



ARL-TR-9687 • MAY 2023



A Large Deformation Multiphase Continuum Mechanics Model for Shock Loading of Lung Parenchyma. Part II: Numerical Methods

by Zachariah T Irwin, Richard A Regueiro, and
John D Clayton

Approved for public release; distribution is unlimited.

NOTICES

Disclaimers

The findings in this report are not to be construed as an official Department of the Army position unless so designated by other authorized documents.

Citation of manufacturer's or trade names does not constitute an official endorsement or approval of the use thereof.

Destroy this report when it is no longer needed. Do not return it to the originator.



A Large Deformation Multiphase Continuum Mechanics Model for Shock Loading of Lung Parenchyma. Part II: Numerical Methods

Zachariah T Irwin
University of Colorado, Boulder

Richard A Regueiro and John D Clayton
DEVCOM Army Research Laboratory

REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

Public reporting burden for this collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing the burden, to Department of Defense, Washington Headquarters Services, Directorate for Information Operations and Reports (0704-0188), 1215 Jefferson Davis Highway, Suite 1204, Arlington, VA 22202-4302. Respondents should be aware that notwithstanding any other provision of law, no person shall be subject to any penalty for failing to comply with a collection of information if it does not display a currently valid OMB control number.

PLEASE DO NOT RETURN YOUR FORM TO THE ABOVE ADDRESS.

1. REPORT DATE (DD-MM-YYYY) May 2023		2. REPORT TYPE Technical Report		3. DATES COVERED (From - To) January 2022–March 2023	
4. TITLE AND SUBTITLE A Large Deformation Multiphase Continuum Mechanics Model for Shock Loading of Lung Parenchyma. Part II: Numerical Methods				5a. CONTRACT NUMBER	
				5b. GRANT NUMBER	
				5c. PROGRAM ELEMENT NUMBER	
6. AUTHOR(S) Zachariah T Irwin, Richard A Regueiro, and John D Clayton				5d. PROJECT NUMBER	
				5e. TASK NUMBER	
				5f. WORK UNIT NUMBER	
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) DEVCOM Army Research Laboratory ATTN: FCDD-RLA-TB Aberdeen Proving Ground, MD 21005-5066				8. PERFORMING ORGANIZATION REPORT NUMBER ARL-TR-9687	
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)				10. SPONSOR/MONITOR'S ACRONYM(S)	
				11. SPONSOR/MONITOR'S REPORT NUMBER(S)	
12. DISTRIBUTION/AVAILABILITY STATEMENT Approved for public release; distribution is unlimited.					
13. SUPPLEMENTARY NOTES ORCID ID: John Clayton, 0000-0003-4107-6282					
14. ABSTRACT A 1-D numerical implementation of a finite-strain theory of a biphasic mixture is described. The theory includes coupled pore fluid flow and solid skeleton deformation for a soft porous material applicable to high strain-rate dynamic loading. The constitutive model is nonlinear elastic and accounts for the compressibility of the pore air. The treatment does not require equivalency of acceleration of pore fluid to that of solid skeleton, but rather allows them to be different. Through implementation of the concept of solid extra stress, the theory is able to distinguish among solid skeleton, pore fluid (air), and total pressures, and similarly among stress tensors for each constituent. General features of the constitutive description are specialized for an application to shock loading of lung parenchyma. This report, which focuses on the finite-element formulation of the theory and constitutive models, is the second in a series of three reports.					
15. SUBJECT TERMS theory of porous media, soft tissue mechanics, shock waves, lung, Sciences of Extreme Materials, Terminal Effects					
16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT UU	18. NUMBER OF PAGES 100	19a. NAME OF RESPONSIBLE PERSON John D Clayton
a. REPORT Unclassified	b. ABSTRACT Unclassified	c. THIS PAGE Unclassified			19b. TELEPHONE NUMBER (Include area code) 410-278-6146

Contents

List of Figures	v
List of Tables	v
Acknowledgments	vi
1. Introduction	1
1.1 Background	1
1.2 Geometric Model	3
1.3 Organization	5
2. 1-D Kinematics	5
3. Variational Formulation	6
3.1 Balance of Mass of the Mixture	6
3.2 Balance of Linear Momentum of the Mixture	10
3.3 Balance of Linear Momentum of the Pore Fluid	13
4. FE Implementation	15
5. Time Integration	20
5.1 Newmark-beta Integrators	21
5.1.1 Linearization of Variational Equations for the (\mathbf{u}) Theory	26
5.1.2 FE Formulation of the (\mathbf{u}) Theory	26
5.1.3 Linearization of Variational Equations for the $(\mathbf{u}-p_f)$ Theory	28
5.1.4 FE formulation of the $(\mathbf{u}-p_f)$ Theory	30
5.1.5 Linearization of Variational Equations for the $(\mathbf{u}-\mathbf{u}_f-p_f)$ Theory	35
5.1.6 FE Formulation of the $(\mathbf{u}-\mathbf{u}_f-p_f)$ Theory	37
5.2 Runge-Kutta Time Integrators	45
5.2.1 Implementation of the (\mathbf{u}) Theory	47
5.2.2 Implementation of the $(\mathbf{u}-p_f)$ Theory	50
5.2.3 Implementation of the $(\mathbf{u}-\mathbf{u}_f-p_f)$ Theory	58

5.3	Central Difference Time Integrator	68
5.3.1	Implementation of the (u) Theory	68
5.3.2	Implementation of the (u - p_i) Theory	70
6.	Stabilization Techniques	79
6.1	Shock Viscosity	80
6.2	Pressure Stabilization	82
7.	Conclusions	83
8.	References	85
	List of Symbols, Abbreviations, and Acronyms	90
	Distribution List	92

List of Figures

Fig. 1	Schematic for a cylindrical mesh used in the 1-D uniaxial strain simplification, where the Q2-Q2-P1 element is highlighted. Here, d corresponds to the interpolated solid skeleton (s) displacement solution, d_f corresponds to the interpolated pore fluid (f) displacement solution, and θ corresponds to the interpolated pore fluid pressure solution at their respective nodes in the FE model.	3
Fig. 2	Schematic for a rectangular mesh used in the 1-D uniaxial strain simplification, where the Q2-Q2-P1 element is highlighted. Here, d corresponds to the interpolated solid skeleton (s) displacement solution, d_f corresponds to the interpolated pore fluid (f) displacement solution, and θ corresponds to the interpolated pore fluid pressure solution at their respective nodes in the FE model.	4
Fig. 3	Illustration of Newton-Raphson method for solution of \ddot{x} , where we show iteration $k = 0, 1, 2$, and convergence likely at iteration $k = 3$. The black curve is a plot of $R(\ddot{x})$	22

List of Tables

Table 1	Acronyms for different element types.	17
---------	--	----

Acknowledgments

The authors acknowledge support from the Army Research Directorate of the US Army Combat Capabilities Development Command (DEVCOM) Army Research Laboratory (ARL). R Regueiro acknowledges support from a Joint Faculty Appointment at ARL. Z Irwin acknowledges the Department of Defense SMART scholarship program for doctoral students. This work utilized resources from the University of Colorado Boulder Research Computing Group, which is supported by the National Science Foundation (awards ACI-1532235 and ACI-1532236), the University of Colorado Boulder, and Colorado State University. This work was supported in part by high-performance computer time and resources from the DoD High Performance Computing Modernization Program.

This report encompasses and extends research recently submitted in abbreviated form by the authors in a refereed article* to the *International Journal for Numerical Methods in Engineering* (2023).

*Irwin Z, Clayton J, Regueiro R. A large deformation multiphase continuum mechanics model for shock loading of soft porous materials. *International Journal for Numerical Methods in Engineering*. Forthcoming 2023.

1. Introduction

1.1 Background

The lung parenchyma consists of alveolar air sacs ranging from 100 to 330 μm in diameter¹ and connective tissues consisting of a complex network of collagen and elastin fibers.² The composition and physical dimensions of the fibers play a great role in determining the stiffness of these networks. Mechanical forces that act upon the extra-cellular matrix (ECM), in which the collagen and elastin fibers reside, can induce the secretion of growth factors that accelerate ECM remodeling and alter the microstructural composition of the fiber networks.²

At high rates of strain, such as those induced by a shock wave, elastin fibers are prone to rupture³ and the remaining stress response percolates through the collagen fibril networks as the collagen fibers start to extend to their uncoiled lengths. This increases the stress response of the entire lung parenchyma.⁴ The complex roles that collagen and elastin fibers play in the dynamics of the lung parenchyma are discussed by Freed et al.³ Dynamic influences of the bronchiole tubes, which are stiffer than the parenchyma, on the response of the impacted lung are observed in experiments by Brannen et al.⁵ The present focus is on the role that 1-D poro-elasto-dynamics plays at the mesoscopic level. Outcomes of the current program of research are intended to be used to inform macroscopic, single-phase models of the lung for 3-D modeling of much larger domains.⁶⁻¹¹

Mixture theory was first established at finite strain by Truesdell and Toupin.¹² Subsequent works by Bowen¹³⁻¹⁵ and others¹⁶⁻¹⁸ applied this rich continuum theory to porous media. Shock waves in mixtures were studied using analytical methods by Bowen, Chen, and Wright¹⁹⁻²³ and more recently, by Clayton.²⁴ The theory of porous media (TPM) is an approximation to a more computationally expensive fluid-structure interaction (FSI) model at the pore length scale. In TPM, interactions are smeared across a continuum material point, simplifying not just the governing mathematical equations but the discretization of the geometry itself.

Lung parenchyma is a complex heterogeneous material.²⁵ Resolution would require representation by a detailed 3-D finite-element (FE) mesh to model the *explicit* FSI. Resolution could be possible via computational fluid dynamics-computational solid mechanics with arbitrary Lagrangian-Eulerian (ALE) modeling. However,

such frameworks would involve costly simulations and mesh generation and bias regarding the structure of the FE mesh.

Shock waves induce large pressure gradients that are thought to directly cause microstructural tearing of the lung parenchyma leading to hemorrhaging.^{25–27} Injury propagations through the lung parenchyma at the ultrastructural (tissue fibers) and microstructural (air sac) levels are not well understood and constitute active areas of research.^{3,8} The fine-scale approach initiated by Freed et al.³ to model the microstructure necessitates a multiscale model of the whole lung tissue, which includes the highly heterogeneous structure of the lung parenchyma. Work is needed to first accurately simulate the deformation of lung parenchyma using a multi-physics approach, before addressing damage and injury pathology at the microscale (i.e., alveolar regime) and linking that to the mesoscale (i.e., parenchymal regime). In the current program,^{28,29} we incorporate TPM to take into account the different response times of the two constituents in the lung parenchyma subjected to shock loading: solid skeleton (s) (lung parenchyma) and the pore fluid (f) (air) that occupies the pore space. Various choices of different explicit time integration schemes are presented and compared in this report.

This report derives the variational equations for balances of mass and linear momentum for a biphasic mixture at large deformations, as well as the FE implementation with various stabilization parameters and time integration schemes. See Irwin et al.²⁸ for notation that is consistently used here and in a follow-up study.²⁹ From the previous report,²⁸ the biphasic model is subject to the following assumptions³⁰:

- saturated solid-fluid mixture:
 $n^s + n^f = 1,$
- materially incompressible solid:
 $\rho^{sR} = \text{const.},$
- materially compressible pore fluid,
- no mass production:
 $\hat{\gamma}^\alpha := 0,$
- non-polar materials:
 $\boldsymbol{\sigma}^\alpha = (\boldsymbol{\sigma}^\alpha)^T,$
- uniform gravitational body force:
 $\mathbf{b}^\alpha = \mathbf{b} = \mathbf{g},$
- mixed constituent temperatures:
 $\theta^s = \theta^f = \theta,$
- no thermo-mechanical coupling:
 $\gamma^s := 0, \quad \gamma^f := 0,$
- nearly-inviscid pore fluid:
 $\boldsymbol{\sigma}_E^f \approx \mathbf{0},$
- neo-Hookean constitutive model for the solid (s) constituent

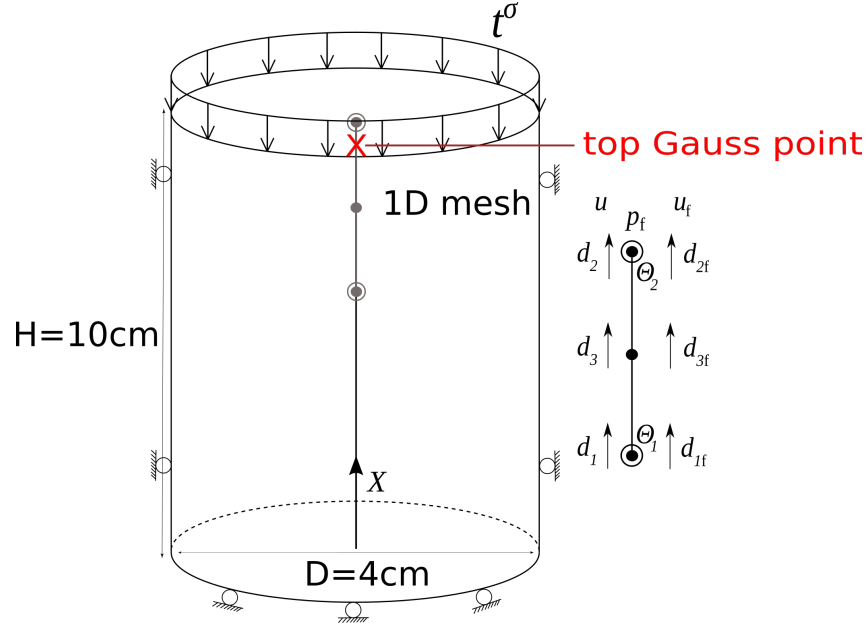


Fig. 1 Schematic for a cylindrical mesh used in the 1-D uniaxial strain simplification, where the Q2-Q2-P1 element is highlighted. Here, d corresponds to the interpolated solid skeleton (s) displacement solution, d_f corresponds to the interpolated pore fluid (f) displacement solution, and θ corresponds to the interpolated pore fluid pressure solution at their respective nodes in the FE model.

1.2 Geometric Model

As first discussed in the previous report,²⁸ we model an excised section of lung parenchyma as either a uniform cylinder, Fig. 1, or uniform rectangular column, Fig. 2. The choice between these two geometries is somewhat arbitrary as long as the motion is 1-D: if we assume uniaxial strain with unidirectional flow (i.e., no permeation of air through the sides of the cylinder), the overall geometry of the lung parenchyma becomes irrelevant besides the (finite) length H . This assumption is appropriate for immediate motions resulting from shock loading in which the magnitude of transverse motions (i.e., perpendicular to the direction of the shock front) are small compared to the motions aligned with the shock velocity. When making comparisons to computational results from other software, such as those obtained using the LS-DYNA code,³¹ we typically assume a rectangular column for ease of meshing. For dimensions shown in Figs. 1 and 2, the rectangular choice has the sole effect of reducing the cross-sectional area (i.e., area of $L^2 = 1 \text{ cm}^2$) when compared to the cylindrical column (i.e., area of $\pi D^2/4 \approx 12.6 \text{ cm}^2$).

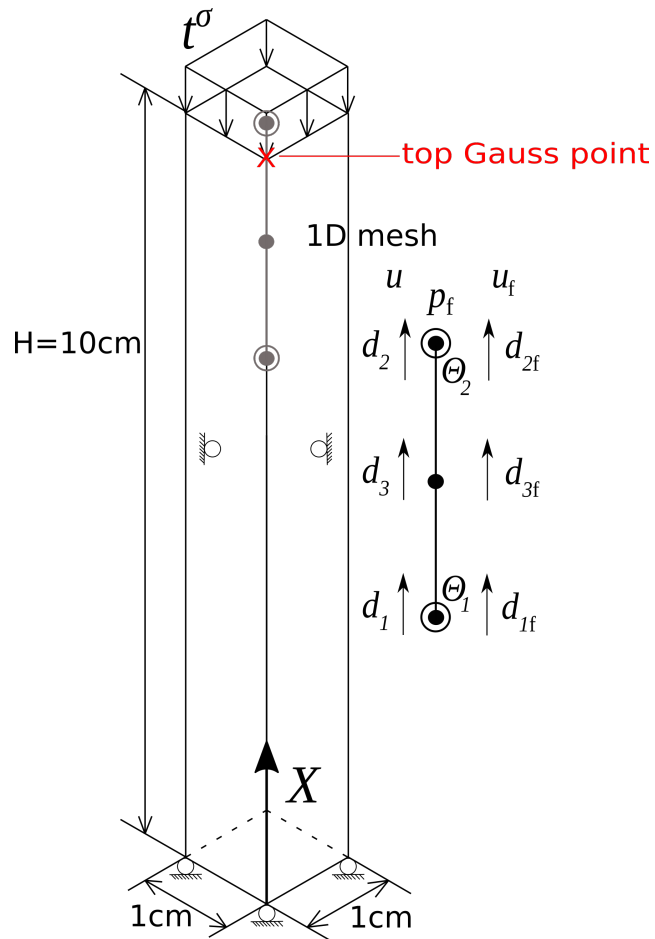


Fig. 2 Schematic for a rectangular mesh used in the 1-D uniaxial strain simplification, where the Q2-Q2-P1 element is highlighted. Here, d corresponds to the interpolated solid skeleton (s) displacement solution, d_f corresponds to the interpolated pore fluid (f) displacement solution, and θ corresponds to the interpolated pore fluid pressure solution at their respective nodes in the FE model.

1.3 Organization

The remainder of this report is organized in the following manner. Section 2 reports the kinematic simplifications for 1-D motion. Section 3 presents the variational forms of the balance equations, wherein we apply the standard Bubnov-Galerkin spatial-discretization procedure for the FE model to solve for the variables of interest: lung parenchyma displacement (and velocity and acceleration), pore fluid pressure and pore fluid displacement (and velocity and acceleration). Section 4 documents the 1-D spatial discretization using the FE method. Section 5 presents a range of time-discretization schemes of the FE equations, including the well-known Newmark-beta (NB) family of time integrators and a fifth-order accurate Runge-Kutta time integrator. Each have their own advantages depending on the magnitude and frequency of the strain-rate loading that is applied. Section 6 discusses numerical dissipation techniques for shock loading. Conclusions summarizing the numerical developments and mentioning past and subsequent applications²⁸⁻³⁰ follow in Section 7. Detailed derivations of the FE equations for the various time-discretization schemes are presented in Appendices of a follow-up report.²⁹

The term “elasticity” is used to refer to a quasi-static setting (i.e., inertial terms and stress waves absent) with linear or nonlinear elastic constitutive behavior. The term “elastodynamics” refers to a fully dynamic setting (i.e., inertial terms retained in the momentum balance, leading to stress waves), again with linear or nonlinear elastic behavior.

2. 1-D Kinematics

One-dimensional uniaxial solid skeleton strain and unidirectional pore fluid flow simplifications are implemented for (s) and (f) phases, respectively. If we assume that the soft porous material only undergoes compression or expansion in one dimension, specifically the vertical direction X , then both the solid skeleton displacements in the transverse directions as well as all shear terms in the deformation gradient go to zero. The solid skeleton deformation gradient becomes

$$\mathbf{F}_s = \begin{bmatrix} 1 + \frac{\partial u}{\partial X} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad J_s = \det \mathbf{F}_s = F_{11(s)} = 1 + \frac{\partial u}{\partial X} \quad (1)$$

where $u(X, t)$ is the axial displacement in the X direction. Similarly, if we assume that pore fluid flow is confined in direction X (i.e., surrounded by an impermeable sleeve, drained or undrained at its ends), then

$$\mathbf{v}_f = \begin{bmatrix} v_f \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{a}_f = \begin{bmatrix} a_f \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{u}_f = \begin{bmatrix} u_f \\ 0 \\ 0 \end{bmatrix} \quad (2)$$

where in the X direction, v_f is the pore fluid velocity, a_f is the pore fluid acceleration, and u_f is the pore fluid displacement. Variational forms that follow are first presented in a general 3-D regime before simplifications to the 1-D regime.

3. Variational Formulation

In this section, we formulate a total Lagrangian implementation such that constituents move relative to the solid skeleton, that is, the reference configuration follows the motion of the solid skeleton; see the prior report by Irwin et al.²⁸ for details. For this reason, we choose to drop $(\cdot)^s$ and $(\cdot)_s$ designations for notational simplicity for variables and parameters associated with the solid phase and solid skeleton.

3.1 Balance of Mass of the Mixture

The strong form of the balance of mass for the biphasic mixture under the assumption of mixed temperatures and barotropic constituents as presented by Irwin et al.²⁸ is written as

$$\frac{J n^f}{K_f^\eta} D_t p_f + D_t J + \frac{J}{K_f^\eta} \text{GRAD}(p_f) \cdot \mathbf{F}^{-1} \cdot (n^f \tilde{\mathbf{v}}_f) + J \text{GRAD}(n^f \tilde{\mathbf{v}}_f) : \mathbf{F}^{-T} = 0 \quad (3)$$

$$p_f(\mathbf{X}, t) = g^p(\mathbf{X}, t) \quad \forall \mathbf{X} \in \Gamma_0^p \quad (4)$$

$$-[J \mathbf{F}^{-1} \cdot (n^f \tilde{\mathbf{v}}_f)] \cdot \mathbf{N} = Q_f(\mathbf{X}, t) \quad \forall \mathbf{X} \in \Gamma_0^{Q_f} \quad (5)$$

where g^p is the prescribed pore fluid pressure on Γ_0^p typically set for the ‘‘drained’’ boundary condition, and Q_f is the prescribed fluid flux (positive inward) on $\Gamma_0^{Q_f}$. We have assumed that the mass supply of the pore fluid (f) constituent $\hat{\gamma}^f$ is negligible.

Let $\mathcal{H}(\mathbf{u}, \mathbf{u}_f, p_f, r)$ be the variational form of Eq. 3, and let r be a set of scalar valued weighting functions associated with pore fluid pressure p_f . We may then

rewrite Eq. 3 in indicial notation as

$$\begin{aligned}
\mathcal{H}(u_i, u_{i(\mathfrak{f})}, p_{\mathfrak{f}}, r) &= \int_{\mathcal{B}_0} r \left(\frac{J n^{\mathfrak{f}}}{K_{\mathfrak{f}}^{\eta}} \dot{p}_{\mathfrak{f}} + \dot{j} \right) dV \\
&+ \int_{\mathcal{B}_0} r \frac{J}{K_{\mathfrak{f}}^{\eta}} \frac{\partial p_{\mathfrak{f}}}{\partial X_I} F_{Ii}^{-1} (n^{\mathfrak{f}} \tilde{v}_{i(\mathfrak{f})}) dV \\
&+ \int_{\mathcal{B}_0} r J \frac{\partial (n^{\mathfrak{f}} \tilde{v}_{i(\mathfrak{f})})}{\partial X_I} F_{Ii}^{-1} dV = 0
\end{aligned} \tag{6}$$

Using chain rule, the last term in Eq. 6 can be rewritten as follows:

$$\begin{aligned}
\int_{\mathcal{B}_0} r J \frac{\partial (n^{\mathfrak{f}} \tilde{v}_{i(\mathfrak{f})})}{\partial X_I} F_{Ii}^{-1} dV &= \int_{\mathcal{B}_0} \frac{\partial (r J (n^{\mathfrak{f}} \tilde{v}_{i(\mathfrak{f})}) F_{Ii}^{-1})}{\partial X_I} dV \\
&- \int_{\mathcal{B}_0} \frac{\partial r}{\partial X_I} J (n^{\mathfrak{f}} \tilde{v}_{i(\mathfrak{f})}) F_{Ii}^{-1} dV \\
&- \int_{\mathcal{B}_0} r (n^{\mathfrak{f}} \tilde{v}_{i(\mathfrak{f})}) \frac{\partial (J F_{Ii}^{-1})}{\partial X_I} dV
\end{aligned} \tag{7}$$

wherein the last term in Eq. 7 goes to zero because of the Piola identity shown by Holzapfel³² (p. 146) or in general curvilinear coordinates by Clayton^{33,34} (p. 42 in the latter). Substitution of Eq. 7 into Eq. 6 gives us

$$\begin{aligned}
\mathcal{H}(u_i, u_{i(\mathfrak{f})}, p_{\mathfrak{f}}, r) &= \int_{\mathcal{B}_0} r \left(\frac{J n^{\mathfrak{f}}}{K_{\mathfrak{f}}^{\eta}} \dot{p}_{\mathfrak{f}} + \dot{j} \right) dV \\
&+ \int_{\mathcal{B}_0} r \frac{J}{K_{\mathfrak{f}}^{\eta}} \frac{\partial p_{\mathfrak{f}}}{\partial X_I} F_{Ii}^{-1} (n^{\mathfrak{f}} \tilde{v}_{i(\mathfrak{f})}) dV \\
&+ \int_{\mathcal{B}_0} \frac{\partial (r J (n^{\mathfrak{f}} \tilde{v}_{i(\mathfrak{f})}) F_{Ii}^{-1})}{\partial X_I} dV \\
&- \int_{\mathcal{B}_0} \frac{\partial r}{\partial X_I} J (n^{\mathfrak{f}} \tilde{v}_{i(\mathfrak{f})}) F_{Ii}^{-1} dV = 0
\end{aligned} \tag{8}$$

Applying the divergence theorem to the third term in Eq. 8, using the boundary conditions in Eq. 4 (where $r \rightarrow 0$ on Γ_0^p) and Eq. 5, the variational form for the

balance of mass of the biphasic mixture becomes

$$\begin{aligned}
\mathcal{H}(u_i, u_{i(f)}, p_f, r) &= \int_{\mathcal{B}_0} r \left(\frac{J n^f}{K_f^\eta} \dot{p}_f + \dot{j} \right) dV \\
&+ \int_{\mathcal{B}_0} r \frac{J}{K_f^\eta} \frac{\partial p_f}{\partial X_I} F_{Ii}^{-1} (n^f \tilde{v}_{i(f)}) dV \\
&- \int_{\mathcal{B}_0} \frac{\partial r}{\partial X_I} J (n^f \tilde{v}_{i(f)}) F_{Ii}^{-1} dV - \int_{\Gamma_0^{Q_f}} r Q_f dA = 0 \quad (9)
\end{aligned}$$

Given Darcy's law (see Irwin et al.^{28,30}), for notational simplicity we denote the separate terms as

$$\mathcal{H} = \mathcal{H}_1^{\text{INT}} + \mathcal{H}_2^{\text{INT}} + \mathcal{H}_3^{\text{INT}} + \mathcal{H}_4^{\text{INT}} - \mathcal{H}^{\text{EXT}} = 0 \quad (10)$$

where

$$\mathcal{H}_1^{\text{INT}} = \int_{\mathcal{B}_0} r \left(\frac{J n^f}{K_f^\eta} \dot{p}_f + \dot{j} \right) dV \quad (11)$$

$$\mathcal{H}_2^{\text{INT}} = \int_{\mathcal{B}_0} r \frac{J}{K_f^\eta} \frac{\partial p_f}{\partial X_I} F_{Ii}^{-1} (n^f \tilde{v}_{i(f)}) dV \quad (12)$$

$$\mathcal{H}_3^{\text{INT}} = \int_{\mathcal{B}_0} \frac{\partial r}{\partial X_I} J F_{Ii}^{-1} \hat{k} \frac{\partial p_f}{\partial X_K} F_{Ki}^{-1} dV \quad (13)$$

$$\mathcal{H}_4^{\text{INT}} = \int_{\mathcal{B}_0} \frac{\partial r}{\partial X_I} J F_{Ii}^{-1} \hat{k} \rho^{\text{fR}} (a_{i(f)} - g_i) dV \quad (14)$$

$$\mathcal{H}^{\text{EXT}} = \int_{\Gamma_0^{Q_f}} r Q_f dA \quad (15)$$

In the $(\mathbf{u}-p_f)$ formulation, where $\mathbf{a}_f \approx \mathbf{a}_s = \mathbf{a}$, the variational form of Eq. 3 is $\mathcal{H}(u_i, p_f, r)$ and Eq. 14 becomes

$$\mathcal{H}_4^{\text{INT}} = \int_{\mathcal{B}_0} \frac{\partial r}{\partial X_I} J F_{Ii}^{-1} \hat{k} \rho^{\text{fR}} (a_i - g_i) dV \quad (16)$$

For poroelasticity, inertia terms are ignored and Eq. 14 becomes

$$\mathcal{H}_4^{\text{INT}} = \int_{\mathcal{B}_0} \frac{\partial r}{\partial X_I} J F_{Ii}^{-1} \hat{k} \rho^{\text{fR}} (-g_i) dV \quad (17)$$

Also note that for poroelasticity, Eq. 12 does not include the inertia term when the Darcy velocity is expanded via Eq. 60 of Irwin et al.²⁸ In the case of 1-D uniaxial strain and unidirectional pore fluid flow (i.e., our underlying assumptions for the FE model presented in Section 2), the above terms simplify to the following:

$$\mathcal{H}_1^{\text{INT}} = \int_0^{X=H} r \left(\frac{J n^{\text{f}}}{K_{\text{f}}^{\eta}} \dot{p}_{\text{f}} + \dot{j} \right) A dX \quad (18)$$

$$\mathcal{H}_2^{\text{INT}} = \int_0^{X=H} r \frac{1}{K_{\text{f}}^{\eta}} \frac{\partial p_{\text{f}}}{\partial X} n^{\text{f}} \tilde{v}_{\text{f}} A dX \quad (19)$$

$$\mathcal{H}_3^{\text{INT}} = \int_0^{X=H} \frac{\partial r}{\partial X} \hat{k} \frac{\partial p_{\text{f}}}{\partial X} F_{11}^{-1} A dX \quad (20)$$

$$\mathcal{H}_4^{\text{INT}} = \int_0^{X=H} \frac{\partial r}{\partial X} \hat{k} \rho^{\text{fR}} (a_{\text{f}} + g) A dX \quad (21)$$

$$\mathcal{H}^{\text{EXT}} = \int_{\Gamma_0^{Q_{\text{f}}}} r Q_{\text{f}} dA = Q_{\text{f}}|_{X=H} A \quad (22)$$

In the $(\mathbf{u}-p_{\text{f}})$ formulation, where $a_{\text{f}} \approx a_{\text{s}} = a$, Eq. 21 becomes

$$\mathcal{H}_4^{\text{INT}} = \int_0^{X=H} \frac{\partial r}{\partial X} \hat{k} \rho^{\text{fR}} (a + g) A dX \quad (23)$$

For poroelasticity, inertia terms are ignored and Eq. 21 becomes

$$\mathcal{H}_4^{\text{INT}} = \int_0^{X=H} \frac{\partial r}{\partial X} \hat{k} \rho^{\text{fR}} g A dX \quad (24)$$

and again for poroelasticity, Eq. 19 does not include the inertia term when the Darcy velocity is expanded via Eq. 60 of Irwin et al.²⁸

3.2 Balance of Linear Momentum of the Mixture

The strong form of the balance of linear momentum using the $(\mathbf{u}-\mathbf{u}_f-p_f)$ formulation for the biphasic mixture following the motion of the solid skeleton as presented in Section 2 of Irwin et al.²⁸ in the mixed-temperature regime is given as

$$\text{DIV } \mathbf{P} + \rho_0 \mathbf{g} - (\rho_0^s \mathbf{a} + \rho_0^f \mathbf{a}_f) = \mathbf{0} \quad (25)$$

$$\mathbf{u}(\mathbf{X}, t) = \mathbf{g}^u(\mathbf{X}, t) \quad \forall \mathbf{X} \in \Gamma_0^u \quad (26)$$

$$\mathbf{P}(\mathbf{X}, t) \cdot \mathbf{N}(\mathbf{X}) = \mathbf{t}^\sigma(\mathbf{X}, t) \quad \forall \mathbf{X} \in \Gamma_0^t \quad (27)$$

Note that in the $(\mathbf{u}-p_f)$ formulation, we make the assumption that $\mathbf{a}_f \approx \mathbf{a}_s = \mathbf{a}$. Thus, Eq. 25 simplifies to

$$\text{DIV } \mathbf{P} + \rho_0 \mathbf{g} - \rho_0 \mathbf{a} = \mathbf{0} \quad (28)$$

Note that for poroelasticity and for elasticity, inertial terms are ignored, and so Eq. 25 simplifies to

$$\text{DIV } \mathbf{P} + \rho_0 \mathbf{g} = \mathbf{0} \quad (29)$$

Returning our focus back to the $(\mathbf{u}-\mathbf{u}_f-p_f)$ formulation, let $\mathcal{G}(u_i, u_{i(f)}, p_f, w_i)$ be the variational form of Eq. 25 and let w_i be a set of vector-valued weighting functions associated with solid skeleton displacement u_i . We may then rewrite Eq. 25 as

$$\begin{aligned} \mathcal{G}(u_i, u_{i(f)}, p_f, w_i) &= \int_{\mathcal{B}_0} w_i \frac{\partial P_{iI}}{\partial X_I} dV + \int_{\mathcal{B}_0} w_i \rho_0 g_i dV \\ &\quad - \int_{\mathcal{B}_0} w_i (\rho_0^s a_i + \rho_0^f a_{i(f)}) dV = 0 \end{aligned} \quad (30)$$

The first term in Eq. 30 can be rewritten as follows:

$$\int_{\mathcal{B}_0} w_i \frac{\partial P_{iI}}{\partial X_I} dV = \int_{\mathcal{B}_0} \frac{\partial (w_i P_{iI})}{\partial X_I} dV - \int_{\mathcal{B}_0} \frac{\partial w_i}{\partial X_I} P_{iI} dV \quad (31)$$

Substitution of Eq. 31 into Eq. 30 gives us

$$\begin{aligned} \mathcal{G}(u_i, u_{i(f)}, p_f, w_i) &= \int_{\mathcal{B}_0} \frac{\partial (w_i P_{iI})}{\partial X_I} dV - \int_{\mathcal{B}_0} \frac{\partial w_i}{\partial X_I} P_{iI} dV \\ &\quad + \int_{\mathcal{B}_0} w_i \rho_0 g_i dV - \int_{\mathcal{B}_0} w_i (\rho_0^s a_i + \rho_0^f a_{i(f)}) dV = 0 \end{aligned} \quad (32)$$

Applying divergence theorem to the first term in Eq. 32, using the boundary conditions in Eq. 26 (where $w_i \rightarrow 0$ on Γ_0^u) and Eq. 27, and applying Eqs. 57 and 58 of Irwin et al.²⁸ with the constitutive form for the first Piola-Kirchhoff effective stress given by Eq. 156 of Irwin et al.,²⁸ the variational form for the balance of linear momentum for the biphasic mixture becomes

$$\begin{aligned} \mathcal{G}(u_i, u_{i(f)}, p_f, w_i) &= \int_{\mathcal{B}_0} w_i (\rho_0^s a_i + \rho_0^f a_{i(f)}) dV + \int_{\mathcal{B}_0} \frac{\partial w_i}{\partial X_I} P_{iI(E)}^s dV \\ &\quad - \int_{\mathcal{B}_0} \frac{\partial w_i}{\partial X_I} J F_{Ii}^{-1} p_f dV - \int_{\mathcal{B}_0} w_i \rho_0 g_i dV \\ &\quad - \left(\int_{\Gamma_0^t} w_i t_i^{\sigma_E^s} dA - \int_{\Gamma_0^t} w_i p_f J F_{Ii}^{-1} N_I dA \right) = 0 \end{aligned} \quad (33)$$

For notational simplicity, we denote the separate terms as

$$\mathcal{G} = \mathcal{G}_1^{\text{INT}} + \mathcal{G}_2^{\text{INT}} + \mathcal{G}_3^{\text{INT}} + \mathcal{G}_4^{\text{INT}} - (\mathcal{G}_1^{\text{EXT}} + \mathcal{G}_2^{\text{EXT}}) = 0 \quad (34)$$

where

$$\mathcal{G}_1^{\text{INT}} = \int_{\mathcal{B}_0} w_i (\rho_0^s a_i + \rho_0^f a_{i(f)}) dV \quad (35)$$

$$\mathcal{G}_2^{\text{INT}} = \int_{\mathcal{B}_0} \frac{\partial w_i}{\partial X_I} P_{iI}^s dV \quad (36)$$

$$\mathcal{G}_3^{\text{INT}} = - \int_{\mathcal{B}_0} \frac{\partial w_i}{\partial X_I} J F_{Ii}^{-1} p_f dV \quad (37)$$

$$\mathcal{G}_4^{\text{INT}} = - \int_{\mathcal{B}_0} w_i \rho_0 g_i dV \quad (38)$$

$$\mathcal{G}_1^{\text{EXT}} = \int_{\Gamma_0^t} w_i t_i^{\sigma^s} dA \quad (39)$$

$$\mathcal{G}_2^{\text{EXT}} = - \int_{\Gamma_0^t} w_i p_f J F_{Ii}^{-1} N_I dA \quad (40)$$

In the $(\mathbf{u}-p_f)$ formulation, where $\mathbf{a}_f \approx \mathbf{a}_s = \mathbf{a}$, the variational form of Eq. 28 is $\mathcal{G}(u_i, p_f, w_i)$, and Eq. 35 becomes

$$\mathcal{G}_1^{\text{INT}} = \int_{\mathcal{B}_0} w_i \rho_0 a_i dV \quad (41)$$

For poroelasticity or elasticity, inertial terms are ignored and thus

$$\mathcal{G}_1^{\text{INT}} = 0 \quad (42)$$

For elastodynamics or elasticity, the variational form of Eq. 28 is $\mathcal{G}(u_i, w_i)$, and for elastodynamics Eq. 35 becomes identical to that of Eq. 41. In the case of 1-D uniaxial strain and unidirectional pore fluid flow (i.e., our underlying assumptions for the FE model), the above terms simplify to the following using the equations presented in Section 2 as well as assuming that the gravitational vector g_i is pointing

in the downward X direction:

$$\mathcal{G}_1^{\text{INT}} = \int_0^{X=H} w (\rho_0^s a_s + \rho_0^f a_f) A dX \quad (43)$$

$$\mathcal{G}_2^{\text{INT}} = \int_0^{X=H} \frac{\partial w}{\partial X} P_{11(E)}^s A dX \quad (44)$$

$$\mathcal{G}_3^{\text{INT}} = - \int_0^{X=H} \frac{\partial w}{\partial X} p_f A dX \quad (45)$$

$$\mathcal{G}_4^{\text{INT}} = \int_0^{X=H} w \rho_0 g A dX \quad (46)$$

$$\mathcal{G}_1^{\text{EXT}} = \int_{\Gamma_0^t} w t^{\sigma^s} dA = t^{\sigma^s} A \quad (47)$$

$$\mathcal{G}_2^{\text{EXT}} = - \int_{\Gamma_0^t} w p_f N dA = -p_f A \quad (48)$$

If we combine Eqs. 47 and 48 we see that

$$\mathcal{G}_1^{\text{EXT}} + \mathcal{G}_2^{\text{EXT}} = A (t^{\sigma^s} - p_f) = A t^\sigma \quad (49)$$

In the $(\mathbf{u}-p_f)$ formulation, where $a_f \approx a_s = a$, Eq. 43 becomes

$$\mathcal{G}_1^{\text{INT}} = \int_0^{X=H} w \rho_0 a A dX \quad (50)$$

For elastodynamics, Eq. 43 becomes identical to Eq. 50.

