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A Large Deformation Multiphase Continuum Mechanics Model for Shock Loading of Lung Parenchyma. Part III: Numerical Simulations

**by Zachariah T Irwin, Richard A Regueiro, and
John D Clayton**

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A Large Deformation Multiphase Continuum Mechanics Model for Shock Loading of Lung Parenchyma. Part III: Numerical Simulations

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14. ABSTRACT Simulations obtained from a 1-D numerical implementation of a finite-strain theory of a biphasic mixture are reported. The theory includes coupled pore fluid flow and solid skeleton deformation for a soft porous material applicable to high strain-rate dynamic loading. The constitutive model is non-linear elastic and accounts for the compressibility of the pore air. General features of the constitutive description are specialized for an application to shock loading of lung parenchyma. This report focuses on representative numerical simulations of the lung, invoking the 1-D finite element formulation of the multiphase theory and nonlinear constitutive models. Verification studies and predictive simulations for dynamic deformation of the lung are reported. Shock wave-type loadings are of prominent interest in this document, which is the third in a series of three reports.					
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This report encompasses and extends research recently submitted in abbreviated form by the authors in a refereed article* to the *International Journal for Numerical Methods in Engineering* (2023).

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1. Introduction

1.1 Background

The current work numerically simulates the 1-D poroelastodynamics as a shock wave propagates through the lung parenchyma, at the mesoscopic level. At high rates of strain, such as those rates induced by a shock wave, the elastin fibers in the parenchyma are prone to rupture,¹ and the remaining stress response percolates through the collagen fibril networks as the collagen fibers extend to their uncoiled lengths. This increases the stress response of the entire lung parenchyma.² The complex roles that collagen and elastin fibers play in the dynamics of the lung parenchyma are discussed by Freed et al.¹ Dynamic influences of the bronchiole tubes, which are stiffer than the parenchyma, on the response of impacted lung have been recently observed in experiments by Brannen et al.³ In the context of trauma and loss of biological function, mechanical forces that act upon the extra-cellular matrix (ECM), in which the collagen and elastin fibers reside, can induce the secretion of growth factors that accelerate ECM remodeling and alter the microstructural composition of the fiber networks.⁴

The theory of porous media (TPM) is an approximation to a more computationally expensive fluid-structure interaction (FSI) model at the pore length scale. In TPM, interactions are smeared across a continuum material point, simplifying not just the governing mathematical equations but the discretization of the geometry itself. Mixture theory was first established at finite strain by Truesdell and Toupin.⁵ Subsequent works by Bowen^{6–8} and others^{9–11} applied this rich continuum theory to porous media (i.e., TPM). Using analytical techniques rather than numerical simulations as in the present work, shock waves in mixtures were studied by Bowen, Chen, and Wright^{12–16} and more recently, by Clayton.¹⁷ Numerical methods allow for solution of more complex governing equations, more general loading protocols, and more sophisticated constitutive equations.

In the current program,^{18,19} we incorporate TPM with advanced numerical methods to consider the different response times of the two constituents in the lung parenchyma subjected to shock loading: solid skeleton (s) (lung parenchyma) and the pore fluid (f) (air) that occupies the pore space. Outcomes of the current program of research are intended to be used to inform macroscopic, single-phase models of the lung for 3-D modeling of much larger domains.^{20–25}

Shock waves induce large pressure gradients that are thought to directly cause microstructural tearing of the lung parenchyma leading to hemorrhaging.^{26–28} Injury propagations through the lung parenchyma at the ultrastructural (tissue fibers) and microstructural (air sac) levels are not well understood and constitute active areas of research.^{1,22} The fine-scale approach initiated by Freed et al.¹ to model the microstructure necessitates a multiscale model of the whole lung tissue, which includes the highly heterogeneous structure of the lung parenchyma.

Research is needed to accurately simulate the deformation of lung parenchyma using a multiphysics approach, before addressing damage and injury pathology at the microscale (i.e., alveolar regime) and linking that to the mesoscale (i.e., parenchymal regime). Resolution of heterogeneous microstructures would require representation by a detailed 3-D finite-element (FE) mesh to model the *explicit* FSI. Resolution could be possible via computational fluid dynamics-computational solid mechanics with arbitrary Lagrangian-Eulerian modeling. Such costly software frameworks are outside the current scope of research.

This report exercises the variational equations of TPM for balances of mass and linear momentum for a biphasic mixture at large deformations through the FE implementation of the previous report.¹⁹ See Irwin et al.¹⁸ for notation that is consistently used throughout this report and the other two reports.^{18,19} The biphasic model is subject to the following assumptions^{18,19,29}:

- saturated solid-fluid mixture:
 $n^s + n^f = 1$,
- materially incompressible solid:
 $\rho^{sR} = \text{const.}$,
- materially compressible pore fluid,
- no mass production:
 $\hat{\gamma}^\alpha := 0$,
- non-polar materials:
 $\boldsymbol{\sigma}^\alpha = (\boldsymbol{\sigma}^\alpha)^T$,
- uniform gravitational body force:
 $\mathbf{b}^\alpha = \mathbf{b} = \mathbf{g}$,
- mixed constituent temperatures:
 $\theta^s = \theta^f = \theta$,
- no thermo-mechanical coupling:
 $\gamma^s := 0, \quad \gamma^f := 0$,
- nearly-inviscid pore fluid:
 $\boldsymbol{\sigma}_E^f \approx \mathbf{0}$,
- neo-Hookean constitutive model for the solid (s) constituent
- dissipative viscosity for shocks in solid material

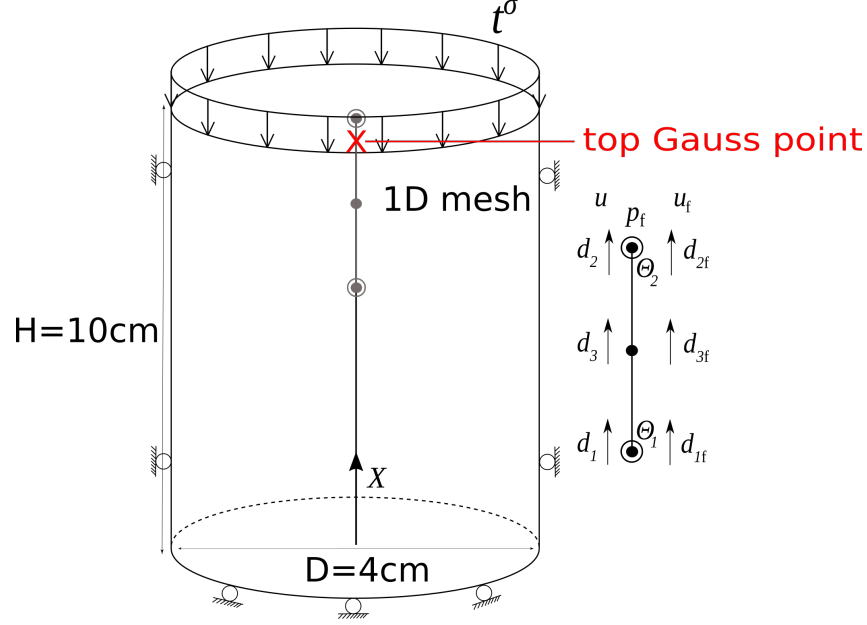


Fig. 1 Schematic for a cylindrical mesh used in the 1-D uniaxial strain simplification, where the Q2-Q2-P1 element¹⁹ is highlighted. Here, d corresponds to the interpolated solid skeleton (s) displacement solution, d_f corresponds to the interpolated pore fluid (f) displacement solution, and θ corresponds to the interpolated pore fluid pressure solution at their respective nodes in the FE model.

1.2 Geometric Model

As discussed in the previous two reports,^{18,19} we model an excised section of lung parenchyma as either a uniform cylinder, Fig. 1, or as a uniform rectangular column, Fig. 2. The choice between these two geometries is somewhat arbitrary as long as the motion is 1-D. For uniaxial strain with unidirectional flow (i.e., no permeation of air through the sides of the cylinder), the overall geometry of the lung parenchyma becomes irrelevant besides the (finite) length H and area A . This assumption is appropriate for immediate motions resulting from shock loading in which the magnitude of transverse motions (i.e., perpendicular to the direction of the shock front) are small compared to the motions aligned with the shock velocity. When making comparisons to computational results from other software, such as those obtained using the LS-DYNA code,^{24,30} we typically assume a rectangular column for ease of meshing. For the domain sizes shown in Figs. 1 and 2, the rectangular choice has the sole effect of reducing the cross-sectional area (i.e., area of $L^2 = 1 \text{ cm}^2$) relative to that of the cylindrical column (i.e., area of $\pi D^2/4 \approx 12.6 \text{ cm}^2$).

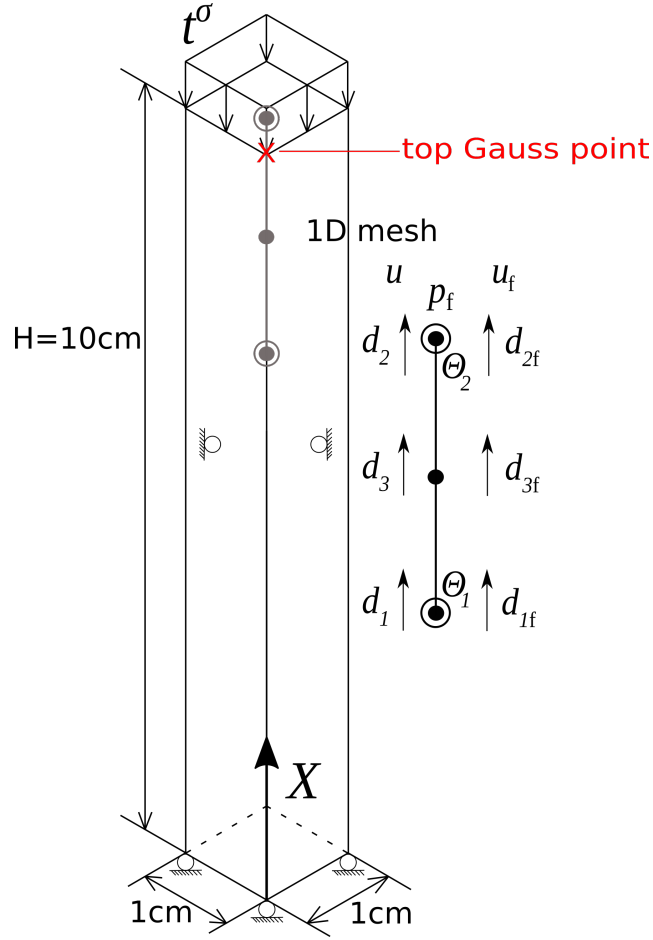


Fig. 2 Schematic for a rectangular mesh used in the 1-D uniaxial strain simplification, where the Q2-Q2-P1 element¹⁹ is highlighted. Here, d corresponds to the interpolated solid skeleton (s) displacement solution, d_f corresponds to the interpolated pore fluid (f) displacement solution, and θ corresponds to the interpolated pore fluid pressure solution at their respective nodes in the FE model.

1.3 Organization

Subsequent content of this report is organized in the following manner. As introduced in a prior report,¹⁹ Section 2 reports the kinematic simplifications for 1-D motion. Since many results are compared with outcomes of a single-phase constitutive model^{22,24,25} implemented in the LS-DYNA code,³⁰ the LS-DYNA implementation is summarized in Section 3.

Results of calculations follow in the next two sections, constituting the bulk of this document. In Section 4, we provide examples to verify the FE formulation¹⁹ of the elastodynamic, poroelastic, and poroelastodynamic governing equations^{18,29} against analytical solutions or other published results. In Section 5, examples consider how lung tissue deforms under shock loading. Simulations predict the response of a column of parenchymal tissue loaded intensely over timescales of milliseconds or less. Comparisons are made from our multiphase model to results from the single-phase model of Clayton, Banton, and Freed^{20–25} implemented in LS-DYNA. Detailed derivations supporting the numerical implementation¹⁹ and examples that follow in this report for the various space-time discretization schemes are presented in Appendices A, B, and C focusing on respective Gateaux derivatives, linearized equations, and FE equations.

The term “elasticity” is used to refer to a quasi-static setting (i.e., inertial terms and stress waves absent) with linear or nonlinear elastic constitutive behavior. The term “elastodynamics” refers to a fully dynamic setting (i.e., inertial terms retained in the momentum balance, leading to stress waves), again with linear or nonlinear elastic behavior.

2. 1-D Kinematics

As introduced by Irwin et al.,^{19,29} 1-D uniaxial solid skeleton strain and unidirectional pore fluid flow simplifications are implemented for (s) and (f) phases, respectively. If we assume that the soft porous material only undergoes compression or expansion in one dimension, specifically the vertical direction X , then both the solid skeleton displacements in the transverse directions as well as all shear terms in the deformation gradient go to zero. The solid skeleton deformation gradient becomes

$$\mathbf{F}_s = \begin{bmatrix} 1 + \frac{\partial u}{\partial X} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad J_s = \det \mathbf{F}_s = F_{11(s)} = 1 + \frac{\partial u}{\partial X} \quad (1)$$

where $u(X, t)$ is the axial displacement in the X direction. Similarly, if we assume that pore fluid flow is confined in direction X (i.e., surrounded by an impermeable sleeve, drained or undrained at its ends), then

$$\mathbf{v}_f = \begin{bmatrix} v_f \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{a}_f = \begin{bmatrix} a_f \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{u}_f = \begin{bmatrix} u_f \\ 0 \\ 0 \end{bmatrix} \quad (2)$$

where in the X direction, v_f is the pore fluid velocity, a_f is the pore fluid acceleration, and u_f is the pore fluid displacement. Simulations with TPM reported subsequently in this report only consider the spatially 1-D regime.

3. Differences Between LS-DYNA Model and Multiphase Model

Before presenting the numerical examples in this report, it is prudent to highlight some key differences between the different numerical models whose results will be compared. LS-DYNA,^{30,31} originally known as DYNA3D, was developed by John Hallquist at Lawrence Livermore National Laboratory in 1976. It is a “general-purpose finite element code for analyzing the large deformation static and dynamic response of structures including structures coupled to fluids”³¹ written in the FORTRAN programming language. The code has been highly optimized for massive multi-parallel explicit dynamics computations. The prominent physical governing equation that LS-DYNA seeks to solve is the balance of linear momentum equa-

tion in the current configuration (e.g., see references on fundamental continuum mechanics^{5,32–35}):

$$\rho \mathbf{a} = \text{div } \boldsymbol{\sigma} + \rho \mathbf{b} \quad (3)$$

wherein the Cauchy stress $\boldsymbol{\sigma}$ is defined by a wide range of available constitutive models. In Section 5, the LS-DYNA material model we employed was a neo-Hookean material model (MAT 45), identical to the one we defined in Section 3 of Irwin et al.¹⁸ A similar neo-Hookean model, albeit also including thermoelastic coupling, was invoked for the solid tissue phase in the analytical study of shock-waves in the tissue-air mixture of the lung by Clayton.¹⁷

There also exists a poroelastodynamics framework in LS-DYNA, which is what we compare to in Section 4. However, its applications are limited. For starters, Darcy’s velocity in LS-DYNA does not account for inertial effects as referenced by the CONTROL_PORE_FLUID input card in the user’s manual³⁰:

$$(n^f \tilde{\mathbf{v}}_f) = -\hat{k} \left(\frac{\partial p_f}{\partial \mathbf{x}} - \rho^f \mathbf{b} \right) \quad (4)$$

Note also that $\hat{k}$ in Eq. 4 is a *constant* hydraulic conductivity parameter. It is not a function of deformation (e.g., no dependence upon porosity change under large deformation). Furthermore, pore fluid pressure boundary conditions can only be specified at the top of the domain, in the z direction, which is limiting for modeling soft porous materials wherein one might want to apply pore fluid pressure boundary conditions at other surfaces on the domain. In our use of LS-DYNA’s poroelastodynamics model, we also found that we could only achieve accurate results for the verification problem in Section 4 by (1) using a linear isotropic elastic constitutive material model for the solid skeleton response, and (2) initializing the solid skeleton effective stress $\boldsymbol{\sigma}_E^s$ due to “geostatic” gravity. The result of (2) is that the solution for the solid skeleton effective stress $\boldsymbol{\sigma}_E^s$ does not match the analytical solution, because the analytical solution ignores body forces.

LS-DYNA also provides the user with the ability to install their own constitutive material model within the preexisting elastodynamics framework. This is what was accomplished for analysis of lung parenchyma tissue by Clayton and colleagues.^{22,24} This is the model we generally refer to as the “single-phase” model in the text or to as “Clayton and Freed” in the figures. In the single-phase model,^{22,23,36}

pore air cannot move relative to the motion of the solid skeleton lung tissue. However, the constitutive model presented by Clayton and Freed^{21–23} is based on the biological tissue constitutive theory, in contrast to our simpler neo-Hookean model. A viscoelastic response is also considered in the single-phase model,^{20,36,37} though more research is needed to determine whether or not lung tissue can be treated as a viscoelastic versus poroelastic material (or both) for high strain-rate loadings. A damage variable is also introduced by Clayton and Freed^{20,22,24} to address local injury. When making comparisons to the single-phase model, we disable damage mechanics by adjusting the associated parameters appropriately. Enabling damage in the multiphase solid skeleton effective stress σ_E^s constitutive model is anticipated to be part of future work.

LS-DYNA provides the user with choices among several time integration schemes to find the solution of Eq. 3: either an implicit Newmark-beta (NB) scheme as referred to Section 5.1 of Irwin et al.¹⁹ or a central difference (CD) scheme as referred to Section 5.3 of Irwin et al.¹⁹ By default, LS-DYNA employs CD time integration scheme with lumped mass matrices (i.e., row-sum lumping technique), where the time step is limited by a local Courant–Friedrichs–Lewy (CFL) condition as referred to in Eq. 440 of Irwin et al.¹⁹ In our LS-DYNA simulations, we use the default time integration scheme. For the linearly interpolated hexahedral elements employed in our LS-DYNA simulations, single-point quadrature rules are applied by default. Thereby, constitutive quantities such as stress are evaluated at the Gauss point located at the centroid of each element.

Lastly, bulk shock viscosity is by default enabled to regularize the shock front propagation³⁸ and in turn stabilize the numerical solution; refer to Section 6.1 of Irwin et al.¹⁹ On this last point, to the best of our knowledge it is *impossible* to disable the bulk shock viscosity in LS-DYNA through direct manipulation of the `CONTROL_BULK_VISCOSITY` and/or `CONTROL_HOURLGLASS` cards. Refer to Chapter 3.1 in the LS-DYNA theory manual³¹ as well as the remark in the `CONTROL_BULK_VISCOSITY` card in the LS-DYNA user’s manual.³⁰ This lack of commercial code flexibility will be exemplified in Section 5.

