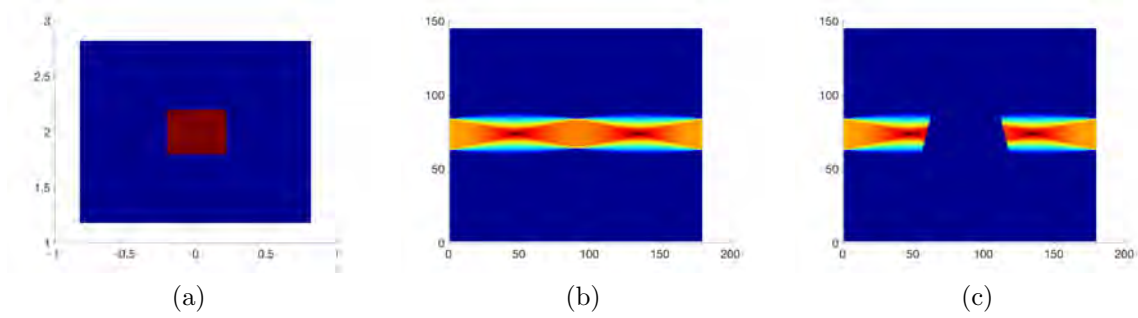


Figure 15: a) Target to be imaged. b) The complete Radon transform data. c) Incomplete Radon transform data as in the 2D case of Figure 14.

3. The data are incomplete, see Figures 15.
4. Lipschitz stability and uniqueness theorem is proven under conditions of items 2,3.
5. A convergent numerical method for the linearized TTTP is developed and numerically implemented. The method works under conditions of items 2,3. The linearization is done near a generic background function.

2.5.2 Statement of the problem

Below $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$. Let numbers $A, \sigma = \text{const.} > 0$. Define the rectangular prism $\Omega \subset \mathbb{R}^3$ as

$$\Omega = \{\mathbf{x} = (x, y, z) : x, y \in (0, 1), z \in (A, A + \sigma)\}. \quad (2.51)$$

Denote parts of the boundary $\partial\Omega$ as

$$B_A = \{\mathbf{x} = (x, y, z) : x, y \in (0, 1), z = A\}, \quad (2.52)$$

$$B_{A+\sigma} = \{\mathbf{x} = (x, y, z) : x, y \in (0, 1), z = A + \sigma\}, \quad (2.53)$$

$$\Gamma = \partial\Omega \setminus (B_A \cup B_{A+\sigma}). \quad (2.54)$$

Let $n(\mathbf{x})$ be the refractive index of the medium at the point $\mathbf{x}$. Hence, $c(\mathbf{x}) = 1/n(\mathbf{x})$ is the sound speed. Denote $m(\mathbf{x}) = n^2(\mathbf{x})$. Let the number $m_0 > 0$ be given. We impose the following assumptions on the function $m(\mathbf{x})$:

$$m(\mathbf{x}) \geq m_0, \quad \mathbf{x} \in \mathbb{R}^3, \quad (2.55)$$

$$m(\mathbf{x}) = 1, \quad \mathbf{x} \in \{z < A\}, \quad (2.56)$$

$$m \in C^2(\mathbb{R}^3), \quad (2.57)$$

$$m_z(\mathbf{x}) \geq 0, \quad \mathbf{x} \in \overline{\Omega}. \quad (2.58)$$

Remark 2.1. We note that the monotonicity condition (2.58) is not an overly restrictive one. Indeed, it can also be found in the end of section 2 of chapter 3 of the book [107]: see formulas (3.24) and (3.24') there. Also, an analogous monotonicity condition was actually imposed in the 1D case in the originating works of Herglotz and Wiechert and Zoeppritz [70, 119], see section 3 of chapter 3 of [107] for a description of their method. We also refer to Figures 5 and 10 in [118] for some geophysical information.

The function $m(\mathbf{x})$ generates the Riemannian metric

$$d\tau = \sqrt{m(\mathbf{x})} |d\mathbf{x}|, |d\mathbf{x}| = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}. \quad (2.59)$$

The travel time from the point $\mathbf{x}_0 \in \mathbb{R}^3$ (source) to the point $\mathbf{x} \in \mathbb{R}^3$ (receiver) is [?]

$$\tau(\mathbf{x}, \mathbf{x}_0) = \int_{\Gamma(\mathbf{x}, \mathbf{x}_0)} \sqrt{m(\mathbf{x}(s))} ds, \quad (2.60)$$

where $\Gamma(\mathbf{x}, \mathbf{x}_0)$ is the geodesic line connecting points $\mathbf{x}$ and $\mathbf{x}_0$ and ds is the euclidean arc length. We assume that the source $\mathbf{x}_0$ runs along an interval L of a straight line located in the plane $\{z = 0\}$,

$$L = \{\mathbf{x} = (x, y, z) : x = \alpha \in (0, 1), y = 1/2, z = 0\}. \quad (2.61)$$

Hence, $\mathbf{x}_0 = \mathbf{x}_\alpha = (\alpha, 1/2, 0)$, $\alpha \in (0, 1)$. Let $\tau(\mathbf{x}, \alpha)$ be the travel time between points $\mathbf{x}$ and $\mathbf{x}_\alpha = (\alpha, 1/2, 0)$. The signal corresponding to this travel time propagates along a geodesic line connecting the point source $\mathbf{x}_\alpha$ with the point $\mathbf{x}$. Thus, we denote $\tau(\mathbf{x}, \alpha) = \tau(\mathbf{x}, \mathbf{x}_\alpha)$. We denote $\Gamma(\mathbf{x}, \alpha)$ the geodesic line connecting points $\mathbf{x}$ and $\mathbf{x}_\alpha$. It is well known [107] that the function $\tau(\mathbf{x}, \alpha)$ satisfies eikonal equation as the function of $\mathbf{x}$,

$$\tau_x^2 + \tau_y^2 + \tau_z^2 = m(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^3, \quad (2.62)$$

$$\tau(\mathbf{x}, \alpha) = O(|\mathbf{x} - \mathbf{x}_\alpha|), \quad \text{as } |\mathbf{x} - \mathbf{x}_\alpha| \rightarrow 0.$$

We use the following assumption:

Regularity of Geodesic Lines. *The function $\tau(\mathbf{x}, \alpha) \in C^2(\mathbb{R}^3 \times [0, 1])$. For any pair of points $(\mathbf{x}, \mathbf{x}_\alpha) \in \bar{\Omega} \times L$ there exists unique geodesic line $\Gamma(\mathbf{x}, \alpha)$ connecting these two points and $\Gamma(\mathbf{x}, \alpha) \cap B_A \neq \emptyset$. In addition, if any geodesic line, which starts at a point $\mathbf{x}_\alpha \in L$, intersects B_A , then it intersects it at a single point. Also, it does not go “downwards” in the z -direction, but instead intersects $\partial\Omega \setminus B_A$ at another single point, see (2.51)-(2.54). In addition, after intersecting $\partial\Omega \setminus B_A$, this line does not “come back” in the domain Ω but rather goes away from this domain. In other words, this line is not reflected back from any point of its intersection with $\partial\Omega$.*

Travel Time Tomography Problem (TTTP). *Suppose that the function $m(\mathbf{x})$ satisfies conditions (2.55)-(2.58). Assume that the following function $f(\mathbf{x}, a)$ is given:*

$$\tau(\mathbf{x}, \alpha) = f(\mathbf{x}, \alpha), \quad \forall \mathbf{x} \in \Gamma \cup B_{A+\sigma}, \quad \forall \alpha \in (0, 1), \quad (2.63)$$

also, see Figure 13. Determine the function $m(\mathbf{x})$ for $\mathbf{x} \in \Omega$.

In other words, by (2.51)-(2.54) and (2.63) the data for the travel time are given for all sources running along the line interval L defined in (2.61) and for the part $\Gamma \cup B_{A+\sigma}$ of the boundary $\partial\Omega$. Hence, the data (2.63) are both non overdetermined and incomplete.

2.6 Convergent numerical method for the linearized TTTP [3, Chapter 12], [17]

Even though a globally convergent convexification-based numerical method for TTTP is developed in [3, Chapter 11], [15, 16] and is presented in section 13, sometimes the

linearization makes sense. Linearized TTTP is suitable for the case when the target structure of a medium is basically known. Nevertheless, one still needs a more accurate evaluation of this property. This linearized problem is also called in the literature the **geodesic X-ray transform problem** [97].

In the linearized case, one assumes that the target coefficient is basically known. But still a small perturbation of it is unknown and is the subject of the solution of the inverse problem. The importance of this problem is specifically pointed out in the classical book of Romanov [107, Chapter 3]. However, a reliable numerical method for this problem was absent until the recent publication of the PI with coauthors [17]. Results of [17] are presented below in section 14 as well as in the book of Klibanov and Li [3, Chapter 12].

A **PIONEERING** numerical method for the linearized TTTP for the d -D case, $d = 2, 3, \dots$ was proposed in [3, Chapter 12], [17]. This method is one of versions of the linear versions of the convexification principle, just as the method of sub-subsection 2.4.3 and, respectively, of section 12 for an important linear integral equation. The PI is unaware about other numerical methods for the linearized TTTP which would have convergence theorems, incomplete data and also would be non overdetermined.

The **BREAKTHROUGH** publications of the PI [3, Chapter 12], [15, 16, 17] have opened the door for truly working numerical methods for the TTTP.

A **REMARKABLE** degree of robustness is demonstrated even for very high levels of noise in the data between 80% and 170% [3, Chapter 12], [17].

2.7 The first globally convergent numerical method for the CIP of the electrical impedance tomography (EIT) [3, Chapter 7], [12]

This is another **BREAKTHROUGH** result of the PI for an enormously popular CIP of Electrical Impedance Tomography.

Electrical impedance tomography (EIT) is a non invasive and diffusive imaging method to recover the electrical conductivity distribution inside an object of interest by using the Dirichlet-to-Neuman map data on the boundary of that object. This modality is safe, portable and also has many clinical imaging applications [71]. EIT is enormously popular among experts in Inverse Problems. There is a vast number of research papers discussing EIT in the past 30 years, see, e.g. [68, 69, 73, 74, ?] and references cited therein. However, a globally convergent convexification-based numerical method of EIT with non overdetermined data was first developed only in the work [12] of the PI with coauthors, also, see the book [3, Chapter 7]. In this subsection we pose EIT. We present the convexification method for this problem in section 15.

Let Ω be an open bounded domain in $\mathbb{R}^d$ ($d = 2, 3$) to be imaged with a smooth boundary $\partial\Omega$. The EIT forward problem is formulated as: For any given input current

$$g_1 \in L_0^2(\partial\Omega) := \{g \in L_2(\partial\Omega) : \int_{\partial\Omega} g \, ds = 0\}$$

and the conductivity distribution $\sigma(x)$, find the function $u(x) \in H^1(\Omega)$ such that

$$\begin{cases} -\nabla \cdot (\sigma(x) \nabla u(x)) = 0 & \text{in } \Omega, \\ \sigma(x) \frac{\partial u}{\partial \nu} = g_1(x) & \text{on } \partial\Omega, \\ \int_{\partial\Omega} u(x) ds = 0, \end{cases} \quad (2.64)$$

where ν is the outward unit normal vector on $\partial\Omega$.

To avoid working with singularities and also to simplify the presentation, we model the point source here by a δ -like function instead of the δ -function. Let $\varepsilon > 0$ be a sufficiently small number. Let the source function $f(x)$ be such that

$$f(x) \in C^\infty(\mathbb{R}^n), f(0) \neq 0, f(x) \geq 0, \forall x \in \mathbb{R}^d, f(x) = 0 \text{ for } |x| > \varepsilon. \quad (2.65)$$

Let $G \subset \mathbb{R}^n$ be a bounded domain with its boundary $\partial G \in C^1$, $\Omega \subset G$ and $\partial\Omega \cap \partial G = \emptyset$. Let $\bar{x} \in \mathbb{R}^{d-1}$ be a fixed point. For $s \in [0, 1]$ denote $x_s = (x_{1s}, \bar{x})$ the position of the point source. Let $I = \{x_s = (x_{1s}, \bar{x}) : s \in [0, 1]\}$ be the interval of the straight line $\{x = (x_1, \bar{x}), x_1 \in \mathbb{R}\}$. Let $I_\varepsilon = \{x \in \mathbb{R}^d : \text{dist}(x, I) < \varepsilon\}$, where $\text{dist}(x, I)$ is the Hausdorff distance between the point x and the set I . We also assume that $I_\varepsilon \subset (G \setminus \bar{\Omega})$, which means that the support of the source function is outside of the domain Ω .

Let the function

$$\sigma(x) \in C^{2+\alpha}(\bar{G}), \sigma(x) = 1 \text{ for } x \in G \setminus \Omega \text{ and } \sigma(x) \geq \sigma_0 = \text{const.} > 0. \quad (2.66)$$

Here $\alpha = \text{const.} \in (0, 1)$ and $C^{k+\alpha}(\bar{G})$ be the Hölder space, where $k \geq 0$ is an integer. Assume first that $\sigma(x)$ is known. For each source position $x_s \in I$ we define the forward boundary value problem for EIT as the problem of finding the function $u(x, s)$ such that

$$\begin{cases} \nabla \cdot (\sigma(x) \nabla u(x, s)) = -f(x - x_s), & x \in G, \forall x_s \in \bar{I} \\ u(x, s)|_{x \in \partial G} = 0, & \forall x_s \in \bar{I}. \end{cases} \quad (2.67)$$

It is well known that for each $x_s \in I$ the problem (2.67) has a unique solution

$$u(x, s) \in C^{3+\alpha}(\bar{G}), \forall x_s \in \bar{I}, \quad (2.68)$$

see, e.g. [66]. We measure both Dirichlet and Neumann boundary conditions of the function u on a part $\Gamma \subseteq \partial\Omega$ of the boundary $\partial\Omega$,

$$u(x, s)|_{x \in \Gamma, x_s \in \bar{I}} = g_0(x, s) \text{ and } \partial_\nu u(x, s)|_{x \in \Gamma, x_s \in \bar{I}} = g_1(x, s). \quad (2.69)$$

We call the Dirichlet and Neumann boundary data (2.69) “restricted DtN data”. “DtN” stands for Dirichlet-to-Neumann map.

If the coefficient $\sigma(x)$ is known, then, having the solution of the forward problem (2.67), one can easily compute functions $g_0(x, s)$ and $g_1(x, s)$. Suppose now that the function $\sigma(x)$ is unknown. Then we arrive at the following inverse problem:

Coefficient Inverse Problem (CIP). *Assume that the function $\sigma(x)$ is unknown for $x \in \Omega$ and also that conditions (2.65), (2.66) hold. Also, assume that functions $g_0(x, s)$ and $g_1(x, s)$ in (2.69) are known for all $x \in \Gamma, x_s \in \bar{I}$. Determine the function $\sigma(x)$.*

Note that in this CIP the number d of free variables in the data equals the number of free variables in the unknown coefficient. Hence, this CIP is non overdetermined.

2.8 Convexification method for a CIP for a parabolic equation [3, Chapter 9], [14]

2.8.1 Preliminaries

The CIP, which is considered here, has applications in heat conduction [41] and in medical optical imaging using the diffuse infrared light [64]. In addition, this CIP has applications in financial mathematics in the search of the volatility coefficient in the Black-Scholes equation using the market data [?]. In the latter case, the volatility coefficient should be assumed to be dependent on the stock price.

Results of this section are taken from [14]. The first step towards the convexification for the CIP of this subsection was made in [85] for a similar CIP. However, there are some problems in [85], which prevent one from a numerical implementation of the idea of [85]. Indeed, although a weighted globally strictly convex Tikhonov-like functional is constructed in [85], the CWF in it is too complicated since it depends on two large parameters rather than on a single one. This means that the CWF of [85] changes too rapidly. The latter does not allow a numerical implementation.

Thus, a new Carleman estimate with a simpler CWF was established in [3, Chapter 9], [14]. This estimate can be used for computations and these computations were performed in [3, Chapter 9], [14], also see section 16.

2.8.2 Statement of the CIP

Below $\mathbf{x} = (x, \bar{x}) \in \mathbb{R}^n$, where $\bar{x} = (x_2, \dots, x_n)$ and $x = x_1$. Let the numbers $A, B > 0$ and $A < B$. We introduce the cube $\Omega \subset \mathbb{R}^n$ and a part Γ of its boundary $\partial\Omega$ as

$$\Omega = \{\mathbf{x} : A < x, x_2, \dots, x_n < B\}, \quad \Gamma = \{x = B, A < x_2, \dots, x_n < B\}. \quad (2.70)$$

Let the number $T > 0$. Denote

$$Q_T^\pm = \Omega \times (-T, T), \quad S_T^\pm = \partial\Omega \times (-T, T), \quad \Gamma_T^\pm = \Gamma \times (-T, T).$$

Below $\alpha \in (0, 1)$, $m \geq 1$ is an integer and $C^{m+\alpha}(\bar{\Omega})$, $C^{2m+\alpha, m+\alpha/2}(\bar{Q}_T^\pm)$ are Hölder spaces [92]. Let

$$b_j(\mathbf{x}), c(\mathbf{x}) \in C^{2+\alpha}(\bar{\Omega}); \quad j = 1, \dots, n.$$

We consider the elliptic operator L in the following form:

$$Lu = \Delta u + \sum_{j=1}^n b_j(\mathbf{x}) u_{x_j} - c(\mathbf{x}) u, \quad \mathbf{x} \in \Omega. \quad (2.71)$$

We assume that

$$c(\mathbf{x}) \geq 0 \quad \text{in } \bar{\Omega}. \quad (2.72)$$

The forward parabolic initial boundary value problem is stated as [92]:

Forward Problem. *Let the initial condition $f(\mathbf{x}) \in C^{4+\alpha}(\bar{\Omega})$. Find a function $u(\mathbf{x}, t) \in C^{4+\alpha, 2+\alpha/2}(\bar{Q}_T^\pm)$ satisfying the following conditions:*

$$u_t = Lu \quad \text{in } Q_T^\pm, \quad (2.73)$$

$$u(\mathbf{x}, -T) = f(\mathbf{x}), \quad (2.74)$$

$$u|_{S_T^\pm} = g_0(\mathbf{x}, t). \quad (2.75)$$

If $n = 3$ and functions $b_j(\mathbf{x}) \equiv 0$ for $j = 1, \dots, n$, then $c(\mathbf{x})$ is the absorption coefficient in the case of medical optical imaging using the diffuse infrared light [64].

If the domain Ω would have its boundary $\partial\Omega \in C^{4+\alpha}$ and if the Dirichlet condition $g_0(\mathbf{x}, t)$ would belong to $C^{4+\alpha, 2+\alpha/2}(\overline{S_T^\pm})$, and also corresponding compatibility conditions would be satisfied [92], then the existence and uniqueness of the solution $u \in C^{4+\alpha, 2+\alpha/2}(\overline{Q_T^\pm})$ of problem (2.71)–(2.75) would be ensured [92]. However, for the convenience of our derivations for the inverse problem, we have chosen the case of a piecewise smooth boundary $\partial\Omega$. Hence, we can only assume the existence of the solution $u \in C^{4+\alpha, 2+\alpha/2}(\overline{Q_T^\pm})$ of problem (2.73)–(2.75). As to its uniqueness, it follows immediately from (2.72) and the maximum principle for parabolic PDEs.

Coefficient Inverse Problem (CIP). *Let the number $t_0 \in (-T, T)$. Suppose that the function $g_0(\mathbf{x}, t)$ in (2.75) is known. Also, assume that the function $f(\mathbf{x})$ in (2.74) as well as the coefficient $c(\mathbf{x})$ are unknown. In addition, let the following two functions $g_1(\bar{x}, t)$ and $f_0(\mathbf{x})$ be known:*

$$u_x|_{\Gamma_T^\pm} = g_1(\bar{x}, t), \quad (2.76)$$

$$u(\mathbf{x}, t_0) = f_0(\mathbf{x}). \quad (2.77)$$

Find the unknown coefficient $c(\mathbf{x})$.

Remarks 2.2:

1. Thus, the data for this CIP is the single pair of functions (g_1, f_0) . These data are non overdetermined ones, so as for all CIPs for which the convexification method works. In other words, the number m of free variables in the data equals the number n of free variables in the unknown coefficient, $m = n$.
2. Assuming that $f_0(\mathbf{x}) \neq 0$ in $\bar{\Omega}$, uniqueness of a very similar CIP was proven for any value of $T > 0$ and for the case of a rather general bounded domain Ω using the method of [53], see, e.g. Theorem 1.10.7 in [45], Theorem 2 in [78], Theorem 3.10 in [79], Theorem 3.4 in [83] and Theorem 3.4.3 in [3]. Furthermore, it was shown in [3, 45, 83] that the uniqueness also holds even if both functions $g_0(\mathbf{x}, t)$ and $g_1(\mathbf{x}, t)$ are given for $(\mathbf{x}, t) \in \Gamma' \times (-T', T')$, where $\Gamma' \subset \partial\Omega$ is a sufficiently arbitrary part of $\partial\Omega$ and $T' \in (0, T]$ is an arbitrary number. We also refer to [120] for the Lipschitz stability estimates for this CIP.
3. As soon as the coefficient $c(\mathbf{x})$ is found via the solution of the above CIP, one can uniquely determine the function $f(\mathbf{x})$ in (2.74) using functions g_0 and f_0 [94]: the knowledge of g_1 is not necessary then, as long as $c(\mathbf{x})$ is known. This is the so-called “parabolic problem with the reversed time”. A recent convergent numerical method for the parabolic problem with the reversed time can be found in [18].

2.9 A TRULY UNIVERSAL NATURE of the convexification

The concept of the convexification has a truly **UNIVERSAL NATURE**. Indeed:

1. As it is clear from sections 2-16 below, the convexification is applicable to a very broad class of Coefficient Inverse Problems. In all cases the convexification results in numerical methods with rigorously proven global convergence (see Definition 1.1 in section 1 for the global convergence). Furthermore, in all cases rigorous accuracy estimates of solutions depending on the noise level are given.
2. Item 1 is **NOT** the case of any other numerical method.
3. It is clear from section 17 that the convexification is applicable not only to inverse problems but to forward problems for **CLASSIC** nonlinear Partial Differential Equations as well! A hard proof of this statement is the recent publication of the PI with coauthors [20] in a highly reputable journal: Journal of Computational Physics. **The impact factor of this journal is 3.553, which is quite a high value for a mathematical journal. “According to the Journal Citation Reports, Journal of Computational Physics has a 2020 impact factor of 3.553, ranking it 6th out of 55 in the category Physics, Mathematical”, https://en.wikipedia.org/wiki/Journal_of_Computational_Physics.**
4. As to item 3, we have applied the convexification method to develop a numerical procedure of computing viscosity solutions of the Hamilton-Jacobi equation [20]. The Hamilton-Jacobi equation is one of the most popular classic equations of the Mathematical Physics. The aim of the paper [20] is to compute viscosity solutions to a large class of Hamilton-Jacobi equations possibly involving non convex Hamiltonians.
5. The publication [20] is a **SURPRISING BREAKTHROUGH** in the context of viscosity solutions to Hamilton-Jacobi equations. It has two **MAJOR ADVANTAGES** from the numerical standpoint. First, it works for very general nonlinear partial differential operators of the Hamilton-Jacobi equation. This is radically better than all other numerical approaches to this equation, all of which impose quite restrictive conditions on those operators. Second, the convexification provides stable solutions with respect to the noise in the boundary data. The convexification is only applicable when a viscosity term is added to the Hamilton-Jacobi equation under consideration. The idea of adding the viscosity term and passing to the limit to obtain viscosity solutions is due to the seminal works [58, 59] published back in 1983,1984.

3 Two Honors of the PI [76]

1. Journal of Ill-Posed and Inverse Problems in issue 1 of volume 30, 2022, <https://doi.org/10.1515/2022-2079> has published an Editorial signed by twelve (12) members of the Editorial Board [76]. **Title:** Research biography of a distinguished expert in the field of inverse problems: Professor Michael Victor Klibanov.
2. The International Conference “Inverse Problems: Modeling and Simulation” to be held in Malta in May 2022 <https://www.ipms-conference.org/ipms2022/index.php>

will have a special minisimposium dedicated to “the 70th anniversary of a distinguished expert in the field of inverse problems Professor Michael V. Klibanov”, <https://www.ipms-conference.org/ipms2022/m9>.

4 Two Books of the PI Acknowledging the Support of This Grant [2, 3]

Two books of PI with the acknowledgment of the support of this grant [2, 3] were published in the time period less than one year. The book [2] is a textbook on Partial Differential Equations. On the other hand, the second book [3] summarizes the best results of the PI and his research team in the forty one (41) years long period after publishing the **GROUND BREAKING** paper [53] of Bukhgeim and Klibanov. In this paper the method of Carleman estimates was introduced in the field of Inverse Problems for the first time, see some related comments in sections 1 and 2.2.2. In particular, various versions of the convexification method for CIPs of the following above sections of this report are discussed in the book [3]:

1. Sub-subsection 2.3.3.
2. Sub-subsection 2.4.2.
3. Sub-subsection 2.5.
4. Subsection 2.6.
5. Subsection 2.7.
6. Subsection 2.8.
7. Section 9.
8. Section 11.
9. Section 13.
10. Section 14.
11. Section 15.
12. Section 16.

Figure 16 depicts the cover page of the book [3].

5 A Brief Outline of the Convexification Principle

Since all above posed CIPs were solved numerically by the convexification method, then it makes sense to briefly outline this method prior going in details of its various versions used to solve CIPs of sections 2.1-3 posed above and published in [3]-[36], [88, 89].

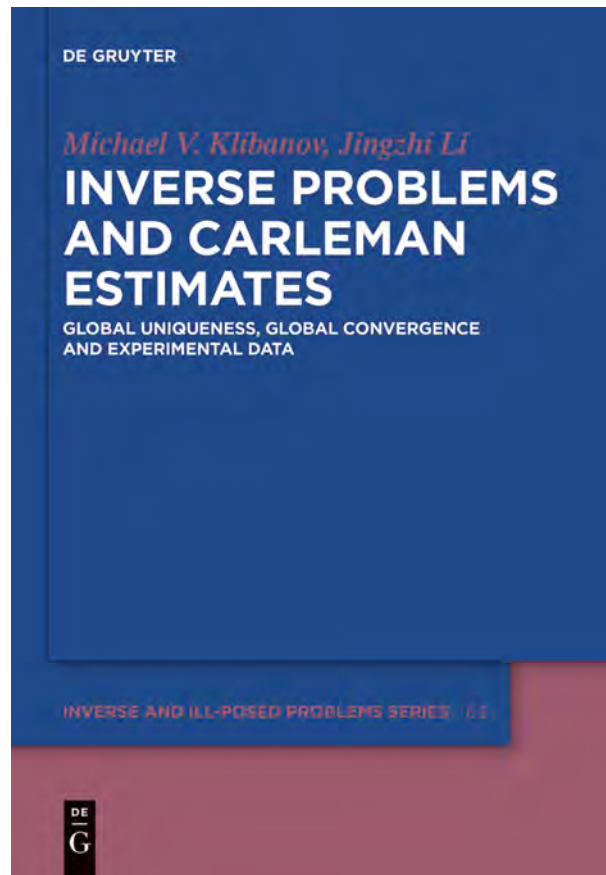


Figure 16: Cover page for the book [3]. The support of the US Army Research Laboratory and Office of Army Research grant W911NF-19-1-0044 is acknowledged in this book. Authors: M.V. Klibanov and J.Li. Title: *Inverse Problems and Carleman Estimates: Global Uniqueness, Global Convergence and Experimental Data*. Publisher: De Gruyter (Germany). Publication year 2021.

The convexification numerical principle, which was originally invented by the PI in [80, 81], is currently the only principle for CIPs, with the *global convergence* property. This was confirmed in many recent publications of the PI and his research team for both computationally simulated and experimentally collected data [3]-[36]. In the multidimensional case of CIPs of waves propagation, this was done only for the data in the frequency domain with the only exceptions in [13, 34]. However, in [13, 34] only transmitted data were considered, whereas we need backscattering data for the above mentioned problems of the standoff detection and identification of buried mine-like targets and targets hidden in inspected buildings. This case will be studied in the future. On the other hand, in the 1D case we have worked with the time dependent data during this reportage period [9, 8, 10, 32]. This is because we need this case for CONSAR.

The first step of the convexification is to obtain a nonlinear boundary value problem (BVP) for a coupled system of nonlinear PDEs. The key feature of this system is that the unknown coefficient is not present in it. To solve this BVP, we construct a weighted least squares cost functional J_λ , where $\lambda \geq 1$ is a parameter. The weight present in J_λ is the *true key* to the convexification. This is the Carleman Weight Function (CWF). The CWF is involved as the weight function in the quite elegant and sophisticated Carleman estimate

for the corresponding PDE operator, see, e.g. [3, 45, 72, 82, 83, 94] for Carleman estimates. Let H^k be an appropriate Hilbert space associated with that BVP and $G(d) \subset H^k$ be an appropriate convex bounded set in H^k of the diameter d , where $d > 0$ is an arbitrary but fixed number. The key theorem of any version of the convexification claims that the functional J_λ is strictly convex on the set $G(d)$ for sufficiently large values of $\lambda \geq 1$. It is proven next that the unique minimizer of J_λ exists on the set $\overline{G(d)}$. The distance between that minimizer and the true solution of the CIP is estimated depending on the level of noise $\sigma > 0$ in the input data. Furthermore, it is proven that the gradient projection method of the minimization of J_λ converges to the true solution of the original CIP as long as σ tends to zero. The starting point of the gradient projection method is an arbitrary point of the set $G(d)$. Furthermore, quite often the global convergence of the simpler to implement gradient descent method can also be prove, see section 6.

Remarks 5.1

1. *Since the starting point of the gradient projection method is an arbitrary point of the set $G(d)$ and since the diameter $d > 0$ of the set $G(d)$ is an arbitrary number, then convexification method is a globally convergent one, see Definition 1.1 in section 1.*
2. *Even though the theory requires that the parameter $\lambda \geq 1$ of the functional J_λ should be sufficiently large, our numerical experience in all above cited publications on the convexification shows that it is sufficient to choose $\lambda \in [1, 3]$.*

6 A New Theorem of Convex Analysis, Which is Useful for the Convexification [5]

With the only exception of the most recent works [5, 8], in almost all other above cited works on the convexification of the PI and his research team, about the convexification the global convergence is proven for only the gradient projection method on the convex bounded set $\overline{G(d)} \subset H^k$ of the diameter $d > 0$. However, it is inconvenient to arrange the projection operator in practical computations. Hence, we have used the simpler gradient descent method in our computations in all those publications. However, it would be good to rigorously justify the use of the gradient descent method. This was done in [5, Theorem 6].

Complex optimization theory is usually concerned with the minimizations of functionals which are strictly convex on the whole Hilbert spaces. The global convergence of the gradient descent method for such a functional is a well known classical result, see, e.g. the book [51]. However, in the convexification we typically have a functional, which is strictly convex on a *bounded* convex set $\overline{G(d)}$. Thus, the global convergence of the gradient descent method on such a set was an open question until our recent works [5, 23]. This is unlike the gradient projection method, which, however, is harder to implement than just gradient descent method. Theorem 6.1 addresses this question positively. Even though the proof of this theorem is rather simple, still no analogs of this result were published before [5].

Remarks 6.1:

1. *The true importance of Theorem 6.1 for the convexification is that it justifies the use of the gradient descent method in all our previous above cited publications about the convexification.*

2. For brevity, we formulate Theorem 6.1 in its simple form when one deals with a ball $G(d) \subset H$ with the center at $\{0\}$. However, it follows from Theorem 6 of [5] that Theorem 6.1 can be generalized exactly for cases which occur in the convexification method.

Let H be a Hilbert space with its norm $\|\cdot\|$. Let the $G(d) \subset H$ be a convex bounded set of the diameter d . Let $J : \overline{G(d)} \rightarrow \mathbb{R}$ be a functional defined on $\overline{G(d)}$. We assume that the Fréchet derivative $J'(x) \in H$ of the functional $J(x)$ exists for all points $x \in \overline{G(d)}$ and is Lipschitz continuous, i.e.

$$\|J'(y) - J'(x)\| \leq D \|y - x\|, \forall x, y \in \overline{G(d)}, \quad (6.1)$$

where $D > 0$ is a constant. We also assume that the functional $J(x)$ is strictly convex on $\overline{G(d)}$,

$$J(y) - J(x) - (J'(x), y - x) \geq \varkappa \|y - x\|^2, \forall x, y \in \overline{G(d)}, \quad (6.2)$$

where $\varkappa = \text{const.} > 0$, where $\varkappa/2$ is the convexity constant.

It is well known that there exists unique minimizer $x_{\min} \in \overline{G(d)}$ of the functional $J(x)$ on the set $\overline{G(d)}$ [43, 96]. Let $A \geq 6$ be a number. We assume that

$$x_{\min} \in G(d/A) \subset G(d). \quad (6.3)$$

Theorem 6.1. Assume that the above conditions (6.1)–(6.3) hold. Let $x_0 \in G(d/A)$ be an arbitrary point. Consider the gradient descent method, which starts iterations at the point x_0 ,

$$\begin{aligned} x_0 &= x_0 \in G(d/A), \\ x_n &= x_{n-1} - \omega J'(x_{n-1}), n = 1, 2, \dots \end{aligned} \quad (6.4)$$

Consider the family of convex sets $G(\beta)$ of the diameter $\beta > 0$ and assume that $G(\beta_1) \subset G(\beta_2)$ if $\beta_1 < \beta_2$. Assume that for every $x \in G(d)$ there exists a number $\beta(x) > 0$ such that

$$x - \omega J'(x) \in G(\beta(x)).$$

Then there exists a number $\omega_0 \in (0, 1)$ such that for every $\gamma \in (0, \gamma_0)$ all points $\{x_n\}_{n=0}^{\infty} \in G(d)$. Furthermore, there exists a number $\theta = \theta(\omega) \in (0, 1)$ such that

$$\|x_n - x_{\min}\| \leq \theta^n \|x_0 - x_{\min}\|, n = 1, 2, \dots$$

7 CONSAR: Convexification Based Nonlinear SAR Imaging. A Breakthrough Joint Invention of ARL and the PI [4, 5]

We have briefly mentioned CONSAR in sub-subsection 2.1.1 and subsection 2.3. We have pointed out that CONSAR is a surprising joint invention of Drs. Anders Sullivan and Lam Nguyen, engineers of ARL and the PI. We have demonstrated in Figure 2 that CONSAR can image well dielectric constants of targets inside of an inspected building using SAR experimental data collected by ARL. In this section we present details about CONSAR and also present result of its performance for the experimental data collected in the microwave laboratory located in the campus of University of North Carolina at Charlotte and led by the PI.

7.1 Setting of SAR data

Below $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$.

We consider the simplest case of SAR data collection, the so-called “stripmap” scenario. Consider first the most popular case of 2-D SAR images. Then the source (antenna) runs along an interval of a straight line, which we denote L_{SAR} . The detector runs along L_{SAR} concurrently with the source. Consider the straight line passing through the point $\mathbf{x}_0$, orthogonal to the line of sources L_{src} and forming the angle $\theta \in (0, \pi/2)$ with the plane $\{z = z_0\}$. The half $CRL(\mathbf{x}_0)$ of this straight line with the end point $\mathbf{x}_0$, which looks towards the domain of our interest Ω , is called “the central line” of the antenna. The angle θ remains the same for all positions of the source, is called the “elevation angle” and it characterizes the direction of the incident wave field. Thus, central lines of all antennas are parallel to each other. Consider the plane P which is passing through all these central lines as well as through the line of sources. P is called *slant range* or *imaging plane*. We remark that in the context of SAR imaging, the domain of interest Ω should be practically located far from the line of source L_{SAR} .

A schematic diagram of collection of the SAR data is presented on Figure 18a). Figure 18b) explains our **BREAKTHROUGH IDEA OF CONSAR** [4, 5]: to solve many 1D CIPs (2.3)-(2.7) along the lines formed by the central lines of antennas $CRL(\mathbf{x}_0)$. Then the 2-D image of the function $\tilde{c}(\mathbf{x})$ is formed in the slant range via combining those 1D images. We call this “slant range” image. Those CIPs are solved by the version of [10] of the convexification method. Here $\tilde{c}(\mathbf{x})$ is related to the dielectric constant $c(\mathbf{x}) = c(\text{target})$ of the target. Our numerical studies of [4, 5] demonstrate that the numbers

$$\max(\tilde{c}(\mathbf{x})) = c(\text{target}) = \text{const. within the target} \quad (7.1)$$

are computed with a good accuracy. In other words, our slant range images provide accurate values of dielectric constants of targets. In addition, these numerical studies show that the sizes of slant range images of targets are about the same as the true sizes of targets.

7.2 The forward problem for the SAR data

This forward problem is:

$$c(\mathbf{x}) u_{tt} = \Delta_{\mathbf{x}} u + W(t, \mathbf{x}, \mathbf{x}_0), \quad \mathbf{x} \in \mathbb{R}^3, t \in (0, T), \quad (7.2)$$

$$u(\mathbf{x}, \mathbf{x}_0, 0) = u_t(\mathbf{x}, \mathbf{x}_0, 0) = 0, \quad \mathbf{x} \in \mathbb{R}^3, \quad (7.3)$$

where $\mathbf{x}_0$ is the center of the antenna, $u(\mathbf{x}, \mathbf{x}_0, t)$ is the amplitude of the time resolved component of the electric field, and the source function W is defined as:

$$W(t, \theta, \mathbf{x} - \mathbf{x}_0) = A(t) e^{-i\omega_0 t} \widetilde{W}(\theta, \mathbf{x} - \mathbf{x}_0), \quad A(t) = \chi_\tau(t) e^{-i\alpha(t-\tau/2)^2}, \quad (7.4)$$

where ω_0 is the carrier frequency, α is the chirp rate, and τ is its duration. In SAR imaging, the expression (7.4) is called “linear modulated pulse” or “chirp” with

$$\chi_\tau(t) = \begin{cases} 1 & \text{for } t \in (0, \tau), \\ 0 & \text{for } t < 0. \end{cases} \quad (7.5)$$

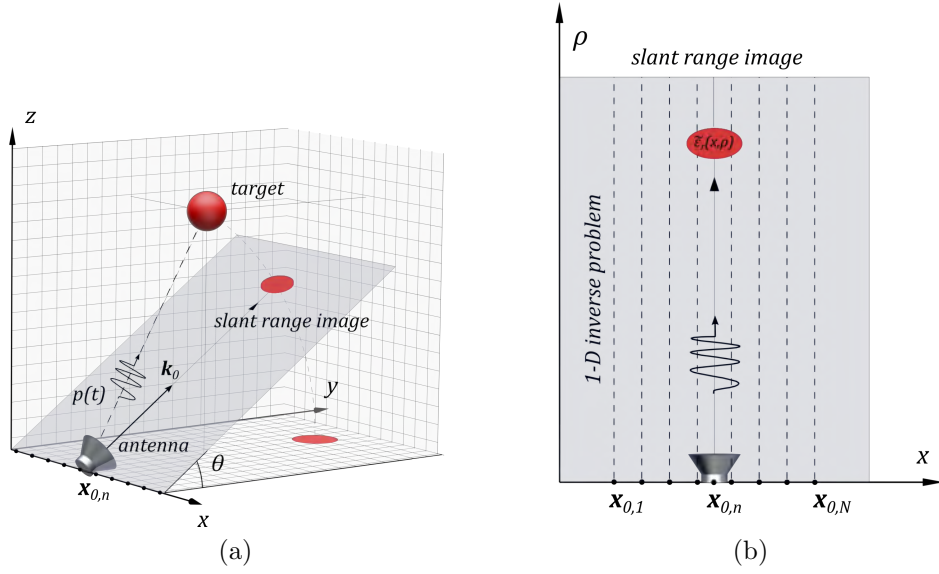


Figure 17: a) Schematic diagram of collection of the 2-D SAR data. The slant range plane, where the resulting image of a certain trace of the target is displayed, is also depicted. b) The schematic diagram of our breakthrough idea of CONSAR of forming images from SAR data on the slant range plane. First, 1-D CIP (2.3)-(2.7) of sub-subsection 2.3.2 is solved for each location $x_{0,n}$ of the source, where $n = 1, \dots, N$. This CIP is solved by the version of [10] of the convexification principle. Next, the image of the function $\tilde{c}(\mathbf{x})$, which is related to the dielectric constant $c(x)$ of the target, is obtained via combining those 1-D images, see (7.1).

The function $\widetilde{W}$ represents the antenna as follows: The antenna is a disk of the diameter $D > 0$ in located in the plane of the antenna, which we denote P_{ant} . Denoted by $S(\mathbf{x}_0, \theta, D)$, this disk is centered at $\mathbf{x}_0$. Now, $\mathbf{x}_0$ lies in the line L_{scr} that intersects with the plane P_{ant} . On the other hand, P_{ant} is orthogonal to the central line $CRL(\mathbf{x}_0)$. Next, we change coordinates via a rotation $\mathbf{x} \rightarrow \mathbf{x}' = (x', y', z')$ such that the central line $CRL(\mathbf{x}_0)$ of the antenna is parallel to the z' -axis. Then $\mathbf{x}_0 = (x_0, y'_0, z'_0) \in L_{\text{SAR}}$.

Given a number $\eta \in (0, D/2)$, we introduce a smooth function $m(\eta, \theta, \mathbf{x}, \mathbf{x}_0, D)$:

$$m(\eta, \theta, \mathbf{x}, \mathbf{x}_0, D) = \begin{cases} 1 & \text{for } |\mathbf{x} - \mathbf{x}_0| < D/2 - \eta, \\ 0 & \text{for } |\mathbf{x} - \mathbf{x}_0| \geq D/2, \\ \in [0, 1] & \text{for } D/2 - \eta \leq |\mathbf{x} - \mathbf{x}_0| < D/2. \end{cases}$$

In this regard, $m(\eta, \theta, \mathbf{x}, \mathbf{x}_0, D) = 0$ outside of the ball centered at $\mathbf{x}_0$ with the radius $D/2$. Henceforth, we define the function $\widetilde{W}$ in (7.4) as

$$\widetilde{W}(\theta, \mathbf{x} - \mathbf{x}_0) = \delta(z' - z'_0) m(\eta, \theta, x - x_0, y' - y'_0, D). \quad (7.6)$$

By this way, due to the presence of m in (7.6), the source function W vanishes outside of our disk antenna $S(\mathbf{x}_0, \theta, D)$.

Inverse Problem for the 2D SAR data. Suppose that conditions (7.2)-(??) are

satisfied. Given the function $\varphi(\mathbf{x}_0, t)$,

$$u(\mathbf{x}_0, \mathbf{x}_0, t) = F(\mathbf{x}_0, t), \quad \mathbf{x}_0 \in L_{\text{SAR}}, t \in (0, T), \quad (7.7)$$

find the function $\tilde{c}(\mathbf{x})$ for points x located in the slant range plane.

We consider in (7.7) only the real part $\text{Re}(u(\mathbf{x}_0, \mathbf{x}_0, t))$ of the function $u(\mathbf{x}_0, \mathbf{x}_0, t)$. The data $F(\mathbf{x}_0, t)$ in (7.7) depends on two variables. Indeed, since the source $\mathbf{x}_0$ runs along the interval L_{SAR} , then $\mathbf{x}_0$ represents only one variable. The time t is the second variable. Thus, we have in (7.7) two free variables in the data. This means, in turn that we can obtain only a 2D image from the SAR data (7.7).

Important Remarks 7.1:

1. *This statement of the inverse problem (7.7) is not rigorous. This is because it is unclear how to define the function $\tilde{c}(\mathbf{x})$. In other words, it is unclear what is the relevance of $\tilde{c}(\mathbf{x})$ to the true dielectric constant of the target $c(\mathbf{x}) = c(\text{target})$ as in (7.1). However, it is CONSAR which finds this relevance*
2. *The distance between the source and detector is much less than the distance between L_{SAR} and targets of interest. Therefore, when working with the backscattering SAR data to solve the corresponding CIP, it is reasonable to assume that the source and detector coincide, form one point, and this point runs along L_{SAR} .*
3. *Furthermore, when working with those backscattering SAR data, it is reasonable to assume that the source function $W(t, \mathbf{x}, \mathbf{x}_0)$ in (7.2) is replaced with $f(\mathbf{x}, \mathbf{x}_0, t) = \delta(\mathbf{x} - \mathbf{x}_0) \delta(t)$, which means that, just for the sake of solution of that CIP by the convexification method, it is reasonable to replace forward problem (7.2)-(7.6) with:*

$$c(\mathbf{x}) u_{tt} = \Delta_{\mathbf{x}} u, \quad \mathbf{x} \in \mathbb{R}^3, t \in (0, T), \quad (7.8)$$

$$u(\mathbf{x}, \mathbf{x}_0, 0) = 0, u_t(\mathbf{x}, \mathbf{x}_0, 0) = \delta(\mathbf{x} - \mathbf{x}_0), \quad \mathbf{x} \in \mathbb{R}^3. \quad (7.9)$$

4. *On the other hand, when generating the data for the forward problem, it is still reasonable to consider the mathematical model (7.2)-(7.6).*
5. *Statements of items 1-4 are confirmed in our numerical studies of [4, 5].*

7.3 Specifics of CONSAR

Let $\mathbf{x}_{0,n} \in L_{\text{SAR}}$ be the source location number $n = \overline{1, N}$. Let $g_{0,n}(t)$ be the real part of the preprocessed data of $F(\mathbf{x}_0, t)$ at $\mathbf{x}_{0,n}$ for $n = \overline{1, N}$, see (7.7). Consider a “pseudo” variable $x \in \mathbb{R}$ and let the function $c(x) \in C^3(\mathbb{R})$. We scale to $x \in (0, 1)$ the length of the interval of our interest, i.e. the length of the interval in the slant range of the central line $CRL(\mathbf{x}_{0,n})$ along which we want to calculate our function $\tilde{c}(\mathbf{x})$. Our idea below is to calculate the slant-range function $\tilde{c}(\mathbf{x})$, $\mathbf{x} \in (\text{slant range plane})$ mentioned in subsection 7.2 via finding N functions $c(x) := c_n(x)$ along each central line $CRL(\mathbf{x}_{0,n})$ and then forming the 2D image in the slant range plane out of these N one-dimensional functions $c_n(x)$, see Figure 18b). This idea was first proposed in [4].

We now specify the 1D CIP (2.3)-(2.7) we work with. Given a number $\bar{c} > 1$, we assume that the function $c_n(x) \in C^3(\mathbb{R})$ and

$$c_n(x) = \begin{cases} \in [1, \bar{c}] & \text{for } x \in (0, 1), \\ 1 & \text{for } x \leq 0 \text{ and } x \geq 1. \end{cases} \quad (7.10)$$

Let $T > 0$. In this scenario, our forward problem is formed as the following initial value (or Cauchy) problem:

$$\begin{cases} c_n(x) u_{tt}^{(n)} = u_{xx}^{(n)} & \text{for } x \in \mathbb{R}, t \in (0, T), \\ u^{(n)}(x, 0) = 0, \quad u_t^{(n)}(x, 0) = \delta(x) & \text{for } x \in \mathbb{R}. \end{cases} \quad (7.11)$$

Coefficient Inverse Problem 7.1 (CIP7.1). *Determine the coefficient $c_n(x)$ for $x \in (0, 1)$ in the PDE of (7.11), provided that the following $g_{0,n}(t)$:*

$$u^{(n)}(0, t) = g_{0,n}(t), \quad t \in (0, T). \quad (7.12)$$

We can prove that

$$u_x^{(n)}(0, t) = g'_{0,n}(t).$$

Hence, CIP 7.1 is exactly the 1D CIP (2.3)-(2.7) of sub-subsection 2.3.2.

Therefore, we find the function $c_n(x)$ via solution of CIP7.1 using the version of [10] of the convexification method. This is how do we form CONSAR images in sub-subsection 2.3.1 and subsection 7.4.

7.4 Performance of CONSAR for SAR experimental data collected in the microwave laboratory located in UNCC campus and led by the PI [5]

We do not possess at the moment a well working device to collect time resolved data. Hence, we have collected frequency resolved data for the frequency interval [1,10] GHz. This was done by Rohde&Schwarz Vector Network Analyzer, which works well in our library. Then we have applied the inverse Fourier transform to these data to obtain the SAR data in the time domain.

Our target was a glass bottle filled water, which is the same as the one on Figure 10a) in sub-subsection 2.3.3. This target was buried in the dry sand which has filled a sandbox. The burial depth was 10 centimeters. Getting a land mine with the 10 cm depth is usually considered in many models of de-mining operations, see, e.g. [63].

Figure 19 displays a photograph of our experimental arrangement. It can be seen on this figure that we have many unwanted obstacles and furniture that can affect the quality of the backscattering data which we measure. Therefore, the fact that we still obtain an accurate image indicate a good stability of CONSAR.

Mounted in front of the sandbox is a standard horn antenna with 20 cm in length, and our detector is a point probe placed behind that transmitter. Besides, the distance between the antenna and the receiver is about 17 cm. In order to obtain the elevation angle, the sandbox is placed and moved along the line, which forms the $\pi/4$ angle with the plane behind the antenna. We have used the concurrent movement along that straight line of both the antenna and detector, see Figure 18a).

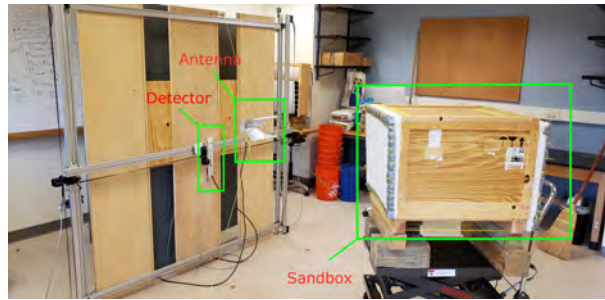


Figure 18: A photograph of our experimental configuration with a fake land mine buried in a sandbox. That sandbox was moved along a straight line, which has formed the $\pi/4$ elevation angle with the wall which can be seen in the photograph. The total length of this movement was 140 cm with the 5 cm step size. We have used the concurrent movement along a straight line of both the antenna and detector, see Figure 18a). This models well the SAR data collection of Figure 18a).

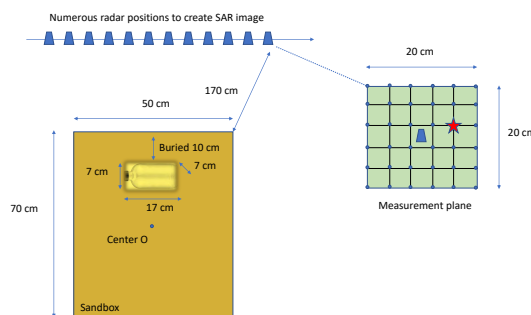


Figure 19: Basic geometry of our collection of experimental data SAR data.

Thus, the above described movement of the sandbox is equivalent with the movement of the antenna/detector pair for the case when the sandbox remains in the same position. The elevation angle is now $\theta = \pi/4$.

First, we have preprocessed experimental data as in [5]. Next, we have applied to the preprocessed experimental SAR data the version of [10] of the convexification method for 1D CIP (2.3)-(2.7) of sub-subsection 2.3.2, which is equivalent to CIP7.1 of subsection 7.3. In other words, we have applied CONSAR.

Figure 19 displays a photograph of our microwave experimental device. Figure 20 displays the basic geometry of our measurements. Figure 21b) displays the computed slant range SAR image, see caption for Figure 21. It is **remarkable** that the dielectric constant of the tested target is computed with a **very high accuracy**.

8 Carleman Contraction Mapping for the Second Generation of the Convexification Method for 1D CIP (2.3)-(2.7) [9]

As it was stated in sub-subsection 2.3.2, several versions of the convexification method for 1D CIP (2.3)-(2.7) were developed in the works the PI with coauthors during the

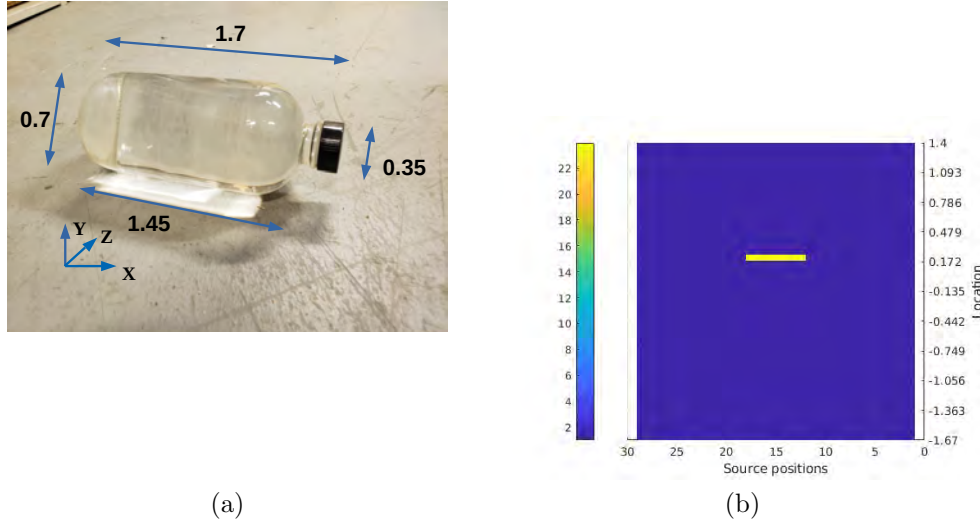


Figure 20: a) A photograph of the glass bottle filled with water, which was buried in the sandbox of Figure 18 with the burial depth of 10 cm. Sizes are given in centimeters. b) The computed slant range SAR image of a), where the location is presented in the dimensionless regime. The computed dielectric constant is 23.9, and it was established in [115] that the true dielectric constant of this object is 23.8. This is a **remarkably high accuracy**. This target mimics well the so-called “Glassmine 43” [60], a non-metallic anti-personnel land mine largely with a glass body that the Germans used to make detection harder in the World War II era.

reportage period [8, 9, 32, 10]. Papers [8, 9, 10] have worked with the experimental data collected by the Forward Looking Radar of ARL [100]. It turns out that the most recent paper [9] has the most advanced version of the convexification, since it works in real time for those experimental data, see results in Table 2 in sub-subsection 2.3.2. Thus, in this section, we outline the method of [9].