3.3 Balance of Linear Momentum of the Pore Fluid

The strong formulation of the balance of linear momentum for a nearly inviscid pore fluid as presented in Section 2 of Irwin et al.²⁸ is given as

$$\rho_0^f \mathbf{a}_f + J n^f \text{GRAD}(p_f) \cdot \mathbf{F}^{-1} + J \frac{(n^f)^2}{\hat{k}} (\mathbf{v}_f - \mathbf{v}) - \rho_0^f \mathbf{g} = \mathbf{0} \quad (51)$$

$$\mathbf{u}_f(\mathbf{X}, t) = \mathbf{g}_{u_f}(\mathbf{X}, t) \quad \forall \mathbf{X} \in \Gamma_0^{u_f} \quad (52)$$

Let $\mathcal{I}(u_i, u_{i(\text{f})}, p_f, q_i)$ be the variational form of Eq. 51, and let q_i be a set of vector-valued weighting functions associated with pore fluid displacement $u_{i(\text{f})}$. We may then rewrite Eq. 51 as follows:

$$\begin{aligned} \mathcal{I}(u_i, u_{i(\text{f})}, p_f, q_i) = & \int_{\mathcal{B}_0} q_i \rho_0^{\text{f}} a_{i(\text{f})} dV + \int_{\mathcal{B}_0} q_i J n^{\text{f}} \frac{\partial p_f}{\partial X_I} F_{Ii}^{-1} dV \\ & + \int_{\mathcal{B}_0} q_i J \frac{(n^{\text{f}})^2}{\hat{k}} (v_{i(\text{f})} - v_i) dV - \int_{\mathcal{B}_0} q_i \rho_0^{\text{f}} g_i dV = 0 \end{aligned} \quad (53)$$

For notational simplicity, we denote the separate terms in Eq. 53 as

$$\mathcal{I} = \mathcal{I}_1^{\text{INT}} + \mathcal{I}_2^{\text{INT}} + \mathcal{I}_3^{\text{INT}} + \mathcal{I}_4^{\text{INT}} = 0 \quad (54)$$

where

$$\mathcal{I}_1^{\text{INT}} = \int_{\mathcal{B}_0} q_i \rho_0^{\text{f}} a_{i(\text{f})} dV \quad (55)$$

$$\mathcal{I}_2^{\text{INT}} = \int_{\mathcal{B}_0} q_i J n^{\text{f}} \frac{\partial p_f}{\partial X_I} F_{Ii}^{-1} dV \quad (56)$$

$$\mathcal{I}_3^{\text{INT}} = \int_{\mathcal{B}_0} q_i J \frac{(n^{\text{f}})^2}{\hat{k}} (v_{i(\text{f})} - v_i) dV \quad (57)$$

$$\mathcal{I}_4^{\text{INT}} = - \int_{\mathcal{B}_0} q_i \rho_0^{\text{f}} g_i dV \quad (58)$$

In the case of 1-D uniaxial strain and unidirectional pore fluid flow, the above terms simplify to the following using the equations presented in Section 2, as well as

assuming that the gravitational vector g_i is pointing in the downward X direction:

$$\mathcal{I}_1^{\text{INT}} = \int_0^{X=H} q\rho_0^f a_f A dX \quad (59)$$

$$\mathcal{I}_2^{\text{INT}} = \int_0^{X=H} qn^f \frac{\partial p_f}{\partial X} A dX \quad (60)$$

$$\mathcal{I}_3^{\text{INT}} = \int_0^{X=H} qJ \frac{(n^f)^2}{\hat{k}} (v_f - v) A dX \quad (61)$$

$$\mathcal{I}_4^{\text{INT}} = \int_0^{X=H} q\rho_0^f g A dX \quad (62)$$

4. FE Implementation

As stated in previous sections, for uniaxial strain and unidirectional flow with axial symmetry, our geometry of a column of porous lung parenchyma is reduced to 1-D. Therefore, we employ a 1-D FE mesh to solve the variational equations of Section 3. For interpolating our solution variables, we use standard Lagrange polynomials up to quadratic order. Displacements of both solid and fluid can either be interpolated using quadratic Lagrange polynomials or linear Lagrange polynomials. Often we choose linear Lagrange polynomials for improved performance and stability, at the cost of reduced accuracy compared to quadratic Lagrange polynomials. For pore fluid pressure, we stick to linear Lagrange polynomials.

Consider then a generic solution variable $y(X(\xi))$ interpolated across the local element coordinate system ξ . For quadratic interpolation, we have

$$y^h(\xi, t) = \sum_{a=1}^3 N_a^y(\xi) y_a^e(t) = \underbrace{\{ \mathbf{N}^{e,y} \}}_{1 \times 3} \cdot \underbrace{\{ \mathbf{y}^e \}}_{3 \times 1} \quad (63)$$

where the shape function matrix is

$$\mathbf{N}^{e,y} := \left\{ \frac{1}{2}\xi(\xi - 1) \quad , \quad \frac{1}{2}\xi(\xi + 1) \quad , \quad 1 - \xi^2 \right\} \quad (64)$$

Here “ e ” refers to local finite element, “ a ” to node number, h^e is the element length,

and \mathbf{y}^e are the nodal values of y . For linear interpolation, we have

$$y^h(\xi, t) = \sum_{a=1}^2 N_a^y(\xi) y_a^e(t) = \underbrace{\left\{ \mathbf{N}^{e,y} \right\}}_{1 \times 2} \cdot \underbrace{\left\{ \mathbf{y}^e \right\}}_{2 \times 1} \quad (65)$$

where

$$\mathbf{N}^{e,y} := \left\{ \frac{1-\xi}{2}, \quad \frac{1+\xi}{2} \right\} \quad (66)$$

Next, consider the gradient of the generic solution variable $y(X)$, such that by the chain rule $\partial y(X(\xi))/\partial X = (\partial y(\xi)/\partial \xi)(\partial \xi/\partial X)$, interpolated across the local coordinate system ξ . For quadratic interpolation, we have

$$\frac{\partial y^h(\xi, t)}{\partial X} = \sum_{a=1}^3 B_a^y(\xi) y_a^e(t) = \underbrace{\left\{ \mathbf{B}^{e,y} \right\}}_{1 \times 3} \cdot \underbrace{\left\{ \mathbf{y}^e \right\}}_{3 \times 1} \quad (67)$$

where

$$\mathbf{B}^{e,y} := \frac{2}{h_0^e} \left\{ \xi - \frac{1}{2}, \quad \xi + \frac{1}{2}, \quad -2\xi \right\} \quad (68)$$

where the jacobian of coordinate transformation $j^e = (\partial X^{h^e}/\partial \xi)^{-1} = 2/h_0^e$ for the initial finite element length h_0^e . For linear interpolation, we have,

$$\frac{\partial y^h(\xi, t)}{\partial X} = \sum_{a=1}^2 B_a^y(\xi) y_a^e(t) = \underbrace{\left\{ \mathbf{B}^{e,y} \right\}}_{1 \times 2} \cdot \underbrace{\left\{ \mathbf{y}^e \right\}}_{2 \times 1} \quad (69)$$

where

$$\mathbf{B}^{e,y} := \frac{2}{h_0^e} \left\{ -\frac{1}{2}, \quad \frac{1}{2} \right\} \quad (70)$$

Hereafter, the lengths of the vectors \mathbf{N} and \mathbf{B} will be determined by the number of degrees of freedom (DOF), or $n_{\text{dof}}^{y,e}$, of the solution variable of interest. Pressure we assume to be evaluated at the endpoints of the element: $n_{\text{dof}}^{p,e} = 2$. However, we allow the local element degrees of freedom of the solid skeleton and pore fluid to vary. We may have $n_{\text{dof}}^{s,e} = 3$ and $n_{\text{dof}}^{f,e} = 2$ (i.e., three degrees of freedom, which will

be interpolated using quadratic shape functions) for the solid skeleton displacement and two degrees of freedom, which will be interpolated using linear shape functions, for the pore fluid displacement. A list of element types with their corresponding degrees of freedom can be found in Table 1.

Table 1. Acronyms for different element types.

Element type	$n_{\text{dof}}^{s,e}$	$n_{\text{dof}}^{f,e}$	$n_{\text{dof}}^{p,e}$
Q1-P1	2	N/A	2
Q2-P1	3	N/A	2
Q2-Q2-P1	3	3	2
Q2-Q1-P1	3	2	2
Q1-Q1-P1	2	2	2

For our solution variables of interest (i.e., u , u_f and p_f) we introduce the corresponding discretizations \mathbf{d} , \mathbf{d}_f and $\boldsymbol{\theta}$, respectively. For solid skeleton displacement \mathbf{d} , we have

$$u^{h^e}(\xi, t) = \sum_{a=1}^{n_{\text{dof}}^{s,e}} N_a^u(\xi) d_a^e(t) = \underbrace{\left\{ \mathbf{N}^{e,u} \right\}}_{1 \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\left\{ \mathbf{d}^e \right\}}_{n_{\text{dof}}^{s,e} \times 1} \quad (71)$$

The solid skeleton velocity and acceleration are defined similarly:

$$v^{h^e}(\xi, t) = \sum_{a=1}^{n_{\text{dof}}^{s,e}} N_a^u(\xi) \dot{d}_a^e(t) = \underbrace{\left\{ \mathbf{N}^{e,u} \right\}}_{1 \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\left\{ \dot{\mathbf{d}}^e \right\}}_{n_{\text{dof}}^{s,e} \times 1} \quad (72)$$

$$a^{h^e}(\xi, t) = \sum_{a=1}^{n_{\text{dof}}^{s,e}} N_a^u(\xi) \ddot{d}_a^e(t) = \underbrace{\left\{ \mathbf{N}^{e,u} \right\}}_{1 \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\left\{ \ddot{\mathbf{d}}^e \right\}}_{n_{\text{dof}}^{s,e} \times 1} \quad (73)$$

Solid skeleton displacement gradient, velocity gradient, and acceleration gradient

are defined as follows, respectively:

$$\frac{\partial u^{h^e}(\xi, t)}{\partial X} = \sum_{a=1}^{n_{\text{dof}}^{s,e}} B_a^u(\xi) d_a^e(t) = \underbrace{\left\{ \mathbf{B}^{e,u} \right\}}_{1 \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\left\{ \mathbf{d}^e \right\}}_{n_{\text{dof}}^{s,e} \times 1} \quad (74)$$

$$\frac{\partial v^{h^e}(\xi, t)}{\partial X} = \sum_{a=1}^{n_{\text{dof}}^{s,e}} B_a^u(\xi) \dot{d}_a^e(t) = \underbrace{\left\{ \mathbf{B}^{e,u} \right\}}_{1 \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\left\{ \dot{\mathbf{d}}^e \right\}}_{n_{\text{dof}}^{s,e} \times 1} \quad (75)$$

$$\frac{\partial a^{h^e}(\xi, t)}{\partial X} = \sum_{a=1}^{n_{\text{dof}}^{s,e}} B_a^u(\xi) \ddot{d}_a^e(t) = \underbrace{\left\{ \mathbf{B}^{e,u} \right\}}_{1 \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\left\{ \ddot{\mathbf{d}}^e \right\}}_{n_{\text{dof}}^{s,e} \times 1} \quad (76)$$

The same procedure also applies to pore fluid displacement \mathbf{d}_f :

$$u_f^{h^e}(\xi, t) = \sum_{a=1}^{n_{\text{dof}}^{f,e}} N_a^{u_f}(\xi) d_{a,f}^e(t) = \underbrace{\left\{ \mathbf{N}^{e,u_f} \right\}}_{1 \times n_{\text{dof}}^{f,e}} \cdot \underbrace{\left\{ \mathbf{d}_f^e \right\}}_{n_{\text{dof}}^{f,e} \times 1} \quad (77)$$

$$v_f^{h^e}(\xi, t) = \sum_{a=1}^{n_{\text{dof}}^{f,e}} N_a^{u_f}(\xi) \dot{d}_{a,f}^e(t) = \underbrace{\left\{ \mathbf{N}^{e,u_f} \right\}}_{1 \times n_{\text{dof}}^{f,e}} \cdot \underbrace{\left\{ \dot{\mathbf{d}}_f^e \right\}}_{n_{\text{dof}}^{f,e} \times 1} \quad (78)$$

$$a_f^{h^e}(\xi, t) = \sum_{a=1}^{n_{\text{dof}}^{f,e}} N_a^{u_f}(\xi) \ddot{d}_{a,f}^e(t) = \underbrace{\left\{ \mathbf{N}^{e,u_f} \right\}}_{1 \times n_{\text{dof}}^{f,e}} \cdot \underbrace{\left\{ \ddot{\mathbf{d}}_f^e \right\}}_{n_{\text{dof}}^{f,e} \times 1} \quad (79)$$

$$\frac{\partial u_f^{h^e}(\xi, t)}{\partial X} = \sum_{a=1}^{n_{\text{dof}}^{f,e}} B_a^{u_f}(\xi) d_{a,f}^e(t) = \underbrace{\left\{ \mathbf{B}^{e,u_f} \right\}}_{1 \times n_{\text{dof}}^{f,e}} \cdot \underbrace{\left\{ \mathbf{d}_f^e \right\}}_{n_{\text{dof}}^{f,e} \times 1} \quad (80)$$

$$\frac{\partial v_f^{h^e}(\xi, t)}{\partial X} = \sum_{a=1}^{n_{\text{dof}}^{f,e}} B_a^{u_f}(\xi) \dot{d}_{a,f}^e(t) = \underbrace{\left\{ \mathbf{B}^{e,u_f} \right\}}_{1 \times n_{\text{dof}}^{f,e}} \cdot \underbrace{\left\{ \dot{\mathbf{d}}_f^e \right\}}_{n_{\text{dof}}^{f,e} \times 1} \quad (81)$$

$$\frac{\partial a_f^{h^e}(\xi, t)}{\partial X} = \sum_{a=1}^{n_{\text{dof}}^{f,e}} B_a^{u_f}(\xi) \ddot{d}_{a,f}^e(t) = \underbrace{\left\{ \mathbf{B}^{e,u_f} \right\}}_{1 \times n_{\text{dof}}^{f,e}} \cdot \underbrace{\left\{ \ddot{\mathbf{d}}_f^e \right\}}_{n_{\text{dof}}^{f,e} \times 1} \quad (82)$$

Similarly for the pore fluid pressure,

$$p_f^{h^e}(\xi, t) = \sum_{a=1}^{n_{\text{dof}}^{p,e}} N_b^p(\xi) \theta_a^e(t) = \underbrace{\left\{ \mathbf{N}^{e,p} \right\}}_{1 \times n_{\text{dof}}^{p,e}} \cdot \underbrace{\left\{ \boldsymbol{\theta}^e \right\}}_{n_{\text{dof}}^{p,e} \times 1} \quad (83)$$

$$\dot{p}_f^{h^e}(\xi, t) = \sum_{a=1}^{n_{\text{dof}}^{p,e}} N_a^p(\xi) \dot{\theta}_a^e(t) = \underbrace{\left\{ \mathbf{N}^{e,p} \right\}}_{1 \times n_{\text{dof}}^{p,e}} \cdot \underbrace{\left\{ \dot{\boldsymbol{\theta}}^e \right\}}_{n_{\text{dof}}^{p,e} \times 1} \quad (84)$$

$$\ddot{p}_f^{h^e}(\xi, t) = \sum_{a=1}^{n_{\text{dof}}^{p,e}} N_a^p(\xi) \ddot{\theta}_a^e(t) = \underbrace{\left\{ \mathbf{N}^{e,p} \right\}}_{1 \times n_{\text{dof}}^{p,e}} \cdot \underbrace{\left\{ \ddot{\boldsymbol{\theta}}^e \right\}}_{n_{\text{dof}}^{p,e} \times 1} \quad (85)$$

$$\frac{\partial p_f^{h^e}(\xi, t)}{\partial X} = \sum_{a=1}^{n_{\text{dof}}^{p,e}} B_a^p(\xi) \theta_a^e(t) = \underbrace{\left\{ \mathbf{B}^{e,p} \right\}}_{1 \times n_{\text{dof}}^{p,e}} \cdot \underbrace{\left\{ \boldsymbol{\theta}^e \right\}}_{n_{\text{dof}}^{p,e} \times 1} \quad (86)$$

$$\frac{\partial \dot{p}_f^{h^e}(\xi, t)}{\partial X} = \sum_{a=1}^{n_{\text{dof}}^{p,e}} B_a^p(\xi) \dot{\theta}_a^e(t) = \underbrace{\left\{ \mathbf{B}^{e,p} \right\}}_{1 \times n_{\text{dof}}^{p,e}} \cdot \underbrace{\left\{ \dot{\boldsymbol{\theta}}^e \right\}}_{n_{\text{dof}}^{p,e} \times 1} \quad (87)$$

$$\frac{\partial \ddot{p}_f^{h^e}(\xi, t)}{\partial X} = \sum_{a=1}^{n_{\text{dof}}^{p,e}} B_a^p(\xi) \ddot{\theta}_a^e(t) = \underbrace{\left\{ \mathbf{B}^{e,p} \right\}}_{1 \times n_{\text{dof}}^{p,e}} \cdot \underbrace{\left\{ \ddot{\boldsymbol{\theta}}^e \right\}}_{n_{\text{dof}}^{p,e} \times 1} \quad (88)$$

The weighting functions for solid skeleton displacement \mathbf{d} and their interpolations including gradients are given as follows:

$$w^{h^e}(\xi) = \underbrace{\left\{ \mathbf{N}^{e,u} \right\}}_{1 \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\left\{ \mathbf{c}^e \right\}}_{n_{\text{dof}}^{s,e} \times 1} = \underbrace{\left\{ \mathbf{c}^e \right\}^T}_{1 \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\left\{ \mathbf{N}^{e,u} \right\}^T}_{n_{\text{dof}}^{s,e} \times 1} \quad (89)$$

$$\frac{\partial w^{h^e}(\xi, t)}{\partial X} = \underbrace{\left\{ \mathbf{B}^{e,u} \right\}}_{1 \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\left\{ \mathbf{c}^e \right\}}_{n_{\text{dof}}^{s,e} \times 1} = \underbrace{\left\{ \mathbf{c}^e \right\}^T}_{1 \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\left\{ \mathbf{B}^{e,u} \right\}^T}_{n_{\text{dof}}^{s,e} \times 1} \quad (90)$$

The weighting functions for the pore fluid displacement \mathbf{d}_f and their interpolations

including gradients are given as follows:

$$q^{h^e}(\xi) = \underbrace{\{N^{e,u_f}\}}_{1 \times n_{\text{dof}}^{f,e}} \cdot \underbrace{\{\omega^e\}}_{n_{\text{dof}}^{f,e} \times 1} = \underbrace{\{\omega^e\}^T}_{1 \times n_{\text{dof}}^{f,e}} \cdot \underbrace{\{N^{e,u_f}\}^T}_{n_{\text{dof}}^{f,e} \times 1} \quad (91)$$

$$\frac{\partial q^{h^e}(\xi, t)}{\partial X} = \underbrace{\{B^{e,u_f}\}}_{1 \times n_{\text{dof}}^{f,e}} \cdot \underbrace{\{\omega^e\}}_{n_{\text{dof}}^{f,e} \times 1} = \underbrace{\{\omega^e\}^T}_{1 \times n_{\text{dof}}^{f,e}} \cdot \underbrace{\{B^{e,u_f}\}^T}_{n_{\text{dof}}^{f,e} \times 1} \quad (92)$$

The weighting functions for pore fluid pressure θ and their interpolations including gradients are given as follows:

$$r^{h^e}(\xi) = \underbrace{\{N^{e,p}\}}_{1 \times n_{\text{dof}}^{p,e}} \cdot \underbrace{\{\alpha^e\}}_{n_{\text{dof}}^{p,e} \times 1} = \underbrace{\{\alpha^e\}^T}_{1 \times n_{\text{dof}}^{p,e}} \cdot \underbrace{\{N^{e,p}\}^T}_{n_{\text{dof}}^{p,e} \times 1} \quad (93)$$

$$\frac{\partial r^{h^e}(\xi, t)}{\partial X} = \underbrace{\{B^{e,p}\}}_{1 \times n_{\text{dof}}^{p,e}} \cdot \underbrace{\{\alpha^e\}}_{n_{\text{dof}}^{p,e} \times 1} = \underbrace{\{\alpha^e\}^T}_{1 \times n_{\text{dof}}^{p,e}} \cdot \underbrace{\{B^{e,p}\}^T}_{n_{\text{dof}}^{p,e} \times 1} \quad (94)$$

5. Time Integration

After applying the Galerkin approximation³⁵ to the coupled variational equations in Section 3, substituting interpolations and their derivatives in Section 4, applying boundary conditions at the nodes, and assembling the FE equations, the general coupled system of matrix-vector equations resembles the following nonlinear form:

$$M\ddot{\mathbf{x}} + C\dot{\mathbf{x}} + \mathbf{F}^{INT}(\ddot{\mathbf{x}}, \mathbf{x}) = \mathbf{F}^{EXT} \quad (95)$$

where M is the mass matrix, C is the viscous matrix, \mathbf{F}^{INT} is the nonlinear internal “force” vector, and \mathbf{F}^{EXT} is the vector of applied external forces and fluxes for the full coupled variational form. Similarly, the accelerations, velocities, and displacements are given by the global DOF vectors $\ddot{\mathbf{x}}$, $\dot{\mathbf{x}}$, and \mathbf{x} , respectively, as

$$\mathbf{x} = \begin{Bmatrix} \mathbf{d} \\ \mathbf{d}_f \\ \boldsymbol{\theta} \end{Bmatrix}, \quad \dot{\mathbf{x}} = \begin{Bmatrix} \dot{\mathbf{d}} \\ \dot{\mathbf{d}}_f \\ \dot{\boldsymbol{\theta}} \end{Bmatrix}, \quad \ddot{\mathbf{x}} = \begin{Bmatrix} \ddot{\mathbf{d}} \\ \ddot{\mathbf{d}}_f \\ \ddot{\boldsymbol{\theta}} \end{Bmatrix} \quad (96)$$

where \mathbf{d} , \mathbf{d}_f , and $\boldsymbol{\theta}$ are the global nodal DOFs of solid skeleton displacement, pore fluid displacement (part of the time integration procedure, not used in any of the

balance equations), and pore fluid pressure, respectively.

5.1 Newmark-beta Integrators

For the dynamic equations that retain inertia terms, we apply the NB method^{35,36} and a nonlinear solver (Newton-Raphson method, refer to Fig. 3 and comments below) for solving Eq. 95 wherein

$$M\ddot{\mathbf{x}}_{n+1} + C\dot{\mathbf{x}}_{n+1} + \mathbf{F}^{\text{INT}}(\ddot{\mathbf{x}}_{n+1}, \mathbf{x}_{n+1}) = \mathbf{F}_{n+1}^{\text{EXT}} \quad (97)$$

$$\mathbf{x}_{n+1} = \mathbf{x}_n + \Delta t \dot{\mathbf{x}}_n + \frac{(\Delta t)^2}{2} [(1 - 2\beta) \ddot{\mathbf{x}}_n + 2\beta \ddot{\mathbf{x}}_{n+1}] \quad (98)$$

$$\dot{\mathbf{x}}_{n+1} = \dot{\mathbf{x}}_n + \Delta t [(1 - \gamma) \ddot{\mathbf{x}}_n + \gamma \ddot{\mathbf{x}}_{n+1}] \quad (99)$$

Here $(\bullet)_{n+1}$ implies a quantity to be solved or updated at current time t_{n+1} , $(\bullet)_n$ implies a quantity known at previous time t_n , β and γ are integration parameters, and the time increment $\Delta t = t_{n+1} - t_n$ may be variable or held constant during the time marching solution.

The predictors are written as

$$\tilde{\mathbf{x}}_{n+1} = \mathbf{x}_n + \Delta t \dot{\mathbf{x}}_n + \frac{(\Delta t)^2}{2} (1 - 2\beta) \ddot{\mathbf{x}}_n \quad (100)$$

$$\dot{\tilde{\mathbf{x}}}_{n+1} = \dot{\mathbf{x}}_n + \Delta t (1 - \gamma) \ddot{\mathbf{x}}_n \quad (101)$$

such that the correctors are

$$\mathbf{x}_{n+1} = \tilde{\mathbf{x}}_{n+1} + \beta(\Delta t)^2 \ddot{\mathbf{x}}_{n+1} \quad (102)$$

$$\dot{\mathbf{x}}_{n+1} = \dot{\tilde{\mathbf{x}}}_{n+1} + \gamma \Delta t \ddot{\mathbf{x}}_{n+1} \quad (103)$$

Writing in residual form for solution via the Newton-Raphson method, this allows us to solve for the accelerations $\ddot{\mathbf{x}}_{n+1}$ at the next time step t_{n+1} as follows:

$$\mathbf{R}(\ddot{\mathbf{x}}_{n+1}) = M\ddot{\mathbf{x}}_{n+1} + C\dot{\mathbf{x}}_{n+1} + \mathbf{F}^{\text{INT}}(\ddot{\mathbf{x}}_{n+1}, \mathbf{x}_{n+1}) - \mathbf{F}_{n+1}^{\text{EXT}} = \mathbf{0} \quad (104)$$

In Eq. 104, the internal forces are nonlinear, therefore requiring us to employ a nonlinear solver.

For simplicity, we use the Newton-Raphson method to find the ‘‘exact’’ solution $\ddot{\mathbf{x}}_{n+1}^*$ such that $\mathbf{R}(\ddot{\mathbf{x}}_{n+1}^*) = 0$ given the applied external force $\mathbf{F}_{n+1}^{\text{EXT}}$. Consider the

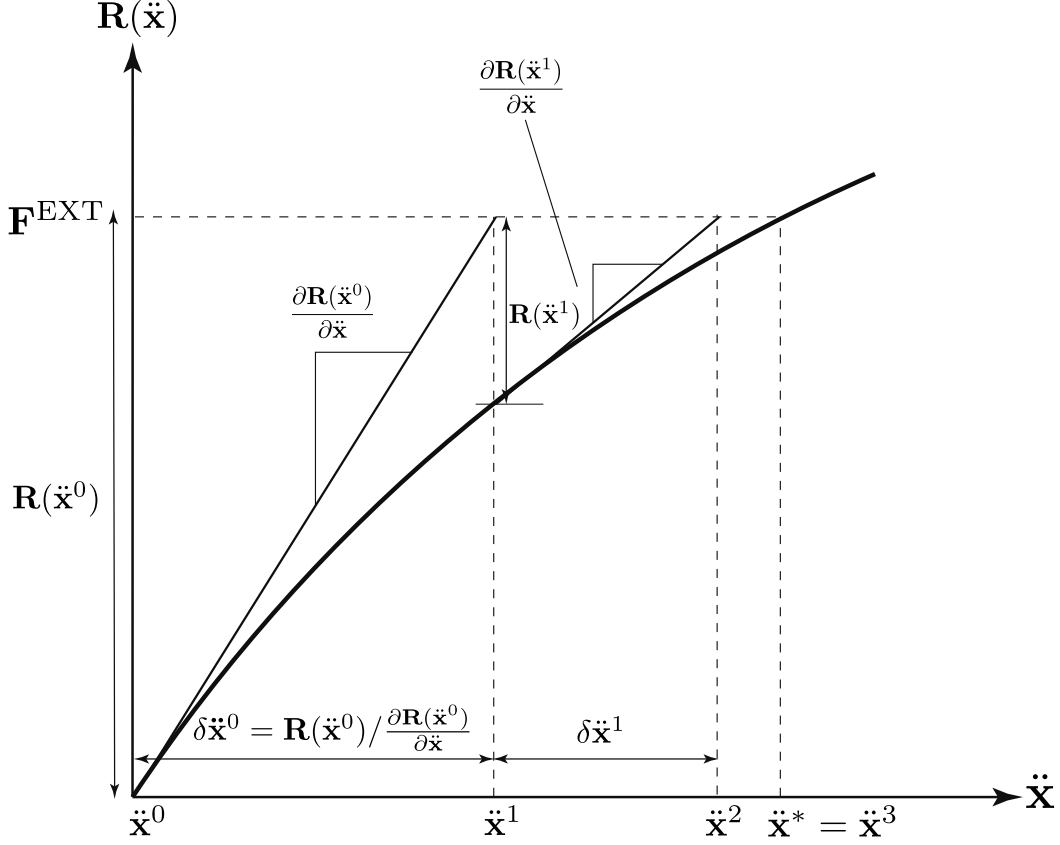


Fig. 3 Illustration of Newton-Raphson method for solution of \ddot{x} , where we show iteration $k = 0, 1, 2$, and convergence likely at iteration $k = 3$. The black curve is a plot of $R(\ddot{x})$.

Taylor-series expansion about the past iterate \ddot{x}^k :

$$\ddot{x}^* = \ddot{x}^k + \delta \ddot{x}^k \quad (105)$$

$$R(\ddot{x}_{n+1}^*) = R(\ddot{x}_{n+1}^k) + \frac{\partial R(\ddot{x}_{n+1}^k)}{\partial \ddot{x}} \delta \ddot{x}_{n+1}^k + \frac{1}{2} \frac{\partial^2 R(\ddot{x}_{n+1}^k)}{\partial \ddot{x}^2} (\delta \ddot{x}_{n+1}^k)^2 + \text{h.o.t.} = 0 \quad (106)$$

Then we “linearize”: let $\partial R(\ddot{x}_{n+1}^k) / \partial \ddot{x}$ be the consistent tangent at iteration k , drop the quadratic and higher order terms (h.o.t.s), and thus $\ddot{x}_{n+1}^* \approx \ddot{x}_{n+1}^{k+1}$:

$$R(\ddot{x}_{n+1}^k) + \frac{\partial R(\ddot{x}_{n+1}^k)}{\partial \ddot{x}} \delta \ddot{x}_{n+1}^k \approx 0; \quad \ddot{x}_{n+1}^* \approx \ddot{x}_{n+1}^{k+1} = \ddot{x}_{n+1}^k + \delta \ddot{x}_{n+1}^k \quad (107)$$

Then, to find the solution to Eq. 104, rearrange Eq. 107₁ as

$$\delta \ddot{\mathbf{x}}^k = -\mathbf{R}(\ddot{\mathbf{x}}_{n+1}^k) \left(\frac{\partial \mathbf{R}(\ddot{\mathbf{x}}_{n+1}^k)}{\partial \ddot{\mathbf{x}}} \right)^{-1} \quad (108)$$

and use Eq. 107₂ to solve for the updated displacement, until convergence is achieved:

$$\begin{aligned} \text{IF } \frac{|\mathbf{R}(\ddot{\mathbf{x}}_{n+1}^{k+1})|}{|\mathbf{R}(\ddot{\mathbf{x}}_{n+1}^0)|} < \text{tol THEN } \ddot{\mathbf{x}}_{n+1}^* &= \ddot{\mathbf{x}}_{n+1}^{k+1} \\ \text{ELSE iterate} \end{aligned}$$

where `tol` is a residual tolerance relative to the initial residual $\mathbf{R}(\ddot{\mathbf{x}}_{n+1}^0)$. The following section is concerned with how to find $\delta \ddot{\mathbf{x}}_{n+1}^k$.