4. Verification

In the following section we provide examples to verify our FE formulation of the elastodynamic, poroelastic, and poroelastodynamic governing equations against either analytical solutions if they exist or against other published results. Validation of our model is beyond the scope of this report. For discussion of the verification and validation procedures, refer to Oberkampf et al.,³⁹ Babuska and Oden,⁴⁰ Schwer,⁴¹ and Sun.⁴²

4.1 Elastodynamics

For this example, we employ the eigenfunction expansion technique used by Eringen and Suhubi,⁴³ which is a modification of the technique proposed by Reissmann,⁴⁴ to solve the following differential equation:

$$c_1^2 \nabla^2 \mathbf{u} - c_2^2 \nabla \times \nabla \times \mathbf{u} + \mathbf{f} = \frac{\partial^2 \mathbf{u}}{\partial t^2} \quad \forall \mathbf{x} \in \Omega \times (0, T) \quad (5)$$

subjected to the boundary conditions

$$\begin{aligned} \mathbf{u}(\mathbf{x}, t) &= \mathbf{g}^u(\mathbf{x}, t) \quad \forall \mathbf{x} \in \Gamma^u, \\ \boldsymbol{\sigma}(\mathbf{x}, t) \cdot \mathbf{n}(\mathbf{x}) &= \mathbf{t}^\sigma(\mathbf{x}, t) \quad \forall \mathbf{x} \in \Gamma^t \end{aligned} \quad (6)$$

and initial the conditions

$$\begin{aligned} \mathbf{u}(\mathbf{x}, 0) &= \mathbf{u}_0(\mathbf{x}), \\ \mathbf{v}(\mathbf{x}, 0) &= \mathbf{v}_0(\mathbf{x}) \end{aligned} \quad (7)$$

Here, c_1 and c_2 are the irrotational and equivoluminal wave velocities, respectively. The solution $\mathbf{u}(\mathbf{x}, t)$ may also be expressed as a summation of an eigenfunction $\Psi(\mathbf{x})$ subjected to homogeneous boundary conditions and a quasi-static displacement field $\Phi(\mathbf{x}, t)$:

$$\mathbf{u}(\mathbf{x}, t) = \Phi(\mathbf{x}, t) + \sum_n \phi_n(t) \Psi_n(\mathbf{x}) \quad (8)$$

wherein the strong formulation for the eigenfunction is given by

$$c_1^2 \nabla^2 \Psi - c_2^2 \nabla \times \nabla \times \Psi + \omega^2 \Psi = \mathbf{0} \quad \mathbf{x} \in \Omega \times (0, T) \quad (9)$$

$$\Psi(\mathbf{x}, t) = \mathbf{0} \quad \forall \mathbf{x} \in \Gamma^u \quad (10)$$

$$\boldsymbol{\sigma}(\mathbf{x}, t) \cdot \mathbf{n}(\mathbf{x}) = \mathbf{0} \quad \forall \mathbf{x} \in \Gamma^t \quad (11)$$

Denoted by ω is the natural frequency (i.e., the eigenvalue), and the strong formulation for the quasi-static displacement field is given by

$$c_1^2 \nabla^2 \Phi - c_2^2 \nabla \times \nabla \times \Phi + \mathbf{f}(\mathbf{x}, t) = \mathbf{0} \quad \mathbf{x} \in \Omega \times (0, T) \quad (12)$$

$$\Phi(\mathbf{x}, t) = \mathbf{g}^u(\mathbf{x}, t) \quad \forall \mathbf{x} \in \Gamma^u \quad (13)$$

$$\boldsymbol{\sigma}(\mathbf{x}, t) \cdot \mathbf{n}(\mathbf{x}) = \mathbf{t}^\sigma(\mathbf{x}, t) \quad \forall \mathbf{x} \in \Gamma^t \quad (14)$$

Substitution of Eq. 8 into Eq. 5 and making use of Eq. 9 gives

$$\sum_n \left(\ddot{\phi}_n + w_n^2 \phi_n \right) \Psi_n(\mathbf{x}) = -\ddot{\Psi}(\mathbf{x}, t) \quad (15)$$

Multiplying Eq. 15 by $\Phi_m(\mathbf{x})$, integrating over Ω and invoking the orthogonality condition on Φ gives

$$\ddot{\phi}_m + \omega_m^2 \phi_m = \ddot{\Phi}_m(t) \quad (16)$$

where

$$\Phi_m(t) = - \int_{\Omega} \phi(\mathbf{x}, t) \cdot \Psi(\mathbf{x}) d\Omega \quad (17)$$

To solve Eq. 16, we require initial conditions, which we can pull from Eq. 7 and substitute into Eq. 8 to find

$$\begin{aligned} \phi_m(0) &= \int_{\Omega} \mathbf{u}_0(\mathbf{x}) \cdot \Phi_m(\mathbf{x}) d\Omega + \Phi_m(0), \\ \dot{\phi}_m(0) &= \int_{\Omega} \mathbf{v}_0(\mathbf{x}) \cdot \Phi_m(\mathbf{x}) d\Omega + \dot{\Phi}_m(0) \end{aligned} \quad (18)$$

Then the solution to Eq. 16 is

$$\begin{aligned}\phi_m(t) = & [\phi_m(0) - \Phi_m(0)] \cos(\omega_m t) + \frac{1}{\omega_m} [\dot{\phi}_m(0) - \dot{\Phi}_m(0)] \sin(\omega_m t) \\ & + \Phi_m(t) - \omega_m \int_0^t \Phi_m(\tau) \sin(\omega_m(t - \tau)) d\tau\end{aligned}\quad (19)$$

Then, Eq. 8 may be rewritten as

$$\begin{aligned}\mathbf{u}(\mathbf{x}, t) = & \sum_m \left[\alpha_m \cos(\omega_m t) + \frac{1}{\omega_m} \beta_m \sin(\omega_m t) \right. \\ & \left. - \omega_m \int_0^t \Phi_m(\tau) \sin(\omega_m(t - \tau)) d\tau \right] \Psi_m(\mathbf{x})\end{aligned}\quad (20)$$

where

$$\phi(\mathbf{x}, t) + \sum_m \Phi_m(t) \Psi_m(\mathbf{x}) \rightarrow 0 \quad (21)$$

and

$$\alpha_m = \int_{\Omega} \mathbf{u}_0(\mathbf{x}) \cdot \Phi(\mathbf{x}) d\Omega, \quad \beta_m = \int_{\Omega} \mathbf{v}_0(\mathbf{x}) \cdot \Phi(\mathbf{x}) d\Omega \quad (22)$$

For this particular verification problem, we presuppose that our domain is 1-D, that it is fixed at its lower end and its sides, and that we apply a sinusoidal traction to the top surface:

$$\begin{aligned}u(X, t) &= 0 \quad \forall X \in \Gamma^u, \\ \mathbf{t}^\sigma(t) &= \frac{t_0^\sigma}{2} [1 - \cos(\omega t)] \quad \forall X \in \Gamma^t\end{aligned}\quad (23)$$

with initial conditions

$$\begin{aligned}u(X, 0) &= 0, \\ v(X, 0) &= 0\end{aligned}\quad (24)$$

Then Eq. 20 reduces to

$$u(X, t) = - \sum_m \omega_m \Psi_m(X) \int_0^t \Phi_m(\tau) \sin(\omega_m(t - \tau)) d\tau \quad (25)$$

It can be shown that

$$\Phi_m(\tau) = \frac{2(2H)^{1/2} t^\sigma(X, t) (-1)^m}{\pi \rho_0 c_1 \omega_m (2m - 1)} \quad (26)$$

where H is the initial height of the column, which allows us to rewrite Eq. 25 as

$$u(X, t) = \frac{4}{\pi \rho_0 c_1} \sum_{m=1}^{\infty} \frac{(-1)^m}{2m - 1} \left[\int_0^t t^\sigma(\tau) \sin\left(\frac{(2m - 1)\pi c_1(t - \tau)}{2H}\right) d\tau \right] \times \sin\left(\frac{(2m - 1)\pi X}{2H}\right) \quad (27)$$

where

$$c_1 = \sqrt{\frac{E \left(\frac{1-\nu}{(1+\nu)(1-2\nu)} \right)}{\rho_0}} = \sqrt{\frac{M}{\rho_0}} \quad (28)$$

and where E , ν , and M are Young's modulus, the Poisson ratio, and the P-wave modulus for the material, respectively.

Values of material and geometric parameters used in this example are given in Tables 1 and 2, respectively. A sinusoidal load in the form of a traction boundary condition, as shown in Fig. 3a, is applied to the top of the single-phase column shown in Fig. 3b. The displacement is probed at four depths, each of which are measured from the base of the column. Results are shown in Fig. 4.

We see excellent agreement between the analytical solution and the various time integrators we have implemented in the custom Python code, as well the default integration scheme (CD with lumped mass) in LS-DYNA. We do note, however, that the implicit integrators perform poorly for this example, particularly as element size is reduced, both in terms of computational cost and convergence. On the contrary, the explicit integrators are computationally inexpensive while maintaining accuracy and stability.

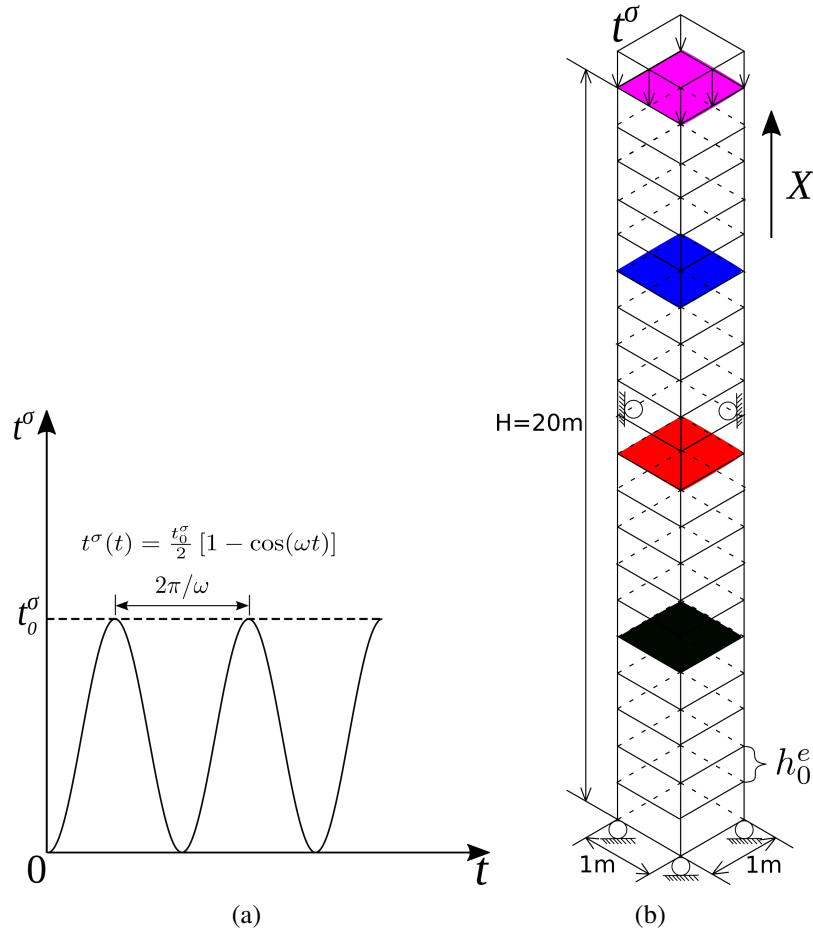


Fig. 3 a) Traction application and b) schematic of column mesh for the elastodynamics verification example

Table 1. Material parameters for elastodynamics verification example

E (MPa)	ν	ρ_0 (kg/m ³)
50	0.3	1986

Table 2. Geometrical and loading parameters for elastodynamics verification example

H (m)	A (m ²)	t_0^σ (kPa)	ω (rad/s)
20	1	40	50

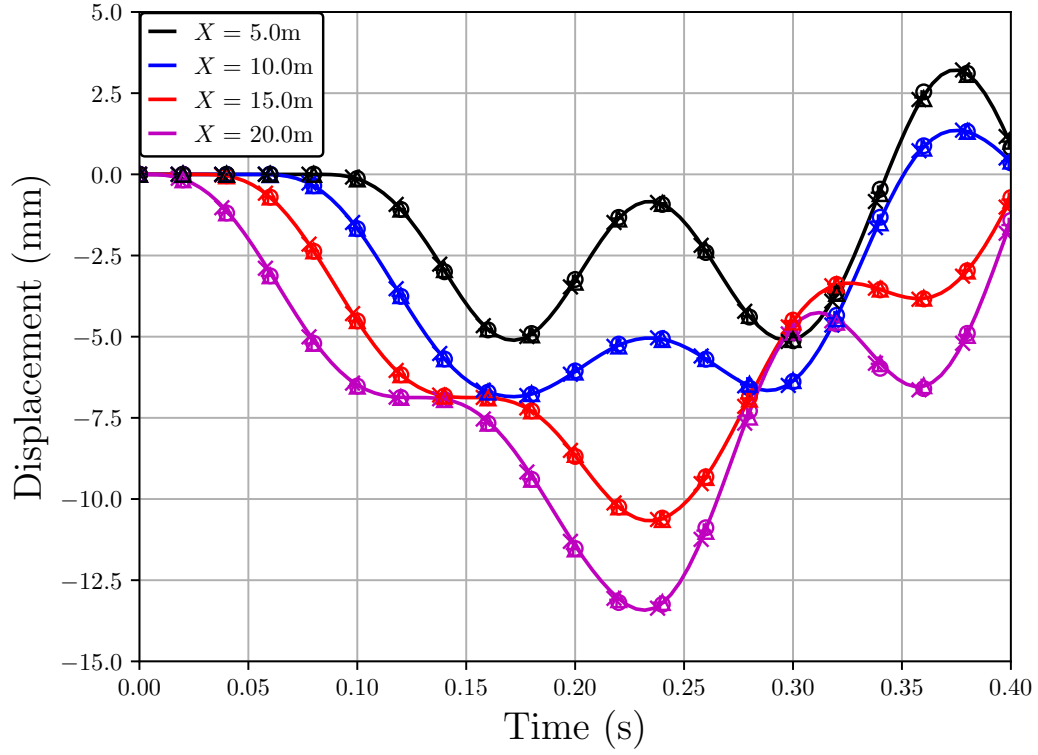


Fig. 4 Results for the displacements for the elastodynamics verification. Here, Q1 elements were used in the Python implementation and Q8 elements were used in LS-DYNA, by default, with mesh size $h_0^e = 1\text{ m}$. Symbols are identified in the mini-table below.

Symbol	Simulation
—	Analytical
○	$(\mathbf{u})$, NB ($\beta = 0.25, \gamma = 0.5$)
×	$(\mathbf{u})$, CD (L. mass)
+	$(\mathbf{u})$, RKFNC (L. mass)
△	$(\mathbf{u})$, LS-DYNA

4.2 Poroelasticity

Recall from the previous report by Irwin et al.¹⁹ the poroelasticity equations for the $(\mathbf{u}-p_f)$ formulation *without* inertia terms. Here, we present results previously published in Li et al.⁴⁵ that demonstrate the differences between small-strain theory and finite-strain theory (i.e., large deformation).

Similar to the elastodynamics example in Section 4.1, a porous column is subjected to an external traction load while its base and sides remain fixed. The analytical solution in small-deformation theory is given by

$$\Delta H = \frac{t_0^\sigma H}{\lambda + 2\mu} \quad (29)$$

where H is the initial height of the column and λ and μ are the first and second Lamé parameters of the solid skeleton, respectively. Values of material and geometric parameters used in this example are given in Tables 3 and 4, respectively. The upper boundary of the fully saturated column is perfectly drained with reference pressure $p_f(X = H, t) = 0$ atm and subjected to a uniform step load with finite rise time, as shown in Fig. 5. Different magnitudes of the maximum applied traction are considered, as listed in Table 4.

Table 3. Material parameters for the poroelasticity verification example

λ (MPa)	μ (MPa)	ρ_0^s (kg/m ³)	ρ_0^f (kg/m ³)	n_0^f	n_0^s
29	7	2700	1000	0.42	0.58

Table 4. Geometrical and loading parameters for the poroelasticity verification example

H (m)	A (m ²)	h_0^e (m)	t_0^σ (MPa)	t_0	t_1
10	1	1	0.04, 2, 4, 8	0.3	1.0

The displacement histories of the top of the column are plotted in Fig. 6 for the milder loading on the order of kilopascals. Owing to the fact that the displacement for our large deformation model is so small, the results line up well with the analytical solution for small-strain theory. However, when load is increased to the order of megapascals, the small-strain theory always predicts deformation magnitudes well-exceeding that of finite strain (i.e., large deformation), as shown in Fig. 7.

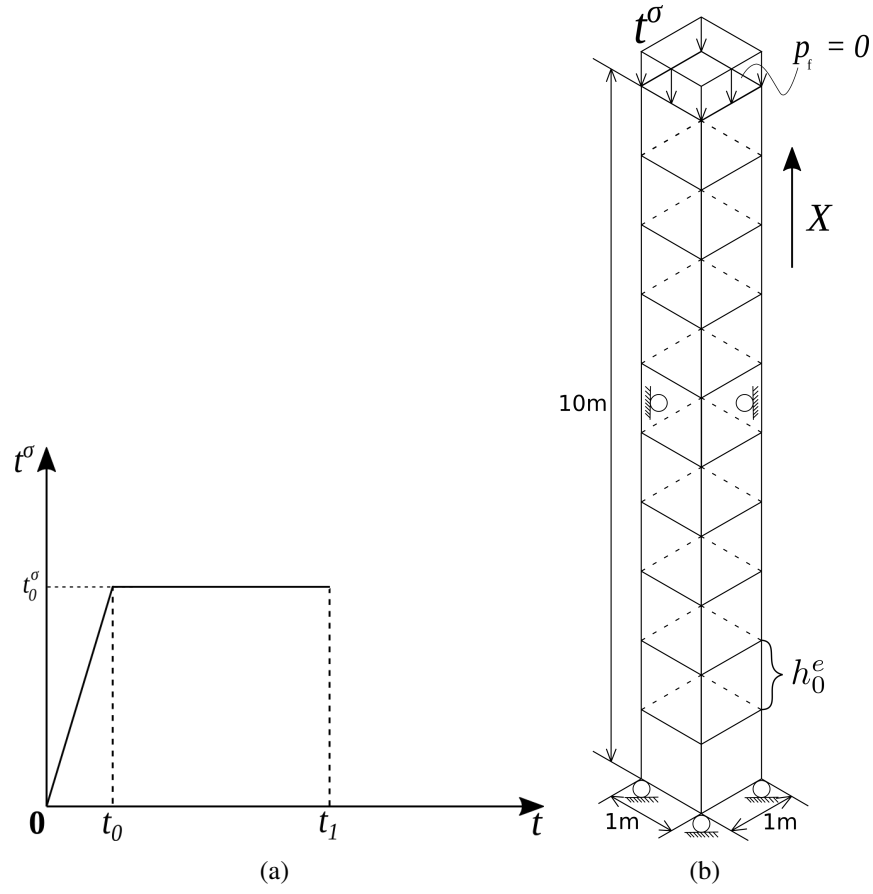


Fig. 5 a) Traction application and b) schematic of column mesh for the poroelasticity verification example

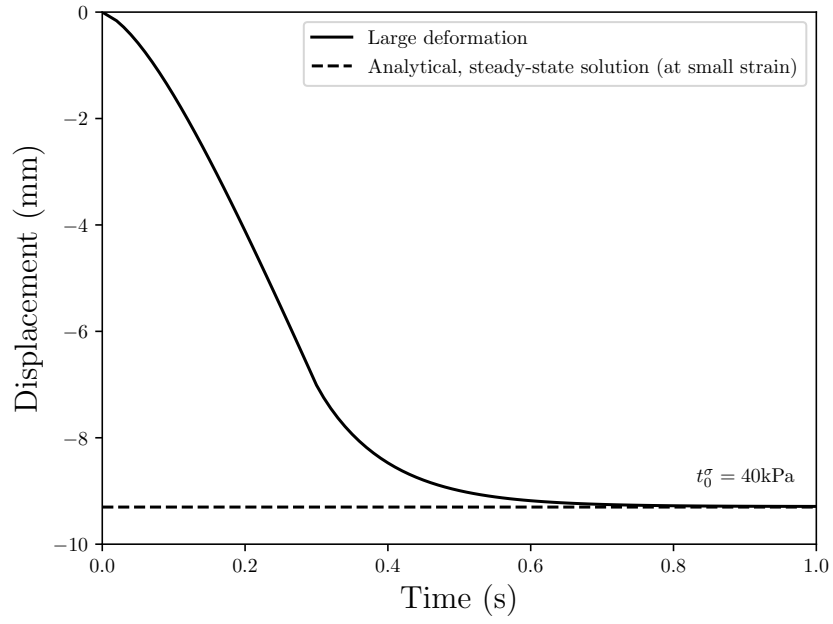


Fig. 6 Porous layer with uniform step load: vertical displacement-time history at topmost node for mild load $t_0^\sigma = 40$ kPa

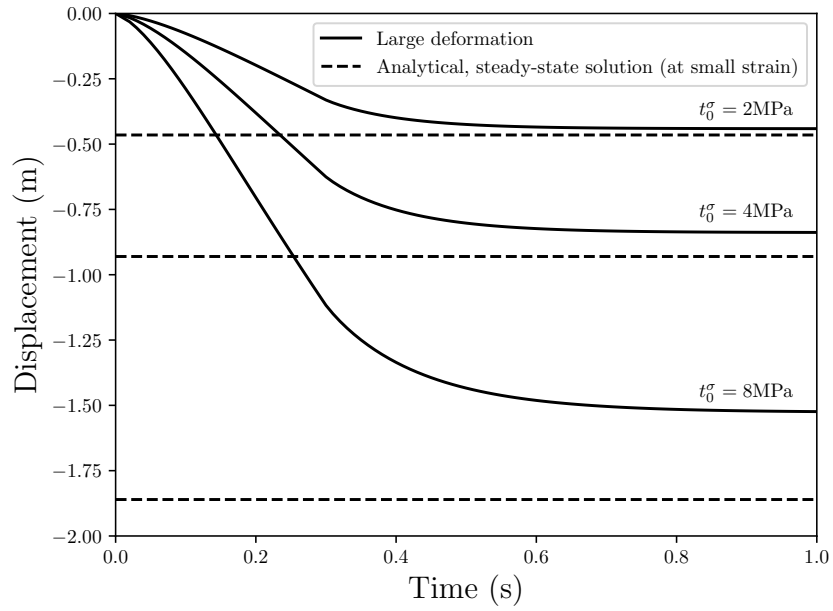


Fig. 7 Porous layer with uniform step load: vertical displacement-time histories at topmost node for severe loads $t_0^\sigma = 2, 4,$ and 8 MPa

4.3 Poroelastodynamics

Recall from the previous report by Irwin et al.¹⁹ the formulations for poroelastodynamics for the $(\mathbf{u}-p_f)$ and $(\mathbf{u}-\mathbf{u}_f-p_f)$ implementations *with* inertia terms. Here, we present results comparing the numerical solutions to an analytical solution developed by de Boer et al.⁴⁶

Similar to the previous examples, a porous column is subjected to an external traction load while its base and sides remain fixed. Values of material and geometric parameters used in this example are given in Tables 5 and 6, respectively, where different load amplitudes and frequencies are included in the latter. The top of the porous column is perfectly drained with reference pressure $p_f(X = H, t) = 0$ atm and subjected to a harmonic loading, as shown in Fig. 8.