8.1 Forward and inverse problems

For the convenience of the reader, we first repeat here the statement of the CIP (2.3)-(2.7). Let $b > 1$ be a known number, $x \in \mathbb{R}$ be the spatial variable and the function $c(x) \in C^3(\mathbb{R})$, represents the spatially distributed dielectric constant. We assume that

$$c(x) \in [1, b], x \in \mathbb{R}, \quad (8.1)$$

$$c(x) = 1 \text{ if } x \in (-\infty, 0] \cup [1, \infty). \quad (8.2)$$

Let T be a positive number. We consider the following Cauchy problem for a 1D hyperbolic PDE with a variable coefficient in the principal part of the operator:

$$c(x)u_{tt} = u_{xx}, \quad x \in \mathbb{R}, t \in (0, T), \quad (8.3)$$

$$u(x, 0) = 0, u_t(x, 0) = \delta(x). \quad (8.4)$$

The problem of finding the function $u(x, t)$ from conditions (8.3), (8.4) is our forward problem.

1D Coefficient Inverse Problem (1D CIP). *Suppose that the following two functions $g_0(t)$, $g_1(t)$ are known:*

$$u(0, t) = g_0(t), \quad u_x(0, t) = g_1(t), \quad t \in (0, T). \quad (8.5)$$

Determine the function $c(x)$ for $x \in (0, 1)$, assuming that the number $b > 1$ in (14.13) is known.

First, we formulate an important property of the function $u(x, t)$: absorbing boundary conditions [10]. We note that in the multidimensional case the absorbing boundary conditions are imposed in the classic work [65]. Unlike this, the absorbing boundary condition for the 1D problem (8.3), (8.4) was proven.

Lemma 8.1 (absorbing boundary conditions [10]). *Let $b > 1$ be the number in (14.13). Let $x_1 \geq b$ and $x_2 < 0$ be two arbitrary numbers. Then the solution $u(x, t)$ of forward problem (8.3), (8.4) satisfies the absorbing boundary conditions at $x = x_1$ and $x = x_2$, i.e.*

$$\begin{aligned} u_x(x_1, t) + u_t(x_1, t) &= 0, \quad t \in (0, T), \\ u_x(x_2, t) - u_t(x_2, t) &= 0, \quad t \in (0, T). \end{aligned}$$

Remark 8.1. *Note that only the function $g_0(t)$ can be measured. As to the function $g_1(t)$, it can be proven that*

$$g_1(t) = g'_0(t). \quad (8.6)$$

We numerically differentiate the noisy function $g_0(t)$ using the Tikhonov regularization method [116]. Since this method is well known, we do not describe it here.

8.2 A boundary value problem for a nonlinear PDE with non-local terms

Let $\tau(x)$ be the travel time needed for the wave to travel from the point source at $\{0\}$ to the point x ,

$$\tau(x) = \int_0^x \sqrt{c(s)} ds \quad (8.7)$$

By (8.7) the following 1D analog of the eikonal equation is valid:

$$\tau'(x) = \sqrt{c(x)}. \quad (8.8)$$

We now introduce a change of variable

$$q(x, t) = u(x, t + \tau(x)). \quad (8.9)$$

It can be proven that

$$\tau''(x) = -\frac{q_x(x, 0)}{2q^3(x, 0)}, \quad (8.10)$$

$$q(x, 0) = \frac{1}{2c^{1/4}(x)} \neq 0. \quad (8.11)$$

Substituting (8.8)-(8.11) in (8.3), we obtain

$$L(q) = q_{xx} - q_{xt} \frac{1}{2q^2(x, 0)} + q_t \frac{q_x(x, 0)}{2q^3(x, 0)} = 0. \quad (8.12)$$

Equation, (8.12) is a nonlinear PDE with respect to the function $q(x, t)$ with nonlocal terms $q_x(x, 0)$ and $q(x, 0)$. The boundary conditions for the function q at $\{x = 0\}$ can be derived from By (8.5) and (8.6) as

$$q(0, t) = g_0(t), q_x(0, t) = 2g'_0(t), \quad t \in (0, T). \quad (8.13)$$

It can also be derived from Lemma 8.1 that

$$q_x(b, t) = 0. \quad (8.14)$$

Define the rectangle Ω ,

$$\Omega = \{(x, t) : x \in (0, b), t \in (0, T)\}. \quad (8.15)$$

It follows from (14.13) and (8.11) that

$$\frac{1}{2b^{1/4}} \leq q(x, 0) \leq \frac{1}{2}, \quad x \in [0, b]. \quad (8.16)$$

In addition, we need $q \in C^2(\overline{\Omega})$ and we also need to bound the norm $\|q\|_{C^2(\overline{\Omega})}$ from the above. Let $R > 0$ be an arbitrary number. We define the set $B(R, g_0)$ as

$$B(R, g_0) = \left\{ \begin{array}{l} q \in H^4(\Omega) : \|q\|_{H^4(\Omega)} < R/2, \\ q(0, t) = g_0(t), q_x(0, t) = 2g'_0(t), \\ q_x(b, t) = 0, \\ \frac{1}{2b^{1/4}} \leq q(x, 0) \leq 1/2, x \in [0, b]. \end{array} \right. \quad (8.17)$$

We assume below that

$$B(R, g_0) \neq \emptyset. \quad (8.18)$$

By embedding theorem $\overline{B(R, g_0)} \subset C^2(\overline{\Omega})$ and

$$\|q\|_{C^2(\overline{\Omega})} \leq KR, \quad \forall q \in \overline{B(R, g_0)}. \quad (8.19)$$

where the constant $K = K(\Omega) > 0$ depends only on the domain Ω . Using (8.16), we define the function $q^0(x, 0)$ as:

$$q^0(x, 0) = \begin{cases} q(x, 0) & \text{if } q(x, 0) \in [1/(2b^{1/4}), 1/2], \\ 1/(2b^{1/4}), & \text{if } q(x, 0) < 1/(2b^{1/4}), \\ 1/2 & \text{if } q(x, 0) > 1/2. \end{cases} \quad \forall q \in \overline{B(R, g_0)}, \quad (8.20)$$

$$\forall x \in [0, b].$$

Then the function $q^0(x, 0)$ is piecewise continuously differentiable in $[0, b]$ and by (8.19) and (8.20)

$$\|q^0(x, 0)\|_{C[0, b]}, \max_{[0, b]} |q^0_x(x, 0)| \leq KR, \quad \forall q \in B(R, g_0), \quad (8.21)$$

$$\frac{1}{2b^{1/4}} \leq q^0(x, 0) \leq \frac{1}{2}, \quad x \in [0, b]. \quad (8.22)$$

It follows from (8.12)-(8.16) and (8.20) that we got the following Boundary Value Problem (BVP) for a nonlinear PDE with non-local terms:

$$q_{xx}(x, t) - q_{xt}(x, t) \frac{1}{2(q^0(x, 0))^2} + q_t(x, t) \frac{q_x(x, 0)}{2(q^0(x, 0))^3} = 0 \text{ in } \Omega, \quad (8.23)$$

$$q(0, t) = g_0(t), \quad q_x(0, t) = 2g'_0(t), \quad q_x(b, t) = 0. \quad (8.24)$$

Thus, we focus below on the numerical solution of the following problem:

Boundary Value Problem 8.1. *Find a function $q \in B(R, g_0)$ satisfying conditions (8.23), (8.24), where the function $q^0(x, 0)$ is defined in (8.20).*

Suppose that we have solved Problem 8.1. Then, using (8.11) and (8.20), we set

$$c(x) = \frac{1}{(2q^0(x, 0))^4}. \quad (8.25)$$

8.3 Numerical method for Boundary Value Problem 8.1

8.3.1 The function $q_0(x, t)$

We now find the first approximation $q_0(x, t)$ for the function $q(x, t)$. Using (8.2), we choose $c(x) \equiv 1$ as the first guess for the function $c(x)$. Hence, by (8.11),

$$q_0(x, 0) \equiv \frac{1}{2}. \quad (8.26)$$

We now need to find the function $q_0(x, t)$. To do this, drop the nonlinear third term in the left hand side of equation (8.23) and, using (8.20) and (8.26), set $1/(2q^0(x, 0))^2 := 2$. Then (8.23) and (8.24) become:

$$q_{0xx}(x, t) - 2q_{0xt}(x, t) = 0 \text{ in } \Omega, \quad (8.27)$$

$$q_0(0, t) = g_0(t), \quad q_{0x}(0, t) = 2g'_0(t), \quad q_{0x}(b, t) = 0. \quad (8.28)$$

BVP (8.27), (8.28) has overdetermined boundary conditions (8.28). Typically, QRM works well for BVPs for PDEs with overdetermined boundary conditions [3, 84]. Therefore, we solve BVP (8.27), (8.28) via the Carleman Quasi-Reversibility Method (CQRM). This means that we consider the following minimization problem:

Minimization Problem Number 0. *Assuming (8.18), minimize the functional $J_{\lambda, \beta}^{(0)} : \overline{B(R, g_0)} \rightarrow \mathbb{R}$*

$$J_{\lambda, \beta}^{(0)}(q_0) = \int_{\Omega} (q_{0xx} - 2q_{0xt})^2 e^{2\lambda\varphi} dxdt + \beta \|q_0\|_{H^4(\Omega)}^2, \quad (8.29)$$

where $e^{2\lambda\varphi}$ is the Carleman Weight Function for the operator $\partial_x - 2\partial_t$ [32]

$$e^{2\lambda\varphi} = e^{-2\lambda(x+\alpha t)}, \quad (8.30)$$

where $\alpha \in (0, 1/2)$ is the parameter, and $\beta \in (0, 1)$ is the regularization parameter. Both parameters will be chosen later.

Theorem 8.2 guarantees that, for appropriate values of parameters λ, β , there exists unique minimizer $q_{0, \min} \in \overline{B(R, g_0)}$ of the functional $J_{\lambda, \beta}^{(0)}(q_0)$.

8.3.2 The function $q_n(x, t)$ for $n \geq 1$

Assume that functionals $J_{\lambda, \beta}^{(m)}(q_m) : \overline{B(R, g_0)} \rightarrow \mathbb{R}$ are defined for $m = 0, \dots, n-1$, and their minimizers functions $\{q_{m, \min}\}_{m=0}^{n-1} \subset \overline{B(R, g_0)}$ are constructed already, all for the same values of parameters λ, α, β . Replace in (8.23) $q(x, t)$ with $q_n(x, t)$, $q^0(x, 0)$ with $q_{n-1, \min}^0(x, 0)$, $q_x(x, t)$ with $\partial_x q_{(n-1), \min}(x, t)$ and $q_t(x, t)$ with $\partial_t q_{(n-1), \min}(x, t)$. Then problem (8.23), (8.24) becomes a linear one with respect to the function $q_n(x, t)$,

$$q_{nxx}(x, t) - \frac{q_{nxt}(x, t)}{2(q_{n-1, \min}^0(x, 0))^2} + \frac{\partial_t q_{(n-1), \min}(x, t) \partial_x q_{(n-1), \min}(x, 0)}{2(q_{n-1, \min}^0(x, 0))^3} = 0 \text{ in } \Omega, \quad (8.31)$$

$$q_n(0, t) = g_0(t), \quad q_{nx}(0, t) = 2g_0'(t), \quad q_{nx}(b, t) = 0. \quad (8.32)$$

To solve problem (8.31), (8.32), we consider the following minimization problem:

Minimization Problem Number n . Assuming (8.18), minimize the functional $J_{\lambda, \beta}^{(n)} : \overline{B(R, g_0)} \rightarrow \mathbb{R}$ on the set $B(R, g_0)$,

$$J_{\lambda, \beta}^{(n)}(q_n) = \int_{\Omega} \left(q_{nxx}(x, t) - \frac{q_{nxt}(x, t)}{2(q_{n-1, \min}^0(x, 0))^2} + \frac{\partial_t q_{(n-1), \min}(x, t) \partial_x q_{(n-1), \min}(x, 0)}{2(q_{n-1, \min}^0(x, 0))^3} \right)^2 e^{2\lambda\varphi} dx dt + \beta \|q_n\|_{H^4(\Omega)}^2. \quad (8.33)$$

Suppose that there exists unique minimizer $q_{n, \min} \in \overline{B(R, g_0)}$ of the functional $J_{\lambda, \beta}^{(n)}(q_n)$. Then, following (8.25), we set

$$c_n(x) = \frac{1}{(2q_{n, \min}^0(x, 0))^4}. \quad (8.34)$$

The rest of the analytical part of section 8 is devoted to the convergence analysis of the iterative numerical method presented in this subsection 8.3.

8.4 The Carleman estimate

In this section we formulate the Carleman estimate, which is the main tool of our construction. This estimate follows from Theorem 3.1 of [8]. Let $q(x, t) \in \overline{B(R, g_0)}$ be an arbitrary function and let the function $q^0(x, 0)$ be constructed from the function $q(x, t)$ as in (??). Consider the operator $L_0 : H^2(\Omega) \rightarrow L_2(\Omega)$,

$$L_0 u = u_{xx}(x, t) - u_{xt}(x, t) \frac{1}{2(q^0(x, 0))^2}, \quad \text{for all } (x, t) \in \Omega.$$

Theorem 8.1 (Carleman estimate [8]). *Let the function $\varphi(x, t) = -(x + \alpha t)$. There exists a number $\alpha_0 = \alpha_0(R, \Omega) > 0$ depending only on R, Ω such that for any $\alpha \in (0, \alpha_0)$ there exists a sufficiently large number $\lambda_0 = \lambda_0(R, \Omega, \alpha) > 1$ depending only on R, Ω, α such that for all $\lambda \geq \lambda_0$ and for all functions $v \in H^2(\Omega)$ the following Carleman estimate holds:*

$$\begin{aligned} & \int_{\Omega} (L_0 v)^2 e^{2\lambda\varphi} dx dt \\ & \geq C \int_{\Omega} (\lambda(v_x^2 + v_t^2) + \lambda^3 v^2) e^{2\lambda\varphi} dx dt + C \int_0^b (\lambda v_x^2(x, 0) + \lambda^3 v^2(x, 0)) e^{-2\lambda x} dx \end{aligned}$$

$$\begin{aligned}
& -C\lambda \int_0^T (v_x^2(0,t) + v_t^2(0,t) + \lambda^2 v^2(0,t)) e^{-2\lambda\alpha t} dt \\
& -C \int_0^b (\lambda v_x^2(x,T) + \lambda^3 v^2(x,T)) e^{-2\lambda(x+\alpha T)} dx.
\end{aligned} \tag{8.35}$$

Remark 8.2. Here and everywhere below in section 8 $C = C(R, \Omega, \alpha) > 0$ denotes different constants depending only on listed parameters.

8.5 Strict convexity of functional (8.33) on the set $B(R, g_0)$, existence and uniqueness of its minimizer

Functional (8.33) is quadratic. We establish in this subsection that it is strictly convex on the set $\overline{B(R, g_0)}$. In addition, we prove existence and uniqueness of its minimizer on this set. Although similar results were proven in many of the above cited publications on the convexification, there are some peculiarities here, which are important for our convergence analysis, see Remarks 8.3 and 8.4 below.

Introduce the subspace $H_0^4(\Omega) \subset H^4(\Omega)$ as:

$$H_0^4(\Omega) = \{v \in H^4(\Omega) : v(0, t) = v_x(0, t) = v_x(b, t) = 0\}. \tag{8.36}$$

Denote $[\cdot, \cdot]$ the scalar product in the space $H^4(\Omega)$. Also, denote

$$A_n(q)(x, t) = q_{xx}(x, t) - q_{xt}(x, t) \frac{1}{2(q_{n-1, \min}^0(x, 0))^2}, \tag{8.37}$$

$$B_n(x, t) = \partial_t q_{(n-1), \min}(x, t) \frac{\partial_x q_{(n-1), \min}(x, 0)}{2(q_{n-1, \min}^0(x, 0))^3}. \tag{8.38}$$

Theorem 8.2. Let $J_{\lambda, \beta}^{(n)}$ be the functional defined in (8.33). Then:

1. For any set of parameters λ, β and for any $q \in \overline{B(R, g_0)}$ this functional has the Fréchet derivative $\left(J_{\lambda, \beta}^{(n)}(q)\right)' \in H_0^4(\Omega)$. The formula for $\left(J_{\lambda, \beta}^{(n)}(q)\right)'$ is:

$$\begin{aligned}
& \left(J_{\lambda, \beta}^{(n)}(q)\right)'(h) \\
& = 2 \int_{\Omega} (A_n(q)(x, t) + B_n(x, t)) A_n(h)(x, t) e^{2\lambda\varphi} dx dt + 2\beta [q, h], \quad \forall h \in H_0^4(\Omega).
\end{aligned} \tag{8.39}$$

2. This derivative is Lipschitz continuous in $\overline{B(R, g_0)}$, i.e. there exists a constant $D > 0$ such that

$$\left\| \left(J_{\lambda, \beta}^{(n)}(q_2)\right)' - \left(J_{\lambda, \beta}^{(n)}(q_1)\right)' \right\|_{H^4(\Omega)} \leq D \|q_2 - q_1\|_{H^4(\Omega)}, \quad \text{for all } q_1, q_2 \in \overline{B(R, g_0)}.$$

3. Let $\alpha_0 = \alpha_0(R, \Omega) > 0$, $\alpha \in (0, \alpha_0)$ and $\lambda_0 = \lambda_0(R, \Omega, \alpha) \geq 1$ be the numbers of Theorem 5.1. Then there exists a sufficiently large constant

$$\lambda_1 = \lambda_1(R, \Omega, \alpha) \geq \lambda_0 > 1 \tag{8.40}$$

depending only on listed parameters such that for all $\lambda \geq \lambda_1$ and for all $\beta \in [2e^{-\lambda\alpha T}, 1)$ the functional $J_{\lambda,\beta}^{(n)}(q)$ is strictly convex on the set $\overline{B(R, g)}$. More precisely, let $q \in \overline{B(R, g)}$ be an arbitrary function and also let the function $q + h \in \overline{B(R, g)}$. Then the following inequality holds:

$$\begin{aligned} J_{\lambda,\beta}^{(n)}(q + h) - J_{\lambda,\beta}^{(n)}(q) - \left(J_{\lambda,\beta}^{(n)}(q) \right)'(h) &\geq C \int_{\Omega} \left[\lambda (h_x^2 + h_t^2) + \lambda^3 h^2 \right] e^{2\lambda\varphi} dx dt \\ &+ C \int_0^b (\lambda h_x^2(x, 0) + \lambda^3 h^2(x, 0)) e^{-2\lambda x} dx + \frac{\beta}{2} \|h\|_{H^4(\Omega)}^2, \quad \forall \lambda \geq \lambda_1. \end{aligned} \quad (8.41)$$

4. For any $\lambda \geq \lambda_1$ there exists unique minimizer

$$q_{n,\min} \in \overline{B(R, g_0)} \quad (8.42)$$

of the functional $J_{\lambda,\beta}^{(n)}(q)$ on the set $\overline{B(R, g_0)}$. Furthermore, the following inequality holds:

$$\left[\left(J_{\lambda,\beta}^{(n)}(q_{n,\min}) \right)', q - q_{n,\min} \right] \geq 0, \quad \forall q \in \overline{B(R, g_0)}. \quad (8.43)$$

Remark 8.3. The peculiarity here is that even though the expression in the right hand side of (8.39) is linear with respect to the function q , we cannot use Riesz theorem here to prove existence and uniqueness of the minimizer $q_{n,\min}$, at which $\left(J_{\lambda,\beta}^{(n)}(q_{n,\min}) \right)' = 0$. Rather, all what we can prove is (8.43). This is because we need to ensure that the function $q_{n,\min} \in \overline{B(R, g_0)}$.

Remark 8.4. Since the functional $J_{\lambda,\beta}^{(n)}(q)$ is quadratic, then its strict convexity on the whole space $H^4(\Omega)$ follows immediately from the presence of the regularization term $\beta \|q_n\|_{H^4(\Omega)}^2$ in it. However, in addition to the claim of its strict convexity, we actually need in our convergence analysis below those terms in the right hand side of the strict convexity estimate (8.41), which are different from the term $\beta \|h\|_{H^4(\Omega)}^2/2$. These terms are provided by Carleman estimate of Theorem 8.1. The condition $\beta \in [2e^{-\lambda\alpha T}, 1)$ of Theorem 8.2 is imposed to dominate the negative term in the last line of (8.35): see (8.41).

8.6 Gradient descent method to find the minimizer $q_{n,\min} \in \overline{B(R, g_0)}$ of the functional $J_{\lambda,\beta}^{(n)}(q)$

Since we search for the minimizer $q_{n,\min} \in \overline{B(R, g_0)}$ of functional (8.33) on the bounded set $\overline{B(R, g_0)}$ rather than on the whole space $H^4(\Omega)$, then we cannot just use Riesz theorem to find this minimizer, also see [43] and [?, Chapter 10, section 3] for the case of finding a minimizer of a strictly convex functional on a bounded set. Theorem 8.2 guarantees existence and uniqueness of this minimizer. We demonstrate in this subsection that we can use the gradient descent method and explore Theorem 6.1 of section 6.

Keeping in mind (8.18), choose an arbitrary function $q_{0,n} \in B(R, g_0)$. We arrange the gradient descent method of the minimization of functional (8.33) as follows:

$$q_{k,n} = q_{(k-1),n} - \eta \left(J_{\lambda,\beta}^{(n)}(q_{(k-1),n}) \right)', \quad k = 1, 2, \dots, \quad (8.44)$$

where $\eta \in (0, 1)$ is a small number, which is chosen later. It is important to note that since functions $J_{\lambda, \beta}^{(n)}(q_{(k-1), n}) \in H_0^4(\Omega)$, then boundary conditions (8.32) are kept the same for all functions $q_{k, n}(x, t)$. Also, using (8.20) and (8.34), we set

$$c_{k, n}(x) = \frac{1}{(2q_{k, n}^0(x, 0))^4}, \quad x \in [\varepsilon, b], \quad (8.45)$$

$$c_{n, \min}(x) = \frac{1}{(2q_{n, \min}^0(x, 0))^4}, \quad x \in [\varepsilon, b]. \quad (8.46)$$

Theorem 8.3 claims the global convergence of the gradient descent method (8.44) to the pair $(q_{n, \min}, c_{n, \min})$ in the case when $q_{\min, n} \in B(R/6, g_0)$. This theorem is based on an analog of Theorem 6.1 of section 6.

Theorem 8.3. *Let the number $\lambda_1 = \lambda_1(R, \Omega, \alpha) \geq \lambda_0 > 1$ be the one defined in (8.40). Let $\lambda \geq \lambda_1$. For this value of λ , let $q_{n, \min} \in \overline{B(R, g_0)}$ be the unique minimizer of the functional $J_{\lambda, \beta}^{(n)}(q_n)$ on the set $\overline{B(R, g_0)}$ with the regularization parameter $\beta \in [2e^{-\lambda\alpha T}, 1)$ (Theorem 8.2). Assume that the function $q_{n, \min} \in B(R/3, g_0)$. For each n , choose the starting point of the gradient descent method (8.44) as $q_{0, n} \in B(R/3, g_0)$. Then, there exists a number $\eta_0 \in (0, 1)$ such that for any $\eta \in (0, \eta_0)$ all functions $q_{k, n} \in B(R, g_0)$. Furthermore, there exists a number $\theta = \theta(\eta) \in (0, 1)$ such that the following convergence estimates are valid:*

$$\|q_{k, n} - q_{n, \min}\|_{H^4(\Omega)} \leq \theta^k \|q_{0, n} - q_{n, \min}\|_{H^4(\Omega)}, \quad k = 1, \dots, \quad (8.47)$$

$$\|c_{k, n} - c_{n, \min}\|_{L_2(0, b)} \leq C\theta^k \|q_{0, n} - q_{n, \min}\|_{H^4(\Omega)}, \quad k = 1, \dots \quad (8.48)$$

8.7 Contraction mapping and global convergence

In this subsection, we formulate the global convergence theorem of the numerical method of subsection 8.3 for solving Boundary Value Problem 8.1 of subsection 8.2. To do this, we introduce first the exact solution of our CIP. Recall that the concept of the existence of the exact solution is one of the main concepts of the theory of ill-posed problems [116]. In particular, an estimate in our global convergence theorem is very similar to the one in the contraction mapping.

Suppose that there exists a function $c^*(x)$ satisfying conditions (14.13), (8.2). Let $u^*(x, t)$ be the solution of problem (8.3), (8.4) with $c := c^*$. We assume that the corresponding data $g_0^*(t), \partial_t g_0^*(t)$ for the CIP are noiseless, see (8.5), (8.6). Let $q^*(x, t)$ be the function $q^*(x, t)$, which is constructed from the function $u^*(x, t)$ as in (8.9). Following (8.17), we assume that

$$q^* \in B(R, g_0^*), \quad (8.49)$$

$$B(R, g_0^*) = \left\{ \begin{array}{l} q \in H^4(\Omega) : \|q\|_{H^4(\Omega)} < R/2, \\ q(0, t) = g_0^*(t), q_x(0, t) = 2(g_0^*)'(t), \\ q_x(b, t) = 0, \\ \frac{1}{2b^{1/4}} \leq q(x, 0) \leq 1/2, x \in [0, b] \end{array} \right\}. \quad (8.50)$$

By (8.12) and (8.13)

$$q_{xx}^*(x, t) - q_{xt}^*(x, t) \frac{1}{2(q^*(x, 0))^2} + q_t^*(x, t) \frac{q_x^*(x, 0)}{2(q^*(x, 0))^3} = 0 \text{ in } \Omega, \quad (8.51)$$

$$q^*(0, t) = g_0^*(t), \quad q_x^*(0, t) = 2\partial_t g_0^*(t), \quad q_x^*(b, t) = 0. \quad (8.52)$$

By (8.16)

$$c^*(x) = \frac{1}{(2q^*(x, 0))^4}. \quad (8.53)$$

It is important in the formulation of Theorem 8.2 that both functions q and $q + h$ should have the same boundary conditions as prescribed in $B(R, g_0)$. However, boundary conditions for functions q_n and q^* are different. Hence, we consider functions $F^*, F \in H^4(\Omega)$ such that (see the second line of (8.50))

$$F^*(0, t) = g_0^*(t), \quad F_x^*(0, t) = 2\partial_t g_0^*(t), \quad F_x^*(b, t) = 0, \quad (8.54)$$

$$F(0, t) = g_0(t), \quad F_x(0, t) = 2\partial_t g_0(t), \quad F_x(b, t) = 0. \quad (8.55)$$

We assume that

$$\|F^*\|_{H^4(\Omega)}, \|F\|_{H^4(\Omega)} < R. \quad (8.56)$$

Also, we introduce the function $p^*(x, t)$ as:

$$p^*(x, t) = q^*(x, t) - F^*(x, t). \quad (8.57)$$

Let λ_1 be the number defined in (8.40). Let $\lambda \geq \lambda_1$ and the function $q_{n,\min} \in \overline{B(R, g_0)}$ be the unique minimizer of the functional $J_{\lambda,\beta}^{(n)}(q_n)$ on the set $\overline{B(R, g_0)}$, the existence of which is established in Theorem 8.2. Following (8.57), denote

$$p_{n,\min}(x, t) = q_{n,\min}(x, t) - F(x, t). \quad (8.58)$$

Let

$$H_0^4(\Omega) = \{v \in H^4(\Omega) : v(0, t) = v_x(0, t) = v_x(b, t) = 0\}, \quad (8.59)$$

$$B_0(2R) = \{p \in H_0^4(\Omega) : \|p\|_{H_0^4(\Omega)} < R\}. \quad (8.60)$$

By (8.54)-(8.60)

$$p_{n,\min}, p^* \in \overline{B_0(R)}. \quad (8.61)$$

Also, it follows from embedding theorem (8.42), (8.49), (8.56) and (8.61) that

$$\|q^*\|_{C^2(\overline{\Omega})}, \|p^*\|_{C^2(\overline{\Omega})}, \|q_{n,\min}\|_{C^2(\overline{\Omega})}, \|p_{n,\min}\|_{C^2(\overline{\Omega})} \leq C. \quad (8.62)$$

We assume that the data g_0, g'_0 for our CIP are given with the noise of the level δ , where the number $\delta > 0$ is sufficiently small. More precisely, we assume that

$$\|F - F^*\|_{H^4(\Omega)} < \delta. \quad (8.63)$$

By (8.51), (8.52), (8.54) and (8.57)

$$p_{xx}^*(x, t) - p_{xt}^*(x, t) \frac{1}{2(q^*(x, 0))^2} + q_t^*(x, t) \frac{q_x^*(x, 0)}{2(q^*(x, 0))^3} + F_{xx}^* - F_{xt}^*(x, t) \frac{1}{2(q^*(x, 0))^2} = 0 \quad (8.64)$$

in Ω and

$$p^*(0, t) = p_x^*(0, t) = p_x^*(b, t) = 0. \quad (8.65)$$

Theorem 8.4 (contraction mapping and the global convergence of the numerical method of subsection 8.3). *Let functions $F, F^* \in H^4(\Omega)$ satisfy conditions (8.54)- (8.56) and (8.63). Let a sufficiently large number $\lambda_1 = \lambda_1(R, \Omega, \alpha) \geq \lambda_0 > 1$ be the one defined in (8.40). Let*

$$\lambda \geq \tilde{\lambda}_1 = \lambda_1(2R, \Omega, \alpha) \geq \lambda_1(R, \Omega, \alpha). \quad (8.66)$$

For this value of λ , let $q_{n,\min} \in \overline{B(R, g_0)}$ be the unique minimizer of the functional $J_{\lambda,\beta}^{(n)}(q_n)$ on the set $\overline{B(R, g_0)}$ with the regularization parameter $\beta \in [2e^{-\lambda\alpha T}, 1)$ (Theorem 8.2). Let

$$\bar{q}_n = q_{n,\min} - q^*, \quad \bar{c}_n = c_{n,\min} - c^*,$$

where $c_{n,\min}$ is defined in (8.46). Then the following convergence estimate holds

$$\begin{aligned} & \int_{\Omega} (\bar{q}_{nx}^2 + \bar{q}_{nt}^2 + \bar{q}_n^2)(x, t) e^{2\lambda\varphi} dx dt + \int_0^b (\bar{q}_{nx}^2 + \bar{q}_n^2)(x, 0) e^{-2\lambda x} dx \\ & \leq \frac{C}{\lambda} \int_{\Omega} (\bar{q}_{(n-1)x}^2 + \bar{q}_{(n-1)t}^2 + \bar{q}_{n-1}^2)(x, t) e^{2\lambda\varphi} dx dt \\ & + \frac{C}{\lambda} \int_0^b (\bar{q}_{(n-1)x}^2(x, 0) + \bar{q}_{n-1}^2(x, 0)) e^{-2\lambda x} dx + C(\delta^2 + \beta), \end{aligned} \quad (8.67)$$

which leads to:

$$\begin{aligned} & \int_{\Omega} (\bar{q}_{nx}^2 + \bar{q}_{nt}^2 + \bar{q}_n^2)(x, t) e^{2\lambda\varphi} dx dt \\ & \leq \frac{C^n}{\lambda^n} \int_{\Omega} (\bar{q}_{0x}^2 + \bar{q}_{0t}^2 + \bar{q}_0^2)(x, t) e^{2\lambda\varphi} dx dt + C(\delta^2 + \beta). \end{aligned} \quad (8.68)$$

In addition,

$$\|\bar{c}_n\|_{H^1(0,b)}^2 \leq \frac{C^n}{\lambda^n} \int_{\Omega} (\bar{q}_{0x}^2 + \bar{q}_{0t}^2 + \bar{q}_0^2)(x, t) e^{2\lambda\varphi} dx dt + C(\delta^2 + \beta). \quad (8.69)$$

Remarks 8.5:

1. *Due to the presence of the term C/λ with a sufficiently large λ , estimate (8.67) is similar to the one in the classic contraction mapping principle, although we do not claim here the existence of the fixed point.*

2. *When computing the unique minimizer $q_{0,\min}$ of functional (8.29) on the set $\overline{B(R, g_0)}$, we do not impose a smallness condition on the number R . Therefore, Theorem 8.4 claims the global convergence of the method of subsection 8.3: see our Definition 1.1 of the global convergence in Introduction.*

8.8 Numerical studies

In subsection 2.3.2 computational results of the numerical method of this section 8 are presented for the experimental data collected by the Forward Looking Radar of ARL. In this subsection we present two numerical results for computationally simulated data.

We add the noise into the data via the formula

$$g_0 = g_0^*(1 + \delta \cdot \text{rand}) \quad (8.70)$$

where δ is the noise level and rand is the function that generates uniformly distributed random numbers in the range $[-1, 1]$. In all numerical tests the noise level $\delta = 0.05$, i.e. 5%. Due to the presence of noise in (8.70), we cannot compute the function $g_1 = g'_0$ by the finite difference method. Hence, the function g'_0 is computed by the Tikhonov regularization method. The version of the Tikhonov regularization method for this problem is well-known. Hence, we do not describe this step here.

Having the data for the function q in hand, we choose

$$\lambda = 2, \alpha = 0.3 \text{ and } \beta = 10^{-11}.$$

These parameters were chosen by a trial-error process that is similar to the one in [10]. First, we choose a reference numerical test in which we know the true solution. We have tried several values of λ , α , and β until the numerical solution to that reference test became acceptable. Then, we have used the same values of these parameters for all other tests, including all five (5) available cases of experimental data.

We write differential operators in the functionals $J_{\lambda,\beta}^{(0)}$ and $J_{\lambda,\beta}^{(n)}$ in the finite differences with the step size in space $\Delta x = 0.0033$ and the step size in time $\Delta t = 0.02$ and minimize resulting functionals with respect to the values of corresponding functions at grid points. In finite differences, we can write a linear system whose solution is the desired critical point. We solve this system by the command “lsqin” of Matlab. The details in implementation by the finite difference method including the discretization, the derivation of the linear system for the critical point, and the use of “lsqin” are very similar to the scheme in [25].

Recall that in our theory, in the definition of the functional $J_{\lambda,\beta}^{(n)}$ acting on q_n , see (8.33), we replaced q_{n-1} with its analog $q_{n-1,\min}$ which belongs to the bounded set $\overline{B(R, g_0)}$, and also replaced $q_{n-1,\min}(x, 0)$ with $q_{n-1,\min}^0(x, 0)$. These replacements are only for the theoretical purpose to avoid the case when q_{n-1} blows up. However, in the numerical studies, these steps can be relaxed. This means that we have minimized the finite difference analog of the functional

$$q_n \mapsto \int_{\Omega} \left(q_{nxx}(x, t) - \frac{q_{nxt}(x, t)}{2(q_{n-1}(x, 0))^2} + \frac{\partial_t q_{n-1}(x, t) \partial_x q_{n-1}(x, 0)}{2(q_{n-1}(x, 0))^3} \right)^2 e^{2\lambda\varphi} dxdt + \beta \|q_n\|_{H^2(\Omega)}^2, \quad (8.71)$$

subject to the boundary conditions in the second line of (8.17). Another numerical simplification is that rather than using the H^4 -norm in the regularization term, we use the H^2 -norm in (8.71). Although the theoretical analysis supporting the above simplifications is missing, we did not experience any difficulties in computing the numerical solutions. All of our numerical results are satisfactory.

8.9 Numerical results for computationally simulated data

Test 1. The true function c in this test has two “inclusions”,

$$c_{\text{true}}(x) = \begin{cases} 1 + 5e^{\frac{(x-0.6)^2}{(x-0.6)^2 - 0.2^2}} & \text{if } |x - 0.6| < 0.2, \\ 1 + 8e^{\frac{(x-1.4)^2}{(x-1.4)^2 - 0.3^2}} & \text{if } |x - 1.4| < 0.3, \\ 1 & \text{otherwise.} \end{cases} \quad (8.72)$$

Numerical results of this test are displayed on Figure 21.

The true maximal values of the dielectric constant of the inclusions in the left and the right are 6 and 9 respectively. The reconstructed maximal values in those inclusions are acceptable. They are 5.31 (relative error 11.5%) and 7.8 (relative error 13.3%). Our iterative procedure converges after 7 iterations, see Figure 21b.

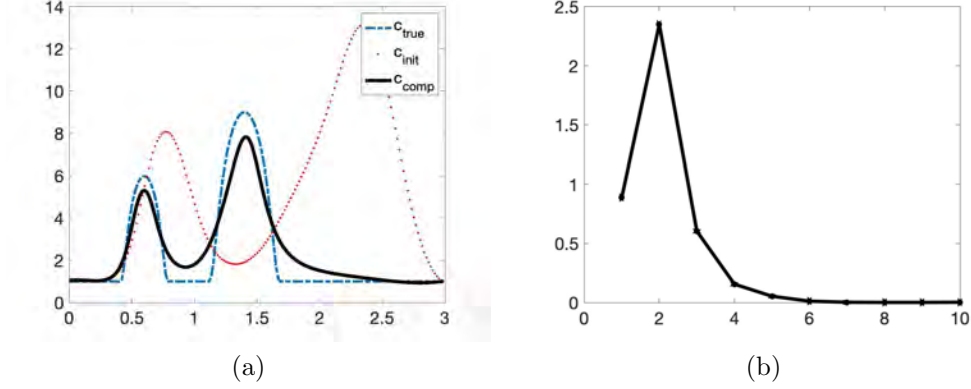


Figure 21: *Test 1. True and reconstructed functions $c(x)$, where the true function $c(x)$ is given in (8.72). c_{true} means the true function $c(x)$ in (8.72). $c_{\text{init}} = c_0$ means the function $c(x)$ computed on the initializing step number 0 when minimizing the functional $J_{\lambda,\beta}^{(0)}(q_0)$ in (8.29). c_{comp} means computed function $c(x)$. a) Functions c_{true} , c_{init} and c_{comp} . b) The consecutive relative error $\|c_n - c_{n-1}\|_{L_\infty(0,b)} / \|c_{n-1}\|_{L_\infty(0,b)}$, $n = 1, \dots, 10$. One can observe on b) that the process has converged after only 7 iterations.*

Test 2 In this test, the function $c_{\text{true}}(x)$ has the following form:

$$c_{\text{true}}(x) = \begin{cases} 3.5 + 0.3 \cdot \sin(\pi(x - 1.35)) & \text{if } |x - 0.9| < 0.5, \\ 8 & \text{if } |x - 2| < 0.37, \\ 1 & \text{otherwise.} \end{cases} \quad (8.73)$$

This test is interesting since the graph of the function (8.73) consists of a curve and an inclusion. The numerical solution for this case is presented in Figure 22.

One can observe from Figure 22a that our method to compute the initial reconstruction via minimizing functional $J_{\lambda,\beta}^{(0)}(q_0)$ in (8.29) not very effective. Nevertheless, after only 6 iterations, good numerical results are obtained. The curve in the first inclusion locally coincides with the true one and the maximal value of the computed dielectric constant within inclusion is quite accurate: it is 7.83 (relative error 2.12%). Our method converges at the iteration number 6, see Figure 22b.

CONCLUSION

It follows from all above tests that our method is robust in solving a highly nonlinear and severely ill-posed CIP (8.3)-(8.5). It provides satisfactory numerical results with a fast convergence. For each test, the computational time to compute the numerical solution is about 29 seconds on a MacBook Pro 2019 with 2.6 GHz processor and 6 Intel i7 cores. This is almost a real time computation.

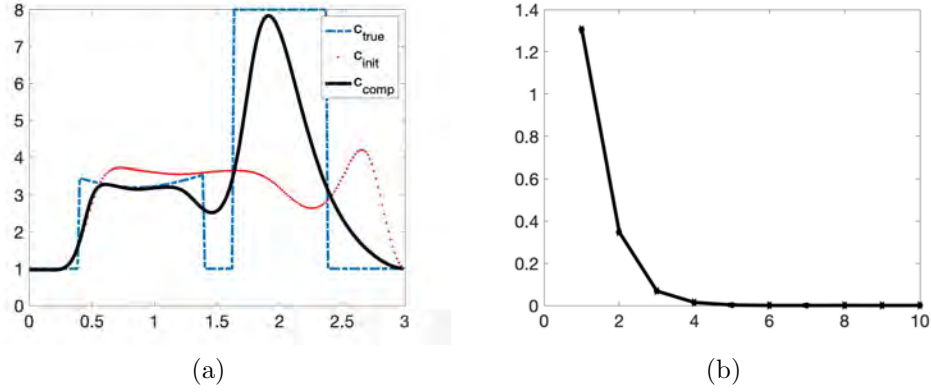


Figure 22: *Test 2. True and reconstructed functions $c(x)$, where the true function $c(x)$ is given in (8.73). c_{true} means the true function $c(x)$ in (8.73). $c_{\text{init}} = c_0$ means the function $c(x)$ computed on the initializing step number 0 when minimizing the functional $J_{\lambda,\beta}^{(0)}(q_0)$ in (8.29). c_{comp} means computed function $c(x)$. a) Functions c_{true} , c_{init} and c_{comp} . b) The consecutive relative error $\|c_n - c_{n-1}\|_{L_\infty(0,b)} / \|c_{n-1}\|_{L_\infty(0,b)}$, $n = 1, \dots, 10$. One can observe on b) that the process has converged after only 6 iterations.*

9 3D CIP for the Helmholtz Equation With Backscattering Data for Moving Source and Fixed Frequency [11]

As to the results for experimental data for this case, we refer to sub-subsection 2.3.3. Those results were published in [3, Chapter 10], [6, 7]. Thus, we present here a numerical result only for computationally simulated data. This is one of results of [11].

For the convenience of the reader, we copy here the problem statement of sub-subsection 2.3.3. The CIP in this section 9 is slightly more general than the one we have considered in [11]. More precisely, we consider here the case when both the dielectric constant and the electric conductivity are unknown. On the other hand only the dielectric constant is unknown in [11]. Nevertheless, the analytical results are the same. As to finding both these functions simultaneously, we have figured this out in a later publication [7].

9.1 Statement of the CIP

Denote $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$ points of the space $\mathbb{R}^3$. Consider a rectangular prism

$$\Omega = (-R, R) \times (-R, R) \times (0, b) \text{ in } \mathbb{R}^3 \quad (9.1)$$

for numbers $R, b > 0$. This prism is our computational domain of interest. Let $c = c(\mathbf{x})$ and $\sigma = \sigma(\mathbf{x})$ be respectively the spatially distributed dielectric constant and the electrical conductivity of the medium [7]. We assume that these functions are smooth and satisfy the following conditions:

$$\begin{cases} c(\mathbf{x}) \geq 1 & \text{in } \Omega, \\ c(\mathbf{x}) = 1 & \text{in } \mathbb{R}^3 \setminus \Omega, \end{cases} \quad \text{and} \quad \begin{cases} \sigma(\mathbf{x}) \geq 0 & \text{in } \Omega, \\ \sigma(\mathbf{x}) = 0 & \text{in } \mathbb{R}^3 \setminus \Omega. \end{cases} \quad (9.2)$$

The second line of (9.2) means that we are assuming to have vacuum outside the domain of interest Ω . Figure 2a represents a schematic diagram of our measurements and Figure 2b is a photograph of our experimental device.

Let the number $d > 0$ and let $a_1 < a_2$. We consider the line L_{src} along which our sources run,

$$L_{\text{src}} = \{(\alpha, 0, -d) : -a \leq \alpha \leq a\}. \quad (9.3)$$

The distance from L_{src} to the xy -plane is d , and the length of the line of sources is $a = (a_2 - a_1)$. Using this setting, for each $\alpha \in [-a, a]$ we arrange the point source $\mathbf{x}_\alpha := (\alpha, 0, -d)$ located on the straight line L_{src} . We also define the near-field measurement site as the lower side of the prism Ω ,

$$\Gamma := \{\mathbf{x} : |x|, |y| < R, z = 0\}. \quad (9.4)$$

We use notations of either α or $\mathbf{x}_\alpha$ to indicate the dependence of a functions from the source position $(\alpha, 0, -d)$. In practice, we measure backscattering data far from the domain Ω at a plane $P = \{\mathbf{x} : |x|, |y| < R, z = -s\}$ for a certain number $s > 0$. However, we use the so-called “data propagation procedure” described in [6] to “move” these far field to the part Γ of the near field plane $\{z = 0\}$.

We assume that the incident electric field is $E = (E_1, E_2, E_3) = (0, u_{\text{inc}}, 0)$. It was shown numerically in [46, 90] and analytically in [109] that the single Helmholtz-like PDE governs well the propagation of the second component $E_2(\mathbf{x}, \alpha) = u(\mathbf{x}, \alpha)$ in the medium. We also refer to the classical textbook of Born and Wolf [50, pages 695,696] for a similar conclusion, which is based on heuristic arguments.

The Helmholtz equation for the function $u(\mathbf{x}, k)$ with the radiation condition is:

$$\Delta u + (k^2 c(\mathbf{x}) - ik\sigma(\mathbf{x})) u = -\delta(\mathbf{x} - \mathbf{x}_\alpha) \quad \text{in } \mathbb{R}^3, \quad (9.5)$$

$$\lim_{r \rightarrow \infty} r(\partial_r u - iku) = 0 \quad \text{for } r = |\mathbf{x} - \mathbf{x}_\alpha|. \quad (9.6)$$

Here, the incident wave is

$$u_{\text{inc}}(\mathbf{x}, \alpha) = u_0(\mathbf{x}, \alpha) = \frac{\exp(ik|\mathbf{x} - \mathbf{x}_\alpha|)}{4\pi|\mathbf{x} - \mathbf{x}_\alpha|}. \quad (9.7)$$

Coefficient Inverse Problem. *Given a fixed value of the wave number $k > 0$, reconstruct both the dielectric constant $c(\mathbf{x})$ and the electric conductivity $\sigma(\mathbf{x})$ for $\mathbf{x} \in \Omega$ from the boundary measurements $F_0(\mathbf{x}, \mathbf{x}_\alpha), F_1(\mathbf{x}, \mathbf{x}_\alpha)$ of the near field data,*

$$F_0(\mathbf{x}, \mathbf{x}_\alpha) = u(\mathbf{x}, \alpha) \quad \text{for } \mathbf{x} \in \Gamma, \mathbf{x}_\alpha \in L_{\text{src}}, \quad (9.8)$$

$$F_1(\mathbf{x}, \mathbf{x}_\alpha) = u_z(\mathbf{x}, \alpha) \quad \text{for } \mathbf{x} \in \Gamma, \mathbf{x}_\alpha \in L_{\text{src}}, \quad (9.9)$$

where $u(\mathbf{x}, \alpha)$ is the total wave generated by the incident wave u_{inc} of (9.7) and the surface Γ is defined in (9.4).

It was explained in sub-subsection 2.3.3 how to obtain both functions F_0, F_1 rather than F_0 only via the so-called data propagation procedure described in [6].

A schematic diagram of data collection is given on Figure 5a and Figure 23.

Physically, to reconstruct the spatially distributed dielectric constant c and the electric conductivity σ of objects in Ω , one sends the incident wave field from the point source $\mathbf{x}_\alpha$,

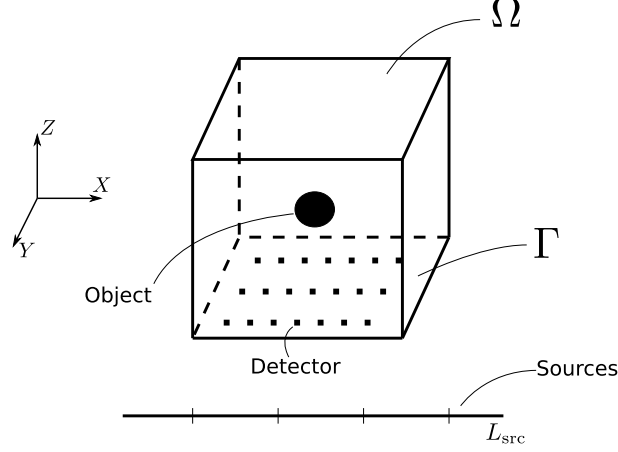


Figure 23: A schematic diagram of data collection for Coefficient Inverse Problem (9.5)-(9.9).

and this wave scatters when it hits the objects. Then, one measures the backscattering wave, which is used as the known measurement data.

Our CIP is a non-overdetermined one since the number of free variables in the data $F_0(\mathbf{x}, \mathbf{x}_\alpha), F_1(\mathbf{x}, \mathbf{x}_\alpha)$ is three (3), which is identical to the number of free variables in the unknown coefficient. In other words, the data (9.8), (9.9) are non-redundant ones.

Everywhere below in sections 9,10 we rely, without further comments, on the following assumption of the regularity of geodesic lines of sub-subsection 2.6.2.

A sufficient condition of the regularity of geodesic lines was derived in [108]. This condition is

$$\sum_{i,j=1}^3 \frac{\partial^2 \ln c(\mathbf{x})}{\partial x_i \partial x_j} \xi_i \xi_j \geq 0, \quad \forall \mathbf{x}, \boldsymbol{\xi} \in \mathbb{R}^3.$$

Lemma 9.1 follows from the formula (3.18) of [86].

Lemma 9.1. *There exists a sufficiently large number $k_0 > 0$ such that the function $u(\mathbf{x}, \alpha) \neq 0$ for all $k \geq k_0$, all $\mathbf{x} \in \bar{\Omega}$ and all $\mathbf{x}_\alpha \in L_{\text{src}}$.*

Below in this section 9 we assume that $k \geq k_0$ and, therefore, Lemma 9.1 is valid.

9.2 An auxiliary problem

Let

$$v_0(\mathbf{x}, \alpha) = \frac{u(\mathbf{x}, \alpha)}{u_0(\mathbf{x}, \alpha)}.$$

Define an auxiliary function $v := v(\mathbf{x}, \alpha) = \log v_0(\mathbf{x}, \alpha)$ for $\mathbf{x} \in \Omega$ and $\alpha \in [-a, a]$. Since we use below only the derivatives of the function $v_0(\mathbf{x}, \alpha)$, then the ambiguity of the logarithm of $v_0(\mathbf{x}, \alpha)$ is irrelevant. Using (9.7), we define $\log u_0(\mathbf{x}, \alpha)$ as

$$\log u_0(\mathbf{x}, \alpha) = ik |\mathbf{x} - \mathbf{x}_\alpha| - \ln(4\pi |\mathbf{x} - \mathbf{x}_\alpha|).$$

We have

$$\nabla v(\mathbf{x}, \alpha) = \frac{\nabla v_0(\mathbf{x}, \alpha)}{v_0(\mathbf{x}, \alpha)}, \quad \Delta v(\mathbf{x}, \alpha) = \frac{\Delta v_0(\mathbf{x}, \alpha)}{v_0(\mathbf{x}, \alpha)} - \left(\frac{\nabla v_0(\mathbf{x}, \alpha)}{v_0(\mathbf{x}, \alpha)} \right)^2$$

for all $\mathbf{x} \in \Omega$ and $\alpha \in [-a, a]$. Thus,

$$\Delta v + (\nabla v)^2 + 2\nabla v \nabla (\log(u_{\text{inc}}(\mathbf{x}, \alpha))) = -k^2(c(\mathbf{x}) - 1) - ik\sigma(\mathbf{x}), \quad \mathbf{x} \in \Omega. \quad (9.10)$$

Hence, differentiating (9.10) with respect to α and using

$$\partial_\alpha [-k^2(c(\mathbf{x}, k) - 1)] \equiv 0,$$

we obtain

$$\Delta \partial_\alpha v + 2\nabla v \cdot \nabla \partial_\alpha v + 2\nabla v_\alpha \nabla \log u_{\text{inc}}(\mathbf{x}, \alpha) \quad (9.11)$$

$$+ 2\nabla v \partial_\alpha \nabla \log u_{\text{inc}}(\mathbf{x}, \alpha) = 0, \quad \mathbf{x} \in \Omega,$$

$$v(\mathbf{x}, \alpha) = h_0(\mathbf{x}, \alpha), \quad v_z(\mathbf{x}, \alpha) = h_1(\mathbf{x}, \alpha), \quad \mathbf{x} \in \Gamma, \quad \mathbf{x}_\alpha \in L_{\text{src}}, \quad (9.12)$$

where functions $h_0(\mathbf{x}, \alpha)$ and $h_1(\mathbf{x}, \alpha)$ are known and are generated by functions $F_0(\mathbf{x}, \alpha)$ and $F_1(\mathbf{x}, \alpha)$ respectively.

We now need to recover an approximation for the function $v(\mathbf{x}, \alpha)$ from conditions (9.11), (9.12).

9.3 A special Fourier basis

Let (c, d) be an arbitrary interval. In his work [87] of 2017, the PI has proposed a new orthonormal basis in $L_2(a, b)$ which was proven to be very effective for numerical solutions of those CIPs in which underlying PDEs do not have initial conditions. Those are various CIPs for Helmholtz equation as well as some other CIPs [3, Chapters 7, 10-12], [11, 6, 7, 12, 15, 16, 17, 19, 21, 23, 31, 33, 35]. We now introduce this basis.

We denote our basis of real valued functions as $\{\Psi_n(\alpha)\}_{n=1}^\infty \subset L_2(-a, a)$. For each $n \in \mathbb{N}$, we define $\varphi_n(\alpha) = \alpha^n e^\alpha$ for $\alpha \in [-a, a]$. Observe that the set $\{\varphi_n(\alpha)\}_{n \in \mathbb{N}}$ is complete in $L_2(-a, a)$. Using the classical Gram-Schmidt orthonormalization procedure, we obtain the orthonormal basis $\{\Psi_n(\alpha)\}_{n \in \mathbb{N}}$ in $L_2(-a, a)$.