The increments of our solution variables within a Newton-Raphson iteration loop are given by

$$\begin{Bmatrix} \delta \mathbf{d} \\ \delta \mathbf{d}_f \\ \delta \boldsymbol{\theta} \end{Bmatrix} = \beta \Delta t^2 \begin{Bmatrix} \delta \ddot{\mathbf{d}} \\ \delta \ddot{\mathbf{d}}_f \\ \delta \ddot{\boldsymbol{\theta}} \end{Bmatrix}, \quad \begin{Bmatrix} \delta \dot{\mathbf{d}} \\ \delta \dot{\mathbf{d}}_f \\ \delta \dot{\boldsymbol{\theta}} \end{Bmatrix} = \gamma \Delta t \begin{Bmatrix} \delta \ddot{\mathbf{d}} \\ \delta \ddot{\mathbf{d}}_f \\ \delta \ddot{\boldsymbol{\theta}} \end{Bmatrix} \quad (109)$$

which then allows us to formulate the linearized forms of the matrix-vector equations, or to directly cast them within the variational form, not presented here (note that in Eq. 109₁, the squaring operation is being applied to the time step Δt ; this notation will be used for convenience in the proceeding equations). For example, for current iteration $k + 1$, the update for solid skeleton acceleration is $\ddot{\mathbf{d}}_{n+1}^{k+1} = \ddot{\mathbf{d}}_{n+1}^k + \delta \ddot{\mathbf{d}}$. With the linearized equations, we can construct the coupled linear equations to solve at each current iteration $k + 1$ as

$$\underbrace{\begin{bmatrix} \mathbf{K}_{u,u} & \mathbf{K}_{u,u_f} & \mathbf{K}_{u,p_f} \\ \mathbf{K}_{u_f,u} & \mathbf{K}_{u_f,u_f} & \mathbf{K}_{u_f,p_f} \\ \mathbf{K}_{p_f,u} & \mathbf{K}_{p_f,u_f} & \mathbf{K}_{p_f,p_f} \end{bmatrix}^k}_{(n_{\text{dof}}^s + n_{\text{dof}}^f + n_{\text{dof}}^p) \times (n_{\text{dof}}^s + n_{\text{dof}}^f + n_{\text{dof}}^p)} \cdot \underbrace{\begin{Bmatrix} \delta \ddot{\mathbf{d}} \\ \delta \ddot{\mathbf{d}}_f \\ \delta \ddot{\boldsymbol{\theta}} \end{Bmatrix}}_{(n_{\text{dof}}^s + n_{\text{dof}}^f + n_{\text{dof}}^p) \times 1} = \underbrace{\begin{Bmatrix} -\mathbf{R}_u \\ -\mathbf{R}_{u_f} \\ -\mathbf{R}_{p_f} \end{Bmatrix}^k}_{(n_{\text{dof}}^s + n_{\text{dof}}^f + n_{\text{dof}}^p) \times 1} \quad (110)$$

Prior to showing the linearized equations, it behooves us to show the linearized variables, the derivations for which are supplied in Appendix A of a complementary report by Irwin et al.²⁹ Starting with the Gateaux derivative of the deformation

gradient, we have

$$\delta(F_{11}) = \beta \Delta t^2 \frac{\partial(\delta a)}{\partial X} = \gamma \Delta t \frac{\partial(\delta v)}{\partial X} \quad (111)$$

For the deformation gradient raised to any power n , Eq. 111 becomes

$$\delta(F_{11}^n) = n F_{11}^{n-1} \beta \Delta t^2 \frac{\partial(\delta a)}{\partial X} = n F_{11}^{n-1} \gamma \Delta t \frac{\partial(\delta v)}{\partial X} \quad (112)$$

In the 1-D approximation, it was shown in Section 2 that the Jacobian J is equivalent to F_{11} and thus,

$$\delta(J) = \delta(F_{11}) = \beta \Delta t^2 \frac{\partial(\delta a)}{\partial X} = \gamma \Delta t \frac{\partial(\delta v)}{\partial X} \quad (113)$$

$$\delta(J^n) = \delta(F_{11}^n) = n J^{n-1} \beta \Delta t^2 \frac{\partial(\delta a)}{\partial X} = n J^{n-1} \gamma \Delta t \frac{\partial(\delta v)}{\partial X} \quad (114)$$

Given the constitutive relation for the second Piola-Kirchhoff stress shown in Section 3 of Irwin et al.,²⁸ we will also require the Gateaux derivative on the $\ln(J)$ terms, namely,

$$\delta(\ln(J)) = \frac{1}{J} \beta \Delta t^2 \frac{\partial(\delta a)}{\partial X} = \frac{1}{J} \gamma \Delta t \frac{\partial(\delta v)}{\partial X} \quad (115)$$

as well as the Gateaux derivative on the inverse Cauchy green tensor component C_{11}^{-1} ,

$$\delta(C_{11}^{-1}) = -2 F_{11}^{-3} \beta \Delta t^2 \frac{\partial(\delta a)}{\partial X} = -2 F_{11}^{-3} \gamma \Delta t \frac{\partial(\delta v)}{\partial X} \quad (116)$$

Now that we have both Eqs. 115 and 116, we can apply the Gateaux derivative to the solid skeleton second Piola-Kirchhoff extra stress component $\delta(S_{11(E)}^s)$ as follows:

$$\begin{aligned} \delta(S_{11(E)}^s) &= (\lambda - 2 [\lambda \ln(J) - \mu]) F_{11}^{-3} \beta \Delta t^2 \frac{\partial(\delta a)}{\partial X} \\ &= (\lambda - 2 [\lambda \ln(J) - \mu]) F_{11}^{-3} \gamma \Delta t \frac{\partial(\delta v)}{\partial X} \end{aligned} \quad (117)$$

Then the Gateaux derivative applied to the solid skeleton first Piola-Kirchhoff extra

stress component $\delta(P_{11(E)}^s)$ is

$$\begin{aligned}\delta(P_{11(E)}^s) &= (\mu + [\lambda - \lambda \ln(J) + \mu] F_{11}^{-2}) \beta \Delta t^2 \frac{\partial(\delta a)}{\partial X} \\ &= (\mu + [\lambda - \lambda \ln(J) + \mu] F_{11}^{-2}) \gamma \Delta t \frac{\partial(\delta v)}{\partial X}\end{aligned}\quad (118)$$

The real mass density of the pore fluid ρ^{fR} is dependent on the pore fluid pressure as defined by Irwin et al.²⁸; therefore,

$$\delta(\rho^{\text{fR}}) = \frac{\rho^{\text{fR}}}{K_f^\eta} \beta \Delta t^2 \delta \ddot{p}_f = \frac{\rho^{\text{fR}}}{K_f^\eta} \gamma \Delta t \delta \dot{p}_f \quad (119)$$

The volume fraction of pore fluid is dependent on the volume fraction of the solid skeleton which is dependent on the deformation of the solid skeleton; therefore,

$$\delta(n^f) = \frac{n^s}{J} \beta \Delta t^2 \frac{\partial(\delta a)}{\partial X} = \frac{n^s}{J} \gamma \Delta t \frac{\partial(\delta v)}{\partial X} \quad (120)$$

The mass density of the pore fluid is likewise dependent on both the volume fraction of the pore fluid as well as the real mass density of the pore fluid; therefore,

$$\begin{aligned}\delta(\rho_0^f) &= \beta \Delta t^2 \left(\rho^{\text{fR}} \frac{\partial(\delta a)}{\partial X} + \frac{J n^f \rho^{\text{fR}}}{K_f^\eta} \delta \ddot{p}_f \right) \\ &= \gamma \Delta t \left(\rho^{\text{fR}} \frac{\partial(\delta v)}{\partial X} + \frac{J n^f \rho^{\text{fR}}}{K_f^\eta} \delta \dot{p}_f \right)\end{aligned}\quad (121)$$

To express the hydraulic conductivity as a function of the porosity we employ the Kozeny-Carman relation as shown in Eq. 150 of Irwin et al.²⁸ such that

$$\begin{aligned}\delta(\hat{k}) &= \frac{\hat{k} n^s}{J} \left[\frac{3}{n^f} + \frac{2n^f}{1 - (n^f)^2} \right] \beta \Delta t^2 \frac{\partial(\delta a)}{\partial X} \\ &= \frac{\hat{k} n^s}{J} \left[\frac{3}{n^f} + \frac{2n^f}{1 - (n^f)^2} \right] \gamma \Delta t \frac{\partial(\delta v)}{\partial X}\end{aligned}\quad (122)$$

Some derivations and equations in the remainder of Section 5 consider aspects of shock viscosity and pressure stabilization, topics that are more formally discussed in this report in respective Sections 6.1 and 6.2. The reader is referred to these sections for background context and further details. Linearization of the referential shock viscosity Q of Eq. 449, $\delta(Q)$, is derived in Eqs. B-7 and B-9 of Appendix B

in Irwin et al.²⁹

5.1.1 Linearization of Variational Equations for the (u) Theory

For elastodynamics, we solve one variational equation, namely, the balance of momentum of the solid, given by Eq. 28. For elasticity, we solve Eq. 29. The linearizations of these equations are given as follows, with additional details being covered in Appendix B of Irwin et al.²⁹ The linearization of Eq. 50 is given by

$$\delta\mathcal{G}_1^{\text{INT}} = \int_0^{X=H} w\rho_0\delta a A dX \quad (123)$$

For static elasticity, Eq. 123 is zero. The linearization of Eq. 44 is given by

$$\delta\mathcal{G}_2^{\text{INT}} = \int_0^{X=H} \frac{\partial w}{\partial X} (\mu + [\lambda - \lambda \ln(J) + \mu] F_{11}^{-2}) (\beta \Delta t^2) \frac{\partial(\delta a)}{\partial X} A dX \quad (124)$$

For linearizations of $\mathcal{G}_2^{\text{INT}}$ with elasticity and viscous damping, refer to Eqs. B-3–B-5 of Irwin et al.²⁹ For linearization of $\mathcal{G}_2^{\text{INT}}$ when shock viscosity is enabled, refer to Eq. B-10 of Irwin et al.²⁹

The linearization of Eq. 45 is zero because pore fluid pressure is not considered for elastodynamics nor for elasticity (i.e., TPM is not applied; a bulk continuum is assumed). The linearization of Eq. 46 is zero under the assumption that the variation of the reference mass density of the material is constant. In addition, the variation of the external force vector given by Eq. 49 is of course unnecessary as this term is simply subtracted from the residual at every Newton-Raphson iteration.

5.1.2 FE Formulation of the (u) Theory

The FE formulation for the elastodynamics variational equation is presented as follows, with additional details covered in Appendix B of Irwin et al.²⁹ The FE formulation for the elastodynamics variational equation is written in block-matrix form as

$$\underbrace{\left[\mathbf{K}_{u,u} \right]}_{n_{\text{dof}}^s \times n_{\text{dof}}^s} \cdot \underbrace{\left\{ \delta \ddot{\mathbf{d}} \right\}}_{n_{\text{dof}}^s \times 1} = \underbrace{\left\{ -\mathbf{R}_u \right\}}_{n_{\text{dof}}^s \times 1} \quad (125)$$

The FE formulation for the elasticity variational equation is written in block-matrix form as

$$\underbrace{\left[\mathbf{K}_{u,u} \right]}_{n_{\text{dof}}^s \times n_{\text{dof}}^s} \cdot \underbrace{\left\{ \delta \dot{\mathbf{d}} \right\}}_{n_{\text{dof}}^s \times 1} = \underbrace{\left\{ -\mathbf{R}_u \right\}}_{n_{\text{dof}}^s \times 1} \quad (126)$$

The global residual for the solid displacement is given as

$$\mathbf{c}^T \cdot \mathbf{R}_u = \mathcal{G}^h = \mathcal{G}_1^{\text{INT},h} + \mathcal{G}_2^{\text{INT},h} + \mathcal{G}_4^{\text{INT},h} - \mathcal{G}^{\text{EXT},h} = 0 \quad (127)$$

where

$$\mathcal{G}_1^{\text{INT},h} = \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{s,e}} \left\{ \mathbf{c}^e \right\}^T \cdot \left(\int_{-1}^1 \underbrace{\left\{ \mathbf{N}^{e,u} \right\}^T}_{n_{\text{dof}}^{s,e} \times 1} \rho_0^{h^e} a^{h^e} A j^e d\xi \right) \quad (128)$$

$$\mathcal{G}_2^{\text{INT},h} = \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{s,e}} \left\{ \mathbf{c}^e \right\}^T \cdot \left(\int_{-1}^1 \underbrace{\left\{ \mathbf{B}^{e,u} \right\}^T}_{n_{\text{dof}}^{s,e} \times 1} P_{11}^{h^e} A j^e d\xi \right) \quad (129)$$

$$\mathcal{G}_4^{\text{INT},h} = \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{s,e}} \left\{ \mathbf{c}^e \right\}^T \cdot \left(\int_{-1}^1 \underbrace{\left\{ \mathbf{N}^{e,u} \right\}^T}_{n_{\text{dof}}^{s,e} \times 1} \rho_0^{h^e} g A j^e d\xi \right) \quad (130)$$

$$\mathcal{G}^{\text{EXT},h} = \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{s,e}} \left\{ \mathbf{c}^e \right\}^T \cdot \begin{cases} \underbrace{\left\{ \mathbf{N}^{e,u}(X=H) \right\}^T}_{n_{\text{dof}}^{s,e} \times 1} t^\sigma A & X = H \\ \mathbf{0} & 0 \leq X < H \end{cases} \quad (131)$$

For quasi-static elasticity, Eq. 128 is zero.

Recall from Eqs. 125 and 126 that the tangent matrix for each iteration must be of the form

$$\mathbf{0} = -\mathbf{R}_u^k = \underbrace{\left[\mathbf{K}_{u,u} \right]}_{n_{\text{dof}}^s \times n_{\text{dof}}^s} \cdot \underbrace{\left\{ \delta \ddot{\mathbf{d}} \right\}}_{n_{\text{dof}}^s \times 1} \quad (132)$$

or

$$\mathbf{0} = \mathbf{R}_u^k = \underbrace{\left[\mathbf{K}_{u,u} \right]}_{n_{\text{dof}}^s \times n_{\text{dof}}^s} \cdot \underbrace{\left\{ \delta \mathbf{d} \right\}}_{n_{\text{dof}}^s \times 1} \quad (133)$$

where

$$\underbrace{\left[\mathbf{K}_{u,u} \right]}_{n_{\text{dof}}^s \times n_{\text{dof}}^s} = \underbrace{\mathbf{A}}_{1 \times n_{\text{dof}}^{s,e}} \underbrace{\left\{ \mathbf{c}^e \right\}^T}_{1 \times n_{\text{dof}}^{s,e}} \cdot \sum_{i=1}^2 \underbrace{\left[\mathbf{k}_{u,u}^{G_i^{\text{INT}},e} \right]}_{n_{\text{dof}}^{s,e} \times n_{\text{dof}}^{s,e}} \quad (134)$$

and where

$$\underbrace{\left[\mathbf{k}_{u,u}^{G_1^{\text{INT}},e} \right]}_{n_{\text{dof}}^{s,e} \times n_{\text{dof}}^{s,e}} = \int_{-1}^1 \rho_0^{h^e} \underbrace{\left\{ \mathbf{N}^{e,u} \right\}^T}_{n_{\text{dof}}^{s,e} \times 1} \underbrace{\left\{ \mathbf{N}^{e,u} \right\}}_{1 \times n_{\text{dof}}^{s,e}} A_j^e d\xi \quad (135)$$

$$\underbrace{\left[\mathbf{k}_{u,u}^{G_2^{\text{INT}},e} \right]}_{n_{\text{dof}}^{s,e} \times n_{\text{dof}}^{s,e}} = \int_{-1}^1 \left(\mu + [\lambda - \lambda \ln(J^{h^e}) + \mu] (F_{11}^{h^e})^{-2} (\beta \Delta t^2) \right) \times \underbrace{\left\{ \mathbf{B}^{e,u} \right\}^T}_{n_{\text{dof}}^{s,e} \times 1} \underbrace{\left\{ \mathbf{B}^{e,u} \right\}}_{1 \times n_{\text{dof}}^{s,e}} A_j^e d\xi \quad (136)$$

For quasi-static elasticity, Eq. 135 is zero. For the formulations of $\mathbf{k}_{u,u}^{G_2^{\text{INT}},e}$ with elasticity and viscous damping refer to Appendix C of Irwin et al.²⁹ Eqs. C-10, C-13, and C-16. For the formulation of $\mathbf{k}_{u,u}^{G_2^{\text{INT}},e}$ when shock viscosity is enabled, refer to Eq. C-19.

5.1.3 Linearization of Variational Equations for the $(u-p_f)$ Theory

For poroelastodynamics $(u-p_f)$ formulation, we solve two variational equations, namely, the balance of momentum of the mixture and the balance of mass of the mixture given by Eqs. 28 and 3, respectively. The linearizations of these equations are given as follows, with additional details being covered in Appendix B of Irwin et al.²⁹; references of the form Eq. B-number refer to equations in that document.

The linearizations of Eqs. 50 and 44 remain unchanged from the elastodynamics/elasticity formulations, the former under the assumption that the Gateaux derivative of the mass density of the biphasic mixture is negligible. Refer to Eq. 123 and

Eq. 124 for $\delta\mathcal{G}_1^{\text{INT}}$ and $\delta\mathcal{G}_2^{\text{INT}}$, respectively. In the context of a multiphase formulation, the single-phase denotation of the first entry of the first Piola-Kirchhoff stress P_{11} in Eq. 124 becomes $P_{11(E)}^s$; refer to Section 2 of Irwin et al.²⁸ for details.

The linearization of Eq. 45 is given by

$$\delta\mathcal{G}_3^{\text{INT}} = - \int_0^{X=H} \frac{\partial w}{\partial X} (\beta\Delta t^2) \delta\ddot{p}_f A dX \quad (137)$$

For poroelasticity, refer to Eq. B-12. The linearization of Eq. 46 is zero under the assumption the Gateaux derivative of the mass density of the biphasic mixture is very small. In addition, the external force vector given by Eq. 49 is unnecessary as this term is simply subtracted from the residual at every Newton-Raphson iteration.

The linearization of Eq. 18 is given by

$$\delta\mathcal{H}_1^{\text{INT}} = \int_0^{X=H} r \left(\left[\frac{\dot{p}_f}{K_f^\eta} (\beta\Delta t^2) + (\gamma\Delta t) \right] \frac{\partial(\delta a)}{\partial X} + \frac{Jn^f}{K_f^\eta} (\gamma\Delta t) \delta\ddot{p}_f \right) A dX \quad (138)$$

For poroelasticity, refer to Eq. B-14. The linearization of Eq. 19 is given by

$$\begin{aligned} \delta\mathcal{H}_2^{\text{INT}} = & \int_0^{X=H} r \left(\left[(n^f \tilde{v}_f) - \hat{k} \frac{\partial p_f}{\partial X} (F_{11})^{-1} \right] (\beta\Delta t^2) \frac{\partial(\delta\ddot{p}_f)}{\partial X} \right. \\ & \left. - \hat{k} \frac{\partial p_f}{\partial X} (a + g) \frac{\rho^{\text{fR}}}{K_f^\eta} (\beta\Delta t^2) \delta\ddot{p}_f \right. \\ & \left. + \frac{\partial p_f}{\partial X} \left[\frac{n^s}{J} \left(\frac{3}{n^f} - \frac{2n^f}{1 - (n^f)^2} \right) (n^f \tilde{v}_f) + \frac{\partial p_f}{\partial X} \hat{k} F_{11}^{-2} \right] (\beta\Delta t^2) \frac{\partial(\delta a)}{\partial X} \right. \\ & \left. - \frac{\partial p_f}{\partial X} \hat{k} \rho^{\text{fR}} \delta a \right) \frac{A}{K_f^\eta} dX \quad (139) \end{aligned}$$

For poroelasticity, refer to Eq. B-18. The linearization of Eq. 20 is given by

$$\delta\mathcal{H}_3^{\text{INT}} = \int_0^{X=H} \frac{\partial r}{\partial X} \left(\left[n^s \left(\frac{3}{n^f} - \frac{2n^f}{1 - (n^f)^2} \right) - 1 \right] \hat{k} \frac{\partial p_f}{\partial X} F_{11}^{-2} \times \right. \\ \left. (\beta\Delta t^2) \frac{\partial(\delta a)}{\partial X} + \hat{k} F_{11}^{-1} (\beta\Delta t^2) \frac{\partial(\delta \ddot{p}_f)}{\partial X} \right) A dX \quad (140)$$

For poroelasticity, refer to Eq. B-21. The linearization of Eq. 23 is given by

$$\delta\mathcal{H}_4^{\text{INT}} = \int_0^{X=H} \frac{\partial r}{\partial X} \hat{k} \rho^{\text{fR}} \left((a + g) \frac{n^s}{J} \left[\frac{3}{n^f} - \frac{2n^f}{1 - (n^f)^2} \right] (\beta\Delta t^2) \frac{\partial(\delta a)}{\partial X} \right. \\ \left. + (a + g) \frac{1}{K_f^{\eta}} (\beta\Delta t^2) \delta \ddot{p}_f + \delta a \right) A dX \quad (141)$$

For poroelasticity, refer to Eq. B-23. The linearization of the external force vector given by Eq. 22 is unnecessary as this term is simply subtracted from the residual at every Newton-Raphson iteration. For pressure stabilization, the linearization of Eq. 453 is given by

$$\delta\mathcal{H}^{\text{stab}} = \int_0^{X=H} \frac{\partial r}{\partial X} \alpha^{\text{stab}} F_{11}^{-1} \left((\gamma\Delta t) \frac{\partial(\delta \ddot{p}_f)}{\partial X} - \frac{\partial \dot{p}_f}{\partial X} F_{11}^{-1} (\beta\Delta t^2) \frac{\partial(\delta a)}{\partial X} \right) A dX \quad (142)$$

For poroelasticity, which often does not require the employment of pressure stabilization, refer to Eq. B-25.

5.1.4 FE formulation of the $(\mathbf{u}-p_f)$ Theory

The FE formulation for the $(\mathbf{u}-p_f)$ poroelastodynamics variational equations is written in block-matrix form as

$$\underbrace{\begin{bmatrix} \mathbf{K}_{u,u} & \mathbf{K}_{p_f,u} \\ \mathbf{K}_{u,p_f} & \mathbf{K}_{p_f,p_f} \end{bmatrix}}_{(n_{\text{dof}}^s + n_{\text{dof}}^p) \times (n_{\text{dof}}^s + n_{\text{dof}}^p)} \cdot \underbrace{\begin{Bmatrix} \delta \mathbf{d} \\ \delta \boldsymbol{\theta} \end{Bmatrix}}_{(n_{\text{dof}}^s + n_{\text{dof}}^p) \times 1} = \underbrace{\begin{Bmatrix} -\mathbf{R}_u \\ -\mathbf{R}_{p_f} \end{Bmatrix}}_{(n_{\text{dof}}^s + n_{\text{dof}}^p) \times 1} \quad (143)$$

The FE formulation for the poroelasticity variational equations is written in block-matrix form as

$$\underbrace{\begin{bmatrix} \mathbf{K}_{u,u} & \mathbf{K}_{p_f,u} \\ \mathbf{K}_{u,p_f} & \mathbf{K}_{p_f,p_f} \end{bmatrix}}_{(n_{\text{dof}}^s+n_{\text{dof}}^p) \times (n_{\text{dof}}^s+n_{\text{dof}}^p)} \cdot \underbrace{\begin{Bmatrix} \delta \mathbf{d} \\ \delta \boldsymbol{\theta} \end{Bmatrix}}_{(n_{\text{dof}}^s+n_{\text{dof}}^p) \times 1} = \underbrace{\begin{Bmatrix} -\mathbf{R}_u \\ -\mathbf{R}_{p_f} \end{Bmatrix}}_{(n_{\text{dof}}^s+n_{\text{dof}}^p) \times 1} \quad (144)$$

The global residual for the solid skeleton displacement is given as

$$\mathbf{c}^T \cdot \mathbf{R}_u = \mathcal{G}^h = \mathcal{G}_1^{\text{INT},h} + \mathcal{G}_2^{\text{INT},h} + \mathcal{G}_3^{\text{INT},h} + \mathcal{G}_4^{\text{INT},h} - \mathcal{G}^{\text{EXT},h} = 0 \quad (145)$$

where

$$\mathcal{G}_1^{\text{INT},h} = \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{s,e}} \underbrace{\{\mathbf{c}^e\}^T}_{n_{\text{dof}}^{s,e} \times 1} \cdot \left(\int_{-1}^1 \underbrace{\{\mathbf{N}^{e,u}\}^T}_{n_{\text{dof}}^{s,e} \times 1} \rho_0^{h^e} a^{h^e} A_j^e d\xi \right) \quad (146)$$

$$\mathcal{G}_2^{\text{INT},h} = \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{s,e}} \underbrace{\{\mathbf{c}^e\}^T}_{n_{\text{dof}}^{s,e} \times 1} \cdot \left(\int_{-1}^1 \underbrace{\{\mathbf{B}^{e,u}\}^T}_{n_{\text{dof}}^{s,e} \times 1} P_{11(E)}^{s,h^e} A_j^e d\xi \right) \quad (147)$$

$$\mathcal{G}_3^{\text{INT},h} = \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{s,e}} \underbrace{\{\mathbf{c}^e\}^T}_{n_{\text{dof}}^{s,e} \times 1} \cdot \left(- \int_{-1}^1 \underbrace{\{\mathbf{B}^{e,u}\}^T}_{n_{\text{dof}}^{s,e} \times 1} p_f^{h^e} A_j^e d\xi \right) \quad (148)$$

$$\mathcal{G}_4^{\text{INT},h} = \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{s,e}} \underbrace{\{\mathbf{c}^e\}^T}_{n_{\text{dof}}^{s,e} \times 1} \cdot \left(\int_{-1}^1 \underbrace{\{\mathbf{N}^{e,u}\}^T}_{n_{\text{dof}}^{s,e} \times 1} \rho_0^{h^e} g A_j^e d\xi \right) \quad (149)$$

$$\mathcal{G}^{\text{EXT},h} = \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{s,e}} \underbrace{\{\mathbf{c}^e\}^T}_{n_{\text{dof}}^{s,e} \times 1} \cdot \begin{cases} \underbrace{\{\mathbf{N}^{e,u}(X=H)\}^T}_{n_{\text{dof}}^{s,e} \times 1} t^\sigma A & X=H \\ \mathbf{0} & 0 \leq X < H \end{cases} \quad (150)$$

For poroelasticity, Eq. 146 is zero.

The global residual for the pore fluid pressure is given as

$$\boldsymbol{\alpha}^T \cdot \mathbf{R}_{p_f} = \mathcal{H}^h = \mathcal{H}_1^{\text{INT},h} + \mathcal{H}_2^{\text{INT},h} + \mathcal{H}_3^{\text{INT},h} + \mathcal{H}_4^{\text{INT},h} - \mathcal{H}_1^{\text{EXT},h} = 0 \quad (151)$$

where

$$\mathcal{H}_1^{\text{INT},h} = \mathbf{A}_e^{n_e} \underbrace{\{\alpha^e\}^T}_{1 \times n_{\text{dof}}^{p,e}} \cdot \left(\int_{-1}^1 \underbrace{\{\mathbf{N}^{e,p}\}^T}_{n_{\text{dof}}^{p,e} \times 1} \left[\frac{J^{h^e} n^{f,h^e}}{K_f^\eta} \dot{p}_f^{h^e} + j^{h^e} \right] A_j^e d\xi \right) \quad (152)$$

$$\mathcal{H}_2^{\text{INT},h} = \mathbf{A}_e^{n_e} \underbrace{\{\alpha^e\}^T}_{1 \times n_{\text{dof}}^{p,e}} \cdot \left(\int_{-1}^1 \underbrace{\{\mathbf{N}^{e,p}\}^T}_{n_{\text{dof}}^{p,e} \times 1} \frac{1}{K_f^\eta} \frac{\partial p_f^{h^e}}{\partial X} (n^{f,h^e} \tilde{v}_f^{h^e}) A_j^e d\xi \right) \quad (153)$$

$$\mathcal{H}_3^{\text{INT},h} = \mathbf{A}_e^{n_e} \underbrace{\{\alpha^e\}^T}_{1 \times n_{\text{dof}}^{p,e}} \cdot \left(\int_{-1}^1 \underbrace{\{\mathbf{B}^{e,p}\}^T}_{n_{\text{dof}}^{p,e} \times 1} \hat{k}^{h^e} \frac{\partial p_f^{h^e}}{\partial X} (F_{11}^{h^e})^{-1} A_j^e d\xi \right) \quad (154)$$

$$\mathcal{H}_4^{\text{INT},h} = \mathbf{A}_e^{n_e} \underbrace{\{\alpha^e\}^T}_{1 \times n_{\text{dof}}^{p,e}} \cdot \left(\int_{-1}^1 \underbrace{\{\mathbf{B}^{e,p}\}^T}_{n_{\text{dof}}^{p,e} \times 1} \hat{k}^{h^e} \rho^{\text{fR},h^e} (a^{h^e} + g) A_j^e d\xi \right) \quad (155)$$

$$\mathcal{H}^{\text{EXT},h} = \mathbf{A}_e^{n_e} \underbrace{\{\alpha^e\}^T}_{1 \times n_{\text{dof}}^{p,e}} \cdot \begin{cases} \underbrace{\{\mathbf{N}^{e,p}(X=H, X=0)\}^T}_{n_{\text{dof}}^{p,e} \times 1} Q_f A & X=0, X=H \\ \mathbf{0} & 0 < X < H \end{cases} \quad (156)$$

For poroelasticity, Eq. 155 becomes

$$\mathcal{H}_4^{\text{INT},h} = \mathbf{A}_e^{n_e} \underbrace{\{\alpha^e\}^T}_{1 \times n_{\text{dof}}^{p,e}} \cdot \left(\int_{-1}^1 \underbrace{\{\mathbf{B}^{e,p}\}^T}_{n_{\text{dof}}^{p,e} \times 1} \hat{k}^{h^e} \rho^{\text{fR},h^e} g A_j^e d\xi \right) \quad (157)$$

When pressure stabilization is enabled, an additional term $\mathcal{H}^{\text{stab}}$ is added to the left-hand side (LHS) of Eq. 151 and is defined as

$$\mathcal{H}^{\text{stab}} = \mathbf{A}_e^{n_e} \underbrace{\{\alpha^e\}^T}_{1 \times n_{\text{dof}}^{p,e}} \cdot \left(\int_{-1}^1 \underbrace{\{\mathbf{B}^{e,p}\}^T}_{n_{\text{dof}}^{p,e} \times 1} \alpha^{\text{stab}} \frac{\partial p_f^{h^e}}{\partial X} (F_{11}^{h^e})^{-1} A_j^e d\xi \right) \quad (158)$$

Recall from Eqs. 143 and 144 that the tangent matrix for poroelastodynamics for

each iteration must be of the form

$$\mathbf{0} = -\mathbf{R}^k = \underbrace{\begin{bmatrix} \mathbf{K}_{u,u} & \mathbf{K}_{u,p_f} \\ \mathbf{K}_{p_f,u} & \mathbf{K}_{p_f,p_f} \end{bmatrix}}_{(n_{\text{dof}}^s+n_{\text{dof}}^p) \times (n_{\text{dof}}^s+n_{\text{dof}}^p)} \cdot \underbrace{\begin{Bmatrix} \delta \dot{\mathbf{d}} \\ \delta \ddot{\boldsymbol{\theta}} \end{Bmatrix}}_{(n_{\text{dof}}^s+n_{\text{dof}}^p) \times 1} \quad (159)$$

or, for poroelasticity,

$$\mathbf{0} = -\mathbf{R}^k = \underbrace{\begin{bmatrix} \mathbf{K}_{u,u} & \mathbf{K}_{u,p_f} \\ \mathbf{K}_{p_f,u} & \mathbf{K}_{p_f,p_f} \end{bmatrix}}_{(n_{\text{dof}}^s+n_{\text{dof}}^p) \times (n_{\text{dof}}^s+n_{\text{dof}}^p)} \cdot \underbrace{\begin{Bmatrix} \delta \dot{\mathbf{d}} \\ \delta \ddot{\boldsymbol{\theta}} \end{Bmatrix}}_{(n_{\text{dof}}^s+n_{\text{dof}}^p) \times 1} \quad (160)$$

where

$$\underbrace{\begin{bmatrix} \mathbf{K}_{u,u} \end{bmatrix}}_{n_{\text{dof}}^s \times n_{\text{dof}}^s} = \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{s,e}} \underbrace{\{\mathbf{c}^e\}^T}_{1 \times n_{\text{dof}}^{s,e}} \cdot \sum_{i=1}^2 \underbrace{\begin{bmatrix} \mathbf{k}_{u,u}^{\mathcal{G}_i^{\text{INT},e}} \end{bmatrix}}_{n_{\text{dof}}^{s,e} \times n_{\text{dof}}^{s,e}} \quad (161)$$

$$\underbrace{\begin{bmatrix} \mathbf{K}_{u,p_f} \end{bmatrix}}_{n_{\text{dof}}^s \times n_{\text{dof}}^p} = \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{s,e}} \underbrace{\{\mathbf{c}^e\}^T}_{1 \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\begin{bmatrix} \mathbf{k}_{u,p_f}^{\mathcal{G}_3^{\text{INT},e}} \end{bmatrix}}_{n_{\text{dof}}^{s,e} \times n_{\text{dof}}^{p,e}} \quad (162)$$

$$\underbrace{\begin{bmatrix} \mathbf{K}_{p_f,u} \end{bmatrix}}_{n_{\text{dof}}^p \times n_{\text{dof}}^s} = \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{p,e}} \underbrace{\{\boldsymbol{\alpha}^e\}^T}_{1 \times n_{\text{dof}}^{p,e}} \cdot \sum_{i=1}^4 \underbrace{\begin{bmatrix} \mathbf{k}_{p_f,u}^{\mathcal{H}_i^{\text{INT},e}} \end{bmatrix}}_{n_{\text{dof}}^{p,e} \times n_{\text{dof}}^{s,e}} \quad (163)$$

$$\underbrace{\begin{bmatrix} \mathbf{K}_{p_f,p_f} \end{bmatrix}}_{n_{\text{dof}}^p \times n_{\text{dof}}^p} = \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{p,e}} \underbrace{\{\boldsymbol{\alpha}^e\}^T}_{1 \times n_{\text{dof}}^{p,e}} \cdot \sum_{i=1}^4 \underbrace{\begin{bmatrix} \mathbf{k}_{p_f,p_f}^{\mathcal{H}_i^{\text{INT},e}} \end{bmatrix}}_{n_{\text{dof}}^{p,e} \times n_{\text{dof}}^{p,e}} \quad (164)$$

The definitions for $\mathbf{K}_{u,u}$ remain unchanged from Section 5.1.2; refer to Eqs. 135 and 136. The skeleton displacement and pore fluid pressure coupling tangent are given as follows:

$$\underbrace{\begin{bmatrix} \mathbf{k}_{u,p_f}^{\mathcal{G}_3^{\text{INT},e}} \end{bmatrix}}_{n_{\text{dof}}^{s,e} \times n_{\text{dof}}^{p,e}} = - \int_{-1}^1 (\beta \Delta t^2) \underbrace{\{\mathbf{B}^{e,u}\}^T}_{n_{\text{dof}}^{s,e} \times 1} \underbrace{\{\mathbf{N}^{e,p}\}}_{1 \times n_{\text{dof}}^{p,e}} A_j^e d\xi \quad (165)$$

The pore fluid pressure and solid skeleton coupling tangents are given as follows:

$$\underbrace{\left[\mathbf{k}_{p_f, u}^{\mathcal{H}_1^{\text{INT}}, e} \right]}_{n_{\text{dof}}^{p, e} \times n_{\text{dof}}^{s, e}} = \int_{-1}^1 \left(\frac{\dot{p}_f^{h^e}}{K_f^\eta} (\beta \Delta t^2) + (\gamma \Delta t) \right) \underbrace{\left\{ \mathbf{N}^{e, p} \right\}^T}_{n_{\text{dof}}^{p, e} \times 1} \underbrace{\left\{ \mathbf{B}^{e, u} \right\}}_{1 \times n_{\text{dof}}^{s, e}} A_j^e d\xi \quad (166)$$

$$\begin{aligned} \underbrace{\left[\mathbf{k}_{p_f, u}^{\mathcal{H}_2^{\text{INT}}, e} \right]}_{n_{\text{dof}}^{p, e} \times n_{\text{dof}}^{s, e}} &= \int_{-1}^1 \underbrace{\left\{ \mathbf{N}^{e, p} \right\}^T}_{n_{\text{dof}}^{p, e} \times 1} \left(\left[\frac{n^{s, h^e}}{J^{h^e}} \left(\frac{3}{n^{f, h^e}} - \frac{2n^{f, h^e}}{1 - (n^{f, h^e})^2} \right) (n^f \tilde{v}_f)^{h^e} \right. \right. \\ &\quad \left. \left. + \hat{k}^{h^e} \frac{\partial p_f^{h^e}}{\partial X} (F_{11}^{h^e})^{-2} \right] (\beta \Delta t^2) \underbrace{\left\{ \mathbf{B}^{e, u} \right\}}_{1 \times n_{\text{dof}}^{s, e}} \right. \\ &\quad \left. - \hat{k}^{h^e} \rho^{\text{fR}, h^e} \underbrace{\left\{ \mathbf{N}^{e, u} \right\}}_{1 \times n_{\text{dof}}^{s, e}} \right) \frac{\partial p_f^{h^e}}{\partial X} \frac{1}{K_f^\eta} A_j^e d\xi \quad (167) \end{aligned}$$

$$\begin{aligned} \underbrace{\left[\mathbf{k}_{p_f, u}^{\mathcal{H}_3^{\text{INT}}, e} \right]}_{n_{\text{dof}}^{p, e} \times n_{\text{dof}}^{s, e}} &= \int_{-1}^1 \hat{k}^{h^e} \frac{\partial p_f^{h^e}}{\partial X} (F_{11}^{h^e})^{-2} \left(n^{s, h^e} \left[\frac{3}{n^{f, h^e}} - \frac{2n^{f, h^e}}{1 - (n^{f, h^e})^2} \right] - 1 \right) \times \\ &\quad (\beta \Delta t^2) \underbrace{\left\{ \mathbf{B}^{e, p} \right\}^T}_{n_{\text{dof}}^{p, e} \times 1} \underbrace{\left\{ \mathbf{B}^{e, u} \right\}}_{1 \times n_{\text{dof}}^{s, e}} A_j^e d\xi \quad (168) \end{aligned}$$

$$\begin{aligned} \underbrace{\left[\mathbf{k}_{p_f, u}^{\mathcal{H}_4^{\text{INT}}, e} \right]}_{n_{\text{dof}}^{p, e} \times n_{\text{dof}}^{s, e}} &= \int_{-1}^1 \underbrace{\left\{ \mathbf{B}^{e, p} \right\}^T}_{n_{\text{dof}}^{p, e} \times 1} \left(\left[\frac{3}{n^{f, h^e}} - \frac{2n^{f, h^e}}{1 - (n^{f, h^e})^2} \right] (a^{h^e} + g) \frac{n^{s, h^e}}{J^{h^e}} \times \right. \\ &\quad \left. (\beta \Delta t^2) \underbrace{\left\{ \mathbf{B}^{e, u} \right\}}_{1 \times n_{\text{dof}}^{s, e}} + \underbrace{\left\{ \mathbf{N}^{e, u} \right\}}_{1 \times n_{\text{dof}}^{s, e}} \right) \hat{k}^{h^e} \rho^{\text{fR}, h^e} A_j^e d\xi \quad (169) \end{aligned}$$

For poroelasticity, Eqs. 166–169 are given by Eqs. C-32, C-40, C-48, and C-56 in Appendix C of Irwin et al.²⁹

When pressure stabilization is enabled, we must add one more pore fluid pressure and solid skeleton coupling tangent to Eq. 163:

$$\underbrace{\left[\mathbf{k}_{p_f, u}^{\mathcal{H}^{\text{stab}}, e} \right]}_{n_{\text{dof}}^{p, e} \times n_{\text{dof}}^{s, e}} = - \int_{-1}^1 \alpha^{\text{stab}} (F_{11}^{h^e})^{-2} \frac{\partial \dot{p}_f^{h^e}}{\partial X} (\beta \Delta t^2) \underbrace{\left\{ \mathbf{B}^{e, p} \right\}^T}_{n_{\text{dof}}^{p, e} \times 1} \underbrace{\left\{ \mathbf{B}^{e, u} \right\}}_{1 \times n_{\text{dof}}^{s, e}} A_j^e d\xi \quad (170)$$

For poroelasticity, Eq. 170 is given by Eq. C-64 of Irwin et al.²⁹ Lastly, the pore fluid pressure tangents are given as follows:

$$\underbrace{\left[\mathbf{k}_{p_f, p_f}^{\mathcal{H}_1^{\text{INT}, e}} \right]}_{n_{\text{dof}}^{p, e} \times n_{\text{dof}}^{p, e}} = \int_{-1}^1 \frac{J^{h^e} n^{\text{f}, h^e}}{K_f^\eta} (\gamma \Delta t) \underbrace{\left\{ \mathbf{N}^{e, p} \right\}^T}_{n_{\text{dof}}^{p, e} \times 1} \underbrace{\left\{ \mathbf{N}^{e, p} \right\}}_{1 \times n_{\text{dof}}^{p, e}} A_j^e d\xi \quad (171)$$

$$\begin{aligned} \underbrace{\left[\mathbf{k}_{p_f, p_f}^{\mathcal{H}_2^{\text{INT}, e}} \right]}_{n_{\text{dof}}^{p, e} \times n_{\text{dof}}^{p, e}} &= \int_{-1}^1 \underbrace{\left\{ \mathbf{N}^{e, p} \right\}^T}_{n_{\text{dof}}^{p, e} \times 1} \left(\left[\left(n^{\text{f}} \tilde{v}_f \right)^{h^e} - \hat{k}^{h^e} \frac{\partial p_f^{h^e}}{\partial X} (F_{11}^h)^{-1} \right] \underbrace{\left\{ \mathbf{B}^{e, p} \right\}}_{1 \times n_{\text{dof}}^{p, e}} \right. \\ &\quad \left. - \hat{k}^{h^e} \frac{\partial p_f^{h^e}}{\partial X} (a^{h^e} + g) \frac{\rho^{\text{fR}, h^e}}{K_f^\eta} \underbrace{\left\{ \mathbf{N}^{e, p} \right\}}_{1 \times n_{\text{dof}}^{p, e}} \right) \frac{1}{K_f^\eta} (\beta \Delta t^2) A_j^e d\xi \end{aligned} \quad (172)$$

$$\underbrace{\left[\mathbf{k}_{p_f, p_f}^{\mathcal{H}_3^{\text{INT}, e}} \right]}_{n_{\text{dof}}^{p, e} \times n_{\text{dof}}^{p, e}} = \int_{-1}^1 \hat{k}^{h^e} (F_{11}^h)^{-1} (\beta \Delta t^2) \underbrace{\left\{ \mathbf{B}^{e, p} \right\}^T}_{n_{\text{dof}}^{p, e} \times 1} \underbrace{\left\{ \mathbf{B}^{e, p} \right\}}_{1 \times n_{\text{dof}}^{p, e}} A_j^e d\xi \quad (173)$$

$$\underbrace{\left[\mathbf{k}_{p_f, p_f}^{\mathcal{H}_4^{\text{INT}, e}} \right]}_{n_{\text{dof}}^{p, e} \times n_{\text{dof}}^{p, e}} = \int_{-1}^1 \hat{k}^{h^e} (a^{h^e} + g) \frac{\rho^{\text{fR}, h^e}}{K_f^\eta} (\beta \Delta t^2) \underbrace{\left\{ \mathbf{B}^{e, p} \right\}^T}_{n_{\text{dof}}^{p, e} \times 1} \underbrace{\left\{ \mathbf{N}^{e, p} \right\}}_{1 \times n_{\text{dof}}^{p, e}} A_j^e d\xi \quad (174)$$

For poroelasticity, Eqs. 171–174 are given by Eqs. C-33, C-41, C-49, and C-57 in Appendix C of Irwin et al.²⁹ When pressure stabilization is enabled, we must add one more pore fluid pressure tangent to Eq. 164:

$$\underbrace{\left[\mathbf{k}_{p_f, p_f}^{\mathcal{H}^{\text{stab}, e}} \right]}_{n_{\text{dof}}^{p, e} \times n_{\text{dof}}^{p, e}} = \int_{-1}^1 \alpha^{\text{stab}} (F_{11}^h)^{-1} (\gamma \Delta t) \underbrace{\left\{ \mathbf{B}^{e, p} \right\}^T}_{n_{\text{dof}}^{p, e} \times 1} \underbrace{\left\{ \mathbf{B}^{e, p} \right\}}_{1 \times n_{\text{dof}}^{p, e}} A_j^e d\xi \quad (175)$$

For poroelasticity, Eq. 175 is given by Eq. C-65 of Irwin et al.²⁹

5.1.5 Linearization of Variational Equations for the $(\mathbf{u}-\mathbf{u}_f-p_f)$ Theory

For poroelastodynamics $(\mathbf{u}-\mathbf{u}_f-p_f)$ formulation, we solve three variational equations, namely, the balance of momentum of the mixture, the balance of momentum of the pore fluid, and the balance of mass of the mixture, given by Eqs. 25, 51, and 3, respectively. The linearization of these equations are given as follows, with additional details covered in Appendix B of Irwin et al.²⁹

The linearization of Eq. 43 is given as follows under the assumption that the Gateaux

derivative of the real mass density of the solid phase is negligible owing to the incompressibility assumption in Section 2 of Irwin et al.²⁸:

$$\delta\mathcal{G}_1^{\text{INT}} = \int_0^{X=H} w \left(\rho_0^s \delta a + a_f (\beta \Delta t^2) \left[\rho^{\text{fR}} \frac{\partial(\delta a)}{\partial X} + \frac{J n^f \rho^{\text{fR}}}{K_f^\eta} \delta \ddot{p}_f \right] + \rho_0^f \delta a_f \right) A dX \quad (176)$$

The linearizations of Eqs. 44 and 45 remain unchanged from the ($\mathbf{u}-p_f$) formulation; refer to Eqs. 124 and 137 for $\delta\mathcal{G}_2^{\text{INT}}$ and $\delta\mathcal{G}_3^{\text{INT}}$, respectively. As before, $\delta\mathcal{G}_4^{\text{INT}} \approx 0$ under the assumption that the variation of mass density of the mixture is negligible.