Table 5. Material parameters for the poroelastodynamics verification example

λ (MPa)	μ (MPa)	ρ_0^s (kg/m ³)	ρ_0^f (kg/m ³)	n_0^f	n_0^s	k_0 (m/s)
5.6	8.4	2700	1000	0.42	0.58	10^{-2}

Table 6. Geometrical and loading parameters for the poroelastodynamics verification example

H (m)	A (m ²)	h_0^e (m)	t_0^σ (kPa)	ω (rad/s)
10	1	1, 0.1	40	50

The analytical solution for this problem is given as follows:

$$u(X, t) = -\frac{1}{\sqrt{a}(\lambda + 2\mu)} \times \int_0^t \left(t^\sigma(t - \tau) \exp \left[-\frac{b}{2a}\tau \right] I_0 \left(\frac{b\sqrt{\tau^2 - aX^2}}{2a} \right) \mathbb{H}(\tau - \sqrt{a}X) \right) d\tau \quad (30)$$

$$u_f(X, t) = \frac{n^s}{n^f \sqrt{a}(\lambda + 2\mu)} \times \int_0^t \left(t^\sigma(t - \tau) \exp \left[-\frac{b}{2a}\tau \right] I_0 \left(\frac{b\sqrt{\tau^2 - aX^2}}{2a} \right) \mathbb{H}(\tau - \sqrt{a}X) \right) d\tau \quad (31)$$

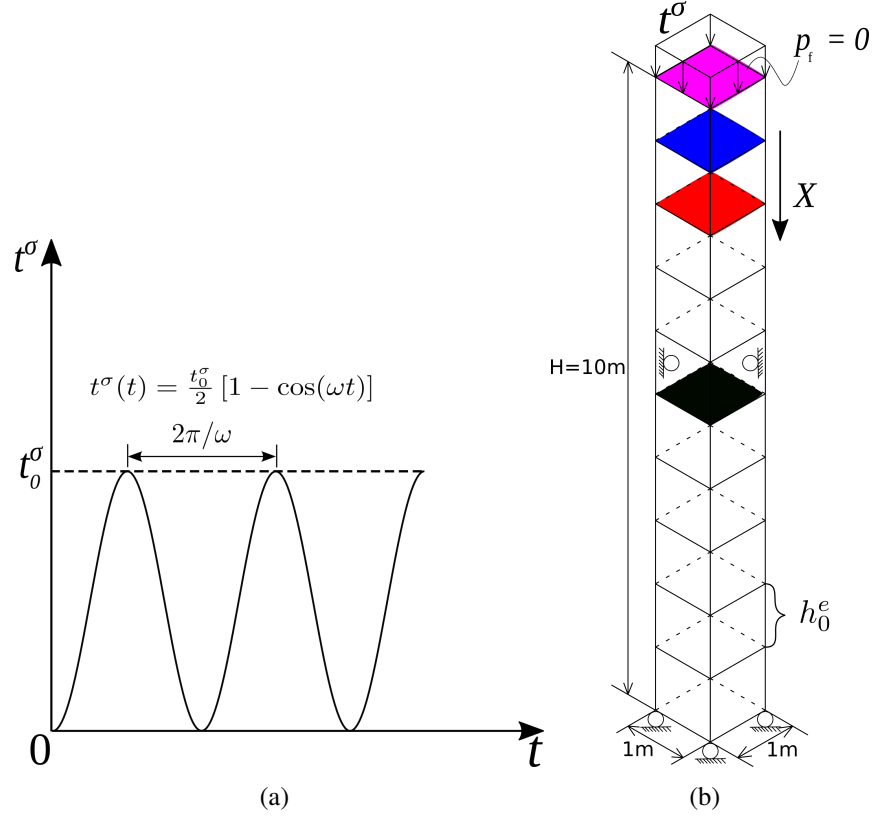


Fig. 8 a) Traction application and b) schematic of column mesh for the poroelastodynamics verification example

$$p_f(X, t) = \frac{1}{(n^f)^2 (\lambda + 2\mu)} \left[n^f \rho^{\text{fr}} \frac{\partial^2 L(X, t)}{\partial t^2} + S_v \frac{\partial L(X, t)}{\partial t} \right] \quad (32)$$

$$\begin{aligned} \sigma_{11}(X, t) = & \frac{b}{2\sqrt{a}} \int_0^t \left(t^\sigma(t - \tau) \exp \left[-\frac{b}{2a} \tau \right] I_1 \left(\frac{b\sqrt{\tau^2 - aX^2}}{2a} \right) \right. \\ & \times \frac{X}{\sqrt{\tau^2 - aX^2}} \mathbb{H}(\tau - \sqrt{a}X) \Big) d\tau \\ & + f(\tau - \sqrt{a}X) \exp \left[-\frac{b}{2\sqrt{a}} \right] X \end{aligned} \quad (33)$$

where

$$L(X, t) = \int_0^t Q(t - \tau) G(X, \tau) d\tau \quad (34)$$

$$Q(t) = \frac{1}{\sqrt{a}} \int_0^t t^\sigma (t - \tau) \exp \left[-\frac{b}{2a} \tau \right] I_0 \left(\frac{b}{2a} \tau \right) d\tau \quad (35)$$

$$G(X, t) = \frac{1}{\sqrt{a}} \exp \left[-\frac{b}{2a} t \right] I_0 \left(\frac{b\sqrt{t^2 - aX^2}}{2a} \right) \mathbb{H}(t - \sqrt{a}X) - \frac{1}{\sqrt{a}} \exp \left[-\frac{b}{2a} t \right] I_0 \left(\frac{b}{2a} t \right) \quad (36)$$

$$S_v = \frac{(n^f)^2 \rho^{\text{fR}} g}{\hat{k}} \quad (37)$$

$$a = \frac{(n^s)^2 \rho^f + (n^f)^2 \rho^s}{(\lambda + 2\mu) (n^s)^2} \quad (38)$$

$$b = \frac{S_v}{(\lambda + 2\mu) (n^f)^2} \quad (39)$$

and where I_m are the modified Bessel functions of order m , $\mathbb{H}$ is the Heaviside function, $t^\sigma(t)$ is the harmonic loading function and λ and μ are the first and second Lamé parameters of the solid skeleton, respectively.

For our custom Python FE implementation,¹⁹ we see excellent agreement between numerical and analytical solutions in Figs. 9b and 10. However, there are slight discrepancies in both solid skeleton displacement and pore fluid pressure in the LS-DYNA poroelastodynamics model.^{30,31} We suspect that this is caused by LS-DYNA ignoring the inertia term in the Darcy velocity constitutive equation, irrespective of whether or not the two-field ($\mathbf{u}$ - p_f) assumption is made.

Furthermore, given that LS-DYNA assumes a ($\mathbf{u}$ - p_f) formulation outright, its poroelastodynamics model cannot be used to verify the displacement of pore fluid. This is because there is no governing equation for the third field variable, which the analytical solution includes: see Fig. 9b. Lastly, we found that we were only able to obtain a close approximation with LS-DYNA by assuming a linear isotropic elastic material and initializing the geostatic stresses prior to starting the simulation. As a result, the resulting solid skeleton effective stress response σ_E^s in LS-DYNA does

not match the analytical solution and is not shown here for this reason.

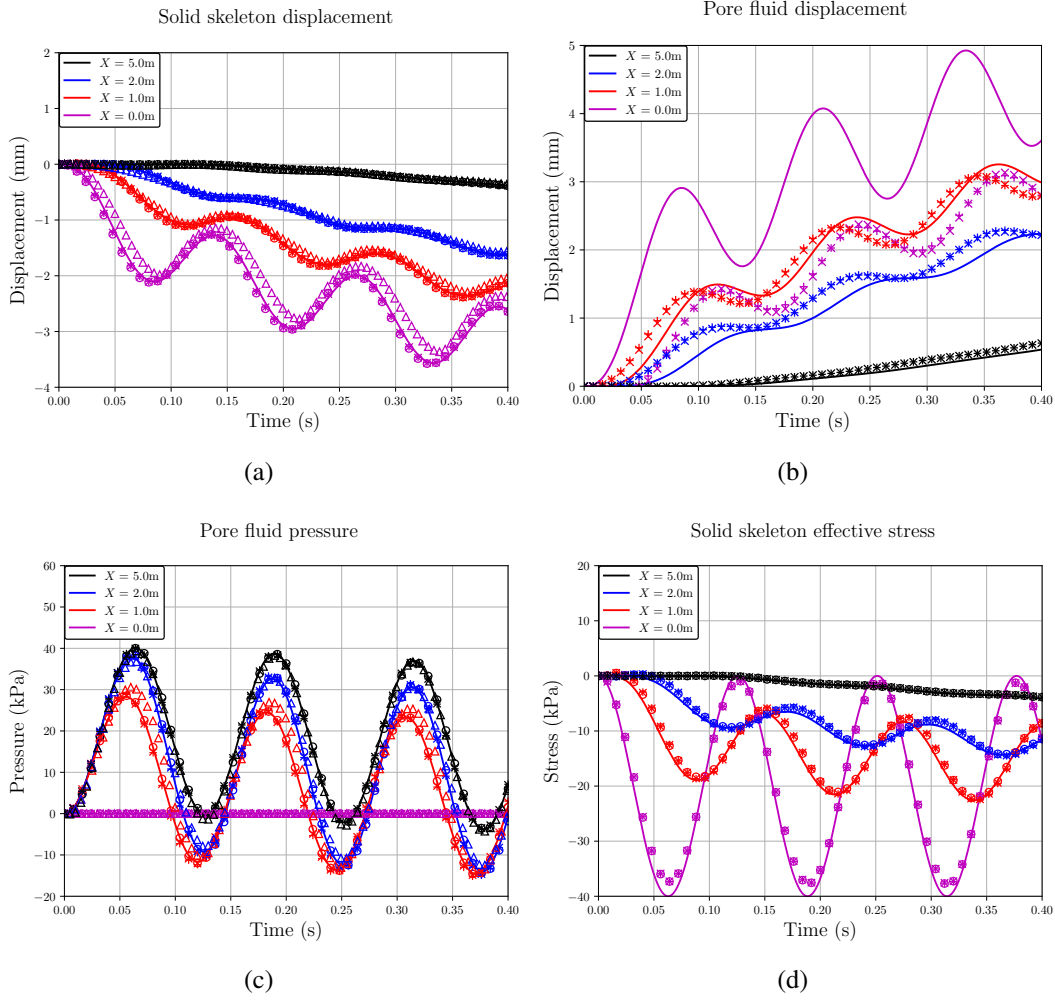


Fig. 9 Verification results for the numerical approximation to the de Boer analytical solution. All element types are stable (i.e., Q2-P1 or Q2-Q2-P1 depending on the formulation). Symbols are identified in the mini-table below.

Symbol	Simulation
—	Analytical
○	$(\mathbf{u}-p_f)$, CD (L. mass)
×	$(\mathbf{u}-\mathbf{u}_f-p_f)$, NB ($\beta = 0.25, \gamma = 0.5$)
+	$(\mathbf{u}-\mathbf{u}_f-p_f)$, RKFNC (L. mass)
△	LS-DYNA

Pore fluid displacement

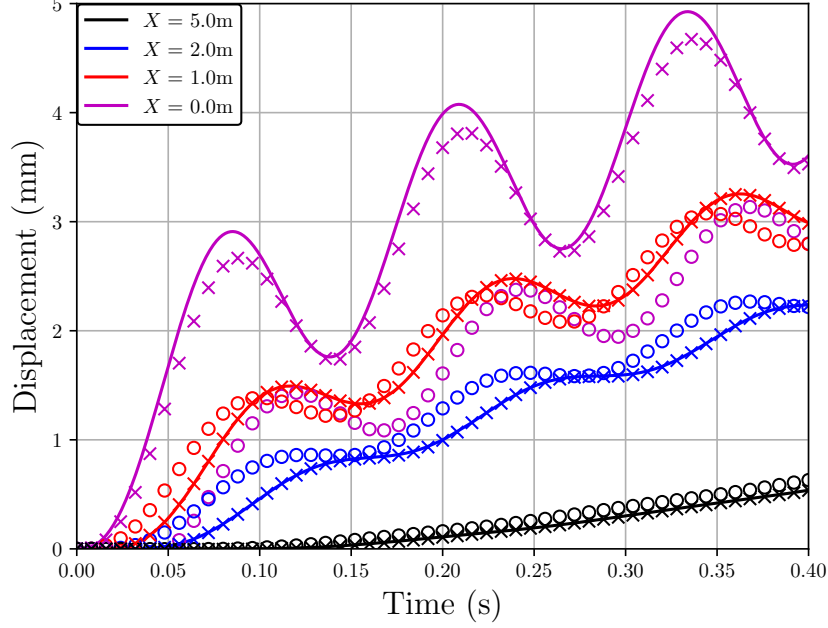


Fig. 10 Comparison between two different mesh resolutions for multiple integrators demonstrating the importance of finer meshes in resolving the pore fluid displacement. Symbols are identified in the mini-table below.

Symbol	Simulation
—	Analytical
○	$(\mathbf{u}-\mathbf{u}_f-p_f)$, NB ($\beta = 0.25, \gamma = 0.5$), $h_0^e = 1$ m
×	$(\mathbf{u}-\mathbf{u}_f-p_f)$, NB ($\beta = 0.25, \gamma = 0.5$), $h_0^e = 0.1$ m

We also observe poor agreement between the analytical solution and our FE solution for the pore fluid displacement in Fig. 9b. However, this is an entirely numerical phenomenon. The same discrepancies were observed by Heider,⁴⁷ who came to the conclusion that for coarse mesh resolution, the pore fluid pressure gradient is numerically unstable at the drained surface. Given that the pressure there is set to 0 atm (i.e., the datum reference pressure) and that a traction load is also applied there, this causes a large pore fluid pressure buildup at the surface, which must quickly dissipate to 0 atm. Thus, a large pore fluid pressure spatial gradient is created, and because of the coupling between pore fluid pressure and pore fluid displacement in the balance of momentum of the pore fluid, this leads to inaccuracy in the FE solution for the pore fluid displacement.

Fortunately, this problem is easily remedied by using mesh refinement. See Fig. 10 for a demonstration. Therefore, in addition to numerical issues with resolving the propagation of a shock wave with coarse meshes, as we shall see momentarily in Section 5, we elect to use finer meshes for the poroelastodynamics simulations of soft porous materials such as lung parenchyma.

The convergence profile for different element types with varying stabilization parameters is shown in Fig. 11. Here we have used the implicit NB integrator for constant acceleration with $\beta = 0.25, \gamma = 0.5$ at fixed time step for variable element lengths.

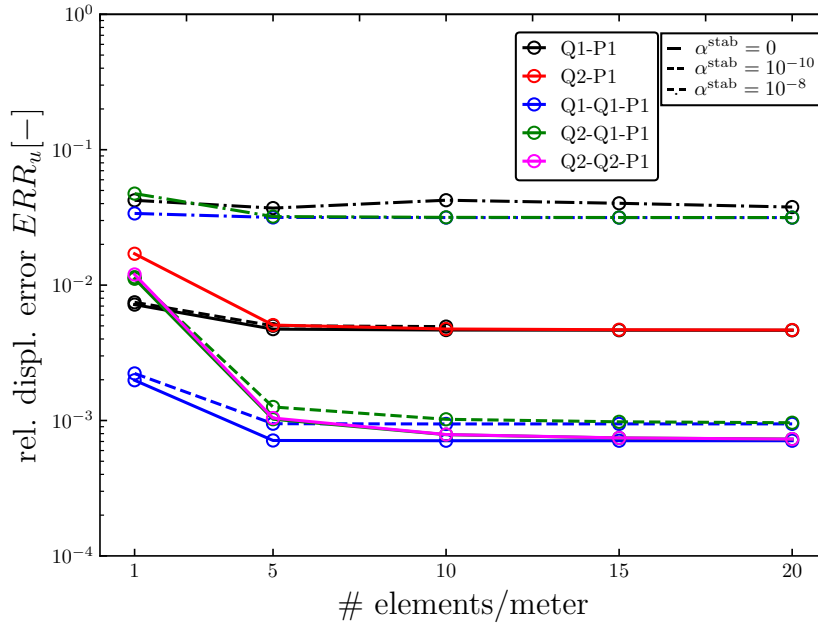


Fig. 11 Relative solid skeleton displacement error (logarithmic scale) at $t = 0.2$ s for fixed $\Delta t = 10^{-6}$ s

The relative displacement error $ERR_u[-]$ is defined as

$$ERR_u[-] := \frac{|u_{\text{analytical}}(X = H, t = 0.2 \text{ s}) - u_{\text{numerical}}(X = H, t = 0.2 \text{ s})|}{|u_{\text{analytical}}(X = H, t = 0.2 \text{ s})|} \quad (40)$$

It is apparent that the $(\mathbf{u}-p_f)$ formulation does not give as accurate a result as the $(\mathbf{u}-\mathbf{u}_f-p_f)$ formulation, especially at the smaller element lengths. We do not observe a strong convergence profile, especially compared to the one obtained by Markert et al.⁴⁸ and Heider.⁴⁷ We suspect that the poor convergence behavior is due to the

abundance of nonlinearities in our model, whereas Markert et al.⁴⁸ and Heider⁴⁷ assumed a linear small-strain theory.

5. Shockwave Loading of the Lung

Of key importance for dynamic injury predictions is how lung tissue deforms under shock loading. In Section 5 we present our findings for a column of lung parenchyma subjected to external traction loadings on the order of milliseconds or less. In all of the simulations, the lung parenchyma column is fixed on its sides and base. Furthermore, the sides and top are assumed to be an impermeable membrane (i.e., “undrained”) such that the pore air is confined solely within the column.

While we do have full control over the boundary conditions at the ends of the column, we have chosen the impermeable boundary condition for comparative purposes to the model developed by Clayton et al.^{20,22–25} in which they assume occluded pore air. One could imagine the experimental analog to this type of boundary condition wherein a small section of lung parenchyma is excised, placed in an impermeable sleeve and then uniaxially loaded. The advantage to our model over the one implemented by Clayton et al.²⁴ is that we could allow for more realistic boundary conditions at the ends of the column of lung parenchyma in our simulations. For example, we could allow an end to remain open to atmospheric pressure (i.e., “drained”) such that experimentally the sleeve need not enclose this end of excised tissue, or we could specify a mixed boundary condition wherein the pore fluid pressure gradient and pore fluid pressure at the end of the column are both free. This would enable experimentally that the sleeve need not enclose the end of excised tissue, and the impacting device would need only to maintain contact with a part of the tissue during loading.

5.1 Effects of Shock Bulk Viscosity

Prior to presenting the results for the multiphase formulation, it is worthwhile to analyze the effect of the viscous bulk shock term Q on reducing numerical noise from the deformation response. Here we consider the material parameters for a single-phase model wherein air is trapped within a column of lung parenchyma and *not* allowed to move relative to it. For the single-phase model in LS-DYNA, we use the standard neo-Hookean hyperelastic material model, MAT 45, so that we can make a one-to-one comparison with our neo-Hookean hyperelastic material

model in Python. Material properties are given in Table 7. Geometrical and loading parameters are given in Table 8.

Table 7. Material parameters for an example on shock viscosity for a column of single-phase, elastodynamic lung parenchyma. Values taken from Clayton et al.²⁴

K (kPa)	G (kPa)	ρ_0 (kg/m ³)
213	3	337

Table 8. Geometrical and loading parameters for an example on shock viscosity for a column of single-phase, elastodynamic lung parenchyma

Overpressure load type	H (cm)	A (cm ²)	h_0^e (cm)	t_0^σ (kPa)	t_0 (ms)	t_1 (ms)
Yen impulse	10	1	1, 0.1	50	0.17	0.34
Friedlander impulse	10	1	1, 0.1	25	10	N/A

The traction is applied to the top of the column in two forms. The first is a linear triangular impulse, hereafter referred to as the Yen impulse, shown in Fig. 12a, which rises to a maximum pressure $t_0^\sigma = 50$ kPa at time $t_0 = 170$ μ s relative to reference pressure, set to 0 atm for these single-phase simulations. It then decays to $t^\sigma = 0$ kPa at time $2t_0 = t_1 = 340$ μ s. The time scales and pressure profile were chosen based on an experimental study of rabbit lung exposed to shock tubes by Yen et al.⁴⁹ Overpressure amplitude was chosen arbitrarily, though we note that Yen et al.⁴⁹ observed that overpressures greater than 2 psi (≈ 14 kPa) resulted in edema of the exposed rabbit lung.

The second traction application is the Friedlander impulse,⁵⁰ which is depicted in Fig. 12b. The profile is of an exponentially decaying function to 0 applied pressure at $t_0 = 10$ ms, whereby it then starts to act as a tensile force relative to atmospheric pressure. This profile is an analytical solution to sound waves diffracting off of a semi-infinite plate. Therefore, the Friedlander wave is more indicative of the overpressure profile of a blast wave⁵¹ (e.g., one resulting from an explosion and impacting against a hard surface than the Yen impulse, which is better suited to describe shock-tube-like experimental loading).

