AFRL-RQ-WP-JA-2023-0001

AN ALTERNATE MEANS TO FORM NON-DIMENSIONAL PRODUCTS IN DIMENSIONAL ANALYSIS

John P. Clark
Aerospace Systems Directorate

MARCH 2023
Interim Report

DISTRIBUTION STATEMENT A. Approved for public release. Distribution is unlimited; AFRL-2021-4550; Cleared 28/12/2021.

AIR FORCE RESEARCH LABORATORY
AEROSPACE SYSTEMS DIRECTORATE
WRIGHT-PATTERSON AIR FORCE BASE, OH 45433-7542
AIR FORCE MATERIEL COMMAND
UNITED STATES AIR FORCE

NOTICE PAGE

REPORT DOCUMENTATION PAGE

PLEASE DO NOT RETURN YOUR FORM TO THE ABOVE ORGANIZATION.

1. REPORT DATE	2. REPORT TYPE	3. DATES COVERED START DATE	END DATE
20-Mar-2023	Interim	21-Mar-2019	20-Mar-2021

4. TITLE AND SUBTITLE
AN ALTERNATE MEANS TO FORM NON-DIMENSIONAL PRODUCTS IN DIMENSIONAL ANALYSIS

5a. CONTRACT NUMBER	5b. GRANT NUMBER	5c. PROGRAM ELEMENT NUMBER
In-House		62203F
5d. PROJECT NUMBER	**5e. TASK NUMBER**	**5f. WORK UNIT NUMBER**
3066		Q28J

6. AUTHOR(S)
John P. Clark

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
Turbomachinery Branch (AFRL/RQTT), Turbine Engine Division,
Aerospace Systems Directorate, Air Force Research Laboratory
Wright-Patterson Air Force Base, OH 45433-7542 USA
Air Force Materiel Command, United States Air Force

8. PERFORMING ORGANIZATION REPORT NUMBER

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)
Aerospace Systems Directorate, Air Force Research Laboratory
Wright-Patterson Air Force Base, OH 45433-7542
Air Force Materiel Command, United States Air Force

10. SPONSOR/MONITOR'S ACRONYM(S)
AFRL/RQTT

11. SPONSOR/MONITOR'S REPORT NUMBER(S)
AFRL-RQ-WP-JA-2023-0001

12. DISTRIBUTION/AVAILABILITY STATEMENT
DISTRIBUTION STATEMENT A. Approved for public release. Distribution is unlimited. AFRL-2021-4550; Cleared 28/12/2021.

13. SUPPLEMENTARY NOTES
PA Clearance Number: AFRL-2021-4550; Clearance Date: 28-Dec-2021; This paper was published in the proceedings of the 2022 ASME Turbo Expo, held in Rotterdam, The Netherlands, June 13 through June 17, 2022.

14. ABSTRACT
Dimensional analysis is taught early in an undergraduate curriculum for mechanical engineering, usually during the very first course in fluid mechanics. Such analysis has its roots in the work of Lord Rayleigh, and the mechanics of the process as typically taught to undergraduates follows directly from the classic paper due to Buckingham on what is now known as the Pi Theorem. This paper is intended to disseminate an alternate technique for forming non-dimensional products that is due to Prof. B. S. Massey of University College, London called the “step-by-step method” to a wider audience and thus to encourage the adoption of the technique for use in undergraduate curricula.

15. SUBJECT TERMS
Dimensional Analysis, Buckingham Pi Theorem, Turbine Scaling, Boundary Layer Transition

16. SECURITY CLASSIFICATION OF: a. REPORT	b. ABSTRACT	C. THIS PAGE	17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES
Unclassified	Unclassified	Unclassified	SAR	8

19a. NAME OF RESPONSIBLE PERSON	19b. PHONE NUMBER *(Include area code)*
John P. Clark	(937) 750-2676

PREVIOUS EDITION IS OBSOLETE.

STANDARD FORM 298 (REV. 5/2020)
Prescribed by ANSI Std. Z39.18

Proceedings of ASME Turbo Expo 2022
Turbomachinery Technical Conference and Exposition
GT2022
June 13-17, 2022, Rotterdam, The Netherlands

GT2022-81558

AN ALTERNATE MEANS TO FORM NON-DIMENSIONAL PRODUCTS IN DIMENSIONAL ANALYSIS

John P. Clark
Air Force Research Laboratory
1950 5th Street, Bldg 18D, Rm 204F
WPAFB, OH
USA
john.clark.38@us.af.mil

ABSTRACT

Dimensional analysis is taught early in an undergraduate curriculum for mechanical engineering, usually during the very first course in fluid mechanics. Such analysis has its roots in the work of Lord Rayleigh, and the mechanics of the process as typically taught to undergraduates follows directly from the classic paper due to Buckingham on what is now known as the Pi Theorem. Students are meant to learn that dimensional analysis is a powerful tool for: developing insight with respect to flow physics, creating new models of physical processes, guiding the performance of experiments and flowfield simulations, aiding the sensible presentation of technical results, and fostering the replication of experiments and simulations. Unfortunately, the usual method for finding non-dimensional products taught to undergraduates requires the selection of scaling variables that can seem arbitrary and the solution of a number of sets of simultaneous equations that is typically tedious and prone to simple errors of arithmetic. As a consequence, students often fail to gain the intended appreciation for the usefulness of dimensional analysis. Fortunately, another technique for forming Π products that does not suffer from these same drawbacks was presented by the late Prof. B. S. Massey of University College London. This paper is intended to disseminate his so-called "step-by-step method" to a wider audience and thus to encourage the adoption of the technique for use in undergraduate curricula.