The linearization of Eq. 59 is given as follows:

$$\delta\mathcal{I}_1^{\text{INT}} = \int_0^{X=H} q \left(a_f (\beta \Delta t^2) \left[\rho^{\text{fR}} \frac{\partial(\delta a)}{\partial X} + \frac{J n^f \rho^{\text{fR}}}{K_f^\eta} \delta \ddot{p}_f \right] + \rho_0^f \delta a_f \right) A dX \quad (177)$$

The linearization of Eq. 60 is given as follows:

$$\delta\mathcal{I}_2^{\text{INT}} = \int_0^{X=H} q \left(\frac{n^s}{J} \frac{\partial p_f}{\partial X} \frac{\partial(\delta a)}{\partial X} + n^f \frac{\partial(\delta \ddot{p}_f)}{\partial X} \right) (\beta \Delta t^2) A dX \quad (178)$$

The linearization of Eq. 61 is given as follows:

$$\delta\mathcal{I}_3^{\text{INT}} = \int_0^{X=H} q \left(\left(1 + \frac{2n^s}{n^f} - n^s \left[\frac{3}{n^f} - \frac{2n^f}{1 - (n^f)^2} \right] \right) \frac{(n^f)^2 \tilde{v}_f}{\hat{k}} \times \right. \\ \left. (\beta \Delta t^2) \frac{\partial \delta a}{\partial X} + \frac{J (n^f)^2}{\hat{k}} (\gamma \Delta t) (\delta a_f - \delta a) \right) A dX \quad (179)$$

The linearization of Eq. 62 is given as follows:

$$\delta\mathcal{I}_4^{\text{INT}} = \int_0^{X=H} q \left(\left[\rho^{\text{fR}} \frac{\partial(\delta a)}{\partial X} + \frac{J n^f \rho^{\text{fR}}}{K_f} \delta \ddot{p}_f \right] (\beta \Delta t^2) \right) g A dX \quad (180)$$

The linearizations of Eqs. 18, 20, and 453 remain unchanged from the ($\mathbf{u}-p_f$) for-

mulation; refer to Eqs. 138, 140, and 142, respectively.

The linearizations of Eqs. 19 and 21 are different from the $(\mathbf{u}-p_f)$ formulation. However, given that the pore fluid acceleration is used in place of the mixture's acceleration (i.e., the solid skeleton acceleration in the $(\mathbf{u}-\mathbf{u}_f-p_f)$ formulation) in the constitutive model for Darcy's law, the following changes to the linearizations of terms that use Darcy's law are given as follows:

$$\begin{aligned} \delta\mathcal{H}_2^{\text{INT}} = & \int_0^{X=H} r \left(\left[(n^f \tilde{v}_f) - \hat{k} \frac{\partial p_f}{\partial X} F_{11}^{-1} \right] (\beta \Delta t^2) \frac{\partial(\delta \ddot{p}_f)}{\partial X} \right. \\ & \left. - \hat{k} \frac{\partial p_f}{\partial X} (a_f + g) \frac{\rho^{\text{fR}}}{K_f} (\beta \Delta t^2) \delta \ddot{p}_f \right. \\ & + \frac{\partial p_f}{\partial X} \left[\frac{n^s}{J} \left(\frac{3}{n^f} - \frac{2n^f}{1 - (n^f)^2} \right) (n^f \tilde{v}_f) + \hat{k} \frac{\partial p_f}{\partial X} F_{11}^{-2} \right] (\beta \Delta t^2) \frac{\partial(\delta a)}{\partial X} \\ & \left. - \frac{\partial p_f}{\partial X} \hat{k} \rho^{\text{fR}} \delta a_f \right) \frac{A}{K_f^\eta} dX \end{aligned} \quad (181)$$

$$\begin{aligned} \delta\mathcal{H}_4^{\text{INT}} = & \int_0^{X=H} \frac{\partial r}{\partial X} \rho^{\text{fR}} \hat{k} \left((a_f + g) \frac{n^s}{J} \left[\frac{3}{n^f} - \frac{2n^f}{1 - (n^f)^2} \right] (\beta \Delta t^2) \frac{\partial(\delta a)}{\partial X} \right. \\ & \left. + (a_f + g) \frac{1}{K_f^\eta} (\beta \Delta t^2) \delta \ddot{p}_f + \delta a_f \right) A dX \end{aligned} \quad (182)$$

5.1.6 FE Formulation of the $(\mathbf{u}-\mathbf{u}_f-p_f)$ Theory

The FE formulation for the $(\mathbf{u}-\mathbf{u}_f-p_f)$ theory is written in block-matrix form as

$$\underbrace{\begin{bmatrix} \mathbf{K}_{u,u} & \mathbf{K}_{u_f,u} & \mathbf{K}_{p_f,u} \\ \mathbf{K}_{u_f,u} & \mathbf{K}_{u_f,u_f} & \mathbf{K}_{u_f,p_f} \\ \mathbf{K}_{p_f,u} & \mathbf{K}_{p_f,u_f} & \mathbf{K}_{p_f,p_f} \end{bmatrix}}_{(n_{\text{dof}}^s + n_{\text{dof}}^f + n_{\text{dof}}^p) \times (n_{\text{dof}}^s + n_{\text{dof}}^f + n_{\text{dof}}^p)} \cdot \underbrace{\begin{Bmatrix} \delta \ddot{\mathbf{d}} \\ \delta \ddot{\mathbf{d}}_f \\ \delta \ddot{\theta} \end{Bmatrix}}_{(n_{\text{dof}}^s + n_{\text{dof}}^f + n_{\text{dof}}^p) \times 1} = \underbrace{\begin{Bmatrix} -\mathbf{R}_u \\ -\mathbf{R}_{u_f} \\ -\mathbf{R}_{p_f} \end{Bmatrix}}_{(n_{\text{dof}}^s + n_{\text{dof}}^f + n_{\text{dof}}^p) \times 1} \quad (183)$$

The global residual for the solid skeleton displacement is given as

$$\mathbf{c}^T \cdot \mathbf{R}_u = \mathcal{G}^h = \mathcal{G}_1^{\text{INT},h} + \mathcal{G}_2^{\text{INT},h} + \mathcal{G}_3^{\text{INT},h} + \mathcal{G}_4^{\text{INT},h} - \mathcal{G}^{\text{EXT},h} = 0 \quad (184)$$

where

$$\mathcal{G}_1^{\text{INT},h} = \mathbf{A}_e^{n_e} \underbrace{\{\mathbf{c}^e\}^T}_{1 \times n_{\text{dof}}^{s,e}} \cdot \left(\int_{-1}^1 \underbrace{\{\mathbf{N}^{e,u}\}^T}_{n_{\text{dof}}^{s,e} \times 1} (\rho_0^{s,h^e} a^{h^e} + \rho_0^{f,h^e} a_f^{h^e}) A j^e d\xi \right) \quad (185)$$

$$\mathcal{G}_2^{\text{INT},h} = \mathbf{A}_e^{n_e} \underbrace{\{\mathbf{c}^e\}^T}_{1 \times n_{\text{dof}}^{s,e}} \cdot \left(\int_{-1}^1 \underbrace{\{\mathbf{B}^{e,u}\}^T}_{n_{\text{dof}}^{s,e} \times 1} P_{11(E)}^{s,h^e} A j^e d\xi \right) \quad (186)$$

$$\mathcal{G}_3^{\text{INT},h} = \mathbf{A}_e^{n_e} \underbrace{\{\mathbf{c}^e\}^T}_{1 \times n_{\text{dof}}^{s,e}} \cdot \left(- \int_{-1}^1 \underbrace{\{\mathbf{B}^{e,u}\}^T}_{n_{\text{dof}}^{s,e} \times 1} p_f^{h^e} A j^e d\xi \right) \quad (187)$$

$$\mathcal{G}_4^{\text{INT},h} = \mathbf{A}_e^{n_e} \underbrace{\{\mathbf{c}^e\}^T}_{1 \times n_{\text{dof}}^{s,e}} \cdot \left(\int_{-1}^1 \underbrace{\{\mathbf{N}^{e,u}\}^T}_{n_{\text{dof}}^{s,e} \times 1} \rho_0^{h^e} g A j^e d\xi \right) \quad (188)$$

$$\mathcal{G}^{\text{EXT},h} = \mathbf{A}_e^{n_e} \underbrace{\{\mathbf{c}^e\}^T}_{1 \times n_{\text{dof}}^{s,e}} \cdot \begin{cases} \underbrace{\{\mathbf{N}^{e,u}(X=H)\}^T}_{n_{\text{dof}}^{s,e} \times 1} t^\sigma A & X = H \\ 0 & 0 \leq X < H \end{cases} \quad (189)$$

The global residual for the pore fluid displacement is given as

$$\boldsymbol{\omega}^T \cdot \mathbf{R}_{u_f} = \mathcal{I}^h = \mathcal{I}_1^{\text{INT},h} + \mathcal{I}_2^{\text{INT},h} + \mathcal{I}_3^{\text{INT},h} + \mathcal{I}_4^{\text{INT},h} = 0 \quad (190)$$

where

$$\mathcal{I}_1^{\text{INT},h} = \mathbf{A}_e^{n_e} \underbrace{\{\boldsymbol{\omega}^e\}^T}_{1 \times n_{\text{dof}}^{f,e}} \cdot \left(\int_{-1}^1 \underbrace{\{\mathbf{N}^{e,u_f}\}^T}_{n_{\text{dof}}^{f,e} \times 1} \rho_0^{f,h^e} a_f^{h^e} A j^e d\xi \right) \quad (191)$$

$$\mathcal{I}_2^{\text{INT},h} = \mathbf{A}_e^{n_e} \underbrace{\{\boldsymbol{\omega}^e\}^T}_{1 \times n_{\text{dof}}^{f,e}} \cdot \left(\int_{-1}^1 \underbrace{\{\mathbf{N}^{e,u_f}\}^T}_{n_{\text{dof}}^{f,e} \times 1} n^{f,h^e} \frac{\partial p_f^{h^e}}{\partial X} A j^e d\xi \right) \quad (192)$$

$$\mathcal{I}_3^{\text{INT},h} = \mathbf{A}_e^{n_e} \underbrace{\{\boldsymbol{\omega}^e\}^T}_{1 \times n_{\text{dof}}^{f,e}} \cdot \left(\int_{-1}^1 \underbrace{\{\mathbf{N}^{e,u_f}\}^T}_{n_{\text{dof}}^{f,e} \times 1} \frac{J^{h^e} (n^{f,h^e})^2 \tilde{v}_f^{h^e}}{\hat{k}^{h^e}} A j^e d\xi \right) \quad (193)$$

$$\mathcal{I}_4^{\text{INT},h} = \mathbf{A}_e^{n_e} \underbrace{\{\boldsymbol{\omega}^e\}^T}_{1 \times n_{\text{dof}}^{f,e}} \cdot \left(\int_{-1}^1 \underbrace{\{\mathbf{N}^{e,u_f}\}^T}_{n_{\text{dof}}^{f,e} \times 1} \rho_0^{f,h^e} g A j^e d\xi \right) \quad (194)$$

The global residual for the pore fluid pressure is given as

$$\boldsymbol{\alpha}^T \cdot \mathbf{R}_{p_f} = \mathcal{H}^h = \mathcal{H}_1^{\text{INT},h} + \mathcal{H}_2^{\text{INT},h} + \mathcal{H}_3^{\text{INT},h} + \mathcal{H}_4^{\text{INT},h} - \mathcal{H}_1^{\text{EXT},h} = 0 \quad (195)$$

where

$$\mathcal{H}_1^{\text{INT},h} = \mathbf{A}_e^{n_e} \underbrace{\{\boldsymbol{\alpha}^e\}^T}_{1 \times n_{\text{dof}}^{p,e}} \cdot \left(\int_{-1}^1 \underbrace{\{\mathbf{N}^{e,p}\}^T}_{n_{\text{dof}}^{p,e} \times 1} \left[\frac{J^{h^e} n^{f,h^e}}{K_f^\eta} \hat{p}_f^{h^e} + j^{h^e} \right] A j^e d\xi \right) \quad (196)$$

$$\mathcal{H}_2^{\text{INT},h} = \mathbf{A}_e^{n_e} \underbrace{\{\boldsymbol{\alpha}^e\}^T}_{1 \times n_{\text{dof}}^{p,e}} \cdot \left(\int_{-1}^1 \underbrace{\{\mathbf{N}^{e,p}\}^T}_{n_{\text{dof}}^{p,e} \times 1} \frac{1}{K_f^\eta} \frac{\partial p_f^{h^e}}{\partial X} (n^{f,h^e} \tilde{v}_f^{h^e}) A j^e d\xi \right) \quad (197)$$

$$\mathcal{H}_3^{\text{INT},h} = \mathbf{A}_e^{n_e} \underbrace{\{\boldsymbol{\alpha}^e\}^T}_{1 \times n_{\text{dof}}^{p,e}} \cdot \left(\int_{-1}^1 \underbrace{\{\mathbf{B}^{e,p}\}^T}_{n_{\text{dof}}^{p,e} \times 1} \hat{k}^{h^e} \frac{\partial p_f^{h^e}}{\partial X} (F_{11}^{h^e})^{-1} A j^e d\xi \right) \quad (198)$$

$$\mathcal{H}_4^{\text{INT},h} = \mathbf{A}_e^{n_e} \underbrace{\{\boldsymbol{\alpha}^e\}^T}_{1 \times n_{\text{dof}}^{p,e}} \cdot \left(\int_{-1}^1 \underbrace{\{\mathbf{B}^{e,p}\}^T}_{n_{\text{dof}}^{p,e} \times 1} \hat{k}^{h^e} \rho^{\text{fR},h^e} (a_{\text{f}}^{h^e} + g) A j^e d\xi \right) \quad (199)$$

$$\mathcal{H}^{\text{EXT},h} = \mathbf{A}_e^{n_e} \underbrace{\{\boldsymbol{\alpha}^e\}^T}_{1 \times n_{\text{dof}}^{p,e}} \cdot \begin{cases} \underbrace{\{\mathbf{N}^{e,p}(X=H, X=0)\}^T}_{n_{\text{dof}}^{p,e} \times 1} Q_{\text{f}} A & X=H, X=0 \\ \mathbf{0} & 0 \leq X \leq H \end{cases} \quad (200)$$

When pressure stabilization is enabled, an additional term $\mathcal{H}^{\text{stab}}$ is added to the LHS of Eq. 195 and is defined as

$$\mathcal{H}^{\text{stab}} = \mathbf{A}_e^{n_e} \underbrace{\{\boldsymbol{\alpha}^e\}^T}_{1 \times n_{\text{dof}}^{p,e}} \cdot \left(\int_{-1}^1 \underbrace{\{\mathbf{B}^{e,p}\}^T}_{n_{\text{dof}}^{p,e} \times 1} \alpha^{\text{stab}} \frac{\partial p_{\text{f}}^{h^e}}{\partial X} (F_{11}^{h^e})^{-1} A j^e d\xi \right) \quad (201)$$

Recall from Eq. 183 that the tangent matrix for each iteration must be of the form

$$\mathbf{0} = -\mathbf{R}^k = \underbrace{\begin{bmatrix} \mathbf{K}_{u,u} & \mathbf{K}_{u,u_{\text{f}}} & \mathbf{K}_{u,p_{\text{f}}} \\ \mathbf{K}_{u_{\text{f}},u} & \mathbf{K}_{p_{\text{f}},u_{\text{f}}} & \mathbf{K}_{u_{\text{f}},p_{\text{f}}} \\ \mathbf{K}_{p_{\text{f}},u} & \mathbf{K}_{p_{\text{f}},u_{\text{f}}} & \mathbf{K}_{p_{\text{f}},p_{\text{f}}} \end{bmatrix}}_{(n_{\text{dof}}^s + n_{\text{dof}}^f + n_{\text{dof}}^p) \times (n_{\text{dof}}^s + n_{\text{dof}}^f + n_{\text{dof}}^p)} \cdot \underbrace{\begin{Bmatrix} \delta \ddot{\mathbf{d}} \\ \delta \ddot{\mathbf{d}}_{\text{f}} \\ \delta \ddot{\boldsymbol{\theta}} \end{Bmatrix}}_{(n_{\text{dof}}^s + n_{\text{dof}}^f + n_{\text{dof}}^p) \times 1} \quad (202)$$

where

$$\underbrace{\mathbf{K}_{u,u}}_{n_{\text{dof}}^s \times n_{\text{dof}}^s} = \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{s,e}} \underbrace{\{\mathbf{c}^e\}^T}_{1 \times n_{\text{dof}}^{s,e}} \cdot \sum_{i=1}^2 \underbrace{\left[\mathbf{k}_{u,u}^{G_i^{\text{INT},e}} \right]}_{n_{\text{dof}}^{s,e} \times n_{\text{dof}}^{s,e}} \quad (203)$$

$$\underbrace{\mathbf{K}_{u,u_f}}_{n_{\text{dof}}^s \times n_{\text{dof}}^f} = \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{s,e}} \underbrace{\{\mathbf{c}^e\}^T}_{1 \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\left[\mathbf{k}_{u,u_f}^{G_1^{\text{INT},e}} \right]}_{n_{\text{dof}}^{s,e} \times n_{\text{dof}}^{f,e}} \quad (204)$$

$$\underbrace{\mathbf{K}_{u,p_f}}_{n_{\text{dof}}^s \times n_{\text{dof}}^p} = \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{s,e}} \underbrace{\{\mathbf{c}^e\}^T}_{1 \times n_{\text{dof}}^{s,e}} \cdot \sum_{i=1,3} \underbrace{\left[\mathbf{k}_{u,p_f}^{G_i^{\text{INT},e}} \right]}_{n_{\text{dof}}^{s,e} \times n_{\text{dof}}^{p,e}} \quad (205)$$

$$\underbrace{\mathbf{K}_{u_f,u}}_{n_{\text{dof}}^f \times n_{\text{dof}}^s} = \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{f,e}} \underbrace{\{\boldsymbol{\omega}^e\}^T}_{1 \times n_{\text{dof}}^{f,e}} \cdot \sum_{i=1}^4 \underbrace{\left[\mathbf{k}_{u_f,u}^{T_i^{\text{INT},e}} \right]}_{n_{\text{dof}}^{f,e} \times n_{\text{dof}}^{s,e}} \quad (206)$$

$$\underbrace{\mathbf{K}_{u_f,u_f}}_{n_{\text{dof}}^f \times n_{\text{dof}}^f} = \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{f,e}} \underbrace{\{\boldsymbol{\omega}^e\}^T}_{1 \times n_{\text{dof}}^{f,e}} \cdot \sum_{i=1,3} \underbrace{\left[\mathbf{k}_{u_f,u_f}^{T_i^{\text{INT},e}} \right]}_{n_{\text{dof}}^{f,e} \times n_{\text{dof}}^{f,e}} \quad (207)$$

$$\underbrace{\mathbf{K}_{u_f,p_f}}_{n_{\text{dof}}^f \times n_{\text{dof}}^p} = \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{f,e}} \underbrace{\{\boldsymbol{\omega}^e\}^T}_{1 \times n_{\text{dof}}^{f,e}} \cdot \sum_{i=1,2,4} \underbrace{\left[\mathbf{k}_{u_f,p_f}^{T_i^{\text{INT},e}} \right]}_{n_{\text{dof}}^{f,e} \times n_{\text{dof}}^{p,e}} \quad (208)$$

$$\underbrace{\mathbf{K}_{p_f,u}}_{n_{\text{dof}}^p \times n_{\text{dof}}^s} = \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{f,e}} \underbrace{\{\boldsymbol{\alpha}^e\}^T}_{1 \times n_{\text{dof}}^{f,e}} \cdot \sum_{i=1}^4 \underbrace{\left[\mathbf{k}_{p_f,u}^{\mathcal{H}_i^{\text{INT},e}} \right]}_{n_{\text{dof}}^{p,e} \times n_{\text{dof}}^{s,e}} \quad (209)$$

$$\underbrace{\mathbf{K}_{p_f,u_f}}_{n_{\text{dof}}^p \times n_{\text{dof}}^f} = \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{f,e}} \underbrace{\{\boldsymbol{\alpha}^e\}^T}_{1 \times n_{\text{dof}}^{f,e}} \cdot \sum_{i=2,4} \underbrace{\left[\mathbf{k}_{p_f,u_f}^{\mathcal{H}_i^{\text{INT},e}} \right]}_{n_{\text{dof}}^{p,e} \times n_{\text{dof}}^{f,e}} \quad (210)$$

$$\underbrace{\mathbf{K}_{p_f,p_f}}_{n_{\text{dof}}^p \times n_{\text{dof}}^p} = \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{f,e}} \underbrace{\{\boldsymbol{\alpha}^e\}^T}_{1 \times n_{\text{dof}}^{f,e}} \cdot \sum_{i=1}^4 \underbrace{\left[\mathbf{k}_{p_f,p_f}^{\mathcal{H}_i^{\text{INT},e}} \right]}_{n_{\text{dof}}^{p,e} \times n_{\text{dof}}^{f,e}} \quad (211)$$

The solid skeleton displacement tangents are given as follows, where the definition for $\mathbf{k}_{u,u}^{G_2^{\text{INT},e}}$ remains unchanged from the (\mathbf{u}) and $(\mathbf{u}-p_f)$ formulations given by Eq. 136:

$$\underbrace{\left[\mathbf{k}_{u,u}^{G_1^{\text{INT},e}} \right]}_{n_{\text{dof}}^{s,e} \times n_{\text{dof}}^{s,e}} = \int_{-1}^1 \underbrace{\left\{ \mathbf{N}^{e,u} \right\}^T}_{n_{\text{dof}}^{s,e} \times 1} \left(\underbrace{\rho_0^{s,h^e}}_{1 \times n_{\text{dof}}^{s,e}} \underbrace{\left\{ \mathbf{N}^{e,u} \right\}}_{1 \times n_{\text{dof}}^{s,e}} + a_f^{h^e} \rho^{\text{fR},h^e} (\beta \Delta t^2) \underbrace{\left\{ \mathbf{B}^{e,u} \right\}}_{1 \times n_{\text{dof}}^{s,e}} \right) A_j^e d\xi \quad (212)$$

The solid skeleton displacement and pore fluid displacement coupling tangent is given as follows:

$$\underbrace{\left[\mathbf{k}_{u,u_f}^{G_1^{\text{INT}},e} \right]}_{n_{\text{dof}}^{s,e} \times n_{\text{dof}}^{f,e}} = \int_{-1}^1 \rho_0^{f,h^e} \underbrace{\left\{ \mathbf{N}^{e,u} \right\}^T}_{n_{\text{dof}}^{s,e} \times 1} \underbrace{\left\{ \mathbf{N}^{e,u_f} \right\}}_{1 \times n_{\text{dof}}^{f,e}} A_j^e d\xi \quad (213)$$

The solid skeleton displacement and pore fluid pressure coupling tangents are given as follows, where $\mathbf{k}_{u,p_f}^{G_3^{\text{INT}},e}$ remains unchanged from the $(\mathbf{u}-p_f)$ formulation given by Eq. 165:

$$\underbrace{\left[\mathbf{k}_{u,p_f}^{G_1^{\text{INT}},e} \right]}_{n_{\text{dof}}^{s,e} \times n_{\text{dof}}^{p,e}} = \int_{-1}^1 \frac{a_f^{h^e} J^{h^e} n^{f,h^e} \rho^{\text{fR},h^e}}{K_f^\eta} (\beta \Delta t^2) \underbrace{\left\{ \mathbf{N}^{e,u} \right\}^T}_{n_{\text{dof}}^{s,e} \times 1} \underbrace{\left\{ \mathbf{N}^{e,p} \right\}}_{1 \times n_{\text{dof}}^{p,e}} A_j^e d\xi \quad (214)$$

The pore fluid displacement and solid skeleton displacement coupling tangents are given as follows:

$$\underbrace{\left[\mathbf{k}_{u_f,u}^{I_1^{\text{INT}},e} \right]}_{n_{\text{dof}}^{f,e} \times n_{\text{dof}}^{s,e}} = \int_{-1}^1 \rho^{\text{fR},h^e} a_f^{h^e} (\beta \Delta t^2) \underbrace{\left\{ \mathbf{N}^{e,u_f} \right\}^T}_{n_{\text{dof}}^{f,e} \times 1} \underbrace{\left\{ \mathbf{B}^{e,u} \right\}}_{1 \times n_{\text{dof}}^{s,e}} A_j^e d\xi \quad (215)$$

$$\underbrace{\left[\mathbf{k}_{u_f,u}^{I_2^{\text{INT}},e} \right]}_{n_{\text{dof}}^{f,e} \times n_{\text{dof}}^{s,e}} = \int_{-1}^1 \frac{n^{s,h^e}}{J^{h^e}} \frac{\partial p_f^{h^e}}{\partial X} (\beta \Delta t^2) \underbrace{\left\{ \mathbf{N}^{e,u_f} \right\}^T}_{n_{\text{dof}}^{f,e} \times 1} \underbrace{\left\{ \mathbf{B}^{e,u} \right\}}_{1 \times n_{\text{dof}}^{s,e}} A_j^e d\xi \quad (216)$$

$$\begin{aligned} \underbrace{\left[\mathbf{k}_{u_f,u}^{I_3^{\text{INT}},e} \right]}_{n_{\text{dof}}^{f,e} \times n_{\text{dof}}^{s,e}} &= \int_{-1}^1 \underbrace{\left\{ \mathbf{N}^{e,u_f} \right\}^T}_{n_{\text{dof}}^{f,e} \times 1} \left(\left[1 + \frac{2n^{s,h^e}}{n^{f,h^e}} - n^{s,h^e} \left(\frac{3}{n^{f,h^e}} - \frac{2n^{f,h^e}}{1 - (n^{f,h^e})^2} \right) \right] \times \right. \\ &\quad \left. \frac{(n^{f,h^e})^2 \tilde{v}_f^{h^e}}{\hat{k}^{h^e}} (\beta \Delta t^2) \underbrace{\left\{ \mathbf{B}^{e,u} \right\}}_{1 \times n_{\text{dof}}^{s,e}} - \frac{J^{h^e} (n^{f,h^e})^2}{\hat{k}^{h^e}} (\gamma \Delta t) \underbrace{\left\{ \mathbf{N}^{e,u} \right\}}_{1 \times n_{\text{dof}}^{s,e}} \right) A_j^e d\xi \end{aligned} \quad (217)$$

$$\underbrace{\left[\mathbf{k}_{u_f,u}^{I_4^{\text{INT}},e} \right]}_{n_{\text{dof}}^{f,e} \times n_{\text{dof}}^{s,e}} = \int_{-1}^1 \rho^{\text{fR},h^e} g (\beta \Delta t^2) \underbrace{\left\{ \mathbf{N}^{e,u_f} \right\}^T}_{n_{\text{dof}}^{f,e} \times 1} \underbrace{\left\{ \mathbf{B}^{e,u} \right\}}_{1 \times n_{\text{dof}}^{s,e}} A_j^e d\xi \quad (218)$$

The pore fluid displacement tangents are given as follows:

$$\underbrace{\left[\mathbf{k}_{u_f, u_f}^{\mathcal{I}_1^{\text{INT}}, e} \right]}_{n_{\text{dof}}^{f,e} \times n_{\text{dof}}^{f,e}} = \int_{-1}^1 \rho_0^{f,h^e} \underbrace{\left\{ \mathbf{N}^{e, u_f} \right\}^T}_{n_{\text{dof}}^{f,e} \times 1} \underbrace{\left\{ \mathbf{N}^{e, u_f} \right\}}_{1 \times n_{\text{dof}}^{f,e}} A_j^e d\xi \quad (219)$$

$$\underbrace{\left[\mathbf{k}_{u_f, u_f}^{\mathcal{I}_3^{\text{INT}}, e} \right]}_{n_{\text{dof}}^{f,e} \times n_{\text{dof}}^{f,e}} = \int_{-1}^1 \frac{J^{h^e} (n^{f,h^e})^2}{\hat{k}^{h^e}} (\gamma \Delta t) \underbrace{\left\{ \mathbf{N}^{e, u_f} \right\}^T}_{n_{\text{dof}}^{f,e} \times 1} \underbrace{\left\{ \mathbf{N}^{e, u_f} \right\}}_{1 \times n_{\text{dof}}^{f,e}} A_j^e d\xi \quad (220)$$

The pore fluid displacement and pore fluid pressure coupling tangents are given as follows:

$$\underbrace{\left[\mathbf{k}_{u_f, p_f}^{\mathcal{I}_1^{\text{INT}}, e} \right]}_{n_{\text{dof}}^{f,e} \times n_{\text{dof}}^{p,e}} = \int_{-1}^1 \frac{a_f^{h^e} \rho_0^{f,h^e}}{K_f^{\eta}} (\beta \Delta t^2) \underbrace{\left\{ \mathbf{N}^{e, u_f} \right\}^T}_{n_{\text{dof}}^{f,e} \times 1} \underbrace{\left\{ \mathbf{N}^{e, p} \right\}}_{1 \times n_{\text{dof}}^{p,e}} A_j^e d\xi \quad (221)$$

$$\underbrace{\left[\mathbf{k}_{u_f, p_f}^{\mathcal{I}_2^{\text{INT}}, e} \right]}_{n_{\text{dof}}^{f,e} \times n_{\text{dof}}^{p,e}} = \int_{-1}^1 n^{f,h^e} (\beta \Delta t^2) \underbrace{\left\{ \mathbf{N}^{e, u_f} \right\}^T}_{n_{\text{dof}}^{f,e} \times 1} \underbrace{\left\{ \mathbf{B}^{e, p} \right\}}_{1 \times n_{\text{dof}}^{p,e}} A_j^e d\xi \quad (222)$$

$$\underbrace{\left[\mathbf{k}_{u_f, p_f}^{\mathcal{I}_4^{\text{INT}}, e} \right]}_{n_{\text{dof}}^{f,e} \times n_{\text{dof}}^{p,e}} = \int_{-1}^1 \frac{J^{h^e} n^{f,h^e} \rho^{\text{fR}, h^e} g}{K_f^{\eta}} (\beta \Delta t^2) \underbrace{\left\{ \mathbf{N}^{e, u_f} \right\}^T}_{n_{\text{dof}}^{f,e} \times 1} \underbrace{\left\{ \mathbf{N}^{e, p} \right\}}_{1 \times n_{\text{dof}}^{p,e}} A_j^e d\xi \quad (223)$$

The pore fluid pressure and solid skeleton coupling tangents are given as follows, where $\mathbf{k}_{p_f, u}^{\mathcal{H}_1^{\text{INT}}, e}$ and $\mathbf{k}_{p_f, u}^{\mathcal{H}_3^{\text{INT}}, e}$ (and $\mathbf{k}_{p_f, u}^{\mathcal{H}^{\text{stab}, e}}$ when pressure stabilization is enabled), remain unchanged from the $(\mathbf{u}-p_f)$ formulation given by Eqs. 166 and 168, respec-

tively, and by Eq. 170 when pressure stabilization is enabled:

$$\underbrace{\left[\mathbf{k}_{p_f, u}^{\mathcal{H}_2^{\text{INT}, e}} \right]}_{n_{\text{dof}}^{p, e} \times n_{\text{dof}}^{s, e}} = \int_{-1}^1 \frac{\partial p_f^{h^e}}{\partial X} \left(\frac{n^{s, h^e}}{J^{h^e}} \left[\frac{3}{n^{f, h^e}} - \frac{2n^{f, h^e}}{1 - (n^{f, h^e})^2} \right] (n^f \tilde{v}_f)^{h^e} \right. \\ \left. + \hat{k}^{h^e} \frac{\partial p_f^{h^e}}{\partial X} (F_{11}^{h^e})^{-2} \right) (\beta \Delta t^2) \underbrace{\left\{ \mathbf{N}^{e, p} \right\}^T}_{n_{\text{dof}}^{p, e} \times 1} \underbrace{\left\{ \mathbf{B}^{e, u} \right\}}_{1 \times n_{\text{dof}}^{s, e}} \frac{1}{K_f^\eta} A_j^e d\xi \quad (224)$$

$$\underbrace{\left[\mathbf{k}_{p_f, u}^{\mathcal{H}_4^{\text{INT}, e}} \right]}_{n_{\text{dof}}^{p, e} \times n_{\text{dof}}^{s, e}} = \int_{-1}^1 \frac{n^{s, h^e} \hat{k}^{h^e} \rho^{\text{fR}, h^e} (a_f^{h^e} + g)}{J^{h^e}} \left[\frac{3}{n^{f, h^e}} - \frac{2n^{f, h^e}}{1 - (n^{f, h^e})^2} \right] \times \\ (\beta \Delta t^2) \underbrace{\left\{ \mathbf{B}^{e, p} \right\}^T}_{n_{\text{dof}}^{p, e} \times 1} \underbrace{\left\{ \mathbf{B}^{e, u} \right\}}_{1 \times n_{\text{dof}}^{s, e}} A_j^e d\xi \quad (225)$$

The pore fluid pressure and pore fluid displacement coupling tangents are given as follows:

$$\underbrace{\left[\mathbf{k}_{p_f, u_f}^{\mathcal{H}_2^{\text{INT}, e}} \right]}_{n_{\text{dof}}^{p, e} \times n_{\text{dof}}^{f, e}} = - \int_{-1}^1 \frac{\partial p_f^{h^e}}{\partial X} \frac{\hat{k}^{h^e} \rho^{\text{fR}, h^e}}{K_f^\eta} \underbrace{\left\{ \mathbf{N}^{e, p} \right\}^T}_{n_{\text{dof}}^{p, e} \times 1} \underbrace{\left\{ \mathbf{N}^{e, u_f} \right\}}_{1 \times n_{\text{dof}}^{f, e}} A_j^e d\xi \quad (226)$$

$$\underbrace{\left[\mathbf{k}_{p_f, u_f}^{\mathcal{H}_4^{\text{INT}, e}} \right]}_{n_{\text{dof}}^{p, e} \times n_{\text{dof}}^{f, e}} = \int_{-1}^1 \hat{k}^{h^e} \rho^{\text{fR}, h^e} \underbrace{\left\{ \mathbf{B}^{e, p} \right\}^T}_{n_{\text{dof}}^{p, e} \times 1} \underbrace{\left\{ \mathbf{N}^{e, u_f} \right\}}_{1 \times n_{\text{dof}}^{f, e}} A_j^e d\xi \quad (227)$$