For both pressure profiles, we will vary the peak overpressure amplitude. As with the de Boer example in Section 4.3, the bottom and sides of the column remain fixed and rigid such that the motion is purely vertical in the X direction. The 1-D FE mesh is shown in Fig. 12c.

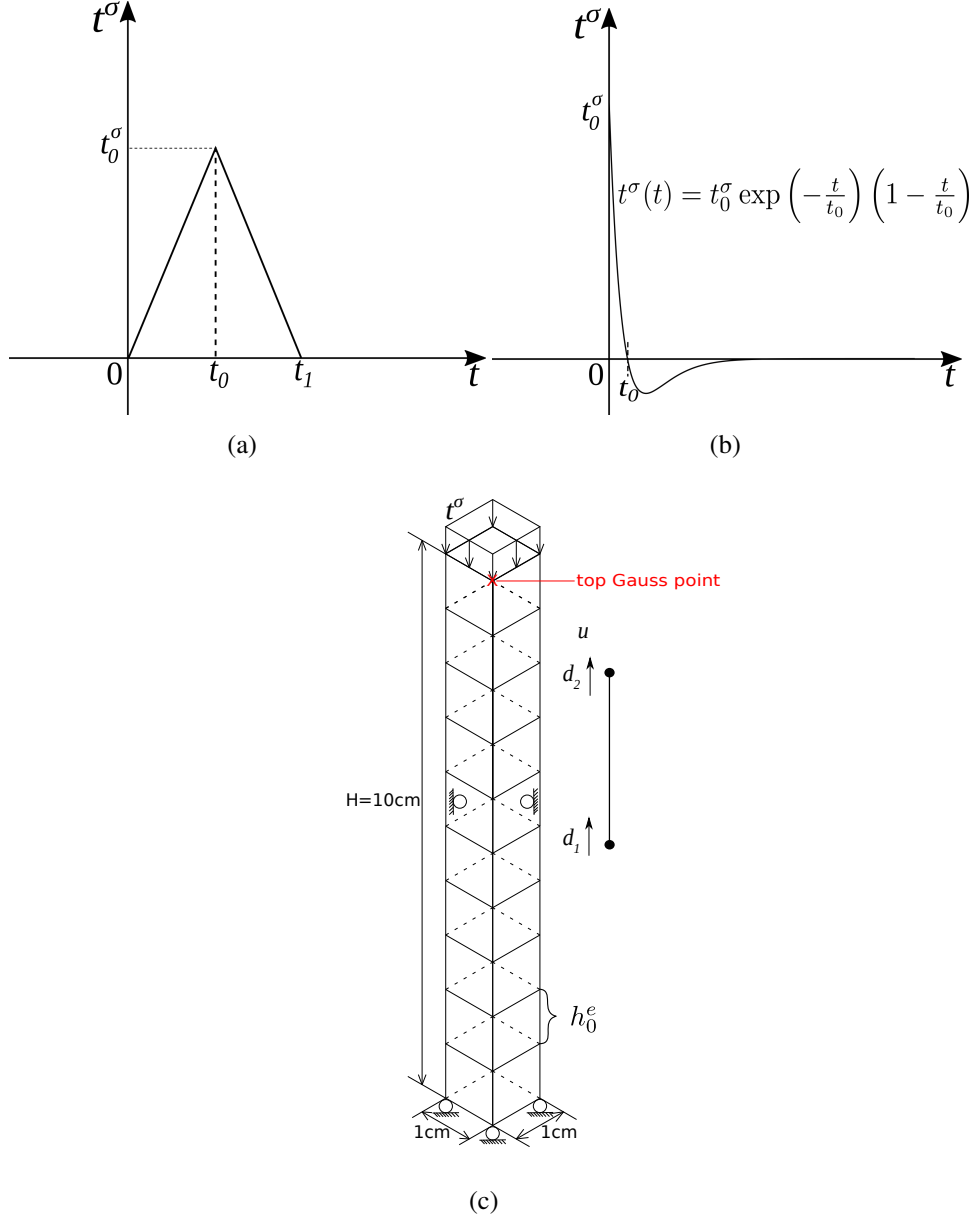


Fig. 12 a) Yen impulse traction application b) Friedlander impulse traction application c) schematic of column mesh for an example on shock viscosity for a column of single-phase/elastodynamic lung parenchyma, highlighting the Q1 element. Note that element size h_0^e is not to scale in c).

Different choices of element length h_0^e and different choices of shock viscosity (i.e., none or default values) are investigated in displacement predictions for the Yen loading protocol in Figs. 13 and 14. Similarly, predictions are compared for the Friedlander loading protocol in Figs. 15 and 16.

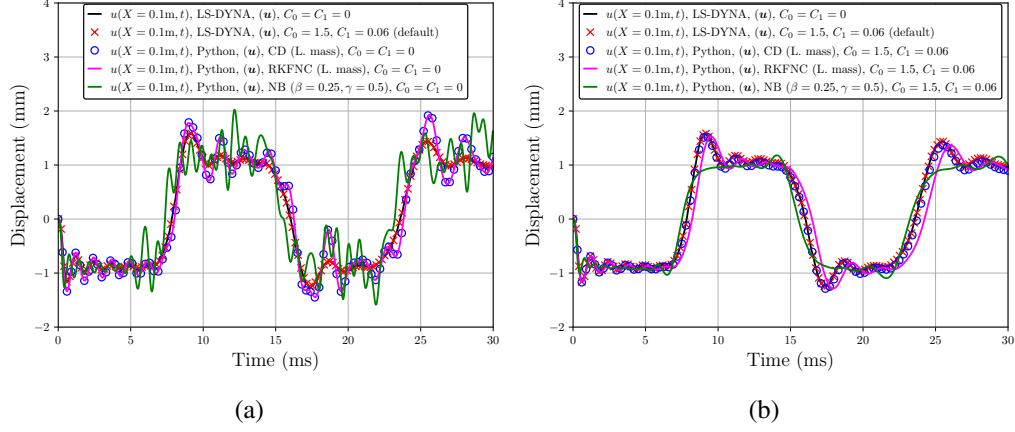


Fig. 13 Comparison of elastodynamical response to Yen impulse loading for peak overpressure of 50 kPa with element size $h_0^e = 1$ cm for a) simulations without shock viscosity enabled b) shock viscosity set to default values

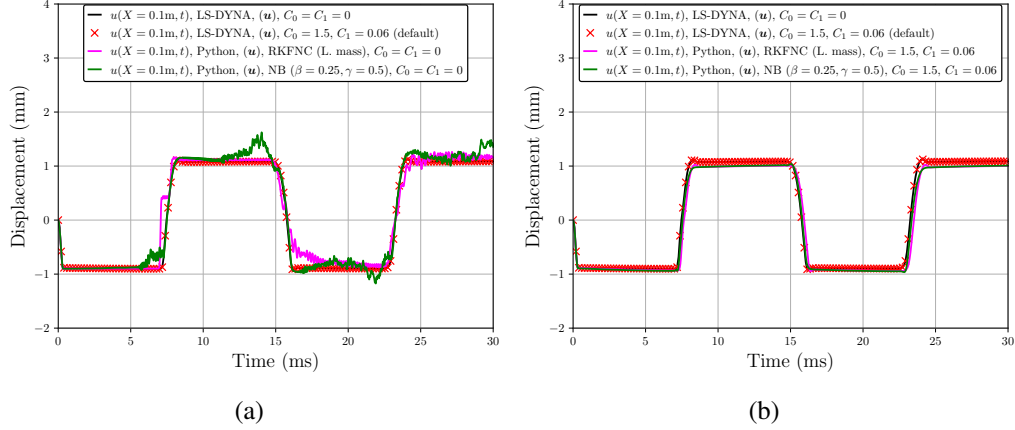


Fig. 14 Comparison of elastodynamical response to Yen impulse loading for peak overpressure of 50 kPa with element size $h_0^e = 0.1$ cm for a) simulations without shock viscosity enabled b) shock viscosity set to default values

For both element lengths h_0^e , it is apparent that without shock viscosity, our integration schemes struggle to constrain the numerical overshoots. This is particularly noticeable for the coarser mesh with $h_0^e = 1$ cm in Fig. 15a for the implicit solver which is taking constant time-steps $\Delta t = 10^{-6}$ s. When some shock viscosity is introduced, the “noise” from the shock is dampened significantly, and we see fairly good agreement with the results from LS-DYNA, which we note does not provide users the option to disable shock viscosity as is evident by the overlapping $C_0 = C_1 = 0.0$ and $C_0 = 1.5, C_1 = 0.06$ curves.

We show further comparisons for the Friedlander impulse⁵⁰ with the fine mesh. Again, we see that the shock viscosity reduces numerical noise and offers stability. This is particularly evident for the CD scheme shown in Fig. 16a, which is starting to go unstable before the simulation is terminated at 30 ms. Therefore, we typically opt to enable shock viscosity for the poroelastodynamics simulations.

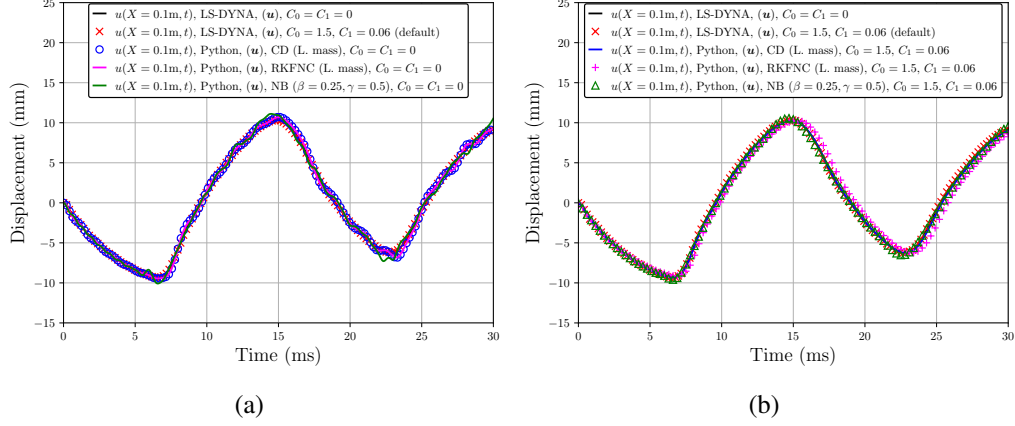


Fig. 15 Comparison of elastodynamical response to Friedlander impulse loading for peak over-pressure of 25 kPa with element size $h_0^e = 1$ cm for a) simulations without shock viscosity enabled b) shock viscosity set to default values

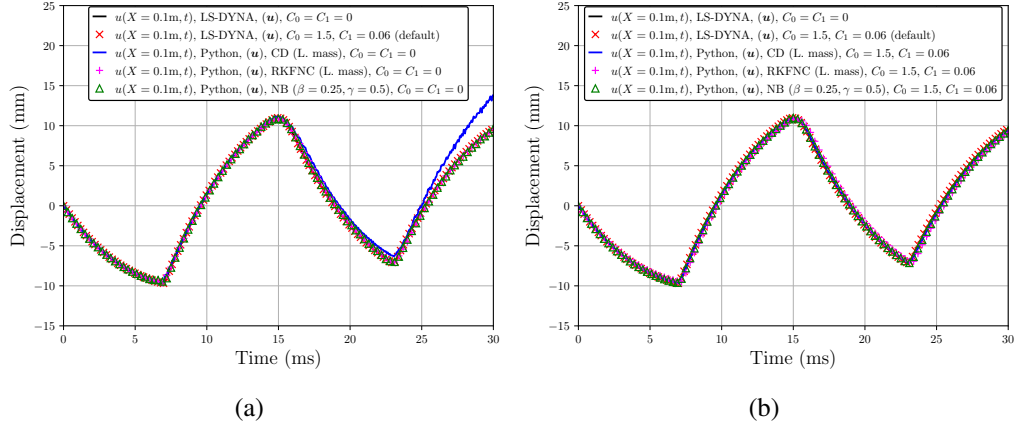


Fig. 16 Comparison of elastodynamical response to Friedlander impulse loading for peak over-pressure of 25 kPa with element size $h_0^e = 0.1$ cm for a) simulations without shock viscosity enabled b) shock viscosity set to default values

In addition, care must be taken when choosing the coefficients C_0 and C_1 ; as the values of these coefficients increase, the shock is smeared across larger portions of the domain (i.e., over more elements). This can be seen in Figs. 17 and 18. Thus, if the constants are too large, we begin to eliminate physical effects that are important

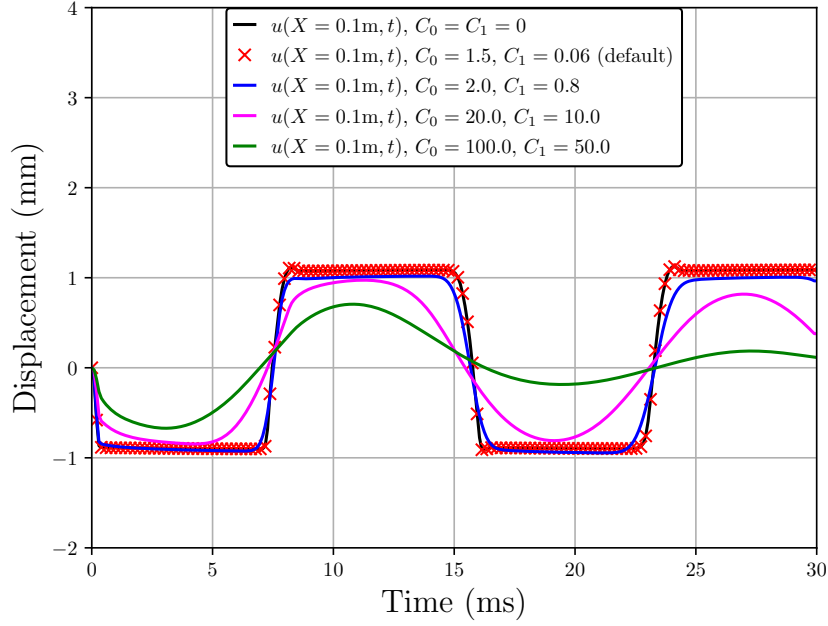


Fig. 17 The effect of changing the shock viscosity coefficients C_0 and C_1 for the Yen impulse at peak overpressure of 50 kPa; here, all simulations were conducted using a neo-Hookean hyperelastic material (UMAT 45) in LS-DYNA with $h_0^e = 0.1$ cm

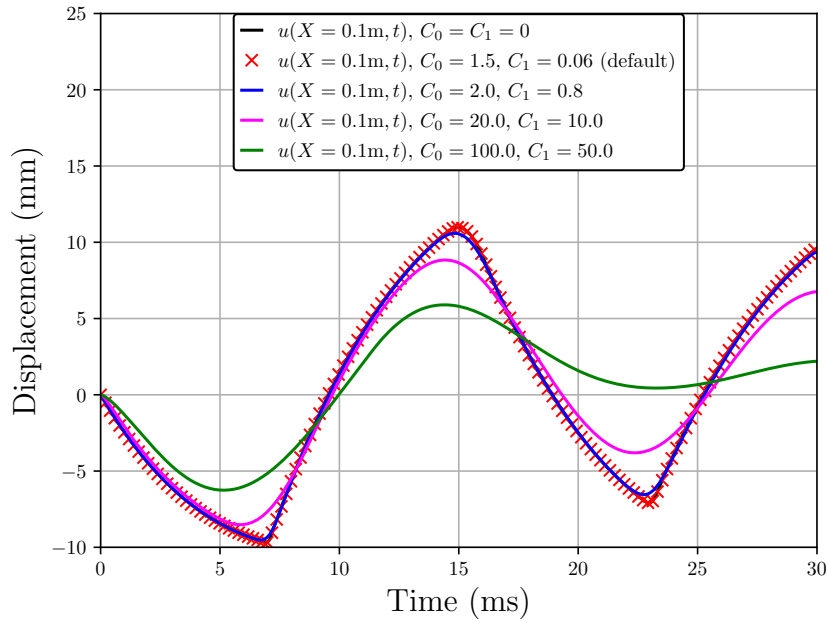


Fig. 18 The effect of changing the shock viscosity coefficients C_0 and C_1 for the Friedlander impulse at peak overpressure of 25 kPa; here, all simulations were conducted using a neo-Hookean hyperelastic material (UMAT 45) in LS-DYNA with $h_0^e = 0.1$ cm

to measure. For that reason, we stick to the default values used by LS-DYNA in our Python implementation of the multiphase mechanics for shock loading.

5.2 Effects of Pressure Stabilization

As with our study of the shock viscosity and how varying the constants C_0 and C_1 can affect the solution, we shall do the same here for pressure stabilization by modifying the stabilizing parameter α^{stab} . In the following investigation, we use the $(\mathbf{u}-p_f)$ formulation and a Q1-P1 element with $h_0^e = 1$ cm. In contrast to the poroelastic and poroelastodynamic verification problems, here we set an impermeable boundary condition at the surface of the column of lung parenchyma: pore air is not allowed to escape the column at any position. Material parameters are listed in Table 9. Geometrical and loading protocols are given in Table 10.

As in previous examples, we apply a pressure load in the form of a traction at the top of the column. However, for the multiphase model, the applied overpressure is relative to 1 atm, as opposed to 0 atm for the single-phase model in Section 5.1. The two pressure profiles we explore are the Yen impulse and Friedlander impulse, as depicted by respective Figs. 19a and 19b. For both pressure profiles, we choose a peak overpressure of 15 kPa.

Table 9. Material parameters for multiphase lung parenchyma simulations. Values taken from Clayton et al. and Lande and Mitzner.^{24,52}

K^{skel} (kPa)	G (kPa)	K_s (kPa)	K_f^η (kPa)	ρ_0^{SR} (kg/m ³)	ρ_0^{fR} (kg/m ³)	n_0^f	$\hat{k}_0$ (m ² /Pa-s)
7.5	3	2.2×10^6	140	1000	1.2	0.664	10^{-5}

Table 10. Geometrical and loading parameters for multiphase lung parenchyma simulations

Overpressure load type	H (cm)	A (cm ²)	h_0^e (cm)	t_0^σ (kPa)	t_0 (ms)	t_1 (ms)
Yen impulse	10	1	0.1	15	0.17	0.34
Friedlander impulse	10	1	0.1	15	10	N/A

It is apparent from Figs. 20 and 21 that as we increase the value of α^{stab} , we introduce numerical error into the solution. We can see that this also results in oscillations in both the pore fluid and solid lung parenchyma pressure (Figs. 22 and 23). Therefore, when enabling pressure stabilization, we choose a value of α^{stab} as close to 0 as possible, typically 10^{-10} m³s²/kg as this shows good overlap with the unstabilized formulation.

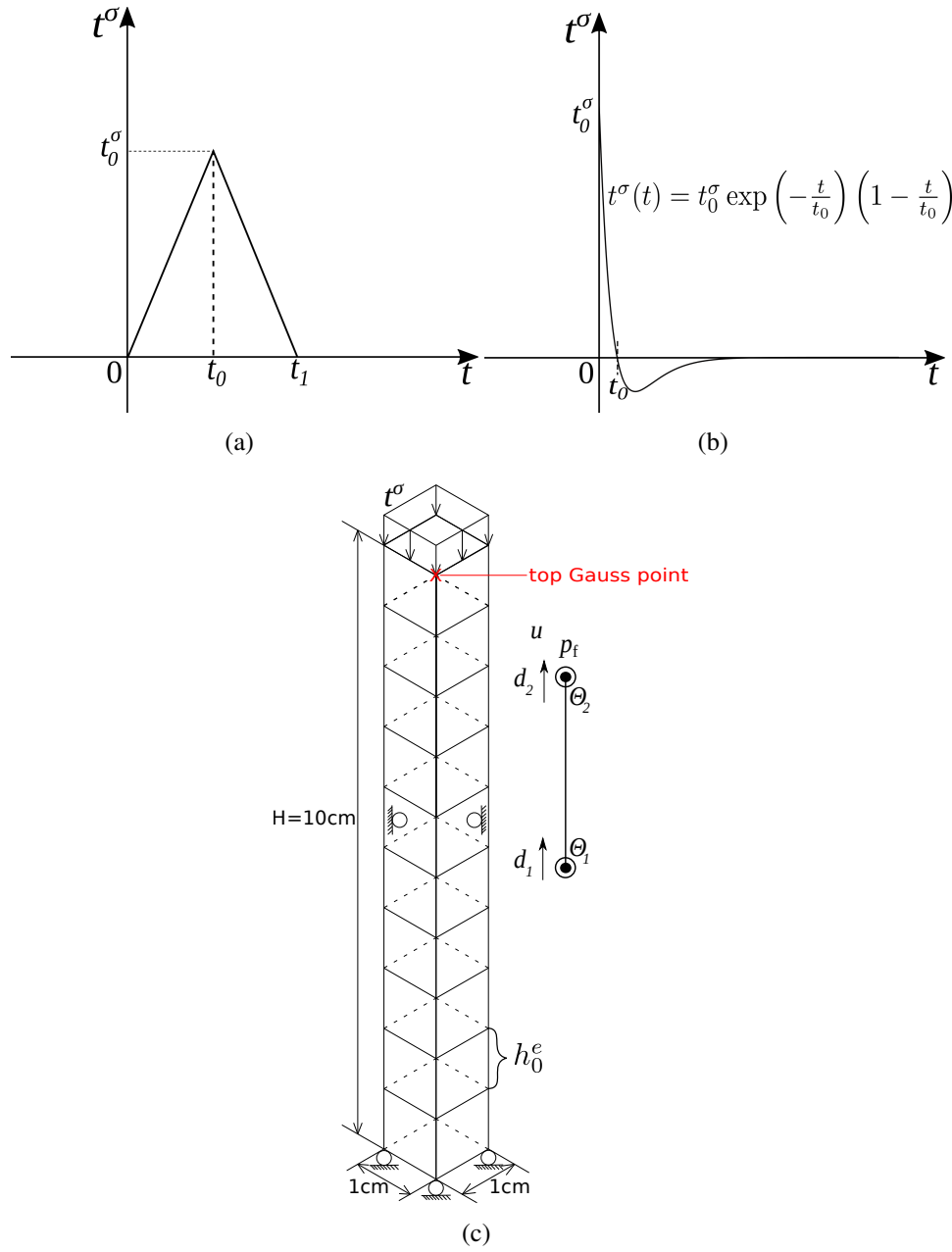


Fig. 19 a) Yen impulse traction application b) Friedlander impulse traction application c) schematic of multiphase column mesh for examples of lung parenchyma deformations, highlighting the Q1-P1 element. Note that element size h_0^e is not to scale in c).