1. This basis possesses the following properties:
2. $\Psi_n \in C^1[-a, a]$ for all $n \in \mathbb{N}$.
3. Let $s_{mn} = \langle \Psi'_n, \Psi_m \rangle$ where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $L_2(-a, a)$. Then the square matrix $S_N = (s_{mn})_{m,n=0}^{N-1}$ is invertible for any $N \in \mathbb{N}$ in the sense that

$$s_{mn} = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{if } n < m. \end{cases}$$

It is worth noting that neither classical orthogonal polynomials nor the classical basis of trigonometric functions do not satisfy the second property. This is simply because in any of these two types the first column of the $N \times N$ matrix S_N would be identically zero. By virtue of this property, the matrix S_N is actually an upper diagonal matrix with

$\det(S_N) = 1$, $S_N = (s_{mn})_{m,n=0}^{N-1}$ is the $N \times N$ matrix that we have constructed above, is schematically represented by

$$S_N = \begin{pmatrix} 1 & * & \cdots & \cdots & * \\ 0 & 1 & * & & \vdots \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & * \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} \in \mathbb{R}^{N \times N}.$$

Hence, the inverse matrix S_N^{-1} exists.

Consider the auxiliary function $v(\mathbf{x}, \alpha)$ in (9.10). Given $N \in \mathbb{N}^*$, our truncated Fourier function for v with respect to α is written as

$$v(\mathbf{x}, \alpha) = \sum_{n=0}^{N-1} v_n(\mathbf{x}) \Psi_n(\alpha) \quad \text{for } \mathbf{x} \in \Omega, \alpha \in [-a, a], \quad (9.13)$$

where

$$v_n(\mathbf{x}) = \int_{-a}^a v(\mathbf{x}, \alpha) \Psi_n(\alpha) d\alpha.$$

We will discuss the choice of the number N below. From now onwards, we substitute (9.13) into (9.11) to get

$$\begin{aligned} & \Delta \left(\sum_{n=0}^{N-1} v_n(\mathbf{x}) \Psi'_n(\alpha) \right) + 2 \nabla \left(\sum_{n=0}^{N-1} v_n(\mathbf{x}) \Psi_n(\alpha) \right) \cdot \nabla \left(\sum_{n=0}^{N-1} v_n(\mathbf{x}) \Psi'_n(\alpha) \right) \\ & + 2 \nabla \left(\sum_{n=0}^{N-1} v_n(\mathbf{x}) \Psi'_n(\alpha) \right) \nabla \log u_{\text{inc}}(\mathbf{x}, \alpha) \\ & + 2 \nabla \left(\sum_{n=0}^{N-1} v_n(\mathbf{x}) \Psi_n(\alpha) \right) \partial_\alpha \nabla \log u_{\text{inc}}(\mathbf{x}, \alpha) = 0. \end{aligned} \quad (9.14)$$

Remark 9.1. *The assumption that the truncated Fourier-like series (9.13) satisfies equation (9.14) forms our approximate mathematical model. Thus, we work below only within the framework of this model. Still, when we computationally simulate the data (??), (??) for our CIP, we solve the full forward problem (??), (??) without any truncation. We point out that we use this model only because the original CIP (9.5)-(9.9) is a very challenging one, which is well known. The latter, as well as the ill-posed nature of our CIP, are the true reasons why we cannot prove convergence of our method as $N \rightarrow \infty$. Also, approximations of this kind are well acceptable in numerical studies.*

Denote

$$V(\mathbf{x}) = (v_0(\mathbf{x}), \dots, v_{N-1}(\mathbf{x}))^T \quad (9.15)$$

the vector of unknown Fourier coefficients of the function $v(\mathbf{x}, \alpha)$. Multiplying both sides of equation (9.14) by the functions $\Psi_m(\alpha)$, $m = 0, \dots, N-1$ and integrating with respect to $\alpha \in (-d, d)$, we obtain the following system of quasilinear elliptic equations:

$$S_N \Delta V(\mathbf{x}) + f(\nabla V(\mathbf{x})) = 0. \quad (9.16)$$

In addition, by (9.14) the following vector functions $g_0(\mathbf{x}), g_1(\mathbf{x})$ are known at Γ :

$$V(\mathbf{x})|_{\Gamma} = g_0(\mathbf{x}), \mathbf{x} \in \Gamma, \quad (9.17)$$

$$V_z(\mathbf{x})|_{\Gamma} = g_1(\mathbf{x}), \mathbf{x} \in \Gamma, \quad (9.18)$$

where the surface Γ is defined in (9.4). In (9.16) the nonlinearity $f = \left((f_m)_{m=0}^{N-1}\right)^T \in \mathbb{R}^N$ stands for

$$f_m(\nabla V(\mathbf{x})) = 2 \sum_{n,l=0}^{N-1} \nabla v_n(\mathbf{x}) \cdot \nabla v_l(\mathbf{x}) \int_{-a}^a \Psi_m(\alpha) \Psi_n(\alpha) \Psi'_l(\alpha) d\alpha$$

Now our focus is on designing a numerical solution of the problem (9.16)-(9.18), which would allow us to compute the vector valued function $V(\mathbf{x})$. If, for example, $f \equiv 0$, then (9.16)-(9.18) is the Cauchy problem for the Laplace equation, which is known to be both ill-posed and very challenging one. This means that the problem (9.16)-(9.18) is a highly ill-posed nonlinear problem.

9.4 The boundary value problem

To complement the lack of data information on the other parts of $\partial\Omega$, we begin with the data completion for (9.16)-(9.18). Specifically, we choose, for each α ,

$$u(\mathbf{x}, \alpha)|_{\partial\Omega} = \begin{cases} F_0(\mathbf{x}, \mathbf{x}_\alpha) & \text{if } \mathbf{x} \in \Gamma, \\ u_0(\mathbf{x}, \alpha) & \text{if } \mathbf{x} \in \partial\Omega \setminus \Gamma, \end{cases} \quad (9.19)$$

where $u_0(\mathbf{x}, \alpha)$ is given in (??) and it is the solution of the problem (??)-(??) for the case $c \equiv 1$. The choice (9.19) is fairly reasonable because of the fact that $\partial\Omega \subset \mathbb{R}^3 \setminus \Omega$. This choice transforms the problem (9.16)-(9.18) in the following one:

$$S_N \Delta V(\mathbf{x}) + f(\nabla V(\mathbf{x})) = 0, \quad (9.20)$$

$$V(\mathbf{x})|_{\Gamma} = g_0(\mathbf{x}), \quad (9.21)$$

$$V(\mathbf{x})|_{\partial\Omega \setminus \Gamma} = 0, \quad (9.22)$$

$$V_z(\mathbf{x})|_{\Gamma} = g_1(\mathbf{x}). \quad (9.23)$$

Thus, we will solve the boundary value problem (9.20)-(9.23). Below for each Banach space B ,

$$B_N = \underbrace{B \times B \times \dots \times B}_N, \text{ } N \text{ times.}$$

Note that by embedding theorem

$$H^4(\Omega) \subset C^2(\overline{\Omega}), \|f\|_{C^2(\overline{\Omega})} \leq C \|f\|_{H^2(\Omega)}, \forall f \in H^4(\Omega), \quad (9.24)$$

where the constant $C > 0$ depends only on the domain Ω .

Suppose that the vector function

$$V_{\text{comp}}(\mathbf{x}) = (v_{0,\text{comp}}(\mathbf{x}), \dots, v_{N-1,\text{comp}}(\mathbf{x}))$$

is computed. Then we substitute it in (9.13) and obtain

$$v_{\text{comp}}(\mathbf{x}, \alpha) = \sum_{n=0}^{N-1} v_{n,\text{comp}}(\mathbf{x}) \Psi_n(\alpha) \quad \text{for } \mathbf{x} \in \Omega, \alpha \in [-a, a].$$

Then we substitute $v_{\text{comp}}(\mathbf{x}, \alpha)$ in (9.10) and obtain the following formulas for computed target coefficients $c_{\text{comp}}(\mathbf{x})$ and $\sigma_{\text{comp}}(\mathbf{x})$:

$$c_{\text{comp}}(\mathbf{x}) = 1 -$$

$$- \frac{1}{2ak^2} \int_{-a}^a \text{Re} \left(\Delta v_{\text{comp}}(\mathbf{x}, \alpha) + (\nabla v_{\text{comp}}(\mathbf{x}, \alpha))^2 - [\nabla (\log u_0(\mathbf{x}, \alpha))]^2 \right) d\alpha, \quad (9.25)$$

$$\sigma_{\text{comp}}(\mathbf{x}) =$$

$$= - \frac{1}{2ak} \int_{-a}^a \text{Im} \left(\Delta v_{\text{comp}}(\mathbf{x}, \alpha) + (\nabla v_{\text{comp}}(\mathbf{x}, \alpha))^2 - [\nabla (\log u_0(\mathbf{x}, \alpha))]^2 \right) d\alpha. \quad (9.26)$$

9.5 Convexification: a globally strictly convex Tikhonov-like functional with the CWF in it

By the convexification approach (section 5), we solve the problem (9.20)-(9.23) via minimizing an appropriate weighted Tikhonov-like functional. The presence of a CWF in it plays the key role. Thus, we define our CWF as

$$\mu_\lambda(z) = \exp[2\lambda(z-r)^2], \quad z \in [-R, R], \quad r = \text{const.} > R. \quad (9.27)$$

Define the subspaces $H_0^2(\Omega)$ and $H_0^4(\Omega)$ of the spaces $H^2(\Omega)$ and $H^4(\Omega)$ as:

$$H_0^2(\Omega) := \{v \in H^2(\Omega) : v|_{\partial\Omega} = 0, \partial_z v|_{\Gamma} = 0\}, \quad (9.28)$$

$$H_0^4(\Omega) := \{v \in H^4(\Omega) : v|_{\partial\Omega} = 0, \partial_z v|_{\Gamma} = 0\}.$$

Theorem 9.1 (Carleman estimate). *Let $\mu_\lambda(z)$ be the function defined in (9.27). Then there exist constants $\lambda_0 = \lambda_0(\Omega, r) \geq 1$ and $C = C(\Omega, r) > 0$ depending only on the domain Ω such that for every function $u \in H_0^2(\Omega)$ and for all $\lambda \geq \lambda_0$ the following Carleman estimate holds:*

$$\int_{\Omega} |\Delta u|^2 \mu_\lambda(z) d\mathbf{x} \geq \frac{C}{\lambda} \sum_{i,j=1}^3 \int_{\Omega} |u_{x_i x_j}|^2 \mu_\lambda(z) d\mathbf{x} + C\lambda \int_{\Omega} [|\nabla u|^2 + \lambda^2 |u|^2] \mu_\lambda(z) d\mathbf{x}. \quad (9.29)$$

For the convenience of the presentation, each N -D complex valued vector function $W = \text{Re } W + i \text{Im } W$ is considered below as the $2N$ -D vector function with real valued components $(\text{Re } W, \text{Im } W) := (W_1, W_2) := W \in \mathbb{R}^{2N}$. All results and proofs below are for these $2N$ -D vector functions. For any number $s \in \mathbb{C}$, its complex conjugate is denoted as $\bar{s}$.

We find an approximate solution of the problem (9.20)-(9.23) via the minimization of an appropriate weighted Tikhonov-like functional with the CWF (9.27) involved in it. Due to (9.20), denote

$$L(V)(\mathbf{x}) = S_N \Delta V(\mathbf{x}) + f(\nabla V(\mathbf{x})). \quad (9.30)$$

Let $\gamma \in (0, 1)$ be the regularization parameter. We now consider the following weighted Tikhonov-like functional $J_{\lambda, \gamma} : H_{2N}^4(\Omega) \rightarrow \mathbb{R}_+$,

$$J_{\lambda, \gamma}(V) = \exp[-2\lambda(R+r)^2] \int_{\Omega} |L(V)|^2 \mu_{\lambda}(z) d\mathbf{x} + \gamma \|V\|_{H_{2N}^4(\Omega)}^2. \quad (9.31)$$

Here $\exp[-2\lambda(R+r)^2]$ is the balancing multiplier: to balance first and second terms in the right hand side of (9.31), since

$$\max_{\Omega} \mu_{\lambda}(z) = \exp[2\lambda(R+r)^2].$$

We use the $H_{2N}^4(\Omega)$ -norm in the regularization term here because of (9.24).

Assuming for a moment that the nonlinear term $f(\nabla V(\mathbf{x}))$ is absent in (9.30), we remark that since the Laplace operator is linear, then one can also find an approximate solution of the problem (9.20)-(9.23) by the regular quasi-reversibility method with $\lambda = 0$ in (9.31) (see, e.g. [3, Chapter 4], [84]). However, if $f(\nabla V(\mathbf{x})) \neq 0$, then the presence of the CWF serves three purposes: first, it controls this nonlinear term; second, it “maximizes” the influence of the important boundary data at $z = -R$; and third, it “convexifies” the cost functional globally. These are the underlying reasons of the convexification idea. Below $(\cdot, \cdot)$ is the scalar product in the space $H_{2N}^4(\Omega)$. Let $M > 0$ be an arbitrary number. We define the set $B(M) \subset H_{2N}^4(\Omega)$ as

$$B(M) = \left\{ \begin{array}{l} V \in H_{2N}^4(\Omega) : \|V\|_{H_{2N}^4(\Omega)} < M/2, \\ V(\mathbf{x})|_{\Gamma} = g_0(\mathbf{x}), V(\mathbf{x})|_{\partial\Omega \setminus \Gamma} = 0, \\ V_z(\mathbf{x})|_{\Gamma} = g_1(\mathbf{x}). \end{array} \right. \quad (9.32)$$

By (9.24), we know that

$$B(M) \subset C_{2N}^2(\overline{\Omega}) \text{ and } \|V\|_{C_{2N}^2(\overline{\Omega})} \leq C_1 \text{ for all } V \in B(M), \quad (9.33)$$

where the constant $C_1 = C_1(\Omega, M) > 0$ depends only on listed parameters.

Minimization Problem. *Minimize the cost functional $J_{\lambda, \gamma}(V)$ on the set $\overline{B(M)}$.*

9.6 Strict convexity on $\overline{B(M)}$ and follow up results

Theorem 9.2 is the central analytical result of this section 9.

Theorem 9.2 (global strict convexity).

1. *The functional $J_{\lambda, \gamma}(V)$ has its Fréchet derivative $J'_{\lambda, \gamma}(V) \in H_{0, N}^4(\Omega)$ at any point $V \in \overline{B(M)}$. Let $\lambda_0 > 1$ be the number of Theorem 9.1. There exists a sufficiently large number $\lambda_1 = \lambda_1(M, N, r, \Omega) \geq \lambda_0 \geq 1$ such that the functional $J_{\lambda, \gamma}(V)$ is strictly convex on $\overline{B(M)}$ for all $\lambda \geq \lambda_1$. More precisely, for all $\lambda \geq \lambda_2$ the following inequality holds*

$$\begin{aligned} & J_{\lambda, \gamma}(V^{(2)}) - J_{\lambda, \gamma}(V^{(1)}) - J'_{\lambda, \gamma}(V^{(1)})(V^{(2)} - V^{(1)}) \\ & \geq C_1 \|V^{(2)} - V^{(1)}\|_{H_{2N}^2(\Omega)}^2 + \gamma \|V^{(2)} - V^{(1)}\|_{H_{2N}^4(\Omega)}^2 \text{ for all } V^{(1)}, V^{(2)} \in \overline{B(M)}. \end{aligned} \quad (9.34)$$

2. There exists unique minimizer $V_{\min, \lambda, \gamma} \in \overline{B(M)}$ of the functional $J_{\lambda, \gamma}(V)$ on the set $\overline{B(M)}$ and the following inequality holds

$$(J'_{\lambda, \gamma}(V_{\min, \lambda, \gamma}), V_{\min, \lambda, \gamma} - Q) \leq 0 \quad \text{for all } Q \in \overline{B(M)}.$$

9.7 The distance between the minimizer and the “ideal” solution

In accordance with the concept of Tikhonov for ill-posed problems [116], assume now that there exists the “ideal” solution V^* of problem (9.20)-(9.23) with the “ideal” noiseless data $g_0^*, \tilde{g}_0, g_1^*$. It makes sense to obtain an estimate of the distance between V^* and the minimizer $V_{\min, \lambda, \gamma}$ of the functional $J_{\lambda, \gamma}(V)$ for the case of noisy data with the noise level $\delta \in (0, 1)$. This is what is done in the current subsection.

To obtain this estimate, we need to “extend” the boundary data $g_0, \tilde{g}_0, g_1$ in (9.32) inside Ω . Thus, we assume there exists a vector function $G(\mathbf{x}) \in H_{2N}^4(\Omega)$ satisfying boundary conditions (9.32),

$$G|_{\Gamma} = g_0(\mathbf{x}), G|_{\partial\Omega \setminus \Gamma} = 0, G_z|_{\Gamma} = g_1(\mathbf{x}). \quad (9.35)$$

On the other hand, the existence of the corresponding vector function $G^*(\mathbf{x}) \in H_{2N}^4(\Omega)$ satisfying boundary conditions with the “ideal” data,

$$G^*|_{\Gamma} = g_0^*(\mathbf{x}), G^*|_{\partial\Omega \setminus \Gamma} = 0, G_z^*|_{\Gamma} = g_1^*(\mathbf{x}) \quad (9.36)$$

follows from the existence of the ideal solution V^* . We assume that

$$\|G - G^*\|_{H_{2N}^4(\Omega)} < \delta. \quad (9.37)$$

In addition, we assume that

$$\|V^*\|_{H_{2N}^4(\Omega)}, \|G^*\|_{H_{2N}^4(\Omega)} < M - \delta. \quad (9.38)$$

Using (9.37), (9.38) and the triangle inequality, we easily see that

$$\|G\|_{H_{2N}^4(\Omega)} < M. \quad (9.39)$$

Our goal now is to estimate $\|V_{\min, \lambda, \gamma} - V^*\|_{H_{2N}^3(\Omega)}$ via the noise parameter δ .

Theorem 9.3 (accuracy and stability of minimizers). *Suppose that conditions (9.35)–(9.38) hold. Suppose that $\gamma = \delta^2$. Then there exists a sufficiently large number $\lambda_2 = \lambda_2(M, N, r, \Omega) > 1$ depending only on listed parameters such that for all $\lambda \geq \lambda_2$ the following accuracy estimate holds*

$$\|V_{\min, \lambda, \gamma} - V^*\|_{H_{2N}^2(\Omega)} \leq C_1 \delta. \quad (9.40)$$

Remark 9.2. *Since the power of δ is 1 in (9.40), then it is natural to call (9.40) “the Lipschitz stability estimate for the minimizers”.*

9.8 Globally convergent gradient descent method

The gradient descent method of minimizing the functional $J_{\lambda,\gamma}(V)$ on the set $B(M)$ is: Let $A \geq 3$ be a number. Choose an arbitrary point $V_0 \in B(M/A)$. Let the number $\omega \in (0, 1)$. Construct the sequence:

$$V_n = V_{n-1} - \omega J'_{\lambda,\gamma}(V_{n-1}), n = 1, \dots, \quad (9.41)$$

Theorem 9.4 follows from Theorem 6.1 (section 6) and Theorem 9.2.

Theorem 9.4. *Let $\lambda \geq \lambda_1$, where λ_1 is the number of Theorem 9.2. For this λ , let $V_{\min}$ be the minimizer found in Theorem 9.2. Assume that $V_{\min} \in B(M/A)$. Then there exists a number $\omega_0 \in (0, 1)$ such that for all $\omega \in (0, \omega_0)$ all points of the sequence (9.41) $\{V_n\}_{n=0}^\infty \in B(M)$. Furthermore, there exists a number $\theta = \theta(\omega) \in (0, 1)$ such that*

$$\|V_n - V_{\min}\|_{H_N^4(\Omega)} \leq \theta^n \|V_0 - V_{\min}\|_{H_N^4(\Omega)}, n = 1, 2, \dots$$

Theorem 9.5. *Assume that conditions of Theorems 9.3 and 9.4 are satisfied. Then the following convergence estimates to the exact solutions are valid:*

$$\|V_n - V^*\|_{H_N^2(\Omega)} \leq C_1 \delta + \theta^n \|V_0 - V_{\min}\|_{H_N^4(\Omega)}, n = 1, 2, \dots,$$

$$\|c_n - c^*\|_{L_2(\Omega)} \leq C_1 \delta + \theta^n \|V_0 - V_{\min}\|_{H_N^4(\Omega)}, n = 1, 2, \dots,$$

$$\|\sigma_n - \sigma^*\|_{L_2(\Omega)} \leq C_1 \delta + \theta^n \|V_0 - V_{\min}\|_{H_N^4(\Omega)}, n = 1, 2, \dots,$$

where functions $c_n(\mathbf{x})$ and $\sigma_n(\mathbf{x})$ are constructed from vector functions $V_n(\mathbf{x})$ via formulas (9.25) and (9.26) respectively. Similarly functions $c^*(\mathbf{x})$ and $\sigma^*(\mathbf{x})$ are constructed from the vector function $V^*(\mathbf{x})$ via formulas (9.25) and (9.26) respectively.

Since the number $M > 0$ is arbitrary in (9.32), then Theorem 9.5 guarantees the **global** convergence of the gradient descent method (9.41) of the minimization of functional (9.31) on the set $B(M)$ in (9.32), see Definition 1.1 in section 1.

9.9 Numerical results

In [11], we have numerically tested the method of this section 9 for five targets for computationally simulated data. As to the images for experimentally collected data for targets buried in the sand by the technique of this section, we refer to sub-subsection 2.3.3, Figures 7-10 and our publications [6, 7]. In our numerical studies in [11] we kept in mind applications to imaging of antipersonnel land mines and improvised explosive devices. We have assumed in numerical studies of computationally simulated data that electric conductivity $\sigma(\mathbf{x}) \equiv 0$. The case of $\sigma(\mathbf{x}) \neq 0$ for experimentally collected data was considered in sub-subsection 2.3.3 and in [7].

We have tested five (5) mine-like targets. To solve the forward problem of data simulation, we have used the Lippmann-Schwinger integral equation [57]. Using trial and error procedure, we have chosen optimal values of parameters:

$$\lambda = 1.1, \gamma = 10^{-4}, N = 4. \quad (9.42)$$

In doing so, we have selected one ball-shaped target and have discovered that parameters (9.42) are optimal for it. Then we have used the same values of parameters for the rest

of four (4) targets. Besides, the value $\lambda = 1.1$ is the same as the one in (2.20) for the experimental data. We have chosen $k = 6.6$, just as the k in experimental data for Target 5 (Figure 10) of sub-subsection 2.3.3. This value of k corresponds to the frequency of 3.15 GHz, see column 5 in Table 3 in sub-subsection 2.3.3.

We have written the differential operator L in (9.31) in the form of finite differences and then minimized the discrete version of functional (9.31) with respect to the values of the vector function V at grid points. We have used in our computations the simpler discrete $\|V\|_{H_{2N}^2(\Omega)}^2$ -norm in (9.31) (subsection 9.5) instead of the more complicated $\|V\|_{H_{2N}^4(\Omega)}^2$ -norm in (9.31).

Remarks 9.3:

1. *It is assumed in all our theoretical results about convexification, such as, e.g. Theorems 8.1-8.4 and 9.1-9.5 that the parameter λ should be sufficiently large. However, the particular choice of λ as in (9.42) demonstrates the validity of the same conclusion as the one drawn in all numerical studies of the convexification, which were cited above: that values $\lambda \in [1, 3]$ are sufficient for computations [3]-[35], [43]. In other words, numerical results are less pessimistic than analytical ones.*
2. *The reason why the simpler $\|V\|_{H_{2N}^2(\Omega)}^2$ -norm works here instead of the more complicated $\|V\|_{H_{2N}^4(\Omega)}^2$ -norm is that all norms in finite dimensional spaces are equivalent. Therefore, as long as the number of grid points is not too large, this equivalence works well computationally.*

Figure 25b) represent a typical example of the images obtained in [11]. The tested target was a vertically stretched ellipsoid (Figure 25a). We point out that this is a *very challenging case of a high inclusion/background contrast*:

$$c = \begin{cases} 5 & \text{inside the target,} \\ 1 & \text{outside the target.} \end{cases}$$

The maximal computed value of c inside the target is 5.19, i.e. only 3.8% error. The ellipsoidal shape of the computed image is also kept well.

10 A Version of the Convexification Method for a 3D CIP for the Helmholtz equation (2.15) With a Fixed Point Source [23]

In section 9, the backscattering data were considered for the case of many sources running along an interval of a straight line and the frequency was fixed. Unlike this, we consider in this section a version of the convexification method for the case of the backscattering data for a fixed point source and the frequency running along a certain frequency interval. The convergence analysis is similar here to the one of section 9. But the way of obtaining an analog of the boundary value problem (9.20)-(9.23) is different from the one of section 9.

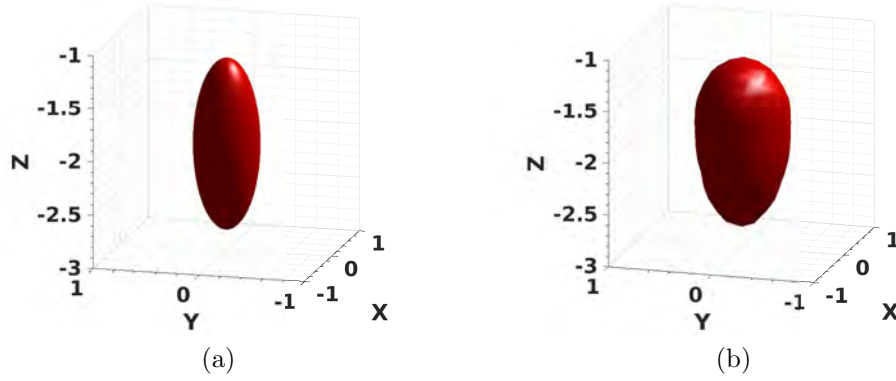


Figure 24: A sample of the numerical performance on computationally simulated data of the version of the verification method of section 9. In (9.5), the spatially dielectric constant $c(\mathbf{x})$ was assumed to be unknown and we took $\sigma(\mathbf{x}) \equiv 0$. The mine-like target of interest was a vertically stretched ellipsoid with $c(\mathbf{x}) = 5$ in it and $c(\mathbf{x}) = 1$ outside. a) The true $c(\mathbf{x})$. b) The computed $c(\mathbf{x})$. The maximal value of the computed $c(\mathbf{x})$ is $\max c(\mathbf{x}) = 5.19$. Thus, we have only 3.8% of error. Both shape and location of the target are imaged with a good accuracy.

10.1 Statements of forward and inverse problems

For the convenience of the reader, we reformulate now once again CIP (2.13), (2.24)-(2.26).

Let $\Omega \subset \mathbb{R}^3$ be a cube and Γ is its lower side,

$$\Omega = (-R, R) \times (-R, R) \times (-R, R), \quad (10.1)$$

$$\Gamma = \{\mathbf{x} : -R < x, y < R, z = -R\}. \quad (10.2)$$

Second and most importantly we assume that the backscattering data at Γ are generated by a single location of the point source

$$\mathbf{x}_0 = (0, 0, -s), s = \text{const.} > R, \quad (10.3)$$

which is located below the cube Ω . The data are available for an interval of wave numbers $k \in (k_1, k_2) \subset (0, \infty)$.

Let the function $c(\mathbf{x})$ be sufficiently smooth and satisfy the following conditions:

$$\begin{cases} c(\mathbf{x}) \geq 1 & \text{in } \Omega, \\ c(\mathbf{x}) = 1 & \text{in } \mathbb{R}^3 \setminus \Omega, \end{cases} \quad (10.4)$$

The Helmholtz equation for the function $u(\mathbf{x}, k)$ with the radiation condition is:

$$\Delta u + k^2 c(\mathbf{x}) u = -\delta(\mathbf{x} - \mathbf{x}_0) \quad \text{in } \mathbb{R}^3, \quad (10.5)$$

$$\lim_{r \rightarrow \infty} r(\partial_r u - iku) = 0 \quad \text{for } r = |\mathbf{x} - \mathbf{x}_0|. \quad (10.6)$$

Here, the incident wave is

$$u_{\text{inc}}(\mathbf{x}, k) = u_0(\mathbf{x}, k) = \frac{\exp(ik|\mathbf{x} - \mathbf{x}_0|)}{4\pi|\mathbf{x} - \mathbf{x}_0|}. \quad (10.7)$$

Coefficient Inverse Problem. *Let conditions (10.1) and (10.2) be in place. Fix a point source x_0 in (10.3) and let a sufficiently smooth dielectric constant $c(\mathbf{x})$ satisfies conditions (10.4). Let the function $u(\mathbf{x}, k)$ be the solution of problem (10.5), (10.6) defined in [57]. Determine the function $c(\mathbf{x})$ for $\mathbf{x} \in \Omega$, given functions $g_0(\mathbf{x}, k), g_1(\mathbf{x}, k)$,*

$$u(\mathbf{x}, k)|_{\Gamma} = F_0(\mathbf{x}, k), \quad u_z(\mathbf{x}, k)|_{\Gamma} = F_1(\mathbf{x}, k), \quad k \in (k_1, k_2). \quad (10.8)$$

It was explained in sub-subsection 2.3.3 how to obtain both functions F_0, F_1 rather than F_0 only via the so-called data propagation procedure described in [6].

To construct the convexification method for CIP (10.4)-(10.8), we proceed similarly with section 9. However, we vary k instead of α .

10.2 A nonlinear boundary value problem

Just as in (9.19), we complement the boundary data (10.8) at Γ with the boundary data at $\partial\Omega \setminus \Gamma$ as

$$u(\mathbf{x}, k)|_{\partial\Omega} = \begin{cases} F_0(\mathbf{x}, k) & \text{if } \mathbf{x} \in \Gamma, \\ u_0(\mathbf{x}, k) & \text{if } \mathbf{x} \in \partial\Omega \setminus \Gamma. \end{cases}$$

For the incident wave (10.7) we set

$$\log(u_{\text{inc}}(\mathbf{x}, k)) = ik|\mathbf{x} - \mathbf{x}_0| - \ln(4\pi|\mathbf{x} - \mathbf{x}_0|). \quad (10.9)$$

Let $\tau(\mathbf{x}, \mathbf{x}_0)$ be the travel time from the point $\mathbf{x}_0$ to the point $\mathbf{x}$, as in (2.60) of subsection 2.6.2,

$$\tau(\mathbf{x}, \mathbf{x}_0) = \int_{\Gamma(\mathbf{x}, \mathbf{x}_0)} \sqrt{c(\mathbf{x}(s))} ds, \quad (10.10)$$

where $\Gamma(\mathbf{x}, \mathbf{x}_0)$ is the geodesic line connecting points $\mathbf{x}$ and $\mathbf{x}_0$ and ds is the euclidean arc length. We again assume the regularity of geodesic lines, see subsection 2.6.2. Then the function $u(\mathbf{x}, k)$ has the following asymptotic behavior [109]:

$$u(\mathbf{x}, k) = e^{ik\tau(\mathbf{x}, \mathbf{x}_0)} \left(A(\mathbf{x}) + O\left(\frac{1}{k}\right) \right), \quad k \rightarrow \infty, \mathbf{x} \in \Omega, \quad (10.11)$$

where the function $A(\mathbf{x}) > 0$. Hence, we uniquely define $\log u(\mathbf{x}, k)$ for sufficiently large values of k as

$$\log u(\mathbf{x}, k) = ik\tau(\mathbf{x}, \mathbf{x}_0) + \ln A(\mathbf{x}) + O\left(\frac{1}{k}\right), \quad k \gg 1, \mathbf{x} \in \Omega. \quad (10.12)$$

In particular, (10.11) implies that $u(\mathbf{x}, k) \neq 0$ for sufficiently large values of k for $\mathbf{x} \in \Omega$.

We now explain why (10.12) defines the function $\log u(\mathbf{x}, k)$ uniquely. Indeed, for any complex number $z \in \mathbb{C}$, $z \neq 0$

$$\log z = \ln|z| + i \arg z + 2in\pi, \quad \arg z \in [0, 2\pi), \quad n \in (-\infty, \infty), \quad (10.13)$$

where n is an integer. On the other hand, obviously

$$|\operatorname{Im}(k\tau(\mathbf{x}, \mathbf{x}_0))| - \left| \operatorname{Im}\left(k\tau(\mathbf{x}, \mathbf{x}_0) + O\left(\frac{1}{k}\right)\right) \right| < 2\pi$$

for sufficiently large k . Therefore, an ambiguity as in (10.13) does not occur.

Thus, assuming that the number k_1 in (10.8) is sufficiently large, we introduce the function $v(\mathbf{x}, kwe)$,

$$v(\mathbf{x}, k) = \frac{1}{k^2} [\log u(\mathbf{x}, k) - \log(u_{\text{inc}}(\mathbf{x}, k))], x \in \Omega, k \in (k_1, k_2). \quad (10.14)$$

Since by (10.1) and (10.3) $\mathbf{x}_0 \notin \bar{\Omega}$, then (10.5), (10.7) and (10.14) lead to

$$\Delta v + k^2 (\nabla v)^2 + 2\nabla v \cdot \nabla (\log(u_{\text{inc}}(\mathbf{x}, \alpha))) = -c(\mathbf{x}) + 1 \quad \text{for } \mathbf{x} \in \Omega. \quad (10.15)$$

Differentiate equation (10.15) with respect to k . We obtain a nonlinear PDE of the third order, in which, however, the unknown coefficient $c(\mathbf{x})$ is not present,

$$\Delta v_k + \partial_k [k^2 (\nabla v)^2 + 2\nabla v \cdot \nabla (\log(u_{\text{inc}}(\mathbf{x}, \alpha)))] = 0. \quad (10.16)$$

Consider now the orthonormal basis $\{\varphi_n(k)\}_{n=1}^\infty$ in $L_2(-k_1, k_2)$, which is the full analog of the orthonormal basis $\{\Psi_n(\alpha)\}_{n=1}^\infty$ in $L_2(-a, a)$ of subsection 9.3. Similarly with (9.13) we set

$$v(\mathbf{x}, k) = \sum_{n=0}^{N-1} v_n(\mathbf{x}) \varphi_n(k) \quad \text{for } \mathbf{x} \in \Omega, k \in [k_1, k_2], \quad (10.17)$$

$$v_n(\mathbf{x}) = \int_{k_1}^{k_2} v(\mathbf{x}, k) \varphi_n(k) dk.$$

Consider the vector function $V(\mathbf{x}) \in \mathbb{R}^N$ of the unknown coefficients of the series (10.17),

$$V(\mathbf{x}) = (v_0(\mathbf{x}) \quad v_1(\mathbf{x}) \quad \cdots \quad v_{N-1}(\mathbf{x}))^T.$$

Then, using the same considerations as in subsection 9.4, we obtain the following analog of the nonlinear boundary value problem (9.20)-(9.23)

$$\tilde{S}_N \Delta V(\mathbf{x}) + \tilde{f}(\nabla V(\mathbf{x})) = 0 \quad \text{for } \mathbf{x} \in \Omega, \quad (10.18)$$

$$V(\mathbf{x}) = \psi_0(\mathbf{x}), V_z(\mathbf{x}) = \psi_1(\mathbf{x}) \quad \text{for } \mathbf{x} \in \Gamma, \quad (10.19)$$

$$V(\mathbf{x}) = 0 \quad \text{for } \mathbf{x} \in \partial\Omega \setminus \Gamma, \quad (10.20)$$

where the invertible $N \times N$ matrix $\tilde{S}_N$ and the vector function $\tilde{f}$ are respectively full analogs of the matrix S_N and the vector function f in (9.20). Also, boundary vector functions $\psi_0(\mathbf{x})$ and $\psi_1(\mathbf{x})$ are known. The vector function $\tilde{f}(\nabla V(\mathbf{x}))$ contains linear and quadratic terms with respect to components of the vector function $\nabla V(\mathbf{x})$.

As soon as this problem is solved, the coefficient $c(\mathbf{x})$ is reconstructed similarly with (9.25) (subsection 9.4) via backwards calculations.

Remarks 10.1:

1. *As soon as this problem is solved, the coefficient $c(\mathbf{x})$ is reconstructed similarly with (9.25) (subsection 9.4) via backwards calculations.*
2. *We choose the number N number numerically. In our numerical studies $N = 7$.*

10.3 Globally strictly convex Tikhonov-like functional

As in (9.27) of subsection 9.5, we choose the Carleman Weight Function as:

$$\mu_\lambda(z) = \exp(2\lambda(z-r)^2) \quad \text{for } z \in [-R, R], \lambda > 0, r = \text{const.} > R. \quad (10.21)$$

Let $M > 0$ be an arbitrary number. Just as in (9.32) of subsection 9.5, we define the set $B(M) \subset H_{2N}^4(\Omega)$ as

$$B(M) = \left\{ \begin{array}{l} V \in H_{2N}^4(\Omega) : \|V\|_{H_{2N}^4(\Omega)} < M/2, \\ V(\mathbf{x})|_{\Gamma} = \psi_0(\mathbf{x}), V(\mathbf{x})|_{\partial\Omega \setminus \Gamma} = 0, \\ V_z(\mathbf{x})|_{\Gamma} = \psi_1(\mathbf{x}). \end{array} \right. \quad (10.22)$$

We introduce the nonlinear operator $\tilde{L}$ similarly with operator L in (9.30) in subsection 9.5,

$$\tilde{L}(V)(\mathbf{x}) = \tilde{S}_N \Delta V(\mathbf{x}) + \tilde{f}(\nabla V(\mathbf{x})).$$

To solve boundary value problem (10.18)-(10.20), we construct the Tikhonov-like cost functional with the CWF (10.21)

$$J_{\lambda,\gamma}(V) = e^{-\lambda(R+r)^2} \int_{\Omega} \left| \tilde{L}(V)(\mathbf{x}) \right|^2 \mu_\lambda(z) d\mathbf{x} + \gamma \|V\|_{H_{2N}^4(\Omega)}^2, \quad (10.23)$$

where the multiplier $e^{-\lambda(R+r)^2}$ is introduced to balance first and second terms in the right hand side of (10.23). We solve the following minimization problem:

Minimization Problem. *Find a minimizer of the functional $J_{\lambda,\gamma}(V)$ in (10.23) on the set $B(M)$ in (10.22).*

10.4 Convergence analysis

Functional (10.23) is completely similar with functional (9.31) in subsection 9.5. For this reason, the convergence analysis of functional (10.23) is completely similar with the one of functional (9.31) in subsections 9.5-9.8. Full analogs of theorems 9.1-9.5 are valid here. Therefore, we omit here the convergence analysis of functional (10.23). Since the number $M > 0$ is arbitrary in (10.23), then the global convergence of the gradient descent method of the minimization of functional (10.23) is guaranteed as long as the starting point of this method belongs to the set $B(M/3)$, see Definition 1.1 in section 1.

10.5 Numerical studies

The version of the convexification of this section 10 was tested in [23] on computationally simulated data for three (3) tests. We present here only one of them. We plan testing on experimentally collected data in our microwave laboratory in the future. We have introduced 10% random noise in our data (10.19). We took in [23]

$$\Omega = \{\mathbf{x} = (x, y, z) : -1 < x, y, z < 1\},$$

$$\lambda = 1.1, N = 7, r = 1.5, \gamma = 10^{-6}, \quad (10.24)$$

$$\mu_\lambda(z) = \exp(2.2(z-1.5)^2) \quad (10.25)$$

Observe that in (10.24) the parameter $\lambda = 1.1$ of the Carleman Weight Function (10.25) is the same as the one in (2.20) in sub-subsection 2.3.3 and in (9.42) in subsection 9.9. The interval of wave numbers is $k \in [\pi, 2\pi]$, which corresponds to the interval of wavelengths $[1.5, 3]$ GHz. We have written the differential operator $\tilde{L}$ in (10.23) in the form of finite differences and then minimized the discrete version of functional (10.23) with respect to the values of the vector function V at grid points. We have used in our computations the discrete $\|V\|_{H_{2N}^2(\Omega)}^2$ -norm in (10.23) (subsection 10.4) instead of the more complicated $\|V\|_{H_{2N}^4(\Omega)}^2$ -norm in (10.23), see the second item of Remarks 9.3 in subsection 9.9.

The first approximation $V_0(\mathbf{x})$ for the solution $V(\mathbf{x})$ of problem boundary value problem (10.18)-(10.20) was obtained in a *truly new way*. More precisely, nonlinear terms of equations (10.18) were dropped and the remaining problem became:

$$\tilde{S}_N \Delta V_0(\mathbf{x}) + K(\nabla V_0(\mathbf{x})) = 0 \quad \text{for } \mathbf{x} \in \Omega, \quad (10.26)$$

$$V_0(\mathbf{x}) = \psi_0(\mathbf{x}), V_{0z}(\mathbf{x}) = \psi_1(\mathbf{x}) \quad \text{for } \mathbf{x} \in \Gamma, \quad (10.27)$$

$$V(\mathbf{x}) = 0 \quad \text{for } \mathbf{x} \in \partial\Omega \setminus \Gamma, \quad (10.28)$$

where the expression $K(\nabla V(\mathbf{x}))$ is linear with respect to the components of the vector function $\nabla V(\mathbf{x})$.

Problem (10.26)-(10.28) was solved by the Carleman Quasi-Reversibility Method. This is similar with the one for the 1D case in subsection 8.3, see (8.33) and [9]. for a survey with the emphasize on Carleman estimates being applied to establish convergence rates. A truly new element of the QRM of [23] is the involvement of the Carleman Weight Function in the numerical scheme of the Quasi-Reversibility Method. More precisely, the following functional was minimized:

$$I_{\lambda,\gamma}(V_0) = e^{-2\lambda(R+r)^2} \int_{\Omega} \left| \tilde{S}_N \Delta V_0(\mathbf{x}) + K(\nabla V_0(\mathbf{x})) \right|^2 e^{2\lambda(z-r)^2} d\mathbf{x} + \gamma \|V_0\|_{H_{2N}^2(\Omega)}^2,$$

subject to boundary conditions (10.27), (10.28). Existence and uniqueness of the minimizer of the functional $I_{\lambda,\beta}(V_0)$ can be proven similarly with [3, Chapter 4], [84]. Convergence rate of this specific version of the QRM can also be established using the Carleman estimate of Theorem 9.1, see [3, Chapter 4], [84] for similar results, in which cases, however, Carleman Weight Functions are not involved in numerical schemes.

Figure 26 displays a sample of computed results. We have imaged a 3D letter ‘O’. We took

$$c(\mathbf{x}) = \begin{cases} 3 & \text{inside the 3D letter } O \\ 1 & \text{otherwise.} \end{cases} \quad (10.29)$$

Remarks 10.2:

1. *We point out that Figures 25 demonstrate that this version of the convexification method is capable to accurately image non-convex structures with voids in them. Furthermore, they are imaged for only a **single** location of the point source. The error of the computation of the dielectric constant is 7% at 10% of random noise in the data.*
2. *Images of non-convex structures for experimental data for many positions of the source running along a straight line are presented on Figures 7,8,9 in subsection 2.3.3.*

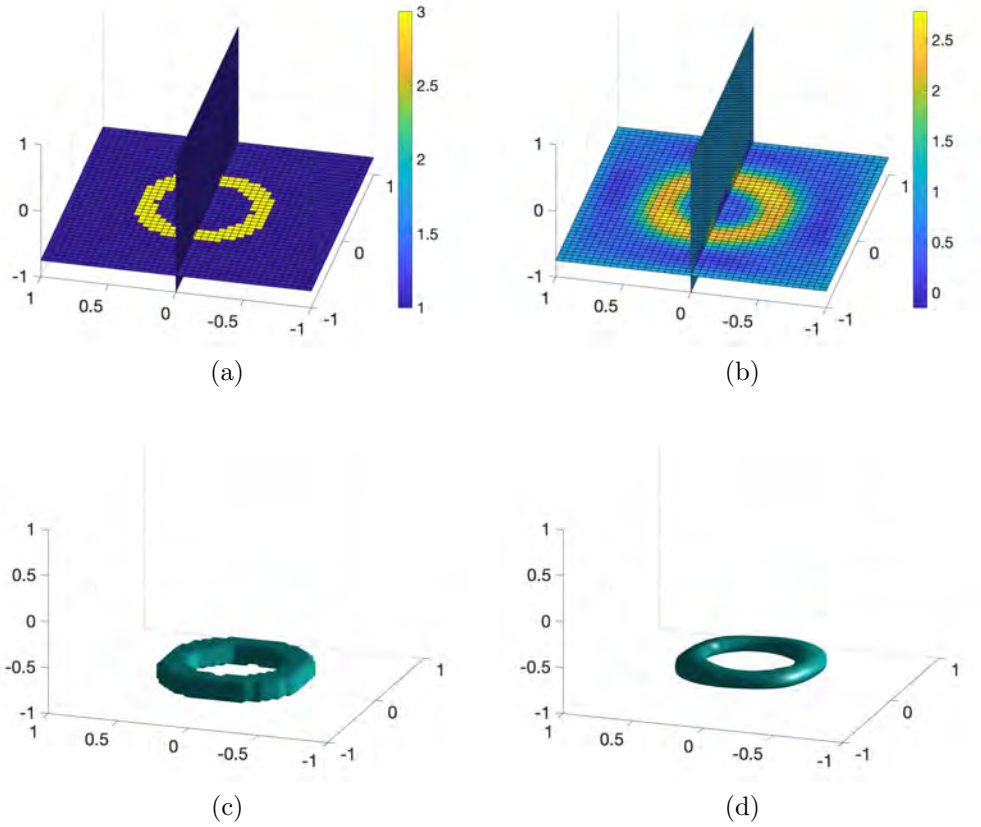


Figure 25: A sample of the performance of the version of the convexification method of section 10, for the case of backscattering data measured for a single location of the source and an interval of frequencies, see Remarks 10.2 for further comments. The data (10.19) contain 10% of random noise. A 3-D letter ‘O’ was imaged. a) A 2D cross-section of the true image. The true dielectric constant of this target is 3, see (10.29). b) The same 2D cross-section of the computed image as on a). The computed dielectric constant of this target is 2.8. Thus, the error of the computation of the dielectric constant is 7% at 10% of random noise in the data. c) The 3D representation of true image. d) The 3D representation of computed image.

11 A Version of the Convexification Method for a 3D CIP for a Wave-Like Equation in Time Domain [3, Chapter 8], [13]

In this section 11 we provide more details for CIP outlined in sub-subsection 2.4.2.

11.1 Statements of forward and inverse problems

Let $A > 0$ be a number. Similarly with (2.21) we set the domain of interest $\Omega \subset \mathbb{R}^3$ is a cube

$$\Omega = \{\mathbf{x} = (x, y, z) : -A/2 < x, y < A/2, z \in (0, A)\}. \quad (11.1)$$

Let the number $a > 0$. We set the single point source we use as $\mathbf{x}_0 = (0, 0, -a)$. Hence, this source is located below the domain Ω . Let Γ_0 be the transmitted side of the boundary $\partial\Omega$ of the domain Ω and Γ_1 be the rest of this boundary,

$$\Gamma_0 = \{\mathbf{x} = (x, y, z) : -A/2 < x, y < A/2, z = A\}, \Gamma_1 = \partial\Omega \setminus \Gamma_0. \quad (11.2)$$

Let the number $T > 0$. Denote

$$S_T = \partial\Omega \times (0, T) \text{ and } \Gamma_{0,T} = \Gamma_0 \times (0, T). \quad (11.3)$$

Figure 12 of sub-subsection 2.4.2 depicts a schematic diagram of measurements.

As the forward problem, we consider the following Cauchy problem for the equation which is connected with the Helmholtz equation via the Fourier transform:

$$c(\mathbf{x}) u_{tt} = \Delta u, \mathbf{x} \in \mathbb{R}^3, t > 0, \quad (11.4)$$

$$u(\mathbf{x}, 0) = 0, u_t(\mathbf{x}, 0) = \delta(\mathbf{x} - \mathbf{x}_0). \quad (11.5)$$

Coefficient Inverse Problem. *Let the domain Ω be as in (11.1). Suppose that the following two functions are given:*

$$u(\mathbf{x}, t) |_{(\mathbf{x}, t) \in S_T} = f_0(\mathbf{x}, t), \partial_z u(\mathbf{x}, t) |_{(\mathbf{x}, t) \in \Gamma_{0,T}} = f_1(\mathbf{x}, t), \quad (11.6)$$

Determine the function $c(\mathbf{x})$ for $\mathbf{x} \in \Omega$.

Let $\tau(\mathbf{x}, \mathbf{x}_0)$ be the travel time between points $\mathbf{x}$ and $\mathbf{x}_0$ defined in (10.10) in subsection 10.2. We again assume the regularity of geodesic lines, see subsection 2.6.2. Since we work with only a single position $\mathbf{x}_0$ of the point source, we omit below indications of dependencies on $\mathbf{x}_0$. It was proven in Theorem 4.1 of [107] that the function $u(\mathbf{x}, t)$ has the form:

$$u(\mathbf{x}, t) = A(\mathbf{x}) \delta(t - \tau(\mathbf{x})) + H(t - \tau(\mathbf{x})) \hat{u}(\mathbf{x}, t), \quad (11.7)$$

where the function $A(\mathbf{x}) > 0, \forall \mathbf{x} \in \mathbb{R}^3$, the function $\hat{u}(\mathbf{x}, t) \in C^2(t \geq \tau(\mathbf{x}))$ and $H(z)$ is the Heaviside function,

$$H(z) = \begin{cases} 1, & z > 0, \\ 0, & z < 0. \end{cases}$$

First, we reduce CIP (11.4)-(11.6) to the boundary value problem for a system of coupled quasilinear elliptic PDEs. This problem is similar with the problem (9.20)-(9.23) of subsection 9.4.

11.2 The function $w(\mathbf{x}, t)$

Integrate equation (11.4) twice with respect to t for points $\mathbf{x} \in \Omega$. Hence, we consider the function $p(\mathbf{x}, t)$,

$$p(\mathbf{x}, t) = \int_0^t dy \int_0^y u(\mathbf{x}, s) ds, \mathbf{x} \in \Omega. \quad (11.8)$$

Hence, $p_{tt}(\mathbf{x}, t) = u(\mathbf{x}, t)$ for $\mathbf{x} \in \Omega$. Since $\mathbf{x}_0 \notin \bar{\Omega}$, then $\delta(\mathbf{x} - \mathbf{x}_0) = 0$ for $\mathbf{x} \in \Omega$. We obtain

$$c(\mathbf{x}) p_{tt}(\mathbf{x}, t) = \Delta p(\mathbf{x}, t) \text{ for } (\mathbf{x}, t) \in \Omega \times (0, T). \quad (11.9)$$

Next, by (11.7) and (11.8)

$$p(\mathbf{x}, t) = A(\mathbf{x}) (t - \tau(\mathbf{x})) H(t - \tau(\mathbf{x})) + O((t - \tau(\mathbf{x}))^2) H(t - \tau^0(\mathbf{x})), \quad (11.10)$$

where

$$\begin{aligned} |O((t - \tau(\mathbf{x}))^2)| &\leq D(t - \tau(\mathbf{x}))^2 \text{ as } t \rightarrow \tau^+(\mathbf{x}), \\ |\partial_t O((t - \tau(\mathbf{x}))^2)| &\leq D(t - \tau(\mathbf{x})) \text{ as } t \rightarrow \tau^+(\mathbf{x}) \end{aligned}$$

with a certain constant $D > 0$ independent on $(\mathbf{x}, t) \in \Omega \times (0, T)$.

Consider the function $w(\mathbf{x}, t)$ defined as

$$w(\mathbf{x}, t) = p(\mathbf{x}, t + \tau(\mathbf{x})), \text{ for } (\mathbf{x}, t) \in \Omega \times (0, T).$$

Then it follows from the above that $w \in C^2(\bar{\Omega} \times [0, T])$,

$$w(\mathbf{x}, 0) = 0,$$

$$w_t(\mathbf{x}, 0) = A(\mathbf{x}) > 0. \quad (11.11)$$

We obtain

$$\Delta w - 2 \sum_{i=1}^3 w_{x_i t} \tau_{x_i} - w_t \Delta \tau = \Delta p - p_{tt} (\nabla \tau)^2.$$

Since $|\nabla \tau(\mathbf{x})|^2 = c(\mathbf{x})$ then by (11.9) $\Delta p - p_{tt} (\nabla \tau)^2 = 0$. Thus,

$$\Delta w - 2 \sum_{i=1}^3 w_{x_i t} \tau_{x_i} - w_t \Delta \tau = 0, \mathbf{x} \in \Omega, t \in (0, T_1), \quad (11.12)$$

$$T_1 = T - \max_{\bar{\Omega}} \tau(\mathbf{x}). \quad (11.13)$$

Denote $\tilde{f}_0(\mathbf{x}, t) = f_0(\mathbf{x}, t + \tau(\mathbf{x}))$ and $\tilde{f}_1(\mathbf{x}, t) = f_1(\mathbf{x}, t + \tau(\mathbf{x}))$. Then

$$w(\mathbf{x}, t) |_{(\mathbf{x}, t) \in S_{T_1}} = \tilde{f}_0(\mathbf{x}, t), \quad \partial_z w(\mathbf{x}, t) |_{(\mathbf{x}, t) \in \Gamma_{0, T_1}} = \tilde{f}_1(\mathbf{x}, t). \quad (11.14)$$

Thus, our goal below is to construct a numerical method, which would approximately find the functions $w(\mathbf{x}, t), \tau(\mathbf{x})$ for $\mathbf{x} \in \Omega, t \in (0, T_1)$ from conditions (11.12)-(11.14). Suppose that these two functions are approximated. Then the corresponding approximation for the target coefficient $c(\mathbf{x})$ can be easily found via the backwards calculation,

$$c(\mathbf{x}) = |\nabla \tau(\mathbf{x})|^2.$$

11.3 The system of coupled quasilinear elliptic PDEs

Lemma 11.1. *Consider the set of functions*

$$\{t, t^2, \dots, t^n, \dots\} = \{t^n\}_{n=1}^{\infty}. \quad (11.15)$$

The set (11.15) is complete in $L_2(0, T_1)$.