Lastly, the pore fluid pressure tangents are given as follows, where $\mathbf{k}_{p_f, p_f}^{\mathcal{H}_1^{\text{INT}, e}}$ and $\mathbf{k}_{p_f, p_f}^{\mathcal{H}_3^{\text{INT}, e}}$ (and $\mathbf{k}_{p_f, p_f}^{\mathcal{H}^{\text{stab}, e}}$ when pressure stabilization is enabled) remain unchanged from the $(\mathbf{u}-p_f)$ formulation given by Eqs. 171 and 173, respectively, and by Eq. 175

when pressure stabilization is enabled:

$$\begin{aligned} \underbrace{\left[\mathbf{k}_{p_f, p_f}^{\mathcal{H}_2^{\text{INT}}, e} \right]}_{n_{\text{dof}}^{p,e} \times n_{\text{dof}}^{p,e}} &= \int_{-1}^1 \underbrace{\left\{ \mathbf{N}^{e,p} \right\}^T}_{n_{\text{dof}}^{p,e} \times 1} \left(\left[\left(n^{f,h^e} \tilde{v}_f^{h^e} \right) - \hat{k}^{h^e} \frac{\partial p_f^{h^e}}{\partial X} (F_{11}^{h^e})^{-1} \right] \underbrace{\left\{ \mathbf{B}^{e,p} \right\}}_{1 \times n_{\text{dof}}^{p,e}} \right. \\ &\quad \left. - \hat{k}^{h^e} \frac{\partial p_f^{h^e}}{\partial X} (a_f^{h^e} + g) \frac{\rho^{\text{fR}, h^e}}{K_f^\eta} \underbrace{\left\{ \mathbf{N}^{e,p} \right\}}_{1 \times n_{\text{dof}}^{p,e}} \right) \frac{1}{K_f^\eta} (\beta \Delta t^2) A_j^e d\xi \end{aligned} \quad (228)$$

$$\underbrace{\left[\mathbf{k}_{p_f, p_f}^{\mathcal{H}_4^{\text{INT}}, e} \right]}_{n_{\text{dof}}^{p,e} \times n_{\text{dof}}^{p,e}} = \int_{-1}^1 \hat{k}^{h^e} (a_f^{h^e} + g) \frac{\rho^{\text{fR}, h^e}}{K_f^\eta} (\beta \Delta t^2) \underbrace{\left\{ \mathbf{B}^{e,p} \right\}^T}_{n_{\text{dof}}^{p,e} \times 1} \underbrace{\left\{ \mathbf{N}^{e,p} \right\}}_{1 \times n_{\text{dof}}^{p,e}} A_j^e d\xi \quad (229)$$

5.2 Runge-Kutta Time Integrators

For explicit time integration of the matrix-vector equations, we apply a generalized adaptive time-stepping Runge-Kutta method for solving Eq. 95, which involves transforming the second-order ordinary differential equations (ODEs) into first-order ODEs by variable substitution:

$$\mathbf{z} := \begin{Bmatrix} \mathbf{z}_x \\ \mathbf{z}_{\dot{x}} \end{Bmatrix} = \begin{Bmatrix} \mathbf{x} \\ \dot{\mathbf{x}} \end{Bmatrix} \quad (230)$$

such that,

$$\dot{\mathbf{z}} = \begin{Bmatrix} \dot{\mathbf{z}}_x \\ \dot{\mathbf{z}}_{\dot{x}} \end{Bmatrix} = \begin{Bmatrix} \dot{\mathbf{x}} \\ \ddot{\mathbf{x}} \end{Bmatrix} \quad (231)$$

For a general nonlinear multi-degree-of-freedom ODE,

$$\dot{\mathbf{z}} = \mathbf{f}(t, \mathbf{z}) \quad (232)$$

where $\mathbf{f}(t, \mathbf{z})$ is in general a nonlinear equation in terms of time t and unknown variable \mathbf{z} . For a general Runge-Kutta method of m^{th} order, the intermediate stages \mathbf{k}_i are defined as follows using standard notation for a Butcher table:

$$\mathbf{k}_i = \mathbf{f} \left(t_n + c_i \Delta t, \mathbf{z}(t_n) + \sum_{j=1}^{i-1} a_{ij} \mathbf{k}_j \right) \quad (233)$$

The higher order solution is given by

$$\mathbf{z}^m(t_{n+1}) = \mathbf{z}(t_n) + \Delta t \sum_{i=1}^{m+1} b_i^m \mathbf{k}_i \quad (234)$$

and the lower order solution is given by

$$\mathbf{z}^{m-1}(t_{n+1}) = \mathbf{z}(t_n) + \Delta t \sum_{i=1}^m b_i^{m-1} \mathbf{k}_i \quad (235)$$

where the b_i^m coefficients are different from the b_i^{m-1} coefficients; refer to the literature^{37,38} for specific values. The difference between the higher and lower order solutions allows us to define a truncation error:

$$\epsilon_{TE} := \left\| \mathbf{z}^m(t_{n+1}) - \mathbf{z}^{m-1}(t_{n+1}) \right\|_{\infty} \quad (236)$$

The adapted time step Δt^* is typically adjusted as follows:

$$\Delta t^* = \text{SF} \times \left(\frac{\epsilon_a}{\epsilon_{TE}} \right)^{1/(m-1)} \Delta t \quad (237)$$

where SF is a safety factor, typically set to 0.9, and ϵ_a is a user-defined absolute tolerance, typically set to $\epsilon_a \in [10^{-8}, 10^{-2}]$. If the absolute error

$$\left(\frac{\epsilon_{TE}}{\epsilon_a} \right)^{1/(m-1)} < 1 \quad (238)$$

then the solution \mathbf{z}^m is accepted with $\Delta t_{n+1} \leftarrow \Delta t^*$. Otherwise, the stages \mathbf{k}_i are recomputed at time t_n with $\Delta t_n \leftarrow \Delta t^*$, new solutions are computed, and a new absolute error is computed until the condition defined by Eq. 238 is met. In the event that the time step starts to approach zero, the simulation is terminated with an error message.

For further reading on Runge-Kutta integrators with adaptive time-stepping schemes based on truncation errors, refer to Cash and Karp,³⁸ Bogacki and Shampine,³⁷ and Press et al.³⁹ In numerical simulations described in a follow-up report,²⁹ we typically employ the fixed-order, 5(4) Runge-Kutta Cash-Karp (RKFNC) scheme.³⁸

Next, we show how the Runge-Kutta integrators are implemented for each branch

of physics (i.e., each theoretical formulation).

5.2.1 Implementation of the (u) Theory

The Runge-Kutta integrator as applied to elastodynamics of Eq. 28 is

$$\{\dot{z}\} := \begin{Bmatrix} \dot{z}_u \\ \dot{z}_v \end{Bmatrix} = \{f(t, z)\} = \begin{Bmatrix} f_v(t, z) \\ f_a(t, z) \end{Bmatrix} \quad (239)$$

such that

$$\{z\} = \begin{Bmatrix} d \\ \dot{d} \end{Bmatrix}, \quad \dot{z} = \begin{Bmatrix} \dot{d} \\ \ddot{d} \end{Bmatrix} \quad (240)$$

The FE formulation for the balance of momentum variational equation is written in block-matrix form as

$$\underbrace{\{\mathbf{R}_u\}}_{n_{\text{dof}}^s \times 1} = \mathbf{0} \quad (241)$$

where the global residual for the solid displacement is given as

$$\mathbf{c}^T \cdot \mathbf{R}_u = \mathcal{G}^h = \mathcal{G}_1^{\text{INT},h} + \mathcal{G}_2^{\text{INT},h} + \mathcal{G}_4^{\text{INT},h} - \mathcal{G}^{\text{EXT},h} = 0 \quad (242)$$

where

$$\mathcal{G}_1^{\text{INT},h} = \mathbf{A}_e \underbrace{\{\mathbf{c}^e\}^T}_{1 \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\left[\mathbf{m}_{u,u}^{\mathcal{G}_1^{\text{INT},e}} \right]}_{n_{\text{dof}}^{s,e} \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\{\ddot{\mathbf{d}}^e\}}_{n_{\text{dof}}^{s,e} \times 1} \quad (243)$$

$$\mathcal{G}_2^{\text{INT},h} = \mathbf{A}_e \underbrace{\{\mathbf{c}^e\}^T}_{1 \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\{\mathbf{f}^{\mathcal{G}_2^{\text{INT},e}}\}}_{n_{\text{dof}}^{s,e} \times 1} \quad (244)$$

$$\mathcal{G}_4^{\text{INT},h} = \mathbf{A}_e \underbrace{\{\mathbf{c}^e\}^T}_{1 \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\{\mathbf{f}^{\mathcal{G}_4^{\text{INT},e}}\}}_{n_{\text{dof}}^{s,e} \times 1} \quad (245)$$

$$\mathcal{G}^{\text{EXT},h} = \mathbf{A}_e \underbrace{\{\mathbf{c}^e\}^T}_{1 \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\{\mathbf{f}^{\mathcal{G}^{\text{EXT},e}}\}}_{n_{\text{dof}}^{s,e} \times 1} \quad (246)$$

and

$$\underbrace{\left\{ \mathbf{f}^{\mathcal{G}_2^{\text{INT}},e} \right\}}_{n_{\text{dof}}^{s,e} \times 1} = \int_{-1}^1 \underbrace{\left\{ \mathbf{B}^{e,u} \right\}^T}_{n_{\text{dof}}^{s,e} \times 1} P_{11}^{h^e} A j^e d\xi \quad (247)$$

$$\underbrace{\left\{ \mathbf{f}^{\mathcal{G}_4^{\text{INT}},e} \right\}}_{n_{\text{dof}}^{s,e} \times 1} = \int_{-1}^1 \underbrace{\left\{ \mathbf{N}^{e,u} \right\}^T}_{n_{\text{dof}}^{s,e} \times 1} \rho_0^{h^e} g A j^e d\xi \quad (248)$$

$$\underbrace{\left\{ \mathbf{f}^{\mathcal{G}^{\text{EXT}},e} \right\}}_{n_{\text{dof}}^{s,e} \times 1} = \begin{cases} \underbrace{\left\{ \mathbf{N}^{e,u}(X=H) \right\}^T}_{n_{\text{dof}}^{s,e} \times 1} t^\sigma A & X = H \\ \mathbf{0} & 0 \leq X < H \end{cases} \quad (249)$$

with the mass matrix associated with the solid acceleration given by

$$\underbrace{\left[\mathbf{m}_{u,u}^{\mathcal{G}_1^{\text{INT}},e} \right]}_{n_{\text{dof}}^{s,e} \times n_{\text{dof}}^{s,e}} = \int_{-1}^1 \rho_0^{h^e} \underbrace{\left\{ \mathbf{N}^{e,u} \right\}^T}_{n_{\text{dof}}^{s,e} \times 1} \underbrace{\left\{ \mathbf{N}^{e,u} \right\}}_{1 \times n_{\text{dof}}^{s,e}} A j^e d\xi \quad (250)$$

Then, returning our attention to Eq. 239, we have

$$\underbrace{\left\{ \dot{\mathbf{z}} \right\}}_{(2 \times n_{\text{dof}}^s) \times 1} = \left\{ \begin{array}{c} \underbrace{\left\{ \dot{\mathbf{d}} \right\}}_{n_{\text{dof}}^s \times 1} \\ \underbrace{\left[\mathbf{M}_{u,u}^{\mathcal{G}_1^{\text{INT}}} \right]^{-1}}_{n_{\text{dof}}^s \times n_{\text{dof}}^s} \cdot \left(- \underbrace{\left\{ \mathbf{F}^{\mathcal{G}_2^{\text{INT}}} \right\}}_{n_{\text{dof}}^s \times 1} - \underbrace{\left\{ \mathbf{F}^{\mathcal{G}_4^{\text{INT}}} \right\}}_{n_{\text{dof}}^s \times 1} + \underbrace{\left\{ \mathbf{F}^{\mathcal{G}^{\text{EXT}}} \right\}}_{n_{\text{dof}}^s \times 1} \right) \end{array} \right\} \quad (251)$$

where

$$\underbrace{\left[M_{u,u}^{\mathcal{G}_1^{\text{INT}}} \right]}_{n_{\text{dof}}^s \times n_{\text{dof}}^s} = \mathbf{A}_e^{n_e} \underbrace{\left[\mathbf{m}_{u,u}^{\mathcal{G}_1^{\text{INT},e}} \right]}_{n_{\text{dof}}^{s,e} \times n_{\text{dof}}^{s,e}} \quad (252)$$

$$\underbrace{\left\{ \mathbf{F}^{\mathcal{G}_2^{\text{INT}}} \right\}}_{n_{\text{dof}}^s \times 1} = \mathbf{A}_e^{n_e} \underbrace{\left\{ \mathbf{f}^{\mathcal{G}_2^{\text{INT},e}} \right\}}_{n_{\text{dof}}^{s,e} \times 1} \quad (253)$$

$$\underbrace{\left\{ \mathbf{F}^{\mathcal{G}_4^{\text{INT}}} \right\}}_{n_{\text{dof}}^s \times 1} = \mathbf{A}_e^{n_e} \underbrace{\left\{ \mathbf{f}^{\mathcal{G}_4^{\text{INT},e}} \right\}}_{n_{\text{dof}}^{s,e} \times 1} \quad (254)$$

$$\underbrace{\left\{ \mathbf{F}^{\mathcal{G}^{\text{EXT}}} \right\}}_{n_{\text{dof}}^s \times 1} = \mathbf{A}_e^{n_e} \underbrace{\left\{ \mathbf{f}^{\mathcal{G}^{\text{EXT},e}} \right\}}_{n_{\text{dof}}^{s,e} \times 1} \quad (255)$$

As described in Section 5.1.6, Eq. 251 is solved for each stage increment i at time $t_n + \Delta t c_i$ (i.e., any variables that are *explicit* functions of time, such as a time-dependent external traction, are to be evaluated at time $t_n + \Delta t c_i$), with stage solution \mathbf{k}_i given by

$$\underbrace{\left\{ \mathbf{k}_i \right\}}_{(2 \times n_{\text{dof}}^s) \times 1} := \underbrace{\left\{ \begin{array}{c} \underbrace{\left\{ \mathbf{k}_{i(v)} \right\}}_{n_{\text{dof}}^s \times 1} \\ \underbrace{\left\{ \mathbf{k}_{i(a)} \right\}}_{n_{\text{dof}}^s \times 1} \end{array} \right\}}_{(2 \times n_{\text{dof}}^s) \times 1} = \underbrace{\left\{ \begin{array}{c} \underbrace{\left\{ \mathbf{z}_v(t_n) \right\}}_{n_{\text{dof}}^s \times 1} + \sum_{j=1}^{i-1} a_{ij} \underbrace{\left\{ \mathbf{k}_{j(v)} \right\}}_{n_{\text{dof}}^s \times 1} \\ \underbrace{\left\{ \mathbf{z}_a(t_n) \right\}}_{n_{\text{dof}}^s \times 1} + \sum_{j=1}^{i-1} a_{ij} \underbrace{\left\{ \mathbf{k}_{j(a)} \right\}}_{n_{\text{dof}}^s \times 1} \end{array} \right\}}_{(2 \times n_{\text{dof}}^s) \times 1} \quad (256)$$

Then, according to Eq. 234, the higher order solution to be accepted or rejected at time t_{n+1} is given by

$$\underbrace{\left\{ \mathbf{z}^m(t_{n+1}) \right\}}_{(2 \times n_{\text{dof}}^s) \times 1} := \underbrace{\left\{ \begin{array}{c} \underbrace{\left\{ \mathbf{z}_u^m(t_{n+1}) \right\}}_{n_{\text{dof}}^s \times 1} \\ \underbrace{\left\{ \mathbf{z}_v^m(t_{n+1}) \right\}}_{n_{\text{dof}}^s \times 1} \end{array} \right\}}_{(2 \times n_{\text{dof}}^s) \times 1} = \underbrace{\left\{ \begin{array}{c} \underbrace{\left\{ \mathbf{z}_u(t_n) \right\}}_{n_{\text{dof}}^s \times 1} + \Delta t \sum_{i=1}^{m+1} b_i^m \underbrace{\left\{ \mathbf{k}_{i(v)} \right\}}_{n_{\text{dof}}^s \times 1} \\ \underbrace{\left\{ \mathbf{z}_v(t_n) \right\}}_{n_{\text{dof}}^s \times 1} + \Delta t \sum_{i=1}^{m+1} b_i^m \underbrace{\left\{ \mathbf{k}_{i(a)} \right\}}_{n_{\text{dof}}^s \times 1} \end{array} \right\}}_{(2 \times n_{\text{dof}}^s) \times 1} \quad (257)$$

According to Eq. 235, the lower order solution at time t_{n+1} is given by

$$\begin{aligned}
\underbrace{\left\{ \mathbf{z}^{m-1}(t_{n+1}) \right\}}_{(2 \times n_{\text{dof}}^s) \times 1} &:= \underbrace{\left\{ \begin{array}{c} \left\{ \mathbf{z}_u^{m-1}(t_{n+1}) \right\} \\ n_{\text{dof}}^s \times 1 \\ \left\{ \mathbf{z}_v^{m-1}(t_{n+1}) \right\} \\ n_{\text{dof}}^s \times 1 \end{array} \right\}} \\
&= \underbrace{\left\{ \begin{array}{c} \left\{ \mathbf{z}_u(t_n) \right\} + \Delta t \sum_{i=1}^m b_i^{m-1} \left\{ \mathbf{k}_{i(v)} \right\} \\ n_{\text{dof}}^s \times 1 \\ \left\{ \mathbf{z}_v(t_n) \right\} + \Delta t \sum_{i=1}^m b_i^{m-1} \left\{ \mathbf{k}_{i(a)} \right\} \\ n_{\text{dof}}^s \times 1 \end{array} \right\}} \quad (258)
\end{aligned}$$

Then, a truncation error is calculated for all values of the solutions $\mathbf{z}(t_{n+1})$ according to Eq. 236, and the new time step Δt^* is adjusted according to Eq. 237. If the absolute error does not meet the condition defined by Eq. 238, the solutions given by Eqs. 257 and 258 are recomputed at time t_{n+1} using the smaller time step Δt^* in place of Δt in Eqs. 256–258.

5.2.2 Implementation of the (u - p_f) Theory

For the (u - p_f) formulation, the Runge-Kutta integrators transform the general solution variables given by Eq. 232 to

$$\left\{ \dot{\mathbf{z}} \right\} := \left\{ \begin{array}{c} \dot{\mathbf{z}}_u \\ \dot{\mathbf{z}}_v \\ \dot{\mathbf{z}}_{p_f} \end{array} \right\} = \left\{ \mathbf{f}(t, \mathbf{z}) \right\} = \left\{ \begin{array}{c} \mathbf{f}_v(t, \mathbf{z}) \\ \mathbf{f}_a(t, \mathbf{z}) \\ \mathbf{f}_{p_f}(t, \mathbf{z}) \end{array} \right\} \quad (259)$$

such that

$$\left\{ \mathbf{z} \right\} = \left\{ \begin{array}{c} \mathbf{d} \\ \dot{\mathbf{d}} \\ \boldsymbol{\theta} \end{array} \right\}, \quad \dot{\mathbf{z}} = \left\{ \begin{array}{c} \dot{\mathbf{d}} \\ \ddot{\mathbf{d}} \\ \dot{\boldsymbol{\theta}} \end{array} \right\} \quad (260)$$

Thus, we require at least one governing equation to solve for the primary unknown $\ddot{\mathbf{d}}$, which when integrated once gives us $\dot{\mathbf{d}}$, and when integrated twice gives us \mathbf{d} ; and at least one governing equation to solve for the primary unknown $\dot{\boldsymbol{\theta}}$, which when integrated once gives us $\boldsymbol{\theta}$. Contrary to the central-difference (CD) scheme, these equations are solved separately (i.e., not in a staggered manner).

The FE formulation for the balance of momentum of the mixture, with variational equation given by Eq. 28, is written in block-matrix form as

$$\underbrace{\{\mathbf{R}_u\}}_{n_{\text{dof}}^s \times 1} = \mathbf{0} \quad (261)$$

where the global residual for the solid skeleton displacement is given as

$$\mathbf{c}^T \cdot \mathbf{R}_u = \mathcal{G}^h = \mathcal{G}_1^{\text{INT},h} + \mathcal{G}_2^{\text{INT},h} + \mathcal{G}_3^{\text{INT},h} + \mathcal{G}_4^{\text{INT},h} - \mathcal{G}^{\text{EXT},h} = 0 \quad (262)$$

where

$$\mathcal{G}_1^{\text{INT},h} = \mathbf{A}_e^{n_e} \underbrace{\{\mathbf{c}^e\}}_{1 \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\left[\mathbf{m}_{u,u}^{\mathcal{G}_1^{\text{INT},e}} \right]}_{n_{\text{dof}}^{s,e} \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\{\ddot{\mathbf{d}}^e\}}_{n_{\text{dof}}^{s,e} \times 1} \quad (263)$$

$$\mathcal{G}_2^{\text{INT},h} = \mathbf{A}_e^{n_e} \underbrace{\{\mathbf{c}^e\}}_{1 \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\{\mathbf{f}^{\mathcal{G}_2^{\text{INT},e}}\}}_{n_{\text{dof}}^{s,e} \times 1} \quad (264)$$

$$\mathcal{G}_3^{\text{INT},h} = \mathbf{A}_e^{n_e} \underbrace{\{\mathbf{c}^e\}}_{1 \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\{\mathbf{f}^{\mathcal{G}_3^{\text{INT},e}}\}}_{n_{\text{dof}}^{s,e} \times 1} \quad (265)$$

$$\mathcal{G}_4^{\text{INT},h} = \mathbf{A}_e^{n_e} \underbrace{\{\mathbf{c}^e\}}_{1 \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\{\mathbf{f}^{\mathcal{G}_4^{\text{INT},e}}\}}_{n_{\text{dof}}^{s,e} \times 1} \quad (266)$$

$$\mathcal{G}^{\text{EXT},h} = \mathbf{A}_e^{n_e} \underbrace{\{\mathbf{c}^e\}}_{1 \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\{\mathbf{f}^{\mathcal{G}^{\text{EXT},e}}\}}_{n_{\text{dof}}^{s,e} \times 1} \quad (267)$$

and

$$\underbrace{\left\{ \mathbf{f}^{\mathcal{G}_2^{\text{INT}},e} \right\}}_{n_{\text{dof}}^{s,e} \times 1} = \int_{-1}^1 \underbrace{\left\{ \mathbf{B}^{e,u} \right\}}_{n_{\text{dof}}^{s,e} \times 1}^T P_{11(E)}^{s,h^e} A_j^e d\xi \quad (268)$$

$$\underbrace{\left\{ \mathbf{f}^{\mathcal{G}_3^{\text{INT}},e} \right\}}_{n_{\text{dof}}^{s,e} \times 1} = - \int_{-1}^1 \underbrace{\left\{ \mathbf{B}^{e,u} \right\}}_{n_{\text{dof}}^{s,e} \times 1}^T p_f^{h^e} A_j^e d\xi \quad (269)$$

$$\underbrace{\left\{ \mathbf{f}^{\mathcal{G}_4^{\text{INT}},e} \right\}}_{n_{\text{dof}}^{s,e} \times 1} = \int_{-1}^1 \underbrace{\left\{ \mathbf{N}^{e,u} \right\}}_{n_{\text{dof}}^{s,e} \times 1}^T \rho_0^{h^e} g A_j^e d\xi \quad (270)$$

$$\underbrace{\left\{ \mathbf{f}^{\mathcal{G}^{\text{EXT}},e} \right\}}_{n_{\text{dof}}^{s,e} \times 1} = \begin{cases} \underbrace{\left\{ \mathbf{N}^{e,u}(X=H) \right\}}_{n_{\text{dof}}^{s,e} \times 1}^T t^\sigma A & X = H \\ \mathbf{0} & 0 \leq X < H \end{cases} \quad (271)$$

The mass matrix associated with the solid skeleton acceleration is given by

$$\underbrace{\left[\mathbf{m}_{u,u}^{\mathcal{G}_1^{\text{INT}},e} \right]}_{n_{\text{dof}}^{s,e} \times n_{\text{dof}}^{s,e}} = \int_{-1}^1 \rho_0^{h^e} \underbrace{\left\{ \mathbf{N}^{e,u} \right\}}_{n_{\text{dof}}^{s,e} \times 1}^T \underbrace{\left\{ \mathbf{N}^{e,u} \right\}}_{1 \times n_{\text{dof}}^{s,e}} A_j^e d\xi \quad (272)$$

The FE formulation for the balance of mass, variational Eq. 3, is written in block-matrix form as

$$\underbrace{\left\{ \mathbf{R}_{p_f} \right\}}_{n_{\text{dof}}^f \times 1} = \mathbf{0} \quad (273)$$

where the global residual for the pore fluid pressure is given as

$$\boldsymbol{\alpha}^T \cdot \mathbf{R}_{p_f} = \mathcal{H}^h = \mathcal{H}_1^{\text{INT},h} + \mathcal{H}_2^{\text{INT},h} + \mathcal{H}_3^{\text{INT},h} + \mathcal{H}_4^{\text{INT},h} - \mathcal{H}_1^{\text{EXT},h} = 0 \quad (274)$$

with

$$\mathcal{H}_1^{\text{INT},h} = \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{p,e}} \underbrace{\{\alpha^e\}^T}_{n_{\text{dof}}^{p,e} \times n_{\text{dof}}^{p,e}} \cdot \left(\underbrace{\left[m_{p_f, p_f}^{\mathcal{H}_1^{\text{INT},e}} \right]}_{n_{\text{dof}}^{p,e} \times n_{\text{dof}}^{p,e}} \cdot \underbrace{\{\dot{\theta}^e\}}_{n_{\text{dof}}^{p,e} \times 1} + \underbrace{\{f^{\mathcal{H}_1^{\text{INT},e}}\}}_{n_{\text{dof}}^{p,e} \times 1} \right) \quad (275)$$

$$\mathcal{H}_2^{\text{INT},h} = \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{p,e}} \underbrace{\{\alpha^e\}^T}_{n_{\text{dof}}^{p,e} \times 1} \cdot \underbrace{\{f^{\mathcal{H}_2^{\text{INT},e}}\}}_{n_{\text{dof}}^{p,e} \times 1} \quad (276)$$

$$\mathcal{H}_3^{\text{INT},h} = \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{p,e}} \underbrace{\{\alpha^e\}^T}_{n_{\text{dof}}^{p,e} \times 1} \cdot \underbrace{\{f^{\mathcal{H}_3^{\text{INT},e}}\}}_{n_{\text{dof}}^{p,e} \times 1} \quad (277)$$

$$\mathcal{H}_4^{\text{INT},h} = \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{p,e}} \underbrace{\{\alpha^e\}^T}_{n_{\text{dof}}^{p,e} \times 1} \cdot \underbrace{\{f^{\mathcal{H}_4^{\text{INT},e}}\}}_{n_{\text{dof}}^{p,e} \times 1} \quad (278)$$

$$\mathcal{H}^{\text{EXT},h} = \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{p,e}} \underbrace{\{\alpha^e\}^T}_{n_{\text{dof}}^{p,e} \times 1} \cdot \underbrace{\{f^{\mathcal{H}^{\text{EXT},e}}\}}_{n_{\text{dof}}^{p,e} \times 1} \quad (279)$$

and

$$\underbrace{\{f^{\mathcal{H}_1^{\text{INT},e}}\}}_{n_{\text{dof}}^{p,e} \times 1} = \int_{-1}^1 \underbrace{\{N^{e,p}\}^T}_{n_{\text{dof}}^{p,e} \times 1} j^{h^e} A_j^e d\xi \quad (280)$$

$$\underbrace{\{f^{\mathcal{H}_2^{\text{INT},e}}\}}_{n_{\text{dof}}^{p,e} \times 1} = \int_{-1}^1 \underbrace{\{N^{e,p}\}^T}_{n_{\text{dof}}^{p,e} \times 1} \frac{1}{K_f^\eta} \frac{\partial p_f^{h^e}}{\partial X} (n^{f,h^e} \tilde{v}_f^{h^e}) A_j^e d\xi \quad (281)$$

$$\underbrace{\{f^{\mathcal{H}_3^{\text{INT},e}}\}}_{n_{\text{dof}}^{p,e} \times 1} = \int_{-1}^1 \underbrace{\{B^{e,p}\}^T}_{n_{\text{dof}}^{p,e} \times 1} \hat{k}^{h^e} \frac{\partial p_f^{h^e}}{\partial X} (F_{11}^{h^e})^{-1} A_j^e d\xi \quad (282)$$

$$\underbrace{\{f^{\mathcal{H}_4^{\text{INT},e}}\}}_{n_{\text{dof}}^{p,e} \times 1} = \int_{-1}^1 \underbrace{\{B^{e,p}\}^T}_{n_{\text{dof}}^{p,e} \times 1} \hat{k}^{h^e} \rho^{\text{fR},h^e} (a^{h^e} + g) A_j^e d\xi \quad (283)$$

$$\underbrace{\{f^{\mathcal{H}^{\text{EXT},e}}\}}_{n_{\text{dof}}^{p,e} \times 1} = \begin{cases} \underbrace{\{N^{e,p}(X=H, X=0)\}^T}_{n_{\text{dof}}^{p,e} \times 1} Q_f A & X=H, X=0 \\ \mathbf{0} & 0 < X < H \end{cases} \quad (284)$$

The mass matrix associated with the pore fluid pressure is given by

$$\underbrace{\left[\mathbf{m}_{p_f, p_f}^{\mathcal{H}_1^{\text{INT}}, e} \right]}_{n_{\text{dof}}^{p, e} \times n_{\text{dof}}^{p, e}} = \int_{-1}^1 \frac{J^{h^e} n^{f, h^e}}{K_f^\eta} \underbrace{\left\{ \mathbf{N}^{e, p} \right\}^T}_{n_{\text{dof}}^{p, e} \times 1} \underbrace{\left\{ \mathbf{N}^{e, p} \right\}}_{1 \times n_{\text{dof}}^{p, e}} A_j^e d\xi \quad (285)$$

When pressure stabilization is enabled, an additional term $\mathcal{H}^{\text{stab}}$ is added to the LHS of Eq. 274 and is defined as

$$\mathcal{H}^{\text{stab}} = \underbrace{\mathbf{A}_e}_{n_e} \alpha^{e, T} \cdot \underbrace{\left[\mathbf{m}_{p_f, p_f}^{\mathcal{H}^{\text{stab}}, e} \right]}_{n_{\text{dof}}^{p, e} \times n_{\text{dof}}^{p, e}} \cdot \underbrace{\left\{ \dot{\boldsymbol{\theta}}^e \right\}}_{n_{\text{dof}}^{p, e} \times 1} \quad (286)$$

where

$$\underbrace{\left[\mathbf{m}_{p_f, p_f}^{\mathcal{H}^{\text{stab}}, e} \right]}_{n_{\text{dof}}^{p, e} \times n_{\text{dof}}^{p, e}} = \int_{-1}^1 \alpha^{\text{stab}} (F_{11}^{h^e})^{-1} \underbrace{\left\{ \mathbf{B}^{e, p} \right\}^T}_{n_{\text{dof}}^{p, e} \times 1} \underbrace{\left\{ \mathbf{B}^{e, p} \right\}}_{1 \times n_{\text{dof}}^{p, e}} A_j^e d\xi \quad (287)$$

Then, returning our attention to Eq. 259, we have

$$\underbrace{\left\{ \dot{\mathbf{z}} \right\}}_{(2 \times n_{\text{dof}}^s + n_{\text{dof}}^p) \times 1} = \left(\begin{array}{c} \underbrace{\left\{ \dot{\mathbf{d}} \right\}}_{n_{\text{dof}}^s \times 1} \\ \underbrace{\left[\mathbf{M}_{u, u}^{\mathcal{G}_1^{\text{INT}}} \right]^{-1}}_{n_{\text{dof}}^s \times n_{\text{dof}}^s} \\ \left(- \underbrace{\left\{ \mathbf{F}^{\mathcal{G}_2^{\text{INT}}} \right\}}_{n_{\text{dof}}^s \times 1} - \underbrace{\left\{ \mathbf{F}^{\mathcal{G}_3^{\text{INT}}} \right\}}_{n_{\text{dof}}^s \times 1} - \underbrace{\left\{ \mathbf{F}^{\mathcal{G}_4^{\text{INT}}} \right\}}_{n_{\text{dof}}^s \times 1} + \underbrace{\left\{ \mathbf{F}^{\mathcal{G}^{\text{EXT}}} \right\}}_{n_{\text{dof}}^s \times 1} \right) \\ \underbrace{\left[\mathbf{M}_{p_f, p_f}^{\mathcal{H}_1^{\text{INT}}} \right]^{-1}}_{n_{\text{dof}}^p \times n_{\text{dof}}^p} \\ \left(- \underbrace{\left\{ \mathbf{F}^{\mathcal{H}_1^{\text{INT}}} \right\}}_{n_{\text{dof}}^p \times 1} - \underbrace{\left\{ \mathbf{F}^{\mathcal{H}_2^{\text{INT}}} \right\}}_{n_{\text{dof}}^p \times 1} - \underbrace{\left\{ \mathbf{F}^{\mathcal{H}_3^{\text{INT}}} \right\}}_{n_{\text{dof}}^p \times 1} - \underbrace{\left\{ \mathbf{F}^{\mathcal{H}_4^{\text{INT}}} \right\}}_{n_{\text{dof}}^p \times 1} + \underbrace{\left\{ \mathbf{F}^{\mathcal{H}^{\text{EXT}}} \right\}}_{n_{\text{dof}}^p \times 1} \right) \end{array} \right) \quad (288)$$

where

$$\underbrace{\left[M_{u,u}^{\mathcal{G}^{\text{INT}}} \right]}_{n_{\text{dof}}^s \times n_{\text{dof}}^s} = \mathbf{A}_e^{n_e} \underbrace{\left[m_{u,u}^{\mathcal{G}^{\text{INT},e}} \right]}_{n_{\text{dof}}^{s,e} \times n_{\text{dof}}^{s,e}} \quad (289)$$

$$\underbrace{\left\{ \mathbf{F}^{\mathcal{G}_2^{\text{INT}}} \right\}}_{n_{\text{dof}}^s \times 1} = \mathbf{A}_e^{n_e} \underbrace{\left\{ \mathbf{f}^{\mathcal{G}_2^{\text{INT},e}} \right\}}_{n_{\text{dof}}^{s,e} \times 1} \quad (290)$$

$$\underbrace{\left\{ \mathbf{F}^{\mathcal{G}_2^{\text{INT}}} \right\}}_{n_{\text{dof}}^s \times 1} = \mathbf{A}_e^{n_e} \underbrace{\left\{ \mathbf{f}^{\mathcal{G}_3^{\text{INT},e}} \right\}}_{n_{\text{dof}}^{s,e} \times 1} \quad (291)$$

$$\underbrace{\left\{ \mathbf{F}^{\mathcal{G}_4^{\text{INT}}} \right\}}_{n_{\text{dof}}^s \times 1} = \mathbf{A}_e^{n_e} \underbrace{\left\{ \mathbf{f}^{\mathcal{G}_4^{\text{INT},e}} \right\}}_{n_{\text{dof}}^{s,e} \times 1} \quad (292)$$

$$\underbrace{\left\{ \mathbf{F}^{\mathcal{G}^{\text{EXT}}} \right\}}_{n_{\text{dof}}^s \times 1} = \mathbf{A}_e^{n_e} \underbrace{\left\{ \mathbf{f}^{\mathcal{G}^{\text{EXT},e}} \right\}}_{n_{\text{dof}}^{s,e} \times 1} \quad (293)$$

$$\underbrace{\left[M_{p_f,p_f}^{\mathcal{H}^{\text{INT}}} \right]}_{n_{\text{dof}}^p \times n_{\text{dof}}^p} = \mathbf{A}_e^{n_e} \underbrace{\left[m_{p_f,p_f}^{\mathcal{H}^{\text{INT},e}} \right]}_{n_{\text{dof}}^{p,e} \times n_{\text{dof}}^{p,e}} \quad (294)$$

$$\underbrace{\left\{ \mathbf{F}^{\mathcal{H}_1^{\text{INT}}} \right\}}_{n_{\text{dof}}^p \times 1} = \mathbf{A}_e^{n_e} \underbrace{\left\{ \mathbf{f}^{\mathcal{H}_1^{\text{INT},e}} \right\}}_{n_{\text{dof}}^{p,e} \times 1} \quad (295)$$

$$\underbrace{\left\{ \mathbf{F}^{\mathcal{H}_2^{\text{INT}}} \right\}}_{n_{\text{dof}}^p \times 1} = \mathbf{A}_e^{n_e} \underbrace{\left\{ \mathbf{f}^{\mathcal{H}_2^{\text{INT},e}} \right\}}_{n_{\text{dof}}^{p,e} \times 1} \quad (296)$$

$$\underbrace{\left\{ \mathbf{F}^{\mathcal{H}_3^{\text{INT}}} \right\}}_{n_{\text{dof}}^p \times 1} = \mathbf{A}_e^{n_e} \underbrace{\left\{ \mathbf{f}^{\mathcal{H}_3^{\text{INT},e}} \right\}}_{n_{\text{dof}}^{p,e} \times 1} \quad (297)$$

$$\underbrace{\left\{ \mathbf{F}^{\mathcal{H}_4^{\text{INT}}} \right\}}_{n_{\text{dof}}^p \times 1} = \mathbf{A}_e^{n_e} \underbrace{\left\{ \mathbf{f}^{\mathcal{H}_4^{\text{INT},e}} \right\}}_{n_{\text{dof}}^{p,e} \times 1} \quad (298)$$

$$\underbrace{\left\{ \mathbf{F}^{\mathcal{H}^{\text{EXT}}} \right\}}_{n_{\text{dof}}^p \times 1} = \mathbf{A}_e^{n_e} \underbrace{\left\{ \mathbf{f}^{\mathcal{H}^{\text{EXT},e}} \right\}}_{n_{\text{dof}}^{p,e} \times 1} \quad (299)$$