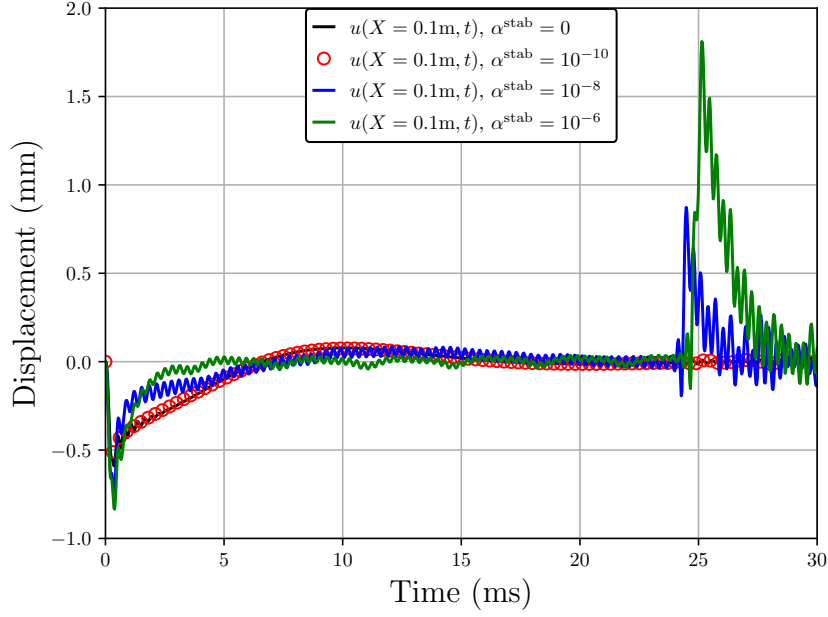


Fig. 20 Comparison of lung parenchyma displacements undergoing overpressure loading from the Yen impulse at 15 kPa between the multiphase model, using RKFNC numerical time integration with consistent mass matrices, tracking the nodes at the top of the column for varying values of α^{stab} . Here, we invoke a Q1-P1 element with $h_0^e = 1$ cm.

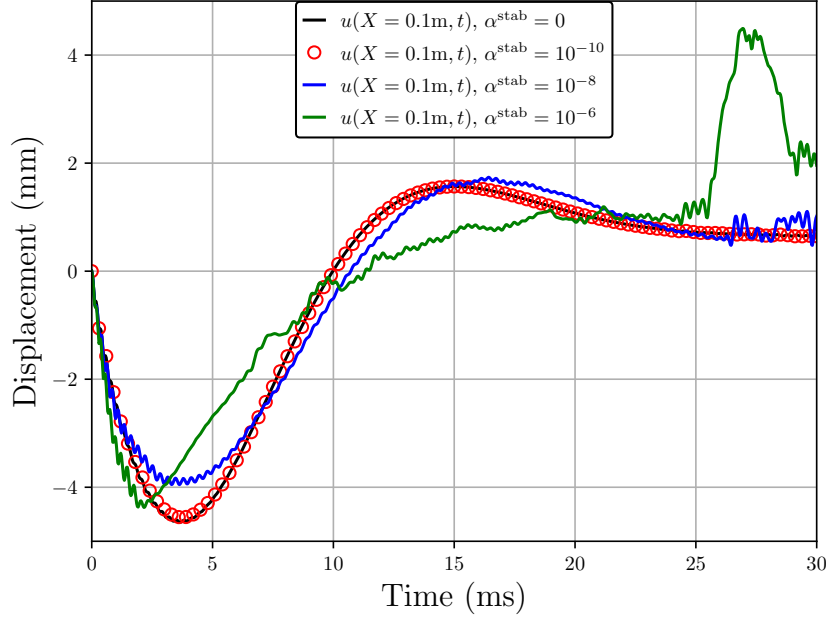


Fig. 21 Comparison of lung parenchyma displacements undergoing overpressure loading from the Friedlander impulse at 15 kPa between the multiphase model, using RKFNC numerical time integration with consistent mass matrices, tracking the nodes at the top of the column for varying values of α^{stab} . Here, we invoke a Q1-P1 element with $h_0^e = 1$ cm.

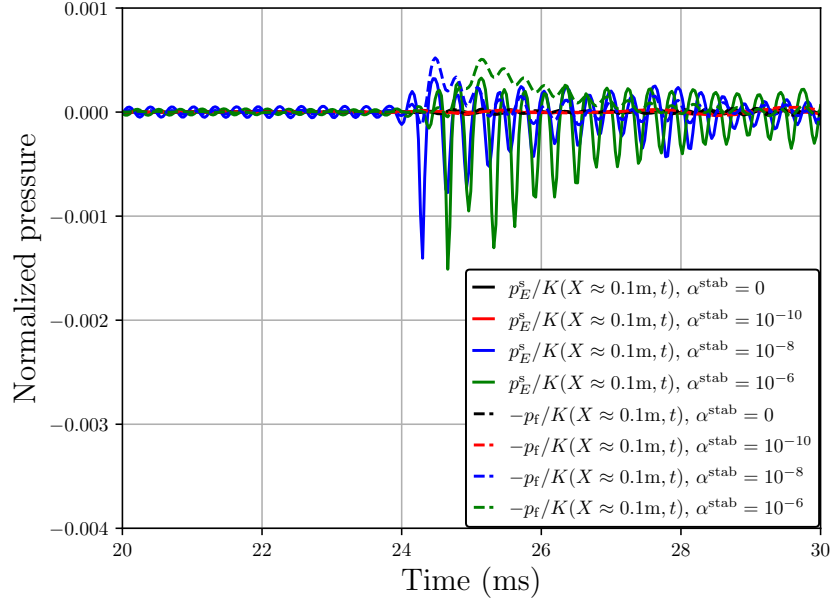


Fig. 22 Comparison of lung parenchyma pressure p_E^s and pore fluid pressure p_f after overpressure loading from the Yen impulse at 15 kPa between the multiphase model, using RKFNC numerical time integration with consistent mass matrices, tracking the Gauss points at the top of the column for varying values of α^{stab} . Here, we invoke a Q1-P1 element with $h_0^e = 1$ cm, and $K = 7.5$ kPa is the bulk modulus of the solid skeleton.

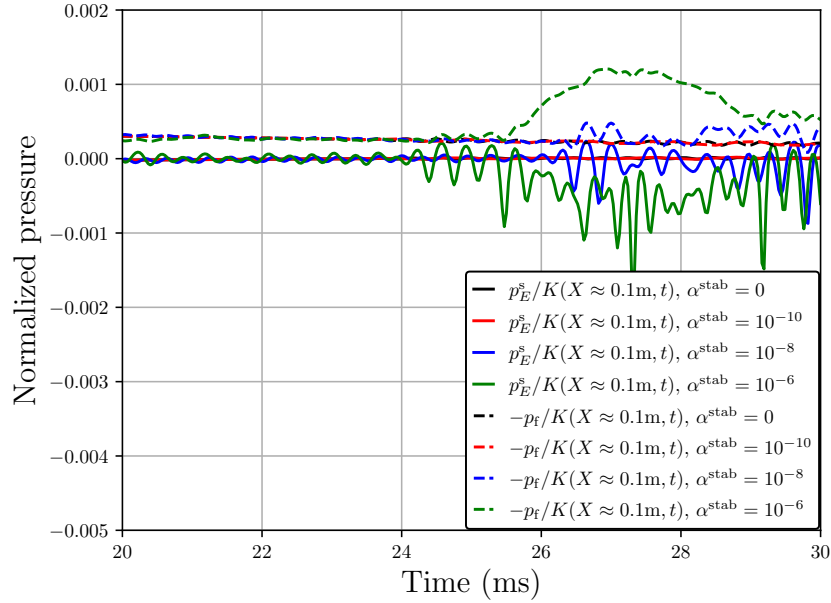


Fig. 23 Comparison of lung parenchyma pressure p_E^s and pore fluid pressure p_f after overpressure loading from the Friedlander impulse at 15 kPa between the multiphase model, using RKFNC numerical time integration with consistent mass matrices, tracking the Gauss points at the top of the column for varying values of α^{stab} . Here, we invoke a Q1-P1 element with $h_0^e = 1$ cm, and $K = 7.5$ kPa is the bulk modulus of the solid skeleton.

5.3 Comparison of Single-phase and Multiphase Models

Next, we present results for the single-phase model developed by Clayton, Banton, and Freed^{20,21,23} and compare this to our multiphase model. The theory of the latter was derived in Section 2 of Irwin et al.,¹⁸ with equations for the FE formulation derived in Irwin et al.¹⁹ Given that the single-phase constitutive model^{20,23} and its numerical implementation^{22,24,25} assume that pore air is trapped within lung parenchyma (i.e., pore fluid [f] is not able to move relative to the solid skeleton [s]), we have assumed for TPM calculations impermeable boundaries at the ends of the column of the lung parenchyma. Thus, pore air is entirely contained within the column, although, in contrast to the single-phase model, pore air is allowed to move freely within the column relative to the solid lung tissue.

As in previous examples, we apply a pressure load in the form of a traction at the top of the column. However, for the multiphase model, the applied overpressure is relative to 1 atm, as opposed to 0 atm for the single-phase model in Section 5.1. Material parameters for this final set of examples are listed in Table 11. Geometrical and loading protocols for this set likewise are given in Table 12.

Table 11. Material parameters for multiphase lung parenchyma simulations. Values taken from Clayton et al. and Lande and Mitzner.^{24,52}

K^{skel} (kPa)	G (kPa)	K_s (kPa)	K_f^η (kPa)	ρ_0^{sR} (kg/m ³)	ρ_0^{fR} (kg/m ³)	n_0^f	$\hat{k}_0$ (m ² /Pa-s)
7.5	3	2.2×10^6	140	1000	1.2	0.664	10^{-5}

Table 12. Geometrical and loading parameters for multiphase lung parenchyma simulations

Overpressure load type	H (cm)	A (cm ²)	h_0^e (cm)	t_0^σ (kPa)	t_0 (ms)	t_1 (ms)
Yen impulse	10	1	0.1	15; 50	0.17	0.34
Friedlander impulse	10	1	0.1	15; 25	10	N/A

The two pressure profiles we explore are the Yen impulse and Friedlander impulse, as depicted by Figs. 24a and 24b. The 1-D FE mesh for uniaxial strain loading is illustrated in Fig. 24c. Recall also examples for the elastodynamics case were provided in Section 5.2. For both pressure profiles, we will vary the peak overpressure amplitude to obtain the following results.

First, we present results for the multiphase model compared to the single-phase model wherein we do *not* enable shock viscosity nor pressure stabilization. In the

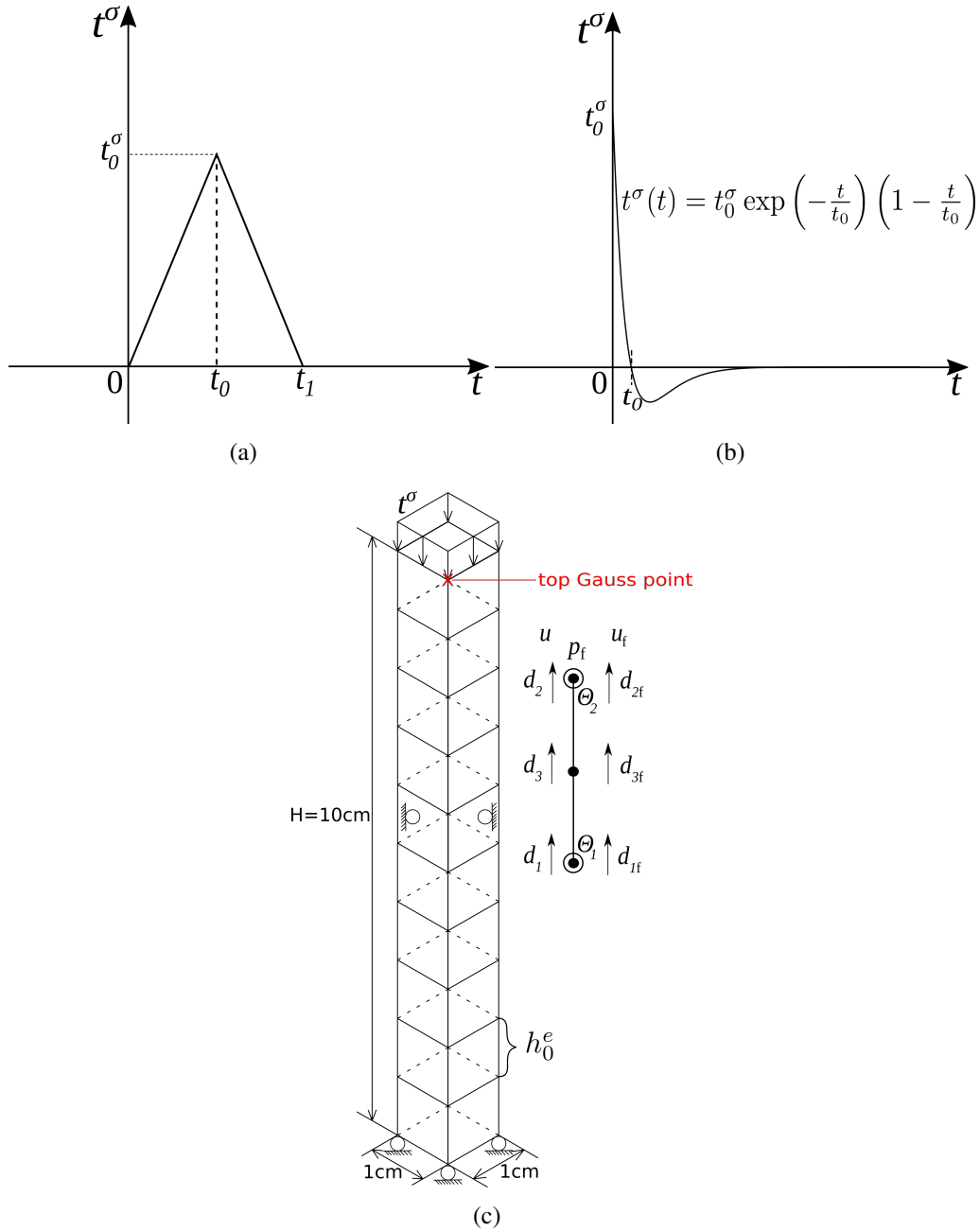


Fig. 24 a) Yen impulse traction application b) Friedlander impulse traction application c) schematic of multiphase column mesh for examples of lung parenchyma deformations, highlighting the Q2-Q2-P1 element. Note that element size h_0^e is not to scale in c).

multiphase model, fully stable elements (i.e., Taylor-Hood elements) are used and, for low overpressure loading magnitudes at high strain-rates, additional stabilization via shock viscosity or pressure stabilization is not necessary to obtain a solution. Figure 25a shows the displacement history of the top of the lung parenchyma

over time. For the single-phase LS-DYNA model (Clayton and Freed), the solution behaves elastodynamically, with little to no dissipation of the shock wave. For the multiphase Python model(s), we can directly observe the effect that the pore air has on the lung parenchyma displacement, with an absorption of the shock wave and eventual dissipation of that energy.

Notice that for the Taylor-Hood elements without any stabilization, numerical oscillations are present in the displacement solution, irrespective of the integration scheme. When shock viscosity and pressure stabilization are introduced, with a Q1-Q1-P1 element, as in Fig. 25b, observe that these displacement solutions are much smoother than their Taylor-Hood counterparts. Furthermore, we found that without the shock viscosity and pressure stabilization techniques, we are unable to achieve greater magnitudes of peak overpressure loadings without numerical problems. Such problems include unstable pore fluid pressure (i.e., pore air pressure $p_f \rightarrow \infty$) and element collapse ($J^e < 0$). Further problems include an inability to achieve convergence in the Newton-Raphson solver for implicit schemes, and the time step $\Delta t \rightarrow 0$ for the adaptive time-stepping explicit schemes (e.g., Runge-Kutta Cash-Karp). Therefore, we choose to enable both stabilization techniques in our simulations of shock loading of the lung parenchyma.

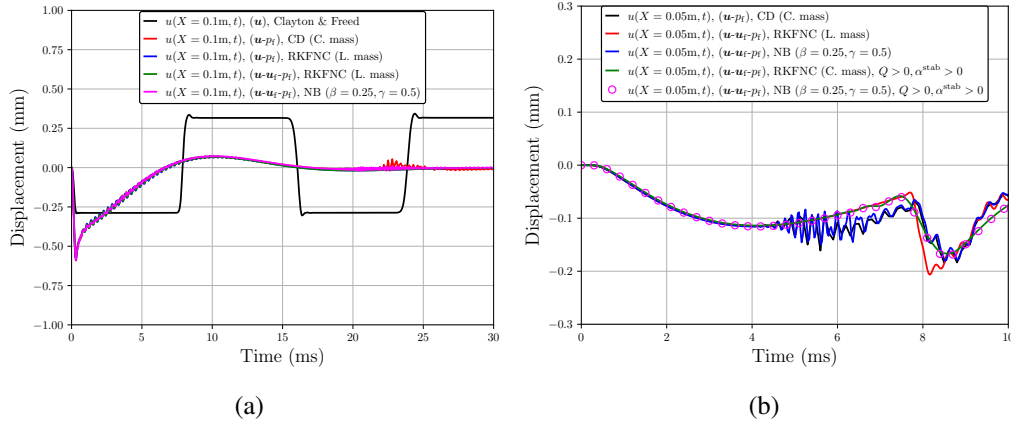


Fig. 25 Comparison of lung parenchyma displacements undergoing overpressure loading from the Yen impulse at 15 kPa between a) the single-phase and multiphase models tracking the nodes at the top of the column, and b) the multiphase models tracking the nodes at the middle of the column with and without stabilization methods. Without shock viscosity Q and pressure stabilization α^{stab} , we invoke a Q2-Q2-P1 element; with shock viscosity Q where $C_0 = 1.5, C_1 = 0.06$ and pressure stabilization where $\alpha^{stab} = 10^{-10}$, we invoke a Q1-Q1-P1 element.

Figure 26 shows the results of applying the stabilization techniques of shock vis-

cosity and pressure stabilization to the Q1-P1 and Q1-Q1-P1 elements. This is for a higher peak overpressure loading magnitude of 50 kPa with the Yen impulse. Again we see that in the multiphase model the pore air significantly dampens the lung parenchyma deformation response to blast-like pressure loading. Even though the strain is still somewhat small (i.e., $\approx 1\%$ nominal strain), we are able to achieve double the strain as compared to the multiphase model without stabilization techniques at lower loads. We also see good overlap between the various multiphase models, irrespective of integration scheme and poroelastodynamics formulation (i.e., $(\mathbf{u}-\mathbf{u}_f-p_f)$ versus $(\mathbf{u}-p_f)$).