Orthonormalize the set (11.15) using the Gram-Schmidt orthonormalization procedure. Then Lemma 11.1 implies that we obtain a basis $\{P_n(t)\}_{n=1}^{\infty}$ of orthonormal polynomials in $L_2(0, T_1)$ such that

$$P_n(0) = 0, \forall n = 1, 2, \dots \quad (11.16)$$

By (11.16) this is not a set of standard orthonormal polynomials.

Let the integer $N \geq 1$. Approximate the function $w(\mathbf{x}, t)$ as

$$w(\mathbf{x}, t) = \sum_{n=1}^N w_n(\mathbf{x}) P_n(t). \quad (11.17)$$

Here and below we use “=” instead of “ $\approx$ ” for convenience. Substitute (11.17) in the left hand side of (11.12) and assume that the resulting left hand side equals zero.

Remark 11.1. *The assumption that the truncated Fourier-like series (11.17) satisfies equation (11.12) forms our approximate mathematical model. The full analog of Remark 9.1 of subsection 9.3 is valid here.*

We obtain for $\mathbf{x} \in \Omega$:

$$\begin{aligned} \sum_{n=1}^N \Delta w_n(\mathbf{x}) P_n(t) - 2 \sum_{i=1}^3 \tau_{x_i}(\mathbf{x}) \sum_{n=1}^N P'_n(t) \partial_{x_i} w_n(\mathbf{x}) \\ - \Delta \tau(\mathbf{x}) \sum_{n=1}^N P'_n(t) w_n(\mathbf{x}) = 0. \end{aligned} \quad (11.18)$$

By (11.11) we can assume that

$$\sum_{n=1}^N P'_n(0) w_n(\mathbf{x}) = A(\mathbf{x}) > 0, \quad \forall \mathbf{x} \in \bar{\Omega}. \quad (11.19)$$

Set in (11.18) $t = 0$. Hence, we obtain the first elliptic equation,

$$\Delta \tau(\mathbf{x}) + 2 \left[\sum_{i=1}^3 \tau_{x_i} \sum_{n=1}^N P'_n(0) \partial_{x_i} w_n(\mathbf{x}) \right] \left[\sum_{n=1}^N P'_n(0) w_n(\mathbf{x}) \right]^{-1} = 0, \quad \mathbf{x} \in \Omega.$$

We rewrite this equation as

$$\Delta \tau = F_1(\nabla \tau, \nabla \widetilde{W}, \widetilde{W}), \quad \mathbf{x} \in \Omega, \quad (11.20)$$

where $\widetilde{W}(x) = (w_1(\mathbf{x}), \dots, w_N(\mathbf{x}))^T$. Next, for $n = 1, \dots, N$ multiply both sides of (11.18) by $P_n(t)$ and integrate with respect to $t \in (0, T_1)$. Replace in the resulting equation $\Delta \tau$ with the right hand side of (11.20). We obtain

$$\Delta \widetilde{W} = F_2(\nabla \tau, \nabla \widetilde{W}, \widetilde{W}), \quad \mathbf{x} \in \Omega. \quad (11.21)$$

Consider the $(N + 1)$ –dimensional vector function

$$W(\mathbf{x}) = (\tau(\mathbf{x}), \widetilde{W}(\mathbf{x}))^T. \quad (11.22)$$

Thus, (11.14), (11.17) and (11.20)-(11.22) lead to the following boundary value problem for a system of coupled quasilinear elliptic equations

$$\Delta W + F(\nabla W, W) = 0, \mathbf{x} \in \Omega, \quad (11.23)$$

$$W|_{\partial\Omega} = q^0(\mathbf{x}), \partial_z W|_{\Gamma_0} = q^1(\mathbf{x}), \quad (11.24)$$

$$\begin{cases} q^0(\mathbf{x}) = (\tau(\mathbf{x}), q_1^0(\mathbf{x}), \dots, q_N^0(\mathbf{x}))^T, \\ q_n^0(\mathbf{x}) = g \int_0^{T_1} \tilde{f}_0(\mathbf{x}, t) P_n(t) dt, \mathbf{x} \in \partial\Omega, \\ q^1(\mathbf{x}) = (\partial_z \tau(\mathbf{x}), q_1^1(\mathbf{x}), \dots, q_N^1(\mathbf{x}))^T, \\ q_n^1(\mathbf{x}) = \int_0^{T_1} \tilde{f}_1(\mathbf{x}, t) P_n(t) dt, \mathbf{x} \in \Gamma_0. \end{cases}$$

Thus, we have obtained the system (11.23) of coupled quasilinear elliptic PDEs with the overdetermined boundary conditions Cauchy data (11.24). Therefore, we solve below the problem (11.23), (11.24) of finding the $(N+1)$ -dimensional vector function $W \in C_{N+1}^2(\overline{\Omega})$. In fact, however, we find below $W \in H_{N+1}^4(\Omega)$. Recall that by embedding theorem

$$H_{N+1}^4(\Omega) \subset C_{N+1}^2(\overline{\Omega}), \|g\|_{C_{N+1}^2(\overline{\Omega})} \leq C \|g\|_{H_{N+1}^4(\Omega)}, \forall g \in H_{N+1}^4(\Omega), \quad (11.25)$$

where the constant $C = C(\Omega, N)$ depends only on listed parameters.

11.4 Globally strictly convex Tikhonov-like functional

Problem (11.23), (11.24) is similar with problem (9.20)-(9.23) of subsection 9.4. The only two differences are:

1. In (11.24) the Neumann condition $\partial_z W|_{\Gamma_0} = q^1(\mathbf{x})$ are given at the upper side $\Gamma_0 \subset \{z = A\}$ of the cube Ω , see (11.1), (11.2). On the other hand, if following (9.20)-(9.23), then the Neumann data would be given at the lower side $\{-A/2 < x, y < A/2, z = 0\}$ of Ω . We handle this by introducing the Carleman Weight Function $\xi_\lambda(z)$, which is a little bit different from the Carleman Weight Function $\mu_\lambda(z)$ in (9.27) in subsection 9.5. More precisely, choose an arbitrary number $a > 0$. Then

$$\xi_\lambda(z) = e^{2\lambda(z+a)^2}. \quad (11.26)$$

2. Condition (11.19) is not a part of problem (9.20)-(9.23). We handle this via inclusion of an analog of (11.19) in the set of vector functions, on which we seek the solution of problem (11.23), (11.24).

Let $M > 0$ be an arbitrary number and $\sigma \in (0, M/2)$ be a sufficiently small number. We define the following analog $B(M, \sigma) \subset H_{N+1}^4(\Omega)$ of the set $B(M)$ in (9.32) of

subsection 9.5;

$$B(M, \sigma) = \left\{ \begin{array}{l} W = (\tau, w_1, \dots, w_N)^T(\mathbf{x}) \in H_{N+1}^4(\Omega) : \\ \quad : \|W\|_{H_{N+1}^4(\Omega)} < M/2, \\ \quad W(\mathbf{x})|_{\partial\Omega} = q^0(\mathbf{x}), \\ \quad W_z(\mathbf{x})|_{\Gamma_0} = q^1(\mathbf{x}), \\ \quad \sum_{n=1}^N P'_n(0) w_n(\mathbf{x}) > \sigma. \end{array} \right. \quad (11.27)$$

The following analog of the Carleman estimate of Theorem 9.1 (subsection 9.5) is valid with the CWF (11.26):

Theorem 11.1 (Carleman estimate). *Let $\xi_\lambda(z)$ be the function defined in (11.26). Then there exist constants $\lambda_0 = \lambda_0(\Omega, a) \geq 1$ and $C = C(\Omega, a) > 0$ depending only on the domain Ω such that for every function $u \in H^2(\Omega)$ satisfying*

$$u|_{\partial\Omega} = u_z|_{\Gamma_0} = 0$$

and for all $\lambda \geq \lambda_0$ the following Carleman estimate holds:

$$\int_{\Omega} |\Delta u|^2 \xi_\lambda(z) d\mathbf{x} \geq \frac{C}{\lambda} \sum_{i,j=1}^3 \int_{\Omega} |u_{x_i x_j}|^2 \xi_\lambda(z) d\mathbf{x} + C\lambda \int_{\Omega} [|\nabla u|^2 + \lambda^2 |u|^2] \xi_\lambda(z) d\mathbf{x}.$$

We now construct the following analog of the Tikhonov-like functional in subsection 9.5:

$$J_{\lambda,\gamma}(V) = \exp[-2\lambda(A+a)^2] \int_{\Omega} (\Delta W + F(\nabla W, W))^2 \xi_\lambda(z) d\mathbf{x} + \gamma \|W\|_{H_{N+1}^4(\Omega)}^2. \quad (11.28)$$

We solve the following problem:

Minimization Problem. *Minimize functional (11.28) on the set $B(M, \sigma)$ defined in (11.27).*

11.5 Convergence analysis

Functional (11.28) is completely similar with functional (9.31) in subsection 9.5. For this reason, the convergence analysis of functional (11.28) is completely similar with the one of functional (9.31) in subsections 9.5-9.8. Full analogs of theorems 9.2-9.5 are valid here. Furthermore, the PI has figured out that convergence estimates in those analogs of Theorems 9.3 and 9.5 are somewhat stronger than convergence estimates of theorem 4.6 of the original publication [13] about this CIP. Therefore, we omit here the convergence analysis of functional (11.28). Since the number $M > 0$ is arbitrary in (11.28), then the global convergence of the gradient descent method of the minimization of functional (11.28) is guaranteed as long as the starting point of this method belongs to the set $B(M/3, \sigma)$, see Definition 1.1 in section 1.

11.6 Numerical studies

The numerical method of this section was tested on six (6) cases in [3, Chapter 8], [13]. In all tests we took $A = 1$ and

$$\lambda = 1, N = 3, \gamma = 0. \quad (11.29)$$

Numbers (11.29) are the optimal values of these parameters here. Just as in subsections 9.9 and 10.5, we have written differential operators in (11.28) in finite differences and then minimized the resulting discrete functional with respect to the values of the vector function W at grid points. Also, just as in subsections 9.9 and 10.5, we have replaced the $H_{N+1}^4(\Omega)$ –norm in (11.28) with the simpler discrete norm $H_{N+1}^2(\Omega)$, see the second item of Remarks 9.3 in subsection 9.9. The forward problem (11.4), (11.5) was solved via finite differences.

We now present an image of a rather complicated structure. We took

$$c(\mathbf{x}) = \begin{cases} \text{varies in the interval } [0.6, 1.7] & \text{inside of the structure,} \\ 1 & \text{otherwise.} \end{cases} \quad (11.30)$$

5% of random noise was added to the data (11.6). Figures 26 represent true and computed function $c(\mathbf{x})$. One can observe that location and shape of the inclusion in (11.30) are computed accurately. The same is true for values of the function $c(\mathbf{x})$ in (11.30).

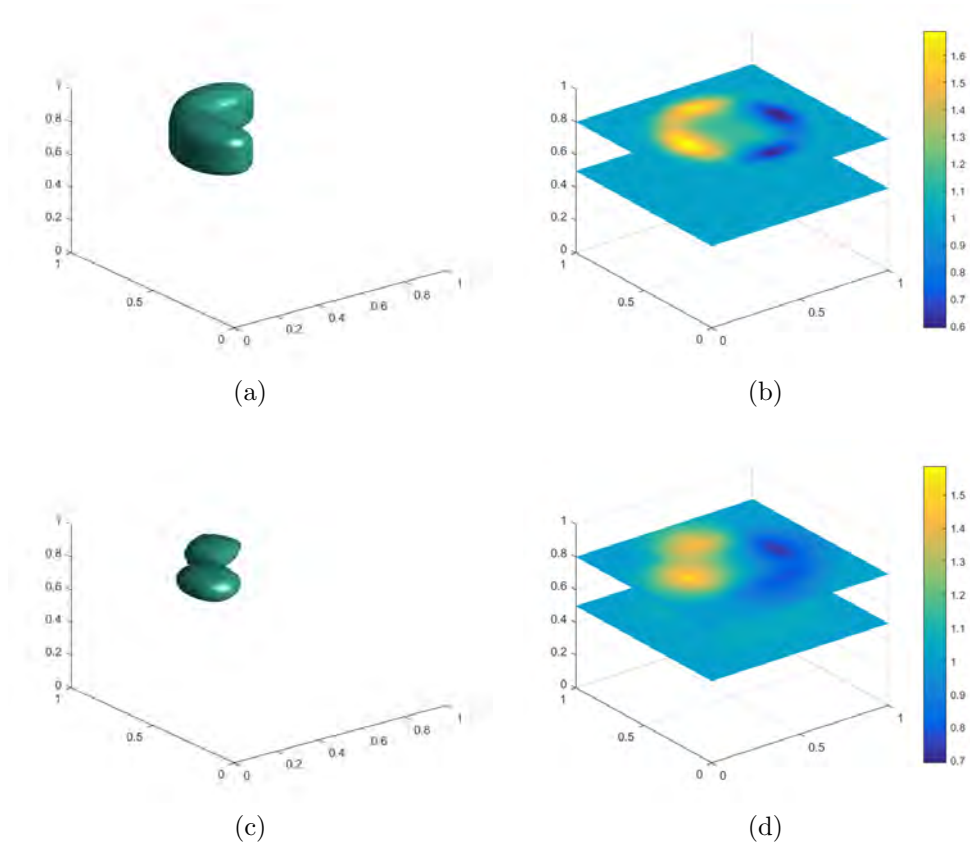


Figure 26: Image of a complicated structure by the version of the globally convergent convexification method of section 11. The function $c(\mathbf{x})$ varies inside of the structure between 0.6 and 1.7, see (11.30). 5% of random noise was added to the data (11.6). a) True 3D image. b) A 2D slice of a). c) Computed 3D image. Both the shape and location are computed accurately. c) A slice of c) by the same horizontal plane as the one of b). Values of $c(\mathbf{x})$ are computed accurately.

12 Solving a Highly Nonlinear 3D CIP for the Wave-Like Equation via a Linear Integral Equation [19]

In this section we explain with some more details the topic of sub-subsection 2.4.3.

12.1 Statements of forward and inverse problems

For the convenience of the reader, we first repeat here some material of sub-subsection 2.4.3: the statement of the CIP as well as of the corresponding linear integral equation. We consider now a 3D CIP with multiple locations of the source and time dependent data measured on either the backscattering plane or on the forward scattering plane, see Figure 13 in sub-subsection 2.4.3 for a schematic diagram of data collection. For brevity we consider in this section only the case of the backscattering data of Figure 13a).

We choose the domain of interest Ω as and the line of sources L_{src} as:

$$\Omega = (-R, R) \times (-R, R) \times (-R, R), \quad (12.1)$$

$$\Gamma = \{\mathbf{x} : -R < x, y < R, z = -R\}, \quad (12.2)$$

$$L_{\text{src}} := \{(\alpha, 0, -R - b) : -a \leq \alpha \leq a\}, \quad (12.3)$$

where $a, b = \text{const.} > 0$ and Γ is the backscattering side on which measurements are conducted.

Let the dielectric constant function $c(\mathbf{x})$ is sufficiently smooth in $\mathbb{R}^3$ and also

$$c(\mathbf{x}) \geq 1 \text{ in } \mathbb{R}^3, \quad (12.4)$$

$$c(\mathbf{x}) = 1 \text{ in } \mathbb{R}^3 \setminus \Omega. \quad (12.5)$$

Consider the following Cauchy problem

$$c(\mathbf{x}) u_{tt} = \Delta u, \quad \mathbf{x} \in \mathbb{R}^3, t > 0, \quad (12.6)$$

$$u(\mathbf{x}, 0) = 0, u_t(\mathbf{x}, 0) = \delta(\mathbf{x} - \mathbf{x}_0). \quad (12.7)$$

Coefficient Inverse Problem. *Let conditions (12.1)-(12.7) be satisfied. Find the function $c(\mathbf{x})$ for $\mathbf{x} \in \Omega$, given the functions $p_0(\mathbf{x}, \mathbf{x}_0, t)$ and $p_1(\mathbf{x}, \mathbf{x}_0, t)$,*

$$u(\mathbf{x}, \mathbf{x}_0, t) = p_0(\mathbf{x}, \mathbf{x}_0, t), \quad (\mathbf{x}, \mathbf{x}_0, t) \in \Gamma \times L_{\text{src}} \times (0, \infty), \quad (12.8)$$

$$\frac{\partial}{\partial n(\mathbf{x})} u(\mathbf{x}, \mathbf{x}_0, t) = p_1(\mathbf{x}, \mathbf{x}_0, t), \quad (\mathbf{x}, \mathbf{x}_0, t) \in \Gamma \times L_{\text{src}} \times (0, \infty), \quad (12.9)$$

where $n = n(\mathbf{x})$ is the outward unit normal vector to Γ at the point $\mathbf{x} \in \Gamma$.

Remark 12.1. *The data (12.8), (12.9) for this CIP are overdetermined ones, i.e. redundant. This is unlike all other CIPs considered above and in all other above cited publication of the PI and his research team about the convexification. Indeed, the number of independent variables in the data $p_0(\mathbf{x}, \mathbf{x}_0, t)$, $p_1(\mathbf{x}, \mathbf{x}_0, t)$ is four (4). On the other hand, the number of independent variables in the unknown coefficient is $3 < 4$.*

12.2 The linear integral equation

It follows from results of [117] that, under some non restrictive conditions imposed on the unknown coefficient $c(\mathbf{x})$, one can apply the Fourier transform with respect to t to the function u ,

$$w(\mathbf{x}, \mathbf{x}_0, k) = \int_0^\infty u(\mathbf{x}, \mathbf{x}_0, t) e^{ikt} dt, k \geq 0. \quad (12.10)$$

Furthermore, one can differentiate the function $w(\mathbf{x}, \mathbf{x}_0, k)$ at least twice with respect to $\mathbf{x}$ and infinitely many times with respect to k , and these derivatives can be calculated under the sign of the integral. In addition, the function w is the solution of the Helmholtz equation with the radiation condition,

$$\Delta_{\mathbf{x}} w + k^2 w + k^2 (c(\mathbf{x}) - 1) w = -\delta(\mathbf{x} - \mathbf{x}_0), \quad (12.11)$$

$$\frac{\partial w}{\partial r} - ikw = o\left(\frac{1}{r}\right), \quad r \rightarrow \infty, r = |\mathbf{x} - \mathbf{x}_0|. \quad (12.12)$$

Denote

$$w_0(\mathbf{x} - \mathbf{x}_0, k) = \frac{\exp(ik|\mathbf{x} - \mathbf{x}_0|)}{4\pi|\mathbf{x} - \mathbf{x}_0|}. \quad (12.13)$$

It is well known that the function $w(\mathbf{x}, \mathbf{x}_0, k)$ satisfies the Lippmann-Schwinger integral equation [57]

$$w(\mathbf{x}, \mathbf{x}_0, k) = w_0(\mathbf{x} - \mathbf{x}_0, k) + k^2 \int_{\Omega} w_0(\mathbf{x} - \boldsymbol{\xi}, k) (c(\boldsymbol{\xi}) - 1) w(\boldsymbol{\xi}, \mathbf{x}_0, k) d\boldsymbol{\xi}. \quad (12.14)$$

Differentiate equation (12.14) twice with respect to k and then set $k = 0$. Denote

$$q(\mathbf{x}) = c(\mathbf{x}) - 1. \quad (12.15)$$

$$w^0(\mathbf{x}, \mathbf{x}_0) = \partial_k^2 w(\mathbf{x}, \mathbf{x}_0, k) |_{k=0}. \quad (12.16)$$

By (12.10) and (12.16)

$$w^0(\mathbf{x}, \mathbf{x}_0) = - \int_0^\infty t^2 u(\mathbf{x}, \mathbf{x}_0, t) dt. \quad (12.17)$$

It follows from (12.14)-(12.17) that

$$\frac{1}{8\pi^2} \int_{\Omega} \frac{q(\boldsymbol{\xi})}{|\mathbf{x} - \boldsymbol{\xi}| |\boldsymbol{\xi} - \mathbf{x}_0|} d\boldsymbol{\xi} = w^0(\mathbf{x}, \mathbf{x}_0) + \frac{|\mathbf{x} - \mathbf{x}_0|}{4\pi}. \quad (12.18)$$

Equation (12.18) with respect to the function $q(\mathbf{x})$ is the desired integral equation.

We now reformulate the CIP (12.8), (12.9) in terms of equation (12.18). Denote

$$g_0(\mathbf{x}, \mathbf{x}_0) = -2\pi \int_0^\infty p_0(\mathbf{x}, \mathbf{x}_0, t) t^2 dt + \frac{|\mathbf{x} - \mathbf{x}_0|}{2}, \quad \text{for } (\mathbf{x}, \mathbf{x}_0) \in \Gamma \times L_{\text{src}}, \quad (12.19)$$

$$g_1(\mathbf{x}, \mathbf{x}_0) = -2\pi \int_0^\infty p_1(\mathbf{x}, \mathbf{x}_0, t) t^2 dt \quad (12.20)$$

$$+ \frac{\partial}{\partial n(\mathbf{x})} \left(\frac{|\mathbf{x} - \mathbf{x}_0|}{4\pi} \right), \text{ for } (\mathbf{x}, \mathbf{x}_0) \in \Gamma \times L_{\text{src}}.$$

Using (12.18)-(12.20), we obtain

$$\frac{1}{4\pi} \int_{\Omega} \frac{q(\boldsymbol{\xi})}{|\mathbf{x} - \boldsymbol{\xi}| |\boldsymbol{\xi} - \mathbf{x}_0|} d\boldsymbol{\xi} = g_0(\mathbf{x}, \mathbf{x}_0), \text{ for } (\mathbf{x}, \mathbf{x}_0) \in \Gamma \times L_{\text{src}}, \quad (12.21)$$

$$\frac{\partial}{\partial n(\mathbf{x})} \left(\frac{1}{4\pi} \int_{\Omega} \frac{q(\boldsymbol{\xi})}{|\mathbf{x} - \boldsymbol{\xi}| |\boldsymbol{\xi} - \mathbf{x}_0|} d\boldsymbol{\xi} \right) = g_1(\mathbf{x}, \mathbf{x}_0), \text{ for } (\mathbf{x}, \mathbf{x}_0) \in \Gamma \times L_{\text{src}}. \quad (12.22)$$

Therefore, we have reduced the CIP (12.8), (12.9) to the following inverse problem:

Inverse Problem. Find the function $q(\mathbf{x})$, $\mathbf{x} \in \Omega$ from functions $g_0(\mathbf{x}, \mathbf{x}_0)$ and $g_1(\mathbf{x}, \mathbf{x}_0)$ given in (12.21)-(12.22).

12.3 A new boundary value problem

Denote

$$v(\mathbf{x}, \mathbf{x}_0) = 2\pi \left[w^0(\mathbf{x}, \mathbf{x}_0) + \frac{|\mathbf{x} - \mathbf{x}_0|}{4\pi} \right].$$

Then

$$v(\mathbf{x}, \mathbf{x}_0) = \frac{1}{4\pi} \int_{\Omega} \frac{q(\boldsymbol{\xi})}{|\mathbf{x} - \boldsymbol{\xi}| |\boldsymbol{\xi} - \mathbf{x}_0|} d\boldsymbol{\xi}. \quad (12.23)$$

Since the function $c(\mathbf{x})$ is sufficiently smooth and since by (12.5) and (12.15) $c(\mathbf{x}) - 1 = q(\mathbf{x}) = 0$ outside of Ω , then applying the Laplace operator to both sides of (12.23), we obtain

$$\Delta_{\mathbf{x}} v(\mathbf{x}, \mathbf{x}_0) = -\frac{q(\mathbf{x})}{|\mathbf{x} - \mathbf{x}_0|}, \quad \mathbf{x} \in \Omega, \mathbf{x}_0 \in L_{\text{src}}. \quad (12.24)$$

Also, by (12.21) and (12.22)

$$v(\mathbf{x}, \mathbf{x}_0) = g_0(\mathbf{x}, \mathbf{x}_0), \quad \mathbf{x} \in \Gamma, \mathbf{x}_0 \in L_{\text{src}}, \quad (12.25)$$

$$\frac{\partial}{\partial n(\mathbf{x})} v(\mathbf{x}, \mathbf{x}_0) = g_1(\mathbf{x}, \mathbf{x}_0), \quad \mathbf{x} \in \Gamma, \mathbf{x}_0 \in L_{\text{src}}. \quad (12.26)$$

Multiply both sides of (12.24) by $|\mathbf{x} - \mathbf{x}_0|$ and denote

$$u(\mathbf{x}, \mathbf{x}_0) = v(\mathbf{x}, \mathbf{x}_0) |\mathbf{x} - \mathbf{x}_0|, \quad \mathbf{x} \in \Omega, \mathbf{x}_0 \in L_{\text{src}}. \quad (12.27)$$

By (12.24)

$$|\mathbf{x} - \mathbf{x}_0| \Delta_{\mathbf{x}} v(\mathbf{x}, \mathbf{x}_0) = -q(\mathbf{x}), \quad \mathbf{x} \in \Omega, \mathbf{x}_0 \in L_{\text{src}}. \quad (12.28)$$

By (12.27) the left hand side of (12.28) can be rewritten as

$$|\mathbf{x} - \mathbf{x}_0| \Delta_{\mathbf{x}} v = \Delta_{\mathbf{x}} (|\mathbf{x} - \mathbf{x}_0| v) - 2\nabla (|\mathbf{x} - \mathbf{x}_0|) \nabla v - \Delta (|\mathbf{x} - \mathbf{x}_0|) v$$

$$= \Delta_{\mathbf{x}} u - 2 \frac{\mathbf{x} - \mathbf{x}_0}{|\mathbf{x} - \mathbf{x}_0|^2} \nabla_{\mathbf{x}} u + \frac{4}{|\mathbf{x} - \mathbf{x}_0|^2} u. \quad (12.29)$$

To improve the stability of our method, we approximately set:

$$\frac{\partial}{\partial n(\mathbf{x})} u(\mathbf{x}, \mathbf{x}_0) = 0 \text{ for } \mathbf{x} \in \partial\Omega \setminus \Gamma, \mathbf{x}_0 \in L_{\text{src}}. \quad (12.30)$$

It follows from (12.23) and (12.27) that condition (12.30) is approximately satisfied if in (12.1) the number R is sufficiently large and the support of $q(\mathbf{x})$ is located far from the boundary $\partial\Omega$.

Next, (12.28) is equivalent with

$$\Delta_{\mathbf{x}} u - \frac{2}{(x - \alpha)^2 + y^2 + z^2} ((x - \alpha) u_x + y u_y + z u_z - 2u) = -\frac{q(x, y, z)}{2\pi}, \quad (12.31)$$

where $(x, y, z) \in \Omega, x_0 \in (-d, d)$. Differentiate both sides of (12.31) with respect to α . We obtain for $(x, y, z) \in \Omega, \alpha \in (-a, a)$:

$$\Delta_{\mathbf{x}} u_{\alpha} - 2 \frac{\partial}{\partial \alpha} \left[\frac{(x - \alpha) u_x + y u_y + z u_z - 2u}{(x - \alpha)^2 + y^2 + z^2} \right] = 0. \quad (12.32)$$

Also, (12.25)-(12.27) and (12.30) lead to the following boundary conditions for the function u :

$$u(\mathbf{x}, \mathbf{x}_0) = g_0(\mathbf{x}, \mathbf{x}_0) |\mathbf{x} - \mathbf{x}_0|, \mathbf{x} \in \Gamma, \mathbf{x}_0 \in L_{\text{src}}, \quad (12.33)$$

$$\frac{\partial}{\partial n(\mathbf{x})} u(\mathbf{x}, \mathbf{x}_0) = |\mathbf{x} - \mathbf{x}_0| g_1(\mathbf{x}, \mathbf{x}_0) \quad (12.34)$$

$$+ g_0(\mathbf{x}, \mathbf{x}_0) \frac{\partial}{\partial n(\mathbf{x})} (|\mathbf{x} - \mathbf{x}_0|), \mathbf{x} \in \Gamma, \mathbf{x}_0 \in L_{\text{src}},$$

$$\frac{\partial}{\partial n(\mathbf{x})} u(\mathbf{x}, \mathbf{x}_0) = 0, \mathbf{x} \in \partial\Omega \setminus \Gamma, \mathbf{x}_0 \in L_{\text{src}}. \quad (12.35)$$

Thus, we now need to find the function $u(\mathbf{x}, \mathbf{x}_0)$ as the solution of boundary value problem (12.32)-(12.35).

12.4 Linear version of the convexification method for boundary value problem (12.32)-(12.35)

Consider the same orthonormal basis $\{\Psi_n(\alpha)\}_{n=0}^{\infty}$ in the space $L_2(-a, a)$ as the one in subsection 9.3. Represent the function $u(\mathbf{x}, \mathbf{x}_0) = u(x, y, z, \alpha)$ as the truncated Fourier series with respect to the basis $\{\Psi_n(\alpha)\}$,

$$u(\mathbf{x}, \alpha) = \sum_{n=0}^{N-1} u_n(\mathbf{x}) \Psi_n(\alpha), \mathbf{x} \in \Omega, \alpha \in (a_1, a_2). \quad (12.36)$$

Substituting (12.36) in (12.32), we obtain for $\mathbf{x} \in \Omega, \mathbf{x}_0 = (\alpha, 0, -R - b) \in L_{\text{src}}$

$$\sum_{n=1}^{N-1} \Delta_{\mathbf{x}} u_n(\mathbf{x}) \Psi'_n(\alpha) \quad (12.37)$$

$$-\frac{\partial}{\partial \alpha} \left\{ \frac{2}{|\mathbf{x} - \mathbf{x}_0|^2} \sum_{n=0}^{N-1} [(\mathbf{x} - \mathbf{x}_0) \nabla u_n(\mathbf{x}) - 2u_n(\mathbf{x})] \Psi_n(\alpha) \right\} = 0.$$

Multiply both sides of (12.37) sequentially by functions $\Psi_m(\alpha)$, $m = 0, \dots, N$ and integrate then with respect to $\alpha \in (-a, a)$. Denote the N -D vector of unknown coefficients in (12.36) as $U(\mathbf{x}) = (u_0, \dots, u_N)^T(\mathbf{x})$. We obtain a system of PDEs,

$$M_N \Delta U + K_1(\mathbf{x}) U_x + K_2(\mathbf{x}) U_y + K_3(\mathbf{x}) U_z + K_0(\mathbf{x}) U = 0, \quad (12.38)$$

where $K_j(\mathbf{x})$ are $N \times N$ matrices, and each element of any of these matrices belongs to the space $C(\overline{\Omega})$.

Since the inverse matrix M_N^{-1} exists (subsection 9.3), then we multiply both sides of (12.38) by the matrix M_N^{-1} from the left. We obtain

$$L(U) = \Delta U + A_1(\mathbf{x}) U_x + A_2(\mathbf{x}) U_y + A_3(\mathbf{x}) U_z + A_0(\mathbf{x}) U = 0, \quad (12.39)$$

where $N \times N$ matrices $A_k(\mathbf{x}) = M_N^{-1} K_j(\mathbf{x})$, $k = 0, 1, 2, 3$. Hence, each element of any matrix $A_k(\mathbf{x})$ belongs to the space $C(\overline{\Omega})$. We see now that the important property of the ellipticity of PDEs in (12.38) is ensured by the existence of the matrix M_N^{-1} .

In addition, it follows from (12.33)-(12.35) that

$$U(\mathbf{x}) = \varphi_0(\mathbf{x}), \quad \frac{\partial}{\partial n(\mathbf{x})} U(\mathbf{x}) = \varphi_1(\mathbf{x}), \quad \mathbf{x} \in \Gamma, \quad (12.40)$$

$$\frac{\partial}{\partial n(\mathbf{x})} U(\mathbf{x}) = 0, \quad \mathbf{x} \in \partial\Omega \setminus \Gamma, \quad (12.41)$$

where boundary vector functions $\varphi_0(\mathbf{x})$, $\varphi_1(\mathbf{x})$ are known.

Assume that there exists a vector function $F(\mathbf{x}) \in H_{2N}^3(\Omega)$ satisfying boundary conditions (12.40), (12.41),

$$F(\mathbf{x}) = \varphi_0(\mathbf{x}), \quad \frac{\partial}{\partial n(\mathbf{x})} F(\mathbf{x}) = \varphi_1(\mathbf{x}), \quad \mathbf{x} \in \Gamma, \quad (12.42)$$

$$\frac{\partial}{\partial n(\mathbf{x})} F(\mathbf{x}) = 0, \quad \mathbf{x} \in \partial\Omega \setminus \Gamma. \quad (12.43)$$

Denote

$$V(\mathbf{x}) = U(\mathbf{x}) - F(\mathbf{x}).$$

Then

$$V(\mathbf{x}) = \frac{\partial}{\partial n(\mathbf{x})} V(\mathbf{x}) = 0, \quad \mathbf{x} \in \Gamma,$$

$$\frac{\partial}{\partial n(\mathbf{x})} V(\mathbf{x}) = 0, \quad \mathbf{x} \in \partial\Omega \setminus \Gamma.$$

Introduce the subspace $H_{2N,0}^3(\Omega)$ of the space $H_{2N}^3(\Omega)$ as:

$$H_{2N,0}^3(\Omega) = \left\{ V \in H_{2N}^3(\Omega) : V(\mathbf{x})|_{\Gamma} = 0, \quad \frac{\partial}{\partial n(\mathbf{x})} V(\mathbf{x})|_{\partial\Omega} = 0 \right\}. \quad (12.44)$$

To solve boundary value problem (12.39)-(12.41), we apply the Quasi-Reversibility Method, see [93] for the first publication about this method as well as [3, Chapter 4] and a survey of the PI [84]. More precisely, we consider the following functional $J_\beta(V)$:

$$J_\beta(V) = \int_{\Omega} |L(V + F)|^2 d\mathbf{x} + \beta \|V\|_{H_N^3(\Omega)}^2, \quad (12.45)$$

where $\beta \in (0, 1)$ is the regularization parameter. We consider the following minimization problem:

Minimization Problem. *Minimize the functional $J_\beta(V)$ on the space $H_{2N,0}^3(\Omega)$.*

12.5 Convergence analysis

As always in the convexification, convergence analysis for this Minimization Problem starts from the Carleman estimate. We use the same Carleman Weight Function as the one in (9.27) (section 9.5)

$$\mu_\lambda(z) = \exp[2\lambda(z - r)^2], \quad z \in [-R, R], \quad r = \text{const.} > R. \quad (12.46)$$

However, the peculiarity here is that the Dirichlet boundary condition $u|_{\partial\Omega} = 0$ on the entire boundary $\partial\Omega$ in (9.28) (subsection 9.5) is replaced now with the Neumann boundary condition $\partial u / \partial n(\mathbf{x})|_{\partial\Omega} = 0$ on the entire boundary $\partial\Omega$ in (12.44). And this peculiarity causes the difference between the Carleman estimate (12.47) and Carleman estimate (9.29) of subsection 9.5: the term with $\|u\|_{H^3(\Omega)}^2$ of (12.47) is not present in (9.29). It is natural, therefore, that proofs of these two Carleman estimates are different.

Theorem 12.1. *There exists constants $\lambda_0 = \lambda_0(R, r) > 1$ and $C = C(R, r) > 0$ depending only on listed parameters such that the following Carleman estimate holds for all functions $u \in H_0^3(\Omega)$ and for all $\lambda \geq \lambda_0$:*

$$\begin{aligned} \int_{\Omega} (\Delta u)^2 \mu_\lambda d\mathbf{x} &\geq \frac{C}{\lambda} \sum_{i,j=1}^3 \int_{\Omega} u_{x_i x_j}^2 \mu_\lambda d\mathbf{x} \\ &+ C\lambda \int_{\Omega} (|\nabla u|^2 + \lambda^2 u^2) \mu_\lambda d\mathbf{x} - C\lambda^3 e^{2\lambda(R-r)^2} \|u\|_{H^3(\Omega)}^2, \end{aligned} \quad (12.47)$$

where $H_0^3(\Omega)$ denotes an obvious analog of the space $H_0^2(\Omega)$ in (9.28) in subsection 9.5.

The variational principle combined with the Riesz theorem leads to Theorem 12.2.

Theorem 12.2. *Let L be the differential operator in (12.39). For any vector function $F(\mathbf{x}) \in H_{2N}^3(\Omega)$ satisfying boundary conditions (12.42), (12.43) and any $\beta > 0$ there exists unique minimizer $V_{\min} \in H_{N,0}^3(\Omega)$ of the functional $J_\beta(V)$, and the following estimate holds*

$$\|V_{\min}\|_{H_N^3(\Omega)} \leq \frac{C}{\sqrt{\beta}} \|F\|_{H_{2N}^3(\Omega)},$$

where the constant $C = C(L, \Omega) > 0$ depends only on listed parameters.

In accordance with the regularization theory [116], the vector function $V_{\min}$ is called the “regularized solution” of the above minimization problem.

Following one of concepts of the regularization theory [116], we assume that there exists exact solution $U^* \in H_N^3(\Omega)$ of equation (12.39) with noiseless boundary data

$\varphi_0^*(\mathbf{x}), \varphi_1^*(\mathbf{x})$ in (12.40) and also boundary condition (12.41). Since such a vector function U^* exists, then there also exists a vector function $F_1^* \in H_N^3(\Omega)$ satisfying noiseless boundary conditions (12.40) as well as the additional zero Neumann boundary conditions (12.41).

Denote

$$V^* = U^* - F^*. \quad (12.48)$$

Then $V^* \in H_{N,0}^3(\Omega)$ and

$$L(V^* + F^*) = 0.$$

Let $\delta \in (0, 1)$ be the level of noise in the data. We assume that

$$\|F - F^*\|_{H_N^2(\Omega)} \leq \delta. \quad (12.49)$$

Let $\varepsilon \in (0, R)$ be an arbitrary number. Denote

$$\Omega_\varepsilon = \{\mathbf{x} = (x, y, z) : -R < x, y < R, z \in (-R, R - \varepsilon)\} \subset \Omega.$$

Theorem 12.3 (convergence rate of regularized solutions). *Let $\beta = \beta(\delta) = \delta^2$. Let $V_{\min, \delta} \in H_{N,0}^3(\Omega)$ be the minimizer of the functional $J_\beta(V)$, which is guaranteed by Theorem 12.2. Denote $U_{\min, \delta} = V_{\min, \delta} + F$. There exists a sufficiently small number $\sigma_0 \in (0, 1)$ and a number $\rho \in (0, 1/2)$ such that for all $\delta \in (0, \delta_0)$ the following convergence rate holds:*

$$\|U_{\min, \delta} - U^*\|_{H_N^2(\Omega_\varepsilon)} \leq B \left(1 + \|U^*\|_{H_N^3(\Omega)} + \|F^*\|_{H_N^3(\Omega)} \right) \delta^\rho,$$

where the constant $B = B(L, \Omega, \varepsilon) > 0$ depends only on listed parameters.

12.6 Numerical study

To computationally simulate the data for the inverse problem, we have numerically solved forward problem (12.6), (12.7) (subsection 12.1) via the implicit scheme of finite differences. Then we have computed the data g_0, g_1 using formulas (12.21), (12.22) (subsection 12.2). For each source position $\mathbf{x}_0 \in L_{\text{src}}$, we have integrated in these formulas for $t \in (0, T)$, where the number $T > 0$ was chosen such that $\|u(\mathbf{x}, \mathbf{x}_0, T)\|_{L^\infty(\Omega)} \leq 10^{-3}$, where the norm is understood in the discrete sense of grid of finite differences. We have found that $T = 10$ is good for all locations of the point source.

We have taken in (12.1), (12.3) and (12.45)

$$R = 0.75, b = -5, a = 1, \beta = 10^{-8},$$

and $N = 6$ in (12.36). The random noise to the data $g_0(\mathbf{x}, \mathbf{x}_0), g_1(\mathbf{x}, \mathbf{x}_0)$ was introduced as:

$$g_{0, \text{noise}}(\mathbf{x}, \mathbf{x}_0) = g_0(\mathbf{x}, \mathbf{x}_0) + \delta \max_{\mathbf{x} \in \Gamma, \mathbf{x}_0 \in L_{\text{src}}} |g_0(\mathbf{x}, \mathbf{x}_0)| \xi_{x_0}, \quad \mathbf{x} \in \Gamma, \mathbf{x}_0 \in L_{\text{src}}, \quad (12.50)$$

$$g_{1, \text{noise}}(\mathbf{x}, \mathbf{x}_0) = g_1(\mathbf{x}, \mathbf{x}_0) + \delta \max_{\mathbf{x} \in \Gamma, \mathbf{x}_0 \in L_{\text{src}}} |g_1(\mathbf{x}, \mathbf{x}_0)| \xi_{x_0}, \quad \mathbf{x} \in \Gamma, \mathbf{x}_0 \in L_{\text{src}}, \quad (12.51)$$

where functions $g_0(\mathbf{x}, \mathbf{x}_0), g_1(\mathbf{x}, \mathbf{x}_0)$ are defined in (12.19), (12.20) (subsection 12.2), δ is the noise level and $\xi_{\mathbf{x}_0}$ is a random variable depending only on the source position $\mathbf{x}_0$ and uniformly distributed on $[-1, 1]$.

In our numerical test we took

$$c(\mathbf{x}) = \begin{cases} 2 & \text{inside the target,} \\ 1 & \text{outside the target.} \end{cases} \quad (12.52)$$

Hence, we had 2:1 inclusion/background contrast. Hence, by (12.15)

$$q(\mathbf{x}) = \begin{cases} 1 & \text{inside the target,} \\ 0 & \text{outside the target.} \end{cases} \quad (12.53)$$

Let $\tilde{q}(\mathbf{x})$ be the function $q(\mathbf{x}) = c(\mathbf{x}) - 1$, computed by our numerical procedure. Then we define the result of our computations $q_{\text{comp}}(\mathbf{x})$ as

$$q_{\text{comp}}(\mathbf{x}) = \begin{cases} \tilde{q}(\mathbf{x}) & \text{if } \tilde{q}(\mathbf{x}) \geq 0.3, \\ 0 & \text{if } \tilde{q}(\mathbf{x}) < 0.3. \end{cases} \quad (12.54)$$

We have found the truncation number 0.3 by the trial and error procedure.

We have tested the reconstruction of a target with a complicated non-convex shape: the shape of the letter ‘C’. Figures 27 depict a sample of reconstruction results.

Remark 12.2. *Artefacts on Figures 27c) and 27d) are due to truncation (12.54). Indeed, this truncation, once established for the reference Test 1 of [19], was uniformly applied to all eight tests of the paper [19]. On the other hand, Results of Test 4 of [19] are depicted on Figures 27.*

13 The First Solution of a 114 Years Old Open Problem [15, 16]

This section is about the so-called Travel Time Tomography Problem (TTTP), also see subsections 2.5, 2.6 and section 14. The PI has solved this problem via two versions of the convexification method in his publications [15, 16] in 2019. Next, the PI and his team solved this problem numerically in the linearized case [17] in 2020, again for the first time in a general 2D case, although the same method works in 3D as well, see section 14 for the linearized case.

Originally TTTP was posed in the pioneering work of a German scientist Herglotz published in 1905 [70] and then by two other German scientists Wiehert and Zoeppritz [119]. They have solved TTTP only in the 1D case, see [107, section 3 of chapter 3] for a detailed description of their solution. Unlike this, the PI’s solution is in n -D, $n = 2, 3$.

The TTTP has immediate applications to the Army needs in the problem of seeing through the walls if using transmitted acoustic/electric signals.

13.1 Preliminaries

Let $\Phi \subset \mathbb{R}^3$ be a bounded domain. Let Γ_0 be a surface/curve and $\Gamma_1 \subseteq \partial\Phi$ be a part of the boundary of the domain Φ . Suppose that there are some point sources $\{x_0\} \subseteq \Gamma_0$ of the acoustic/electric signals. Suppose that at each point $x \in \Gamma_1$ one is measuring the first time $\tau(x, x_0)$ of arrival of those signals from points $x_0 \in \Gamma_0$. The TTTP is: Having $\tau(x, x_0)$ for $x_0 \in \Gamma_0, x \in \Gamma$, find the spatially distributed speed of sound in the acoustic case and spatially distributed dielectric constants in the electric case.

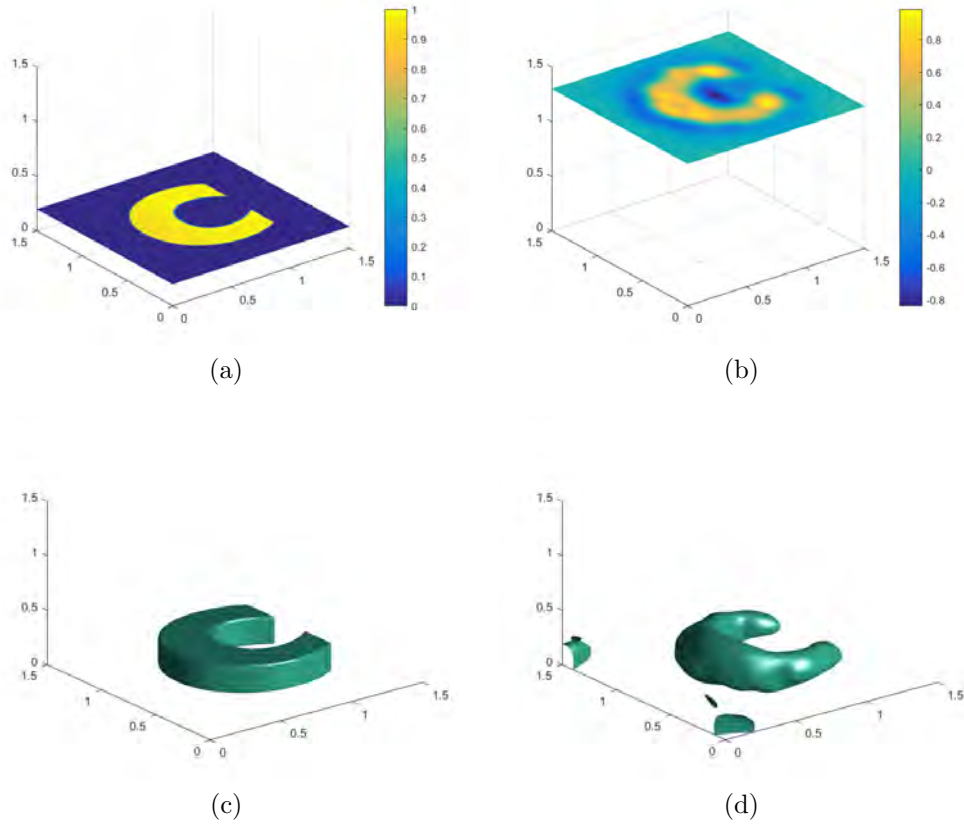


Figure 27: A sample of the image by the method of section 12. We have imaged an inclusion with a complicated non-convex shape of the letter ‘C’. By (12.52) we had 2:1 inclusion/background contrast. Images of the function $q(\mathbf{x}) = c(\mathbf{x}) - 1$ are presented. 5% random noise in the data was introduced as in (12.50), (12.51) with $\delta = 0.05$. a) A 2D slice of the true $q(\mathbf{x})$. b) The 2D slice of the computed function $q_{\text{comp}}(\mathbf{x})$ by the same horizontal plane. c) The 3D image of the true $q(\mathbf{x})$. d) The 3D image of the of the computed function $q_{\text{comp}}(\mathbf{x})$, see Remark 12.2 about artefacts.

Another important application of the TTTP is in Geophysics, in which case Φ is the Earth, $\Gamma_0 \subseteq \partial\Phi$ is the surface where explosions/earthquakes are and $\Gamma \subseteq \partial\Phi$ is the surface where seismic stations are located.

114 years prior publications [15, 16] on TTTP of the PI, in 1905, a German scientist Herglotz and then in 1907 German scientists Wiechert and Zoeppritz have posed and addressed the TTTP in the 1D case [70, 119]. They kept in mind Geophysical applications. Since then many efforts of many mathematicians were invested in solving the TTTP in the n -D case, $n = 2, 3$, see, e.g. [98, 99, 104, ?, 111, 113, 121]. However, *most importantly*, globally convergent numerical methods were not developed. In addition, all works in 3D assume that the data are over determined. In other words, the number m of free variables in the data exceeds the number n of free variables in the unknown coefficient, $m = 4 > n = 3$.

Also, all previous publications work with complete data. On the other hand, we work with incomplete data. Complete data for the TTTP are generated by the point source running along the entire surface of the boundary of the domain of interest Φ . Unlike these,

our point source runs along an interval of a straight line located outside of the domain of interest. This means that the data are incomplete.

Figure 14 displays the schematic diagram of the sources/detectors configuration in our case. Curves indicate geodesic lines. The purpose of Figures 15 is to illustrate the incompleteness of our data for the simplest case when the geodesic lines are straight lines.

Two recent publications of the PI [15, 16] are the first ones where:

1. The globally convergent numerical methods for the TTTP were developed.
2. The data are non overdetermined ones in the n -D case, $n = 2, 3$.
3. The data are incomplete, see Figures 15.
4. Lipschitz stability and uniqueness theorem is proven under conditions of items 2,3.

In this section we briefly describe the result of [15].

13.2 Statement of the problem

Below $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$. Let numbers $A, \sigma = \text{const.} > 0$. Define the rectangular prism $\Omega \subset \mathbb{R}^3$ as

$$\Omega = \{\mathbf{x} = (x, y, z) : x, y \in (0, 1), z \in (A, A + \sigma)\}. \quad (13.1)$$

Denote parts of the boundary $\partial\Omega$ as

$$B_A = \{\mathbf{x} = (x, y, z) : x, y \in (0, 1), z = A\}, \quad (13.2)$$

$$B_{A+\sigma} = \{\mathbf{x} = (x, y, z) : x, y \in (0, 1), z = A + \sigma\}, \quad (13.3)$$

$$\Gamma = \partial\Omega \setminus (B_A \cup B_{A+\sigma}). \quad (13.4)$$

Let $n(\mathbf{x})$ be the refractive index of the medium at the point $\mathbf{x}$. Hence, $c(\mathbf{x}) = 1/n(\mathbf{x})$ is the sound speed. Denote $m(\mathbf{x}) = n^2(\mathbf{x})$. Let the number $m_0 > 0$ be given. We impose the following assumptions on the function $m(\mathbf{x})$:

$$m(\mathbf{x}) \geq m_0, \quad \mathbf{x} \in \mathbb{R}^3, \quad (13.5)$$

$$m(\mathbf{x}) = 1, \quad \mathbf{x} \in \{z < A\}, \quad (13.6)$$

$$m \in C^2(\mathbb{R}^3), \quad (13.7)$$

$$m_z(\mathbf{x}) \geq 0, \quad \mathbf{x} \in \overline{\Omega}. \quad (13.8)$$

Remark 13.1. We note that the monotonicity condition (13.8) is not an overly restrictive one. Indeed, it can also be found in the end of section 2 of chapter 3 of the book [107]: see formulas (3.24) and (3.24') there. Also, an analogous monotonicity condition was actually imposed in the 1D case in the originating works of Herglotz and Wiechert and Zoeppritz [70, 119], see section 3 of chapter 3 of [107] for a description of their method. We also refer to Figures 5 and 10 in [118] for some geophysical information.

The function $m(\mathbf{x})$ generates the Riemannian metric

$$d\tau = \sqrt{m(\mathbf{x})} |d\mathbf{x}|, \quad |d\mathbf{x}| = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}. \quad (13.9)$$

The travel time from the point $\mathbf{x}_0 \in \mathbb{R}^3$ (source) to the point $\mathbf{x} \in \mathbb{R}^3$ (receiver) is [?]

$$\tau(\mathbf{x}, \mathbf{x}_0) = \int_{\Gamma(\mathbf{x}, \mathbf{x}_0)} \sqrt{m(\mathbf{x}(s))} ds, \quad (13.10)$$

where $\Gamma(\mathbf{x}, \mathbf{x}_0)$ is the geodesic line connecting points $\mathbf{x}$ and $\mathbf{x}_0$ and ds is the euclidean arc length. We assume that the source $\mathbf{x}_0$ runs along an interval L_{src} of a straight line located in the plane $\{z = 0\}$,

$$L_{\text{src}} = \{\mathbf{x} = (x, y, z) : x = \alpha \in (0, 1), y = 1/2, z = 0\}. \quad (13.11)$$

Hence, $\mathbf{x}_0 = \mathbf{x}_\alpha = (\alpha, 1/2, 0)$, $\alpha \in (0, 1)$. Let $\tau(\mathbf{x}, \alpha)$ be the travel time between points $\mathbf{x}$ and $\mathbf{x}_\alpha = (\alpha, 1/2, 0)$. Thus, we denote $\tau(\mathbf{x}, \alpha) = \tau(\mathbf{x}, \mathbf{x}_\alpha)$. We denote $\Gamma(\mathbf{x}, \alpha)$ the geodesic line connecting points $\mathbf{x}$ and $\mathbf{x}_\alpha$. It is well known [107] that the function $\tau(\mathbf{x}, \alpha)$ satisfies eikonal equation as the function of $\mathbf{x}$,

$$\tau_x^2 + \tau_y^2 + \tau_z^2 = m(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^3, \quad (13.12)$$

$$\tau(\mathbf{x}, \alpha) = O(|\mathbf{x} - \mathbf{x}_\alpha|), \quad \text{as } |\mathbf{x} - \mathbf{x}_\alpha| \rightarrow 0.$$

Just as in sub-subsection 2.6.2, we again assume in this section the regularity property of geodesic lines.