If pressure stabilization is enabled, then we must invert a summation of matrices in

Eq. 288:

$$\underbrace{\left\{ \dot{\mathbf{z}} \right\}}_{(2 \times n_{\text{dof}}^s + n_{\text{dof}}^p) \times 1} = \left\{ \begin{array}{c} \underbrace{\left\{ \dot{\mathbf{d}} \right\}}_{n_{\text{dof}}^s \times 1} \\ \underbrace{\left[\mathbf{M}_{u,u}^{\mathcal{G}^{\text{INT}}} \right]^{-1}}_{n_{\text{dof}}^s \times n_{\text{dof}}^s} \\ \left(- \underbrace{\left\{ \mathbf{F}^{\mathcal{G}^{\text{INT}}}_2 \right\}}_{n_{\text{dof}}^s \times 1} - \underbrace{\left\{ \mathbf{F}^{\mathcal{G}^{\text{INT}}}_3 \right\}}_{n_{\text{dof}}^s \times 1} - \underbrace{\left\{ \mathbf{F}^{\mathcal{G}^{\text{INT}}}_4 \right\}}_{n_{\text{dof}}^s \times 1} + \underbrace{\left\{ \mathbf{F}^{\mathcal{G}^{\text{EXT}}} \right\}}_{n_{\text{dof}}^s \times 1} \right) \\ \underbrace{\left(\left[\mathbf{M}_{p_f, p_f}^{\mathcal{H}^{\text{INT}}} \right] + \left[\mathbf{M}_{p_f, p_f}^{\mathcal{H}^{\text{stab}}} \right] \right)^{-1}}_{n_{\text{dof}}^p \times n_{\text{dof}}^p} \\ \left(- \underbrace{\left\{ \mathbf{F}^{\mathcal{H}^{\text{INT}}}_1 \right\}}_{n_{\text{dof}}^p \times 1} - \underbrace{\left\{ \mathbf{F}^{\mathcal{H}^{\text{INT}}}_2 \right\}}_{n_{\text{dof}}^p \times 1} - \underbrace{\left\{ \mathbf{F}^{\mathcal{H}^{\text{INT}}}_3 \right\}}_{n_{\text{dof}}^p \times 1} - \underbrace{\left\{ \mathbf{F}^{\mathcal{H}^{\text{INT}}}_4 \right\}}_{n_{\text{dof}}^p \times 1} + \underbrace{\left\{ \mathbf{F}^{\mathcal{H}^{\text{EXT}}} \right\}}_{n_{\text{dof}}^p \times 1} \right) \end{array} \right\} \quad (300)$$

where

$$\underbrace{\left[\mathbf{M}_{p_f, p_f}^{\mathcal{H}^{\text{stab}}} \right]}_{n_{\text{dof}}^p \times n_{\text{dof}}^p} = \underbrace{\mathbf{A}_e}_{n_e} \underbrace{\left[\mathbf{m}_{p_f, p_f}^{\mathcal{H}^{\text{stab}, e}} \right]}_{n_{\text{dof}}^{p, e} \times n_{\text{dof}}^{p, e}} \quad (301)$$

As described in Section 5.1.6, Eq. 288 or Eq. 300 is solved for each stage increment i at time $t_n + \Delta tc_i$ (i.e., any variables that are *explicit* functions of time, such as a time-dependent external traction, are to be evaluated at time $t_n + \Delta tc_i$), with stage solution \mathbf{k}_i given by

$$\underbrace{\left\{ \mathbf{k}_i \right\}}_{(2 \times n_{\text{dof}}^s + n_{\text{dof}}^p) \times 1} := \left\{ \begin{array}{c} \underbrace{\left\{ \mathbf{k}_{i(v)} \right\}}_{n_{\text{dof}}^s \times 1} \\ \underbrace{\left\{ \mathbf{k}_{i(a)} \right\}}_{n_{\text{dof}}^s \times 1} \\ \underbrace{\left\{ \mathbf{k}_{i(p_f)} \right\}}_{n_{\text{dof}}^p \times 1} \end{array} \right\} = \left\{ \begin{array}{c} \underbrace{\left\{ \mathbf{z}_v(t_n) \right\}}_{n_{\text{dof}}^s \times 1} + \sum_{j=1}^{i-1} a_{ij} \underbrace{\left\{ \mathbf{k}_{j(v)} \right\}}_{n_{\text{dof}}^s \times 1} \\ \underbrace{\left\{ \mathbf{z}_a(t_n) \right\}}_{n_{\text{dof}}^s \times 1} + \sum_{j=1}^{i-1} a_{ij} \underbrace{\left\{ \mathbf{k}_{j(a)} \right\}}_{n_{\text{dof}}^s \times 1} \\ \underbrace{\left\{ \mathbf{z}_{p_f}(t_n) \right\}}_{n_{\text{dof}}^p \times 1} + \sum_{j=1}^{i-1} a_{ij} \underbrace{\left\{ \mathbf{k}_{j(p_f)} \right\}}_{n_{\text{dof}}^p \times 1} \end{array} \right\} \quad (302)$$

Then, according to Eq. 234, the higher order solution to be accepted or rejected at

time t_{n+1} is given by

$$\underbrace{\left\{ \mathbf{z}^m(t_{n+1}) \right\}}_{(2 \times n_{\text{dof}}^s + n_{\text{dof}}^p) \times 1} := \underbrace{\left\{ \begin{array}{c} \underbrace{\left\{ \mathbf{z}_u^m(t_{n+1}) \right\}}_{n_{\text{dof}}^s \times 1} \\ \underbrace{\left\{ \mathbf{z}_v^m(t_{n+1}) \right\}}_{n_{\text{dof}}^s \times 1} \\ \underbrace{\left\{ \mathbf{z}_{p_f}^m(t_{n+1}) \right\}}_{n_{\text{dof}}^p \times 1} \end{array} \right\}}_{(2 \times n_{\text{dof}}^s + n_{\text{dof}}^p) \times 1} = \underbrace{\left\{ \begin{array}{c} \underbrace{\left\{ \mathbf{z}_u(t_n) \right\}}_{n_{\text{dof}}^s \times 1} + \Delta t \sum_{i=1}^{m+1} b_i^m \underbrace{\left\{ \mathbf{k}_{i(v)} \right\}}_{n_{\text{dof}}^s \times 1} \\ \underbrace{\left\{ \mathbf{z}_v(t_n) \right\}}_{n_{\text{dof}}^s \times 1} + \Delta t \sum_{i=1}^{m+1} b_i^m \underbrace{\left\{ \mathbf{k}_{i(a)} \right\}}_{n_{\text{dof}}^s \times 1} \\ \underbrace{\left\{ \mathbf{z}_{p_f}(t_n) \right\}}_{n_{\text{dof}}^p \times 1} + \Delta t \sum_{i=1}^{m+1} b_i^m \underbrace{\left\{ \mathbf{k}_{i(p_f)} \right\}}_{n_{\text{dof}}^p \times 1} \end{array} \right\}}_{(2 \times n_{\text{dof}}^s + n_{\text{dof}}^p) \times 1} \quad (303)$$

and, according to Eq. 235, the lower order solution at time t_{n+1} is given by

$$\underbrace{\left\{ \mathbf{z}^{m-1}(t_{n+1}) \right\}}_{(2 \times n_{\text{dof}}^s + n_{\text{dof}}^p) \times 1} := \underbrace{\left\{ \begin{array}{c} \underbrace{\left\{ \mathbf{z}_u^{m-1}(t_{n+1}) \right\}}_{n_{\text{dof}}^s \times 1} \\ \underbrace{\left\{ \mathbf{z}_v^{m-1}(t_{n+1}) \right\}}_{n_{\text{dof}}^s \times 1} \\ \underbrace{\left\{ \mathbf{z}_{p_f}^{m-1}(t_{n+1}) \right\}}_{n_{\text{dof}}^p \times 1} \end{array} \right\}}_{(2 \times n_{\text{dof}}^s + n_{\text{dof}}^p) \times 1} = \underbrace{\left\{ \begin{array}{c} \underbrace{\left\{ \mathbf{z}_u(t_n) \right\}}_{n_{\text{dof}}^s \times 1} + \Delta t \sum_{i=1}^m b_i^{m-1} \underbrace{\left\{ \mathbf{k}_{i(v)} \right\}}_{n_{\text{dof}}^s \times 1} \\ \underbrace{\left\{ \mathbf{z}_v(t_n) \right\}}_{n_{\text{dof}}^s \times 1} + \Delta t \sum_{i=1}^m b_i^{m-1} \underbrace{\left\{ \mathbf{k}_{i(a)} \right\}}_{n_{\text{dof}}^s \times 1} \\ \underbrace{\left\{ \mathbf{z}_{p_f}(t_n) \right\}}_{n_{\text{dof}}^p \times 1} + \Delta t \sum_{i=1}^m b_i^{m-1} \underbrace{\left\{ \mathbf{k}_{i(p_f)} \right\}}_{n_{\text{dof}}^p \times 1} \end{array} \right\}}_{(2 \times n_{\text{dof}}^s + n_{\text{dof}}^p) \times 1} \quad (304)$$

Then, a truncation error is calculated for all values of the solutions $\mathbf{z}(t_{n+1})$ according to Eq. 236, and the new time step Δt^* is adjusted according to Eq. 237. If the absolute error does not meet the condition defined by Eq. 238, the solutions given by Eqs. 303 and 304 are recomputed for time t_{n+1} using the smaller time step Δt^* in place of Δt (i.e., $\Delta t_n = \Delta t \leftarrow \Delta t^*$) in Eqs. 302–304.

5.2.3 Implementation of the $(\mathbf{u}-\mathbf{u}_f-p_f)$ Theory

For the $(\mathbf{u}-\mathbf{u}_f-p_f)$ formulation, the Runge-Kutta integrators transform the general solution variables given by Eq. 232 to

$$\{\dot{\mathbf{z}}\} := \begin{Bmatrix} \dot{z}_u \\ \dot{z}_v \\ \dot{z}_{u_f} \\ \dot{z}_{v_f} \\ \dot{z}_{p_f} \end{Bmatrix} = \{\mathbf{f}(t, \mathbf{z})\} = \begin{Bmatrix} \mathbf{f}_v(t, \mathbf{z}) \\ \mathbf{f}_a(t, \mathbf{z}) \\ \mathbf{f}_{v_f}(t, \mathbf{z}) \\ \mathbf{f}_{a_f}(t, \mathbf{z}) \\ \mathbf{f}_{p_f}(t, \mathbf{z}) \end{Bmatrix} \quad (305)$$

such that

$$\{\mathbf{z}\} = \begin{Bmatrix} \mathbf{d} \\ \dot{\mathbf{d}} \\ \mathbf{d}_f \\ \dot{\mathbf{d}}_f \\ \boldsymbol{\theta} \end{Bmatrix}, \quad \dot{\mathbf{z}} = \begin{Bmatrix} \dot{\mathbf{d}} \\ \ddot{\mathbf{d}} \\ \dot{\mathbf{d}}_f \\ \ddot{\mathbf{d}}_f \\ \dot{\boldsymbol{\theta}} \end{Bmatrix} \quad (306)$$

Thus, we require at least one governing equation to solve for the primary unknown $\ddot{\mathbf{d}}$, which when integrated once gives us $\dot{\mathbf{d}}$, and when integrated twice gives us \mathbf{d} ; at least one governing equation to solve for the primary unknown $\ddot{\mathbf{d}}_f$, which when integrated once gives us $\dot{\mathbf{d}}_f$, and when integrated twice gives us \mathbf{d}_f ; and at least one governing equation to solve for the primary unknown $\dot{\boldsymbol{\theta}}$, which when integrated once gives us $\boldsymbol{\theta}$.

The FE formulation for the balance of momentum of the mixture, variational Eq. 25, is written in block-matrix form as

$$\underbrace{\{\mathbf{R}_u\}}_{n_{\text{dof}}^s \times 1} = \mathbf{0} \quad (307)$$

where the global residual for the solid skeleton displacement is given as

$$\mathbf{c}^T \cdot \mathbf{R}_u = \mathcal{G}^h = \mathcal{G}_1^{\text{INT},h} + \mathcal{G}_2^{\text{INT},h} + \mathcal{G}_3^{\text{INT},h} + \mathcal{G}_4^{\text{INT},h} - \mathcal{G}^{\text{EXT},h} = 0 \quad (308)$$

where

$$\mathcal{G}_1^{\text{INT},h} = \mathbf{A}_e \underbrace{\left\{ \mathbf{c}^e \right\}^T}_{1 \times n_{\text{dof}}^{s,e}} \cdot \left(\underbrace{\left[\mathbf{m}_{u,u}^{\mathcal{G}_1^{\text{INT},e}} \right]}_{n_{\text{dof}}^{s,e} \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\left\{ \ddot{\mathbf{d}}^e \right\}}_{n_{\text{dof}}^{s,e} \times 1} + \underbrace{\left\{ \mathbf{f}^{\mathcal{G}_1^{\text{INT},e}} \right\}}_{n_{\text{dof}}^{s,e} \times 1} \right) \quad (309)$$

$$\mathcal{G}_2^{\text{INT},h} = \mathbf{A}_e \underbrace{\left\{ \mathbf{c}^e \right\}^T}_{1 \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\left\{ \mathbf{f}^{\mathcal{G}_2^{\text{INT},e}} \right\}}_{n_{\text{dof}}^{s,e} \times 1} \quad (310)$$

$$\mathcal{G}_3^{\text{INT},h} = \mathbf{A}_e \underbrace{\left\{ \mathbf{c}^e \right\}^T}_{1 \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\left\{ \mathbf{f}^{\mathcal{G}_3^{\text{INT},e}} \right\}}_{n_{\text{dof}}^{s,e} \times 1} \quad (311)$$

$$\mathcal{G}_4^{\text{INT},h} = \mathbf{A}_e \underbrace{\left\{ \mathbf{c}^e \right\}^T}_{1 \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\left\{ \mathbf{f}^{\mathcal{G}_4^{\text{INT},e}} \right\}}_{n_{\text{dof}}^{s,e} \times 1} \quad (312)$$

$$\mathcal{G}^{\text{EXT},h} = \mathbf{A}_e \underbrace{\left\{ \mathbf{c}^e \right\}^T}_{1 \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\left\{ \mathbf{f}^{\mathcal{G}^{\text{EXT},e}} \right\}}_{n_{\text{dof}}^{s,e} \times 1} \quad (313)$$

with

$$\underbrace{\left\{ \mathbf{f}^{\mathcal{G}_1^{\text{INT},e}} \right\}}_{n_{\text{dof}}^{s,e} \times 1} = \int_{-1}^1 \underbrace{\left\{ \mathbf{N}^{e,u} \right\}^T}_{n_{\text{dof}}^{s,e} \times 1} \rho_0^{f,h^e} a_{\text{f}}^{h^e} A_j^e d\xi \quad (314)$$

$$\underbrace{\left\{ \mathbf{f}^{\mathcal{G}_2^{\text{INT},e}} \right\}}_{n_{\text{dof}}^{s,e} \times 1} = \int_{-1}^1 \underbrace{\left\{ \mathbf{B}^{e,u} \right\}^T}_{n_{\text{dof}}^{s,e} \times 1} P_{11(E)}^{s,h^e} A_j^e d\xi \quad (315)$$

$$\underbrace{\left\{ \mathbf{f}^{\mathcal{G}_2^{\text{INT},e}} \right\}}_{n_{\text{dof}}^{s,e} \times 1} = - \int_{-1}^1 \underbrace{\left\{ \mathbf{B}^{e,u} \right\}^T}_{n_{\text{dof}}^{s,e} \times 1} p_{\text{f}}^{h^e} A_j^e d\xi \quad (316)$$

$$\underbrace{\left\{ \mathbf{f}^{\mathcal{G}_4^{\text{INT},e}} \right\}}_{n_{\text{dof}}^{s,e} \times 1} = \int_{-1}^1 \underbrace{\left\{ \mathbf{N}^{e,u} \right\}^T}_{n_{\text{dof}}^{s,e} \times 1} \rho_0^{h^e} g A_j^e d\xi \quad (317)$$

$$\underbrace{\left\{ \mathbf{f}^{\mathcal{G}^{\text{EXT},e}} \right\}}_{n_{\text{dof}}^{s,e} \times 1} = \begin{cases} \underbrace{\left\{ \mathbf{N}^{e,u}(X=H) \right\}^T}_{n_{\text{dof}}^{s,e} \times 1} t^\sigma A & X = H \\ \mathbf{0} & 0 \leq X < H \end{cases} \quad (318)$$

The mass matrix associated with the solid skeleton acceleration is given by

$$\underbrace{\left[\mathbf{m}_{u,u}^{G_{1,u}^{\text{INT},e}} \right]}_{n_{\text{dof}}^{s,e} \times n_{\text{dof}}^{s,e}} = \int_{-1}^1 \rho_0^{s,h^e} \underbrace{\left\{ \mathbf{N}^{e,u} \right\}^T}_{n_{\text{dof}}^{s,e} \times 1} \underbrace{\left\{ \mathbf{N}^{e,u} \right\}}_{1 \times n_{\text{dof}}^{s,e}} A_j^e d\xi \quad (319)$$

The FE formulation for the balance of mass variational Eq. 3 is written in block-matrix form as

$$\underbrace{\left\{ \mathbf{R}_{p_f} \right\}}_{n_{\text{dof}}^f \times 1} = \mathbf{0} \quad (320)$$

where the global residual for the pore fluid pressure is given as

$$\boldsymbol{\alpha}^T \cdot \mathbf{R}_{p_f} = \mathcal{H}^h = \mathcal{H}_1^{\text{INT},h} + \mathcal{H}_2^{\text{INT},h} + \mathcal{H}_3^{\text{INT},h} + \mathcal{H}_4^{\text{INT},h} - \mathcal{H}_1^{\text{EXT},h} = 0 \quad (321)$$

where

$$\mathcal{H}_1^{\text{INT},h} = \underbrace{\mathbf{A}}_e \underbrace{\left\{ \boldsymbol{\alpha}^e \right\}^T}_{1 \times n_{\text{dof}}^{p,e}} \cdot \left(\underbrace{\left[\mathbf{m}_{p_f,p_f}^{\mathcal{H}_1^{\text{INT},e}} \right]}_{n_{\text{dof}}^{p,e} \times n_{\text{dof}}^{p,e}} \cdot \underbrace{\left\{ \dot{\boldsymbol{\theta}}^e \right\}}_{n_{\text{dof}}^{p,e} \times 1} + \underbrace{\left\{ \mathbf{f}^{\mathcal{H}_1^{\text{INT},e}} \right\}}_{n_{\text{dof}}^{p,e} \times 1} \right) \quad (322)$$

$$\mathcal{H}_2^{\text{INT},h} = \underbrace{\mathbf{A}}_e \underbrace{\left\{ \boldsymbol{\alpha}^e \right\}^T}_{1 \times n_{\text{dof}}^{p,e}} \cdot \underbrace{\left\{ \mathbf{f}^{\mathcal{H}_2^{\text{INT},e}} \right\}}_{n_{\text{dof}}^{p,e} \times 1} \quad (323)$$

$$\mathcal{H}_3^{\text{INT},h} = \underbrace{\mathbf{A}}_e \underbrace{\left\{ \boldsymbol{\alpha}^e \right\}^T}_{1 \times n_{\text{dof}}^{p,e}} \cdot \underbrace{\left\{ \mathbf{f}^{\mathcal{H}_3^{\text{INT},e}} \right\}}_{n_{\text{dof}}^{p,e} \times 1} \quad (324)$$

$$\mathcal{H}_4^{\text{INT},h} = \underbrace{\mathbf{A}}_e \underbrace{\left\{ \boldsymbol{\alpha}^e \right\}^T}_{1 \times n_{\text{dof}}^{p,e}} \cdot \underbrace{\left\{ \mathbf{f}^{\mathcal{H}_4^{\text{INT},e}} \right\}}_{n_{\text{dof}}^{p,e} \times 1} \quad (325)$$

$$\mathcal{H}^{\text{EXT},h} = \underbrace{\mathbf{A}}_e \underbrace{\left\{ \boldsymbol{\alpha}^e \right\}^T}_{1 \times n_{\text{dof}}^{p,e}} \cdot \underbrace{\left\{ \mathbf{f}^{\mathcal{H}^{\text{EXT},e}} \right\}}_{n_{\text{dof}}^{p,e} \times 1} \quad (326)$$

with

$$\underbrace{\left\{ \mathbf{f}^{\mathcal{H}_1^{\text{INT}},e} \right\}}_{n_{\text{dof}}^{p,e} \times 1} = \int_{-1}^1 \underbrace{\left\{ \mathbf{N}^{e,p} \right\}^T}_{n_{\text{dof}}^{p,e} \times 1} j^{h^e} A_j^e d\xi \quad (327)$$

$$\underbrace{\left\{ \mathbf{f}^{\mathcal{H}_2^{\text{INT}},e} \right\}}_{n_{\text{dof}}^{p,e} \times 1} = \int_{-1}^1 \underbrace{\left\{ \mathbf{N}^{e,p} \right\}^T}_{n_{\text{dof}}^{p,e} \times 1} \frac{1}{K_f^\eta} \frac{\partial p_f^{h^e}}{\partial X} (n^{f,h^e} \tilde{v}_f^{h^e}) A_j^e d\xi \quad (328)$$

$$\underbrace{\left\{ \mathbf{f}^{\mathcal{H}_3^{\text{INT}},e} \right\}}_{n_{\text{dof}}^{p,e} \times 1} = \int_{-1}^1 \underbrace{\left\{ \mathbf{B}^{e,p} \right\}^T}_{n_{\text{dof}}^{p,e} \times 1} \hat{k}^{h^e} \frac{\partial p_f^{h^e}}{\partial X} (F_{11}^{h^e})^{-1} A_j^e d\xi \quad (329)$$

$$\underbrace{\left\{ \mathbf{f}^{\mathcal{H}_4^{\text{INT}},e} \right\}}_{n_{\text{dof}}^{p,e} \times 1} = \int_{-1}^1 \underbrace{\left\{ \mathbf{B}^{e,p} \right\}^T}_{n_{\text{dof}}^{p,e} \times 1} \hat{k}^{h^e} \rho^{\text{fR},h^e} (a_f^{h^e} + g) A_j^e d\xi \quad (330)$$

$$\underbrace{\left\{ \mathbf{f}^{\mathcal{H}^{\text{EXT}},e} \right\}}_{n_{\text{dof}}^{p,e} \times 1} = \begin{cases} \underbrace{\left\{ \mathbf{N}^{e,p}(X=H, X=0) \right\}^T}_{n_{\text{dof}}^{p,e} \times 1} Q_f A & X=H, X=0 \\ \mathbf{0} & 0 < X < H \end{cases} \quad (331)$$

The mass matrix associated with the pore fluid pressure is given by

$$\underbrace{\left[\mathbf{m}_{p_f, p_f}^{\mathcal{H}_1^{\text{INT}},e} \right]}_{n_{\text{dof}}^{p,e} \times n_{\text{dof}}^{p,e}} = \int_{-1}^1 \frac{J^{h^e} n^{f,h^e}}{K_f^\eta} \underbrace{\left\{ \mathbf{N}^{e,p} \right\}^T}_{n_{\text{dof}}^{p,e} \times 1} \underbrace{\left\{ \mathbf{N}^{e,p} \right\}}_{1 \times n_{\text{dof}}^{p,e}} A_j^e d\xi \quad (332)$$

When pressure stabilization is enabled, an additional term $\mathcal{H}^{\text{stab}}$ is added to the LHS of Eq. 321 and is defined as

$$\mathcal{H}^{\text{stab}} = \underbrace{\mathbf{A}^e}_{n_{\text{dof}}^{p,e} \times n_{\text{dof}}^{p,e}} \alpha^{e,T} \cdot \underbrace{\left[\mathbf{m}_{p_f, p_f}^{\mathcal{H}^{\text{stab}},e} \right]}_{n_{\text{dof}}^{p,e} \times n_{\text{dof}}^{p,e}} \cdot \underbrace{\left\{ \dot{\boldsymbol{\theta}}^e \right\}}_{n_{\text{dof}}^{p,e} \times 1} \quad (333)$$

where

$$\underbrace{\left[\mathbf{m}_{p_f, p_f}^{\mathcal{H}^{\text{stab}},e} \right]}_{n_{\text{dof}}^{p,e} \times n_{\text{dof}}^{p,e}} = \int_{-1}^1 \alpha^{\text{stab}} (F_{11}^{h^e})^{-1} \underbrace{\left\{ \mathbf{B}^{e,p} \right\}^T}_{n_{\text{dof}}^{p,e} \times 1} \underbrace{\left\{ \mathbf{B}^{e,p} \right\}}_{1 \times n_{\text{dof}}^{p,e}} A_j^e d\xi \quad (334)$$

The FE formulation for the balance of momentum of the fluid variational Eq. 51 is written in block-matrix form as

$$\underbrace{\{\mathbf{R}_{u_f}\}}_{n_{\text{dof}}^f \times 1} = \mathbf{0} \quad (335)$$

where the global residual for the pore fluid displacement is given as

$$\boldsymbol{\omega}^T \cdot \mathbf{R}_{u_f} = \mathcal{I}^h = \mathcal{I}_1^{\text{INT},h} + \mathcal{I}_2^{\text{INT},h} + \mathcal{I}_3^{\text{INT},h} + \mathcal{I}_4^{\text{INT},h} = 0 \quad (336)$$

where

$$\mathcal{I}_1^{\text{INT},h} = \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{f,e}} \underbrace{\{\boldsymbol{\omega}^e\}^T}_{n_{\text{dof}}^{f,e} \times n_{\text{dof}}^{f,e}} \cdot \underbrace{\left[\mathbf{m}_{u_f, u_f}^{\mathcal{I}_1^{\text{INT},e}} \right]}_{n_{\text{dof}}^{f,e} \times n_{\text{dof}}^{f,e}} \cdot \underbrace{\{\mathbf{d}_f^e\}}_{n_{\text{dof}}^{f,e} \times 1} \quad (337)$$

$$\mathcal{I}_2^{\text{INT},h} = \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{f,e}} \underbrace{\{\boldsymbol{\omega}^e\}^T}_{n_{\text{dof}}^{f,e} \times 1} \cdot \underbrace{\{\mathbf{f}^{\mathcal{I}_2^{\text{INT},e}}\}}_{n_{\text{dof}}^{f,e} \times 1} \quad (338)$$

$$\mathcal{I}_3^{\text{INT},h} = \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{f,e}} \underbrace{\{\boldsymbol{\omega}^e\}^T}_{n_{\text{dof}}^{f,e} \times 1} \cdot \underbrace{\{\mathbf{f}^{\mathcal{I}_3^{\text{INT},e}}\}}_{n_{\text{dof}}^{f,e} \times 1} \quad (339)$$

$$\mathcal{I}_4^{\text{INT},h} = \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{f,e}} \underbrace{\{\boldsymbol{\omega}^e\}^T}_{n_{\text{dof}}^{f,e} \times 1} \cdot \underbrace{\{\mathbf{f}^{\mathcal{I}_4^{\text{INT},e}}\}}_{n_{\text{dof}}^{f,e} \times 1} \quad (340)$$

with

$$\underbrace{\{\mathbf{f}^{\mathcal{I}_2^{\text{INT},e}}\}}_{n_{\text{dof}}^{f,e} \times 1} = \int_{-1}^1 \underbrace{\{\mathbf{N}^{e, u_f}\}^T}_{n_{\text{dof}}^{f,e} \times 1} n^{f, h^e} \frac{\partial p_f^{h^e}}{\partial X} A_j^e d\xi \quad (341)$$

$$\underbrace{\{\mathbf{f}^{\mathcal{I}_3^{\text{INT},e}}\}}_{n_{\text{dof}}^{f,e} \times 1} = \int_{-1}^1 \underbrace{\{\mathbf{N}^{e, u_f}\}^T}_{n_{\text{dof}}^{f,e} \times 1} \frac{J^{h^e} (n^{f, h^e})^2}{K_f^\eta} (v_f^{h^e} - v^{h^e}) A_j^e d\xi \quad (342)$$

$$\underbrace{\{\mathbf{f}^{\mathcal{I}_4^{\text{INT},e}}\}}_{n_{\text{dof}}^{f,e} \times 1} = \int_{-1}^1 \underbrace{\{\mathbf{N}^{e, u_f}\}^T}_{n_{\text{dof}}^{f,e} \times 1} \rho_0^{f, h^e} g A_j^e d\xi \quad (343)$$

The mass matrix associated with the pore fluid acceleration is given by

$$\underbrace{\left[\mathbf{m}_{u_f, u_f}^{\mathcal{I}_1^{\text{INT}, e}} \right]}_{n_{\text{dof}}^{f,e} \times n_{\text{dof}}^{f,e}} = \int_{-1}^1 \rho_0^{f,h^e} \underbrace{\left\{ \mathbf{N}^{e, u_f} \right\}^T}_{n_{\text{dof}}^{f,e} \times 1} \underbrace{\left\{ \mathbf{N}^{e, u_f} \right\}}_{1 \times n_{\text{dof}}^{f,e}} A_j^e d\xi \quad (344)$$

Then, returning our attention to Eq. 305, we have

$$\underbrace{\left\{ \dot{\mathbf{z}} \right\}}_{(2 \times n_{\text{dof}}^s + 2 \times n_{\text{dof}}^f + n_{\text{dof}}^p) \times 1} = \left\{ \begin{array}{c} \underbrace{\left\{ \dot{\mathbf{d}} \right\}}_{n_{\text{dof}}^s \times 1} \\ \underbrace{\left[\mathbf{M}_{u, u}^{\mathcal{G}_1^{\text{INT}}} \right]^{-1}}_{n_{\text{dof}}^s \times n_{\text{dof}}^s} \cdot \\ \left(- \underbrace{\left\{ \mathbf{F}^{\mathcal{G}_1^{\text{INT}}} \right\}}_{n_{\text{dof}}^s \times 1} - \underbrace{\left\{ \mathbf{F}^{\mathcal{G}_2^{\text{INT}}} \right\}}_{n_{\text{dof}}^s \times 1} - \underbrace{\left\{ \mathbf{F}^{\mathcal{G}_3^{\text{INT}}} \right\}}_{n_{\text{dof}}^s \times 1} - \right. \\ \left. \underbrace{\left\{ \mathbf{F}^{\mathcal{G}_4^{\text{INT}}} \right\}}_{n_{\text{dof}}^s \times 1} + \underbrace{\left\{ \mathbf{F}^{\mathcal{G}^{\text{EXT}}} \right\}}_{n_{\text{dof}}^s \times 1} \right) \\ \underbrace{\left\{ \dot{\mathbf{d}}_f \right\}}_{n_{\text{dof}}^f \times 1} \\ \underbrace{\left[\mathbf{M}_{u_f, u_f}^{\mathcal{I}_1^{\text{INT}}} \right]^{-1}}_{n_{\text{dof}}^f \times n_{\text{dof}}^f} \cdot \\ \left(- \underbrace{\left\{ \mathbf{F}^{\mathcal{I}_2^{\text{INT}}} \right\}}_{n_{\text{dof}}^f \times 1} - \underbrace{\left\{ \mathbf{F}^{\mathcal{I}_3^{\text{INT}}} \right\}}_{n_{\text{dof}}^f \times 1} - \underbrace{\left\{ \mathbf{F}^{\mathcal{I}_4^{\text{INT}}} \right\}}_{n_{\text{dof}}^f \times 1} \right) \\ \underbrace{\left[\mathbf{M}_{p_f, p_f}^{\mathcal{H}_1^{\text{INT}}} \right]^{-1}}_{n_{\text{dof}}^p \times n_{\text{dof}}^p} \cdot \left(- \underbrace{\left\{ \mathbf{F}^{\mathcal{H}_1^{\text{INT}}} \right\}}_{n_{\text{dof}}^p \times 1} - \underbrace{\left\{ \mathbf{F}^{\mathcal{H}_2^{\text{INT}}} \right\}}_{n_{\text{dof}}^p \times 1} - \underbrace{\left\{ \mathbf{F}^{\mathcal{H}_3^{\text{INT}}} \right\}}_{n_{\text{dof}}^p \times 1} - \right. \\ \left. \underbrace{\left\{ \mathbf{F}^{\mathcal{H}_4^{\text{INT}}} \right\}}_{n_{\text{dof}}^p \times 1} + \underbrace{\left\{ \mathbf{F}^{\mathcal{H}^{\text{EXT}}} \right\}}_{n_{\text{dof}}^p \times 1} \right) \end{array} \right\} \quad (345)$$

where

$$\underbrace{\left[M_{u,u}^{\mathcal{G}^{\text{INT}}} \right]}_{n_{\text{dof}}^s \times n_{\text{dof}}^s} = \mathbf{A}_e^{n_e} \underbrace{\left[m_{u,u}^{\mathcal{G}^{\text{INT},e}} \right]}_{n_{\text{dof}}^{s,e} \times n_{\text{dof}}^{s,e}} \quad (346)$$

$$\underbrace{\left\{ \mathbf{F}^{\mathcal{G}_1^{\text{INT}}} \right\}}_{n_{\text{dof}}^s \times 1} = \mathbf{A}_e^{n_e} \underbrace{\left\{ \mathbf{f}^{\mathcal{G}_1^{\text{INT},e}} \right\}}_{n_{\text{dof}}^{s,e} \times 1} \quad (347)$$

$$\underbrace{\left\{ \mathbf{F}^{\mathcal{G}_2^{\text{INT}}} \right\}}_{n_{\text{dof}}^s \times 1} = \mathbf{A}_e^{n_e} \underbrace{\left\{ \mathbf{f}^{\mathcal{G}_2^{\text{INT},e}} \right\}}_{n_{\text{dof}}^{s,e} \times 1} \quad (348)$$

$$\underbrace{\left\{ \mathbf{F}^{\mathcal{G}_2^{\text{INT}}} \right\}}_{n_{\text{dof}}^s \times 1} = \mathbf{A}_e^{n_e} \underbrace{\left\{ \mathbf{f}^{\mathcal{G}_3^{\text{INT},e}} \right\}}_{n_{\text{dof}}^{s,e} \times 1} \quad (349)$$

$$\underbrace{\left\{ \mathbf{F}^{\mathcal{G}_4^{\text{INT}}} \right\}}_{n_{\text{dof}}^s \times 1} = \mathbf{A}_e^{n_e} \underbrace{\left\{ \mathbf{f}^{\mathcal{G}_4^{\text{INT},e}} \right\}}_{n_{\text{dof}}^{s,e} \times 1} \quad (350)$$

$$\underbrace{\left\{ \mathbf{F}^{\mathcal{G}^{\text{EXT}}} \right\}}_{n_{\text{dof}}^s \times 1} = \mathbf{A}_e^{n_e} \underbrace{\left\{ \mathbf{f}^{\mathcal{G}^{\text{EXT},e}} \right\}}_{n_{\text{dof}}^{s,e} \times 1} \quad (351)$$

$$\underbrace{\left[M_{u_f, u_f}^{\mathcal{T}^{\text{INT}}} \right]}_{n_{\text{dof}}^f \times n_{\text{dof}}^f} = \mathbf{A}_e^{n_e} \underbrace{\left[m_{u_f, u_f}^{\mathcal{T}^{\text{INT},e}} \right]}_{n_{\text{dof}}^{f,e} \times n_{\text{dof}}^{f,e}} \quad (352)$$

$$\underbrace{\left\{ \mathbf{F}^{\mathcal{T}_2^{\text{INT}}} \right\}}_{n_{\text{dof}}^f \times 1} = \mathbf{A}_e^{n_e} \underbrace{\left\{ \mathbf{f}^{\mathcal{T}_2^{\text{INT},e}} \right\}}_{n_{\text{dof}}^{f,e} \times 1} \quad (353)$$

$$\underbrace{\left\{ \mathbf{F}^{\mathcal{T}_2^{\text{INT}}} \right\}}_{n_{\text{dof}}^f \times 1} = \mathbf{A}_e^{n_e} \underbrace{\left\{ \mathbf{f}^{\mathcal{T}_3^{\text{INT},e}} \right\}}_{n_{\text{dof}}^{f,e} \times 1} \quad (354)$$

$$\underbrace{\left\{ \mathbf{F}^{\mathcal{T}_4^{\text{INT}}} \right\}}_{n_{\text{dof}}^f \times 1} = \mathbf{A}_e^{n_e} \underbrace{\left\{ \mathbf{f}^{\mathcal{T}_4^{\text{INT},e}} \right\}}_{n_{\text{dof}}^{f,e} \times 1} \quad (355)$$

$$\underbrace{\left[\mathbf{M}_{p_f, p_f}^{\mathcal{H}_1^{\text{INT}}} \right]}_{n_{\text{dof}}^p \times n_{\text{dof}}^p} = \underbrace{\mathbf{A}_e}_{n_e} \underbrace{\left[\mathbf{m}_{p_f, p_f}^{\mathcal{H}_1^{\text{INT}, e}} \right]}_{n_{\text{dof}}^{p, e} \times n_{\text{dof}}^{p, e}} \quad (356)$$

$$\underbrace{\left\{ \mathbf{F}_1^{\mathcal{H}_1^{\text{INT}}} \right\}}_{n_{\text{dof}}^p \times 1} = \underbrace{\mathbf{A}_e}_{n_e} \underbrace{\left\{ \mathbf{f}_1^{\mathcal{H}_1^{\text{INT}, e}} \right\}}_{n_{\text{dof}}^{p, e} \times 1} \quad (357)$$

$$\underbrace{\left\{ \mathbf{F}_2^{\mathcal{H}_2^{\text{INT}}} \right\}}_{n_{\text{dof}}^p \times 1} = \underbrace{\mathbf{A}_e}_{n_e} \underbrace{\left\{ \mathbf{f}_2^{\mathcal{H}_2^{\text{INT}, e}} \right\}}_{n_{\text{dof}}^{p, e} \times 1} \quad (358)$$

$$\underbrace{\left\{ \mathbf{F}_3^{\mathcal{H}_3^{\text{INT}}} \right\}}_{n_{\text{dof}}^p \times 1} = \underbrace{\mathbf{A}_e}_{n_e} \underbrace{\left\{ \mathbf{f}_3^{\mathcal{H}_3^{\text{INT}, e}} \right\}}_{n_{\text{dof}}^{p, e} \times 1} \quad (359)$$

$$\underbrace{\left\{ \mathbf{F}_4^{\mathcal{H}_4^{\text{INT}}} \right\}}_{n_{\text{dof}}^p \times 1} = \underbrace{\mathbf{A}_e}_{n_e} \underbrace{\left\{ \mathbf{f}_4^{\mathcal{H}_4^{\text{INT}, e}} \right\}}_{n_{\text{dof}}^{p, e} \times 1} \quad (360)$$

$$\underbrace{\left\{ \mathbf{F}^{\mathcal{H}^{\text{EXT}}} \right\}}_{n_{\text{dof}}^p \times 1} = \underbrace{\mathbf{A}_e}_{n_e} \underbrace{\left\{ \mathbf{f}^{\mathcal{H}^{\text{EXT}, e}} \right\}}_{n_{\text{dof}}^{p, e} \times 1} \quad (361)$$

If pressure stabilization is enabled, refer to the addition given by Eq. 301 and insert into Eq. 345 as necessary. This was done in Eq. 300.