Keywords: Dimensional Analysis, Similitude, Scaling

NOMENCLATURE

Latin

a	sonic speed (m/s)
C	column of an indicial matrix
c_p	constant pressure specific heat (J/kg/K)
d	cylinder diameter (m)
Ec	Eckert number (1)
f	mathematical function
H	dimension of heat
h	convective heat transfer coefficient (W/m^2/K)
I	indicial matrix
K	acceleration parameter (1)
k	thermal conductivity (W/m/K)
L	dimension of length
M	dimension of mass
Ma	Mach number (1)
Nu	Nusselt number (1)
n	frequency of vortex shedding (Hz)
Pr	Prandtl number (1)
R	row of an indicial matrix
Re	Reynolds number (1)
r	rank of an indicial matrix
S	surface distance (m)
Sr	Strouhal number (1)
St	Stanton number (1)
T	dimension of temperature
T_g	driving temperature for heat transfer (K)
T_w	wall temperature (K)
t	dimension of time
u'	fluctuating velocity (m/s)
V	velocity (m/s)

Greek

θ	boundary layer momentum thickness (m)
λ	turbulent length scale (m)
μ	dynamic viscosity (kg/m/s)
Π	non-dimensional parameter (1)
ρ	density (kg/m^3)

1. INTRODUCTION

In three of the more widely used texts that introduce the student to fluid mechanics [1-3], instruction on the topic of dimensional analysis follows the seminal work of Buckingham [4] very closely. This is true as well in many other introductory titles that are extant [5-7] as well as in reference works [8] and graduate level texts [9, 10]. Typically, the Pi Theorem is stated in a manner consistent with that of Buckingham himself, and then the "method of repeating variables" [1] is presented as the most suitable technique for determining the non-dimensional products. This method is also sometimes called the "method of exponents [7]" or the "method of indices [9]," and as Massey [11] states, the technique really has its origins in the work of Lord Rayleigh. It was later utilized by Buckingham when he presented the theorem that bears his name and adopted the symbol Π to represent the non-dimensional variables that result from a formal dimensional analysis.

A critical step in the method of repeating variables involves the selection of scaling parameters that are used in the formation of each non-dimensional parameter [1]. Unfortunately, this selection can seem mysterious and arbitrary to the student encountering dimensional analysis for the first time. From personal experience this can leave the student thinking that dimensional analysis is all well and good so long as one already knows the answer to a problem (e.g. that the drag on a cylinder in crossflow depends on the Reynolds number). However, an alternate means of forming Π products called the "step-by-step method" exists, and that is due to Massey [11]. The technique is quite similar to one presented earlier by Ipsen [12], but it appears that Prof. Massey came upon it independently. Massey's method is also accomplished in a strikingly different way than that of Ipsen, and this makes it really much more useful for complex problems involving many variables. This becomes quite evident later in the paper.

Discussion of the technique is absent from standard undergraduate textbooks on fluid mechanics [1-3, 5-7], although the Massey text is cited in [1]. The citation to Massey disappears in a later edition of the same text [13] wherein the method of Ipsen [12] is described. Ipsen's method is also utilized in [7], but the only text on fluid mechanics where Massey's method is properly treated is by the author himself [14]. This situation is unfortunate, and this paper is an attempt to introduce Massey's method to a wider audience. The technique requires no prior selection of scaling parameters. It is also far less tedious and less prone to simple errors than the method of repeating variables. Accordingly, it encourages the student to explore the various non-dimensional formulations that can serve as solutions to a problem and thereby fosters learning. It is therefore ideal for use in an undergraduate course of study. Moreover, it is amenable to easy implementation in a computer scripting language, and this makes it as well useful for those wishing to apply dimensional analysis in anger during research and development projects.

For context, the method of repeating variables is presented below with reference to the process of vortex shedding from a circular cylinder. Then, the method of Massey [11] is briefly described right along with Ipsen's [12] technique, and both are compared to the more usual method of indices. Then, further illustrations of Massey's step-by-step technique are given with respect to the variation of the heat-transfer distribution along a flat plate and the development of a model for the onset of transition to turbulence in turbomachinery flows. Needless to say, the form of the latter was not known at the outset, and dimensional analysis was essential to the ultimate formulation of the model and its successful application. An implementation of Massey's technique in Matlab is available for distribution to interested parties.

2. METHOD OF REPEATING VARIABLES

It is well known that the frequency of vortex shedding, n, in incompressible flow over a circular cylinder depends only on the freestream values of the density, ρ, and dynamic viscosity, μ, the cylinder diameter, d, and the magnitude of the freestream velocity, V. This relationship is expressed as

$$n = f_1(\rho, \mu, d, V) \tag{1}$$

Let [M], [L], and [t] represent the fundamental dimensions of mass, length, and time, respectively. Then, the dimensional formulae associated with each variable are $[n] = [t^{-1}]$, $[\rho] = [M\,L^{-3}]$, $[\mu] = [M\,L^{-1}\,t^{-1}]$, $[d]=[L]$, and $[V] = [L\,t^{-1}]$. The indicial matrix, I_1, associated with the physical process described by eqn. 1 is thus

$$I_1 = \begin{array}{c} \\ n \\ \rho \\ \mu \\ d \\ V \end{array} \begin{array}{c} \begin{array}{ccc} M & L & t \end{array} \\ \begin{bmatrix} 0 & 0 & -1 \\ 1 & -3 & 0 \\ 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \end{array} \tag{2}$$