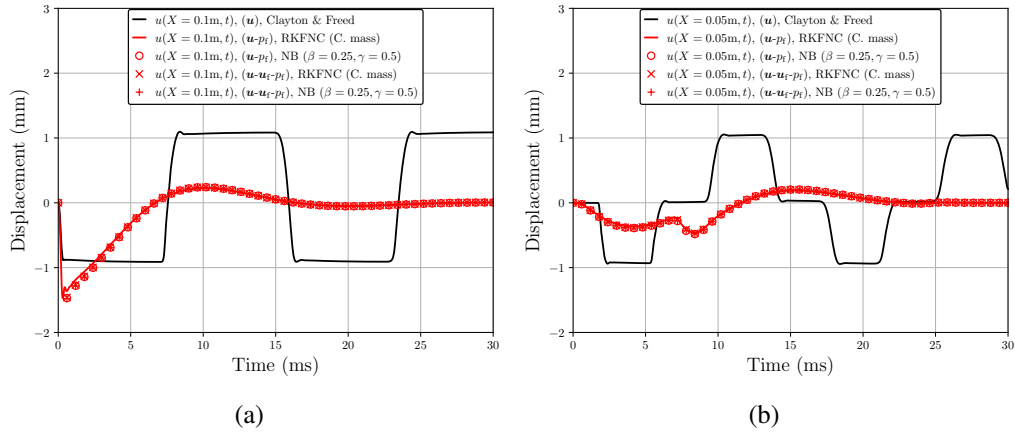


Fig. 26 Comparison of lung parenchyma displacements between the single-phase and multiphase models undergoing overpressure loading from the Yen impulse at 50 kPa tracking the nodes at a) the top of the column and b) the middle of the column. Here, shock viscosity with $C_0 = 1.5, C_1 = 0.06$ and pressure stabilization for the multiphase model with $\alpha^{\text{stab}} = 10^{-10}$ have been enabled.

An additional advantage of the $(\mathbf{u}-\mathbf{u}_f-p_f)$ to both the single-phase model and the $(\mathbf{u}-p_f)$ formulation is that we can directly calculate the pore air displacement, velocity and acceleration relative to those of the lung parenchyma, as shown in Fig. 27. The pore air displacement is plotted at $X = H/2$, rather than at $X = H$, because the impermeable boundary condition negates the effects of relative motion between pore air and solid lung tissue at the latter location.

Figure 27a shows the relative motion of the pore air to that of the lung parenchyma, wherein we observe that the shock wave propagates through the air faster than the lung parenchyma given that the pore fluid displaces earlier and more than the lung parenchyma. This is also apparent in the plots of pore air acceleration, shown in Figs. 27b and 27c. We also see the necessity for a full $(\mathbf{u}-\mathbf{u}_f-p_f)$ formulation as

compared to the $(\mathbf{u}-p_f)$ formulation. In the $(\mathbf{u}-p_f)$ model, it is inaccurately assumed for shock loading that $\mathbf{a}_f \approx \mathbf{a}_s = \mathbf{a}$, which is clearly not the case here as the pore air acceleration can be up to two orders of magnitude larger than the lung parenchyma acceleration. Note that the ordinate is on a logarithmic scale.

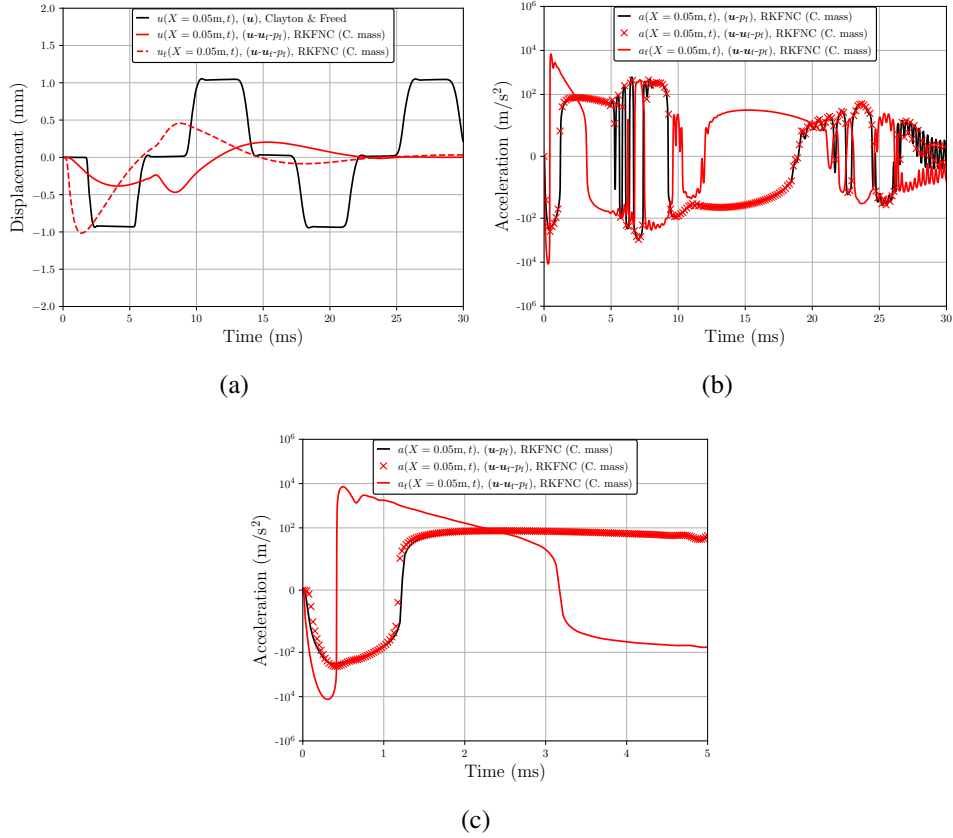


Fig. 27 Overpressure loading from the Yen impulse at 50 kPa showing a) differences in displacements of solid skeleton lung parenchyma vs. pore air b) differences in accelerations of solid skeleton lung parenchyma vs. pore air and c) zoomed-in version of b). Here, shock viscosity with $C_0 = 1.5$, $C_1 = 0.06$ and pressure stabilization for the multiphase model with $\alpha^{\text{stab}} = 10^{-10}$ have been enabled.

Another advantage of the multiphase models over the single-phase model is that in the former we can directly compute the lung parenchyma and pore air pressures separate from one another, and we can separate computation of the solid stress and total stress. In the single-phase model, these metrics are lumped together.

We see fairly good correlation between the total pressures and total stresses between the single-phase and multiphase model, shown in Figs. 28a and 28b, respectively. We believe that the slight discrepancies between the two models are due to the

differences in Gauss integration used. As mentioned previously, LS-DYNA (i.e., the single-phase model) assumes 1-point Gauss quadrature, whereas here we use a Q1-Q1-P1 element with 2-point quadrature for the multiphase model.

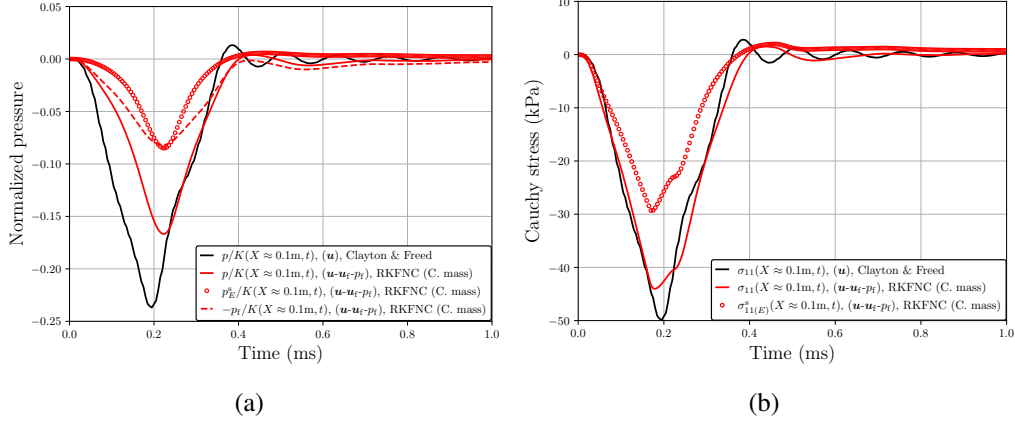


Fig. 28 Overpressure loading from the Yen impulse at 50 kPa showing a comparison of a) the total pressure between the single-phase and multiphase model, as well as the distinct lung parenchyma and pore air pressures for the multiphase model, all of which are normalized by the single-phase bulk modulus $K = 213$ kPa, and b) the total Cauchy stress between the single-phase and multiphase model, as well as the lung parenchyma Cauchy stress for the multiphase model. Here, shock viscosity with $C_0 = 1.5, C_1 = 0.06$ and pressure stabilization for the multiphase model with $\alpha^{\text{stab}} = 10^{-10}$ have been enabled.

Finally, we present the results for the Friedlander impulse, with traction profile shown in Fig. 24b. For this particular type of blast wave, we are unable to obtain peak overpressures greater than 15 kPa without any sort of stabilization techniques. However, the numerical oscillations in the “unstable” simulations are not as obvious as they are for the Yen impulse (Fig. 29). Nor can we presently simulate peak overpressures greater than 25 kPa with shock viscosity and pressure stabilization enabled, which is markedly lower than those we can achieve with the Yen impulse. This likely has to do with how long the Friedlander impulse is applied: 10 ms of compressive force, versus 340 μs for the Yen impulse, followed by a tensile force relative to the reference pressure (i.e., atmospheric pressure), versus no force for the Yen impulse. However, with numerical stabilization, we are able to achieve strains on the order of 7%-8% nominal axial strain, around $7\times$ greater than those for the Yen impulse, as shown in Fig. 30.

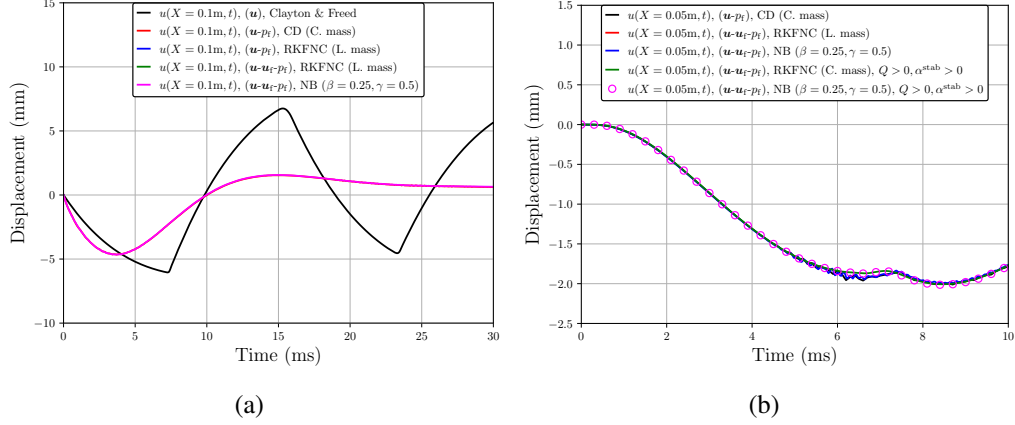


Fig. 29 Comparison of lung parenchyma displacements undergoing overpressure loading from the Friedlander impulse at 15 kPa between a) the single-phase and multiphase models tracking the nodes at the top of the column, and b) the multiphase models tracking the nodes at the middle of the column with and without stabilization methods. Without shock viscosity Q and pressure stabilization α^{stab} , we invoke a Q2-Q2-P1 element; with shock viscosity Q where $C_0 = 1.5$, $C_1 = 0.06$ and pressure stabilization where $\alpha^{\text{stab}} = 10^{-10}$, we invoke a Q1-Q1-P1 element.

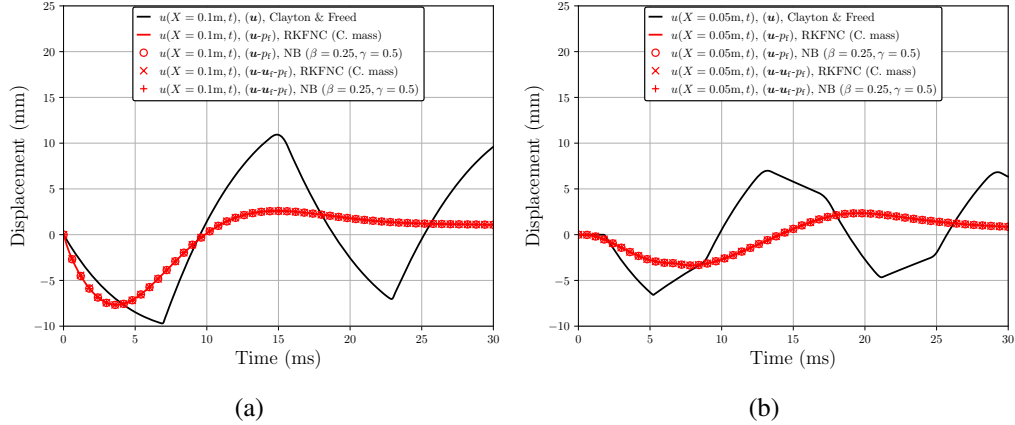


Fig. 30 Comparison of displacements of solid skeleton lung parenchyma between the single-phase and multiphase models undergoing overpressure loading from the Friedlander impulse at 25 kPa tracking the nodes at a) the top of the column and b) the middle of the column. Here, shock viscosity with $C_0 = 1.5$, $C_1 = 0.06$ and pressure stabilization for the multiphase model with $\alpha^{\text{stab}} = 10^{-10}$ have been enabled.

As with the resulting deformations from the applied Yen impulse, we are also able to see the relative motion of the pore air to the lung parenchyma, shown in Fig. 31a. Again we see the necessity of the full $(u-u_f-p_f)$ formulation in accounting for the differences in accelerations between the lung parenchyma and pore air, demonstrated by results in Figs. 31b and 31c.

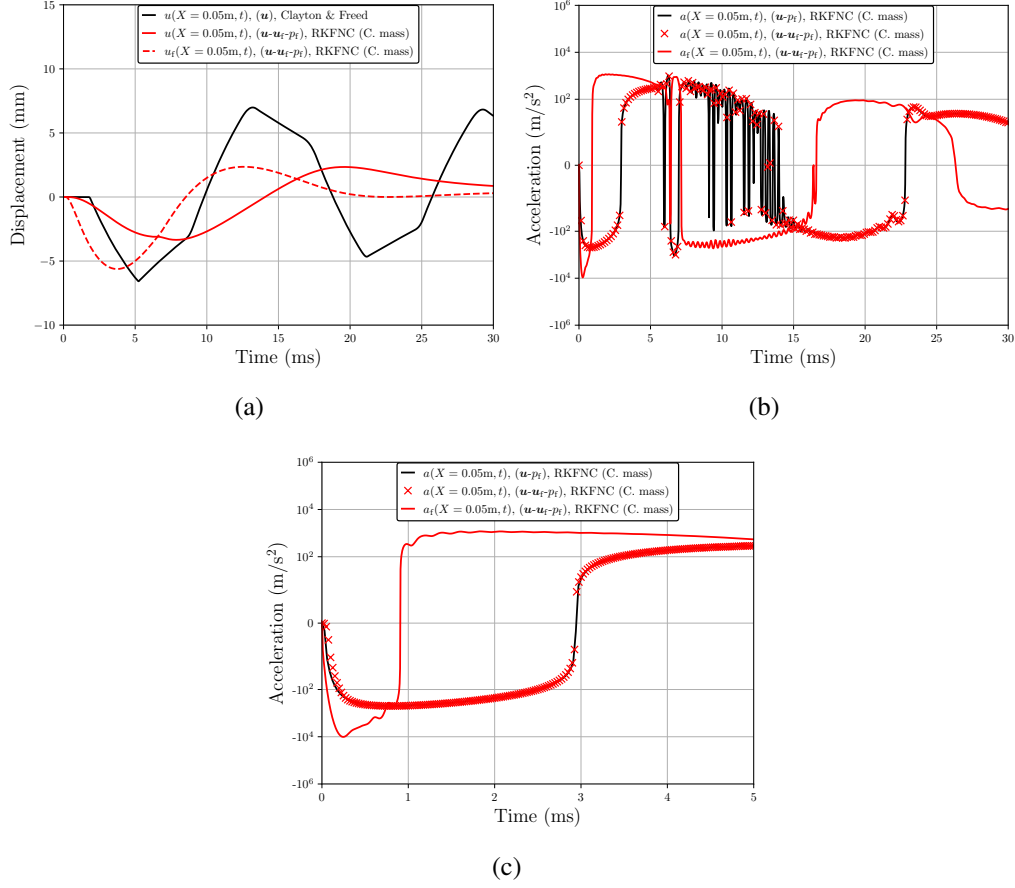


Fig. 31 Overpressure loading from the Friedlander impulse at 25 kPa showing a) differences in displacements of solid skeleton lung parenchyma vs. pore air b) differences in accelerations of solid skeleton lung parenchyma vs. pore air and c) zoomed-in version of b). Here, shock viscosity with $C_0 = 1.5$, $C_1 = 0.06$ and pressure stabilization for the multiphase model with $\alpha^{\text{stab}} = 10^{-10}$ have been enabled.

In contrast to the Yen impulse, however, we see a little bit better agreement between the total pressures and total stresses, shown in Figs. 32a and 32b, respectively. Of great interest is the lung parenchyma stress going into tension sooner in the multiphase model than in the single-phase model. Fung has argued that excessive tensile forces in the walls of the lung alveoli can lead to rupturing of the alveoli and local injury.⁵³ Thus, we might be able to predict injury sooner by accounting for the relative motion of air to the lung parenchyma in the multiphase model. Further work is needed to quantify damage and local injury in the multiphase model to be certain.

Finally, from a computational expense standpoint, we note that the explicit time integration methods are better suited for the high strain-rate loading examples con-

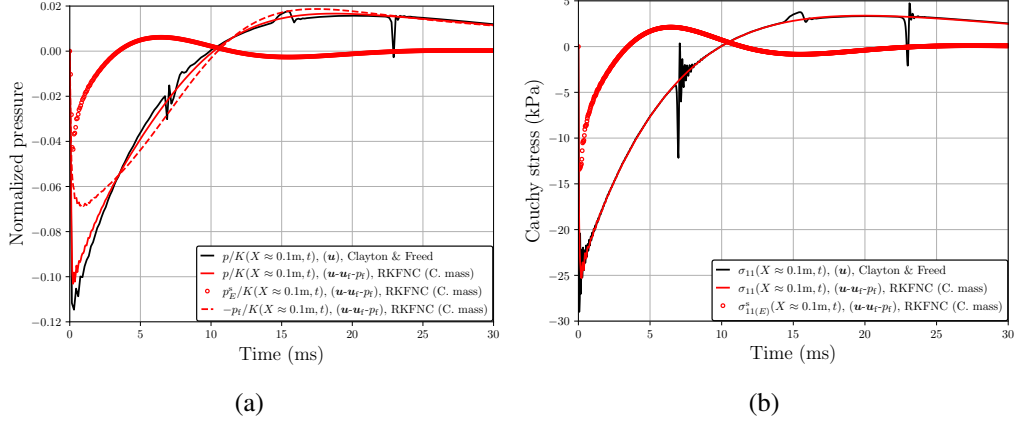


Fig. 32 Overpressure loading from the Friedlander impulse at 25 kPa showing a comparison of a) the total pressure between the single-phase and multiphase model, as well as the distinct lung parenchyma and pore air pressures for the multiphase model, all of which are normalized by the single-phase bulk modulus $K = 213$ kPa, and b) the total Cauchy stress between the single-phase and multiphase model, as well as the lung parenchyma Cauchy stress for the multiphase model. Here, shock viscosity with $C_0 = 1.5, C_1 = 0.06$ and pressure stabilization for the multiphase model with $\alpha^{\text{stab}} = 10^{-10}$ have been enabled.

sidered here than the implicit integration methods, which is not surprising. The simulations were conducted in serial on four 2x Intel Xeon Gold 6230 processors with 192 GB of RAM.

In Tables 13 and 14, we present wall clock times for the various formulations with different time integration schemes for the low strain-rate loading example (i.e., the de Boer verification of Section 5.2) and for the high strain-rate loading example of the Friedlander shock waveform at 15 kPa overpressure. In Fig. 33, we present the time steps for the Friedlander shock waveform at 25 kPa overpressure. These tend to hover around the maximum time step constrained by the Courant-Friedrichs-Lewy (CFL) condition for a single-phase material.

Table 13. Computational cost of de Boer simulations for element length $h_0^e = 1$ m. “N/A” denotes that simulation did not finish. Implicit time-stepping schemes used fixed $\Delta t = 10^{-4}$ s.