Regularity of Geodesic Lines. *The function $\tau(\mathbf{x}, \alpha) \in C^2(\mathbb{R}^3 \times [0, 1])$. For any pair of points $(\mathbf{x}, \mathbf{x}_\alpha) \in \overline{\Omega} \times L_{\text{src}}$ there exists unique geodesic line $\Gamma(\mathbf{x}, \alpha)$ connecting these two points and $\Gamma(\mathbf{x}, \alpha) \cap B_A \neq \emptyset$. In addition, if any geodesic line, which starts at a point $\mathbf{x}_\alpha \in L$, intersects B_A , then it intersects it at a single point. Also, it does not go “downwards” in the z -direction, but instead intersects $\partial\Omega \setminus B_A$ at another single point, see (2.51)-(2.54). In addition, after intersecting $\partial\Omega \setminus B_A$, this line does not “come back” in the domain Ω but rather goes away from this domain. In other words, this line is not reflected back from any point of its intersection with $\partial\Omega$.*

Travel Time Tomography Problem (TTTP). *Suppose that the function $m(\mathbf{x})$ satisfies conditions (13.5)-(13.8). Assume that the following function $f(\mathbf{x}, a)$ is given:*

$$\tau(\mathbf{x}, \alpha) = f(\mathbf{x}, \alpha), \quad \forall \mathbf{x} \in \Gamma \cup B_{A+\sigma}, \forall \alpha \in (0, 1), \quad (13.13)$$

also, see Figure 13. Determine the function $m(\mathbf{x})$ for $\mathbf{x} \in \Omega$.

In other words, by (13.11) and (13.13) the data for the travel time are given for all sources running along the line interval L_{src} and for the part $\Gamma \cup B_{A+\sigma}$ of the boundary $\partial\Omega$. Hence, the data (13.13) are both non overdetermined and incomplete.

13.3 Estimates of $\tau_z^2(\mathbf{x}, \alpha)$, $\tau_z(\mathbf{x}, \alpha)$ from the below

Lemma 13.1. *Assume that conditions (13.5)-(13.8) hold. Then*

$$\tau_z^2(\mathbf{x}, \alpha) \geq \frac{A^2}{A^2 + 2}, \quad \forall \mathbf{x} \in \overline{\Omega}, \forall \alpha \in [0, 1]. \quad (13.14)$$

Also,

$$\tau_z(\mathbf{x}, \alpha) > 0, \quad \forall \mathbf{x} \in \overline{\Omega} \cup \{z \in (0, A]\}, \forall \alpha \in [0, 1].$$

Thus,

$$\tau_z(\mathbf{x}, \alpha) \geq \frac{A}{\sqrt{A^2 + 2}}, \quad \forall \mathbf{x} \in \overline{\Omega}, \forall \alpha \in [0, 1]. \quad (13.15)$$

13.4 A nonlinear integral differential equation

Using Lemma 13.1, change variables as

$$u(\mathbf{x}, \alpha) = \tau_z^2(\mathbf{x}, \alpha), \mathbf{x} \in \Omega, \alpha \in (0, 1).$$

By (13.15)

$$\tau_z(\mathbf{x}, \alpha) = \sqrt{u(\mathbf{x}, \alpha)}, \mathbf{x} \in \Omega. \quad (13.16)$$

Hence, (13.13) and (13.16) imply that for all $\alpha \in (0, 1), (x, y, z) \in \Omega$

$$\begin{aligned} \tau(x, y, z, \alpha) &= - \int_z^{A+\sigma} \sqrt{u(x, y, t, \alpha)} dt + f(x, y, A + \sigma, \alpha), \\ \tau_x(x, y, z, \alpha) &= - \int_z^{A+\sigma} \left(\frac{u_x}{2\sqrt{u}} \right) (x, y, t, \alpha) dt + f_x(x, y, A + \sigma, \alpha), \\ \tau_y(x, y, z, \alpha) &= - \int_z^{A+\sigma} \left(\frac{u_y}{2\sqrt{u}} \right) (x, y, t, \alpha) dt + f_y(x, y, A + \sigma, \alpha). \end{aligned}$$

Substituting these in eikonal equation (13.12), we obtain the following equation for $(x, y, z) \in \Omega, \alpha \in (0, 1)$:

$$\begin{aligned} &u(x, y, z, \alpha) \\ &+ \left[- \int_z^{A+\sigma} \left(\frac{u_x}{2\sqrt{u}} \right) (x, y, t, \alpha) dt + f_x(x, y, A + \sigma, \alpha) \right]^2 \\ &+ \left[- \int_z^{A+\sigma} \left(\frac{u_y}{2\sqrt{u}} \right) (x, y, t, \alpha) dt + f_y(x, y, A + \sigma, \alpha) \right]^2 = m(x, y, z). \end{aligned} \quad (13.17)$$

Differentiating this equation with respect to α and using $\partial_\alpha m(x, y, z) \equiv 0$, we obtain for $(x, y, z) \in \Omega, \alpha \in (0, 1)$

$$\begin{aligned} &u_\alpha(x, y, z, \alpha) \\ &+ \frac{\partial}{\partial \alpha} \left[- \int_z^{A+\sigma} \left(\frac{u_x}{2\sqrt{u}} \right) (x, y, t, \alpha) dt + f_x(x, y, A + \sigma, \alpha) \right]^2 \\ &+ \frac{\partial}{\partial \alpha} \left[- \int_z^{A+\sigma} \left(\frac{u_y}{2\sqrt{u}} \right) (x, y, t, \alpha) dt + f_y(x, y, A + \sigma, \alpha) \right]^2 = 0. \end{aligned} \quad (13.18)$$

13.5 Boundary value problem for a system of coupled integral differential equations

Using (13.14), denote for $(x, y, \alpha) \in [0, 1]^3$

$$u_0(\mathbf{x}, \alpha) = u(x, y, A, \alpha) = \frac{A^2}{(x - \alpha)^2 + (y - 1/2)^2 + A^2} \geq \frac{A^2}{A^2 + 2}.$$

We seek the function $u(\mathbf{x}, \alpha)$ in the form

$$u(\mathbf{x}, \alpha) = u_0(\mathbf{x}, \alpha) + v(\mathbf{x}, \alpha), \mathbf{x} \in \overline{\Omega}, \alpha \in [0, 1],$$

$$v(x, y, A, \alpha) = 0, (x, y, \alpha) \in [0, 1]^3,$$

where the function $v(\mathbf{x}, \alpha)$ is unknown for $\mathbf{x} \in \Omega, \alpha \in (0, 1)$. We need to obtain zero boundary condition at Γ for a function associated with the function v . To do this, we assume first that there exists a function $g(\mathbf{x}, \alpha) \in H^1(\Omega)$ for every $\alpha \in [0, 1]$ such that

$$g(\mathbf{x}, \alpha) = (f_z)^2(\mathbf{x}, \alpha) - u_0(\mathbf{x}, \alpha), \forall \mathbf{x} \in \Gamma, \forall \alpha \in [0, 1],$$

$$g(x, y, A, \alpha) = 0.$$

Introduce the function $w(\mathbf{x}, \alpha)$,

$$w(\mathbf{x}, \alpha) = v(\mathbf{x}, \alpha) - g(\mathbf{x}, \alpha), \mathbf{x} \in \overline{\Omega}, \alpha \in [0, 1].$$

Then

$$u(\mathbf{x}, \alpha) = u_0(\mathbf{x}, \alpha) + w(\mathbf{x}, \alpha) + g(\mathbf{x}, \alpha), \mathbf{x} \in \overline{\Omega}, \alpha \in [0, 1],$$

$$w(\mathbf{x}, \alpha) = 0, \forall \mathbf{x} \in \Gamma, \forall \alpha \in [0, 1], \quad (13.19)$$

$$w(x, y, A, \alpha) = 0, (x, y, \alpha) \in [0, 1]^3. \quad (13.20)$$

Let $\{\Psi_n(\alpha)\}_{n=0}^\infty, \alpha \in (0, 1)$ be the special orthonormal basis in $L_2(0, 1)$ which was introduced in subsection 9.3. We assume that both functions $w(\mathbf{x}, \alpha)$ and $g(\mathbf{x}, \alpha)$ have the form of the truncated Fourier series with respect to this basis:

$$w(\mathbf{x}, \alpha) = \sum_{n=0}^{N-1} w_n(\mathbf{x}) \Psi_n(\alpha), \mathbf{x} \in \overline{\Omega}, \alpha \in [0, 1], \quad (13.21)$$

$$g(\mathbf{x}, \alpha) = \sum_{n=0}^{N-1} g_n(\mathbf{x}) \Psi_n(\alpha), \mathbf{x} \in \overline{\Omega}, \alpha \in [0, 1]. \quad (13.22)$$

Here, coefficients $w_n(\mathbf{x})$ are unknown and coefficients $g_n(\mathbf{x})$ are known. And similarly for $w_x, w_y, w_{x\alpha}, w_{y\alpha}$ and the same derivatives of the function g . Furthermore, we assume that these functions, being substituted in equation (13.18), give us zero in its right hand side for $\mathbf{x} \in \Omega, \alpha \in (0, 1)$. By (13.19)-(13.22)

$$w_n(\mathbf{x})|_{\Gamma} = 0, w_n(x, y, A) = 0. \quad (13.23)$$

Denote

$$W(\mathbf{x}) = (w_1, \dots, w_N)^T(\mathbf{x}), \quad (13.24)$$

$$G(\mathbf{x}) = (g_1, \dots, g_N)^T(\mathbf{x}), \quad (13.25)$$

Let

$$f(x, y, A + \sigma, \alpha) = \sum_{n=0}^{N-1} f_n(x, y, A + \sigma) \Psi_n(\alpha), (x, y, \alpha) \in (0, 1)^3, \quad (13.26)$$

$$F(x, y, A + \sigma) = (f_0, f_1, \dots, f_{N-1})^T(x, y, A + \sigma), (x, y) \in (0, 1)^2, \quad (13.27)$$

Keeping in mind (13.21)-(13.23), we define the spaces $C_N^1(\bar{\Omega})$, $C_{N,0}^1(\bar{\Omega})$ of N -D vector functions $W(\mathbf{x})$ as

$$C_{x,y,N}^1(\bar{\Omega}) = \left\{ \begin{array}{l} W(\mathbf{x}) : \|W\|_{C_{x,y,N}^1(\bar{\Omega})} \\ = \max_{n \in [0, N-1]} \left(\|w_n\|_{C(\bar{\Omega})} + \|w_{nx}\|_{C(\bar{\Omega})} + \|w_{ny}\|_{C(\bar{\Omega})} \right) < \infty \end{array} \right\},$$

$$C_{x,y,N,0}^1(\bar{\Omega}) = \{W \in C_{x,y,N}^1(\bar{\Omega}) : W|_{\Gamma} = W(x, y, A) = 0\}.$$

Keeping in mind (13.21)-(13.27), substitute functions w , g and their corresponding first derivatives with respect to x, y, α in equation (13.18). Next, multiply the resulting equation sequentially by functions $\Psi_n(\alpha)$, $n = 1, \dots, N$ and integrate with respect to $\alpha \in (0, 1)$. Then multiply both sides of obtained system of nonlinear integral differential equations by the matrix S_N^{-1} , where the matrix S_N was introduced in section 3. We obtain

$$W(\mathbf{x}) = M_N^{-1} P(W, W_x, W_y, G, G_x, G_y, F_x, F_y, \mathbf{x}), \mathbf{x} \in \Omega, \quad (13.28)$$

$$W|_{\Gamma} = W(x, y, A) = 0, \quad (13.29)$$

where P is the N -D vector function,

$$P = (P_1, \dots, P_N)^T(W, W_x, W_y, G_x, G_y, F_x, F_y, \mathbf{x}), \mathbf{x} \in \Omega, \quad (13.30)$$

$$P_n(W, W_x, W_y, G_x, G_y, F_x, F_y, \mathbf{x}) = \quad (13.31)$$

$$\begin{aligned} &= \int_0^1 \Psi_n(\alpha) \frac{\partial}{\partial \alpha} \left(- \int_z^{A+\sigma} \frac{u_{0x} + w_x + g_x}{2\sqrt{u_0 + w + g}} dt + f_x(x, y, A, \alpha) \right)^2 d\alpha \\ &+ \int_0^1 \Psi_n(\alpha) \frac{\partial}{\partial \alpha} \left(- \int_z^{A+\sigma} \frac{u_{0y} + w_y + g_y}{2\sqrt{u_0 + w + g}} dt + f_y(x, y, A, \alpha) \right)^2 d\alpha. \end{aligned}$$

Thus, we have obtained the desired boundary value problem (13.28)-(13.31) for the system of nonlinear coupled integral differential equations. Below we focus on this problem.

13.6 The positivity of the function $(u_0 + w + g)(\mathbf{x}, \alpha)$

It follows from (13.31) that we need the function $(u_0 + w + g)(\mathbf{x}, \alpha)$ to be positive. We discuss this issue in the current section.

Define the set of functions K as:

$$K = \{w(\mathbf{x}, \alpha) := (w + g)(\mathbf{x}, \alpha) > 0, (13.21) \text{ holds, } W \in C_{x,y,N,0}^1(\overline{\Omega})\}, \quad (13.32)$$

where $(\mathbf{x}, \alpha) \in \overline{\Omega} \times [0, 1]$. Then

$$(u_0 + w + g)(\mathbf{x}, \alpha) \geq \frac{A^2}{A^2 + 2}, \forall w \in K, (\mathbf{x}, \alpha) \in \overline{\Omega} \times [0, 1]. \quad (13.33)$$

To obtain a sufficient condition guaranteeing (13.32) in terms of the vector function W , we need Lemma 13.3.

Consider the function $q(\alpha)$ in the following form:

$$q(\alpha) = \sum_{n=0}^{N-1} q_n \Psi_n(\alpha), \quad q_n = \int_0^1 q(\alpha) \Psi_n(\alpha) d\alpha. \quad (13.34)$$

Below we need to impose such a sufficient condition on the vector of coefficients

$q^N = (q_0, \dots, q_{N-1})^T$ in the Fourier expansion (13.34) which would guarantee that the function $q(\alpha)$ is positive for all $\alpha \in [0, 1]$. Recall that in subsection 9.3 we have formed the orthonormal basis $\{\Psi_n(\alpha)\}_{n=0}^\infty$ via the Gram-Schmidt orthogonalization procedure from functions $\{\varphi_n(\alpha)\}_{n=0}^\infty = \{\alpha^n e^\alpha\}_{n=0}^\infty$. Consider vector functions $\varphi^N(\alpha) = (\varphi_1, \dots, \varphi_N)^T(\alpha)$ and $\Psi^N(\alpha) = (\Psi_1, \dots, \Psi_{N-1})^T(\alpha)$.

Lemma 13.2. *Let the $N \times N$ matrix X_N transforms the vector function $\xi^N(\alpha)$ in the vector function $\Psi^N(\alpha)$ via the Gram-Schmidt orthonormalization procedure, i.e. $X_N \cdot \xi^N(\alpha) = \Psi^N(\alpha)$. Let the matrix X_N^T be the transpose of X_N . Let $X_N^T q^N = \tilde{q}^N = (\tilde{q}_0, \dots, \tilde{q}_{N-1})^T$. Suppose that all numbers $\tilde{q}_0, \dots, \tilde{q}_{N-1} > 0$. Then in (13.34) the function $q(\alpha) > 0$ for all $\alpha \in [0, 1]$.*

Lemma 13.3. *Let (13.21) and (13.22) hold. Consider the vector function $v^N(\mathbf{x}) = (w_1 + g_1, w_2 + g_2, \dots, w_N + g_N)^T(\mathbf{x})$. Let X_N be the $N \times N$ matrix of Lemma 13.2. Consider the vector function $\tilde{v}^N(\mathbf{x}) = X_N^T \cdot v^N(\mathbf{x})$. Let $\tilde{v}^N(\mathbf{x}) = (v_1, \dots, v_N)^T(\mathbf{x})$. Suppose that all functions $v_n(\mathbf{x}) > 0$ in $\overline{\Omega}$. Then the function $w \in K$ and, therefore, (13.33) holds. Also, the set K is convex.*

13.7 Problem (13.28)-(13.31) in the form of partial finite differences

We now rewrite equation (13.31) in the form of finite differences with respect to variables x, y while keeping the continuous derivative with respect to z . For brevity we keep the same grid step size $h > 0$ in both directions x, y . Consider partitions of the intervals $x \in (0, 1), y \in (0, 1)$ in small subintervals of the same length h with $B = 1/h$ and corresponding semidiscrete sub-domains of the domains Ω and $\overline{\Omega}$,

$$0 = x_0 < x_1 < \dots < x_{B-1} < x_B = 1, x_i - x_{i-1} = h, \quad (13.35)$$

$$0 = y_0 < y_1 < \dots < y_{B-1} < y_B = 1, y_i - y_{i-1} = h, \quad (13.36)$$

$$\Omega^h = \left\{ \mathbf{x}^h(z) = \{(x_i, y_j, z)\}_{i,j=1}^{B-1}, z \in (A, A + \sigma) \right\}. \quad (13.37)$$

$$\Omega_1^h = \left\{ \mathbf{x}^h(z) = \{(x_i, y_j, z)\}_{i,j=0}^B, z \in (A, A + \sigma) \right\}. \quad (13.38)$$

Hence, $\mathbf{x}^h(z)$ is a z -dependent vector. Its dimension is $(B-2)^2$ in the case of Ω^h and $(B+1)^2$ in the case of Ω_1^h . By (13.35)-(13.38) only those points $(x_i, y_j, z) \in \Omega^h$, which are corresponding interior points of the domain Ω . On the other hand, in addition to points of Ω^h , the semidiscrete domain Ω_1^h contains boundary points which belong to the part Γ of the boundary $\partial\Omega$. We assume below that

$$h \in [h_0, 1), h_0 = \text{const.} \in (0, 1). \quad (13.39)$$

Remark 13.2. *Unlike classical forward problems for PDEs, we do not let the grid step size h tend to zero. This is typical for numerical methods for many inverse problems: due to their ill-posed nature, see, e.g. [6, 7].*

Consider the N -D vector function $Q(\mathbf{x}) = (Q_1, \dots, Q_N)^T(\mathbf{x})$ with $Q_n \in C(\overline{\Omega})$. We denote $Q^h(\mathbf{x}^h(z))$ the trace of this vector function on the set $\overline{\Omega}_1^h$. Thus,

$$Q^h(\mathbf{x}^h(z)) = (Q_0^h, \dots, Q_{N-1}^h)^T(\mathbf{x}^h(z)) = \left((Q_0^{i,j}(z))_{i,j=0}^B, \dots, (Q_{N-1}^{i,j}(z))_{i,j=0}^B \right)^T \quad (13.40)$$

is the matrix depending on the variable $z \in [A, A + \sigma]$. Here $Q_k^{i,j}(z) = Q_k(x_i, y_j, z)$, where $k = 1, \dots, N$. Hence, the finite difference analogs of functions $w(\mathbf{x}, \alpha)$, $g(\mathbf{x}, \alpha)$, $f(x, y, A + \sigma)$ are:

$$w^h(\mathbf{x}^h(z), \alpha) = \sum_{n=1}^N w_n^h(\mathbf{x}^h(z)) \Psi_n(\alpha), \quad \mathbf{x}^h(z) \in \overline{\Omega}_1^h, \alpha \in [0, 1],$$

$$g^h(\mathbf{x}^h(z), \alpha) = \sum_{n=1}^N g_n^h(\mathbf{x}^h(z)) \Psi_n(\alpha), \quad \mathbf{x}^h(z) \in \overline{\Omega}_1^h, \alpha \in [0, 1].$$

$$f^h(\mathbf{x}^h(A + \sigma), \alpha) = \sum_{n=1}^N f_n^h(\mathbf{x}^h(A + \sigma)) \Psi_n(\alpha), \quad \mathbf{x}^h(A + \sigma) \in \overline{\Omega}_1^h, \alpha \in [0, 1].$$

Next, the finite difference analogs of matrices W , G and F are

$$W^h(\mathbf{x}^h(z)) = (w_1^h, \dots, w_N^h)^T(\mathbf{x}^h(z)), \quad \mathbf{x}^h(z) \in \overline{\Omega}_1^h,$$

$$G^h(\mathbf{x}^h(z)) = (g_1^h, \dots, g_N^h)^T(\mathbf{x}^h(z)), \quad \mathbf{x}^h(z) \in \overline{\Omega}_1^h,$$

$$F^h(\mathbf{x}^h(z)) = (f_1^h, f_2^h, \dots, f_N^h)^T(\mathbf{x}^h(z)), \quad z = A + \sigma, \mathbf{x}^h(A + \sigma) \in \overline{\Omega}_1^h.$$

For an arbitrary number $z \in [A, A + \sigma]$ denote

$$\Omega_z^h = \left\{ \mathbf{x}^h(z) = (x_i, y_j, z)_{i,j=1}^{B-1}, \quad z \text{ is fixed} \right\}.$$

We introduce semidiscrete functional spaces for matrices like $Q^h(\mathbf{x}(z))$ in (13.40),

$$C_N^h(\overline{\Omega}_z^h) = \left\{ Q^h(\mathbf{x}^h(z)) : \|Q^h\|_{C_N^h(\overline{\Omega}_z^h)}(z) = \max_{k \in [1, N]} \max_{i,j=1, \dots, B-1} |Q_k^{i,j}(z)| < \infty \right\},$$

$$\begin{aligned}
C_N^h(\bar{\Omega}^h) &= \left\{ Q^h(\mathbf{x}^h(z)) : \|Q^h\|_{C_N^h(\bar{\Omega}^h)} = \max_{z \in [A, A+\sigma]} \|Q^h\|_{C_N^h(\bar{\Omega}_z^h)}(z) < \infty \right\}, \\
C_N^h(\bar{\Omega}_{1,z}^h) &= \left\{ Q^h(\mathbf{x}^h(z)) : \|Q^h\|_{C^h(\bar{\Omega}_{1,z}^h)}(z) = \max_{k \in [0, N-1]} \max_{i,j=0,\dots,B} |Q_k^{i,j}(z)| < \infty \right\}, \\
C_N^h(\bar{\Omega}_1^h) &= \left\{ Q^h(\mathbf{x}^h(z)) : \|Q^h\|_{C_N^h(\bar{\Omega}_1^h)} = \max_{z \in [A, A+\sigma]} \|Q^h\|_{C_N^h(\bar{\Omega}_{1z}^h)}(z) < \infty \right\}, \\
C_{N,0}^h(\bar{\Omega}_1^h) &= \left\{ Q^h(\mathbf{x}^h(z)) \in C_N^h(\bar{\Omega}_1^h) : Q^h(\mathbf{x}^h(z)) = 0 \text{ for } \mathbf{x}^h(z) \in \Gamma \cup \{z = A\} \right\}, \\
L_{2,N}^h(\Omega_z^h) &= \left\{ Q^h(\mathbf{x}^h(z)) : \|Q^h\|_{L_2^h(\Omega_z^h)}^2(z) = h^2 \sum_{k=0}^{N-1} \sum_{i,j=1}^{B-1} (Q_k^{i,j}(z))^2 < \infty \right\}, \\
L_{2,N}^h(\Omega^h) &= \left\{ Q^h(\mathbf{x}^h(z)) : \|Q^h\|_{L_{2,N}^h(\Omega^h)}^2 = \int_A^{A+\sigma} \|Q^h\|_{L_{2,N}^h(\Omega_\theta^h)}^2(z) dz < \infty \right\}, \\
H_{0,N}^{1,h}(\Omega_1^h) &= \left\{ \begin{array}{l} Q^h(\mathbf{x}^h(z)) : Q^h(\mathbf{x}^h(z))|_{\Gamma \cup B_A} = 0, \\ \|Q^h\|_{H_{0,N}^{1,h}(\Omega_1^h)}^2 = \\ = \int_A^{A+\sigma} [\|Q^h\|_{L_{2,N}^h(\Omega_z^h)}^2(z) + \|\partial_z Q^h\|_{L_{2,N}^h(\Omega_z^h)}^2(z)] dz < \infty \end{array} \right\}.
\end{aligned}$$

We approximate x, y derivatives of the vector function $W(\mathbf{x})$ via central finite differences. It is convenient to write this in the equivalent form for the matrix function $W^h(\mathbf{x})$ as

$$\begin{aligned}
W_{k,x}^{i,j}(z) &= \frac{W_k^{i+1,j}(z) - W_k^{i-1,j}(z)}{2h}; i, j = 1, \dots, B-1, \\
W_{k,y}^{i,j}(z) &= \frac{W_k^{i,j+1}(z) - W_k^{i,j-1}(z)}{2h}, i, j = 1, \dots, B-1, \\
W_x^h(\mathbf{x}^h(z)) &= \left((W_{0,x}^{i,j}(z))_{i,j=0}^B, \dots, (W_{B,x}^{i,j}(z))_{i,j=0}^B \right)^T, \\
W_y^h(\mathbf{x}^h(z)) &= \left((W_{0,y}^{i,j}(z))_{i,j=0}^B, \dots, (W_{B,y}^{i,j}(z))_{i,j=0}^B \right)^T.
\end{aligned}$$

We obtain that there exists a constant $C_2 = C_2(h_0, N, \Omega^h) > 0$ depending only on listed parameters such that

$$\|Q_x^h\|_{C_N^h(\bar{\Omega}^h)}, \|Q_y^h\|_{C_N^h(\bar{\Omega}^h)} \leq C_2 \|Q^h\|_{C_N^h(\bar{\Omega}_1^h)}, \quad \forall Q^h \in C_N^h(\bar{\Omega}_1^h). \quad (13.41)$$

In addition, by embedding theorem $H_{0,N}^{1,h}(\Omega_1^h) \subset C_{N,0}^h(\bar{\Omega}_1^h)$ and there exists a constant $C = C(A, \sigma, h_0, N) > 0$ such that

$$\|Q^h\|_{C_N^h(\bar{\Omega}_1^h)} \leq C \|Q^h\|_{H_{0,N}^{1,h}(\Omega_1^h)}, \quad \forall Q^h \in H_{0,N}^{1,h}(\Omega_1^h).$$

Thus, we obtain the following partial finite difference analog of problem (13.28)-(13.31):

$$\begin{aligned}
&M_N W^h(\mathbf{x}^h(z)) = \\
&= P(W^h, W_x^h, W_y^h, G^h, G_x^h, G_y^h, F^h, F_x^h, F_y^h, \mathbf{x}^h(z)), \mathbf{x}^h(z) \in \Omega^h,
\end{aligned} \quad (13.42)$$

$$W^h(\mathbf{x}^h(z)) \in H_{0,N}^{1,h}(\Omega_1^h). \quad (13.43)$$

Remark 13.3. Note that problem (13.42), (13.43) is not just a nonlinear system of Volterra integral equations. Indeed, along with the Volterra integrals from z to $A + \sigma$, we also have the boundary condition $W^h(\mathbf{x}^h(A)) = 0$.

Also, we assume that the vector functions $G^h(\mathbf{x}^h(z))$ and $F^h(\mathbf{x}^h(z))$ are such that

$$G^h(\mathbf{x}^h(z)), F^h(\mathbf{x}^h(z)) \in C_N^h(\overline{\Omega}_1^h). \quad (13.44)$$

Let $R > 0$ be an arbitrary number. We now introduce the finite difference analog K^h of set (13.32) as well as its subset $K^h(R)$,

$$K^h = \left\{ \begin{array}{l} w^h = w^h(\mathbf{x}^h(z), \alpha) := (w^h + g^h)(\mathbf{x}^h(z), \alpha) > 0, \\ \forall (\mathbf{x}^h(z), \alpha) \in \overline{\Omega}_1^h \times [0, 1], \\ W^h(\mathbf{x}^h(z)) \in H_{0,N}^{1,h}(\Omega_1^h) \end{array} \right\}, \quad (13.45)$$

$$K^h(R) = \left\{ W^h(\mathbf{x}^h(z)) : w^h \in K^h, \|W^h\|_{H_{0,N}^{1,h}(\Omega_1^h)} < R \right\}. \quad (13.46)$$

Hence, $K^h(R) \subset K^h \subset C_{N,0}^h(\overline{\Omega}_1^h)$.

Lemma 13.4 is a discrete analog of Lemma 13.3. It provides a sufficient condition imposed on the components of the matrix $w^h(\mathbf{x}^h(z), \alpha)$, which guarantees that $w^h(\mathbf{x}^h(z), \alpha) \in K^h$.

Lemma 13.4. Let the matrix $w^h = w^h(\mathbf{x}^h(z), \alpha) \in K^h$. Select a triple (i, j, z) with $i, j = 0, \dots, B, z \in [A, A + \sigma]$ and consider the vector

$v(i, j, z) = (w_1 + g_1, w_2 + g_2, \dots, w_N + g_N)^T(i, j, z)$. Let X_N be the $N \times N$ matrix of Lemma 3.1. Consider the vector $\tilde{v}(i, j, z) = X_N^T \cdot v(i, j, z)$. Let

$\tilde{v}(i, j, z) = (\tilde{v}_1, \dots, \tilde{v}_N)^T(i, j, z)$. Suppose that all numbers $\tilde{v}_n(i, j, z) > 0$ for all $i, j = 0, \dots, B, z \in [A, A + \sigma]$. Then the function $w^h \in K$ and, therefore, the following analog of (16.59) holds for $\mathbf{x}^h(A + \sigma), \mathbf{x}^h(z) \in \overline{\Omega}_1^h$:

$$u_0^h(\mathbf{x}(A + \sigma)) + w^h(\mathbf{x}^h(z), \alpha) + g^h(\mathbf{x}^h(z), \alpha) > \frac{A^2}{A^2 + 2}.$$

Also, the set $K^h(R)$ is convex.

13.8 Lipschitz stability and uniqueness of problem (13.42), (13.43)

Denote

$$\Phi^h(\mathbf{x}^h(z)) = (G^h, G_x^h, G_y^h, F^h, F_x^h, F_y^h)(\mathbf{x}^h(z)). \quad (13.47)$$

Theorem 13.1 (Lipschitz stability and uniqueness). Let $R > 0$ be an arbitrary number. Consider two matrices $W^{(1),h}(\mathbf{x}^h(z)), W^{(2),h}(\mathbf{x}^h(z)) \in K^h(R)$. Suppose that these matrices generate two pairs of matrices $G^{(i),h}(\mathbf{x}^h(z))$ and $F^{(i),h}(\mathbf{x}^h(A + \sigma))$, $i = 1, 2$, which satisfy (13.42). Let $\Phi^{(i),h}(\mathbf{x}^h(z))$, $i = 1, 2$ be the corresponding matrices as in (13.47). Let $m^{(1),h}(\mathbf{x}(z))$ and $m^{(2),h}(\mathbf{x}(z))$ be the corresponding finite difference versions of the coefficient $m(x)$, which are reconstructed via $W^{(1),h}(\mathbf{x}^h(z)), W^{(2),h}(\mathbf{x}^h(z))$ and

the finite difference analog of equation (13.31). Assume that functions $m^{(1),h}(\mathbf{x}(z))$ and $m^{(2),h}(\mathbf{x}(z))$ are independent on α . Then there exists a constant

$$C_3 = C_3 \left(h_0, A, \sigma, R, N, \Omega^h, \|\Phi^{(1),h}\|_{C_{2N}^h(\overline{\Omega}_1^h)}, \|\Phi^{(2),h}\|_{C_{2N}^h(\overline{\Omega}_1^h)} \right) > 0, \quad (13.48)$$

depending only on listed parameters such that the following Lipschitz stability estimate holds

$$\|m^{(1),h} - m^{(2),h}\|_{C_1^h(\overline{\Omega}^h)} \leq C_3 \|\Phi^{(1),h} - \Phi^{(2),h}\|_{C_{2N}^h(\overline{\Omega}_1^h)}. \quad (13.49)$$

In particular, if $\Phi^{(1),h}(\mathbf{x}^h(z)) \equiv \Phi^{(2),h}(\mathbf{x}^h(z))$, then $m^{(1),h}(\mathbf{x}^h(z)) \equiv m^{(2),h}(\mathbf{x}^h(z))$, i.e. uniqueness holds.

13.9 The Carleman estimate for an integral operator

Let $\lambda > 0$ be the parameter to be chosen later. We choose the “integral analog” of the CWF as

$$\varphi_\lambda(z) = e^{2\lambda z}.$$

Lemma 13.5 (Carleman estimate for an integral operator [15]). *The following estimate holds for all $\lambda > 0$ and for every function $p \in L_2(A, A + \sigma)$:*

$$\int_A^{A+\sigma} e^{2\lambda z} \left(\int_z^{A+\sigma} |p(y)| dy \right) dz \leq \frac{1}{2\lambda} \int_A^{A+\sigma} |p(z)| e^{2\lambda z} dz.$$

It was the first time in [15] when a Carleman estimate was derived for an integral operator. All known Carleman estimates are for differential operators only.

13.10 Weighted globally strictly convex Tikhonov-like functional for problem (13.42), (13.43)

To solve problem (13.42), (13.43) numerically, we consider the following minimization problem:

Minimization Problem. Fix an arbitrary number $R > 0$ as well as the grid step size $h \in [h_0, 1)$. Let $\gamma > 0$ be the regularization parameter. Minimize the functional $J_{\lambda,\gamma}(W^h)$ on the closed set $\overline{K^h(R)}$, where

$$\begin{aligned} J_{\lambda,\gamma}(W^h) = & e^{-2\lambda A} \left\| [M_N W^h(\mathbf{x}^h(z)) - P(W^h, W_x^h, W_y^h, \Phi^h, \mathbf{x}^h(z))] e^{\lambda z} \right\|_{L_2^h(\Omega^h)}^2 \\ & + \gamma \|W^h\|_{H_0^{1,h}(\Omega_1^h)}^2. \end{aligned} \quad (13.50)$$

We use the multiplier $e^{-2\lambda A}$ in order to balance two terms in the right hand side of (13.50). Note that since $R > 0$ is an arbitrary number, then we do not impose a smallness condition on the set $\overline{K^h(R)}$ where we search for the solution of problem (13.42), (13.43). This is why we are talking below about the **global** strict convexity and the **globally** convergent numerical method.

Theorems 13.2 and 13.3 were proved in the work of the PI [15]. Theorems 13.4 and 13.5 were proved using Theorem 6.1 of section 6 as well as Theorems 13.2 and 13.3. In [15]

direct analogs of Theorems 13.4 and 13.5 were proven for the gradient projection method rather than for a simpler gradient descent method.

Theorem 13.2 (global strict convexity). *Let h_0 be the number defined in (13.39) and let $h \in [h_0, 1]$. At every point $W^h \in K^h(2R)$ and for all $\lambda > 0, \gamma \in (0, 1)$ the functional $J_{\lambda, \gamma}(W^h)$ has the Frechét derivative $J'_{\lambda, \gamma}(W^h) \in H_0^{1, h}(\Omega_1^h)$. Furthermore, this derivative is Lipschitz continuous on $K^h(2R)$, i.e. there exists a constant*

$$C_2 = C_2 \left(h_0, A, \sigma, R, N, \Omega^h, \|G^h\|_{C_N^h(\overline{\Omega_1^h})}, \|F^h\|_{C_N^h(\overline{\Omega_1^h})} \right) > 0 \quad (13.51)$$

depending only on parameters listed in (13.51) such that for all $W^{(1), h}, W^{(2), h} \in \overline{K^h(2R)}$

$$\|J'_{\lambda, \gamma}(W^{(2), h}) - J'_{\lambda, \gamma}(W^{(1), h})\|_{H_0^{1, h}(\Omega_1^h)} \leq \overline{C} \|W^{(2), h} - W^{(1), h}\|_{H_0^{1, h}(\Omega_1^h)}, \quad (13.52)$$

where the constant $\overline{C} > 0$ depends on the same parameters as ones listed in (13.51) as well as on λ . In addition, there exists a sufficiently large number $\lambda_0 > 1$ depending on the same parameters as those listed in (13.51) such that for every $\lambda \geq \lambda_0$ and for every $\gamma \in (0, 1)$ the functional $J_{\lambda, \gamma}(W^h)$ is strictly convex on the closed set $\overline{K^h(R)}$. More precisely, the following estimate holds for all $W^{(1), h}, W^{(2), h} \in \overline{K^h(2R)}$

$$\begin{aligned} & J_{\lambda, \gamma}(W^{(2), h}) - J_{\lambda, \gamma}(W^{(1), h}) - J'_{\lambda, \gamma}(W^{(1), h})(W^{(2), h} - W^{(1), h}) \\ & \geq \frac{1}{8} \|W^{(2), h} - W^{(1), h}\|_{L_2^h(\Omega^h)}^2 + \gamma \|W^{(2), h} - W^{(1), h}\|_{H_0^{1, h}(\Omega_1^h)}^2. \end{aligned} \quad (13.53)$$

Below C_2 denotes different constants depending on parameters listed in (13.51).

Theorem 13.3. *Suppose that conditions of Theorem 13.3 are in place. Then for every $\lambda \geq \lambda_0$ and for every $\gamma \in (0, 1)$ there exists a single minimizer $W_{\min}^h \in \overline{K^h(2R)}$ of the functional $J_{\lambda, \gamma}(W^h)$ on the set $\overline{K^h(R)}$. Furthermore,*

$$J'_{\lambda, \gamma}(W_{\min}^h)(W_{\min}^h - W^h) \leq 0, \quad \forall W^h \in \overline{K^h(R)}. \quad (13.54)$$

Let $\theta \in (0, 1/3)$ be an arbitrary number. Consider an arbitrary point $W^{0, h} \in K^h(2R\theta)$. And minimize the functional $J_{\lambda, \gamma}(W^h)$ by the gradient descent method, which starts its iterations at the point $W^{0, h}$,

$$W_n^h = W_{n-1}^h - \rho J'_{\lambda, \gamma}(W_{n-1}^h), \quad n = 1, 2, \dots \quad (13.55)$$

Here the number $\rho \in (0, 1)$.

Theorem 13.4. *Suppose that conditions of Theorem 13.2 are in place, $\lambda \geq \lambda_0$ and $\gamma \in (0, 1)$. Let $W_{\min}^h \in \overline{K^h(2R\theta)}$ be the minimizer listed in Theorem 13.2. Then there exists a sufficiently small number $\rho_0 \in (0, 1)$ depending on the same parameters as ones listed in (13.51) such that for every $\rho \in (0, \rho_0)$ there exists a number $\eta = \eta(\rho) \in (0, 1)$ such that all terms of the sequence (13.55) $W_n^h \in K^h(2R)$, this sequence converges to $W_{\min}^h$ and the following rate of convergence holds:*

$$\|W_{\min}^h - W_n^h\|_{H_{0, N}^{1, h}(\Omega_1^h)} \leq \eta^n \|W_{\min}^h - W_0^h\|_{H_{0, N}^{1, h}(\Omega_1^h)}. \quad (13.56)$$

In the regularization theory, the minimizer $W_{\min}^h$ of functional (13.50) is called “regularized solution”, see, e.g. [116]. We now need to show that regularized solutions converge

to the exact one when the noise in the data tends to zero. Following the regularization theory, we assume that there exists an exact, i.e. idealized, solution $W^{*,h} \in K^h(2R)$ of problem (13.42), (13.43) with the noiseless data $\Phi^{*,h}(\mathbf{x}(z))$,

$$G^{*,h}, F^{*,h} \in C_N^h(\bar{\Omega}_1^h), \quad (13.57)$$

where $\mathbf{x}^h(z) \in \bar{\Omega}^h$. As it is always the case in the theory of ill-posed problems [116], we impose the assumption that there exists the exact, i.e. idealized function $m^{*,h}(\mathbf{x}^h(z))$, $\mathbf{x}^h(z) \in \bar{\Omega}^h$, which produces the data (13.57).

Let the number $\delta \in (0, 1)$ be the level of the error in the data G^h, F^h , i.e.

$$\|G^{*,h} - G^h\|_{C_N^h(\bar{\Omega}_1^h)}, \|F^{*,h} - F^h\|_{C_N^h(\bar{\Omega}_1^h)} < \delta. \quad (13.58)$$

Denote $\tilde{G}^h = G^{*,h} - G^h$, $\tilde{F}^h = F^{*,h} - F^h$. Then (13.58) implies that with a constant $C_2 = C_2(h_0, N, \Omega^h) > 0$ depending only on listed parameters the following inequalities hold:

$$\|\tilde{G}^h\|_{C_N^h(\bar{\Omega}_1^h)}, \|\tilde{G}_x^h\|_{C_N^h(\bar{\Omega}^h)}, \|\tilde{G}_y^h\|_{C_N^h(\bar{\Omega}^h)} < C_2\delta, \quad (13.59)$$

$$\|\tilde{F}^h\|_{C_N^h(\bar{\Omega}_1^h)}, \|\tilde{F}_x^h\|_{C_N^h(\bar{\Omega}^h)}, \|\tilde{F}_y^h\|_{C_N^h(\bar{\Omega}^h)} < C_2\delta. \quad (13.60)$$

Since $\delta \in (0, 1)$ and (13.58)-(13.60) hold, then, using the triangle inequality, we replace below the dependence of the constant $C_2 > 0$ on $\|\Phi^h\|_{C_{2N}^h(\bar{\Omega}_1^h)}$ in (13.51) with the dependence of C_2 on $\|\Phi^{*,h}\|_{C_{2N}^h(\bar{\Omega}_1^h)}$. Thus, below $C_2 > 0$ denotes different constants depending on the same parameters as ones listed in (13.51), except that $\|\Phi^h\|_{C_{2N}^h(\bar{\Omega}_1^h)}$ is replaced with $\|\Phi^{*,h}\|_{C_{2N}^h(\bar{\Omega}_1^h)}$.

Consider now the right hand side of equation (13.42) for the case when W^h is replaced with $W^{*,h}$, whereas other arguments remain the same. In addition, we use now (13.57)-(13.60). Thus, we obtain

$$\begin{aligned} & M_N P(W^{*,h}, W_x^{*,h}, W_y^{*,h}, \Phi^h, \mathbf{x}^h(z)) \\ &= P(W^{*,h}, W_x^{*,h}, W_y^{*,h}, \Phi^{*,h}, \mathbf{x}^h(z)) + \hat{P}(\mathbf{x}^h(z)), \mathbf{x}^h(z) \in \bar{\Omega}^h, \end{aligned} \quad (13.61)$$

$$\|\hat{P}\|_{C_N^h(\bar{\Omega}^h)} \leq C_2\delta. \quad (13.62)$$

Theorem 13.5 (global convergence of the gradient descent method to the exact solution). *Suppose that conditions of Theorem 13.2 are in place and also that there exists an exact solution $W^{*,h} \in K^h(2R)$ of problem (13.42), (13.43) with the noiseless data $\Phi^{*,h}$. Let the $\lambda_0 > 1$ be the number chosen in Theorem 13.1. Fix an arbitrary number $\lambda = \lambda_1 \geq \lambda_0$ in the functional $J_{\lambda,\gamma}(W^h) = J_{\lambda_1,\gamma}(W^h)$. Just as in the regularization theory, set $\gamma = \gamma(\delta) = \delta^2$. Let the numbers $\rho \in (0, \rho_0)$, $\rho_0, \eta = \eta(\rho) \in (0, 1)$ be the same as in Theorem 13.4. Let the number $\theta \in (0, 1/3)$. Assume that $W_{\min}^h \in K^h(2R\theta)$. Consider an arbitrary point $W^{0,h} \in K^h(2R\theta)$. Then all all terms of the sequence (13.55) $W_n^h \in K^h(2R)$ and the following rates of convergence hold:*

$$\|W^{h,*} - W_{\min}^h\|_{L_{2,N}^h(\Omega^h)} \leq C_4\delta, \quad (13.63)$$

$$\|W^{*,h} - W_n^h\|_{L_{2,N}^h(\Omega^h)} \leq C_4\delta + \eta^n \|W_{\min}^h - W^{0,h}\|_{H_{0,N}^{1,h}(\Omega_1^h)}. \quad (13.64)$$

$$\|m^{*,h} - m_n^h\|_{L_{2,1}^h(\Omega^h)} \leq C_4\delta + \eta^n \|W_{\min}^h - W^{0,h}\|_{H_{0,N}^{1,h}(\Omega_1^h)}, \quad (13.65)$$

where the functions $m_n^h(\mathbf{x}^h(z))$ are constructed from matrices W_n^h .

Remark 13.4. Since smallness conditions are not imposed on the number $R > 0$ and since the starting point of iterations of the gradient method (13.55), $W^{0,h}$, is an arbitrary point of the set $K^h(2R\theta)$, then convergence estimates (13.63)-(13.65) imply the global convergence of the gradient descent method (13.55) to the exact solution of the discrete version of the TTTP.

14 Convergent Numerical Method and Numerical Results for the Linearized Travel Time Tomography Problem [3, Chapter 12], [17]

14.1 Preliminaries

Even though a globally convergent convexification-based numerical method for TTTP is developed in [3, Chapter 11], [15, 16] and is presented in the previous section 13, sometimes the linearized TTTP makes sense as well. The linearization makes sense in those applied scenarios, in which the target property of interest of a medium is basically known, but one needs a more accurate evaluation of this property. This linearized problem is also called the *geodesic X-ray transform problem*.

In the linearized case, one assumes that the target coefficient is basically known. But still a small perturbation of it is unknown and is the subject of the solution of the inverse problem. The importance of this problem is specifically pointed out in the classical book of Romanov [107, Chapter 3]. However, a reliable numerical method for this problem was absent until the recent publication of the PI with coauthors [17]. Results of [17] are presented below in this section 14. They are also described in the book of Klibanov and Li [3, Chapter 12].

We have developed in [3, Chapter 12], [17] a **PIONEERING** numerical method for the linearized TTTP for the d -D case, $d = 2, 3, \dots$. The PI is unaware about other numerical methods for the linearized TTTP which would have convergence theorems, incomplete data and also would be non overdetermined. Below in this section 14 $d \geq 2$ is the dimension of the space $\mathbb{R}^d$. Points of this space are denoted as $\mathbf{x} \in \mathbb{R}^d$.

The **BREAKTHROUGH** publications of the PI [15, 16] have opened the door for truly working numerical methods for the TTTP. However, they are about the nonlinear case, which is a quite complicated one. Thus, the PI has decided to work on the linearized case, as a preliminary step towards the numerical implementation of the full nonlinear methods of [15, 16].

In particular, we demonstrate that our method provides good accuracy of images of complicated objects with 5% noise in the data. Furthermore, a good accuracy of images is demonstrated even for very high levels of noise between 80% and 170%. The latter clearly manifests a **REMARKABLE ROBUSTNESS** of our approach.

Let $\mathbf{n}(\mathbf{x})$ be the refractive index. To linearize, one should assume that $\mathbf{n}(\mathbf{x}) = \mathbf{n}_0(\mathbf{x}) + \mathbf{n}_1(\mathbf{x})$, where $\mathbf{n}_0(\mathbf{x})$ is the known background function and $\mathbf{n}_1(\mathbf{x})$ with $|\mathbf{n}_1(\mathbf{x})| \ll \mathbf{n}_0(\mathbf{x})$

is its unknown perturbation, which is the subject to the solution of the linearized TTTP. Thus, one assumes that the refractive index is basically known, whereas its small perturbation $\mathbf{n}_1$ is unknown.

In our derivation, we end up with an over determined boundary value problem for a system of coupled linear PDEs of the first order. It is well known that the quasi-reversibility method (QRM) is an effective tool for numerical solutions of boundary value problems for PDEs with overdetermined boundary conditions. In fact, we have use QRM in section 12. Also, the PI has published a survey on QRM in [84]. In particular, it was shown in [3, Chapter 4], [84] that while it is rather easy to prove, using Riesz theorem, the existence and uniqueness of the minimizer of a certain functional related to this method, the proof of convergence of those minimizers to the correct solution requires a stronger tool of Carleman estimates.

Just as in subsection 13.7, we assume in this section that a certain partial differential operator is written in finite differences. In this representation we bound from the below the grid step size $h_x \geq h_0 > 0$ with respect to all spatial variables, except of one. Here where the number h_0 is fixed and is not allowed to tend to zero, see (13.39).

14.2 The linearization

Consider numbers R, a, b such that $R > 1$ and $0 < a < b$. Set

$$\Omega = (-R, R)^{d-1} \times (a, b) \subset \mathbb{R}^d. \quad (14.1)$$

Denote $\mathbf{n}(\mathbf{x}) = \bar{c}/c(\mathbf{x})$, where $c(\mathbf{x})$ is the speed of sound propagation and $n(\mathbf{x})$ is the refractive index. Let the function $\mathbf{n}_0(\mathbf{x})$ be the known refractive index of the background. We assume that

$$\mathbf{n}_0, \mathbf{n} \in C^2(\mathbb{R}^d); \mathbf{n}_0^2(\mathbf{x}), \mathbf{n}^2(\mathbf{x}) \geq 1 \text{ in } \Omega, \quad (14.2)$$

$$\mathbf{n}_0^2(\mathbf{x}) = \mathbf{n}^2(\mathbf{x}) = 1 \text{ for } \mathbf{x} \in \mathbb{R}^d \setminus \Omega. \quad (14.3)$$

For any two points $\mathbf{x}_1$ and $\mathbf{x}_2$ in $\mathbb{R}^d$, define the geodesic line generated by $\mathbf{n}_0$ connecting $\mathbf{x}_1$ and $\mathbf{x}_2$ as:

$$\Gamma_0(\mathbf{x}_1, \mathbf{x}_2) = \operatorname{argmin} \left\{ \int_{\gamma} \mathbf{n}_0(\boldsymbol{\xi}) d\sigma(\boldsymbol{\xi}) \text{ where } \gamma : [0, 1] \rightarrow \mathbb{R}^d \right. \\ \left. \text{is a smooth map with } \gamma(0) = \mathbf{x}_1, \gamma(1) = \mathbf{x}_2 \right\}.$$

Here $d\sigma(\boldsymbol{\xi})$ is the elementary arc length. Note that by (??) the geodesic line $\Gamma_0(\mathbf{x}_1, \mathbf{x}_2)$ connects points $\mathbf{x}_1$ and $\mathbf{x}_2$. Let

$$\mathbf{a}_0(\mathbf{x}) = \mathbf{n}_0^2(\mathbf{x}) \text{ for all } \mathbf{x} \in \mathbb{R}^d. \quad (14.4)$$

The corresponding travel time between $\mathbf{x}_1$ and $\mathbf{x}_2$ is the integral

$$\int_{\Gamma_0(\mathbf{x}_1, \mathbf{x}_2)} \mathbf{n}_0(\boldsymbol{\xi}) d\sigma(\boldsymbol{\xi}) = \int_{\Gamma_0(\mathbf{x}_1, \mathbf{x}_2)} \sqrt{\mathbf{a}_0(\boldsymbol{\xi})} d\sigma(\boldsymbol{\xi}).$$

Introduce the line of sources L_{src} located on the x_1 -axis as

$$L_{\text{src}} = [-\bar{\alpha}, \bar{\alpha}] \times \{(0, 0, \dots, 0)\}, \quad (14.5)$$

where $\bar{\alpha}$ is a fixed positive number. It follows from (14.1) and (14.5) that

$$\bar{\Omega} \cap L_{\text{src}} = \emptyset. \quad (14.6)$$

For $\mathbf{x}_\alpha \in L_{\text{src}}$, the travel time along $\Gamma_0(\mathbf{x}, \mathbf{x}_\alpha)$ of the wave from $\mathbf{x}_\alpha$ to $\mathbf{x}$ is

$$u_0(\mathbf{x}, \mathbf{x}_\alpha) = \int_{\Gamma_0(\mathbf{x}, \mathbf{x}_\alpha)} \sqrt{\mathbf{a}_0(\boldsymbol{\xi})} d\sigma(\boldsymbol{\xi}), \quad \mathbf{x} \in \mathbb{R}^d. \quad (14.7)$$

Just as in sub-subsection 2.6.2 and subsection 13.2, we assume in this section 14 that the regularity of geodesic lines in the following sense: for each point $\mathbf{x}$ of the closed domain $\bar{\Omega}$ and for each point $\mathbf{x}_\alpha$ of the line of sources L_{src} there exists a single geodesic line $\Gamma_0(\mathbf{x}, \mathbf{x}_\alpha)$ connecting them.