The rank of the indicial matrix, r = 3, and since the physical process is described fully by 5 variables, the Buckingham Pi Theorem indicates that the physical process described by eqn. (1) is as well represented by a relationship between 5 - r = 2 non-dimensional Π products, thus

$$\Pi_1 = f_{11}(\Pi_2) \tag{3}$$

To form the Π products by the more standard method of repeating variables, one chooses r = 3 scaling parameters that do not themselves form a non-dimensional quantity and then selects each of the remaining 5 - r = 2 variables in turn to form the Π products. Here, we select ρ, μ, and V for the scaling parameters and form the first Π product by declaring the frequency of vortex shedding, n, as the dependent variable in the process. We have

$$\Pi_1 = \rho^a\, V^b\, \mu^c\, n^1 \tag{4}$$

In terms of dimensional formulae, then, we have

$$1 = [M^0\, L^0\, t^0] = [M\, L^{-3}]^a\, [L\, t^{-1}]^b\, [M\, L^{-1}\, t^{-1}]^c\, [t^{-1}] \tag{5}$$

This gives a system of equations that one can solve simultaneously for the unknown powers on the scaling parameters. In matrix form, this is

$$\begin{bmatrix} 1 & 0 & 1 \\ -3 & 1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad (6)$$

Solving via matrix inversion gives a=-1, b=-2 and c=1. So, the first non-dimensional product, Π_1, becomes

$$\Pi_1 = \rho^{-1} V^{-2} \mu^1 n^1 = \mu\, n / (\rho V^2) \quad (7)$$

We treat the diameter of the cylinder, d, similarly to form the second non-dimensional product. We have

$$\Pi_2 = \rho^a V^b \mu^c d^1 \quad (8)$$

In terms of dimensional formulae, then, we have

$$1 = [M^0 L^0 t^0] = [M L^{-3}]^a \; [L t^{-1}]^b [M L^{-1} t^{-1}]^c [L]^1 \quad (9)$$

This also gives a system of equations one can solve simultaneously for the unknown powers on the scaling parameters. In matrix form, this is

$$\begin{bmatrix} 1 & 0 & 1 \\ -3 & 1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \quad (10)$$

Solving the system of equations gives a=b=1 and c=-1. Accordingly, the second non-dimensional product, Π_2, is the Reynolds number based on the cylinder diameter, $Re_d=\rho V d / \mu$.

It is always permissible to perform algebraic operations on the non-dimensional products that arise from the method of repeating variables so long as one maintains the correct number predicted by the Buckingham Pi Theorem. Knowing that, we take

$$\Pi_2 \Pi_1 = (\rho V d / \mu)\{\mu n / (\rho V^2)\} = n d / V \quad (11)$$

Where the right hand size of eqn. 11 is the Strouhal number, $Sr=n d / V$. Thus, equation 3 becomes the well-known result

$$Sr = f_{11} (Re_d) \quad (12)$$

The selection of scaling parameters in this example is obvious for an experienced researcher, and this is especially true if one has already come across the use of the unit Reynolds number, $Re_u=\rho V/\mu$, with respect to wind-tunnel considerations. This is more mysterious to undergraduates encountering dimensional analysis for the first time. Additionally, the tedium required to determine non-dimensional products via this method and the associated potential for errors in arithmetic are apparent even in this simple example. Fortunately, a better method is available both for green undergraduates and for experienced engineers alike.

3. STEP-BY-STEP METHODS

At this point it is important to stress that considerations of the Buckingham Pi Theorem are critical to any dimensional analysis. The Pi Theorem allows one at once to determine rigorously the required number of non-dimensional parameters to specify fully the physical process under consideration. Accordingly, it is essential to define the indicial matrix of the process and determine its rank before proceeding. That said, it is possible to bypass that step and proceed directly with the method of Ipsen [12]. He showed that one could as well form non-dimensional parameters step-by-step, starting by selecting a single variable containing one of the fundamental dimensions, and using it to eliminate that dimension from all other variables in which it appears. The process is repeated for each fundamental dimension until the non-dimensional products are formed.

For the physical process eqn. 1, one might eliminate the dimension [M] by selecting the variable for density and eliminating it from both itself and the dynamic viscosity, μ. In terms of dimensional formulae, we have

$$[\rho] / [\rho] = 1 \quad (13)$$

and

$$[\mu] / [\rho] = [M L^{-1} t^{-1}] / [M L^{-3}] = [L^2 t^{-1}] \quad (14)$$

Because eqn. 13 is now a pure number, the density is removed from consideration in the analysis. This allows us to express the same physical process in eqn. 1 by another function that is also consistent with the principle of dimensional homogeneity, viz.

$$n = f_2 (\mu / \rho, d, V) \quad (15)$$

Massey [11] appears to have come upon the same concept independently. However, he made an additional observation that allows one to proceed in a clever way. Massey noted that this operation was consistent with using the row containing the density, R2, in the initial indicial matrix of eqn.2 to zero out the column containing the dimension [M], C1, by using elementary row operations on all rows with non-zero values in that column. For the matrix in eqn. 2 we have

$$R2 - R2 = [\,1\ {-3}\ \ 0\,] - [\,1\ {-3}\ \ 0\,] = [\,0\ \ 0\ \ 0\,] \quad (16)$$

and

$$R3 - R2 = [\,1\ {-1}\ {-1}\,] - [\,1\ {-3}\ \ 0\,] = [\,0\ \ 2\ {-1}\,] \quad (17)$$

The new indicial matrix that is now consistent with the physical process described by eqn. 15 becomes

$$I_2 = \begin{matrix} \\ n \\ \rho/\rho \\ \mu/\rho \\ d \\ V \end{matrix} \begin{matrix} \begin{matrix} M & L & t \end{matrix} \\ \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \end{matrix} \quad (18)$$