As described in Section 5.1.6, Eq. 345 is solved for each stage increment i at time $t_n + \Delta t c_i$ (i.e., any variables that are *explicit* functions of time, such as a time-dependent external traction, are to be evaluated at time $t_n + \Delta t c_i$), with stage solu-

tion \mathbf{k}_i given by

$$\underbrace{\left\{ \mathbf{k}_i \right\}}_{(2 \times n_{\text{dof}}^s + 2 \times n_{\text{dof}}^f + n_{\text{dof}}^p) \times 1} := \left\{ \begin{array}{l} \underbrace{\left\{ \mathbf{k}_{i(v)} \right\}}_{n_{\text{dof}}^s \times 1} \\ \underbrace{\left\{ \mathbf{k}_{i(a)} \right\}}_{n_{\text{dof}}^s \times 1} \\ \underbrace{\left\{ \mathbf{k}_{i(v_f)} \right\}}_{n_{\text{dof}}^f \times 1} \\ \underbrace{\left\{ \mathbf{k}_{i(a_f)} \right\}}_{n_{\text{dof}}^f \times 1} \\ \underbrace{\left\{ \mathbf{k}_{i(\dot{p}_f)} \right\}}_{n_{\text{dof}}^p \times 1} \end{array} \right\} = \left\{ \begin{array}{l} \underbrace{\left\{ \mathbf{z}_v(t_n) \right\}}_{n_{\text{dof}}^s \times 1} + \sum_{j=1}^{i-1} a_{ij} \underbrace{\left\{ \mathbf{k}_{j(v)} \right\}}_{n_{\text{dof}}^s \times 1} \\ \underbrace{\left\{ \mathbf{z}_a(t_n) \right\}}_{n_{\text{dof}}^s \times 1} + \sum_{j=1}^{i-1} a_{ij} \underbrace{\left\{ \mathbf{k}_{j(a)} \right\}}_{n_{\text{dof}}^s \times 1} \\ \underbrace{\left\{ \mathbf{z}_{v_f}(t_n) \right\}}_{n_{\text{dof}}^f \times 1} + \sum_{j=1}^{i-1} a_{ij} \underbrace{\left\{ \mathbf{k}_{j(v_f)} \right\}}_{n_{\text{dof}}^f \times 1} \\ \underbrace{\left\{ \mathbf{z}_{a_f}(t_n) \right\}}_{n_{\text{dof}}^f \times 1} + \sum_{j=1}^{i-1} a_{ij} \underbrace{\left\{ \mathbf{k}_{j(a_f)} \right\}}_{n_{\text{dof}}^f \times 1} \\ \underbrace{\left\{ \mathbf{z}_{\dot{p}_f}(t_n) \right\}}_{n_{\text{dof}}^p \times 1} + \sum_{j=1}^{i-1} a_{ij} \underbrace{\left\{ \mathbf{k}_{j(\dot{p}_f)} \right\}}_{n_{\text{dof}}^p \times 1} \end{array} \right\} \quad (362)$$

Then, according to Eq. 234, the higher order solution to be accepted or rejected at time t_{n+1} is given by

$$\underbrace{\left\{ \mathbf{z}^m(t_{n+1}) \right\}}_{(2 \times n_{\text{dof}}^s + 2 \times n_{\text{dof}}^f + n_{\text{dof}}^p) \times 1} := \left\{ \begin{array}{l} \underbrace{\left\{ \mathbf{z}_u^m(t_{n+1}) \right\}}_{n_{\text{dof}}^s \times 1} \\ \underbrace{\left\{ \mathbf{z}_v^m(t_{n+1}) \right\}}_{n_{\text{dof}}^s \times 1} \\ \underbrace{\left\{ \mathbf{z}_{u_f}^m(t_{n+1}) \right\}}_{n_{\text{dof}}^f \times 1} \\ \underbrace{\left\{ \mathbf{z}_{v_f}^m(t_{n+1}) \right\}}_{n_{\text{dof}}^f \times 1} \\ \underbrace{\left\{ \mathbf{z}_{\dot{p}_f}^m(t_{n+1}) \right\}}_{n_{\text{dof}}^p \times 1} \end{array} \right\} = \left\{ \begin{array}{l} \underbrace{\left\{ \mathbf{z}_u(t_n) \right\}}_{n_{\text{dof}}^s \times 1} + \Delta t \sum_{i=1}^{m+1} b_i^m \underbrace{\left\{ \mathbf{k}_{i(v)} \right\}}_{n_{\text{dof}}^s \times 1} \\ \underbrace{\left\{ \mathbf{z}_v(t_n) \right\}}_{n_{\text{dof}}^s \times 1} + \Delta t \sum_{i=1}^{m+1} b_i^m \underbrace{\left\{ \mathbf{k}_{i(a)} \right\}}_{n_{\text{dof}}^s \times 1} \\ \underbrace{\left\{ \mathbf{z}_{u_f}(t_n) \right\}}_{n_{\text{dof}}^f \times 1} + \Delta t \sum_{i=1}^{m+1} b_i^m \underbrace{\left\{ \mathbf{k}_{i(v_f)} \right\}}_{n_{\text{dof}}^f \times 1} \\ \underbrace{\left\{ \mathbf{z}_{v_f}(t_n) \right\}}_{n_{\text{dof}}^f \times 1} + \Delta t \sum_{i=1}^{m+1} b_i^m \underbrace{\left\{ \mathbf{k}_{i(a_f)} \right\}}_{n_{\text{dof}}^f \times 1} \\ \underbrace{\left\{ \mathbf{z}_{\dot{p}_f}(t_n) \right\}}_{n_{\text{dof}}^p \times 1} + \Delta t \sum_{i=1}^{m+1} b_i^m \underbrace{\left\{ \mathbf{k}_{i(\dot{p}_f)} \right\}}_{n_{\text{dof}}^p \times 1} \end{array} \right\} \quad (363)$$

and, according to Eq. 235, the lower order solution at time t_{n+1} is given by

$$\underbrace{\left\{ \mathbf{z}^{m-1}(t_{n+1}) \right\}}_{(2 \times n_{\text{dof}}^s + 2 \times n_{\text{dof}}^f + n_{\text{dof}}^p) \times 1} := \left\{ \begin{array}{l} \underbrace{\left\{ \mathbf{z}_u^{m-1}(t_{n+1}) \right\}}_{n_{\text{dof}}^s \times 1} \\ \underbrace{\left\{ \mathbf{z}_v^{m-1}(t_{n+1}) \right\}}_{n_{\text{dof}}^s \times 1} \\ \underbrace{\left\{ \mathbf{z}_{u_f}^{m-1}(t_{n+1}) \right\}}_{n_{\text{dof}}^f \times 1} \\ \underbrace{\left\{ \mathbf{z}_{v_f}^{m-1}(t_{n+1}) \right\}}_{n_{\text{dof}}^f \times 1} \\ \underbrace{\left\{ \mathbf{z}_{p_f}^{m-1}(t_{n+1}) \right\}}_{n_{\text{dof}}^p \times 1} \end{array} \right\} \quad (364)$$

$$= \left\{ \begin{array}{l} \underbrace{\left\{ \mathbf{z}_u(t_n) \right\}}_{n_{\text{dof}}^s \times 1} + \Delta t \sum_{i=1}^m b_i^{m-1} \underbrace{\left\{ \mathbf{k}_{i(v)} \right\}}_{n_{\text{dof}}^s \times 1} \\ \underbrace{\left\{ \mathbf{z}_v(t_n) \right\}}_{n_{\text{dof}}^s \times 1} + \Delta t \sum_{i=1}^m b_i^{m-1} \underbrace{\left\{ \mathbf{k}_{i(a)} \right\}}_{n_{\text{dof}}^s \times 1} \\ \underbrace{\left\{ \mathbf{z}_{u_f}(t_n) \right\}}_{n_{\text{dof}}^f \times 1} + \Delta t \sum_{i=1}^m b_i^{m-1} \underbrace{\left\{ \mathbf{k}_{i(v_f)} \right\}}_{n_{\text{dof}}^f \times 1} \\ \underbrace{\left\{ \mathbf{z}_{v_f}(t_n) \right\}}_{n_{\text{dof}}^f \times 1} + \Delta t \sum_{i=1}^m b_i^{m-1} \underbrace{\left\{ \mathbf{k}_{i(a_f)} \right\}}_{n_{\text{dof}}^f \times 1} \\ \underbrace{\left\{ \mathbf{z}_{p_f}(t_n) \right\}}_{n_{\text{dof}}^p \times 1} + \Delta t \sum_{i=1}^m b_i^{m-1} \underbrace{\left\{ \mathbf{k}_{i(p_f)} \right\}}_{n_{\text{dof}}^p \times 1} \end{array} \right\} \quad (365)$$

Then, a truncation error is calculated for all values of the solutions $\mathbf{z}(t_{n+1})$ according to Eq. 236, and the new time step Δt^* is adjusted according to Eq. 237. If the absolute error does not meet the condition defined by Eq. 238, the solutions given by Eqs. 363 and 304 are recomputed at time t_{n+1} using the smaller time step Δt^* in place of Δt in Eqs. 362–364.

5.3 Central Difference Time Integrator

For the CD in time integrator, the discretized balance equations take the similar general form as

$$\mathbf{M}\ddot{\mathbf{x}}_{n+1} + \mathbf{C}\dot{\mathbf{x}}_{n+1} + \mathbf{F}^{\text{INT}}(\ddot{\mathbf{x}}_{n+1}, \mathbf{x}_{n+1}) = \mathbf{F}_{n+1}^{\text{EXT}} \quad (366)$$

where it is assumed that $\ddot{\mathbf{x}}_{n+1}$ is to be solved at time t_{n+1} , and \mathbf{M} , \mathbf{C} , \mathbf{F}^{INT} and \mathbf{F}^{EXT} are known at time t_{n+1} from values at t_n . For a CD scheme,

$$\mathbf{x}_{n+1} = \mathbf{x}_n + \Delta t \dot{\mathbf{x}}_n + \frac{(\Delta t)^2}{2} \ddot{\mathbf{x}}_n \quad (367)$$

$$\dot{\mathbf{x}}_{n+1} = \dot{\mathbf{x}}_n + \frac{\Delta t}{2} (\ddot{\mathbf{x}}_n + \ddot{\mathbf{x}}_{n+1}) \quad (368)$$

Next, we show how these are implemented for the (\mathbf{u}) and $(\mathbf{u}-p_f)$ formulations; as we will show in a subsequent report,²⁹ the CD integrator as applied to the $(\mathbf{u}-p_f)$ formulation is unstable for shock loading use; thus, we did not formulate the integrator for the $(\mathbf{u}-\mathbf{u}_f-p_f)$ formulation which only adds more complexity and potential instability.

5.3.1 Implementation of the (u) Theory

The FE formulation for the balance of momentum of the solid, variational Eq. 28, is written in block-matrix form as

$$\underbrace{\left\{ \mathbf{R}_u \right\}}_{n_{\text{dof}}^s \times 1} = \mathbf{0} \quad (369)$$

where this global residual for the solid displacement is given as

$$\mathbf{c}^T \cdot \mathbf{R}_u = \mathcal{G}^h = \mathcal{G}_1^{\text{INT},h} + \mathcal{G}_2^{\text{INT},h} + \mathcal{G}_4^{\text{INT},h} - \mathcal{G}^{\text{EXT},h} = 0 \quad (370)$$

where

$$\mathcal{G}_1^{\text{INT},h} = \mathbf{A}_e^{n_e} \underbrace{\left\{ \mathbf{c}^e \right\}^T}_{1 \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\left[\mathbf{m}_{u,u}^{\mathcal{G}_1^{\text{INT},e}} \right]}_{n_{\text{dof}}^{s,e} \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\left\{ \ddot{\mathbf{d}}^e \right\}}_{n_{\text{dof}}^{s,e} \times 1} \quad (371)$$

$$\mathcal{G}_2^{\text{INT},h} = \mathbf{A}_e^{n_e} \underbrace{\left\{ \mathbf{c}^e \right\}^T}_{1 \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\left\{ \mathbf{f}^{\mathcal{G}_2^{\text{INT},e}} \right\}}_{n_{\text{dof}}^{s,e} \times 1} \quad (372)$$

$$\mathcal{G}_4^{\text{INT},h} = \mathbf{A}_e^{n_e} \underbrace{\left\{ \mathbf{c}^e \right\}^T}_{1 \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\left\{ \mathbf{f}^{\mathcal{G}_4^{\text{INT},e}} \right\}}_{n_{\text{dof}}^{s,e} \times 1} \quad (373)$$

$$\mathcal{G}^{\text{EXT},h} = \mathbf{A}_e^{n_e} \underbrace{\left\{ \mathbf{c}^e \right\}^T}_{1 \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\left\{ \mathbf{f}^{\mathcal{G}^{\text{EXT},e}} \right\}}_{n_{\text{dof}}^{s,e} \times 1} \quad (374)$$

and

$$\underbrace{\left\{ \mathbf{f}^{\mathcal{G}_2^{\text{INT},e}} \right\}}_{n_{\text{dof}}^{s,e} \times 1} = \int_{-1}^1 \underbrace{\left\{ \mathbf{B}^{e,u} \right\}^T}_{n_{\text{dof}}^{s,e} \times 1} P_{11}^{h^e} A j^e d\xi \quad (375)$$

$$\underbrace{\left\{ \mathbf{f}^{\mathcal{G}_4^{\text{INT},e}} \right\}}_{n_{\text{dof}}^{s,e} \times 1} = \int_{-1}^1 \underbrace{\left\{ \mathbf{N}^{e,u} \right\}^T}_{n_{\text{dof}}^{s,e} \times 1} \rho_0^{h^e} g A j^e d\xi \quad (376)$$

$$\underbrace{\left\{ \mathbf{f}^{\mathcal{G}^{\text{EXT},e}} \right\}}_{n_{\text{dof}}^{s,e} \times 1} = \begin{cases} \underbrace{\left\{ \mathbf{N}^{e,u}(X=H) \right\}^T}_{n_{\text{dof}}^{s,e} \times 1} t^\sigma A & X=H \\ \mathbf{0} & 0 \leq X < H \end{cases} \quad (377)$$

The mass matrix associated with the solid acceleration is given by

$$\underbrace{\left[\mathbf{m}_{u,u}^{\mathcal{G}_1^{\text{INT},e}} \right]}_{n_{\text{dof}}^{s,e} \times n_{\text{dof}}^{s,e}} = \int_{-1}^1 \rho_0^{h^e} \underbrace{\left\{ \mathbf{N}^{e,u} \right\}^T}_{n_{\text{dof}}^{s,e} \times 1} \underbrace{\left\{ \mathbf{N}^{e,u} \right\}}_{1 \times n_{\text{dof}}^{s,e}} A j^e d\xi \quad (378)$$

Thus, the solution is given by

$$\underbrace{\left\{ \ddot{\mathbf{d}}_{m+1} \right\}}_{n_{\text{dof}}^s \times 1} = \underbrace{\left[\mathbf{M}_{u,u}^{\mathcal{G}_1^{\text{INT}}} \right]^{-1}}_{n_{\text{dof}}^s \times n_{\text{dof}}^s} \cdot \left(- \underbrace{\left\{ \mathbf{F}^{\mathcal{G}_2^{\text{INT}}} \right\}}_{n_{\text{dof}}^s \times 1} - \underbrace{\left\{ \mathbf{F}^{\mathcal{G}_4^{\text{INT}}} \right\}}_{n_{\text{dof}}^s \times 1} + \underbrace{\left\{ \mathbf{F}^{\mathcal{G}^{\text{EXT}}} \right\}}_{n_{\text{dof}}^s \times 1} \right) \quad (379)$$

where

$$\underbrace{\left[M_{u,u}^{\mathcal{G}^{\text{INT}}} \right]}_{n_{\text{dof}}^s \times n_{\text{dof}}^s} = \mathbf{A}_e^{n_e} \underbrace{\left[\mathbf{m}_{u,u}^{\mathcal{G}^{\text{INT},e}} \right]}_{n_{\text{dof}}^{s,e} \times n_{\text{dof}}^{s,e}} \quad (380)$$

$$\underbrace{\left\{ \mathbf{F}^{\mathcal{G}^{\text{INT}}} \right\}}_{n_{\text{dof}}^s \times 1} = \mathbf{A}_e^{n_e} \underbrace{\left\{ \mathbf{f}^{\mathcal{G}^{\text{INT},e}} \right\}}_{n_{\text{dof}}^{s,e} \times 1} \quad (381)$$

$$\underbrace{\left\{ \mathbf{F}^{\mathcal{G}^{\text{INT}}} \right\}}_{n_{\text{dof}}^s \times 1} = \mathbf{A}_e^{n_e} \underbrace{\left\{ \mathbf{f}^{\mathcal{G}^{\text{INT},e}} \right\}}_{n_{\text{dof}}^{s,e} \times 1} \quad (382)$$

$$\underbrace{\left\{ \mathbf{F}^{\mathcal{G}^{\text{EXT}}} \right\}}_{n_{\text{dof}}^s \times 1} = \mathbf{A}_e^{n_e} \underbrace{\left\{ \mathbf{f}^{\mathcal{G}^{\text{EXT},e}} \right\}}_{n_{\text{dof}}^{s,e} \times 1} \quad (383)$$

The solid velocity and solid displacement updates are recovered by

$$\underbrace{\left\{ \dot{\mathbf{d}}_{n+1} \right\}}_{n_{\text{dof}}^s \times 1} = \underbrace{\left\{ \dot{\mathbf{d}}_n \right\}}_{n_{\text{dof}}^s \times 1} + \frac{\Delta t}{2} \left(\underbrace{\left\{ \ddot{\mathbf{d}}_n \right\}}_{n_{\text{dof}}^s \times 1} + \underbrace{\left\{ \ddot{\mathbf{d}}_{n+1} \right\}}_{n_{\text{dof}}^s \times 1} \right) \quad (384)$$

$$\underbrace{\left\{ \mathbf{d}_{n+1} \right\}}_{n_{\text{dof}}^s \times 1} = \underbrace{\left\{ \mathbf{d}_n \right\}}_{n_{\text{dof}}^s \times 1} + \Delta t \underbrace{\left\{ \dot{\mathbf{d}}_n \right\}}_{n_{\text{dof}}^s \times 1} + \frac{\Delta t^2}{2} \underbrace{\left\{ \ddot{\mathbf{d}}_{n+1} \right\}}_{n_{\text{dof}}^s \times 1} \quad (385)$$

5.3.2 Implementation of the (\mathbf{u} - p_f) Theory

For the two-field poroelastodynamics (\mathbf{u} - p_f) formulation, we solve two variational equations, namely, the balance of linear momentum of the mixture and the balance of mass of the mixture. These variational equations are given by respective Eqs. 28 and 3.

The FE equations integrated in time using CD for the poroelastodynamic equations are solved in a staggered manner. A solid skeleton displacement update is computed first. There are no unknowns related to pressure in the balance of linear momentum of the mixture. Then the solid displacements are substituted into the corresponding equations to compute a pore fluid pressure update; see Algorithm 1. The advantage of such a procedure is that there are no coupling matrices that would introduce off-diagonal entries in a block matrix of the system when trying to solve the solid skeleton acceleration and second time derivative on pore fluid pressure updates. This reduces computational cost at the expense of accuracy when compared to the

explicit Runge-Kutta methods and the implicit NB methods.

Algorithm 1 General concept of staggered solution process for $(u-p_f)$ formulation with CD time integration for a given time t_n

- 1: Update t_n
 - 2: Update external force vector(s)
 - 3: Compute deformations and stresses at time t_{n+1} from values at t_n
 - 4: Assemble internal force vector and mass matrix associated with \mathcal{G}^h from values at t_n
 - 5: Compute a^h at time t_{n+1}
 - 6: Assemble internal force vector and mass matrix associated with \mathcal{H}^h from values at t_n
 - 7: Compute \ddot{p}_f^h at time t_{n+1}
-

The FE formulation for the balance of momentum of the mixture variational equation is written in block-matrix form as

$$\underbrace{\{\mathbf{R}_u\}}_{n_{\text{dof}}^s \times 1} = \mathbf{0} \quad (386)$$

This global residual for the solid skeleton displacement is given as

$$\mathbf{c}^T \cdot \mathbf{R}_u = \mathcal{G}^h = \mathcal{G}_1^{\text{INT},h} + \mathcal{G}_2^{\text{INT},h} + \mathcal{G}_3^{\text{INT},h} + \mathcal{G}_4^{\text{INT},h} - \mathcal{G}^{\text{EXT},h} = 0 \quad (387)$$

where

$$\mathcal{G}_1^{\text{INT},h} = \underbrace{\mathbf{A}}_e \underbrace{\{\mathbf{c}^e\}}_{1 \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\left[\mathbf{m}_{u,u}^{\mathcal{G}_1^{\text{INT},e}} \right]}_{n_{\text{dof}}^{s,e} \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\{\ddot{\mathbf{d}}^e\}}_{n_{\text{dof}}^{s,e} \times 1} \quad (388)$$

$$\mathcal{G}_2^{\text{INT},h} = \underbrace{\mathbf{A}}_e \underbrace{\{\mathbf{c}^e\}}_{1 \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\{\mathbf{f}_{\mathcal{G}_2^{\text{INT},e}}\}}_{n_{\text{dof}}^{s,e} \times 1} \quad (389)$$

$$\mathcal{G}_3^{\text{INT},h} = \underbrace{\mathbf{A}}_e \underbrace{\{\mathbf{c}^e\}}_{1 \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\{\mathbf{f}_{\mathcal{G}_3^{\text{INT},e}}\}}_{n_{\text{dof}}^{s,e} \times 1} \quad (390)$$

$$\mathcal{G}_4^{\text{INT},h} = \underbrace{\mathbf{A}}_e \underbrace{\{\mathbf{c}^e\}}_{1 \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\{\mathbf{f}_{\mathcal{G}_4^{\text{INT},e}}\}}_{n_{\text{dof}}^{s,e} \times 1} \quad (391)$$

$$\mathcal{G}^{\text{EXT},h} = \underbrace{\mathbf{A}}_e \underbrace{\{\mathbf{c}^e\}}_{1 \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\{\mathbf{f}_{\mathcal{G}^{\text{EXT},e}}\}}_{n_{\text{dof}}^{s,e} \times 1} \quad (392)$$

and

$$\underbrace{\left\{ \mathbf{f}^{\mathcal{G}_2^{\text{INT}},e} \right\}}_{n_{\text{dof}}^{s,e} \times 1} = \int_{-1}^1 \underbrace{\left\{ \mathbf{B}^{e,u} \right\}^T}_{n_{\text{dof}}^{s,e} \times 1} P_{11(E)}^{s,h^e} A_j^e d\xi \quad (393)$$

$$\underbrace{\left\{ \mathbf{f}^{\mathcal{G}_3^{\text{INT}},e} \right\}}_{n_{\text{dof}}^{s,e} \times 1} = - \int_{-1}^1 \underbrace{\left\{ \mathbf{B}^{e,u} \right\}^T}_{n_{\text{dof}}^{s,e} \times 1} p_{f,n} A_j^e d\xi \quad (394)$$

$$\underbrace{\left\{ \mathbf{f}^{\mathcal{G}_4^{\text{INT}},e} \right\}}_{n_{\text{dof}}^{s,e} \times 1} = \int_{-1}^1 \underbrace{\left\{ \mathbf{N}^{e,u} \right\}^T}_{n_{\text{dof}}^{s,e} \times 1} \rho_0^{h^e} g A_j^e d\xi \quad (395)$$

$$\underbrace{\left\{ \mathbf{f}^{\mathcal{G}^{\text{EXT}},e} \right\}}_{n_{\text{dof}}^{s,e} \times 1} = \begin{cases} \underbrace{\left\{ \mathbf{N}^{e,u}(X=H) \right\}^T}_{n_{\text{dof}}^{s,e} \times 1} t^\sigma A & X = H \\ \mathbf{0} & 0 \leq X < H \end{cases} \quad (396)$$

The mass matrix associated with the solid skeleton acceleration is given by

$$\underbrace{\left[\mathbf{m}_{u,u}^{\mathcal{G}_1^{\text{INT}},e} \right]}_{n_{\text{dof}}^{s,e} \times n_{\text{dof}}^{s,e}} = \int_{-1}^1 \rho_0^{h^e} \underbrace{\left\{ \mathbf{N}^{e,u} \right\}^T}_{n_{\text{dof}}^{s,e} \times 1} \underbrace{\left\{ \mathbf{N}^{e,u} \right\}}_{1 \times n_{\text{dof}}^{s,e}} A_j^e d\xi \quad (397)$$

The FE formulation for the balance of mass variational equation is written in block-matrix form as

$$\underbrace{\left\{ \mathbf{R}_{p_f} \right\}}_{n_{\text{dof}}^f \times 1} = \mathbf{0} \quad (398)$$

where the global residual for the pore fluid pressure is given as

$$\boldsymbol{\alpha}^T \cdot \mathbf{R}_{p_f} = \mathcal{H}^h = \mathcal{H}_1^{\text{INT},h} + \mathcal{H}_2^{\text{INT},h} + \mathcal{H}_3^{\text{INT},h} + \mathcal{H}_4^{\text{INT},h} - \mathcal{H}_1^{\text{EXT},h} = 0 \quad (399)$$

where

$$\mathcal{H}_1^{\text{INT},h} = \mathbf{A}_e^{n_e} \underbrace{\{\alpha^e\}^T}_{1 \times n_{\text{dof}}^{p,e}} \cdot \left(\underbrace{\left[\mathbf{m}_{p_f, p_f}^{\mathcal{H}_1^{\text{INT},e}} \right]}_{n_{\text{dof}}^{p,e} \times n_{\text{dof}}^{p,e}} \cdot \underbrace{\{\ddot{\theta}^e\}}_{n_{\text{dof}}^{p,e} \times 1} + \underbrace{\left[\mathbf{k}_{p_f, u}^{\mathcal{H}_1^{\text{INT},e}} \right]}_{n_{\text{dof}}^{p,e} \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\{\ddot{d}^e\}}_{n_{\text{dof}}^{s,e} \times 1} + \underbrace{\{\mathbf{f}^{\mathcal{H}_1^{\text{INT},e}}\}}_{n_{\text{dof}}^{p,e} \times 1} \right) \quad (400)$$

$$\mathcal{H}_2^{\text{INT},h} = \mathbf{A}_e^{n_e} \underbrace{\{\alpha^e\}^T}_{1 \times n_{\text{dof}}^{p,e}} \cdot \left(\underbrace{\left[\mathbf{k}_{p_f, u}^{\mathcal{H}_2^{\text{INT},e}} \right]}_{n_{\text{dof}}^{p,e} \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\{\ddot{d}^e\}}_{n_{\text{dof}}^{s,e} \times 1} + \underbrace{\{\mathbf{f}^{\mathcal{H}_2^{\text{INT},e}}\}}_{n_{\text{dof}}^{p,e} \times 1} \right) \quad (401)$$

$$\mathcal{H}_3^{\text{INT},h} = \mathbf{A}_e^{n_e} \underbrace{\{\alpha^e\}^T}_{1 \times n_{\text{dof}}^{p,e}} \cdot \underbrace{\{\mathbf{f}^{\mathcal{H}_3^{\text{INT},e}}\}}_{n_{\text{dof}}^{p,e} \times 1} \quad (402)$$

$$\mathcal{H}_4^{\text{INT},h} = \mathbf{A}_e^{n_e} \underbrace{\{\alpha^e\}^T}_{1 \times n_{\text{dof}}^{p,e}} \cdot \left(\underbrace{\left[\mathbf{k}_{p_f, u}^{\mathcal{H}_4^{\text{INT},e}} \right]}_{n_{\text{dof}}^{p,e} \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\{\ddot{d}^e\}}_{n_{\text{dof}}^{s,e} \times 1} + \underbrace{\{\mathbf{f}^{\mathcal{H}_4^{\text{INT},e}}\}}_{n_{\text{dof}}^{p,e} \times 1} \right) \quad (403)$$

$$\mathcal{H}^{\text{EXT},h} = \mathbf{A}_e^{n_e} \underbrace{\{\alpha^e\}^T}_{1 \times n_{\text{dof}}^{p,e}} \cdot \underbrace{\{\mathbf{f}^{\mathcal{H}^{\text{EXT},e}}\}}_{n_{\text{dof}}^{p,e} \times 1} \quad (404)$$

with

$$\underbrace{\{\mathbf{f}^{\mathcal{H}_1^{\text{INT}},e}\}}_{n_{\text{dof}}^{p,e} \times 1} = \int_{-1}^1 \underbrace{\{\mathbf{N}^{e,p}\}}_{n_{\text{dof}}^{p,e} \times 1}^T \left[\frac{J^{h^e} n^{f,h^e}}{K_f^\eta} \left(\dot{p}_{f,n}^{h^e} + \frac{\Delta t}{2} \ddot{p}_{f,n}^{h^e} \right) + \left(\frac{\partial v_n^{h^e}}{\partial X} + \frac{\Delta t}{2} \frac{\partial a_n^{h^e}}{\partial X} \right) \right] A_j^e d\xi \quad (405)$$

$$\underbrace{\{\mathbf{f}^{\mathcal{H}_2^{\text{INT}},e}\}}_{n_{\text{dof}}^{p,e} \times 1} = - \int_{-1}^1 \underbrace{\{\mathbf{N}^{e,p}\}}_{n_{\text{dof}}^{p,e} \times 1}^T \frac{1}{K_f^\eta} \frac{\partial p_{f,n}^{h^e}}{\partial X} \hat{k}^{h^e} \left(\frac{\partial p_{f,n}^{h^e}}{\partial X} (F_{11}^{h^e})^{-1} - \rho^{\text{fR},h^e} g \right) A_j^e d\xi \quad (406)$$

$$\underbrace{\{\mathbf{f}^{\mathcal{H}_3^{\text{INT}},e}\}}_{n_{\text{dof}}^{p,e} \times 1} = \int_{-1}^1 \underbrace{\{\mathbf{B}^{e,p}\}}_{n_{\text{dof}}^{p,e} \times 1}^T \hat{k}^{h^e} \frac{\partial p_{f,n}^{h^e}}{\partial X} (F_{11}^{h^e})^{-1} A_j^e d\xi \quad (407)$$

$$\underbrace{\{\mathbf{f}^{\mathcal{H}_4^{\text{INT}},e}\}}_{n_{\text{dof}}^{p,e} \times 1} = \int_{-1}^1 \underbrace{\{\mathbf{B}^{e,p}\}}_{n_{\text{dof}}^{p,e} \times 1}^T \hat{k}^{h^e} \rho^{\text{fR},h^e} g A_j^e d\xi \quad (408)$$

$$\underbrace{\{\mathbf{f}^{\mathcal{H}^{\text{EXT}},e}\}}_{n_{\text{dof}}^{p,e} \times 1} = \begin{cases} \underbrace{\{\mathbf{N}^{e,p}(X = H, X = 0)\}}_{n_{\text{dof}}^{p,e} \times 1}^T Q_f A & X = H, X = 0 \\ \mathbf{0} & 0 < X < H \end{cases} \quad (409)$$

The coupling tangents that get multiplied by the updated solid skeleton acceleration at t_{n+1} are given by

$$\underbrace{\left[\mathbf{k}_{p_{f,u}}^{\mathcal{H}_1^{\text{INT}},e} \right]}_{n_{\text{dof}}^{p,e} \times n_{\text{dof}}^{s,e}} = \int_{-1}^1 \frac{\Delta t}{2} \underbrace{\{\mathbf{N}^{e,p}\}}_{n_{\text{dof}}^{p,e} \times 1}^T \underbrace{\{\mathbf{B}^{e,u}\}}_{1 \times n_{\text{dof}}^{s,e}} A_j^e d\xi \quad (410)$$

$$\underbrace{\left[\mathbf{k}_{p_{f,u}}^{\mathcal{H}_2^{\text{INT}},e} \right]}_{n_{\text{dof}}^{p,e} \times n_{\text{dof}}^{s,e}} = - \int_{-1}^1 \frac{1}{K_f^\eta} \frac{\partial p_{f,n}^{h^e}}{\partial X} \hat{k}^{h^e} \rho^{\text{fR},h^e} \underbrace{\{\mathbf{N}^{e,p}\}}_{n_{\text{dof}}^{p,e} \times 1}^T \underbrace{\{\mathbf{N}^{e,u}\}}_{1 \times n_{\text{dof}}^{s,e}} A_j^e d\xi \quad (411)$$

$$\underbrace{\left[\mathbf{k}_{p_{f,u}}^{\mathcal{H}_4^{\text{INT}},e} \right]}_{n_{\text{dof}}^{p,e} \times n_{\text{dof}}^{s,e}} = \int_{-1}^1 \hat{k}^{h^e} \rho^{\text{fR},h^e} \underbrace{\{\mathbf{B}^{e,p}\}}_{n_{\text{dof}}^{p,e} \times 1}^T \underbrace{\{\mathbf{N}^{e,u}\}}_{1 \times n_{\text{dof}}^{s,e}} A_j^e d\xi \quad (412)$$

and the mass matrix associated with the pore fluid pressure is given by

$$\underbrace{\left[\mathbf{m}_{p_f, p_f}^{\mathcal{H}_1^{\text{INT}, e}} \right]}_{n_{\text{dof}}^{p, e} \times n_{\text{dof}}^{p, e}} = \int_{-1}^1 \frac{J^{h^e} n^{\text{f}, h^e} \Delta t}{K_f^\eta} \frac{1}{2} \underbrace{\left\{ \mathbf{N}^{e, p} \right\}^T}_{n_{\text{dof}}^{p, e} \times 1} \underbrace{\left\{ \mathbf{N}^{e, p} \right\}}_{1 \times n_{\text{dof}}^{p, e}} A_j^e d\xi \quad (413)$$

When pressure stabilization is enabled, an additional term $\mathcal{H}^{\text{stab}}$ is added to the LHS of Eq. 399 and is defined as

$$\mathcal{H}^{\text{stab}} = \mathbf{A}_e \boldsymbol{\alpha}^{e, T} \cdot \left(\underbrace{\left[\mathbf{m}_{p_f, p_f}^{\mathcal{H}^{\text{stab}, e}} \right]}_{n_{\text{dof}}^{p, e} \times n_{\text{dof}}^{p, e}} \cdot \underbrace{\left\{ \ddot{\boldsymbol{\theta}}^e \right\}}_{n_{\text{dof}}^{p, e} \times 1} + \underbrace{\left\{ \mathbf{f}^{\mathcal{H}^{\text{stab}, e}} \right\}}_{n_{\text{dof}}^{p, e} \times 1} \right) \quad (414)$$

where

$$\underbrace{\left[\mathbf{m}_{p_f, p_f}^{\mathcal{H}^{\text{stab}, e}} \right]}_{n_{\text{dof}}^{p, e} \times n_{\text{dof}}^{p, e}} = \int_{-1}^1 \alpha^{\text{stab}} (F_{11}^{h^e})^{-1} \frac{\Delta t}{2} \underbrace{\left\{ \mathbf{B}^{e, p} \right\}^T}_{n_{\text{dof}}^{p, e} \times 1} \underbrace{\left\{ \mathbf{B}^{e, p} \right\}}_{1 \times n_{\text{dof}}^{p, e}} A_j^e d\xi \quad (415)$$

$$\underbrace{\left\{ \mathbf{f}^{\mathcal{H}^{\text{stab}, e}} \right\}}_{n_{\text{dof}}^{p, e} \times 1} = \int_{-1}^1 \underbrace{\left\{ \mathbf{B}^{e, p} \right\}^T}_{n_{\text{dof}}^{p, e} \times 1} \alpha^{\text{stab}} (F_{11}^{h^e})^{-1} \left(\frac{\partial \dot{p}_{f, n}^{h^e}}{\partial X} + \frac{\Delta t}{2} \frac{\partial \ddot{p}_{f, n}^{h^e}}{\partial X} \right) A_j^e d\xi \quad (416)$$