Assume that the function $\mathbf{a}(\mathbf{x}) = \mathbf{n}^2(\mathbf{x})$ contains a perturbation term of the background function $\mathbf{a}_0(\mathbf{x}) = \mathbf{n}_0^2(\mathbf{x})$. In other words,

$$\mathbf{a}(\mathbf{x}) = \mathbf{a}_0(\mathbf{x}) + 2\epsilon \sqrt{\mathbf{a}_0(\mathbf{x})} p(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d, \quad (14.8)$$

where $\epsilon > 0$ is a sufficiently small number. Here, the function $p \in C(\mathbb{R}^d)$ and $p(\mathbf{x}) = 0$ for $\mathbf{x} \notin \bar{\Omega}$. Hence, by (14.6) $p(\mathbf{x}) = 0$ for points $\mathbf{x}$ in a small neighborhood of the line of sources L_{src} . Denote

$$u_{\mathbf{n}}(\mathbf{x}, \mathbf{x}_\alpha) = \int_{\Gamma_{\mathbf{n}}(\mathbf{x}, \mathbf{x}_\alpha)} \mathbf{n}(\boldsymbol{\xi}) d\sigma(\boldsymbol{\xi})$$

the travel time from the point $\mathbf{x}_\alpha \in L_s$ to the point $\mathbf{x} \in \Omega$, where $\Gamma_{\mathbf{n}}(\mathbf{x}, \mathbf{x}_\alpha)$ is the geodesic line generated by the function $\mathbf{n}(\mathbf{x})$. It is well-known that $u_{\mathbf{n}}(\mathbf{x}, \mathbf{x}_\alpha)$ satisfies the Eikonal equation

$$|\nabla_{\mathbf{x}} u_{\mathbf{n}}(\mathbf{x}, \mathbf{x}_\alpha)|^2 = \mathbf{a}(\mathbf{x}), \quad \mathbf{x} \in \Omega, \mathbf{x}_\alpha \in L_s. \quad (14.9)$$

Let $u_0(\mathbf{x}, \mathbf{x}_\alpha)$ be the travel time function in (14.7) corresponding to the background $\mathbf{a}_0$. Then

$$|\nabla_{\mathbf{x}} u_0(\mathbf{x}, \mathbf{x}_\alpha)|^2 = \mathbf{a}_0(\mathbf{x}), \quad \mathbf{x} \in \Omega, \mathbf{x}_\alpha \in L_s. \quad (14.10)$$

Due to (14.8) we represent $\nabla_{\mathbf{x}} u_{\mathbf{n}}(\mathbf{x}, \mathbf{x}_\alpha)$ as

$$\nabla_{\mathbf{x}} u_{\mathbf{n}}(\mathbf{x}, \mathbf{x}_\alpha) = \nabla_{\mathbf{x}} u_0(\mathbf{x}, \mathbf{x}_\alpha) + \epsilon \nabla_{\mathbf{x}} u^{(1)}(\mathbf{x}, \mathbf{x}_\alpha). \quad (14.11)$$

Hence, ignoring the term with ϵ^2 , we obtain

$$|\nabla_{\mathbf{x}} u_{\mathbf{n}}(\mathbf{x}, \mathbf{x}_\alpha)|^2 \approx |\nabla_{\mathbf{x}} u_0(\mathbf{x}, \mathbf{x}_\alpha)|^2 + 2\epsilon \nabla_{\mathbf{x}} u_0(\mathbf{x}, \mathbf{x}_\alpha) \nabla_{\mathbf{x}} u^{(1)}(\mathbf{x}, \mathbf{x}_\alpha). \quad (14.12)$$

Denoting

$$u^{(1)} := u \quad (14.13)$$

and comparing (14.12) with (14.8) and (14.10), we obtain

$$\frac{\nabla_{\mathbf{x}} u_0(\mathbf{x}, \mathbf{x}_\alpha)}{\sqrt{\mathbf{a}_0(\mathbf{x})}} \cdot \nabla_{\mathbf{x}} u(\mathbf{x}, \mathbf{x}_\alpha) = p(\mathbf{x}). \quad (14.14)$$

Thus, equation (14.14) is the “linearization” of the nonlinear equation (14.9). By (14.10) $|\nabla_{\mathbf{x}} u_0(\mathbf{x}, \mathbf{x}_\alpha)| / \sqrt{\mathbf{a}_0(\mathbf{x})} \equiv 1$. Hence, this is a unit vector, which is tangent to the curve $\Gamma_0(\mathbf{x}, \mathbf{x}_\alpha)$ at the point $\mathbf{x}$ [107, Chapter 3]. Hence, the left hand side of (14.14) is the derivative of the function $u(\mathbf{x}, \mathbf{x}_\alpha)$ along the curve $\Gamma_0(\mathbf{x}, \mathbf{x}_\alpha)$. Thus, integrating, we obtain [107, Chapter 3]

$$u(\mathbf{x}, \mathbf{x}_\alpha) = \int_{\Gamma_0(\mathbf{x}, \mathbf{x}_\alpha)} p(\boldsymbol{\xi}) d\sigma(\boldsymbol{\xi}). \quad (14.15)$$

Let $\partial\Omega_{\text{sm}}$ be the smooth part of the boundary $\partial\Omega$ of the domain Ω . For each $\alpha \in (-\bar{\alpha}, \bar{\alpha})$, define

$$\begin{aligned} \partial\Omega_\alpha^- &= \{\mathbf{x} \in \partial\Omega_{\text{sm}} : \nabla_{\mathbf{x}} u_0(\mathbf{x}, \mathbf{x}_\alpha) \cdot \nu(\mathbf{x}) < 0\}, \\ \partial\Omega_\alpha^+ &= \{\mathbf{x} \in \partial\Omega_{\text{sm}} : \nabla_{\mathbf{x}} u_0(\mathbf{x}, \mathbf{x}_\alpha) \cdot \nu(\mathbf{x}) > 0\}, \end{aligned}$$

where $\mathbf{x}_\alpha = (\alpha, 0, \dots, 0) \in L_{\text{src}}$ and $\nu(\mathbf{x})$ is the outward looking unit normal vector at the point $\mathbf{x} \in \partial\Omega_{\text{sm}}$. If $\mathbf{n}_0 \equiv 1$, then $\Gamma_0(\mathbf{x}_1, \mathbf{x}_2)$ is the line segment connecting these two points. Hence, it follows from (??), (14.5), (14.6), (14.11) and (14.13) that

$$u(\mathbf{x}, \mathbf{x}_\alpha) = 0, \mathbf{x} \in \partial\Omega_\alpha^-. \quad (14.16)$$

The aim of this paper is to solve the following inverse problem:

Problem 14.1 (linearized travel time tomography problem). *Let the function $u = u(\mathbf{x}, \mathbf{x}_\alpha) \in C^1(\bar{\Omega} \times [-\bar{\alpha}, \bar{\alpha}])$ be the solution of boundary value problem (14.14), (14.16). Given the data $f(\mathbf{x}, \mathbf{x}_\alpha)$,*

$$f(\mathbf{x}, \mathbf{x}_\alpha) = \begin{cases} u(\mathbf{x}, \mathbf{x}_\alpha), & \mathbf{x} \in \partial\Omega_\alpha^+, \mathbf{x}_\alpha \in L_s, \\ 0, & \mathbf{x} \in \partial\Omega_\alpha^-, \mathbf{x}_\alpha \in L_s, \end{cases} \quad (14.17)$$

determine the function $p(\mathbf{x})$, $\mathbf{x} \in \Omega$.

Note that the data (14.17) are non-redundant ones. Indeed, the source $\mathbf{x}_\alpha \in L_s$ depends on one variable and the point $\mathbf{x} \in \partial\Omega_\alpha^+$ depends on $d - 1$ variables. Hence the function $f(\mathbf{x}, \mathbf{x}_\alpha)$ depends on d variables, so as the target unknown function $p(\mathbf{x})$.

From now on, to separate the coordinate number d of the point $\mathbf{x}$, we write $\mathbf{x} = (x_1, \dots, x_{d-1}, z)$. The transport equation in (14.14) is read as

$$\frac{\partial_z u_0(\mathbf{x}, \mathbf{x}_\alpha)}{\sqrt{\mathbf{a}_0(\mathbf{x})}} \partial_z u(\mathbf{x}, \mathbf{x}_\alpha) + \sum_{i=1}^{d-1} \frac{\partial_{x_i} u_0(\mathbf{x}, \mathbf{x}_\alpha)}{\sqrt{\mathbf{a}_0(\mathbf{x})}} \partial_{x_i} u(\mathbf{x}, \mathbf{x}_\alpha) = p(\mathbf{x}), \quad (14.18)$$

which is equivalent to

$$\partial_z u_0(\mathbf{x}, \mathbf{x}_\alpha) \partial_z u(\mathbf{x}, \mathbf{x}_\alpha) + \sum_{i=1}^{d-1} \partial_{x_i} u_0(\mathbf{x}, \mathbf{x}_\alpha) \partial_{x_i} u(\mathbf{x}, \mathbf{x}_\alpha) = \sqrt{\mathbf{a}_0(\mathbf{x})} p(\mathbf{x}) \quad (14.19)$$

for all $\mathbf{x} \in \Omega$, $\mathbf{x}_\alpha \in L_{\text{src}}$.

14.3 A boundary value problem for a system of coupled PDEs of the first order

This subsection aims to derive a system of partial differential equations, which can be stably solved by the quasi-reversibility method in the semi-finite difference scheme. The solution of this system yields the desired numerical solution to Problem 14.1. Recall that Problem 14.1 is the linearized travel time tomography problem, and it is labeled this way by its title.

Consider now the same orthonormal basis $\{\Psi_n(\alpha)\}_{n=0}^\infty$ in the space $L_2(-\bar{\alpha}, \bar{\alpha})$ as the one in subsections 9.3, 12.4 and 13.5. This basis was originally constructed in [87].

Using this basis, we derive a system of partial differential equations for the Fourier coefficients of the function

$$w(\mathbf{x}, \mathbf{x}_\alpha) = u(\mathbf{x}, \mathbf{x}_\alpha) \partial_z u_0(\mathbf{x}, \mathbf{x}_\alpha) \quad \mathbf{x} \in \Omega, \mathbf{x}_\alpha \in L_{\text{src}} \quad (14.20)$$

with respect to the basis $\{\Psi_n\}_{n=1}^\infty$. Differentiate (14.19) with respect to α . We obtain

$$\frac{\partial}{\partial \alpha} \left[\partial_z u_0(\mathbf{x}, \mathbf{x}_\alpha) \partial_z u(\mathbf{x}, \mathbf{x}_\alpha) + \sum_{i=1}^{d-1} \partial_{x_i} u_0(\mathbf{x}, \mathbf{x}_\alpha) \partial_{x_i} u(\mathbf{x}, \mathbf{x}_\alpha) \right] = 0 \quad (14.21)$$

for all $\mathbf{x} \in \Omega, \mathbf{x}_\alpha \in L_s$. From now on, we impose the following condition.

Assumption. *The travel time function u_0 , defined in (14.7) with n replaced by n_0 , is strictly increasing with respect to z . In other words, $\partial_z u_0(\mathbf{x}, \mathbf{x}_\alpha) > 0$ for all $x = (x_1, \dots, x_{d-1}, z) \in \Omega$ and for all $\mathbf{x}_\alpha \in L_{\text{src}}$.*

This Assumption means that the higher in the z -direction, the longer the traveling time is. A sufficient condition for this Assumption to be true is formulated in (14.22) of Lemma 14.1. A similar monotonicity condition can be found in formulas (3.24) and (3.24') of the book [107, section 2 of Chapter 3]. Besides, figures 5 and 10 of [118] justify this condition from the geophysical standpoint. Recall that by (14.2) and (14.3) $\mathbf{a}_0 \in C^2(\mathbb{R}^d)$ and $\mathbf{a}_0(\mathbf{x}) \geq 1$ in $\mathbb{R}^d$. Therefore, the following lemma follows immediately from Lemma 4.1 of [15]:

Lemma 14.1. *Let conditions (14.2) and (14.3) hold. Also, let*

$$\partial_z \mathbf{a}_0(\mathbf{x}) \geq 0 \text{ for all } \mathbf{x} \in \bar{\Omega}. \quad (14.22)$$

Then

$$\partial_z u_0(\mathbf{x}, \mathbf{x}_\alpha) \geq \frac{a}{\sqrt{a^2 + 2}} \text{ for all } \mathbf{x} \in \bar{\Omega}, \alpha \in [-\bar{\alpha}, \bar{\alpha}].$$

Let $w(\mathbf{x}, \mathbf{x}_\alpha)$ be the function defined in (14.20). Then

$$\begin{aligned} \partial_z u_0(\mathbf{x}, \mathbf{x}_\alpha) \partial_z u(\mathbf{x}, \mathbf{x}_\alpha) &= \partial_z w(\mathbf{x}, \mathbf{x}_\alpha) - u(\mathbf{x}, \alpha) \partial_{zz} u_0(\mathbf{x}, \mathbf{x}_\alpha) \\ &= \partial_z w(\mathbf{x}, \mathbf{x}_\alpha) - w(\mathbf{x}, \alpha) \frac{\partial_{zz} u_0(\mathbf{x}, \mathbf{x}_\alpha)}{\partial_z u_0(\mathbf{x}, \mathbf{x}_\alpha)}. \end{aligned} \quad (14.23)$$

Also, for $i = 1, \dots, d-1$

$$\begin{aligned} \partial_{x_i} u(\mathbf{x}, \mathbf{x}_\alpha) &= \frac{\partial}{\partial x_i} \left(\frac{w(\mathbf{x}, \mathbf{x}_\alpha)}{\partial_z u_0(\mathbf{x}, \mathbf{x}_\alpha)} \right) \\ &= \frac{\partial_{x_i} w(\mathbf{x}, \mathbf{x}_\alpha) \partial_z u_0(\mathbf{x}, \mathbf{x}_\alpha) - w(\mathbf{x}, \mathbf{x}_\alpha) \partial_{zx_i} u_0(\mathbf{x}, \mathbf{x}_\alpha)}{(\partial_z u_0(\mathbf{x}, \mathbf{x}_\alpha))^2} \end{aligned} \quad (14.24)$$

for all $\mathbf{x} \in \Omega$, $\mathbf{x}_\alpha \in L_{\text{src}}$. Combining (14.21), (14.23) and (14.24), we obtain

$$\begin{aligned} & \frac{\partial}{\partial \alpha} \left[\partial_z w(\mathbf{x}, \mathbf{x}_\alpha) - w(\mathbf{x}, \mathbf{x}_\alpha) \frac{\partial_{zz} u_0(\mathbf{x}, \mathbf{x}_\alpha)}{\partial_z u_0(\mathbf{x}, \mathbf{x}_\alpha)} \right. \\ & \quad \left. + \sum_{i=1}^{d-1} \frac{\partial_{x_i} w(\mathbf{x}, \mathbf{x}_\alpha) \partial_z u_0(\mathbf{x}, \mathbf{x}_\alpha) - w(\mathbf{x}, \mathbf{x}_\alpha) \partial_{zx_i} u_0(\mathbf{x}, \mathbf{x}_\alpha)}{(\partial_z u_0(\mathbf{x}, \mathbf{x}_\alpha))^2} \partial_{x_i} u_0(\mathbf{x}, \mathbf{x}_\alpha) \right] = 0. \end{aligned} \quad (14.25)$$

This is equivalent to

$$\begin{aligned} & \partial_{\alpha z} w(\mathbf{x}, \mathbf{x}_\alpha) - \frac{\partial_{zz} u_0(\mathbf{x}, \mathbf{x}_\alpha)}{\partial_z u_0(\mathbf{x}, \mathbf{x}_\alpha)} \partial_\alpha w(\mathbf{x}, \mathbf{x}_\alpha) - \frac{\partial}{\partial \alpha} \left(\frac{\partial_{zz} u_0(\mathbf{x}, \mathbf{x}_\alpha)}{\partial_z u_0(\mathbf{x}, \mathbf{x}_\alpha)} \right) w(\mathbf{x}, \mathbf{x}_\alpha) \\ & \quad + \sum_{i=1}^{d-1} \left[\frac{\partial_{x_i} u_0(\mathbf{x}, \mathbf{x}_\alpha)}{\partial_z u_0(\mathbf{x}, \mathbf{x}_\alpha)} \partial_{\alpha x_i} w(\mathbf{x}, \mathbf{x}_\alpha) + \frac{\partial}{\partial \alpha} \left(\frac{\partial_{x_i} u_0(\mathbf{x}, \mathbf{x}_\alpha)}{\partial_z u_0(\mathbf{x}, \mathbf{x}_\alpha)} \right) \partial_{x_i} w(\mathbf{x}, \mathbf{x}_\alpha) \right. \\ & \quad \left. - \frac{\partial_{zx_i} u_0(\mathbf{x}, \mathbf{x}_\alpha) \partial_{x_i} u_0(\mathbf{x}, \mathbf{x}_\alpha)}{(\partial_z u_0(\mathbf{x}, \mathbf{x}_\alpha))^2} \partial_\alpha w(\mathbf{x}, \mathbf{x}_\alpha) - \frac{\partial}{\partial \alpha} \left(\frac{\partial_{zx_i} u_0(\mathbf{x}, \mathbf{x}_\alpha) \partial_{x_i} u_0(\mathbf{x}, \mathbf{x}_\alpha)}{(\partial_z u_0(\mathbf{x}, \mathbf{x}_\alpha))^2} \right) w(\mathbf{x}, \mathbf{x}_\alpha) \right] = 0. \end{aligned} \quad (14.26)$$

We recall now the orthonormal basis $\{\Psi_n\}_{n=1}^\infty$ constructed at the beginning of this section. For each $\mathbf{x} \in \Omega$ and for all $\mathbf{x}_\alpha \in L_{\text{src}}$, we write

$$w(\mathbf{x}, \mathbf{x}_\alpha) = \sum_{n=1}^\infty w_n(\mathbf{x}) \Psi_n(\alpha) \approx \sum_{n=1}^N w_n(\mathbf{x}) \Psi_n(\alpha), \quad (14.27)$$

$$w_n(\mathbf{x}) = \int_{-\bar{\alpha}}^{\bar{\alpha}} w(\mathbf{x}, \mathbf{x}_\alpha) \Psi_n(\alpha) d\alpha. \quad (14.28)$$

The “cut-off” number N is chosen numerically. Following our approximate mathematical model introduced in (14.27), we assume that the approximation $\approx$ in (14.27) is an equality as well as

$$\partial_\alpha w(\mathbf{x}, \mathbf{x}_\alpha) = \sum_{n=1}^N w_n(\mathbf{x}) \Psi'_n(\alpha). \quad (14.29)$$

Plugging (14.27) and (14.29) into (14.26) gives

$$\begin{aligned} & \sum_{n=1}^N \partial_z w_n(\mathbf{x}) \Psi'_n(\alpha) - \frac{\partial_{zz} u_0(\mathbf{x}, \mathbf{x}_\alpha)}{\partial_z u_0(\mathbf{x}, \mathbf{x}_\alpha)} \sum_{n=1}^N w_n(\mathbf{x}) \Psi'_n(\alpha) - \frac{\partial}{\partial \alpha} \left(\frac{\partial_{zz} u_0(\mathbf{x}, \mathbf{x}_\alpha)}{\partial_z u_0(\mathbf{x}, \mathbf{x}_\alpha)} \right) \sum_{n=1}^N w_n(\mathbf{x}) \Psi_n(\alpha) \\ & \quad + \sum_{i=1}^{d-1} \left[\frac{\partial_{x_i} u_0(\mathbf{x}, \mathbf{x}_\alpha)}{\partial_z u_0(\mathbf{x}, \mathbf{x}_\alpha)} \sum_{n=1}^N \partial_{x_i} w_n(\mathbf{x}) \Psi'_n(\alpha) + \frac{\partial}{\partial \alpha} \left(\frac{\partial_{x_i} u_0(\mathbf{x}, \mathbf{x}_\alpha)}{\partial_z u_0(\mathbf{x}, \mathbf{x}_\alpha)} \right) \sum_{n=1}^N \partial_{x_i} w_n(\mathbf{x}) \Psi_n(\alpha) \right. \\ & \quad \left. - \frac{\partial_{zx_i} u_0(\mathbf{x}, \mathbf{x}_\alpha) \partial_{x_i} u_0(\mathbf{x}, \mathbf{x}_\alpha)}{(\partial_z u_0(\mathbf{x}, \mathbf{x}_\alpha))^2} \sum_{n=1}^N w_n(\mathbf{x}) \Psi'_n(\alpha) - \frac{\partial}{\partial \alpha} \left(\frac{\partial_{zx_i} u_0(\mathbf{x}, \mathbf{x}_\alpha) \partial_{x_i} u_0(\mathbf{x}, \mathbf{x}_\alpha)}{(\partial_z u_0(\mathbf{x}, \mathbf{x}_\alpha))^2} \right) \sum_{n=1}^N w_n(\mathbf{x}) \Psi_n(\alpha) \right] = 0. \end{aligned}$$

For each $m \in \{1, \dots, N\}$, multiply the latter equation by $\Psi_m(\alpha)$ and then integrate the resulting equation with respect to α . We get

$$\sum_{n=1}^N s_{mn} \partial_z w_n(\mathbf{x}) + \sum_{n=1}^N a_{mn}(\mathbf{x}) w_n(\mathbf{x}) + \sum_{n=1}^N \sum_{i=1}^{d-1} b_{mn,i}(\mathbf{x}) \partial_{x_i} w_n(\mathbf{x}) = 0 \quad (14.30)$$

for all $\mathbf{x} \in \Omega$ where s_{mn} is defined as in Proposition ??,

$$a_{mn}(\mathbf{x}) = \int_{-\bar{\alpha}}^{\bar{\alpha}} \left[-\frac{\partial_{zz}u_0(\mathbf{x}, \mathbf{x}_\alpha)}{\partial_z u_0(\mathbf{x}, \mathbf{x}_\alpha)} \Psi'_n(\alpha) - \frac{\partial}{\partial \alpha} \left(\frac{\partial_{zz}u_0(\mathbf{x}, \mathbf{x}_\alpha)}{\partial_z u_0(\mathbf{x}, \mathbf{x}_\alpha)} \right) \Psi_n(\alpha) \right. \\ \left. - \sum_{i=1}^{d-1} \frac{\partial}{\partial \alpha} \left(\frac{\partial_{zx_i}u_0(\mathbf{x}, \mathbf{x}_\alpha) \partial_{x_i}u_0(\mathbf{x}, \mathbf{x}_\alpha)}{(\partial_z u_0(\mathbf{x}, \mathbf{x}_\alpha))^2} \right) \Psi'_n(\alpha) \right. \\ \left. - \sum_{i=1}^{d-1} \frac{\partial}{\partial \alpha} \left(\frac{\partial_{zx_i}u_0(\mathbf{x}, \mathbf{x}_\alpha) \partial_{x_i}u_0(\mathbf{x}, \mathbf{x}_\alpha)}{(\partial_z u_0(\mathbf{x}, \mathbf{x}_\alpha))^2} \right) \Psi_n(\alpha) \right] \Psi_m(\alpha) d\alpha$$

and for $i = 1, \dots, d-1$

$$b_{mn,i}(\mathbf{x}) = \int_{-\bar{\alpha}}^{\bar{\alpha}} \left[\frac{\partial_{x_i}u_0(\mathbf{x}, \mathbf{x}_\alpha)}{\partial_z u_0(\mathbf{x}, \mathbf{x}_\alpha)} \Psi'_n(\alpha) + \frac{\partial}{\partial \alpha} \left(\frac{\partial_{x_i}u_0(\mathbf{x}, \mathbf{x}_\alpha)}{\partial_z u_0(\mathbf{x}, \mathbf{x}_\alpha)} \right) \Psi_n(\alpha) \right] \Psi_m(\alpha) d\alpha, \quad (14.31)$$

for all $\mathbf{x} \in \Omega$. For each $\mathbf{x} \in \Omega$, let $W(\mathbf{x}) = (w_1(\mathbf{x}), \dots, w_N(\mathbf{x}))^T$, $S = (s_{mn})_{m,n=1}^N$, $A(\mathbf{x}) = (a_{mn}(\mathbf{x}))_{m,n=1}^N$ and $B_i(\mathbf{x}) = (b_{mn,i}(\mathbf{x}))_{m,n=1}^N$ for $i = 1, \dots, d-1$. Since (14.30) holds true for every $m = 1, \dots, N$, it can be rewritten as

$$S_N \partial_z W(\mathbf{x}) + A(\mathbf{x}) W(\mathbf{x}) + \sum_{i=1}^{d-1} B_i(\mathbf{x}) \partial_{x_i} W(\mathbf{x}) = 0. \quad (14.32)$$

Since the matrix S_N is invertible ??, then (14.32) implies the following system of transport equations

$$\partial_z W(\mathbf{x}) + S_N^{-1} A(\mathbf{x}) W(\mathbf{x}) + \sum_{i=1}^{d-1} S_N^{-1} B_i(\mathbf{x}) \partial_{x_i} W(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega. \quad (14.33)$$

The boundary data for W are:

$$W|_{\partial\Omega} = F(\mathbf{x}) = (f_n)_{n=1}^N, \quad f_n(\mathbf{x}) = \int_{-\bar{\alpha}}^{\bar{\alpha}} f(\mathbf{x}, \mathbf{x}_\alpha) \partial_z u_0(\mathbf{x}, \mathbf{x}_\alpha) \Psi_n(\alpha) d\alpha \quad (14.34)$$

where f is the given data, see (14.17).

Remark 14.1 (The approximation context). *Due to the truncation in (14.27), equation (14.33) is within the framework of our approximate mathematical model mentioned in Introduction. Since this paper is concerned with computational rather than theoretical results, then this model is acceptable. Our approximation leads to good numerical results in the simulation section.*

Remark 14.2. *Problem 14.1 is reduced to the problem of finding the vector valued function W satisfying the system (14.33) and the boundary condition (14.34). Assume this vector function is computed and denote it as $W^{\text{comp}} = (w_1^{\text{comp}}, \dots, w_n^{\text{comp}})$. Then, we can compute the function $w^{\text{comp}}(\mathbf{x}, \mathbf{x}_\alpha)$ and then the function $u^{\text{comp}}(\mathbf{x}, \mathbf{x}_\alpha)$ sequentially via (14.27) and (??). The computed target function $p^{\text{comp}}(\mathbf{x})$ is given by (14.18).*

We find an approximate solution of the boundary value problem (14.33)–(14.34) by the Quasi-Reversibility Method. More precisely, we minimize the functional

$$J_\epsilon(W) = \int_{\Omega} \left| \partial_z W(\mathbf{x}) + \sum_{i=1}^{d-1} S_N^{-1} B_i(\mathbf{x}) \partial_{x_i} W(\mathbf{x}) + S_N^{-1} A(\mathbf{x}) W(\mathbf{x}) \right|^2 d\mathbf{x} + \epsilon \|W\|_{H^1(\Omega)^N}^2$$

on the set of vector functions $W \in H^1(\Omega)^N$ satisfying the boundary constraint (14.34). Here the space $H^1(\Omega)^N = \underbrace{H^1(\Omega) \times \cdots \times H^1(\Omega)}_N$ with the commonly defined norm.

14.4 The quasi-reversibility method in finite differences

For brevity, we describe and analyze here the Quasi-Reversibility Method only in the case when $d = 2$. The arguments for higher dimensions can be done in the same manner. In 2D, $\Omega = (-R, R) \times (a, b)$. We arrange an $N_x \times N_z$ grid of points on $\bar{\Omega}$

$$\mathcal{G} = \{(x_i, z_j) : x_i = -R + (i-1)h_x, z_j = a + (j-1)h_z, \\ i = 1, \dots, N_x, j = 1, \dots, N_z\},$$

where $h_x \in [h_0, \beta_x)$ and $h_z \in (0, \beta_z)$ are grid step sizes in the x and z directions respectively and $h_0, \beta_x, \beta_z > 0$ are certain numbers. Here, N_x and N_z are two positive integers. Let $\mathbf{h} = (h_x, h_z)$. We define the discrete set $\Omega^{\mathbf{h}}$ as the set of those points of the set (??) which are interior points of the rectangle Ω and $\partial\Omega^{\mathbf{h}}$ is the set of those points of the set (??) which are located on the boundary of Ω ,

$$\begin{aligned} \Omega^{\mathbf{h}} &= \{(x_i, z_j) : x_i = -R + (i-1)h_x, z_j = a + (j-1)h_z : \\ &\quad i = 2, \dots, N_x - 1; j = 2, \dots, N_z - 1\} \\ \partial\Omega^{\mathbf{h}} &= \{(\pm R, z_j) : j = 1, \dots, N_z\} \cup \{(x_i, z) : i = 1, \dots, N_x, z \in \{a, b\}\}, \\ \bar{\Omega}^{\mathbf{h}} &= \Omega^{\mathbf{h}} \cup \partial\Omega^{\mathbf{h}}. \end{aligned}$$

For any continuous function v defined on Ω its finite difference version is $v^{\mathbf{h}} = v|_{\mathcal{G}}$. Here, $\mathbf{h}$ denotes the pair (h_x, h_z) . The partial derivatives of the function v are given via forward finite differences as

$$\begin{aligned} \partial_x^{h_x} v^{\mathbf{h}}(x_i, z_j) &= \frac{v^{\mathbf{h}}(x_{i+1}, z_j) - v^{\mathbf{h}}(x_i, z_j)}{h_x} \\ \partial_z^{h_z} v^{\mathbf{h}}(x_i, z_j) &= \frac{v^{\mathbf{h}}(x_i, z_{j+1}) - v^{\mathbf{h}}(x_i, z_j)}{h_z} \end{aligned} \quad (14.35)$$

for $i = 0, \dots, N_x - 1$ and $j = 0, \dots, N_z - 1$. We denote the finite difference analogs of the spaces $L^2(\Omega)$ and $H^1(\Omega)$ as $L^{2,\mathbf{h}}(\Omega)$ and $H^{1,\mathbf{h}}(\Omega)$. Norms in these spaces are defined as

$$\begin{aligned} \|v^{\mathbf{h}}\|_{L^{2,\mathbf{h}}(\Omega^{\mathbf{h}})} &= \left[h_x h_z \sum_{i=0}^{N_x} \sum_{j=0}^{N_z} [v^{\mathbf{h}}(x_i, z_j)]^2 \right]^{1/2}, \\ \|v^{\mathbf{h}}\|_{H^{1,\mathbf{h}}(\Omega^{\mathbf{h}})} &= \left[\|v^{\mathbf{h}}\|_{L^{2,\mathbf{h}}(\Omega^{\mathbf{h}})}^2 + h_x h_z \sum_{i=0}^{N_x-1} \sum_{j=0}^{N_z-1} [\partial_x^{h_x} v^{\mathbf{h}}(x_i, z_j)]^2 + [\partial_z^{h_z} v^{\mathbf{h}}(x_i, z_j)]^2 \right]^{1/2}. \end{aligned}$$

Let $F^{\mathbf{h}} = F|_{\partial\Omega^{\mathbf{h}}}$. The problem (14.33)–(14.34) becomes

$$L^h(W^{\mathbf{h}}) = \partial_z^{h_z} W^{\mathbf{h}}(x_i, z_j) + S_N^{-1} B_1(\mathbf{x}_i, z_j) \partial_x^{h_x} W^{\mathbf{h}}(x_i, z_j) + S_N^{-1} A(x_i, z_j) W^{\mathbf{h}}(x_i, z_j) = 0$$

for $i = 0, \dots, N_x - 1; j = 0, \dots, N_z - 1$ and

$$W^{\mathbf{h}}|_{\partial\Omega^{\mathbf{h}}} = F^{\mathbf{h}}. \quad (14.36)$$

To solve problem (??)–(14.36) numerically, we introduce the finite difference version of the functional J_ϵ , defined in (??),

$$\begin{aligned} J_\epsilon^{\mathbf{h}}(W^{\mathbf{h}}) = h_x h_z \sum_{i=0}^{N_x-1} \sum_{j=0}^{N_z-1} & \left| \partial_z^{h_z} W^{\mathbf{h}}(x_i, z_j) + S_N^{-1} B_1(\mathbf{x}_i, z_j) \partial_x^{h_x} W^{\mathbf{h}}(x_i, z_j) \right. \\ & \left. + S_N^{-1} A(x_i, z_j) W^{\mathbf{h}}(x_i, z_j) \right|^2 + \epsilon \|W^{\mathbf{h}}\|_{H_N^{1,\mathbf{h}}(\Omega^{\mathbf{h}})}^2, \end{aligned}$$

where $H_N^{1,\mathbf{h}}(\Omega^{\mathbf{h}}) = [H^{1,\mathbf{h}}(\Omega^{\mathbf{h}})]^N$ and similarly for $L_N^{2,\mathbf{h}}(\Omega^{\mathbf{h}})$. We consider the following problem:

Problem 14.2 (Minimization Problem). *Minimize the functional $J_\epsilon^{\mathbf{h}}(W^{\mathbf{h}})$ on the set of such vector functions $W^{\mathbf{h}} \in H_N^{1,\mathbf{h}}(\Omega^{\mathbf{h}})$ that satisfy boundary condition (14.36).*

The convergence theory for this problem is formulated in Theorems 14.1 and 14.2. Proofs of these theorems follow closely the arguments of [31]. Theorem 14.1 guarantees the existence and uniqueness of the minimizer of $J_\epsilon^{\mathbf{h}}(W^{\mathbf{h}})$, and this result can be proven on the basis of Riesz theorem. The next natural, although a more difficult question, is about the convergence of regularized solutions (i.e. minimizers) to the exact one when the level of the noise in the data tends to zero, i.e. Theorem 14.2. A close analog of Theorem 14.2 is proven in [31, Section 5] via applying a new discrete Carleman estimate: recall that conventional Carleman estimates are in the continuous form. In other words, these two theorems confirm the effectiveness of our proposed numerical method for solving Problem 14.1.

Theorem 14.1 (existence and uniqueness of the minimizer). *For any $\mathbf{h} = (h_x, h_z)$ with $h_x \in [h_0, \beta_x)$, $h_z \in (0, \beta_z)$, any $\epsilon > 0$ and for any matrix $F^{\mathbf{h}}$ of boundary conditions there exists unique minimizer $W_{\min, \epsilon}^{\mathbf{h}} \in H_N^{1,\mathbf{h}}(\Omega^{\mathbf{h}})$ of the functional satisfying boundary condition (14.36).*

As it is always the case in the regularization theory, assume now that there exists an “ideal” solution $W_*^{\mathbf{h}} \in H_N^{1,\mathbf{h}}(\Omega^{\mathbf{h}})$ of problem (??)–(14.36) satisfying the following boundary condition:

$$W_*^{\mathbf{h}}|_{\partial\Omega^{\mathbf{h}}} = F_*^{\mathbf{h}}, \quad (14.37)$$

where $F_*^{\mathbf{h}}$ is the “ideal” noiseless boundary data. Since $W_*^{\mathbf{h}}$ exists, then (14.37) implies that there exists an extension $G_*^{\mathbf{h}} \in H_N^{1,\mathbf{h}}(\Omega^{\mathbf{h}})$ with $G_*^{\mathbf{h}}|_{\partial\Omega^{\mathbf{h}}} = F_*^{\mathbf{h}}$ of the matrix $F_*^{\mathbf{h}}$ in $\Omega^{\mathbf{h}}$. As to the data $F^{\mathbf{h}}$ in (14.36), we assume now that there exists an extension $G^{\mathbf{h}} \in H_N^{1,\mathbf{h}}(\Omega^{\mathbf{h}})$ with $G^{\mathbf{h}}|_{\partial\Omega^{\mathbf{h}}} = F^{\mathbf{h}}$ of $F^{\mathbf{h}}$ in $\Omega^{\mathbf{h}}$. Let $\delta > 0$ be the level of the noise in $G^{\mathbf{h}}$, see Remark 5.1. We assume that

$$\|G^{\mathbf{h}} - G_*^{\mathbf{h}}\|_{H_N^{1,\mathbf{h}}(\Omega^{\mathbf{h}})} < \delta. \quad (14.38)$$

It is convenient to replace the above notation of the minimizer $W_{\min, \epsilon}^{\mathbf{h}}$ with $W_{\min, \epsilon, \delta}^{\mathbf{h}}$, thus, indicating its dependence on δ . A direct analog of Theorem 14.2 (formulated below) was proven in [31] using a new Carleman estimate for the finite difference operator $\partial_z^{h_z} v$. The Carleman Weight Function of this estimate depends only on the discrete variable z . The value of this function at the point $z_j = a + (j - 1)h_z$ is $e^{2\lambda(j-1)h_z}$, where $\lambda > 0$ is a parameter. This estimate is valid only if $\lambda h_z < 1$ (Lemma 4.7 of [31]). The latter explains the condition of Theorem 14.2 imposed on the grid step size h_z in the z -direction.

We now explain why do we impose the condition that the grid step size h_x in the x -direction must be bounded from below as $h_x \geq h_0 = \text{const.} > 0$. Indeed, this bound guarantees that $\|\partial_x^{h_x} W^{\mathbf{h}}\|_{L^{2, \mathbf{h}}(\Omega^{\mathbf{h}})} \leq C \|\partial_x^{h_x} W^{\mathbf{h}}\|_{L^{2, \mathbf{h}}(\Omega^{\mathbf{h}})}$, which is exactly inequality (4.8) of [31]. Note that proofs of convergence results in [31] use the latter inequality quite essentially.

Recall that the minimizer of functional (??) is called *regularized solution* of boundary value problem (14.33), (14.34). Just as we have done for all above presented numerical methods of this report, we now formulate both the **KEY** and the **MOST CHALLENGING** analytical result of the convergence analysis. This result guarantees convergence of the regularized solutions to the exact one when the noise δ in the data tends to zero, see (14.38). Furthermore, this theorem provides an accuracy estimate of the regularized solution depending on δ .

Theorem 14.2 (convergence of regularized solutions). *Let conditions (14.37) and (14.38) be valid. Let $L^{\mathbf{h}}$ be the operator in (??). Let $W_{\min, \epsilon, \delta}^{\mathbf{h}} \in H_N^{1, \mathbf{h}}(\Omega^{\mathbf{h}})$ be the minimizer of the functional $J_{\epsilon}^{\mathbf{h}}(W^{\mathbf{h}})$ with boundary condition (14.36). Then there exists a sufficiently small number $\bar{h}_z > 0$ depending only on $h_0, a, b, R, N, L^{\mathbf{h}}$ such that the following estimate is valid for all $(h_x, h_z) \in [h_0, \beta_x) \times (0, \bar{h}_z)$ and all $\epsilon, \delta > 0$ with a constant $C > 0$ independent on ϵ, δ*

$$\|W_{\min, \epsilon, \delta}^{\mathbf{h}} - W_*^{\mathbf{h}}\|_{L_N^{2, \mathbf{h}}(\Omega^{\mathbf{h}})} \leq C \left(\delta + \sqrt{\epsilon} \|W_*^{\mathbf{h}}\|_{H_N^{1, \mathbf{h}}(\Omega^{\mathbf{h}})} \right).$$

14.5 Numerical implementation

In this section, we solve Problem 14.1 in the 2D case. The domain Ω is

$$\Omega = (-1, 1) \times (1, 3). \quad (14.39)$$

The line of sources L_s is set to be $(-\bar{\alpha}, \bar{\alpha})$ with $\bar{\alpha} = 3$.

We solve the forward problem to compute the simulated data as follows. Given the background function $\mathbf{n}_0$, instead of solving the nonlinear Eikonal equation (14.10), we find $u_0(\mathbf{x}, \mathbf{x}_\alpha)$ using (14.7). To do this, we first find the geodesic line $\Gamma_0(\mathbf{x}, \mathbf{x}_\alpha)$ in (14.7) connecting points $\mathbf{x} \in \Omega$ and $\mathbf{x}_\alpha \in L_s$. We do the latter by using the 2D Fast Marching toolbox which is built in Matlab. The Fast Marching is very similar to the Dijkstra algorithm to find the shortest paths on graphs. We refer the reader to [?] for more details about Fast Marching. Next, with this geodesic line $\Gamma_0(\mathbf{x}, \mathbf{x}_\alpha)$ in hand, we compute the function $u(\mathbf{x}, \mathbf{x}_\alpha)$ via (14.15). It is clear that this function u solves (14.14). The point $\mathbf{x}_\alpha$ above is chosen as $(\alpha_i, 0)$ where $\alpha_i = 2(i-1)\bar{\alpha}/N_\alpha$. We set in our computations $N_\alpha = 209$.

Remark 14.3. Denote by $f^*(\mathbf{x}, \mathbf{x}_\alpha)$ the noiseless data $u(\mathbf{x}, \mathbf{x}_\alpha)$, $\mathbf{x} \in \partial\Omega$, $\mathbf{x}_\alpha \in L_s$. The corresponding noisy data at the noise level $\delta > 0$ are set as

$$f^\delta(\mathbf{x}, \mathbf{x}_\alpha) = f^*(\mathbf{x}, \mathbf{x}_\alpha)(1 + \delta \text{rand}(\mathbf{x}, \mathbf{x}_\alpha)), \quad \mathbf{x} \in \partial\Omega_\alpha^+, \mathbf{x}_\alpha \in L_s, \quad (14.40)$$

where rand is the uniformly distributed function of random numbers taking values in the range $[-1, 1]$. Recall that by (14.17) $f^*(\mathbf{x}, \mathbf{x}_\alpha) = 0$ for $\mathbf{x} \in \partial\Omega_\alpha^-$. This noise generates a noise in the boundary condition F^h in (14.36). Hence, using (14.37), we obtain $F^h = F_*^h(1 + \sigma^h)$, where σ^h is generated by the noisy part of (14.40).

The choice of appropriate values of parameters is always a difficult task. We have selected an appropriate cut-off number N in (14.27) by a trial and error procedure. More precisely, we took Test 4 as a reference test and have selected such a value of N , which gave us the best reconstruction result. We have selected $N = 35$ this way. Then we have used the same $N = 35$ for all other tests.

14.6 Computing W^{comp}

We arrange the grid $\mathcal{G}$ in $\bar{\Omega}$ as in (??). For simplicity, we choose $N_{\mathbf{x}} = N_x = N_z$. The step size $h = h_x = h_z = 2R/(N_{\mathbf{x}} - 1)$. We observe numerically that the matrix S_N^{-1} , present in the definition of J_ϵ in (??), contains some large numbers. This causes some unwanted errors in computations. Therefore, using (14.32), we slightly modify the functional J_ϵ of (??) as:

$$I_\epsilon(W) = \int_{\Omega} |S_N \partial_z W(\mathbf{x}) + A(\mathbf{x})W(\mathbf{x}) + \sum_{i=1}^{d-1} B_i(\mathbf{x}) \partial_{x_i} W(\mathbf{x})|^2 d\mathbf{x} \\ + \epsilon \|W\|_{H^1(\Omega)^N}^2 + \epsilon \left(\|W_{xx}\|_{L^2(\Omega)^N}^2 + \|W_{zz}\|_{L^2(\Omega)^N}^2 \right).$$

We have numerically observed that the additional regularization term with the second derivatives in (14.41) is crucial. If this term is absent, then our numerical results do not meet our expectations.

The procedure of computing $p(\mathbf{x})$ is summarized in Algorithm 1.

In all tests with all noise level in the data, we choose $\epsilon = 10^{-8}$ by a trial and error procedure. The finite difference version of the functional I_ϵ for $d = 2$ is

$$I_\epsilon^h(W) = h^2 \sum_{m=1}^N \sum_{i,j=1}^{N_{\mathbf{x}}-1} \left| \sum_{n=1}^N \left[\frac{s_{mn}[w_n(x_i, z_{j+1}) - w_n(x_i, z_j)]}{h} \right. \right. \\ \left. \left. + a_{mn}(x_i, z_j)w(x_i, y_j) + \frac{b_{mn}(x_i, z_j)(w(x_{i+1}, z_j) - w(x_i, z_j))}{h} \right] \right|^2 \\ + \epsilon h^2 \sum_{n=1}^N \sum_{i,j=0}^{N_{\mathbf{x}}} |w_n(x_i, z_j)|^2 + \epsilon h^2 \sum_{n=1}^N \sum_{i,j=0}^{N_{\mathbf{x}}-1} [|\partial_x^h w_n(x_i, z_j)|^2 + |\partial_z^h w_n(x_i, z_j)|^2] \\ + \epsilon h^2 \sum_{n=1}^N \sum_{i,j=1}^{N_{\mathbf{x}}-1} |\partial_{xx} w_n(x_i, z_j)|^2 + \epsilon h^2 \sum_{n=1}^N \sum_{i,j=1}^{N_{\mathbf{x}}-1} |\partial_{zz}^h w_n(x_i, z_j)|^2$$

where a_{mn} and $b_{mn} = b_{mn,1}$ in (??) and (14.31) respectively. The partial derivatives ∂_x^h and ∂_z^h are as in (14.35). The second derivatives in finite difference are understood as usual.

Algorithm 1 The procedure to solve Problem 14.1

- 1: Choose the cut-off number $N = 35$. Find $\{\Psi_n\}_{n=1}^N$.
- 2: Compute the boundary data of the vector valued function $W(\mathbf{x})$.
- 3: Minimize the functional $I_\epsilon(W)$ subjected to the boundary condition (14.34) to obtain $W^{\text{comp}}(\mathbf{x})$, $\mathbf{x} \in \Omega$.
- 4: Set $w^{\text{comp}}(\mathbf{x}, \mathbf{x}_\alpha) = \sum_{n=1}^N w_n^{\text{comp}} \Psi_n(\alpha)$, $\mathbf{x} \in \Omega$, $\alpha \in [-\bar{\alpha}, \bar{\alpha}]$.
- 5: Set $u^{\text{comp}} = w^{\text{comp}} / \partial_z u_0$. Compute p^{comp} by the average of the left hand side of (14.19), namely

$$p^{\text{comp}} = \frac{1}{2\bar{\alpha}\sqrt{\mathbf{a}_0(\mathbf{x})}} \int_{-\bar{\alpha}}^{\bar{\alpha}} \left[\partial_z u_0(\mathbf{x}, \mathbf{x}_\alpha) \partial_z u^{\text{comp}}(\mathbf{x}, \mathbf{x}_\alpha) + \sum_{i=1}^{d-1} \partial_{x_i} u_0(\mathbf{x}, \mathbf{x}_\alpha) \partial_{x_i} u^{\text{comp}}(\mathbf{x}, \mathbf{x}_\alpha) \right] d\alpha. \quad (14.41)$$

Remark 14.4 (Postprocessing). *In Step 5 of Algorithm 1 when computing p^{comp} , which involves ∇u^{comp} , we smooth out u^{comp} by replacing the value of $u^{\text{comp}}(x, y, \alpha)$ $\alpha \in [-\bar{\alpha}, \bar{\alpha}]$ by the average of u^{comp} on the rectangle of 5×5 points around the point (x, y) . We also apply the same smoothing technique to the function p^{comp} .*

14.7 Numerical tests

We perform four (4) numerical tests. When indicating dependence of any function below on x, z , we assume that $(x, z) \in \Omega$, where the domain Ω is defined in (14.39). In all our tests, the noise level δ is as in (14.40).

Test 1. The true source function p is given by

$$p^{\text{true}}(x, z) = \begin{cases} 8 & (x - 0.5)^2 + (z - 2)^2 < 0.24^2, \\ 5 & (x + 0.5)^2 + (z - 2)^2 < 0.22^2, \\ 0 & \text{otherwise.} \end{cases}$$

The background function $\mathbf{a}_0$ is

$$\mathbf{a}_0(x, z) = \begin{cases} 1 + 0.3(1 - x^2)(z^2 - 2) & \text{if } z^2 - 2 > 0, \\ 1 & \text{otherwise.} \end{cases}$$

The numerical results of this test are displayed in Figure 27.

The support of p^{true} in Test 1 consists of two discs. The value of the function p in the right disc is higher than the value in the left disc. Our method detects both these inclusions very well, see Figures 27. There are some unwanted artifacts near $\partial\Omega$ where we measure the noisy data. The higher level of noisy data, the more artifacts present. When the noise level 5% ($\delta = 0.05$), the computed maximal value of p^{comp} in the left inclusion is 4.97 (relative error 0.6%) and the computed maximal value of p^{comp} in the right inclusion is 7.79 (relative error 2.62%). When the noise level 170% ($\delta = 1.7$), the computed maximal value of p^{comp} in the left inclusion is 4.327 (relative error 13.46%) and the computed maximal value of p^{comp} in the right inclusion is 7.811 (relative error 2.36%).

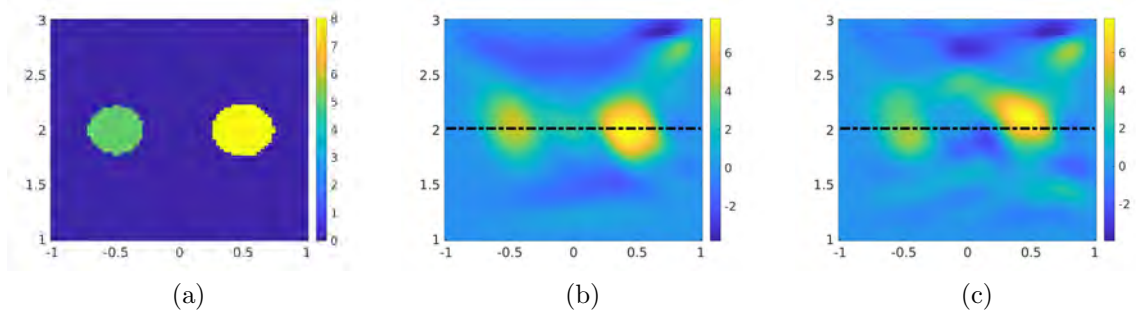


Figure 28: Test 1. The true function p^{true} in (14.14) and reconstructed function p^{comp} . The reconstruction was performed by Algorithm 1 of this section 14. See comments in the text about the accuracy of reconstructed values of the function p^{comp} . a) The true function p^{true} in (14.14). b) The reconstructed function p^{comp} , the noise in the data is 5% ($\delta = 0.05$). c) The reconstructed function p^{comp} , the noise in the data is 170% ($\delta = 1.7$).

Test 2. We test a complicated case when the support of p^{true} looks like a ring. In this test,

$$p^{\text{true}}(x, z) = \begin{cases} 2 & 0.55^2 < r^2 = x^2 + (z - 2)^2 < 0.75^2, \\ 0 & \text{otherwise.} \end{cases}$$

The background function $\mathbf{a}_0$ is given by

$$\mathbf{a}_0(x, z) = \begin{cases} 1 + 0.25(x - 0.5)^2 \ln(z) & z > 1, \\ 1 & \text{otherwise.} \end{cases}$$

The numerical results of this test are displayed in Figure 28.

It is evident that the reconstructed “ring” is acceptable, see Figures 28b),c). The position of the ring is detected quite well. When the noise level is 5% ($\delta = 0.05$), the reconstructed maximal value of p^{comp} in the ring is 2.078 (relative error 3.9%). When the noise level is 100% ($\delta = 1$), the reconstructed maximal value of p^{comp} in the ring is 2.329 (relative error 16.45%).

Test 3. We test an interesting and complicated case of the up-side-down letter Y having both positive and negative values. In this test, the function p^{true} is given by

$$p^{\text{true}}(x, z) = \begin{cases} 2.5 & |x - (z - 2)| < 0.35, \max\{|x|, |z - 2|\} < 0.7, z < 2, x < 0, \\ -2.5 & |x + (z - 2)| < 0.2, \max\{|x|, |z - 2|\} < 0.7, z < 2, x > 0, \\ 2.5 & |x| < 0.2, \max\{|x|, |z - 2|\} < 0.8, z > 2, x < 0, \\ -2.5 & |x| < 0.2, \max\{|x|, |z - 2|\} < 0.8, z > 2, x > 0. \end{cases}$$

The background function $\mathbf{a}_0$ is given by

$$\mathbf{a}_0(x, z) = \begin{cases} 1 + 0.5(x + 0.5)^2 \ln(z) & z > 1, \\ 1 & \text{otherwise.} \end{cases}$$

The numerical results of this test are displayed in Figure 29.

It is clear from Figure 29 that both positive and negative parts of the function $p(x, z)$ are successfully identified. When the noise level is 5% ($\delta = 0.05$), the reconstructed maximal value of the positive part of p^{comp} is 2.186 (relative error 12.56%) and the reconstructed minimal value of p^{comp} of the negative part is -2.482 (relative error 0.72%).

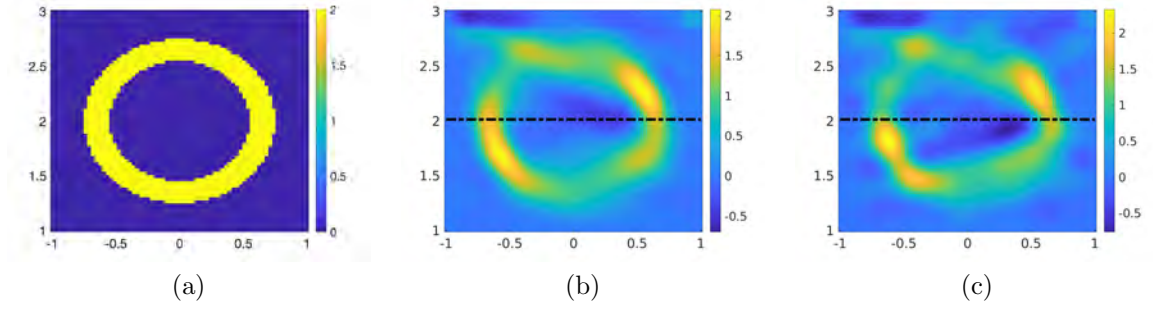


Figure 29: Test 2. A ring-like target. The true function p^{true} in (14.14) and reconstructed function p^{comp} . The reconstruction was performed by Algorithm 1 of this section 14. See comments in the text about the accuracy of reconstructed values of the function p^{comp} . a) The true function p^{true} in (14.14). b) The reconstructed function p^{comp} , the noise in the data is 5% ($\delta = 0.05$). c) The reconstructed function p^{comp} , the noise in the data is 100% ($\delta = 1$).

When the noise level is 100% ($\delta = 1$), the reconstructed maximal value of p^{comp} of the positive part is 2.492 (relative error 0.32%) and the reconstructed minimal value of p^{comp} of the negative part is -3.327 (relative error 33.08%).

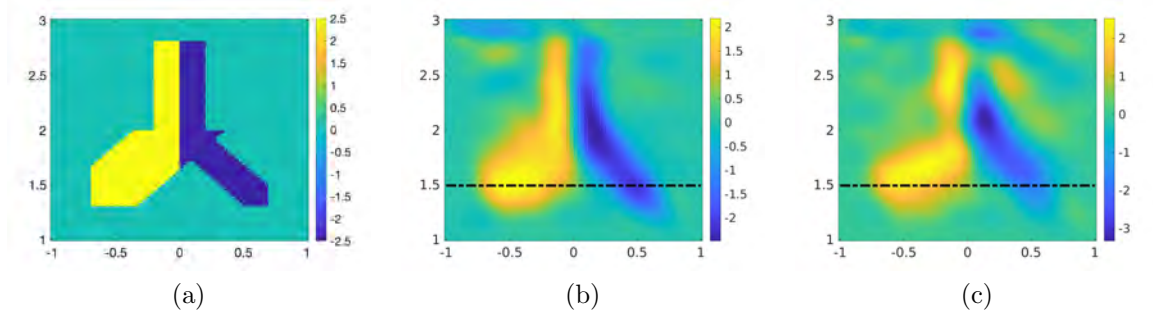


Figure 30: Test 3. The target is the up-side-down letter Y. The true function p^{true} in (14.14) and reconstructed function p^{comp} . The reconstruction was performed by Algorithm 1 of this section 14. See comments in the text about the accuracy of reconstructed values of the function p^{comp} . a) The true function p^{true} in (14.14). b) The reconstructed function p^{comp} , the noise in the data is 5% ($\delta = 0.05$). c) The reconstructed function the noise in the data is 100% ($\delta = 1$).