Using the methods of both Ipsen [12] and Massey [11] next to eliminate the time dimension, [t], using the variable, V, we have

$$[V] / [V] = 1 \quad (19)$$

and

$$[\,\mu / \rho\,] / [\,V\,] = [\,L^2 t^{-1}\,] \;/\; [\,L t^{-1}\,] = [L] \qquad (20)$$

and

$$[\,n\,] / [\,V\,] = [t^{-1}] / [\,L t^{-1}\,] = [L^{-1}] \qquad (21)$$

This results in a new expression for the physical process, viz.

$$n / V \;= f_3 \{\, \mu / (\,\rho V\,)\,,\, d \,\} \qquad (22)$$

Again, this is equivalent to the following elementary row operations applied to the indicial matrix of eqn. 18

$$R1 - R5 = [\;0\;\;0\;\;-1\;] - [\;0\;\;1\;-1\;] = [\;0\;-1\;\;0\;] \qquad (23)$$

$$R3 - R5 = [\;0\;\;2\;\;-1\;] - [\;0\;\;1\;-1\;] = [\;0\;\;1\;\;0\;] \qquad (24)$$

$$R5 - R5 = [\;0\;\;1\;\;-1\;] - [\;0\;\;1\;-1\;] = [\;0\;\;0\;\;0\;] \qquad (25)$$

And the indicial matrix associated with the physical process defined by eqn. 22 is

$$I_3 = \begin{array}{c} \\ n/V \\ \rho/\rho \\ \mu/(\rho V) \\ d \\ V/V \end{array} \begin{array}{c} \begin{array}{ccc} M & L & t \end{array} \\ \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \qquad (26)$$

Finally, one can use the variable for the diameter, d, to eliminate the dimension of length, [L] from all variables. Using the method of Ipsen [12] this is

$$[\,d\,] / [\,d\,] = 1 \qquad (27)$$

and

$$[\,\mu / (\,\rho V\,)\,] / [\,d\,] = [L] / [L] = 1 \qquad (28)$$

and

$$[\,n \;/ V\,] \;[\,d\,] = [L^{-1}]\,[L] = 1 \qquad (29)$$

This gives a final result of

$$Sr = n\,d / V = f_4 \{\, \mu / (\,\rho V d\,)\,\} = f_4\,(1 / Re_d\,) \qquad (30)$$

This is of course equivalent to the following elementary row operations applied to the indicial matrix of eqn. 26

$$R1 + R4 = [\;0\;\;-1\;\;0\;] + [\;0\;\;1\;\;0\;] = [\;0\;\;0\;\;0\;] \qquad (31)$$

and

$$R3 - R4 = [\;0\;\;1\;\;0\;] - [\;0\;\;1\,0\;] = [\;0\;\;0\;\;0\;] \qquad (32)$$

and

$$R4 - R4 = [\;0\;\;1\;\;0\;] - [\;0\;\;1\;\;0\;] = [\;0\;\;0\;\;0\;] \qquad (33)$$

Finally, the indicial matrix corresponding to eqn. 30 is

$$I_4 = \begin{array}{c} \\ n\,d/V \\ \rho/\rho \\ \mu/(\rho V d) \\ d/d \\ V/V \end{array} \begin{array}{c} \begin{array}{ccc} M & L & t \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \qquad (34)$$

Note that the Π products are found in two rows of the final indicial matrix, R1 and R3. All other rows correspond to pure numbers equal to 1. So, again, the end result of Massey's analysis is exactly that found in eqn. 30, above.

So, the methods of Ipsen [12] and Massey [11] produce equivalent results. In both cases, the physical process described in the beginning by eqn. 1 is represented ultimately by a function of two non-dimensional Π products as in eqn. 30. One could argue that neither of these two methods is more readily understood by new undergraduates than the other. However, because Massey's method is represented by elementary row operations applied to the indicial matrix that describes a physical process, it is readily coded in a modern scripting language. Such an embodiment was produced in this study. The code is available for distribution to interested individuals in academia, industry, and government. One very useful feature of the code is the ability to eliminate the dimensions from the indicial matrix using any possible combination of variables. Accordingly, the student can experiment with the formulation of various non-dimensional parameters for a given problem. This allows the student to focus on the results of a dimensional analysis as opposed to getting weighed down in the often tedious arithmetic associated with the method of repeating variables. One example of that flexibility is found in the next section.

4. THE HEAT TRANSFER DISTRIBUTION OVER A FLAT PLATE

The step-by-step method of Massey [11] is stated succinctly as follows: First, for any given physical process, collect all relevant variables and list them in terms of their fundamental dimensions. Then, construct the indicial matrix that describes the physical process and determine its rank. Applying the Buckingham Pi Theorem, determine the number of non-dimensional parameters required to describe the physical process completely. Next, select each dimension (i.e. each column of the indicial matrix) in turn and using a selected variable (i.e. a selected row of the matrix) for that dimension, eliminate it from the indicial matrix using elementary row operations. When a null matrix is achieved, all Π products are known. Algebraic manipulation of the resulting Π products is acceptable provided the total number of non-dimensional parameters according to the theorem of Buckingham is preserved.