Returning our attention to Eq. 366, we see now that we can write it without introducing nonlinearity if we solve the equations in a staggered manner as described in Algorithm 1. Therefore,

$$\underbrace{\left\{ \ddot{\mathbf{d}}_{n+1} \right\}}_{n_{\text{dof}}^s \times 1} = \underbrace{\left[\mathbf{M}_{u, u}^{\mathcal{G}_1^{\text{INT}}} \right]^{-1}}_{n_{\text{dof}}^s \times n_{\text{dof}}^s} \cdot \left(- \underbrace{\left\{ \mathbf{F}^{\mathcal{G}_2^{\text{INT}}} \right\}}_{n_{\text{dof}}^s \times 1} - \underbrace{\left\{ \mathbf{F}^{\mathcal{G}_3^{\text{INT}}} \right\}}_{n_{\text{dof}}^s \times 1} - \underbrace{\left\{ \mathbf{F}^{\mathcal{G}_4^{\text{INT}}} \right\}}_{n_{\text{dof}}^s \times 1} + \underbrace{\left\{ \mathbf{F}^{\mathcal{G}^{\text{EXT}}} \right\}}_{n_{\text{dof}}^s \times 1} \right) \quad (417)$$

where

$$\underbrace{\left[M_{u,u}^{\mathcal{G}^{\text{INT}}} \right]}_{n_{\text{dof}}^s \times n_{\text{dof}}^s} = \mathbf{A}_e^{n_e} \underbrace{\left[\mathbf{m}_{u,u}^{\mathcal{G}^{\text{INT},e}} \right]}_{n_{\text{dof}}^{s,e} \times n_{\text{dof}}^{s,e}} \quad (418)$$

$$\underbrace{\left\{ \mathbf{F}^{\mathcal{G}_2^{\text{INT}}} \right\}}_{n_{\text{dof}}^s \times 1} = \mathbf{A}_e^{n_e} \underbrace{\left\{ \mathbf{f}^{\mathcal{G}_2^{\text{INT},e}} \right\}}_{n_{\text{dof}}^{s,e} \times 1} \quad (419)$$

$$\underbrace{\left\{ \mathbf{F}^{\mathcal{G}_3^{\text{INT}}} \right\}}_{n_{\text{dof}}^s \times 1} = \mathbf{A}_e^{n_e} \underbrace{\left\{ \mathbf{f}^{\mathcal{G}_3^{\text{INT},e}} \right\}}_{n_{\text{dof}}^{s,e} \times 1} \quad (420)$$

$$\underbrace{\left\{ \mathbf{F}^{\mathcal{G}_4^{\text{INT}}} \right\}}_{n_{\text{dof}}^s \times 1} = \mathbf{A}_e^{n_e} \underbrace{\left\{ \mathbf{f}^{\mathcal{G}_4^{\text{INT},e}} \right\}}_{n_{\text{dof}}^{s,e} \times 1} \quad (421)$$

$$\underbrace{\left\{ \mathbf{F}^{\mathcal{G}^{\text{EXT}}} \right\}}_{n_{\text{dof}}^s \times 1} = \mathbf{A}_e^{n_e} \underbrace{\left\{ \mathbf{f}^{\mathcal{G}^{\text{EXT},e}} \right\}}_{n_{\text{dof}}^{s,e} \times 1} \quad (422)$$

and the solid velocity and solid displacement updates are recovered by

$$\underbrace{\left\{ \dot{\mathbf{d}}_{n+1} \right\}}_{n_{\text{dof}}^s \times 1} = \underbrace{\left\{ \dot{\mathbf{d}}_n \right\}}_{n_{\text{dof}}^s \times 1} + \frac{\Delta t}{2} \left(\underbrace{\left\{ \ddot{\mathbf{d}}_n \right\}}_{n_{\text{dof}}^s \times 1} + \underbrace{\left\{ \ddot{\mathbf{d}}_{n+1} \right\}}_{n_{\text{dof}}^s \times 1} \right) \quad (423)$$

$$\underbrace{\left\{ \mathbf{d}_{n+1} \right\}}_{n_{\text{dof}}^s \times 1} = \underbrace{\left\{ \mathbf{d}_n \right\}}_{n_{\text{dof}}^s \times 1} + \Delta t \underbrace{\left\{ \dot{\mathbf{d}}_n \right\}}_{n_{\text{dof}}^s \times 1} + \frac{\Delta t^2}{2} \underbrace{\left\{ \ddot{\mathbf{d}}_{n+1} \right\}}_{n_{\text{dof}}^s \times 1} \quad (424)$$

Then, when solving for pore fluid pressure, we use the result given by Eq. 417 for all values of $\ddot{\mathbf{d}}_{n+1}$ in Eqs. 400–403. Therefore,

$$\underbrace{\left\{ \ddot{\boldsymbol{\theta}}_{n+1} \right\}}_{n_{\text{dof}}^p \times 1} = \underbrace{\left[\mathbf{M}_{p_f, p_f}^{\mathcal{H}_1^{\text{INT}}} \right]^{-1}}_{n_{\text{dof}}^p \times n_{\text{dof}}^p} \cdot \left(- \underbrace{\left\{ \mathbf{F}^{\mathcal{H}_1^{\text{INT}}} \right\}}_{n_{\text{dof}}^p \times 1} - \underbrace{\left\{ \mathbf{F}^{\mathcal{H}_2^{\text{INT}}} \right\}}_{n_{\text{dof}}^p \times 1} - \underbrace{\left\{ \mathbf{F}^{\mathcal{H}_3^{\text{INT}}} \right\}}_{n_{\text{dof}}^p \times 1} \right. \\ \left. - \underbrace{\left\{ \mathbf{F}^{\mathcal{H}_4^{\text{INT}}} \right\}}_{n_{\text{dof}}^p \times 1} + \left[\underbrace{\left[\mathbf{K}_{p_f, u}^{\mathcal{H}_1^{\text{INT}}} \right]}_{n_{\text{dof}}^p \times n_{\text{dof}}^s} + \underbrace{\left[\mathbf{K}_{p_f, u}^{\mathcal{H}_2^{\text{INT}}} \right]}_{n_{\text{dof}}^p \times n_{\text{dof}}^s} + \underbrace{\left[\mathbf{K}_{p_f, u}^{\mathcal{H}_4^{\text{INT}}} \right]}_{n_{\text{dof}}^p \times n_{\text{dof}}^s} \right] \cdot \underbrace{\left\{ \ddot{\mathbf{d}}_{n+1} \right\}}_{n_{\text{dof}}^s \times 1} \right. \\ \left. + \underbrace{\left\{ \mathbf{F}^{\mathcal{H}^{\text{EXT}}} \right\}}_{n_{\text{dof}}^p \times 1} \right) \quad (425)$$

where

$$\underbrace{\left[M_{p_f, p_f}^{\mathcal{H}_1^{\text{INT}}} \right]}_{n_{\text{dof}}^p \times n_{\text{dof}}^p} = \mathbf{A}_e^{n_e} \underbrace{\left[\mathbf{m}_{p_f, p_f}^{\mathcal{H}_1^{\text{INT}, e}} \right]}_{n_{\text{dof}}^{p, e} \times n_{\text{dof}}^{p, e}} \quad (426)$$

$$\underbrace{\left[K_{p_f, u}^{\mathcal{H}_1^{\text{INT}}} \right]}_{n_{\text{dof}}^p \times n_{\text{dof}}^s} = \mathbf{A}_e^{n_e} \underbrace{\left[\mathbf{m}_{p_f, u}^{\mathcal{H}_1^{\text{INT}, e}} \right]}_{n_{\text{dof}}^{p, e} \times n_{\text{dof}}^{s, e}} \quad (427)$$

$$\underbrace{\left[K_{p_f, u}^{\mathcal{H}_2^{\text{INT}}} \right]}_{n_{\text{dof}}^p \times n_{\text{dof}}^s} = \mathbf{A}_e^{n_e} \underbrace{\left[\mathbf{m}_{p_f, u}^{\mathcal{H}_2^{\text{INT}, e}} \right]}_{n_{\text{dof}}^{p, e} \times n_{\text{dof}}^{s, e}} \quad (428)$$

$$\underbrace{\left[K_{p_f, u}^{\mathcal{H}_4^{\text{INT}}} \right]}_{n_{\text{dof}}^p \times n_{\text{dof}}^s} = \mathbf{A}_e^{n_e} \underbrace{\left[\mathbf{m}_{p_f, u}^{\mathcal{H}_4^{\text{INT}, e}} \right]}_{n_{\text{dof}}^{p, e} \times n_{\text{dof}}^{s, e}} \quad (429)$$

$$\underbrace{\left\{ \mathbf{F}_{p_f}^{\mathcal{H}_1^{\text{INT}}} \right\}}_{n_{\text{dof}}^p \times 1} = \mathbf{A}_e^{n_e} \underbrace{\left\{ \mathbf{f}_{p_f}^{\mathcal{H}_1^{\text{INT}, e}} \right\}}_{n_{\text{dof}}^{p, e} \times 1} \quad (430)$$

$$\underbrace{\left\{ \mathbf{F}_{p_f}^{\mathcal{H}_2^{\text{INT}}} \right\}}_{n_{\text{dof}}^p \times 1} = \mathbf{A}_e^{n_e} \underbrace{\left\{ \mathbf{f}_{p_f}^{\mathcal{H}_2^{\text{INT}, e}} \right\}}_{n_{\text{dof}}^{p, e} \times 1} \quad (431)$$

$$\underbrace{\left\{ \mathbf{F}_{p_f}^{\mathcal{H}_3^{\text{INT}}} \right\}}_{n_{\text{dof}}^p \times 1} = \mathbf{A}_e^{n_e} \underbrace{\left\{ \mathbf{f}_{p_f}^{\mathcal{H}_3^{\text{INT}, e}} \right\}}_{n_{\text{dof}}^{p, e} \times 1} \quad (432)$$

$$\underbrace{\left\{ \mathbf{F}_{p_f}^{\mathcal{H}_4^{\text{INT}}} \right\}}_{n_{\text{dof}}^p \times 1} = \mathbf{A}_e^{n_e} \underbrace{\left\{ \mathbf{f}_{p_f}^{\mathcal{H}_4^{\text{INT}, e}} \right\}}_{n_{\text{dof}}^{p, e} \times 1} \quad (433)$$

$$\underbrace{\left\{ \mathbf{F}_{p_f}^{\mathcal{H}^{\text{EXT}}} \right\}}_{n_{\text{dof}}^p \times 1} = \mathbf{A}_e^{n_e} \underbrace{\left\{ \mathbf{f}_{p_f}^{\mathcal{H}^{\text{EXT}, e}} \right\}}_{n_{\text{dof}}^{p, e} \times 1} \quad (434)$$

The first time derivative on pore fluid pressure and the pore fluid pressure updates are recovered by

$$\underbrace{\left\{ \dot{\boldsymbol{\theta}}_{n+1} \right\}}_{n_{\text{dof}}^p \times 1} = \underbrace{\left\{ \dot{\boldsymbol{\theta}}_n \right\}}_{n_{\text{dof}}^p \times 1} + \frac{\Delta t}{2} \left(\underbrace{\left\{ \ddot{\boldsymbol{\theta}}_n \right\}}_{n_{\text{dof}}^p \times 1} + \underbrace{\left\{ \ddot{\boldsymbol{\theta}}_{n+1} \right\}}_{n_{\text{dof}}^p \times 1} \right) \quad (435)$$

$$\underbrace{\left\{ \boldsymbol{\theta}_{n+1} \right\}}_{n_{\text{dof}}^p \times 1} = \underbrace{\left\{ \boldsymbol{\theta}_n \right\}}_{n_{\text{dof}}^p \times 1} + \Delta t \underbrace{\left\{ \dot{\boldsymbol{\theta}}_n \right\}}_{n_{\text{dof}}^p \times 1} + \frac{\Delta t^2}{2} \underbrace{\left\{ \ddot{\boldsymbol{\theta}}_{n+1} \right\}}_{n_{\text{dof}}^p \times 1} \quad (436)$$

If pressure stabilization is enabled, then we invert a summation of matrices in Eq. 425 and must subtract the force vector associated with pressure stabilization along with the other force vectors:

$$\begin{aligned}
\underbrace{\{\ddot{\boldsymbol{\theta}}_{n+1}\}}_{n_{\text{dof}}^p \times 1} &= \left(\underbrace{\left[\mathbf{M}_{p_f, p_f}^{\mathcal{H}_1^{\text{INT}}} \right]}_{n_{\text{dof}}^p \times n_{\text{dof}}^p} + \underbrace{\left[\mathbf{M}_{p_f, p_f}^{\mathcal{H}^{\text{stab}}} \right]}_{n_{\text{dof}}^p \times n_{\text{dof}}^p} \right)^{-1} \cdot \left(- \underbrace{\left\{ \mathbf{F}^{\mathcal{H}_1^{\text{INT}}} \right\}}_{n_{\text{dof}}^p \times 1} - \underbrace{\left\{ \mathbf{F}^{\mathcal{H}_2^{\text{INT}}} \right\}}_{n_{\text{dof}}^p \times 1} - \underbrace{\left\{ \mathbf{F}^{\mathcal{H}_3^{\text{INT}}} \right\}}_{n_{\text{dof}}^p \times 1} \right. \\
&\quad - \underbrace{\left\{ \mathbf{F}^{\mathcal{H}_4^{\text{INT}}} \right\}}_{n_{\text{dof}}^p \times 1} - \underbrace{\left\{ \mathbf{F}^{\mathcal{H}^{\text{stab}}} \right\}}_{n_{\text{dof}}^p \times 1} + \left[\underbrace{\left[\mathbf{K}_{p_f, u}^{\mathcal{H}_1^{\text{INT}}} \right]}_{n_{\text{dof}}^p \times n_{\text{dof}}^s} + \underbrace{\left[\mathbf{K}_{p_f, u}^{\mathcal{H}_2^{\text{INT}}} \right]}_{n_{\text{dof}}^p \times n_{\text{dof}}^s} + \underbrace{\left[\mathbf{K}_{p_f, u}^{\mathcal{H}_4^{\text{INT}}} \right]}_{n_{\text{dof}}^p \times n_{\text{dof}}^s} \right] \cdot \underbrace{\left\{ \ddot{\mathbf{d}}_{n+1} \right\}}_{n_{\text{dof}}^s \times 1} \\
&\quad \left. + \underbrace{\left\{ \mathbf{F}^{\mathcal{H}^{\text{EXT}}} \right\}}_{n_{\text{dof}}^p \times 1} \right)
\end{aligned} \tag{437}$$

where

$$\underbrace{\left[\mathbf{M}_{p_f, p_f}^{\mathcal{H}^{\text{stab}}} \right]}_{n_{\text{dof}}^p \times n_{\text{dof}}^p} = \mathbf{A}_e^{n_e} \underbrace{\left[\mathbf{m}_{p_f, p_f}^{\mathcal{H}^{\text{stab}, e}} \right]}_{n_{\text{dof}}^{p, e} \times n_{\text{dof}}^{p, e}} \tag{438}$$

$$\underbrace{\left\{ \mathbf{F}^{\mathcal{H}^{\text{stab}}} \right\}}_{n_{\text{dof}}^p \times 1} = \mathbf{A}_e^{n_e} \underbrace{\left\{ \mathbf{f}^{\mathcal{H}^{\text{stab}, e}} \right\}}_{n_{\text{dof}}^{p, e} \times 1} \tag{439}$$

We also typically employ an adaptive time-stepping scheme for the CD time integrator that is loosely based on a localized Courant-Fredrichs-Lewy (CFL) condition. Recall for the critical time step,

$$\Delta t_{cr}^e \leq \frac{\Delta x^e}{c^e} \tag{440}$$

where $\Delta x^e = h^e$ is the shortest element length in the current configuration. The local P-wave speed c^e for uniaxial strain, assuming linear isotropic elasticity as an

approximation, is usually determined as

$$c^e = \max \left\{ \begin{array}{l} \sqrt{\frac{K^{\text{skel}} + \frac{4}{3}G^{\text{skel}}}{\rho^e}} \\ \sqrt{\frac{n^{s,e}K_s + n^{f,e}K_f^\eta + \frac{4}{3}G^{\text{skel}}}{\rho^e}} \end{array} \right. \quad (441)$$

For single-phase materials, the current total mass density is $\rho = J\rho_0$, and for multiphase materials $\rho = (n^s\rho^{sR} + n^f\rho^{fR})$. Then, the following time step is chosen:

$$\Delta t_{n+1} = \min_e \Delta t_{cr}^e \quad (442)$$

where we can either manually constrain the time step to not exceed some global maximum should the CFL condition at time t_n produce a time step too large for calculations at time t_{n+1} , resulting in a greater likelihood of error buildup, or, more commonly, we multiply Eq. 440 by a safety factor, $SF \in [0.2, 0.9]$.

6. Stabilization Techniques

Oftentimes for dynamical problems, particularly at high strain rates, the stability of the numerical integration scheme is dependent on artificial damping introduced into the time-discretized balance equations. Of the integrators introduced in Section 5, only the NB integration scheme with added dissipation (i.e., NB method with the parameters $\beta = 0.3025$ and $\gamma = 0.6$) has some algorithmic damping. Therefore, whether or not the other numerical integration schemes remain stable is dependent upon tolerances (relative and/or absolute) and the length of the time steps. We found that for shock loading problems, tight tolerances and small time steps, either fixed, or calculated through an adaptive time-stepping scheme with tight tolerance for truncation error (e.g., the Runge-Kutta methods discussed in Section 5.1.6) are not enough to control spurious pressure oscillations or in some cases, negative Jacobians of deformation (i.e., “inverted” elements) that make the simulation go unstable. Furthermore, for consistent element types, at low-to-moderate permeabilities, we require element-stabilization techniques to satisfy the well-known *inf-sup* condition.

To address these issues, we have explored adding a canonical “shock viscosity”

term to the solid skeleton stress, as well as a pressure stabilization term to the balance of mass of the mixture, both of which are discussed below. As will be shown in examples of shock loadings in a subsequent report,²⁹ we have found that a combination of both shock viscosity and pressure stabilization allows us to obtain results for lung parenchyma deformations at greater overpressure loading amplitudes, and with less computational expense given that we are able to take larger time-steps which are comparable to those suggested by a CFL condition.

6.1 Shock Viscosity

The history of the artificial shock viscosity can be traced back to the seminal work by von Neumann and Richtmyer⁴⁰ who proposed that a viscous term q be added to an otherwise inviscid fluid's momentum equation for shocks propagating in 1-D:

$$q := c_0^2 \rho (\Delta x)^2 \left(\frac{\partial \dot{x}}{\partial x} \right)^2 \quad (443)$$

where $q = 0$ for expanding motions ($\partial \dot{x} / \partial x \geq 0$, i.e., rarefaction), x is the coordinate in the direction of motion, Δx is the grid spacing, ρ is the material density and c_0 is a constant ≈ 2 . Landshoff⁴¹ introduced a q term that is linear in the velocity gradient:

$$q := c_L \rho \Delta x c \left| \frac{\partial \dot{x}}{\partial x} \right| \quad (444)$$

Regarding notation, c_L is a constant ≈ 1 and c is the local sound speed (i.e., the square root of the local P-wave modulus divided by the local density). For further information on the subject, we refer the interested reader to reviews by Benson⁴² and Margolin and Lloyd-Ronning.⁴³

In this work, we use the artificial viscosity given by Wilkins,⁴⁴ which is also used in LS-DYNA.⁴⁵ This approach combines linear and quadratic terms into one form:

$$q = \begin{cases} \rho l \left(C_0 l \dot{\varepsilon}_{kk}^2 - C_1 c \dot{\varepsilon}_{kk} \right) & \text{if } \dot{\varepsilon}_{kk} < 0 \\ 0 & \text{if } \dot{\varepsilon}_{kk} \geq 0 \end{cases} \quad (445)$$

where l is a characteristic length scale (in 1-D this reduces to the local element length), C_0 and C_1 are constants typically taken to be 1.5 and 0.06, respectively,

and $\dot{\epsilon}_{ij}$ is the symmetric part of the strain-rate tensor:

$$\dot{\epsilon}_{ij} := \frac{1}{2}(l_{ij} + l_{ji}), \quad l_{ij} := \frac{\partial v_i}{\partial x_j} \quad (446)$$

The extent to which the shock viscosity affects the solution depends on the magnitude of the constants C_0 and C_1 . The latter is responsible for damping out oscillations behind the front, but large values of either constant tend to overly smooth out the shock over multiple elements; examples of shock loadings with varying ranges of the constants that demonstrate this phenomenon are shown in a future report.²⁹

The shock viscosity is essentially an opposing pressure term added on to the solid skeleton stress such that the augmented effective solid skeleton Cauchy stress is written as

$$\tilde{\sigma}_E^s = \sigma_E^{s,\text{dev}} - (P + q)\mathbf{1} \quad (447)$$

which, using the common definitions for solid skeleton deviatoric stress $\sigma_E^{s,\text{dev}}$ and solid skeleton hydrostatic pressure P , we may also write as

$$\tilde{\sigma}_{ij(E)}^s = \begin{cases} \sigma_{ij(E)}^s - \underbrace{\rho^s h [C_0 h l_{kk}^2 - C_1 c l_{kk}]}_{:=q} \delta_{ij} & \text{if } l_{kk} < 0 \text{ (compression)} \\ \sigma_{ij(E)}^s & \text{if } l_{kk} \geq 0 \text{ (tension)} \end{cases} \quad (448)$$

In the reference configuration,

$$\tilde{P}_{iI(E)}^s = \begin{cases} P_{iI(E)}^s - J \rho^s h \left[C_0 h \left(\frac{\partial v_k}{\partial X_K} F_{Kk}^{-1} \right)^2 - C_1 c \frac{\partial v_k}{\partial X_K} F_{Kk}^{-1} \right] F_{Ii}^{-1} & \text{if } \frac{\partial v_k}{\partial X_K} < 0 \text{ (compression)} \\ P_{iI(E)}^s & \text{if } \frac{\partial v_k}{\partial X_K} \geq 0 \text{ (tension)} \end{cases} \quad (449)$$

where the F_{Kk}^{-1} scaling of the velocity gradient is omitted from the conditions because it does not affect the sign of the velocity gradient, which is what determines whether or not the shock viscosity is applied. In the 1-D uniaxial strain regime,

Eq. 449 reduces to the following expressions along the direction of motion:

$$\tilde{P}_{11(E)}^s = \begin{cases} P_{11(E)}^s - \underbrace{\rho_0 h_0 \left(C_0 J h_0 F_{11}^{-2} \left[\frac{\partial v}{\partial X} \right]^2 - C_1 c F_{11}^{-1} \frac{\partial v}{\partial X} \right)}_{:=Q_{1-D}} & \text{if } \frac{\partial v}{\partial X} < 0 \\ P_{11(E)}^s & \text{if } \frac{\partial v}{\partial X} \geq 0 \end{cases} \quad (450)$$

where the F_{11}^{-1} scaling of the velocity gradient is omitted from the conditions for aforementioned reasons. For explicit and semi-implicit integration methods, the shock viscosity terms are simply added to the stress as a force residual since we assume that solid skeleton stress is evaluated explicitly. For implicit integration methods, we require additional terms be added to the consistent tangent when shock viscosity is enabled. Consulting the follow-up report by Irwin et al.,²⁹ refer to Appendix B for the formulation of the linearized term and Appendix C for the FE formulation.

6.2 Pressure Stabilization

For stabilizing the linear Q2-Q1-P1, Q1-Q1-P1 and Q1-P1 elements, we follow the approach of Truty and Zimmerman,⁴⁶ which is based on the method of Brezzi and Pitkaranta.⁴⁷ The stabilization term acts like an opposing pore fluid pressure flux in the variational equation of the balance of mass, wherein

$$\mathcal{H} = \left(\sum_i^{n_{\text{INT}}} \mathcal{H}_i^{\text{INT}} \right) - \mathcal{H}^{\text{EXT}} + \mathcal{H}^{\text{stab}} = 0 \quad (451)$$

$$\mathcal{H}^{\text{stab}} := \int_{B_0} \alpha^{\text{stab}} \frac{\partial r}{\partial X_I} F_{Ii}^{-1} \frac{\partial \dot{p}_f}{\partial X_K} F_{Ki}^{-1} J dV \quad (452)$$

The value of n_{INT} depends on the governing physics: in the present work, $n_{\text{INT}} = 4$. In the 1-D uniaxial strain regime, the stabilization term reduces to

$$\mathcal{H}^{\text{stab}} = \int_0^{X=H} \alpha^{\text{stab}} \frac{\partial r}{\partial X} F_{11}^{-1} \frac{\partial \dot{p}_f}{\partial X} A dX \quad (453)$$

While Truty and Zimmerman⁴⁶ related the pressure stabilization parameter α^{stab} to material geometry and simulation time step, such an approach would be difficult to

derive for compressible pore fluid, large deformations, and high strain-rate loadings. Therefore, we have chosen α^{stab} on an ad-hoc basis. Typically, the smaller the value of α^{stab} , the more stable the simulation and the closer the results become to those from the already stable (for low-pressure amplitude loadings) Q2-Q2-P1 and Q2-P1 mixed elements. “Large” values of $\alpha^{\text{stab}} \geq 10^{-6}$ give rise to numerical instabilities or otherwise inaccurate results; $\alpha^{\text{stab}} = 10^{-10} \text{ m}^3\text{s}^2/\text{kg}$ appears to give the best results. A detailed discussion on the effect of pressure stabilization as it applies to a numerical verification example is provided in a follow-up report.²⁹

For both explicit and implicit integration methods, the inclusion of the stabilization term requires an additional tangent matrix. For explicit methods, this is because of the inclusion of the time derivative on the pore fluid pressure gradient, which is unknown at solution time t_{n+1} . This tangent also limits us to use consistent “mass” matrices in the weak formulation of the balance of mass given that the gradient shape functions for linear-order interpolations produce constant off-diagonal terms. These terms do not lead to cancellation at the Gauss points as they otherwise would for linear-order and quadratic-order interpolations (i.e., no gradients) using a row-sum lumping technique. This tangent is derived, respectively, in Appendix B of a follow-up report²⁹ and formulated for finite elements in Appendix C of that report.²⁹

7. Conclusions

We derived the numerical implementation of a finite-strain framework of a biphasic mixture (i.e., coupled pore fluid flow and solid skeleton deformation) of a soft porous material for high strain-rate dynamic loading. The constitutive theory, discussed in detail in a prior report,²⁸ is nonlinear elastic and accounts for the compressibility of the pore air. The formulation herein did not make assumptions regarding the equivalency of acceleration of pore fluid to that of solid skeleton, but rather allowed them to be different.

For shock loadings, the addition of both a bulk shock viscosity term⁴⁰ in the constitutive stress response of the solid skeleton and a pressure stabilization term^{46,47} in the balance of mass of the mixture are implemented for a linear, equal-order element interpolation. This is expected to allow simulations of deformations for higher magnitudes of peak overpressure loadings, to be verified in a subsequent report.²⁹ The current work succeeds in deriving novel methodologies for computationally

modest, 1-D space-time numerical simulations of smeared pore-scale FSI in soft porous materials loaded to high strain-rates.

8. References

1. Sobin S, Fung Y, Tremmer H. Collagen and elastin fibers in human pulmonary alveolar walls. *Journal of Applied Physiology*. 1988;64:1659-1675.
2. Suki B, Ito S, Stamenović D, Lutchen K, Ingenito E. Biomechanics of the lung parenchyma: critical roles of collagen and mechanical forces. *Journal of Applied Physiology*. 2005;98:1892-1899.
3. Freed A, Zamana S, Paul S, Clayton J. A dodecahedral model for alveoli. Part I. theory and numerical methods. DEVCOM Army Research Laboratory (US); 2021. Report No.: ARL-TR-9148.
4. Suki B, Bates J. Lung tissue mechanics as an emergent phenomenon. *Journal of Applied Physiology*. 2011;110:1111-1118.
5. Brannen M, Kang G, Dutrisac S, Banton R, Clayton J, Petel O. The influence of the tertiary bronchi on dynamic lung deformation. *Journal of the Mechanical Behavior of Biomedical Materials*. 2022;130:105181.
6. Clayton J, Freed A. A continuum mechanical model of the lung. DEVCOM Army Research Laboratory (US); 2019. Report No.: ARL-TR-8859.
7. Clayton J, Banton R, Freed A. A nonlinear thermoelastic-viscoelastic continuum model of lung mechanics for shock wave analysis. In: *Shock Compression of Condensed Matter*; Vol. 2272. Lane J, editor. AIP Conference Proceedings; 2020. p. 040001.
8. Clayton J. Modeling lung tissue dynamics and injury under pressure and impact loading. *Biomechanics and Modeling in Mechanobiology*. 2020;19:2603-2626.
9. Clayton JD, Freed AD. A constitutive model for lung mechanics and injury applicable to static, dynamic, and shock loading. *Mechanics of Soft Materials*. 2020;2.
10. Clayton J, Banton R, Goertz A. A continuum model of the human lung: Implementation and parameterization. DEVCOM Army Research Laboratory (US); 2021. Report No.: ARL-TR-9138.

11. Freed A, Clayton J. Coordinate indexing strategies for the laplace stretch in two and three dimensions. DEVCOM Army Research Laboratory (US); 2022. Report No.: ARL-TR-9530.
12. Truesdell C, Toupin R. The classical field theories. In: *Handbuch der Physik*; Vol. III. Flugge S, editor. Springer; 1960. p. 226–793.
13. Bowen R. Theory of mixtures. In: *Continuum Physics*; Vol. 3. Eringen A, editor. Academic Press; 1976. p. 1–127.
14. Bowen R. Incompressible porous media models by use of the theory of mixtures. *International Journal of Engineering Science*. 1980;18:1129–1148.
15. Bowen R. Compressible porous media models by use of the theory of mixtures. *International Journal of Engineering Science*. 1982;20:697–735.
16. Ehlers W. Foundations of multiphasic and porous materials. In: *Porous media: theory, experiments and numerical applications*. Ehlers W, Bluhm J, editors. Springer; 2002. p. 3–86.
17. Coussy O. *Poromechanics*. John Wiley and Sons; 2004.
18. de Boer R. *Trends in continuum mechanics of porous media: Theory and applications of transport in porous media*. Springer; 2005.
19. Bowen R, Chen P. Shock waves in a mixture of linear elastic materials. *Rendiconti del Circolo Matematico di Palermo*. 1972;21:267–283.
20. Bowen R, Wright T. On the growth and decay of wave fronts in a mixture of linear elastic materials. *Rendiconti del Circolo Matematico di Palermo*. 1972;21:209–234.
21. Bowen R, Wright T. On wave propagation in a mixture of linear elastic materials. US Army Ballistic Research Laboratory; 1972. Report No.: 1581.
22. Bowen R, Chen P. Shock waves in ideal fluid mixtures with several temperatures. *Archive for Rational Mechanics and Analysis*. 1974;53:277–294.
23. Bowen R, Chen P, Nunziato J. Shock waves in a mixture of chemically reacting materials with memory. *Acta Mechanica*. 1975;21:1–11.

24. Clayton J. Analysis of shock waves in a mixture theory of a thermoelastic solid and fluid with distinct temperatures. *International Journal of Engineering Science*. 2022;175:103675.
25. Tsokos M, Paulsen F, Petri S, Madea B, Püschel K, Türk E. Histologic, immunohistochemical, and ultrastructural findings in human blast lung injury. *American Journal of Respiratory and Critical Care Medicine*. 2003;168:549-555.
26. Cooper G, Townend D, Cater S, Pearce B. The role of stress waves in thoracic visceral injury from blast loading: Modification of stress transmission by foams and high-density materials. *Journal of Biomechanics*. 1991;24:273-285.
27. Cooper G. Protection of the lung from blast overpressure by thoracic stress wave decouplers. *The Journal of Trauma: Injury, Infection, and Critical Care*. 1996;40:105S-110S.
28. Irwin Z, Regueiro R, Clayton J. A large deformation multiphase continuum mechanics model for shock loading of lung parenchyma. Part I: Theory. DEVCOM Army Research Laboratory (US); 2023. Report No.: ARL-TR-9686.
29. Irwin Z, Regueiro R, Clayton J. A large deformation multiphase continuum mechanics model for shock loading of lung parenchyma. Part III: Numerical simulations. DEVCOM Army Research Laboratory (US); 2023. Report No.: ARL-TR-9688.
30. Irwin Z, Clayton J, Regueiro R. A large deformation multiphase continuum mechanics model for shock loading of soft porous materials. *International Journal for Numerical Methods in Engineering*. Forthcoming 2023.
31. Hallquist J. LS-DYNA user's manual R12.0. Livermore Software Technology Corporation; 2020.
32. Holzapfel G. *Nonlinear solid mechanics: A continuum approach for engineering*. John Wiley and Sons; 2000.
33. Clayton J. *Nonlinear mechanics of crystals*. Springer; 2011.
34. Clayton J. *Differential geometry and kinematics of continua*. World Scientific; 2014.

35. Hughes T. The finite element method: Linear static and dynamic finite element analysis. Dover Publications, Inc.; 2000.
36. Newmark N. A method of computation for structural dynamics. *Journal of the Engineering Mechanics Divison*. 1959;85:67-94.
37. Bogacki P, Shampine L. A 3(2) pair of Runge-Kutta formulas. *Applied Mathematics Letters*. 1989;2:321-325.
38. Cash JR, Karp A. A variable order runge-kutta method for initial value problems with rapidly varying right-hand sides. *ACM Transactions on Mathematical Software*. 1990;16:201–222.
39. Press W, Teukolsky S, Vetterling W, Flannery B. *Numerical recipes in c: The art of scientific computing*. 2nd ed. Cambridge University Press; 1992.
40. von Neumann J, Richtmyer R. A method for the numerical calculation of hydrodynamical shocks. *Journal of Applied Physics*. 1950;21:232-237.
41. Landshoff R. A numerical method for treating fluid flow in the presence of shocks. Los Alamos Scientific Laboratory; 1955. Report No.: 1930.
42. Benson D. Explicit finite element methods for large deformation problems in solid mechanics. In: *Encyclopedia of Computational Mechanics*; Vol. 2. Stein E, de Borst R, Hughes T, editors. John Wiley and Sons; 2007. p. 43.
43. Margolin L, Lloyd-Ronning N. Artificial viscosity—then and now. *arXiv*. 2022;2202:11084.
44. Wilkins M. Use of artificial viscosity in multidimensional fluid dynamic calculations. *Journal of Computational Physics*. 1980;36:281–303.
45. Hallquist J. *LS-DYNA theory manual*. Livermore Software Technology Corporation; 2020.
46. Truty A, Zimmerman T. Stabilized mixed finite element formulations for materially nonlinear partially saturated two-phase media. *Computer Methods in Applied Mechanical Engineering*. 2006;195:1517-1546.
47. Brezzi F, Pitkäranta J. On the stabilization of finite element approximations of the Stokes equations. In: *Efficient solutions of elliptic systems: proceedings of*

a GAMM seminar. Hackbusch W, editor. Vieweg+Teubner Verlag; 1984. p. 11–19.

List of Symbols, Abbreviations, and Acronyms

TERMS:

1-D	one-dimensional
3-D	three-dimensional
ARL	Army Research Laboratory
CD	central difference
CFD	computational fluid dynamics
CFL	Courant-Friedrichs-Lewy
DEVCOM	US Army Combat Capabilities Development Command
DOF	degree(s) of freedom
ECM	extra-cellular matrix
f	fluid
FE	finite element
FSI	fluid-structure interaction
LHS	left-hand side
NB	Newmark-beta
ODE	ordinary differential equation
s	solid
SF	safety factor
TPM	theory of porous media

MATHEMATICAL SYMBOLS:

e	internal energy per unit mass
\mathbf{F}, F_{iJ}	deformation gradient
p	Cauchy pressure
t	time
\mathbf{u}, u_k	displacement

V	volume
\mathbf{x}, x_k	Cartesian spatial coordinates
\mathbf{X}, X_K	Cartesian reference coordinates
η	entropy per unit mass
ψ	free energy per unit mass
θ	temperature
ρ	mass density
$\boldsymbol{\sigma}, \sigma_{ij}$	Cauchy stress
\mathbf{v}, v_k	velocity

1 DEFENSE TECHNICAL
(PDF) INFORMATION CTR
DTIC OCA

1 DEVCOM ARL
(PDF) FCDD RLB CI
TECH LIB

36 DEVCOM ARL
(PDF) FCDD RLA B
R BECKER
C HOPPEL
P GILLICH
A TONGE
FCDD RLA HC
J CRONE
J KNAP
FCDD RLA MB
G GAZONAS
FCDD RLA T
B LOVE
M FERREN-COKER
S SCHOENFELD
FCDD RLA TA
S TURNAGE
C WILLIAMS
FCDD RLA TB
S ALEXANDER
R BANTON
T BAUMER
D CASEM
J CLAYTON
B FAGAN
A GOERTZ
C HAMPTON
R KARGUS
M KLEINBERGER
D KRAYTERMAN
J MCDONALD
C MEREDITH
T PLAISTED
K RAFAELS
R REGUEIRO
S SATAPATHY
T WEERASOORIYA
C WEAVER
S WOZNIAK
T ZHANG
FCDD RLA TE
J LLOYD
T SCHARF
FCDD RLA TF
J CAZAMIAS