Test 4. In this test, we reconstruct the letter ‘ λ ’. The function p^{true} is given by

$$p^{\text{true}}(x, z) = \begin{cases} 2 & |x - (z - 2)| < 0.325, \max\{|x|, |z - 2|\} < 0.7 \text{ and } x < -0.03, \\ 2 & |x + (z - 2)| < 0.2 \text{ and } \max\{|x|, |z - 2|\} < 0.7, \\ 0 & \text{otherwise.} \end{cases}$$

In this test, we choose $\mathbf{a}_0$ as

$$\mathbf{a}_0(x, z) = \begin{cases} 1 + x^2 \ln(z) & z > 1, \\ 1 & \text{otherwise.} \end{cases}$$

The numerical results of this test are displayed in Figure 30.

The letter ‘ λ ’ and the values of the function p^{true} are successfully reconstructed. The computed position of the target is a quite accurate one, see Figures 30b),c). When the noise level 5% ($\delta = 0.05$), the computed maximal value of p^{comp} is 2.24 (relative error 12.0%). When the noise level 80% ($\delta = 0.8$) the computed maximal value of p^{comp} is 2.375 (relative error 18.75%).

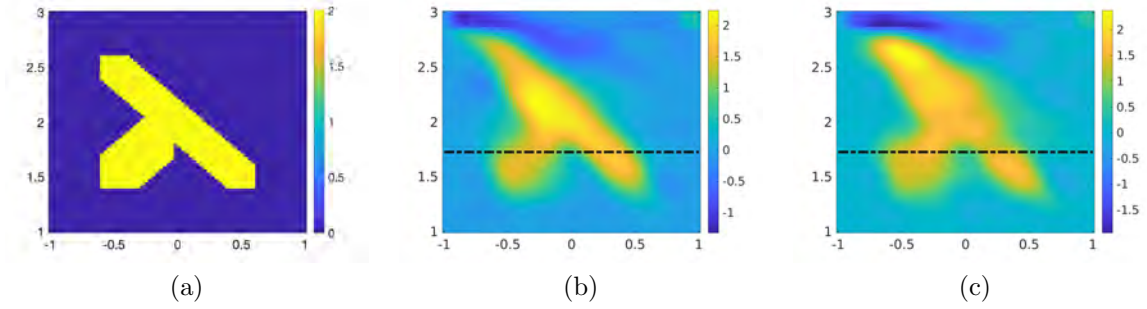


Figure 31: The target is the letter λ . The true function p^{true} in (14.14) and reconstructed function p^{comp} . The reconstruction was performed by Algorithm 1 of this section 14. See comments in the text about the accuracy of reconstructed values of the function p^{comp} . a) The true function p^{true} in (14.14). b) The reconstructed function p^{comp} , the noise in the data is 5% ($\delta = 0.05$). b) The reconstructed function the noise in the data is 80% ($\delta = 0.8$).

14.8 Concluding Remarks

The PI has presented in this section 14 a **PIONEERING** convergent numerical method of [3, Chapter 12], [17] for the solution of the linearized Travel Time Tomography Problem with non overdetermined incomplete data. Just as in all other sections of this report, convergence analysis is based on the convexification idea, which, in turn is based on the powerful tool of Carleman estimates. Analytical dependence on the noise in the data of accuracy estimates of numerical solutions are presented.

Numerical studies have demonstrated a high degree of stability of this method. Indeed, images with acceptable quality are obtained even for very high noise levels between 80% and 170%.

15 The First Globally Convergent Numerical Method for the CIP of the Electrical Impedance Tomography (EIT) [3, Chapter 7], [12]

15.1 Statement of Coefficient Inverse Problem of EIT

We now repeat the statement of the CIP, which was posed in subsection 2.7.

Let Ω be an open bounded domain in $\mathbb{R}^d$ ($d = 2, 3$) of our interest with a smooth boundary $\partial\Omega$.

The EIT Forward Problem. For any given input current

$$g_1 \in L_0^2(\partial\Omega) := \{g \in L^2(\Omega) : \int_{\partial\Omega} g \, ds = 0\}$$

and the conductivity distribution $\sigma(x)$, find the function $u(x) \in H^1(\Omega)$ such that

$$\begin{cases} -\nabla \cdot (\sigma(x) \nabla u(x)) = 0 & \text{in } \Omega, \\ \sigma(x) \frac{\partial u}{\partial \nu} = g_1(x) & \text{on } \partial\Omega, \\ \int_{\partial\Omega} u(x) \, ds = 0, \end{cases}$$

where ν is the outward unit normal vector on $\partial\Omega$. Denote $g_0(x) = u|_{\partial\Omega}$. Then the inverse EIT problem is to recover the internal conductivity function $\sigma(x)$ from the Dirichlet-to-Neumann map data (DtN) $\Lambda : g_0 \rightarrow g_1$.

We consider the EIT Coefficient Inverse Problem in the case when the sources are located outside of the domain of interest and the so-called restricted DtN data measured on the boundary of the domain of interest, as described below. “Restricted DtN data” contain much less information than the conventional DtN data, see [3, section 6.2], [87] for restricted DtN data. In particular, restricted DtN data are always non overdetermined. On the other hand, the standard DtN data are overdetermined in the 3D case [114].

To avoid working with singularities and also to simplify the presentation, we model the point source here by a δ -like function instead of the δ -function. Let $\varepsilon > 0$ be a sufficiently small number. Let the source function $f(x)$ be such that

$$f(x) \in C^\infty(\mathbb{R}^n), f(0) \neq 0, f(x) \geq 0, \forall x \in \mathbb{R}^d, f(x) = 0 \text{ for } |x| > \varepsilon. \quad (15.1)$$

Let $G \subset \mathbb{R}^n$ be a bounded domain with its boundary $\partial G \in C^1$, $\Omega \subset G$ and $\partial\Omega \cap \partial G = \emptyset$. Let $\bar{x} \in \mathbb{R}^{d-1}$ be a fixed point. For $s \in [0, 1]$ denote $x_s = (x_{1s}, \bar{x})$ the position of the point source. Let $I = \{x_s = (x_{1s}, \bar{x}) : s \in [0, 1]\}$ be the interval of the straight line $\{x = (x_1, \bar{x}), x_1 \in \mathbb{R}\}$. Let $I_\varepsilon = \{x \in \mathbb{R}^d : \text{dist}(x, I) < \varepsilon\}$, where $\text{dist}(x, I)$ is the Hausdorff distance between the point x and the set I . We also assume that $I_\varepsilon \subset (G \setminus \bar{\Omega})$, which means that the support of the source function is outside of the domain Ω .

Let the function

$$\sigma(x) \in C^{2+\alpha}(\bar{G}), \sigma(x) = 1 \text{ for } x \in G \setminus \Omega \text{ and } \sigma(x) \geq \sigma_0 = \text{const.} > 0. \quad (15.2)$$

Here $\alpha = \text{const.} \in (0, 1)$ and $C^{k+\alpha}(\bar{G})$ be the Hölder space, where $k \geq 0$ is an integer. Assume first that $\sigma(x)$ is known. For each source position $x_s \in I$ we define the forward boundary value problem for EIT as the problem of finding the function $u(x, s)$ such that

$$\begin{cases} \nabla \cdot (\sigma(x) \nabla u(x, s)) = -f(x - x_s), & x \in G, \forall x_s \in \bar{I} \\ u(x, s)|_{x \in \partial G} = 0, & \forall x_s \in \bar{I}. \end{cases} \quad (15.3)$$

It is well known that for each $x_s \in I$ the problem (15.3) has a unique solution

$$u(x, s) \in C^{3+\alpha}(\bar{G}), \forall x_s \in \bar{I}, \quad (15.4)$$

see, e.g. [66]. We measure both Dirichlet and Neumann boundary conditions of the function u on a part $\Gamma \subseteq \partial\Omega$ of the boundary $\partial\Omega$,

$$u(x, s)|_{x \in \Gamma, x_s \in \bar{I}} = g_0(x, s) \text{ and } \partial_\nu u(x, s)|_{x \in \Gamma, x_s \in \bar{I}} = g_1(x, s). \quad (15.5)$$

We call the Dirichlet and Neumann boundary data (15.5) “restricted DtN data”.

If the coefficient $\sigma(x)$ is known, then, having the solution of the forward problem (15.3), one can easily compute functions $g_0(x, s)$ and $g_1(x, s)$. Suppose now that the function $\sigma(x)$ is unknown. Then we arrive at the following inverse problem:

Coefficient Inverse Problem (CIP). *Assume that the function $\sigma(x)$ is unknown for $x \in \Omega$ and also that conditions (15.1), (15.2) hold. Also, assume that functions $g_0(x, s)$ and $g_1(x, s)$ in (15.5) are known for all $x \in \Gamma, x_s \in \bar{I}$. Determine the function $\sigma(x)$.*

Note that in this CIP the number d of free variables in the data equals the number of free variables in the unknown coefficient.

15.2 An equivalent problem

In this subsection, we transform the above CIP to an inverse problem for a quasilinear PDE. First, introduce the well known change of variables

$$u_1 = \sqrt{\sigma}u, \quad (15.6)$$

where $u(x, s)$ is the solution of problem (15.3). Then

$$\begin{cases} \Delta u_1(x, s) + a_0(x)u_1(x, s) &= -f(x - x_s), & \forall x_s \in \bar{I}, \\ u_1(x, s)|_{x \in \partial G} &= 0, & \forall x_s \in \bar{I}, \end{cases} \quad (15.7)$$

where

$$a_0(x) = -\frac{\Delta(\sqrt{\sigma(x)})}{\sqrt{\sigma(x)}}. \quad (15.8)$$

Recalling that $\sigma = 1$ on $\partial\Omega$, we obtain from (15.5)

$$u_1(x, s)|_{x \in \Gamma, s \in [0, 1]} = g_0(x, s) \text{ and } \partial_n u_1(x, s)|_{x \in \Gamma, s \in [0, 1]} = g_1(x, s). \quad (15.9)$$

If we would recover the function $a_0(x)$ for $x \in \Omega$ from the conditions (15.7) and (15.9), then, assuming that 0 is not an eigenvalue of the elliptic operator $\Delta + a_0(x)$ with the Dirichlet boundary condition either on $\partial\Omega$ or on ∂G , we would recover the function $\sigma(x)$ via solving the elliptic equation (15.8) either in the domain Ω with the Dirichlet boundary condition $\sigma|_{\partial\Omega} = 1$, or in the domain G with the Dirichlet boundary condition $\sigma|_{\partial G} = 1$. Hence, we focus below on the recovery of the function $a_0(x)$ for $x \in \Omega$ from conditions (15.7), (15.9).

It follows from (15.1), (15.3), (15.6) and the maximum principle for elliptic equations [66] that $u_1(x, s) > 0$ for all $x \in \bar{\Omega}$ and all $s \in [0, 1]$. Hence, we can consider the function $v(x, s)$,

$$v(x, s) = \ln u_1(x, s). \quad (15.10)$$

Then $u_1(x, s) = e^{v(x, s)}$ and (15.7) imply that

$$\Delta v(x, s) + (\nabla v(x, s))^2 = -a_0(x), \quad x \in \Omega, \forall s \in [0, 1]. \quad (15.11)$$

Here we use (15.1) and the fact that $I_\varepsilon \subset (G \setminus \overline{\Omega})$. In addition, using (15.9), we obtain

$$v(x, s)|_{x \in \Gamma, s \in [0, 1]} = \tilde{g}_0(x, s) \text{ and } \partial_\nu v(x, s)|_{x \in \Gamma, s \in [0, 1]} = \tilde{g}_1(x, s), \quad (15.12)$$

where

$$\tilde{g}_0(x, s) = \ln g_0(x, s) \text{ and } \tilde{g}_1(x, s) = \frac{g_1(x, s)}{g_0(x, s)}.$$

Differentiating equation (15.11) with respect to s and noting that the function $a_0(x)$ is independent on s , we obtain

$$\Delta v_s + 2\nabla v_s \cdot \nabla v = 0, \quad x \in \Omega, \forall s \in [0, 1]. \quad (15.13)$$

Now the above CIP is reduced to the following problem:

Reduced Problem. *Recover the function $v(x, s)$ from the equation (15.13), given the boundary measurements $\tilde{g}_0(x, s)$ and $\tilde{g}_1(x, s)$ in (15.12).*

If the function $v(x, s)$ is approximated, then the approximate coefficient $a_0(x)$ can be found via (15.11). Thus, our focus below is on the solution of Reduced Problem.

15.3 Cauchy problem for a system of coupled quasilinear elliptic equations

We now again use the orthonormal basis of [87]. Consider now the same orthonormal basis $\{\Psi_n(\alpha)\}_{n=0}^\infty$ in the space $L_2(0, 1)$ as the one in subsections 9.3, 12.4, 13.5 and 14.3. This basis was originally constructed in [87], also, see [3, section 6.2]. Fix an integer $N \geq 1$. Denote $\Psi(N) = \{\Psi_n(s)\}_{n=0}^{N-1}$. We approximate the function $v(x, s)$ in (15.10) as:

$$v(x, s) = \sum_{n=0}^{N-1} v_n(x) \Psi_n(s), \quad x \in \Omega, \forall s \in [0, 1]. \quad (15.14)$$

Here and below we use “=” instead of “ $\approx$ ” for convenience. Note that the functions $v_n(x)$ are unknown and should be determined and they are targets of our considerations below.

We assume that the approximation (15.14) of the true function $v(x, s)$ satisfies equation (15.11). This is our approximate mathematical model.

The derivative $v_s(x, s)$ of the function $v(x, s)$ in (??) is

$$v_s(x, s) = \sum_{n=0}^{N-1} v_n(x) \Psi'_n(s), \quad x \in \Omega, \forall s \in [0, 1] \quad (15.15)$$

By (15.4) and (15.6) it is reasonable to assume that the functions $v_n(x)$ are such that

$$v_n \in C^3(\overline{\Omega}), n = 0, \dots, N-1. \quad (15.16)$$

It is likely that (15.16) can be proven using the classical theory of elliptic PDEs [66]. However, we are not doing this here for brevity.

Substituting (15.14) and (15.15) in (15.13), we obtain

$$\sum_{n=0}^{N-1} \Delta v_n(x) \Psi'_n(s) + \sum_{n,k=0}^{N-1} \nabla v_n(x) \nabla v_k(x) \Psi'_n(s) \Psi_k(s) = 0, \quad x \in \Omega, \forall s \in [0, 1]. \quad (15.17)$$

Consider the vector function of unknown coefficients $v_n(x)$ in the expansion (15.14),

$$V(x) = (v_0(x), \dots, v_{N-1}(x))^T. \quad (15.18)$$

For $m = 0, \dots, N-1$ multiply both sides of (15.17) by the function $\Psi_m(s)$ and then integrate with respect to $s \in (0, 1)$. Using (15.16) and (15.18), we obtain

$$S_N \Delta V - \tilde{F}(\nabla V) = 0, \quad x \in \Omega, V \in C^3(\overline{\Omega}), \quad (15.19)$$

where the N -dimensional vector function $\tilde{F}$ is quadratic with respect to the first derivatives $\partial_{x_j} v_k(x)$, $j = 1, \dots, d$; $k = 0, \dots, N-1$. Since the matrix S_N is invertible [3, Theorem 6.2.1], [87], then multiplying both sides of (15.19) by the inverse matrix S_N^{-1} , we obtain a system of coupled quasilinear elliptic equations,

$$\Delta V - F(\nabla V) = 0, \quad x \in \Omega, V \in C^3(\overline{\Omega}), \quad (15.20)$$

$$F(\nabla V) = S_N^{-1} \tilde{F}(\nabla V). \quad (15.21)$$

Since the vector function $\tilde{F}$ is quadratic with respect to the first derivatives $\partial_{x_j} v_k(x)$, then (15.21) implies that the vector function F is also quadratic. In addition, using (15.12), we obtain Cauchy data for the vector function $V(x)$ on Γ ,

$$V(x)|_{\Gamma} = p_0(x), \quad \partial_\nu V(x)|_{\Gamma} = p_1(x). \quad (15.22)$$

If we would solve the Cauchy problem (15.20), (15.22), then we would find the coefficients $v_n(x)$ in (15.14). Next, we would substitute (15.14) in (15.11) and obtain the following approximate formula for the function $a_0(x)$:

$$a_0(x) = - \sum_{n=0}^{N-1} \Delta v_n(x) \Psi_n(s) + \left(\sum_{n=0}^{N-1} \nabla v_n(x) \Psi_n(s) \right)^2, \quad x \in \Omega, s \in (0, 1). \quad (15.23)$$

As to the value of the parameter s for which the function $a_0(x)$ should be calculated in (15.23), it should be chosen numerically. Hence, we develop below a numerical method for solving problem (15.20), (15.22).

15.4 Two Carleman estimates

Since in our numerical examples the domain $\Omega \subset \mathbb{R}^2$ is a disk, we formulate in this subsection a new Carleman estimate for the Laplace operator, which is specifically used for the disk in the 2D case and for the ball in the 3D case. We work with the case when $\Gamma = \partial\Omega$ since this is done in our numerical experiments. In principle, Carleman estimates are known for this kind of domains, see, e.g. [81]. However, the CWF in [81] has a rather complicated form and changes too rapidly, which is inconvenient for computations. This is the reason of presenting here two new Carleman estimates with a simple CWF.

15.4.1 The 3D case

We derive in this section a new Carleman estimate for the 3D case when the domain Ω is a ball of the radius ρ ,

$$\Omega = \{x \in \mathbb{R}^3 : |x| < \rho\}. \quad (15.24)$$

Let $\mu \in (0, \rho)$ be a number. Define the domain Ω_μ as

$$\Omega_\mu = \{x \in \mathbb{R}^3 : \mu < |x| < \rho\} \subset \Omega. \quad (15.25)$$

Consider the spherical coordinates

$$r = |x| \in (\mu, \rho), \varphi \in (0, 2\pi), \theta \in (0, \pi),$$

$$x_1 = r \cos \varphi \sin \theta, x_2 = r \sin \varphi \sin \theta, x_3 = r \cos \theta.$$

Also, denote

$$S_\rho = \{r = \rho\}, S_\mu = \{r = \mu\}.$$

The Laplace operator in the spherical coordinates is

$$\Delta_{\text{sp}} w = w_{rr} + \frac{1}{r^2 \sin^2 \theta} w_{\varphi\varphi} + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta w_\theta) + \frac{2}{r} w_r.$$

Everywhere below $C = C(\Omega_\mu) > 0$ denotes different constants depending only on the domain Ω_μ defined in (15.25). Let

$$\nabla w = (w_{x_1}, w_{x_2}, w_{x_3})^T.$$

Note that since $w_\varphi = -w_{x_1} \sin \varphi \sin \theta + w_{x_2} \cos \varphi \sin \theta$, then the function $w_\varphi / \sin \theta$ does not have a singularity. Introduce the subspace $H_0^m(\Omega_\mu)$ of the Hilbert space $H^m(\Omega_\mu)$ as

$$H_0^m(\Omega_\mu) = \{u \in H^m(\Omega_\mu) : u|_{S_\rho} = u_r|_{S_\rho} = 0\}, \quad m = 2, 3.$$

Theorem 15.1 (Carleman estimate in the 3D case). *There exists a number $\lambda_0 = \lambda_0(\Omega_\mu) \geq 1$ and a number $C = C(\Omega_\mu) > 0$, both depending only on the domain Ω_μ , such that for any function $w \in H^2(\Omega_\mu)$ and for all $\lambda \geq \lambda_0$ the following Carleman estimate with the CWF $e^{2\lambda r}$ holds:*

$$\begin{aligned} \int_{\Omega_\mu} (\Delta w)^2 e^{2\lambda r} dx &\geq \frac{1}{2} \int_{\Omega_\mu} (\Delta w)^2 e^{2\lambda r} dx + C\lambda \int_{\Omega_\mu} (\nabla w)^2 e^{2\lambda r} dx + C\lambda^3 \int_{\Omega_\mu} w^2 e^{2\lambda r} dx \\ &\quad - C\lambda e^{2\lambda \rho} \int_{S_\rho} w_r^2 dS - C\lambda^3 e^{2\lambda \rho} \int_{S_\rho} w^2 dS - C\lambda^3 e^{2\lambda \mu} \|w\|_{H^2(\Omega_\mu)}^2. \end{aligned} \quad (15.26)$$

In particular, if $w \in H_0^2(\Omega_\mu)$, then

$$\begin{aligned} \int_{\Omega_\mu} (\Delta w)^2 e^{2\lambda r} dx &\geq \frac{1}{2} \int_{\Omega_\mu} (\Delta w)^2 e^{2\lambda r} dx + C\lambda \int_{\Omega_\mu} (\nabla w)^2 e^{2\lambda r} dx + C\lambda^3 \int_{\Omega_\mu} w^2 e^{2\lambda r} dx \\ &\quad - C\lambda^3 e^{2\lambda \mu} \|w\|_{H^2(\Omega_\mu)}^2. \end{aligned} \quad (15.27)$$

15.4.2 The 2D case

In this case we keep the same notations for the domains Ω, Ω_μ as those in subsection 15.4.1. We assume that they are the domains in $\mathbb{R}^2$. Polar coordinates are

$$r = |x| \in (\mu, \rho), \varphi \in (0, 2\pi),$$

$$x_1 = r \cos \varphi, x_2 = r \sin \varphi.$$

The Laplace operator in polar coordinates is

$$\Delta_p w = w_{rr} + \frac{1}{r^2} w_{\varphi\varphi} + \frac{1}{r} w_r.$$

Theorem 15.2 (Carleman estimate in the 2D case). *There exists a number $\lambda_0 = \lambda_0(\Omega_\mu) \geq 1$ depending only on the domain Ω_μ such that for any function $w \in H^2(\Omega_\mu)$ and for all $\lambda \geq \lambda_0$ the following Carleman estimate holds:*

$$\begin{aligned} \int_{\Omega_\mu} (\Delta w)^2 e^{2\lambda r} dx &\geq \frac{1}{2} \int_{\Omega_\mu} (\Delta w)^2 e^{2\lambda r} dx + C\lambda \int_{\Omega_\mu} (\nabla w)^2 e^{2\lambda r} dx + C\lambda^3 \int_{\Omega_\mu} w^2 e^{2\lambda r} dx \\ &\quad - C\lambda e^{2\lambda\rho} \int_{S_\rho} w_r^2 dS - C\lambda^3 e^{2\lambda\rho} \int_{S_\rho} w^2 dS - C\lambda e^{2\lambda\mu} \|w\|_{H^2(\Omega_\mu)}^2. \end{aligned}$$

In particular, if $w \in H_0^2(\Omega_\mu)$, then

$$\begin{aligned} \int_{\Omega_\mu} (\Delta w)^2 e^{2\lambda r} dx &\geq \frac{1}{2} \int_{\Omega_\mu} (\Delta w)^2 e^{2\lambda r} dx + C\lambda \int_{\Omega_\mu} (\nabla w)^2 e^{2\lambda r} dx + C\lambda^3 \int_{\Omega_\mu} w^2 e^{2\lambda r} dx \\ &\quad - C\lambda^3 e^{2\lambda\mu} \|w\|_{H^2(\Omega_\mu)}^2. \end{aligned}$$

15.5 Hölder stability and uniqueness of the Cauchy problem (15.20), (15.22)

We establish in this subsection the Hölder stability estimate for problem (15.20), (15.22). Uniqueness follows immediately from this estimate. We work here only with the 3D case. Theorem 15.2 implies that the 2D case can be handled almost exactly the same way. Thus, in this subsection the domain Ω is as in (15.24), and in (15.22) $\Gamma = \partial\Omega = S_\rho = \{r = \rho\}$.

Suppose that there exist two solutions of problem (15.20), (15.22) $V_1, V_2 \in H_N^2(\Omega) \cap C_N^1(\overline{\Omega})$ such that

$$V_1|_{S_\rho} = p_0, V_2|_{S_\rho} = p_{0,\delta}, V_{1,r}|_{S_\rho} = p_1, V_{2,r}|_{S_\rho} = p_{1,\delta}, \quad (15.28)$$

where

$$\|p_{0,\delta} - p_0\|_{L^2(S_\rho)} \leq \delta, \|p_{1,\delta} - p_1\|_{L^2(S_\rho)} \leq \delta, \quad (15.29)$$

where $\delta \in (0, 1)$ is a sufficiently small number which is interpreted as the level of the noise in the data. Denote

$$\tilde{V} = V_1 - V_2, \tilde{p} = p_0 - p_{0,\delta}, \tilde{p}_1 = p_1 - p_{1,\delta}. \quad (15.30)$$

Recalling that the function F in (15.20) is quadratic with respect to the derivatives $\partial_{x_j} v_k(x)$, we obtain from (15.20), (15.22)

$$\Delta \tilde{V} = Q(\nabla V_1, \nabla V_2) \nabla \tilde{V}, \quad x \in \Omega, \tilde{V} \in H_N^2(\Omega) \cap C_N^1(\overline{\Omega}), \quad (15.31)$$

$$\tilde{V}|_{S_\rho} = \tilde{p}_0, \tilde{V}_r|_{S_\rho} = \tilde{p}_1, \quad (15.32)$$

where the vector function $Q(\nabla V_1, \nabla V_2)$ is linear with respect to components of the vector functions $\nabla V_1, \nabla V_2$.

Theorem 15.3 (Hölder stability estimate). *For any two vector functions $V_1, V_2 \in H_N^2(\Omega) \cap C_N^1(\overline{\Omega})$ introduced in this section, let $\|V_1\|_{C^1(\overline{\Omega}_N)}, \|V_2\|_{C^1(\overline{\Omega}_N)} \leq A$, where $A = \text{const.} > 0$. Let (15.28)-(15.30) hold. Choose a number $\eta \in (0, \rho - \mu)$ and let $\Omega_{\mu+\eta} = \{x : \mu + \eta < |x| < \rho\} \subset \Omega_\mu$. Then there exists a number $C_1 = C_1(\Omega_\mu, \eta, F, \Psi(N), A) > 0$ and a sufficiently small number $\delta_0 = \delta_0(\Omega_\mu, \eta, F, \Psi(N), A) \in (0, 1)$ such that, for all $\delta \in (0, \delta_0)$ the following Hölder stability estimate holds:*

$$\|\tilde{V}\|_{H_N^1(\Omega_{\mu+\eta})} \leq C_1 \delta^\gamma, \quad \text{where } \gamma = \eta / (4\rho).$$

All above numbers depend only on listed parameters.

15.6 Convexification

To solve the Cauchy problem (15.20), (15.22) numerically, we construct in this subsection a weighted Tikhonov-like functional with the CWF $e^{2\lambda r}$ in it. For brevity, we construct the Tikhonov-like functional only for the 3D case. The 2D case is completely similar and direct analogs of Theorems 15.4-15.7 (below) are valid in 2D.

We assume that in (15.22)

$$\Gamma = \partial\Omega = S_\rho; \quad p_0, p_1 \in C^3(S_\rho). \quad (15.33)$$

Let $\eta \in (0, \rho - \mu)$ be the number which was chosen in Theorem 15.3. Our weighted Tikhonov-like functional is:

$$J_{\lambda, \beta}(V) = e^{-2\lambda(\mu+\eta)} \int_{\Omega_\mu} [\Delta V - F(\nabla V)]^2 e^{2\lambda r} dx + \beta \|V\|_{H_N^4(\Omega_\mu)}^2. \quad (15.34)$$

Here $\beta \in (0, 1)$ is the regularization parameter and the multiplier $e^{-2\lambda(\mu+\eta)}$ is introduced to balance two terms in the right hand side of (15.34). Let $R > 0$ be an arbitrary number. Define the set $B(R) \subset H^3(\Omega_\mu)$ of the diameter $2R$ as:

$$B(R) = \left\{ V \in H_N^4(\Omega_\mu) : V|_{S_\rho} = p_0, V_r|_{S_\rho} = p_1, \|V\|_{H_N^4(\Omega_\mu)} < R \right\}. \quad (15.35)$$

We consider the following minimization problem:

Minimization Problem. *Minimize the functional $J_{\lambda, \beta}(V)$ on the closed set $\overline{B(R)}$. Define the subspace $H_0^4(\Omega_\mu)$ of the space $H^4(\Omega_\mu)$ as:*

$$H_0^4(\Omega_\mu) = \{u \in H^4(\Omega_\mu) : u|_{S_\rho} = u_r|_{S_\rho} = 0\}.$$

15.7 Convergence analysis

The central analytical result of this section 15 is Theorem 5.4.

Theorem 15.4: *Choose a number $\eta \in (0, \rho - \mu)$.*

1. *The functional $J_{\lambda,\beta}(V)$ has the Fréchet derivative $J'_{\lambda,\beta}(V) \in H_{0N}^4(\Omega_\mu)$ at every point $V \in H_N^3(\Omega_\mu)$. Furthermore, there exist numbers $\lambda_1 = \lambda_1(\mu, \eta, F, \Psi(N), P, R) \geq \lambda_0 > 0$ and $C_1 = C_1(\mu, \eta, F, \Psi(N), P, R) > 0$ depending only on listed parameters such that $2e^{-\lambda_2\eta} < 1$ and for all $\lambda \geq \lambda_1$ the functional $J_{\lambda,\beta}(V)$ is strictly convex on $\overline{B(R)}$ for the choice of β as*

$$\beta \in (2e^{-\lambda\eta}, 1). \quad (15.36)$$

More precisely, the following inequality holds:

$$\begin{aligned} & J_{\lambda,\beta}(V_2) - J_{\lambda,\beta}(V_1) - J'_{\lambda,\beta}(V_1)(V_2 - V_1) \\ & \geq C_1 \|\Delta(V_2 - V_1)\|_{L_{2,N}(\Omega_{\mu+\eta})} + C_1 \|V_2 - V_1\|_{H_N^1(\Omega_{\mu+\eta})}^2 + \frac{\beta}{2} \|V_2 - V_1\|_{H_N^4(\Omega_\mu)}^2, \quad (15.37) \\ & \forall V_1, V_2 \in \overline{B(R)}. \end{aligned}$$

2. *For each $\lambda \geq \lambda_1$ and for each β as in (15.36) there exists unique minimizer $V_{\min} \in \overline{B(R)}$ of the functional $J_{\lambda,\beta}(V)$ on the set $\overline{B(R)}$ and*

$$[J'_{\lambda,\beta}(V_{\min,\lambda,\beta}), V_{\min,\lambda,\beta} - V] \leq 0, \quad \forall V \in \overline{B(R)},$$

where $[\cdot, \cdot]$ is the scalar product in the space of real valued $N-D$ vector functions $H_N^4(\Omega_\mu)$.

Note that, allowing the regularization parameter $\beta \in (2e^{-\lambda\eta}, 1)$, we actually allow β to be sufficiently small.

Consider now the gradient descent method of the minimization of the functional $J_{\lambda,\beta}$ on the set $B(R)$. This is similar with (6.4) of section 6. Let $\alpha \in (0, 1/3)$ be an arbitrary number. Consider an arbitrary point $V_0 \in B(\alpha R)$ and construct the sequence of the gradient method:

$$V_n = V_{n-1} - \zeta J'_{\lambda,\beta}(V_{n-1}), \quad n = 1, 2, \dots, \quad (15.38)$$

where $\zeta \in (0, 1)$ is a sufficiently small number. Note that since $J'_{\lambda,\beta}(V_{n-1}) \in H_{0N}^4(\Omega_\mu)$ by Theorem 15.4, then all functions V_n have the same boundary conditions as ones in (15.35). Theorem 15.5 follows from a combination of Theorem 15.4 with Theorem 6.1 of section 6.

Theorem 15.5. *Choose a number $\eta \in (0, \rho - \mu)$. Let $\lambda_1 = \lambda_1(\mu, \eta, F, N, P, R) \geq \lambda_0 > 0$ be the number of Theorem 15.4. Choose a number $\lambda \geq \lambda_2$ and choose the regularization parameter $\beta \in (2e^{-\lambda\eta}, 1)$. Let $V_{\min,\lambda,\beta} \in \overline{B(R)}$ be the unique minimizer of the functional $J_{\lambda,\beta}(V)$ claimed in Theorem 15.4. Assume that $V_{\min,\lambda,\beta} \in B(\alpha R)$. Then there exists a sufficiently small number $\zeta_0 = \zeta_0(\mu, \eta, F, \Psi(N), P, R, \lambda, \beta) \in (0, 1)$ depending only on listed parameters such that for every $\zeta \in (0, \zeta_0)$ sequence (15.38) converges to the minimizer $V_{\min,\lambda,\beta}$ and the following estimate of the convergence rate holds:*

$$\|V_n - V_{\min,\lambda,\beta}\|_{H_N^4(\Omega_{\mu+\eta})} \leq \omega^n \|V_0 - V_{\min,\lambda,\beta}\|_{H_N^4(\Omega_\mu)}, \quad n = 1, 2, \dots,$$

where the number $\omega = \omega(\zeta) \in (0, 1)$ depends only on the parameter ζ .

Even though Theorem 15.5 guarantees the convergence of the gradient method to the unique minimizer of the functional $J_{\lambda,\beta}(V)$, it is not yet clear how far this minimizer is

from the exact solution. To address this question, we assume, as it is commonly accepted in the theory of ill-posed problems [116], that there exists an exact solution

$$V^* \in B^*(R) = \left\{ V \in H_N^4(\Omega_\mu) : V|_{S_\rho} = p_0^*, V_r|_{S_\rho} = p_1^*, \|V\|_{H_N^4(\Omega_\mu)} < R \right\}. \quad (15.39)$$

of the problem (15.20), (15.22) with the noiseless data p_0^*, p_1^* .

Let $\delta \in (0, 1)$ be a sufficiently small number characterizing the level of the noise in the data. Suppose that there exists two vector functions $P, P^* \in H_N^4(\Omega_\mu)$ such that

$$P|_{S_\rho} = p_0, P_r|_{S_\rho} = p_1, \quad (15.40)$$

$$P^*|_{S_\rho} = p_0^*, P_r^*|_{S_\rho} = p_1^*, \quad (15.41)$$

$$\|P - P^*\|_{H_N^4(\Omega_\mu)} < \delta. \quad (15.42)$$

Theorem 15.6. *Let $\lambda_1 \geq \lambda_0 > 0$ and $C_1 > 0$ be numbers of Theorem 15.4. Choose the number $\delta_1 > 0$ so small that $\delta_1 < \min(e^{-4\rho\lambda_1}, 3^{-4\rho/\eta})$ and let $\delta \in (0, \delta_1)$. Set*

$$\lambda = \lambda(\delta) = \ln \delta^{-1/(4\rho)} \text{ and } \beta = \beta(\delta) = 3\delta^{\eta/(4\rho)} = 3e^{-\lambda\eta} \in (2e^{-\lambda\eta}, 1).$$

Let (15.40)-(15.42) be true. Also, assume that the vector function $V^* \in B^*(R)$, where the set was defined in (15.39). Let $V_{\min, \lambda(\delta), \beta(\delta)} \in \overline{B(R)}$ be the minimizer of functional $J_{\lambda, \beta}(V)$, which is guaranteed by Theorem 5.3, where the set $B(R)$ is defined in (15.35). Also, let the number $\zeta \in (0, \zeta_0)$ in (15.38) be the same as in Theorem 15.5, so as the number $\omega \in (0, 1)$. Then the following estimates hold:

$$\|V^* - V_{\min, \lambda(\delta), \beta(\delta)}\|_{H_N^1(\Omega_{\mu+\eta})} \leq C_1 \delta^{\eta/(8\rho)}, \quad (15.43)$$

$$\|\Delta V^* - \Delta V_{\min, \lambda(\delta), \beta(\delta)}\|_{L_{2N}(\Omega_{\mu+\eta})} \leq C_1 \delta^{\eta/(8\rho)}, \quad (15.44)$$

$$\|V^* - V_n\|_{H_N^1(\Omega_{\mu+\eta})} \leq C_1 \delta^{\eta/(8\rho)} + \omega^n \|V_0 - V_{\min, \lambda(\delta), \beta(\delta)}\|_{H_N^4(\Omega_\mu)}, \quad n = 1, 2, \dots, \quad (15.45)$$

$$\|\Delta V^* - \Delta V_n\|_{L_{2,N}(\Omega_{\mu+\eta})} \leq C_1 \delta^{\eta/(8\rho)} + \omega^n \|V_0 - V_{\min, \lambda(\delta), \beta(\delta)}\|_{H_N^4(\Omega_\mu)}, \quad n = 1, 2, \dots \quad (15.46)$$

In the case of noiseless data with $\delta = 0$ one should replace in (15.43)-(15.46) $\delta^{\eta/(8\rho)}$ with $\sqrt{\beta}$, where $\beta = 3e^{-\lambda\eta}$ and $\lambda \geq \lambda_1$.

While (15.43)-(15.46) are convergence estimates for the vector function $V^*(x)$, we still need to obtain a convergence estimate for our target coefficient $a_0(x)$ in equation (15.11). This is done in Theorem 15.7. Let $a_0^*(x)$ be the exact coefficient $a_0(x)$ which corresponds to $V^*(x)$ via (15.23), i.e.

$$a_0^*(x) = - \sum_{k=0}^{N-1} \Delta v_k^*(x) \Psi_k(\bar{s}) + \left(\sum_{k=0}^{N-1} \nabla v_k^*(x) \Psi_k(\bar{s}) \right)^2, \quad x \in \Omega. \quad (15.47)$$

Next, let $V_{\min, \lambda(\delta), \beta(\delta)}(x) = (v_{0, \min, \lambda(\delta), \beta(\delta)}(x), \dots, v_{N-1, \min, \lambda(\delta), \beta(\delta)}(x))^T$ and let

$$a_{0, \min, \lambda(\delta), \beta(\delta)}(x) = - \sum_{k=0}^{N-1} \Delta v_{k, \min, \lambda(\delta), \beta(\delta)}(x) \Psi_k(\bar{s}) \quad (15.48)$$

$$+ \left(\sum_{k=0}^{N-1} \nabla v_{k,\min,\lambda(\delta),\beta(\delta)}(x) \Psi_k(\bar{s}) \right)^2, \quad x \in \Omega.$$

Let $V_n(x) = \left(v_0^{(n)}(x), \dots, v_{N-1}^{(n)}(x) \right)^T$. Define the function $a_{0,n}(x)$ as

$$a_{0,n}(x) = - \sum_{k=0}^{N-1} \Delta v_k^{(n)}(x) \Psi_k(\bar{s}) + \left(\sum_{k=0}^{N-1} \nabla v_k^{(n)}(x) \Psi_k(\bar{s}) \right)^2, \quad x \in \Omega, \quad (15.49)$$

where $\bar{s} \in (0, 1)$ is a certain fixed number.

Theorem 15.7. *Assume that the conditions of Theorem 15.6 hold. Then the following analogs of estimates (15.43)-(15.46) are in place:*

$$\|a_0^* - a_{0,\min,\lambda(\delta),\beta(\delta)}\|_{L_2(\Omega_{\mu+\eta})} \leq C_1 \delta^{\eta/(8\rho)}, \quad (15.50)$$

$$\|a_0^* - a_{0,n}\|_{L_2(\Omega_{\mu+\eta})} \leq C_1 \delta^{\eta/(8\rho)} + \omega^n \|V_0 - V_{\min,\lambda(\delta),\beta(\delta)}\|_{H_N^4(\Omega_\mu)}, \quad n = 1, 2, \dots, \quad (15.51)$$

where the functions a_0^* , $a_{0,\min,\lambda(\delta),\beta(\delta)}$ and $a_{0,n}$ are defined in (15.47)-(15.49). In the case of noiseless data with $\delta = 0$ one should replace in (15.50) and (15.51) $\delta^{\eta/(8\rho)}$ with $\sqrt{\beta}$, where $\beta = 3e^{-\lambda\eta} \in (2e^{-\lambda\eta}, 1)$ and $\lambda \geq \lambda_1$.

15.8 Numerical studies

We have applied the above technique to numerical studies of the inverse EIT problem in the 2D case. Recall that even though theorems 15.4-15.7 are formulated only in the 3D case, their direct analogs are also valid in the 2D case due to the Carleman estimate of Theorem 15.2. In this subsection we describe our numerical results. Hence, in this section

$$\Omega = \{r \in (0, \rho)\} \subset \mathbb{R}^2, \Omega_\mu = \{r \in (\mu, \rho)\} \subset \Omega.$$

We have found in our computations that the influence of the regularization parameter β in (15.34) is not essential. Hence, we set $\beta := 0$ in our computational examples.

Let $G \subset \mathbb{R}^2$ be the bounded domain introduced in subsection 15.1. We solve our forward problem (2.67) in G . Our domain of interest $\Omega \subset G$. In all our numerical examples

$$G = \{x_1^2 + x_2^2 < 5\}, \Omega = \{x_1^2 + x_2^2 \leq 1\} \text{ and } \Omega_\mu = \{r \in (0.01, 1)\} \subset \Omega.$$

We measure the data on the whole boundary $\partial\Omega = S_1$. The source runs over the circle $C^{(s)} = \{x_1^2 + x_2^2 = 4\} \subset G$. In other words, in polar coordinates

$$x_s = (r, s) = (2, s), \quad s = \varphi \in (0, 2\pi), \quad x_s \in C^{(s)}. \quad (15.52)$$

However, when constructing the required orthonormal basis $\{\Psi_n(s)\}_{n=0}^\infty$, we still have used functions $\{s^n e^s\}_{n=0}^\infty$, i.e. we did not impose the periodicity condition on this basis. The source function $f(x)$ in our case is the bump function below:

$$f(x - x^{(s)}) = \begin{cases} \frac{1}{\varepsilon} \exp\left(-\frac{1}{1-|x-x_s|^2/\varepsilon}\right), & \text{if } (x - x_s)^2 < \varepsilon, \\ 0, & \text{otherwise.} \end{cases}$$

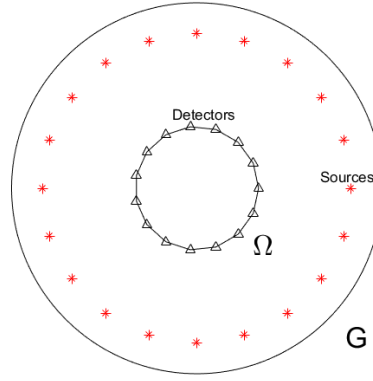


Figure 32: A schematic diagram of data collection in the EIT problem for numerical tests of subsection 15.7.

We have chosen $\varepsilon = 0.01$.

We use 32 sources and 32 detectors. In numerical examples both sources and detectors are uniformly distributed over the whole circle $\{x_1^2 + x_2^2 = 4\}$ and the whole circle $S_1 = \{x_1^2 + x_2^2 = 1\}$ respectively.

To solve the forward problem (2.67), we have used the standard FEM. However, to minimize functional (15.34), we have written the differential operators in it via finite differences. Thus, we have not committed “inverse crime”. To use the finite differences, we have discretized the domain Ω_μ in polar coordinates using the uniform finite difference mesh. Next, we have used the gradient descent method to minimize functional (15.34) with respect to the values of the vector function $V(r, \varphi)$ at grid points. As the basis ψ_k is not periodic over $[0, 2\pi]$, we treat numerically $s = 0$ and $s = 2\pi$ as two different discrete points.

As to the choice of the parameter λ , even though the above theory works only for sufficiently large values of λ , we have established in our computational experiments that the choice

$$\lambda = 1$$

is sufficient for all numerical tests we have performed. This choice of λ corresponds well with all other similar choices in the above sections of this report. We have also tested three different values of the number N terms in the series (15.14):

$$N = 4, 6, 8.$$

Our computational results indicate that $N = 8$ is the best choice out of these three. All computational results below are for the case $N = 8$.

We have found in our computational experiments that the gradient descent method for our weighted Tikhonov-like functional (15.34) converges rapidly on a coarse mesh. This provides us with a rough image. Hence, we have implemented a multi-level method [122]. Let $M_{h_1} \subset M_{h_2} \dots \subset M_{h_K}$ be nested finite difference meshes, i.e. M_{h_k} is a refinement of $M_{h_{k-1}}$ for $k \leq K$. Let P_{h_k} be the corresponding finite difference functional space. On the first level M_{h_1} , we solve the discrete optimization problem. In other words, let $V_{h_1, \min}$ be

the minimizer of the following functional which is found via the gradient descent method

$$J_{\lambda}^{(h_1)}(V_{h_1}) = e^{-2\lambda(\mu+\eta)} \int_{\Omega_{\mu}} [\Delta V_{h_1} - F(\nabla V_{h_1})]^2 e^{2\lambda r} dx, \quad (15.53)$$

where the integral is understood in the discrete sense. Then we interpolate the minimizer $V_{h_1, \min, \lambda}$ on the finer mesh M_{h_2} and take the resulting vector function $V_{h_2, \text{int}}$ as the starting point of the gradient descent method of the optimization of the direct analog of functional (15.53) in which h_1 is replaced with h_2 and V_{h_1} is replaced with V_{h_2} . This process was repeated until we got the minimizer $V_{h_K, \min}$ on the K_{th} level on the mesh M_{h_K} .

Since $(r, \varphi) \in (0, 1) \times (0, 2\pi)$, then our first level M_{h_1} is set to be the uniform mesh with the mesh size in the r direction to be $1/4$ and the mesh size in the φ direction to be $2\pi/8$. For each mesh refinement, we will refine the mesh in both r direction and φ direction in a way that we set the mesh size of the refined mesh **in both directions** to be $1/2$ of the previous mesh sizes. On each level M_{h_k} , as soon as we see that $\|\nabla J_{\lambda}^{(h_k)}(V_{h_k})\| < 2 \times 10^{-2}$, we refine the mesh and compute the solution on the next level $M_{h_{k+1}}$. In the end, we compute $a_0(x)$ using the relation (??) with $s = 0$.

Our starting point $W^{(0)}(r, \varphi)$ for the vector function $V(r, \varphi)$ for the gradient descent method on the coarse mesh M_{h_1} is set to be the background solution $V_{\text{bckgr}}^{(0)}(r, \varphi, 1)$ which corresponds to the solution of the problem (2.67) with $\sigma(x) \equiv 1$. Hence, our starting point is not located in a small neighborhood of the exact solution.

15.9 Numerical testing

In the tests of this sub-subsection, we demonstrate the efficiency of our numerical method for imaging of small inclusions as well as for imaging of a smoothly varying function $\sigma(x)$, i.e. a “stretched” inclusion with a wide range of change of the conductivity inside of it. In particular, we test the case of a rather high contrast 5:1 of the inclusion. In all tests the background value of the conductivity is $\sigma_{\text{bckgr}} = 1$. All necessary derivatives of the data were calculated using finite differences. We have not observed instabilities due to the differentiation, most likely because the step sizes of finite differences were not too small.

Test 1: In this example, we test the reconstruction method for a single inclusion with a rather high inclusion/background contrast 5:1. The results are shown on Figure 32.

Test 2. We now test our method for a quite challenging case when the function $\sigma(x)$ is smoothly varying within an abnormality and with a wide range of variations between 0.4 and 1.6. The noise is added for $x \in S_1$ and for the source $s \in [0, 2\pi]$:

$$g_{0, \text{noise}}(x, s) = g_0(x, s)(1 + \delta \xi_s) \text{ and } g_{1, \text{noise}}(x, s) = g_1(x, s)(1 + \delta \xi_s), \quad (15.54)$$

where δ is the noise level and ξ_s is the independent random variable depending only on the source position s and uniformly distributed on $[-1, 1]$. The results are shown in Figure 33. Thus, our method can accurately image not only “sharp” inclusions as in Test 1, but smoothly varying functions as well.

Test 3. Here we present another challenging case when the function $\sigma(x)$ is in the form of a ring shape with $\sigma = 1.6$ inside the ring and $\sigma = 1$ otherwise. The results are shown in Figure 34. Thus, we image this case quite accurately, including the maximal computed value of $\sigma \approx 1.6$.

Our conclusion from Figures 32-34 is that our method works quite well numerically.

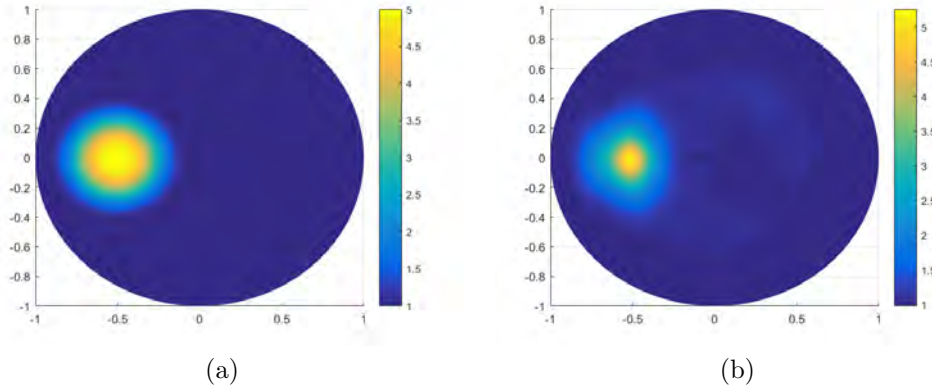


Figure 33: *Test 1 for the EIT problem. There is a quite high inclusion/background contrast 5:1. a) Correct image. b) Computed image. The high 5:1 inclusion/background contrast is computed accurately.*

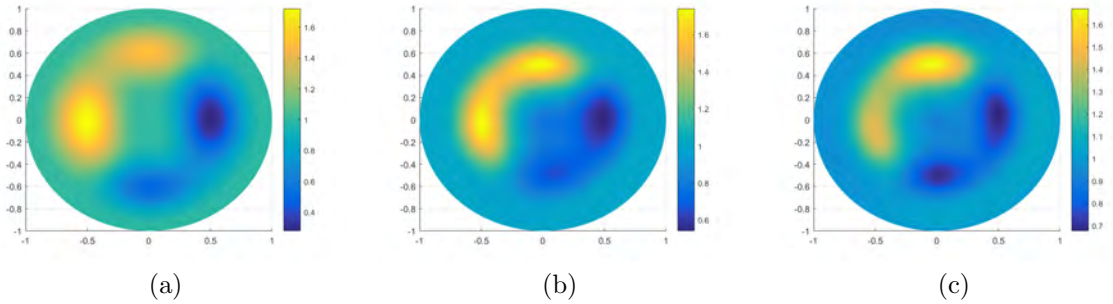


Figure 34: *Test 2 for the EIT problem. This is a quite challenging case when the function $\sigma(x)$ is smoothly varying within an abnormality and with a wide range of variations between 0.4 and 1.6. A random noise of 1% and 10% is added to the data as in (15.54) with $\delta = 0.01$ and $\delta = 0.1$ respectively. a) Correct image. b) Computed image for the case of 1% of noise. c) Computed image for the case of 10% of noise. Thus, our method can accurately image not only “sharp” inclusions as in Test 1, but smoothly varying functions as well.*

16 Convexification for a Coefficient Inverse Problem for a Parabolic Equation [3, Chapter 9], [14]

In this section, we construct a globally convergent numerical method for a Coefficient Inverse Problem (CIP) for a parabolic PDE. This method is based on the so-called *convexification* concept. Both convergence analysis and numerical results are presented. The CIP, which is considered here, has applications in heat conduction [41] and in medical optical imaging using the diffuse infrared light [64].

In this section, we describe the version of the convexification method for a CIP for a parabolic PDE. Results are taken from [14]. The first step towards this goal was made in [85] for a similar CIP. However, there are some problems in [?], which prevent one from a numerical implementation of the idea of [85]. Indeed, although a weighted globally strictly convex Tikhonov-like functional is constructed in [85], the CWF in it is too complicated

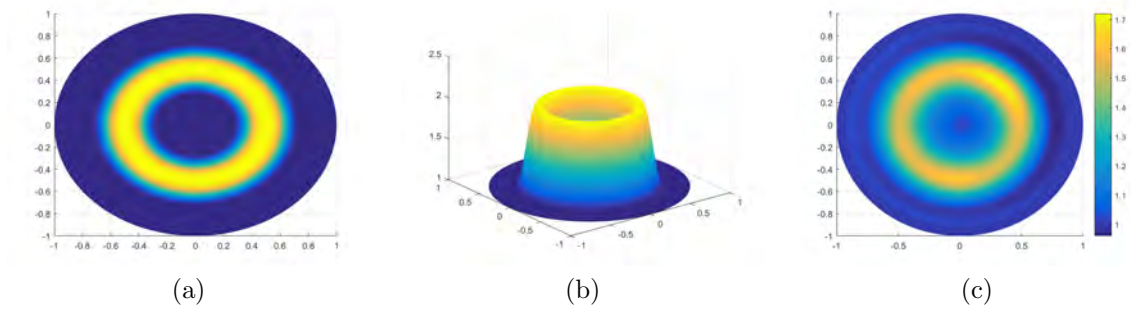


Figure 35: *Test 2 for the EIT problem. This is a quite challenging case when the function $\sigma(x)$ has a ring shape with $\sigma = 1.6$ inside the ring and $\sigma = 1$ otherwise. a) Correct image. b) 3D representation of a). c) Computed image. d) 3D representation of c). Thus, we image this case quite accurately, including the maximal computed value of $\sigma \approx 1.6$.*

since it depends on two large parameters rather than on a single one. This means that the CWF of [85] changes too rapidly. The latter does not allow a numerical implementation.

Thus, we have established in [14] a new Carleman estimate with a simpler CWF, which can be used for computations. Our central result is the global strict convexity of our weighted Tikhonov-like functional. Next, we establish the existence and uniqueness of its minimizer, estimate the distance between that minimizer and the exact solution and establish the global convergence of the gradient projection method to the exact solution. Finally, we present results of our numerical experiments.