So, with these steps in mind, let us consider the distribution of the heat transfer coefficient over the surface of a flat plate immersed in constant velocity flow with an elevated freestream temperature relative to the surface. It is well known [15] that as a boundary layer develops over the plate, the local heat transfer coefficient, h, depends on the local fluid density, ρ, and dynamic viscosity, μ, the distance downstream from the start of the viscous layer, x, the magnitude of the freestream velocity, V, and both the specific heat at constant pressure, c_p, and the thermal conductivity of the fluid, k. This relationship is expressed as

$$h = f_1\,(\rho, \mu, x, V, c_p, k) \qquad (35)$$

In problems involving heat transfer the fundamental dimension of temperature, [T], also occurs. So, in terms of dimensional formulae, the variables in eqn. 35 are $[h]=[M\ t^{-3}\ T^{-1}]$, $[\rho]=[M\ L^{-3}]$, $[\mu] = [M\ L^{-1}\ t^{-1}]$, $[x]=[L]$, $[V] = [L\ t^{-1}]$, $[c_p] = [L^2\ t^{-2}\ T^{-1}]$, and $[k] = [M\ L\ t^{-3}\ T^{-1}]$. The indicial matrix becomes

$$\begin{array}{c} \\ h \\ \rho \\ \mu \\ x \\ V \\ c_p \\ k \end{array} \begin{array}{c} \begin{array}{cccc} M & L & t & T \end{array} \\ \begin{bmatrix} 1 & 0 & -3 & -1 \\ 1 & -3 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & -2 & -1 \\ 1 & 1 & -3 & -1 \end{bmatrix} \end{array} \tag{36}$$

The rank of the indicial matrix is 4, so the number of non-dimensional parameters required to describe the physical process is 3. The forms of the Π products are revealed by applying the method of Massey via the following four steps: using k to eliminate temperature, [T], μ to eliminate mass, [M], V to eliminate time, [t], and x to eliminate [L]. The following null matrix, where the variables are as given in the output of the dimensional-analysis code, obtains

$$\begin{array}{c} \\ h\,k^{-1}\,x \\ \rho\,\mu^{-1}\,V\,x \\ \mu\,\mu^{-1} \\ x\ x^{-1} \\ V\,V^{-1} \\ c_p\ k^{-1}\,\mu \\ k\,k^{-1} \end{array} \begin{array}{c} \begin{array}{cccc} M & L & t & T \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \tag{37}$$

The non-dimensional parameters are found in the first, second, and sixth rows of the null matrix, and these are the Nusselt number, $Nu_x=hx/k$, the Reynolds number, $Re_x=\rho Vx/\mu$, and the Prandtl number, $Pr=\mu\ c_p/k$. The physical process of eqn. 35 is thus represented by the following relation among non-dimensional variables

$$Nu_x = f_2\ (Re_x, Pr) \tag{38}$$

Equation 38 is a very well-known result, and an example of experimental data [16] for a boundary layer undergoing transition to turbulence on a flat plate in compressible flow is shown in fig. 1, below. The correlations for laminar and turbulent heat transfer that are plotted on the figure are from Kays and Crawford [15].

Of course some authors prefer to plot the Stanton number, $St=h/(\rho V\ c_p)$, versus Reynolds number since it decreases as the boundary layer develops with increasing distance downstream, and this makes sense intuitively. Of course it is permissible to perform algebraic operations on the resulting Π products of a dimensional analysis, and one notes that $St=Nu_x/(Re_x\,Pr)$. This result is likewise derivable directly from the step-by-step technique. In this case, one starts with the indicial matrix of eqn. 36 and forms of the Π products via the following four steps: using ρ to eliminate mass, [M], c_p to eliminate temperature, [T], V to eliminate time, [t], and x to eliminate [L]. The following null matrix results

$$\begin{array}{c} \\ h\,\rho^{-1}\,c_p^{-1}\,V^{-1} \\ \rho\,\rho^{-1} \\ \mu\,\rho^{-1}\,V^{-1}\,x^{-1} \\ x\ x^{-1} \\ V\,V^{-1} \\ c_p\,c_p^{-1} \\ k\,\rho^{-1}\,c_p^{-1}\,V^{-1}\,x^{-1} \end{array} \begin{array}{c} \begin{array}{cccc} M & L & t & T \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \tag{39}$$

The non-dimensional parameters are now found in the first, third, and seventh rows of the null matrix. The Stanton number, $St=h/(\rho V\ c_p)$, is in the first row, while the third row is the inverse of the Reynolds number, $1/Re_x=\mu/(\rho Vx)$. The Prandtl number, $Pr=\mu\ c_p/k$, is formed by dividing the third row by the seventh row. So, the physical process of eqn. 35 is thus as well represented by the following relation among non-dimensional variables

$$St = f_3\ (Re_x, Pr) \tag{40}$$

This flexibility in the formation of non-dimensional parameters is one of the great advantages that the method of Massey presents to the student that is newly acquainted with dimensional analysis.

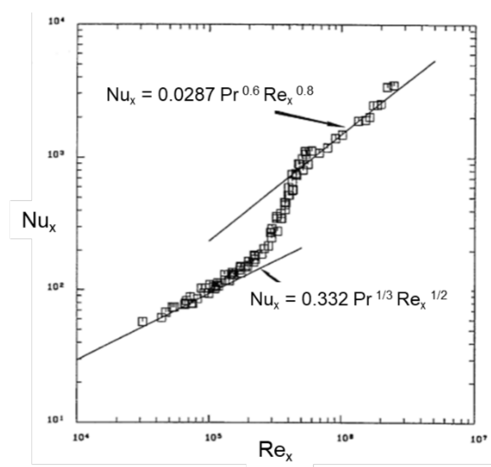

FIGURE 1: FLAT PLATE HEAT TRANSFER DATA PLOTTED ACCORDING TO EQN. 38.