Element type	α^{stab}	C_0, C_1	Integration scheme	Wall clock time (min)
Q1-P1	0	0,0	N.B. ($\gamma = 0.25, \beta = 0.5$)	1.21
Q1-P1	10^{-10}	0,0	N.B. ($\gamma = 0.25, \beta = 0.5$)	1.26
Q1-P1	10^{-8}	0,0	N.B. ($\gamma = 0.25, \beta = 0.5$)	1.41
Q1-P1	0	0,0	RKFNC (C. mass)	154.07
Q1-P1	10^{-10}	0,0	RKFNC (C. mass)	5.88
Q1-P1	10^{-8}	0,0	RKFNC (C. mass)	6.02
Q2-P1	0	0,0	N.B. ($\gamma = 0.25, \beta = 0.5$)	1.45
Q2-P1	0	0,0	RKFNC (C. mass)	221.97
Q1-Q1-P1	0	0,0	N.B. ($\gamma = 0.25, \beta = 0.5$)	2.08
Q1-Q1-P1	10^{-10}	0,0	N.B. ($\gamma = 0.25, \beta = 0.5$)	2.29
Q1-Q1-P1	10^{-8}	0,0	N.B. ($\gamma = 0.25, \beta = 0.5$)	2.23
Q1-Q1-P1	0	0,0	RKFNC (C. mass)	196.86
Q1-Q1-P1	10^{-10}	0,0	RKFNC (C. mass)	12.25
Q1-Q1-P1	10^{-8}	0,0	RKFNC (C. mass)	3.73
Q2-Q1-P1	0	0,0	N.B. ($\gamma = 0.25, \beta = 0.5$)	2.52
Q2-Q1-P1	10^{-10}	0,0	N.B. ($\gamma = 0.25, \beta = 0.5$)	2.80
Q2-Q1-P1	10^{-8}	0,0	N.B. ($\gamma = 0.25, \beta = 0.5$)	2.67
Q2-Q1-P1	0	0,0	RKFNC (C. mass)	N/A
Q2-Q1-P1	10^{-10}	0,0	RKFNC (C. mass)	N/A
Q2-Q1-P1	10^{-8}	0,0	RKFNC (C. mass)	3.73
Q2-Q2-P1	0	0,0	N.B. ($\gamma = 0.25, \beta = 0.5$)	2.50
Q2-Q2-P1	0	0,0	RKFNC (C. mass)	319.77

Table 14. Computational cost of Friedlander impulse simulations for element length $h_0^e = 1$ mm at 15 kPa overpressure. “N/A” denotes that simulation did not finish. Implicit time-stepping schemes used fixed $\Delta t = 10^{-6}$ s.

Element type	α^{stab}	C_0, C_1	Integration scheme	Wall clock time (min)
Q1-P1	0	0, 0	N.B. ($\gamma = 0.25, \beta = 0.5$)	112.74
Q1-P1	10^{-10}	1.5, 0.06	N.B. ($\gamma = 0.25, \beta = 0.5$)	105.6
Q1-P1	0	0, 0	RKFNC (C. mass)	2280.65
Q1-P1	10^{-10}	1.5, 0.06	RKFNC (C. mass)	11.14
Q2-P1	0	0,0	N.B. ($\gamma = 0.25, \beta = 0.5$)	124.47
Q2-P1	0	0,0	RKFNC (C. mass)	N/A
Q1-Q1-P1	0	0,0	N.B. ($\gamma = 0.25, \beta = 0.5$)	221.21
Q1-Q1-P1	10^{-10}	1.5, 0.06	N.B. ($\gamma = 0.25, \beta = 0.5$)	175.23
Q1-Q1-P1	0	0,0	RKFNC (C. mass)	3539.34
Q1-Q1-P1	10^{-10}	1.5, 0.06	RKFNC (C. mass)	25.23
Q2-Q2-P1	0	0,0	N.B. ($\gamma = 0.25, \beta = 0.5$)	201.45
Q2-Q2-P1	0	0,0	RKFNC (C. mass)	N/A

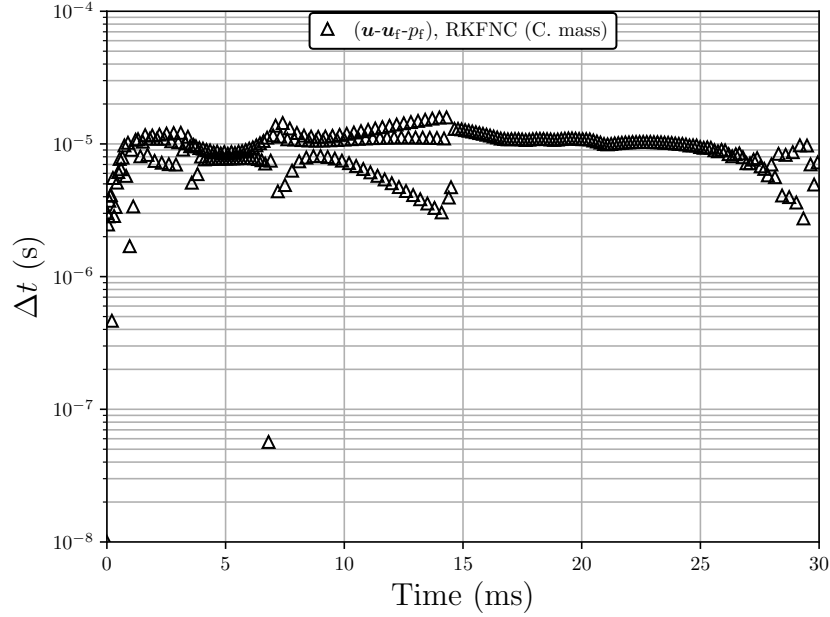


Fig. 33 Time steps over the course of the Friedlander shock loading example at 25 kPa overpressure

6. Conclusions

Simulations have invoked a novel numerical implementation¹⁹ of a finite-strain framework of a biphasic mixture (i.e., coupled pore fluid flow and solid skeleton deformation) of a soft porous material for high strain-rate dynamic loading. The constitutive theory, discussed in detail in a prior report,¹⁸ is non-linear elastic and accounts for the compressibility of the pore air. The formulation herein did not make assumptions regarding the equivalency of acceleration of pore fluid to that of solid skeleton, but rather allowed them to be different.

For shock loadings, it was shown that the addition of both a bulk shock viscosity term⁵⁴ in the constitutive stress response of the solid skeleton and a pressure stabilization term^{55,56} in the balance of mass of the mixture for a linear, equal-order element interpolation allowed us to achieve deformations for higher magnitudes of peak overpressure loadings.

In a verification example, we showcased the effectiveness of different time integration schemes, both implicit and explicit. It was shown that accounting for the inertia term in Darcy's constitutive law gave more accurate results with respect to the analytical solution than a simpler poroelastic model which neglected those physics

in its application of Darcy's law. Furthermore, spatial discretization plays a role in determining the accuracy of the numerical solution: poor resolution can lead to numerical instabilities resulting from large changes in the pore fluid pressure solution. It was also demonstrated that for this low strain-rate loading, an implicit time integration scheme is more cost effective than an explicit time integration scheme.

The role of the bulk viscosity term is significant for high strain-rate loading applications. For elastodynamics, it is clear that the bulk viscosity term damps out numerical oscillations caused by overshoots when the numerical time integration scheme is trying to resolve the shock front. We also showed that poor spatial discretization cannot accurately capture the shock front, which motivated the use of more refined FE meshes for poroelastodynamics. Refinement also sought to avoid any instabilities in the pore fluid pressure gradient.

For the multiphase model, we found that a linear-linear, equal-order element type, with both bulk viscosity and pressure stabilization enabled, produced a smoother solution than a fully stable Taylor-Hood element without any additional artificial stabilization. In addition, using a 5(4) Runge-Kutta integration scheme with adaptive time-stepping based on local truncation error resulted in lower computational cost compared to traditional implicit methods. This Runge-Kutta scheme yielded higher accuracy compared to the standard second-order CD method, the latter which is often unstable for poroelastodynamics applications at high strain rates.

When comparing to a single-phase model, it is apparent that accounting for relative motion between pore fluid and solid skeleton drastically alters the deformation response of the soft porous material (i.e., the lung parenchyma). In particular, deformations of the solid skeleton lung parenchyma are dampened out by the oscillating pore air, whereas the single-phase model behaves elastodynamically with minimal dissipation of the shock wave. It was also shown that accounting for differences in accelerations between the two constituents was effective in demonstrating the necessity for the more complex poroelastodynamics formulation for high strain-rate loading applications. To this extent, it was shown that for a shock wave resulting from an explosion, the multiphase model can pick up tensile stresses in the solid skeleton lung parenchyma sooner than in the single-phase model, which may be indicative of tissue rupture.

This work offers new insight into computationally inexpensive modeling of smeared

pore-scale FSI in soft porous materials at high strain-rates. For modeling of lung parenchyma, further research is needed to: (1) develop a more complex constitutive response for the lung parenchyma that can take into account the stiffening of the tissue as it is stretched and compressed, analogous to what was done by Clayton, Banton, and Freed^{20–22,25}; (2) consider other physiological boundary conditions that might mimic *in-vivo* conditions; (3) develop a solid skeleton damage variable to help quantify local injury; (4) implement thermomechanical coupling between pore air and solid lung tissue; and (5) address the microstructural response of individual alveoli to deformations in a multiscale framework.

7. References

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Appendix A. Derivation of Gateaux Derivatives

Gateaux derivatives are used in the numerical implementation of TPM established by Irwin et al.,^{1,2} wherein notation is defined explicitly. Kinematic derivatives are obtained as follows:

$$\begin{aligned}\delta(F_{11}) &= \delta \left(1 + \frac{\partial u}{\partial X} \right) = \delta \left(\frac{\partial u}{\partial X} \right) = \frac{\partial(\delta u)}{\partial X} \\ &= \beta \Delta t^2 \frac{\partial(\delta a)}{\partial X} = \gamma \Delta t \frac{\partial(\delta v)}{\partial X}\end{aligned}\tag{A-1}$$

$$\begin{aligned}\delta(F_{11}^n) &= \delta \left(1 + \frac{\partial u}{\partial X} \right)^n = n F_{11}^{n-1} \delta(F_{11}) \\ &= n F_{11}^{n-1} \beta \Delta t^2 \frac{\partial(\delta a)}{\partial X} = n F_{11}^{n-1} \gamma \Delta t \frac{\partial(\delta v)}{\partial X}\end{aligned}\tag{A-2}$$

$$\delta(\ln(J)) = \frac{1}{J} \delta(J) = \frac{1}{J} \beta \Delta t^2 \frac{\partial(\delta a)}{\partial X} = \frac{1}{J} \gamma \Delta t \frac{\partial(\delta v)}{\partial X}\tag{A-3}$$

$$\begin{aligned}\delta(C_{11}^{-1}) &= -C_{11}^{-2} \delta(C_{11}) = -C_{11}^{-2} \delta(F_{11}^2) = -2 C_{11}^{-2} F_{11} \frac{\partial(\delta u)}{\partial X} \\ &= -2 F_{11}^{-3} \beta \Delta t^2 \frac{\partial(\delta a)}{\partial X} = -2 F_{11}^{-3} \gamma \Delta t \frac{\partial(\delta v)}{\partial X}\end{aligned}\tag{A-4}$$

Recall that the solid skeleton model is that of a neo-Hookean hyperelastic material. Thus, the following derivations of the variations of the solid skeleton extra stresses are not generalizable to other material models:

$$\begin{aligned}\delta(S_{11(E)}^s) &= \delta (\mu + [\lambda \ln(J) - \mu] C_{11}^{-1}) \\ &= \lambda \delta (\ln(J)) C_{11}^{-1} + [\lambda \ln(J) - \mu] \delta (C_{11}^{-1}) \\ &= (\lambda - 2 [\lambda \ln(J) - \mu]) F_{11}^{-3} \beta \Delta t^2 \frac{\partial(\delta a)}{\partial X} \\ &= (\lambda - 2 [\lambda \ln(J) - \mu]) F_{11}^{-3} \gamma \Delta t \frac{\partial(\delta v)}{\partial X}\end{aligned}\tag{A-5}$$

¹Irwin Z, Regueiro R, Clayton J. A large deformation multiphase continuum mechanics model for shock loading of lung parenchyma. Part I: Theory. DEVCOM Army Research Laboratory (US); 2023. Report No.: ARL-TR-9686.

²Irwin Z, Regueiro R, Clayton J. A large deformation multiphase continuum mechanics model for shock loading of lung parenchyma. Part II: Numerical methods. DEVCOM Army Research Laboratory (US); 2023. Report No.: ARL-TR-9687.

$$\begin{aligned}
\delta(P_{11(E)}^s) &= \delta(F_{11}S'_{11}) = \delta(F_{11})S_{11(E)}^s + F_{11}\delta(S_{11(E)}^s) \\
&= (\mu + [\lambda \ln(J) - \mu] F_{11}^{-2}) \frac{\partial(\delta u)}{\partial X} \\
&\quad + (\lambda - 2[\lambda \ln(J) - \mu] F_{11}^{-2}) \frac{\partial(\delta u)}{\partial X} \\
&= (\mu + [\lambda - \lambda \ln(J) + \mu] F_{11}^{-2}) \beta \Delta t^2 \frac{\partial(\delta a)}{\partial X} \\
&= (\mu + [\lambda - \lambda \ln(J) + \mu] F_{11}^{-2}) \gamma \Delta t \frac{\partial(\delta v)}{\partial X} \tag{A-6}
\end{aligned}$$

$$\begin{aligned}
\delta(S_{11(E),\text{vis}}^s) &= \nu_0 [\lambda + 2(\mu - \ln F_{11})] \left(\delta(F_{11}^{-3}) \frac{\partial v}{\partial X} + F_{11}^{-3} \frac{\partial \delta(v)}{\partial X} \right) \\
&\quad + \nu_0 F_{11}^{-3} \frac{\partial v}{\partial X} (-2\delta(\ln F_{11})) \\
&= \nu_0 [\lambda + 2(\mu - \ln F_{11})] \left(-3F_{11}^{-4} \frac{\partial \delta(u)}{\partial X} \frac{\partial v}{\partial X} + F_{11}^{-3} \frac{\partial \delta(v)}{\partial X} \right) \\
&\quad + \nu_0 F_{11}^{-3} \frac{\partial v}{\partial X} \left(\frac{-2}{F_{11}} \frac{\partial \delta(u)}{\partial X} \right) \\
&= \nu_0 F_{11}^{-3} \left[(\lambda + 2(\mu - \ln F_{11})) \left(\gamma \Delta t - 3F_{11}^{-1} \frac{\partial v}{\partial X} \beta \Delta t^2 \right) \right. \\
&\quad \left. - 2F_{11}^{-1} \frac{\partial v}{\partial X} \beta \Delta t^2 \right] \frac{\partial(\delta a)}{\partial X} \\
&= \nu_0 F_{11}^{-3} \left[(\lambda + 2(\mu - \ln F_{11})) \left(1 - 3F_{11}^{-1} \frac{\partial v}{\partial X} \gamma \Delta t \right) \right. \\
&\quad \left. - 2F_{11}^{-1} \frac{\partial v}{\partial X} \gamma \Delta t \right] \frac{\partial(\delta v)}{\partial X} \tag{A-7}
\end{aligned}$$

$$\begin{aligned}
\delta(F_{11}S_{11(E),\text{vis}}^s) &= \alpha F_{11}^{-3} \frac{\partial v}{\partial X} [\lambda + 2(\mu - \ln F_{11})] \beta \Delta t^2 \frac{\partial(\delta a)}{\partial X} \\
&\quad + \nu_0 F_{11}^{-2} \left[(\lambda + 2(\mu - \ln F_{11})) \left(\gamma \Delta t - 3F_{11}^{-1} \frac{\partial v}{\partial X} \beta \Delta t^2 \right) \right. \\
&\quad \quad \left. - 2F_{11}^{-1} \frac{\partial v}{\partial X} \beta \Delta t^2 \right] \frac{\partial(\delta a)}{\partial X} \\
&= \nu_0 F_{11}^{-3} \frac{\partial v}{\partial X} [\lambda + 2(\mu - \ln F_{11})] \gamma \Delta t \frac{\partial(\delta v)}{\partial X} \\
&\quad + \nu_0 F_{11}^{-2} \left[(\lambda + 2(\mu - \ln F_{11})) \left(1 - 3F_{11}^{-1} \frac{\partial v}{\partial X} \gamma \Delta t \right) \right. \\
&\quad \quad \left. - 2F_{11}^{-1} \frac{\partial v}{\partial X} \gamma \Delta t \right] \frac{\partial(\delta v)}{\partial X}
\end{aligned} \tag{A-8}$$

Together, Eqs. A-6 and A-8 give

$$\begin{aligned}
\delta(P_{11(E)}^s) &= F_{11}^{-2} \left([\lambda + 2\mu] \left(\nu_0 \gamma \Delta t - 2\nu_0 F_{11}^{-1} \frac{\partial v}{\partial X} \beta \Delta t^2 \right) \right. \\
&\quad \left. - \ln F_{11} \left(2\nu_0 \gamma \Delta t - \left[4\nu_0 F_{11}^{-1} \frac{\partial v}{\partial X} - \lambda \right] \beta \Delta t^2 \right) \right. \\
&\quad \left. + \left([\lambda + \mu + \mu F_{11}^2] - 2\nu_0 F_{11}^{-1} \frac{\partial v}{\partial X} \right) \beta \Delta t^2 \right) \frac{\partial(\delta a)}{\partial X} \\
&= F_{11}^{-2} \left([\lambda + 2\mu] \left(\nu_0 - 2\nu_0 F_{11}^{-1} \frac{\partial v}{\partial X} \gamma \Delta t \right) \right. \\
&\quad \left. - \ln F_{11} \left(2\nu_0 - \left[4\nu_0 F_{11}^{-1} \frac{\partial v}{\partial X} - \lambda \right] \gamma \Delta t \right) \right. \\
&\quad \left. + \left([\lambda + \mu + \mu F_{11}^2] - 2\nu_0 F_{11}^{-1} \frac{\partial v}{\partial X} \right) \gamma \Delta t \right) \frac{\partial(\delta v)}{\partial X}
\end{aligned} \tag{A-9}$$

$$\begin{aligned}
\delta(P_{11(E)}^s) &= F_{11}^{-2} \left([\lambda + 2\mu] \left(\nu_0 \gamma \Delta t - 2\nu_0 F_{11}^{-1} \frac{\partial v}{\partial X} \beta \Delta t^2 \right) \right. \\
&\quad \left. - \ln F_{11} \left(2\nu_0 \gamma \Delta t - \left[4\nu_0 F_{11}^{-1} \frac{\partial v}{\partial X} - \lambda \right] \beta \Delta t^2 \right) \right. \\
&\quad \left. + \left([\lambda + \mu + \mu F_{11}^2] - 2\nu_0 F_{11}^{-1} \frac{\partial v}{\partial X} \right) \beta \Delta t^2 \right) \frac{\partial(\delta a)}{\partial X} \\
&= F_{11}^{-2} \left([\lambda + 2\mu] \left(\nu_0 - 2\nu_0 F_{11}^{-1} \frac{\partial v}{\partial X} \gamma \Delta t \right) \right. \\
&\quad \left. - \ln F_{11} \left(2\nu_0 - \left[4\nu_0 F_{11}^{-1} \frac{\partial v}{\partial X} - \lambda \right] \gamma \Delta t \right) \right. \\
&\quad \left. + \left([\lambda + \mu + \mu F_{11}^2] - 2\nu_0 F_{11}^{-1} \frac{\partial v}{\partial X} \right) \gamma \Delta t \right) \frac{\partial(\delta v)}{\partial X} \quad (\text{A-10})
\end{aligned}$$

$$\begin{aligned}
\delta(\rho^{\text{fR}}) &= \delta \left(\rho_0^{\text{fR}} \exp \left[\frac{p_{\text{f}} - p_{\text{f},0}}{K_{\text{f}}^{\eta}} \right] \right) = \rho_0^{\text{fR}} \exp \left[\frac{p_{\text{f}} - p_{\text{f},0}}{K_{\text{f}}^{\eta}} \right] \delta \left(\frac{p_{\text{f}}}{K_{\text{f}}^{\eta}} \right) \\
&= \frac{\rho^{\text{fR}}}{K_{\text{f}}^{\eta}} \delta(p_{\text{f}}) = \frac{\rho^{\text{fR}}}{K_{\text{f}}^{\eta}} \beta \Delta t^2 \delta \ddot{p}_{\text{f}} = \frac{\rho^{\text{fR}}}{K_{\text{f}}^{\eta}} \gamma \Delta t \delta \dot{p}_{\text{f}} \quad (\text{A-11})
\end{aligned}$$

$$\begin{aligned}
\delta(n^{\text{f}}) &= \delta(1 - n^{\text{s}}) = -\delta(n^{\text{s}}) = -\delta \left(\frac{n_0^{\text{s}}}{J} \right) = \frac{n_0^{\text{s}}}{J^2} \delta(J) \\
&= \frac{n^{\text{s}}}{J} \beta \Delta t^2 \frac{\partial(\delta a)}{\partial X} = \frac{n^{\text{s}}}{J} \gamma \Delta t \frac{\partial(\delta v)}{\partial X} \quad (\text{A-12})
\end{aligned}$$

$$\begin{aligned}
\delta(\rho_0^{\text{f}}) &= \delta(J n^{\text{f}} \rho^{\text{fR}}) = n^{\text{f}} \rho^{\text{fR}} \delta(J) + J \rho^{\text{fR}} \delta(n^{\text{f}}) + J n^{\text{f}} \delta(\rho^{\text{fR}}) \\
&= n^{\text{f}} \rho^{\text{fR}} \frac{\partial(\delta u)}{\partial X} + \rho^{\text{fR}} n^{\text{s}} \frac{\partial(\delta u)}{\partial X} + \frac{J n^{\text{f}} \rho^{\text{fR}}}{K_{\text{f}}^{\eta}} \delta p_{\text{f}} \\
&= \beta \Delta t^2 \left(\rho^{\text{fR}} \frac{\partial(\delta a)}{\partial X} + \frac{J n^{\text{f}} \rho^{\text{fR}}}{K_{\text{f}}^{\eta}} \delta \ddot{p}_{\text{f}} \right) \\
&= \gamma \Delta t \left(\rho^{\text{fR}} \frac{\partial(\delta v)}{\partial X} + \frac{J n^{\text{f}} \rho^{\text{fR}}}{K_{\text{f}}^{\eta}} \delta \dot{p}_{\text{f}} \right) \quad (\text{A-13})
\end{aligned}$$

$$\begin{aligned}
\delta(\hat{k}) &= \delta \left(\frac{\varkappa \mathcal{F}(n^f)}{\eta_f \mathcal{F}(n_0^f)} \right) = \frac{\partial \hat{k}}{\partial n^f} \delta(n^f) = \frac{\partial \left(\frac{\varkappa}{\eta_f} \frac{1}{\mathcal{F}(n_0^f)} \frac{(n^f)^3}{(1-(n^f)^2)} \right)}{\partial n^f} \delta(n^f) \\
&= \frac{\varkappa \mathcal{F}(n^f)}{\eta_f \mathcal{F}(n_0^f)} \left[\frac{3}{n^f} + \frac{2n^f}{1-(n^f)^2} \right] \delta(n^f) = \hat{k} \left[\frac{3}{n^f} + \frac{2n^f}{1-(n^f)^2} \right] \delta(n^f) \\
&= \frac{\hat{k} n^s}{J} \left[\frac{3}{n^f} + \frac{2n^f}{1-(n^f)^2} \right] \beta \Delta t^2 \frac{\partial(\delta a)}{\partial X} \\
&= \frac{\hat{k} n^s}{J} \left[\frac{3}{n^f} + \frac{2n^f}{1-(n^f)^2} \right] \gamma \Delta t \frac{\partial(\delta v)}{\partial X}
\end{aligned} \tag{A-14}$$