16.1 Statement of the Coefficient Inverse Problem

Below $\mathbf{x} = (x, \bar{x}) \in \mathbb{R}^n$, where $\bar{x} = (x_2, \dots, x_n)$ and $x = x_1$. Let the numbers $A, B > 0$ and $A < B$. We introduce the cube $\Omega \subset \mathbb{R}^n$ and a part Γ of its boundary $\partial\Omega$ as

$$\Omega = \{\mathbf{x} : A < x, x_2, \dots, x_n < B\}, \quad \Gamma = \{x = B, A < x_2, \dots, x_n < B\}. \quad (16.1)$$

Let the number $T > 0$. Denote

$$Q_T^\pm = \Omega \times (-T, T), \quad S_T^\pm = \partial\Omega \times (-T, T), \quad \Gamma_T^\pm = \Gamma \times (-T, T).$$

Below $\alpha \in (0, 1)$, $m \geq 1$ is an integer and $C^{m+\alpha}(\bar{\Omega})$, $C^{2m+\alpha, m+\alpha/2}(\bar{Q}_T^\pm)$ are Hölder spaces [92]. Let

$$b_j(\mathbf{x}), c(\mathbf{x}) \in C^{2+\alpha}(\bar{\Omega}); \quad j = 1, \dots, n.$$

We consider the elliptic operator L in the following form:

$$Lu = \Delta u + \sum_{j=1}^n b_j(\mathbf{x}) u_{x_j} - c(\mathbf{x}) u, \quad \mathbf{x} \in \Omega. \quad (16.2)$$

We assume that

$$c(\mathbf{x}) \geq 0 \quad \text{in } \bar{\Omega}. \quad (16.3)$$

The forward parabolic initial boundary value problem is stated as [92]:

Forward Problem. Let the initial condition $f(\mathbf{x}) \in C^{4+\alpha}(\overline{\Omega})$. Find a function $u(\mathbf{x}, t) \in C^{4+\alpha, 2+\alpha/2}(\overline{Q_T^\pm})$ satisfying the following conditions:

$$u_t = Lu \text{ in } Q_T^\pm, \quad (16.4)$$

$$u(\mathbf{x}, -T) = f(\mathbf{x}), \quad (16.5)$$

$$u|_{S_T^\pm} = g_0(\mathbf{x}, t). \quad (16.6)$$

If $n = 3$ and functions $b_j(\mathbf{x}) \equiv 0$ for $j = 1, \dots, n$, then $c(\mathbf{x})$ is the absorption coefficient in the case of medical optical imaging using the diffuse infrared light [64].

If the domain Ω would have its boundary $\partial\Omega \in C^{4+\alpha}$ and if the Dirichlet condition $g_0(\mathbf{x}, t)$ would belong to $C^{4+\alpha, 2+\alpha/2}(\overline{S_T^\pm})$ and also corresponding compatibility conditions would be satisfied [92], then the existence and uniqueness of the solution $u \in C^{4+\alpha, 2+\alpha/2}(\overline{Q_T^\pm})$ of problem (16.2)–(16.6) would be ensured [92]. However, for the convenience of our derivations for the inverse problem, we have chosen the case of a piecewise smooth boundary $\partial\Omega$. Hence, we can only assume the existence of the solution $u \in C^{4+\alpha, 2+\alpha/2}(\overline{Q_T^\pm})$ of problem (16.4)–(16.6). As to its uniqueness, it follows immediately from (16.3) and the maximum principle for parabolic PDEs.

Coefficient Inverse Problem (CIP). Let the number $t_0 \in (-T, T)$. Suppose that the function $g_0(\mathbf{x}, t)$ in (16.6) is known. Also, assume that the function $f(\mathbf{x})$ in (16.5) as well as the coefficient $c(\mathbf{x})$ are unknown. In addition, let the following two functions $g_1(\overline{x}, t)$ and $f_0(\mathbf{x})$ be known:

$$u_x|_{\Gamma_T^\pm} = g_1(\overline{x}, t), \quad (16.7)$$

$$u(\mathbf{x}, t_0) = f_0(\mathbf{x}). \quad (16.8)$$

Find the unknown coefficient $c(\mathbf{x})$.

Remarks 16.1:

1. Thus, the data for this CIP is the single pair of functions (g_1, f_0) . These data are non overdetermined ones, so as for all CIPs for which the convexification method works. In other words, the number m of free variables in the data equals the number n of free variables in the unknown coefficient, $m = n$.
2. Assuming that $f_0(\mathbf{x}) \neq 0$ in $\overline{\Omega}$, uniqueness of a very similar CIP was proven for any value of $T > 0$ and for the case of a rather general bounded domain Ω using the method of [53], see, e.g. theorem 1.10.7 in [45], theorem 2 in [78], theorem 3.10 in [79], theorem 3.4 in [83] and theorem 3.4.3 in [3]. We also refer to [120] for the Lipschitz stability estimates for this CIP.
3. As soon as the coefficient $c(\mathbf{x})$ is found via the solution of the above CIP, one can uniquely determine the function $f(\mathbf{x})$ in (16.5) using functions g_0 and f_0 [94]: the knowledge of g_1 is not necessary then, as long as $c(\mathbf{x})$ is known. This is the so-called “parabolic problem with the reversed time”. Since this problem is outside of our the scope, we now only provide two short comments. A recent convergent numerical method for the parabolic problem with the reversed time can be found in [18].

16.2 Weighted globally strictly convex Tikhonov-like functional

We assume below that there exists a number $\mu > 0$ such that

$$f(\mathbf{x}) \geq \mu, \quad \forall \mathbf{x} \in \bar{\Omega}, \quad (16.9)$$

$$g_0(\mathbf{x}, t) \geq \mu, \quad \forall (\mathbf{x}, t) \in \overline{S_T^\pm}. \quad (16.10)$$

Then (16.3), (16.9), (16.10) and the maximum principle for parabolic PDEs [92] imply that

$$u(\mathbf{x}, t) \geq \mu \text{ in } \overline{Q_T^\pm}. \quad (16.11)$$

16.3 Nonlinear integral differential equation

Using (16.11), we introduce a new function $v(\mathbf{x}, t)$,

$$v(\mathbf{x}, t) = \ln u(\mathbf{x}, t) \rightarrow u = e^v. \quad (16.12)$$

Substituting (16.12) in (16.4)–(16.8), we obtain in $Q_T^\pm$:

$$v_t - \Delta v - (\nabla v)^2 - \sum_{k=1}^n b_j(\mathbf{x}) v_{x_j} = c(\mathbf{x}), \quad (16.13)$$

$$v|_{S_T^\pm} = \ln g_0(\mathbf{x}, t), \quad v_x|_{\Gamma_T^\pm} = \frac{g_1(\bar{x}, t)}{g_0((B, \bar{x}), t)}, \quad (16.14)$$

$$v(\mathbf{x}, t_0) = \ln f_0(\mathbf{x}) := \tilde{f}_0(\mathbf{x}). \quad (16.15)$$

For brevity we set below $t_0 := 0$. The case $t_0 \neq 0$ can be considered along the same lines. Differentiate both sides of the nonlinear equation (16.13) with respect to t and denote $w(\mathbf{x}, t) = v_t(\mathbf{x}, t)$. Since the function $c(\mathbf{x})$ is independent on t , then the right hand side of the resulting equation will be zero. By (16.15)

$$v(\mathbf{x}, t) = \int_0^t w(\mathbf{x}, \tau) d\tau + \tilde{f}_0(\mathbf{x}), \quad (\mathbf{x}, t) \in Q_T^\pm. \quad (16.16)$$

Thus, (16.14)–(16.16) lead to a nonlinear integral differential PDE with Volterra integrals, supplied by the lateral Cauchy data,

$$N(w) = w_t - Mw - 2\nabla w \int_0^t \nabla w(\mathbf{x}, \tau) d\tau = 0, \quad (\mathbf{x}, t) \in Q_T^\pm, \quad (16.17)$$

$$w|_{S_T^\pm} = p_0(\mathbf{x}, t), \quad w_x|_{\Gamma_T^\pm} = p_1(\bar{x}, t), \quad (16.18)$$

where $p_0(\mathbf{x}, t) = (g_{0t}/g_0)(\mathbf{x}, t)$ and $p_1(\bar{x}, t) = \partial_t(g_1(\bar{x}, t)/g_0(B, \bar{x}, t))$, where

$$Mw = Lw + 2\nabla w \nabla \tilde{f}_0 = \Delta w + \sum_{j=1}^n b_j(\mathbf{x}) w_{x_j} + 2\nabla w \nabla \tilde{f}_0. \quad (16.19)$$

16.4 The functional

There are a number of possible choices of the CWF for the parabolic operator, see, e.g. [3, 45, 83, 94, 120]. However, among all these choices, we should select such a CWF which would be simple and would work well computationally. Indeed, the CWFs of [3, 45, 83, 94, 120] depend on two large parameters, which means that they change too rapidly. As it was stated in Introduction, that rapid change is inconvenient for a implementation. Recall that by (16.1) $\mathbf{x} = (x, x_2, \dots, x_n)$. Thus, we have chosen the CWF $\varphi_\lambda(x, t)$ as:

$$\varphi_\lambda(x, t) = \exp(2\lambda(x^2 - t^2)), \quad (16.20)$$

where $\lambda \geq 1$ is a parameter. This means that we need to prove the Carleman estimate with this CWF.

Let $[(n+1)/2]$ be the maximal integer which does not exceed $(n+1)/2$. Denote $k_n = [(n+1)/2] + 3$. For example, we have for the most popular cases of $n = 1, 2, 3$:

$$k_n = \begin{cases} 4 & \text{if } n = 1, 2, \\ 5 & \text{if } n = 3. \end{cases}$$

We have chosen the number k_n in such a way that

$$H^{k_n}(Q_T^\pm) \subset C^2(\overline{Q_T^\pm}), \|q\|_{C^2(\overline{Q_T^\pm})} \leq C_0 \|q\|_{H^{k_n}(Q_T^\pm)}, \forall q \in H^{k_n}(Q_T^\pm), \quad (16.21)$$

where the number $C_0 = C_0(Q_T^\pm) > 0$ depends only on the domain $Q_T^\pm$. Relations (16.21) follow from the embedding theorem.

Let $R > 0$ be an arbitrary number. We define the bounded set of functions $B(R, p_0, p_1)$, on which we find the solution w of the problem (16.17)–(16.18), as follows:

$$B(R, p_0, p_1) = \left\{ w \in H^{k_n}(Q_T^\pm) : \|w\|_{H^{k_n}(Q_T^\pm)} < R, w|_{S_T^\pm} = p_0, w_x|_{\Gamma_T^\pm} = p_1 \right\}. \quad (16.22)$$

Let $\beta > 0$ be a small regularization parameter and $N(w)$ be the nonlinear integral differential operator defined in (16.17). We construct our weighted Tikhonov-like functional with the CWF (16.20) in it as:

$$J_{\lambda, \beta}(w) = e^{-2\lambda B^2} \int_{Q_T^\pm} (N(w))^2 \varphi_\lambda d\mathbf{x}dt + \beta \|w\|_{H^{k_n}(Q_T^\pm)}^2. \quad (16.23)$$

Since $\max_{\overline{Q_T^\pm}} \varphi_\lambda = e^{2\lambda B^2}$, then the multiplier $e^{-2\lambda B^2}$ is introduced in (16.23) to balance two terms in the right hand side of (16.23).

Minimization Problem. *Minimize the functional $J_{\lambda, \beta}(w)$ on the set $\overline{B(R)}$ defined in (16.22).*

Assume for a moment that a minimizer $w_{\min, \lambda, \beta}(\mathbf{x}, t)$ of functional (16.23) exists and is computed. Then we substitute $w_{\min, \lambda, \beta}(\mathbf{x}, t)$ in the integral of (16.16). Let the function $v_{\text{comp}}(\mathbf{x}, t)$ be the resulting left hand side of (16.16). Next, substituting $w_{\min, \lambda, \beta}(\mathbf{x}, t)$ in equation (16.13), we calculate an approximation for the target unknown coefficient $c(\mathbf{x})$. However, due to the inevitable computational errors as well as the noise in the

data, the resulting left hand side of (16.13) would depend on t . Hence, to calculate an approximation $c_{\text{comp}}(\mathbf{x})$ for $c(\mathbf{x})$, we set

$$c_{\text{comp}}(\mathbf{x}) = \frac{1}{2\gamma T} \int_{-\gamma T}^{\gamma T} \left(\partial_t v_{\text{comp}} - \Delta v_{\text{comp}} - (\nabla v_{\text{comp}})^2 - \sum_{k=1}^n b_j(\mathbf{x}) \partial_{x_j} v_{\text{comp}} \right) dt, \quad (16.24)$$

where the number $\gamma \in (0, 1/\sqrt{3})$ is chosen in subsection 16.6. Thus, we focus below on the Minimization Problem.

16.5 Convergence analysis

Introduce the subspaces $H_0^{2,1}(Q_T^\pm) \subset H^{2,1}(Q_T^\pm)$ and $H_0^{k_n}(Q_T^\pm) \subset H^{k_n}(Q_T^\pm)$ as

$$H_0^{2,1}(Q_T^\pm) = \left\{ u \in H^{2,1}(Q_T^\pm) : u|_{S_T^\pm} = 0, u_x|_{\Gamma_T^\pm} = 0 \right\}, \quad (16.25)$$

$$H_0^{k_n}(Q_T^\pm) = \left\{ u \in H^{k_n}(Q_T^\pm) : u|_{S_T^\pm} = 0, u_x|_{\Gamma_T^\pm} = 0 \right\}.$$

Since it is well known that any Carleman estimate depends only on the principal part of the operator, see, e.g. [83, 94], then we consider in Theorem 13 only the principal part $\partial_t - \Delta$ of the parabolic operator $\partial_t - L$.

Theorem 16.1 (Carleman estimate). *Suppose that the domain Ω and the CWF $\varphi_\lambda(x, t)$ are the same as in (16.1) and (16.20) respectively. Then there exist numbers λ_0, C ,*

$$\lambda_0 = \lambda_0(\Omega) \geq 1, C = C(\Omega, T) > 0 \quad (16.26)$$

depending only on listed parameters such that the following Carleman estimate holds

$$\begin{aligned} \int_{Q_T^\pm} (u_t - \Delta u)^2 \varphi_\lambda d\mathbf{x} dt &\geq \frac{C}{\lambda} \int_{Q_T^\pm} \left(u_t^2 + \sum_{i,j=1}^n u_{x_i x_j}^2 \right) \varphi_\lambda d\mathbf{x} dt \\ &\quad + C\lambda \int_{Q_T^\pm} [(\nabla u)^2 + \lambda^2 u^2] \varphi_\lambda d\mathbf{x} dt \\ &\quad - C \exp(2\lambda(B^2 - T^2)) \int_{\Omega} \lambda^2 u^2(\mathbf{x}, T) d\mathbf{x} \\ &\quad - C \exp(2\lambda(B^2 - T^2)) ((\nabla u)^2 + \lambda^2 u^2)(\mathbf{x}, -T), \end{aligned} \quad (16.27)$$

for all $\lambda \geq \lambda_0$ and $u \in H_0^{2,1}(Q_T^\pm)$.

For brevity, let K denotes below the following vector of numbers:

$$K = \left(\max_j \|b_j\|_{C(\bar{\Omega})}, \mu \right). \quad (16.28)$$

Theorem 16.2 (the central theorem of this section 16). *Let $\mu, R > 0$ be two numbers. Assume that condition (16.11) holds. Let $B(R, p_0, p_1)$ be the set defined in (16.22). Then:*

1. The functional $J_{\lambda,\beta}(w)$ has the Fréchet derivative $J'_{\lambda,\beta}(w) \in H_0^{k_n}(Q_T^\pm)$ for all $\lambda, \beta > 0$ and for all $w \in H^{k_n}(Q_T^\pm)$ satisfying boundary conditions (16.18). Then there exist constants

$$\lambda_1 = \lambda_1(R, T, \Omega, K) \geq \lambda_0, \quad (16.29)$$

$$C_1 = C_1(R, T, \Omega, K) > 0 \quad (16.30)$$

depending only on listed parameters such that if $\lambda \geq \lambda_1$ and the regularization parameter $\beta \in (2e^{-\lambda T^2}, 1)$, then the functional $J_{\lambda,\beta}(w)$ is strictly convex on the set $\overline{B(R, p_0, p_1)}$ for all $\lambda \geq \lambda_1$, i.e. for all $w_1, w_2 \in \overline{B(R, p_0, p_1)}$ and for all $\lambda \geq \lambda_1$ the following estimate is valid:

$$\begin{aligned} & J_{\lambda,\beta}(w_2) - J_{\lambda,\beta}(w_1) - J'_{\lambda,\beta}(w_1)(w_2 - w_1) \\ & \geq \frac{C_1}{\lambda} \exp(-2\lambda(T^2 + B^2 - A^2)) \|w_2 - w_1\|_{H^{2,1}(Q_T^\pm)}^2 + \frac{\beta}{2} \|w_2 - w_1\|_{H^{k_n}(Q_T^\pm)}^2. \end{aligned} \quad (16.31)$$

2. For every value of $\lambda \geq \lambda_1$ and for every value of the regularization parameter $\beta \in (2e^{-\lambda T^2}, 1)$ there exists unique minimizer $w_{\min,\lambda,\beta} \in \overline{B(R, p_0, p_1)}$ of the functional $J_{\lambda,\beta}(w)$ on the set $\overline{B(R, p_0, p_1)}$. Furthermore, the following inequality holds:

$$J'_{\lambda,\beta}(w_{\min,\lambda,\beta})(w_{\min,\lambda,\beta} - w) \leq 0, \quad \forall w \in \overline{B(R, p_0, p_1)}. \quad (16.32)$$

Everywhere below in this section $C > 0$ and $C_1 > 0$ denote different constants depending only on parameters listed in (16.26) and (16.30) respectively, also see (16.28) for K .

By one of main concepts of the regularization theory [116], we assume now that there exists an ideal, the so-called ‘exact’ solution $c^*(\mathbf{x}) \in C^{2+\alpha}(\overline{\Omega})$ of the CIP (2.72), (16.4)–(16.8), where the data (16.6)–(16.8) are noiseless. Having the function $c^*(\mathbf{x})$, one can consider the noise free solution $w^* \in H^{k_n}(Q_T^\pm)$ of equation (16.17) with the noiseless boundary data p_0^*, p_1^* in (16.18) and the noiseless function $\tilde{f}_0^*(\mathbf{x})$ in (16.17).

Let $c_{\text{comp}}(\mathbf{x})$ be the coefficient $c(\mathbf{x})$ reconstructed from the minimizer $w_{\min,\lambda,\beta}(\mathbf{x}, t)$ via backwards calculations as in (16.24). We now want to estimate how the distances between the minimizer $w_{\min,\lambda,\beta}$ and the function w^* as well as between coefficients $c^*(\mathbf{x})$ and $c_{\text{comp}}(\mathbf{x})$ depend on the level of noise

$$\delta \in (0, \min(1, \mu/2, R)) \quad (16.33)$$

in the data. Naturally, we assume that the number δ is sufficiently small.

The regularization theory [116] says that one can assume that the exact solution of an ill-posed problem belongs to a given bounded set in an appropriate Banach space. Hence, we assume that

$$w^* \in B(R - \delta, p_0^*, p_1^*). \quad (16.34)$$

Hence,

$$\|f_0^*\|_{C^1(\overline{\Omega})} < C_0 R, \quad \min_{\overline{\Omega}} f_0^* \geq \mu > 0, \quad (16.35)$$

where the number μ is the same as in (16.11) and is independent on δ .

To obtain the desired estimate of the distance between $w_{\min,\lambda,\beta}$ and w^* , we arrange zero boundary conditions in an analog of (16.18). We assume that there exists an extension $G \in H^{k_n}(Q_T^\pm)$ inside of the cylinder $Q_T^\pm$ of the boundary conditions (16.18),

$$G|_{S_T^\pm} = p_0(\mathbf{x}, t), \quad G_x|_{\Gamma_T^\pm} = p_1(\mathbf{x}, t). \quad (16.36)$$

Since

$$w^*|_{S_T^\pm} = p_0^*(\mathbf{x}, t), w_x^*|_{\Gamma_T^\pm} = p_1^*(\bar{x}, t), \quad (16.37)$$

then there also exists a function $G^* \in H^{k_n}(Q_T^\pm)$ satisfying the latter boundary conditions. Since $R > 0$ is an arbitrary number, then, taking into account (??), it is reasonable to assume that

$$\|G\|_{H^{k_n}(Q_T^\pm)} < R, \quad \|G^*\|_{H^{k_n}(Q_T^\pm)} < R. \quad (16.38)$$

Thus, (16.36)–(16.38) lead to:

$$G \in B(R, p_0, p_1), G^* \in B(R, p_0^*, p_1^*). \quad (16.39)$$

It follows from (16.36) and (16.37) that functions $G(\mathbf{x}, t)$ and $G^*(\mathbf{x}, t)$ can be treated as the first part of noisy and noiseless data respectively for our CIP. The second part of such data are functions f_0 and f_0^* . Thus, we assume below the presence of the noise of the level δ in functions G and f_0 ,

$$\|G - G^*\|_{H^{k_n}(Q_T^\pm)} < \delta. \quad (16.40)$$

$$\|f_0 - f_0^*\|_{C^1(\bar{\Omega})} < \delta. \quad (16.41)$$

Hence, (16.35) and (16.41) imply that

$$\min_{\bar{\Omega}} f_0(\mathbf{x}) \geq \frac{\mu}{2}, \quad \|f_0\|_{C^1(\bar{\Omega})} < 2C_0 R.$$

Denote

$$W = w - G, W^* = w^* - G^*. \quad (16.42)$$

Similarly with (16.22) denote

$$B_0(2R) = \left\{ W \in H^{k_n}(Q_T^\pm) : \|W\|_{H^{k_n}(Q_T^\pm)} < 2R, W|_{S_T^\pm} = W_x|_{\Gamma_T^\pm} = 0 \right\}. \quad (16.43)$$

Then (16.34)–(16.42) imply that

$$W \in B_0(2R), \forall w \in B(R, p_0, p_1) \text{ and also } W^* \in B_0(2R - \delta), \quad (16.44)$$

$$W + G \in B(3R, p_0, p_1), \quad \forall W \in B_0(2R). \quad (16.45)$$

Due to (16.45), it is convenient to denote below $\lambda_1(3R), \lambda(3R)$, which means that the values of the parameters λ_1 and $\lambda \geq \lambda_1$ correspond to $B(3R, p_0, p_1)$ in Theorem 15 and, in particular, R is replaced with $3R$ in (16.29) and (16.30).

Consider the functional $I_{\lambda, \beta}(W)$,

$$I_{\lambda, \beta}(W) = J_{\lambda, \beta}(W + G) \text{ for } W \in B_0(2R). \quad (16.46)$$

Theorem 16.3. *Let $\mu, R > 0$ be two numbers. Assume that condition (16.11) holds. Let parameters λ_1 and β be the same in Theorem 16.2, except that R is replaced with $3R$ in (16.29). Then the functional $I_{\lambda, \beta}(W)$ is strictly convex on the ball $\overline{B_0(2R)}$ for such*

values of the parameter λ . More precisely, the following analog of (16.31) holds for all $\lambda \geq \lambda_1(3R, T, \Omega, K)$, all $\beta \in (2e^{-\lambda T^2}, 1)$ and for all $W_1, W_2 \in \overline{B_0(2R)}$:

$$\begin{aligned} & I_{\lambda, \beta}(W_2) - I_{\lambda, \beta}(W_1) - I'_{\lambda, \beta}(W_1)(W_2 - W_1) \\ & \geq \frac{C_1}{\lambda} \exp(-2\lambda(T^2 + B^2 - A^2)) \|W_2 - W_1\|_{H^{2,1}(Q_T^\pm)}^2 + \frac{\beta}{2} \|W_2 - W_1\|_{H^{k_n}(Q_T^\pm)}^2, \end{aligned} \quad (16.47)$$

where $I'_{\lambda, \beta}(W) \in H_0^{k_n}(Q_T^\pm)$ is the Fréchet derivative of the functional $I_{\lambda, \beta}(W)$ at the point W , which exists due to Theorem 16.2 and (16.46). Furthermore, there exists unique minimizer $W_{\min, \lambda, \beta} \in \overline{B_0(2R)}$ of the functional $I_{\lambda, \beta}(W)$ and the following analog of inequality (16.32) holds:

$$I'_{\lambda, \beta}(W_{\min, \lambda, \beta})(W_{\min, \lambda, \beta} - W) \leq 0, \quad \forall W \in \overline{B_0(2R)}. \quad (16.48)$$

The proof of Theorem 16.4 is based on Theorem 16.3 and, in particular, on inequality (16.48).

Theorem 16.4 (accuracy estimates). *Let $\mu, R > 0$ be two numbers. Assume that condition (16.11) holds. Suppose that conditions (16.34), (16.35), (16.40)–(16.41) and (16.42) hold and also let $T > \sqrt{3(B^2 - A^2)}$. Choose a number $\gamma \in (0, 1/\sqrt{3})$ such that $T^2(1 - 3\gamma^2) - 3(B^2 - A^2) > 0$. Denote*

$$\eta_1 = \gamma^2 T^2 + B^2 - A^2, \eta_2 = (1 - 3\gamma^2) T^2 - 3(B^2 - A^2), \rho = \min\left(\frac{1}{2}, \frac{\eta_2}{6\eta_1}\right). \quad (16.49)$$

Let $\lambda_1 = \lambda_1(3R, T, \Omega, K)$ be the number of Theorem 16.3. Choose a number $\lambda' = \lambda'(\gamma T, \Omega)$ such that

$$\frac{1}{\lambda} \geq \exp[-\lambda(\gamma^2 T^2 + B^2 - A^2)], \quad \forall \lambda \geq \lambda'. \quad (16.50)$$

Denote

$$\lambda_2 = \lambda_2(3R, T, \gamma, \Omega, K) = \max(\lambda_1, \lambda'). \quad (16.51)$$

Choose a sufficiently small number $\delta_0 > 0$ satisfying the following inequality:

$$\ln\left(\delta_0^{-1/(3\eta_1)}\right) \geq \lambda_2. \quad (16.52)$$

For each $\delta \in (0, \delta_0)$, choose $\lambda = \lambda(\delta) > \lambda_2$ as:

$$\lambda = \lambda(\delta) = \ln\left(\delta^{-1/(3\eta_1)}\right) > \ln\left(\delta_0^{-1/(3\eta_1)}\right).$$

Let the regularization parameter $\beta = \beta(\delta, T) = 3e^{-\lambda(\delta)T^2}$ (see Theorem 16.2). Let

$$w_{\min, \lambda(\delta), \beta(\delta, T)} = W_{\min, \lambda(\delta), \beta(\delta, T)} + G. \quad (16.53)$$

(Theorem 16.2) and let $c_{\text{comp}}(\mathbf{x})$ be the function $c(\mathbf{x})$ computed from $w_{\min, \lambda(\delta), \beta(\delta, T)}(\mathbf{x}, t)$ by the procedure described in the last paragraph of subsection 16.4. Then the following accuracy estimates are valid

$$\|w^* - w_{\min, \lambda(\delta), \beta(\delta, T)}\|_{H^2(Q_{\gamma T}^\pm)} \leq C_1 \delta^\rho, \quad (16.54)$$

$$\|c^* - c_{\min, \lambda(\delta), \beta(\delta, T)}\|_{L_2(\Omega)} \leq C_1 \delta^\rho. \quad (16.55)$$

We now construct the gradient descent method of the minimization of the functional $J_{\lambda, \beta}(V)$ defined in (16.23) on the set $B(R, p_0, p_1)$ defined in (16.22). This construction is similar with (6.4) of section 6 and (15.38) of subsection 15.6.

Let $\alpha \in (0, 1/3)$ be an arbitrary number. Choose an arbitrary point $w_0 \in B(\alpha R, p_0, p_1)$. Let the number $\omega \in (0, 1)$. We arrange the gradient descent method of the minimization of the functional $J_{\lambda, \beta}(V)$ as:

$$w_n = w_{n-1} - \omega J'_{\lambda, \beta}(w_{n-1}), \quad n = 1, 2, \dots \quad (16.56)$$

Note that since by Theorem 16.2 $J'_{\lambda, \beta}(w_{n-1}) \in H_0^{k_n}(Q_T^\pm)$, then the function all functions V_n have the same boundary conditions as ones in the definition of the set $B(R, p_0, p_1)$ in (16.22).

Theorem 16.5 (global convergence of the gradient descent method). Let $\mu, R > 0$ be two numbers. Assume that condition (16.11) holds. Let the parameter $\lambda_1(3R, T, \Omega, K)$ be the same as in Theorem 16.2, except that R is replaced with $3R$ in (16.29). Let $\lambda \geq \lambda_1(3R, T, \Omega, K)$. Let $\beta \in (2e^{-\lambda T^2}, 1)$ be the same as in Theorem 16.2. For these values of parameters λ and β , let $w_{\min, \lambda, \beta}$ be the unique minimizer of the functional $J_{\lambda, \beta}(w)$ on the set $\overline{B(R, p_0, p_1)}$, which is guaranteed by Theorem 16.2. Assume that $w_{\min, \lambda, \beta} \in B(\alpha R, p_0, p_1)$. Then there exists a sufficiently small number $\omega_0 = \omega_0(\Omega, R, T, \lambda)$ such that for any $\omega \in (0, \omega_0)$ there exists a number $\theta = \theta(\omega) \in (0, 1)$ such that the sequence all terms of the sequence (16.56) $w_n \in B(R, p_0, p_1)$, this sequence converges to $w_{\min, \lambda, \beta}$ in the norm of the space $H^{k_n}(Q_T^\pm)$ and the following convergence rate holds:

$$\|w_{\min, \lambda, \beta} - w_n\|_{H^{k_n}(Q_T^\pm)} \leq \theta^n \|w_{\min, \lambda, \beta} - w_0\|_{H^{k_n}(Q_T^\pm)}. \quad (16.57)$$

Even though Theorem 16.5 claims the global convergence of the sequence (16.56) $\{w_n\}_{n=0}^\infty$ of the gradient descent method to the unique minimizer $w_{\min, \lambda, \beta} \in B(\alpha R, p_0, p_1)$ of the functional $J_{\lambda, \beta}(w)$ on the set $\overline{B(R, p_0, p_1)}$, it is still important to know the convergence rate of this sequence to the exact function w^* as well as the convergence rate of corresponding target coefficients $\{c_n(\mathbf{x})\}_{n=0}^\infty$ to the exact coefficient $c^*(\mathbf{x})$. Theorem 16.6 addresses these questions.

Theorem 16.6 (global convergence to the exact solution of the gradient descent method). Suppose that conditions of both Theorems 16.4 and 16.5 hold. Let $\{w_n\}_{n=0}^\infty$ be the sequence of the gradient descent method (16.56) with $w_0 \in B(\alpha R, p_0, p_1)$. Let $c_{n, \text{comp}}(\mathbf{x})$ be the function $c(\mathbf{x})$ obtained from the function $w_n(\mathbf{x}, t)$ by the procedure outlined in the last paragraph of subsection 16.4. Let $\gamma \in (0, 1/\sqrt{3})$ be the number in (16.24). Then there exists a sufficiently small number $\omega_1 = \omega_1(\Omega, R, T, \lambda, \gamma) \in (0, \omega_0]$ such that for any $\omega \in (0, \omega_1)$ there exists a number $\theta = \theta(\omega) \in (0, 1)$ such that the the following convergence estimates are valid for $n = 1, 2, \dots$

$$\|w^* - w_n\|_{H^{2,1}(Q_{\gamma T}^\pm)} \leq C_1 \delta^\rho + \theta^n \|w_{\min, \lambda(\delta), \beta(\delta, T)} - w_0\|_{H^{k_n}(Q_T^\pm)}, \quad (16.58)$$

$$\|c^* - c_{n, \text{comp}}\|_{L_2(\Omega)} \leq C_1 \delta^\rho + \theta^n \|w_{\min, \lambda(\delta), \beta(\delta, T)} - w_0\|_{H^{k_n}(Q_T^\pm)}. \quad (16.59)$$

Remark 16.2 Since the starting point $w_0 \in B(\alpha R, p_0, p_1)$ of the iterative process (16.56) is an arbitrary point of the set $B(\alpha R, p_0, p_1)$ and since $R > 0$ and $\alpha \in (0, 1/3)$

are arbitrary numbers, then Theorem 16.6 ensures the global convergence of the gradient projection method (16.56) to the correct solution as long as the noise level δ tends to zero, see Definition 1.1 in section 1 of the global convergence.

16.6 Numerical testing

In the following tests, we set the domain $\Omega = (1, 2) \times (1, 2)$ and also

$$Lu = \Delta u - c(\mathbf{x})u.$$

To solve the inverse problem, we should first computationally simulate the data (16.7), (16.8) via the numerical solution of the forward problem (16.4)–(16.6). To solve problem (16.4)–(16.6), computationally, we have used the standard finite difference method. The spatial mesh size is $1/640 \times 1/640$ while the temporal one $T/512$. For the forward problem, we use the implicit scheme to compute the data needed for the inverse problem.

In computations of the inverse problem, the spatial mesh size is $1/16 \times 1/16$ and the temporal one $T/16$. When minimizing the functional $J_{\lambda,\beta}(w)$ in the discrete sense, we formulate the right hand side of (16.23) via finite differences and minimize with respect to the values of the function w at grid points. To minimize the discretized functional, we use Matlab's built-in function **fminunc** with its option of *quasi-newton algorithm*. This procedure calculates the gradient $\nabla J_{\lambda,\beta}(w)$ automatically and iterations stop when the condition $|\nabla J_{\lambda,\beta}(w)| < 10^{-2}$ holds.

As to (16.24), we have numerically discovered that rather than taking an average over $t \in [-\gamma T, \gamma T]$, better to use (16.13) at $\{t = t_0\}$. In numerical tests below, we took

$$\lambda = 1, \quad k_n = 3, \quad \beta = 0.01, \quad T = 1, \quad t_0 = 0. \quad (16.60)$$

In the process of the minimization of the functional $J_{\lambda,\beta}(w)$, the starting point w_0 of iterations is always chosen as:

$$w_0 \equiv 0.$$

Remarks 16.3:

1. The choice (16.60) $\lambda = 1$ coincides once again in our consistent observations in all other our numerical works on the convexification cited above, including the majority of the above sections of this report. This observation is that even though the convexification theory requires λ to be sufficiently large, our computational experience shows that $\lambda \in [1, 3]$ is sufficient.
2. On the other hand, it was demonstrated numerically in the recent work of the PI with coauthors [32] that $\lambda = 0$ does not result in a good solution, at least for the CIP considered in [32]. In other words, the CWF must be present in the functional involved in the convexification.

In the following two tests, we show the results of the recovery of the coefficients $c(\mathbf{x})$ with sophisticated structures. We choose the tested coefficients $c(\mathbf{x})$ having the shapes of the letters 'A' and ' Ω ' and set

$$c(\mathbf{x}) = \begin{cases} 1 & \text{inside of the target,} \\ 0 & \text{outside of the target.} \end{cases} \quad (16.61)$$

We measure the function $g_1(x_1, x_2, t)$ on 16×32 detectors uniformly distributed on the rectangle $\Gamma_T^\pm$ and ‘measure’ the function $f_0(x_1, x_2, t_0)$ on 16×16 detectors uniformly distributed on the square $(1, 2) \times (1, 2) \times \{t = t_0\}$. As initial and Dirichlet boundary conditions for the data simulations in (16.5), (16.6), we took

$$u(\mathbf{x}, -T) = 1 + \sin(\pi(x_1 - 1)) \sin(\pi(x_2 - 1)) \text{ and } u|_{S_T^\pm} = 1.$$

We allow in our reconstruction results the function $c(\mathbf{x})$ to be both positive and negative. Indeed, we have imposed the positivity condition (16.3) only to ensure that the function $u(\mathbf{x}, t) \neq 0$ in $\overline{Q_T^\pm}$, see (16.11). However, we have not observed any zeros of this function in our numerical studies.

In Tests 1,2 below we add 5% relative random noise to each detector on $\Gamma_T^\pm$ as well as on $(1, 2) \times (1, 2) \times \{t = 0\}$, i.e. we work now with the noisy data,

$$u_x^{noise}|_{\Gamma_T^\pm} = g_1(\mathbf{x}, t) + \sigma \xi_{\mathbf{x},t} g_1(\mathbf{x}, t), \quad (16.62)$$

$$u^{noise}(\mathbf{x}, t_0) = f_0(\mathbf{x}) + \sigma \xi_{\mathbf{x}} f_0(\mathbf{x}). \quad (16.63)$$

Here $\sigma = 0.05$, i.e. 5% is the noise level, $\xi_{\mathbf{x},t}$ and $\xi_{\mathbf{x}}$ are independent normally distributed random variables. To preprocess the noisy data, we use the thin plate spline smoother developed in [55]. The algorithm proposed in [55] provides a good approximation to the true function without knowing neither the noise level nor any other a priori information of the true function to be approximated. Then the cubic B-splines are employed to approximate the first and second order derivatives of the noisy data. In this test, we ‘measure’ $g_1(x_1, x_2, t)$ on 16×32 detectors uniformly distributed on the plane $\Gamma_T^\pm$ and also ‘measure’ $f_0(x_1, x_2, t_0)$ on 160×160 detectors uniformly distributed on the plane $(1, 2) \times (1, 2) \times \{t = 0\}$.

Since letters represent non convex shapes, which are usually hard to image, we test in our numerical studies reconstructions of letters ‘A’ and ‘Ω’.

Test 1. We reconstruct the letter ‘A’. Results are presented on Figures 35.

Test 2. We reconstruct the letter ‘Ω’. Results are presented on Figures 36.

One can see on Figures 35 and 36 that shapes of targets are reconstructed well, regardless on the difficulties of reconstructions of non-convex shapes. Also, comparing Figures 35b) and 36b) with (16.61), one can see that the maximal values of the function $c(\mathbf{x})$ are reconstructed well. Thus, our method works very well in the mild noisy case.

17 Convexification for the Classic Hamilton-Jacobi Equation [20]

The convexification is a very general and a very powerful concept. The convexification is applicable not only to inverse problems but to forward problems for **CLASSIC** non-linear Partial Differential Equations as well! A hard proof of this statement is the recent publication of the PI with coauthors [20] in a quite reputable journal: Journal of Computational Physics. The impact factor of this journal is 3.553, which is quite a high value for a mathematical journal. “**According to the Journal Citation Reports, Journal of Computational Physics has a 2020 impact factor of 3.553, ranking it 6th out of 55 in the category Physics, Mathematical**”,

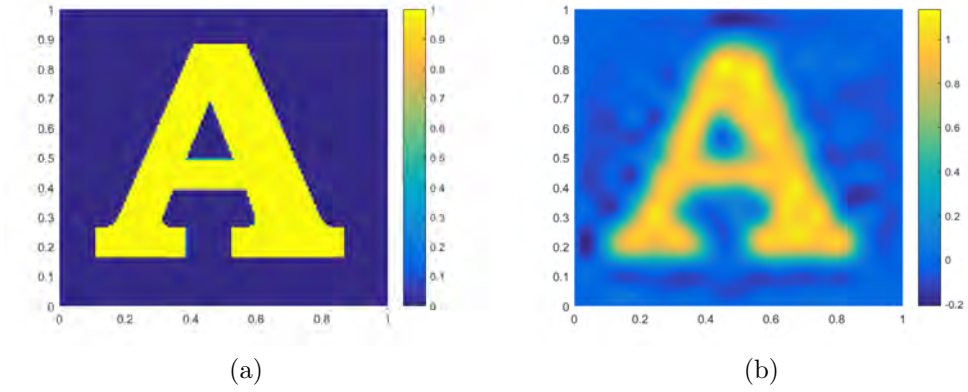


Figure 36: *Test 1 of subsection 15.6. 5% noise in the data is added as in (16.62), (16.63) with $\sigma = 0.05$. The target has the shape of the letter ‘A’ with $c(\mathbf{x})$ as in (16.61). It is well known that non-convex shapes are challenging for the reconstructions. a) Correct image. b) Reconstructed image. Thus, both the shape of the target and the maximal value of the function $c(\mathbf{x})$ in it are reconstructed well.*

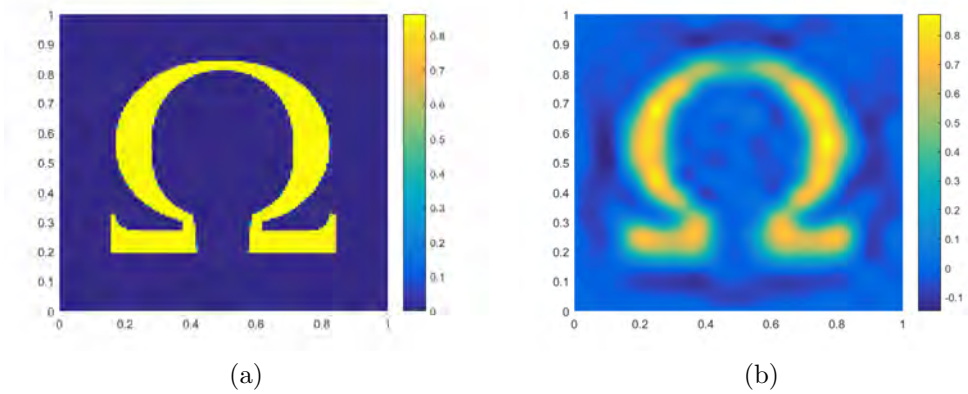


Figure 37: *Test 2 of subsection 15.6. 5% noise in the data is added as in (16.62), (16.63) with $\sigma = 0.05$. The target has the shape of the letter ‘Ω’ with $c(\mathbf{x})$ as in (16.61). It is well known that non-convex shapes are challenging for the reconstructions. a) Correct image. b) Reconstructed image. Thus, both the shape of the target and the maximal value of the function $c(\mathbf{x})$ in it are reconstructed well.*

https://en.wikipedia.org/wiki/Journal_of_Computational_Physics.

Indeed, we have applied the convexification method to develop a numerical procedure of computing viscosity solutions of the Hamilton-Jacobi equation [20]. The Hamilton-Jacobi equation is one of the most popular classic equations of the Mathematical Physics. The aim of the paper [20] is to compute viscosity solutions to a large class of Hamilton-Jacobi equations possibly involving non convex Hamiltonians. The **key ingredient** for reaching this goal is the use of a new Carleman estimate and the convexification method. This method is only applicable when a viscosity term is added to the Hamilton-Jacobi equation under consideration. The idea of adding the viscosity term and passing to the limit to obtain viscosity solutions is due to the seminal works [58, 59] published back in 1983,1984.

Let $R > 0$ and

$$\Omega = (-R, R)^n \subset \mathbb{R}^n. \quad (17.1)$$

Let $F : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ and $f : \overline{\Omega} \rightarrow \mathbb{R}$ be functions of the class C^2 . We have proposed in [20] a globally convergent convexification-based numerical method to solve the Hamilton-Jacobi equation

$$F(\mathbf{x}, u(\mathbf{x}), \nabla u(\mathbf{x})) = 0, \quad \text{for all } \mathbf{x} \in \Omega \quad (17.2)$$

with the Dirichlet boundary condition

$$u(\mathbf{x}) = f(\mathbf{x}), \quad \text{for all } \mathbf{x} \in \partial\Omega. \quad (17.3)$$

A number of numerical methods were proposed for problem (17.2), (17.3), see, e.g. [49, 52, 112] for a non exhaustive list of references. However, all of them work with either very restrictive conditions imposed on the function F or with optimization techniques, which again suffer from the problem of local minima and ravines of optimized least squares cost functionals.

Unlike this, we have proposed in [20] a to use the convexification method, which does not rely on the above assumptions. This method is globally convergent in the sense that:

1. It delivers good approximation of the true solution without any advance knowledge of the true solution even when the given data are noisy.
2. The claim of item 1 is rigorously proved and numerically verified.

To the best of our knowledge, our method is a **SURPRISING BREAKTHROUGH** in the context of viscosity solutions to Hamilton-Jacobi equations. It has two **MAJOR ADVANTAGES** from the numerical standpoint. First, it works for some quite general $F(\mathbf{x}, u, \nabla u)$, and without imposing any special condition on the structure of the function F . Second, it is stable even with some noise level on the boundary data, which occurs naturally in applications.

Consider the upper part Γ^+ of the boundary $\partial\Omega$ of the domain Ω in (17.1),

$$\Gamma^+ = \{\mathbf{x} = (x_1, x_2, \dots, x_{n-1}, z = R) : |x_i| \leq R, 1 \leq i \leq n-1\} \subset \partial\Omega.$$

We assume that, in addition to (17.3), we know the function $g(x_1, x_2, \dots, x_{n-1})$,

$$u_z|_{x \in \Gamma^+} = g(x_1, x_2, \dots, x_{n-1}), |x_i| \leq R, 1 \leq i \leq n-1. \quad (17.4)$$

The function g can be recovered from (17.2), if, for example,

$$\frac{\partial}{\partial u_z} F(\mathbf{x}, f(\mathbf{x}), f_{x_1}(\mathbf{x}), \dots, f_{x_{n-1}}(\mathbf{x}), u_z(\mathbf{x})) \neq 0 \text{ for } \mathbf{x} \in \Gamma^+.$$

Definition 17.1.

1. We say that $u \in C^1(\overline{\Omega})$ is a viscosity subsolution to (17.2), (17.3) if for any test function $\varphi \in C^1(\overline{\Omega})$ such that $u - \varphi$ has a strict maximum at $\mathbf{x}_0 \in \overline{\Omega}$, then either

$$F(\mathbf{x}_0, u(\mathbf{x}_0), \nabla \varphi(\mathbf{x}_0)) \leq 0 \quad \text{if } \mathbf{x}_0 \in \Omega,$$

or

$$\max \{F(\mathbf{x}_0, u(\mathbf{x}_0), \nabla \varphi(\mathbf{x}_0)), u(\mathbf{x}_0) - f(\mathbf{x}_0)\} \leq 0 \quad \text{if } \mathbf{x}_0 \in \partial\Omega.$$

2. We say that the function $u \in C^1(\overline{\Omega})$ is a viscosity supersolution to (17.2), (17.3) if for any test function $\varphi \in C^1(\overline{\Omega})$ such that $u - \varphi$ has a strict minimum at $x_0 \in \overline{\Omega}$, then either

$$F(\mathbf{x}_0, u(\mathbf{x}_0), \nabla \varphi(\mathbf{x}_0)) \geq 0 \quad \text{if } \mathbf{x}_0 \in \Omega,$$

or

$$\max \{F(\mathbf{x}_0, u(\mathbf{x}_0), \nabla \varphi(\mathbf{x}_0)), u(\mathbf{x}_0) - f(\mathbf{x}_0)\} \geq 0 \quad \text{if } \mathbf{x}_0 \in \partial\Omega.$$

3. We say that the function $u \in C^1(\overline{\Omega})$ is a viscosity solution to (17.2), (17.3) if it is both its viscosity subsolution and its viscosity supersolution.

We write $F = F(\mathbf{x}, s, \mathbf{p})$. Let $\nabla_{\mathbf{x}}F$ and $\nabla_{\mathbf{p}}F$ denote the gradient vector of F with respect to $\mathbf{x}$ and $\mathbf{p}$ respectively, and $\partial_s F = F_s$ denotes the partial derivative of F with respect to the second variable. Here is one set of conditions that often occurs in the literature:

Assumption 17.1:

1. There exists $\alpha > 0$ such that $\alpha \leq F_s(\mathbf{x}, s, \mathbf{p}) \leq 1/\alpha$ for all $(\mathbf{x}, s, \mathbf{p}) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n$.
2. F is coercive in p in the sense that

$$\lim_{|\mathbf{p}| \rightarrow \infty} \inf_{(\mathbf{x}, s) \in \overline{\Omega} \times \mathbb{R}} (F(\mathbf{x}, s, \mathbf{p})^2 + \partial_s F(\mathbf{x}, s, \mathbf{p})|\mathbf{p}|^2 + \nabla_{\mathbf{x}}F(\mathbf{x}, s, \mathbf{p}) \cdot \mathbf{p}) = +\infty.$$

3. There exists $\phi \in C^2(\overline{\Omega})$ such that $\phi = f$ on $\partial\Omega$, and

$$F(\mathbf{x}, \phi(\mathbf{x}), \nabla \phi(\mathbf{x})) < 0 \quad \text{in } \Omega.$$

The following fundamental theorem is well known [59]:

Theorem 17.1. Suppose that Assumption 17.1 holds. Then, problem (17.2), (17.3) has a unique viscosity solution, denoted by $u(\mathbf{x})$. Moreover, for $\varepsilon > 0$ the following problem

$$-\varepsilon \Delta u^\varepsilon + F(\mathbf{x}, u^\varepsilon(\mathbf{x}), \nabla u^\varepsilon(\mathbf{x})) = 0, \tag{17.5}$$

$$u^\varepsilon|_{\partial\Omega} = f \tag{17.6}$$

has unique solution $u^\varepsilon \in C^2(\overline{\Omega})$ and

$$\|u^\varepsilon - u\|_{C(\overline{\Omega})} \leq C\sqrt{\varepsilon}, \text{ as } \varepsilon \rightarrow 0.$$

Thus, our idea of [20] is to use (17.4)-(17.6) to consider the following Boundary Value Problem (BVP)

$$-\varepsilon \Delta u^\varepsilon + F(\mathbf{x}, u^\varepsilon(\mathbf{x}), \nabla u^\varepsilon(\mathbf{x})) = 0, \tag{17.7}$$

$$u^\varepsilon|_{\partial\Omega} = f, \tag{17.8}$$

$$u_z|_{x \in \Gamma^+} = g(x_1, x_2, \dots, x_{n-1}), |x_i| \leq R, 1 \leq i \leq n-1. \tag{17.9}$$

Since (17.7) is an elliptic PDE, then BVP (17.7)-(17.9) is very similar with BVP (9.20)-(9.23) of subsection 9.4. Therefore, an analog of functional $J_{\lambda, \gamma}(V)$ in (9.31) (subsection 9.5) was constructed in [20] for BVP (17.7)-(17.9). Next, analogs of convexification Theorems 9.2-9.5 of subsections 9.6-9.8 were proven in [20] for that analog.

As one of numerical examples, consider the 2-D case of $\Omega = (-1, 1) \times (-1, 1)$,

$$F(\mathbf{x}, s, \mathbf{p}) = \frac{1}{150}s + |\mathbf{p}| + \frac{1}{150}(x^2 + z^2) - 2\sqrt{x^2 + z^2} \quad \text{for all } \mathbf{x} \in \Omega, s \in \mathbb{R}, \mathbf{p} \in \mathbb{R}^2 \quad (17.10)$$

and the boundary data are given by

$$u(\mathbf{x}) = f(\mathbf{x}) = -(x^2 + z^2) \quad \text{for all } \mathbf{x} = (x, z) \in \partial\Omega \quad (17.11)$$

and

$$u_z(\mathbf{x}) = g(\mathbf{x}) = -2z \quad \text{for all } \mathbf{x} = (x, z) \in \Gamma^+. \quad (17.12)$$

The true solution of problem (17.7)-(17.12) is $u_{\text{true}}(x, z) = -(x^2 + z^2)$. We took $\varepsilon = 10^{-3}$ in (17.7). Figure 37 displays true and computed solutions of this problem. A variety of quite more complicated functions F and u was also successfully tested numerically in [20].

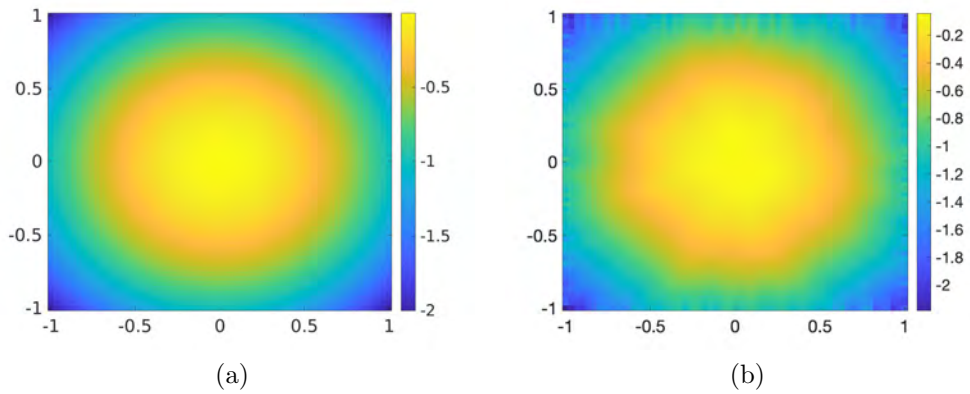


Figure 38: *Converification method for computations of viscosity solutions of classic Hamilton-Jacobi equation [20]. As a numerical example, problem (17.7)-(17.12) was solved by the converification method. We took in (17.7) $\varepsilon = 10^{-3}$ and added 10% of random noise to the data (17.11). a) The true solution of problem (17.7)-(17.12). b) The computed solution of problem (17.7)-(17.12).*

18 Presentations

The PI and his research team have presented results of this project in:

1. Two talks in North Carolina State University (2020).
2. Two talks on the 5th annual Meeting of SIAM Central States Section at Iowa State University (2020).
3. Three talks on an annual meeting of American Mathematical Society (2021).

Although a number of other presentations was planned, but all of them were cancelled due to the pandemic.

19 Publications

We now briefly outline the topic of publications. A more detailed information about this topic is given in “References” below.

1. The support of this grant was acknowledged in thirty four (33) publications [2]-[34], i.e. $33/3 = 11$ publications annually as the average.
2. In particular, two books were published with the acknowledgment of the support of this grant [2, 3].
3. Seven (7) joint papers of the PI’s team with ARL with the acknowledgments of the support of this grant were published during the reportage period of [4]-[10]. The total number of joint publications of the PI’s team with ARL reflecting their long term ten (10) years collaboration is thirteen (13) during the period of 2012-2022 [4]-[10], [35]-[40].

References

- [1] Total thirty three (33) publications [2]-[34] with the acknowledgment of the support of this grant. Two books [2, 3] are among them. Total seven (7) joint papers [4]-[10] with engineers of ARL Drs. Anders Sullivan and Lam Nguyen acknowledge the support of this grant. Total thirteen (13) joint papers of the PI with these people were published in the ten (10) years period of 2012-2022 [4]-[10], [35]-[40].
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