5. A MODEL FOR BYPASS TRANSITION

So far the examples of the application of the step-by-step method of Massey were confined to situations where the results were already known from available textbooks. The analysis of such problems was necessary to clarify the method and to demonstrate its positive attributes. Here, we consider another example for a situation where the outcome was not predetermined, and that concerns the onset of bypass transition in turbomachinery flows. This problem was discussed at length in [17]. However, only a cursory description of the dimensional analysis that guided the development of the transition onset model was given at the time, and the utility of the method of Massey was scarcely mentioned.

In the previous paper [17], an effort was made to develop a model for bypass transition by considering all information available in a 2-equation RANS analysis that might conceivably have an influence on the onset of the process. All variables were considered locally, and these included the momentum thickness of the boundary layer, θ, the density, ρ, the dynamic viscosity, μ, the sonic speed, a, the driving temperature for heat transfer, T_g, the wall temperature, T_w, the magnitude of the freestream velocity, V, as well as its streamwise gradient, dV/dS, the specific heat at constant pressure, c_p, and thermal conductivity of the fluid, k, and both the magnitude of the fluctuating velocity, u′, and the length scale, λ. This relationship is expressed as

$$\theta = f_1\,(\rho, \mu, a, T_g, T_w, V, dV/dS, k, c_p, u', \lambda) \qquad (41)$$

Altogether, there are 12 variables in eqn. 41, so it likely requires 8 non-dimensional parameters to describe the physical process. However, if one argues that there is little conversion of mechanical energy into thermal energy in this situation, then it is possible to treat heat, [H] as a fundamental dimension [11]. This restricts the applicability of the analysis to low supersonic Mach numbers, but one argues that this is an appropriate assumption for turbomachinery flows. However, it also has the potential to reduce the required number of Π products by one. So, in terms of dimensional formulae we have $[\theta]=[L]$, $[\rho]=[M\,L^{-3}]$, $[\mu] = [M\,L^{-1}\,t^{-1}]$, $[a]=[L\,t^{-1}]$, $[T_g]=[T]$, $[T_w]=[T]$, $[V] = [L\,t^{-1}]$, $[dV/dS] = [t^{-1}]$, $[k] = [H\,L^{-1}\,t^{-1}\,T^{-1}]$, $[c_p] =[H\,M^{-1}\,T^{-1}]$, $[u']=[L\,t^{-1}]$, and $[\lambda]=[L]$. The indicial matrix becomes

$$\begin{array}{r} \\ \theta \\ \rho \\ \mu \\ a \\ T_g \\ T_w \\ V \\ dV/dS \\ k \\ c_p \\ u' \\ \lambda \end{array}
\begin{array}{c} \begin{array}{ccccc} M & L & t & T & H \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -3 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 1 \\ -1 & 0 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array} \qquad (42)$$

Sure enough, the rank of the matrix in eqn. 42 equals 5, so the physical process of eqn. 41 is described by 12 – r = 7 non-dimensional parameters. The forms of the Π products are determined using the method of Massey with the following five steps: using k to eliminate heat, [H], T_w to eliminate temperature, [T], μ to eliminate mass, [M], V to eliminate time, [t], and θ to eliminate [L]. This results in the null matrix of eqn. 43 where the Π products are found in rows 2, 4, 5, 8, and 10-12.

$$\begin{array}{r} \\ \theta\,\theta^{-1} \\ \rho\,\mu^{-1}\,V\,\theta \\ \mu\,\mu^{-1} \\ a\,V^{-1} \\ T_g\,T_w^{-1} \\ T_w\,T_w^{-1} \\ V\,V^{-1} \\ dV/dS\,V^{-1}\,\theta \\ k\,k^{-1} \\ c_p\,k^{-1}\,\mu \\ u'\,V^{-1} \\ \lambda\,\theta^{-1} \end{array}
\begin{array}{c} \begin{array}{ccccc} M & L & t & T & H \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -3 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 1 \\ -1 & 0 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array} \qquad (43)$$

The resulting relation among the non-dimensional parameters becomes

$$Re_\theta = f_2\,\{Ma, T_g/T_w, (\theta/V)\,dV/ds, Pr, u'/V, \lambda/\theta\} \qquad (44)$$

The more familiar acceleration parameter, K, is formed by

$$K = \{(\theta\,/V)\,dV/dS\}/\,Re_\theta = \{\mu/(\rho V^2)\}\,dV/dS \qquad (45)$$

Similarly, one could form Thwaites' parameter by

$$K\ \ Re_\theta^2 = \{\mu\,/(\rho V^2)\}\,(dV/dS)\,Re_\theta^2 = (\rho\theta^2/\mu)\,dV/dS \qquad (46)$$

Nevertheless, substituting eqn. 45 in eqn. 44, we have

$$Re_\theta = f_3\,\{Ma, T_g/T_w, K, Pr, u'/V, \lambda/\theta\} \qquad (47)$$

In the previous transition modeling study [17] a database of 57 experimental transition-onset events that occurred over cascade airfoil surfaces was supplemented by flowfield simulations to obtain all the necessary information to evaluate the parameters in eqn. 47. When this was done, a very strong correlation was found for the Reynolds number based on momentum thickness at transition onset as a function of the local freestream turbulence intensity, Tu=100 u′/V multiplied by the ratio of the local momentum thickness to the turbulent length scale at the edge of the boundary layer. The results of that effort are plotted in fig. 2, below. The data for the 57 transition onset events are very well represented by the relation

$$Re_\theta = 7.0\,\{\,Tu\,(\theta/\lambda)\}^{-1} \qquad (48)$$

and this correlation is presented on the figure as the solid line. Also plotted on the figure is a set of validation data for transition onset over a flat-plate surface immersed in a heated flow in a low-speed wind tunnel via a combination of infrared thermography and hot-wire anemometry.

This result is striking because the non-dimensional parameters in eqn. 47 vary by extremely wide ranges for the transition onset events captured by the database. As seen in the list of parameters below, the database covers a range of flow speeds from incompressible through the transonic regime. Also, the dataset includes both adiabatic flows and those with substantial heat transfer where the stability of the boundary layer is markedly affected. It also includes both strong accelerations in excess of the re-laminarization limit ($K>3x10^{-6}$) and large decelerations in excess of the typical value given to indicate separation via Thwaites' parameter ($K\ Re_\theta^2 < -0.09$). The range of turbulence intensities includes both levels found in relatively quiet wind tunnels and those consistent with local levels found in multi-row low pressure turbines. Accordingly, it is expected that eqn. 48 is applicable to a wide range of turbomachinery flowfields.

$$73 < Re_\theta < 860$$
$$0.05 < Ma < 1.24$$
$$1.0 < T_g / T_w < 1.4$$
$$-1.9 < K \times 10^6 < 4.8$$
$$0.11 < Tu < 5.1$$
$$Pr = 0.71$$
$$4 < \lambda / \theta < 66$$
$$-0.15 < K\ Re_\theta^2 < 0.057$$

Given the form of eqn. 48, the physical process involved in the onset of bypass transition in turbomachinery flows is actually described by

$$\theta = f_4\ (\rho, \mu, u', \lambda) \tag{49}$$

Accordingly, the indicial matrix of the process is

$$\begin{array}{c} \\ \theta \\ \rho \\ \mu \\ u' \\ \lambda \end{array} \begin{array}{c} \begin{array}{ccc} M & L & t \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & -3 & 0 \\ 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix} \end{array} \tag{50}$$

and the Π products are formed by using μ to eliminate mass, [M], u′ to eliminate time, [t], and θ to eliminate [L]. This results in the following null matrix

$$\begin{array}{c} \\ \theta\,\theta^{-1} \\ \rho\,\mu^{-1}u'\,\theta \\ \mu\,\mu^{-1} \\ u'\,u'^{-1} \\ \lambda\,\theta^{-1} \end{array} \begin{array}{c} \begin{array}{ccc} M & L & t \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \tag{51}$$

and the correlation of eqn. 48 is rewritten as

$$\rho\, u'\, \theta / \mu = 0.07\, \lambda / \theta \tag{52}$$

Now, one can as well write eqn. 52 as

$$(u' / \lambda)\ (\rho\, \theta^2 / \mu) = 0.07 \pm 0.01 \tag{53}$$

where the first term on the left side of eqn. 53 is the inverse of the timescale associated with large eddies at the edge of the boundary layer, while the second is the laminar diffusion time. The right hand side represents the mean and the standard deviation of all transition events in the database. So, over a very large range of non-dimensional conditions consistent with turbomachinery, transition to turbulence occurs when the ratio of timescales associated with the growth of the laminar boundary layer to that of the large eddies at the edge of the viscous layer reaches a critical value. This is a surprising result, and it follows directly from a consideration of the physical process associated with transition onset in terms of the Buckingham Pi Theorem and the step-by-step method of Massey.

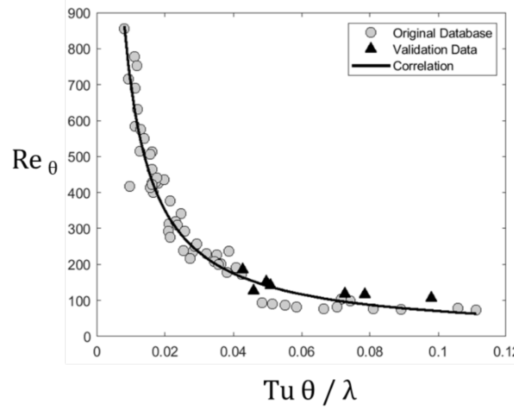

FIGURE 2: MODEL DATA AND EXPERIMENTAL RESULTS FOR TRANSITION ONSET PLOTTED ACCORDING TO EQN. 38

Comparing eqn. 53 to eqn. 47 (or eqn. 49 to eqn. 41) it is possible in hindsight to argue that the effect of compressibility is fully captured by the inclusion of density in the analysis. Similarly, the effect of heat transfer is fully captured by the temperature dependence of the dynamic viscosity. Also, one notes that the momentum thickness is an accumulation of the history of the boundary layer, and that is greatly influenced by the pressure gradients over which it develops. So, perhaps it was possible to write equation 49 directly at the outset. However, the elusiveness of a predictive model for transition onset over many decades would tend to discourage such a leap of imagination.

Finally, in keeping with the character of turbomachinery flows, it was assumed from the outset that there was not a substantial conversion between mechanical and thermal energies during the transition to turbulence, and this allowed us to introduce the dimension of heat, [H]. This potentially limits the validity of the correlations of eqn. 48, 52, and 53 for application below hypersonic flow speeds. However, if the analysis beginning with eqn. 41 is repeated without invoking a dimension of heat, then one additional Π product is obtained in eqn. 47, and that is the Eckert number, $Ec = V^2/(c_p\, T_g)$. Of course, it is not surprising that the Eckert number might play a vital role in the modeling of transition in hypersonic flows.

6. CONCLUSION

There is a robust technique for the formation of non-dimensional parameters during a dimensional analysis that is consistently overlooked by authors of textbooks that introduce students to the Buckingham Pi Theorem. The method is original to the late Prof. B. S. Massey from University College London. Massey's method was presented here with reference to physical processes describing vortex shedding from a circular cylinder, the heat transfer distribution over a flat plate in constant velocity flow, and the onset of bypass transition in turbomachinery flows. The method requires no prior selection of scaling parameters to form the Π products, and its simplicity encourages the student to attempt various formulations of non-dimensional parameters during the solution to problems. Accordingly, the method is recommended for teaching to first time students of fluid mechanics, and it is also a useful technique for practicing engineers and researchers alike. Its use is especially recommended for tackling problems involving a great many dimensional variables such as one encounters in film cooling [18]. Finally, the method is well suited to implementation in computer scripting languages like Matlab and Python, and such a code was written in the former for distribution to interested parties. Please contact this author if interested in applying the code to specific problems.

ACKNOWLEDGEMENTS

Thanks to Dr. Shichuan Ou for his assistance with the collection of the experimental data for transition onset that is plotted in fig. 2. Also, the transition modeling work presented here as an illustration of Massey's method was originally accomplished in collaboration with Dr. Tom Praisner of Pratt & Whitney, albeit with a slightly different presentation of results in [17]. The author is indebted to the late Prof. T. V. Jones who taught him truly to appreciate the use of dimensional analysis in fluid mechanics. Prof. Jones applied the Buckingham Pi Theorem to the solution of problems in Engineering Science with great enthusiasm, and the author first learned from him that there are cases where one might profitably treat heat as a fundamental dimension. Additionally, casting the model for bypass transition in terms of two time scales follows directly from knowledge gleaned via discussions with Prof. Jones many years ago. Finally, it was through the interest in dimensional analysis obtained as a student of Prof. Jones that this author stumbled upon the text of Massey [11].

A brief note about Prof. Massey is likewise in order. Prof. Bernard Stanford Massey's career at University College London spanned 32 years, and in that time he authored three textbooks in Engineering Science, he became an Honorary Member of the Royal Corps of Naval Constructors, and he had a profound and positive effect on a great many students [19]. In addition to his contributions to engineering education he was also a scholar of hymnology as well as a composer and arranger of music for the Church of England. Prof. Massey also served for many years as the editor of the *Bulletin of the Hymn Society of Great Britain and Ireland* [20].

REFERENCES

[1] White, F. M., 1994, *Fluid Mechanics*, 3rd Ed., McGraw-Hill, New York, NY.

[2] Young, D. F., Munson, B. R., and Okiishi, T. H., 1997, *A Brief Introduction to Fluid Mechanics*, Wiley, New York, NY.

[3] Fox, R. W., Pritchard, P. J., McDonald, A. T., 2009, *Introduction to Fluid Mechanics*, 7th Ed., Wiley, New York, NY.

[4] Buckingham, E., 1914, "On Physically Similar Systems; Illustrations of the Use of Dimensional Equations," *Physical Review*, Vol. 4, No. 4, pp. 345-376.

[5] Shames, I. H., 1992, *Mechanics of Fluids*, 3rd Ed., McGraw-Hill, New York, NY.

[6] Franzini, J. B. and Finnemore, E. J., 1997, *Fluid Mechanics with Engineering Applications*, 9th Ed., McGraw-Hill, New York, NY.

[7] Roberson, J. A. and Crowe, C. T., 1997, *Engineering Fluid Mechanics*, 6th Ed., Wiley, New York, NY.

[8] Panton, R. L., 1998, "Dimensional Analysis," in *The Handbook of Fluid Dynamics*, ed. Johnson, R. W., CRC Press, Boca Raton, FL.

[9] Schlichting, H., 1979, *Boundary-Layer Theory*, 7th Ed., McGraw-Hill, New York, NY.

[10] White, F. M., 1991, *Viscous Fluid Flow*, 2nd Ed., McGraw-Hill, New York, NY.

[11] Massey, B. S., 1986, *Measures in Science and Engineering*, Ellis-Horwood, Ltd., Chichester, UK.

[12] Ipsen, D. C., 1960, *Units, Dimensions, and Dimensional Numbers*, McGraw-Hill, New York, NY.

[13] White, F. M., 2011, *Fluid Mechanics*, 7th Ed., McGraw-Hill, New York, NY.

[14] Massey, B. S., 1989, *Mechanics of Fluids*, 6th Ed., Van Nostrand Reinhold, London, U.K.

[15] Kays, W. M. and Crawford, M. E., 1993, *Convective Heat and Mass Transfer*, 3rd Ed., McGraw-Hill, New York, NY.

[16] Clark, J. P., 1993, "A Study of Turbulent Spot Propagation in Turbine Representative Flows," DPhil Thesis, University of Oxford, U.K.

[17] Praisner, T. J. and Clark, J. P., 2007, "Predicting Transition in Turbomachinery, Part I – A Review and New Model Development," *ASME J. Turbomach.*, Vol. 129, pp. 1-13, January.

[18] Ornano, F. and Povey, T., 2020, "Theory of Dimensionless Groups in Film Effectiveness Studies," *ASME J. Turbomach.*, Vol. 142, pp. 04100-1:20, April.

[19] Pindelski, T., 2017, "My Years in College: 1970-1973," accessed December 13, 2021, http://pindelski.org/Photography/2017/12/16/years-college-1970-1973-2/

[20] "Bernard Stanford Massey," 2011, *The Canterbury Dictionary of Hymnology*, Canterbury Press, accessed December 13, 2021, http://www.hymnology.co.uk/b/bernard-stanford-massey.