Appendix B. Derivation of Linearized Equations

B.1 Linearization of the (u) Formulation Variational Equations

Refer to Section 5.1.1 of Irwin et al.¹ The linearization of the variational equations given by Irwin et al.¹ commences as follows:

$$\delta \mathcal{G}_1^{\text{INT}} = \int_0^{X=H} w \delta(\rho_0 a) A dX = \int_0^{X=H} w \rho_0 \delta a A dX \quad (\text{B-1})$$

For quasi-static elasticity, Eq. B-1 is zero. Next,

$$\begin{aligned} \delta \mathcal{G}_2^{\text{INT}} &= \int_0^{X=H} \frac{\partial w}{\partial X} \delta(P_{11}) A dX \\ &= \int_0^{X=H} \frac{\partial w}{\partial X} (\mu + [\lambda - \lambda \ln(J) + \mu] F_{11}^{-2}) (\beta \Delta t^2) \frac{\partial(\delta a)}{\partial X} A dX \end{aligned} \quad (\text{B-2})$$

When viscous damping is considered, where ν_0 is the viscous dampening coefficient (unit (s)), Eq. B-2 becomes

$$\begin{aligned} \delta \mathcal{G}_2^{\text{INT}} &= \int_0^{X=H} \frac{\partial w}{\partial X} F_{11}^{-2} \left([\lambda + 2\mu] \left[\nu_0(\gamma \Delta t) - 2\nu_0 F_{11}^{-1} \frac{\partial v}{\partial X} (\beta \Delta t^2) \right] \right. \\ &\quad \left. - \ln(F_{11}) \left[2\nu_0(\gamma \Delta t) - \left(4\nu_0 F_{11}^{-1} \frac{\partial v}{\partial X} - \lambda \right) (\beta \Delta t^2) \right] \right. \\ &\quad \left. + \left[(\lambda + \mu + \mu F_{11}^2) - 2\nu_0 F_{11}^{-1} \frac{\partial v}{\partial X} \right] (\beta \Delta t^2) \right) \frac{\partial(\delta a)}{\partial X} A dX \end{aligned} \quad (\text{B-3})$$

For elasticity, Eq. B-2 becomes

$$\delta \mathcal{G}_2^{\text{INT}} = \int_0^{X=H} \frac{\partial w}{\partial X} (\mu + [\lambda - \lambda \ln(J) + \mu] F_{11}^{-2}) (\gamma \Delta t) \frac{\partial(\delta v)}{\partial X} A dX \quad (\text{B-4})$$

¹Irwin Z, Regueiro R, Clayton J. A large deformation multiphase continuum mechanics model for shock loading of lung parenchyma. Part II: Numerical methods. DEVCOM Army Research Laboratory (US); 2023. Report No.: ARL-TR-9687.

When viscous damping is considered, Eq. B-4 becomes

$$\begin{aligned}\delta\mathcal{G}_2^{\text{INT}} = & \int_0^{X=H} \frac{\partial w}{\partial X} F_{11}^{-2} \left([\lambda + 2\mu] \left[\nu_0 - 2\nu_0 F_{11}^{-1} \frac{\partial v}{\partial X} (\gamma \Delta t) \right] \right. \\ & \left. - \ln(F_{11}) \left[2\nu_0 - \left(4\nu_0 F_{11}^{-1} \frac{\partial v}{\partial X} - \lambda \right) (\gamma \Delta t) \right] \right. \\ & \left. + \left[(\lambda + \mu + \mu F_{11}^2) - 2\nu_0 F_{11}^{-1} \frac{\partial v}{\partial X} \right] (\gamma \Delta t) \right) \frac{\partial(\delta v)}{\partial X} dX\end{aligned}\quad (\text{B-5})$$

When shock viscosity is enabled, linearization with the additions given by Irwin et al.¹ proceeds as follows:

$$\delta\mathcal{G}_2^{\text{INT}} = \int_0^{X=H} \frac{\partial w}{\partial X} (\delta(P_{11}) - \delta(Q)) A dX \quad (\text{B-6})$$

where

$$\begin{aligned}\delta(Q) = & \rho_0 h_0 \left(C_0 h_0 \delta J F_{11}^{-2} \left[\frac{\partial v}{\partial X} \right]^2 + C_0 J h_0 \delta F_{11}^{-2} \left[\frac{\partial v}{\partial X} \right]^2 \right. \\ & + C_0 J h_0 F_{11}^{-2} \delta \left[\frac{\partial v}{\partial X} \right]^2 - C_1 \delta c F_{11}^{-2} \frac{\partial v}{\partial X} \\ & \left. - C_1 c \delta F_{11}^{-1} \frac{\partial v}{\partial X} - C_1 c F_{11}^{-1} \frac{\partial \delta v}{\partial X} \right)\end{aligned}\quad (\text{B-7})$$

and

$$\begin{aligned}\delta c = & \delta \sqrt{\frac{K}{\rho}} = -\frac{1}{2} K^{1/2} \rho^{-3/2} \delta \rho = -\frac{1}{2} K^{1/2} \rho^{-3/2} \delta \left(\frac{\rho_0}{J} \right) \\ = & \frac{1}{2} K^{1/2} \rho^{-3/2} \frac{\rho_0}{J^2} \delta J = \frac{1}{2} K^{1/2} \frac{\rho^{-1/2}}{J} \delta J = \frac{c}{2J} \delta J\end{aligned}\quad (\text{B-8})$$

where K denotes the solid bulk modulus (solid skeleton in the multiphase regime).

Thus, after some algebra,

$$\delta(Q) = \rho_0 h_0 \left(C_0 h_0 F_{11}^{-2} \frac{\partial v}{\partial X} \left[(1 - 2F_{11}^{-1}) \frac{\partial v}{\partial X} (\beta \Delta t^2) + 2(\gamma \Delta t) \right] - C_1 F_{11}^{-1} c \left[\frac{3}{2} F_{11}^{-1} \frac{\partial v}{\partial X} (\beta \Delta t^2) - (\gamma \Delta t) \right] \right) \frac{\partial \delta a}{\partial X} \quad (\text{B-9})$$

Then, combining the results from Eq. B-2 with B-9, Eq. B-6 becomes

$$\begin{aligned} \delta \mathcal{G}_2^{\text{INT}} = & \int_0^{X=H} \frac{\partial w}{\partial X} \left([\mu + (\lambda - \lambda \ln(J) + \mu) F_{11}^{-2}] (\beta \Delta t^2) \right. \\ & - \rho_0 h_0 \left[C_0 h_0 F_{11}^{-2} \frac{\partial v}{\partial X} \left((1 - 2F_{11}^{-1}) \frac{\partial v}{\partial X} (\beta \Delta t^2) + 2(\gamma \Delta t) \right) \right. \\ & \left. \left. - C_1 F_{11}^{-1} c \left(\frac{3}{2} F_{11}^{-1} \frac{\partial v}{\partial X} (\beta \Delta t^2) - (\gamma \Delta t) \right) \right] \right) \frac{\partial(\delta a)}{\partial X} A dX \end{aligned} \quad (\text{B-10})$$

Note that it would be unnecessary to derive the linearization of $\mathcal{G}_2^{\text{INT}}$ with shock viscosity and without inertia terms. We found that for overall stability of the numerical solution, the shock viscosity is usually only necessary when the inertia terms are enabled, and so a quasi-static formulation of the linearization of $\mathcal{G}_2^{\text{INT}}$ with shock viscosity is omitted here.

B.2 Linearization of the $(u-p_f)$ Formulation Variational Equations

Refer to Section 5.1.3 of Irwin et al.¹ The linearizations $\delta \mathcal{G}_1^{\text{INT}}$ and $\delta \mathcal{G}_2^{\text{INT}}$ remain unchanged from elastodynamics and elastics for the $(u-p_f)$ formulation.

The first required linearization is given by

$$\delta \mathcal{G}_3^{\text{INT}} = - \int_0^{X=H} \frac{\partial w}{\partial X} \delta(p_f) A dX = - \int_0^{X=H} \frac{\partial w}{\partial X} (\beta \Delta t^2) \delta \ddot{p}_f A dX \quad (\text{B-11})$$

For poroelasticity, Eq. B-11 becomes

$$\delta \mathcal{G}_3^{\text{INT}} = - \int_0^{X=H} \frac{\partial w}{\partial X} (\gamma \Delta t) \delta \dot{p}_f A dX \quad (\text{B-12})$$

The next linearization proceeds as follows:

$$\begin{aligned}
\delta \mathcal{H}_1^{\text{INT}} &= \int_0^{X=H} r \delta \left(\frac{J n^f}{K_f^\eta} \dot{p}_f + \dot{J} \right) A dX \\
&= \int_0^{X=H} r \left(\frac{n^f \dot{p}_f}{K_f^\eta} \delta J + \frac{J \dot{p}_f}{K_f^\eta} \delta n^f + \frac{J n^f}{K_f^\eta} \delta \dot{p}_f + \delta \dot{J} \right) A dX \\
&= \int_0^{X=H} r \left(\left[\frac{n^f \dot{p}_f}{K_f^\eta} (\beta \Delta t^2) + \frac{J \dot{p}_f}{K_f^\eta} \frac{n^s}{J} (\beta \Delta t^2) + (\gamma \Delta t) \right] \frac{\partial(\delta a)}{\partial X} \right. \\
&\quad \left. + \frac{J n^f}{K_f^\eta} (\gamma \Delta t) \delta \ddot{p}_f \right) A dX \\
&= \int_0^{X=H} r \left(\left[\frac{\dot{p}_f}{K_f^\eta} (\beta \Delta t^2) + (\gamma \Delta t) \right] \frac{\partial(\delta a)}{\partial X} + \frac{J n^f}{K_f^\eta} (\gamma \Delta t) \delta \ddot{p}_f \right) A dX \quad (\text{B-13})
\end{aligned}$$

For poroelasticity, Eq. B-13 becomes

$$\delta \mathcal{H}_1^{\text{INT}} = \int_0^{X=H} r \left(\left[\frac{\dot{p}_f}{K_f^\eta} (\gamma \Delta t) + 1 \right] \frac{\partial(\delta v)}{\partial X} + \frac{J n^f}{K_f^\eta} \delta \dot{p}_f \right) A dX \quad (\text{B-14})$$

The next linearization proceeds as follows:

$$\begin{aligned}
\delta \mathcal{H}_2^{\text{INT}} &= \int_0^{X=H} r \delta \left(\frac{1}{K_f^\eta} \frac{\partial p_f}{\partial X} n^f \tilde{v}_f \right) A dX \\
&= \int_0^{X=H} r \left(n^f \tilde{v}_f \frac{\partial(\delta p_f)}{\partial X} + \frac{\partial p_f}{\partial X} \delta(n^f \tilde{v}_f) \right) \frac{A}{K_f^\eta} dX \\
&= \int_0^{X=H} r \left(n^f \tilde{v}_f (\beta \Delta t^2) \frac{\partial(\delta \ddot{p}_f)}{\partial X} + \frac{\partial p_f}{\partial X} \underbrace{\left[- \left(\frac{\partial p_f}{\partial X} F_{11}^{-1} + \rho^{\text{fR}}(a + g) \right) \delta \hat{k} \right.} \right. \\
&\quad \left. \left. \underbrace{\left. - \hat{k} \left(\frac{\partial(\delta p_f)}{\partial X} F_{11}^{-1} + \frac{\partial p_f}{\partial X} \delta F_{11}^{-1} + (a + g) \delta \rho^{\text{fR}} + \rho^{\text{fR}} \delta a \right) \right] \right)}_{\text{Term 2}} \right) \frac{A}{K_f^\eta} dX \quad (\text{B-15})
\end{aligned}$$

$$\begin{aligned}
\mathbf{Term\ 1} &= -\left(\frac{\partial p_f}{\partial X} F_{11}^{-1} + \rho^{\text{fR}}(a+g)\right) \frac{\hat{k} n^s}{J} \left(\frac{3}{n^f} - \frac{2n^f}{1-(n^f)^2}\right) (\beta \Delta t^2) \frac{\partial(\delta a)}{\partial X} \\
&= \frac{n^s}{J} \left(\frac{3}{n^f} - \frac{2n^f}{1-(n^f)^2}\right) (n^f \tilde{v}_f) (\beta \Delta t^2) \frac{\partial(\delta a)}{\partial X} \\
\mathbf{Term\ 2} &= -\hat{k} \left(F_{11}^{-1}(\beta \Delta t^2) \frac{\partial(\delta \ddot{p}_f)}{\partial X} - \frac{\partial p_f}{\partial X} F_{11}^{-2}(\beta \Delta t^2) \frac{\partial \delta a}{\partial X} \right. \\
&\quad \left. + (a+g) \frac{\rho^{\text{fR}}}{K_f^\eta} (\beta \Delta t^2) \delta \ddot{p}_f + \rho^{\text{fR}} \delta a \right)
\end{aligned}$$

Combining terms leaves us with

$$\begin{aligned}
\delta \mathcal{H}_2^{\text{INT}} &= \int_0^{X=H} r \left(n^f \tilde{v}_f (\beta \Delta t^2) \frac{\partial(\delta \ddot{p}_f)}{\partial X} + \frac{\partial p_f}{\partial X} \left[\frac{n^s}{J} \left(\frac{3}{n^f} - \frac{2n^f}{1-(n^f)^2} \right) \times \right. \right. \\
&\quad \left. \left. (n^f \tilde{v}_f) (\beta \Delta t^2) \frac{\partial(\delta a)}{\partial X} - \hat{k} \left(F_{11}^{-1}(\beta \Delta t^2) \frac{\partial(\delta \ddot{p}_f)}{\partial X} - \frac{\partial p_f}{\partial X} F_{11}^{-2}(\beta \Delta t^2) \frac{\partial(\delta a)}{\partial X} \right. \right. \right. \\
&\quad \left. \left. \left. + (a+g) \frac{\rho^{\text{fR}}}{K_f^\eta} (\beta \Delta t^2) \delta \ddot{p}_f + \rho^{\text{fR}} \delta a \right) \right] \right) \frac{A}{K_f^\eta} dX \quad (\text{B-16})
\end{aligned}$$

Rearranging like terms, we arrive at

$$\begin{aligned}
\delta \mathcal{H}_2^{\text{INT}} &= \int_0^{X=H} r \left(\left[(n^f \tilde{v}_f) - \hat{k} \frac{\partial p_f}{\partial X} F_{11}^{-1} \right] (\beta \Delta t^2) \frac{\partial(\delta \ddot{p}_f)}{\partial X} - \hat{k} \frac{\partial p_f}{\partial X} (a+g) \times \right. \\
&\quad \left. \frac{\rho^{\text{fR}}}{K_f^\eta} (\beta \Delta t^2) \delta \ddot{p}_f + \frac{\partial p_f}{\partial X} \left[\frac{n^s}{J} \left(\frac{3}{n^f} - \frac{2n^f}{1-(n^f)^2} \right) (n^f \tilde{v}_f) + \hat{k} \frac{\partial p_f}{\partial X} F_{11}^{-2} \right] \times \right. \\
&\quad \left. \left. (\beta \Delta t^2) \frac{\partial(\delta a)}{\partial X} - \frac{\partial p_f}{\partial X} \hat{k} \rho^{\text{fR}} \delta a \right) \frac{A}{K_f^\eta} dX \quad (\text{B-17}) \right.
\end{aligned}$$

For poroelasticity, Eq. B-17 becomes,

$$\begin{aligned} \delta\mathcal{H}_2^{\text{INT}} = & \int_0^{X=H} r \left(\left[(n^f \tilde{v}_f) - \hat{k} \frac{\partial p_f}{\partial X} F_{11}^{-1} \right] (\gamma \Delta t) \frac{\partial(\delta \dot{p}_f)}{\partial X} \right. \\ & \left. - \hat{k} \frac{\partial p_f}{\partial X} g \frac{\rho^{\text{fR}}}{K_f^\eta} (\gamma \Delta t) \delta \dot{p}_f \right. \\ & \left. + \frac{\partial p_f}{\partial X} \left[\frac{n^s}{J} \left(\frac{3}{n^f} - \frac{2n^f}{1 - (n^f)^2} \right) (n^f \tilde{v}_f) + \hat{k} \frac{\partial p_f}{\partial X} F_{11}^{-2} \right] (\gamma \Delta t) \frac{\partial(\delta v)}{\partial X} \right) \frac{A}{K_f^\eta} dX \quad (\text{B-18}) \end{aligned}$$

The next linearization proceeds as follows:

$$\begin{aligned} \delta\mathcal{H}_3^{\text{INT}} = & \int_0^{X=H} \frac{\partial r}{\partial X} \delta \left(\hat{k} \frac{\partial p_f}{\partial X} F_{11}^{-1} \right) A dX \\ = & \int_0^{X=H} \frac{\partial r}{\partial X} \left(\frac{\partial p_f}{\partial X} F_{11}^{-1} \delta \hat{k} + \hat{k} F_{11}^{-1} \frac{\partial(\delta p_f)}{\partial X} + \hat{k} \frac{\partial p_f}{\partial X} \delta F_{11}^{-1} \right) A dX \\ = & \int_0^{X=H} \frac{\partial r}{\partial X} \left(\frac{\partial p_f}{\partial X} F_{11}^{-1} \frac{\hat{k} n^s}{J} \left[\frac{3}{n^f} - \frac{2n^f}{1 - (n^f)^2} \right] (\beta \Delta t^2) \frac{\partial(\delta a)}{\partial X} \right. \\ & \left. + \hat{k} F_{11}^{-1} (\beta \Delta t^2) \frac{\partial(\delta \ddot{p}_f)}{\partial X} - \hat{k} \frac{\partial p_f}{\partial X} F_{11}^{-2} (\beta \Delta t^2) \frac{\partial(\delta a)}{\partial X} \right) A dX \quad (\text{B-19}) \end{aligned}$$

Rearranging like terms, we arrive at

$$\begin{aligned} \delta\mathcal{H}_3^{\text{INT}} = & \int_0^{X=H} \frac{\partial r}{\partial X} \left(\left[n^s \left(\frac{3}{n^f} - \frac{2n^f}{1 - (n^f)^2} \right) - 1 \right] \hat{k} \frac{\partial p_f}{\partial X} F_{11}^{-2} \times \right. \\ & \left. (\beta \Delta t^2) \frac{\partial(\delta a)}{\partial X} + \hat{k} F_{11}^{-1} (\beta \Delta t^2) \frac{\partial(\delta \ddot{p}_f)}{\partial X} \right) A dX \quad (\text{B-20}) \end{aligned}$$

For poroelasticity, Eq. B-20 becomes

$$\begin{aligned} \delta \mathcal{H}_3^{\text{INT}} = & \int_0^{X=H} \frac{\partial r}{\partial X} \left(\left[n^s \left(\frac{3}{n^f} - \frac{2n^f}{1 - (n^f)^2} \right) - 1 \right] \hat{k} \frac{\partial p_f}{\partial X} F_{11}^{-2} \times \right. \\ & \left. (\gamma \Delta t) \frac{\partial(\delta v)}{\partial X} + \hat{k} F_{11}^{-1} (\gamma \Delta t) \frac{\partial(\delta \dot{p}_f)}{\partial X} \right) A dX \end{aligned} \quad (\text{B-21})$$

The next linearization proceeds as

$$\begin{aligned} \delta \mathcal{H}_4^{\text{INT}} = & \int_0^{X=H} \frac{\partial r}{\partial X} \delta(\hat{k} \rho^{\text{fR}}(a + g)) A dX \\ = & \int_0^{X=H} \frac{\partial r}{\partial X} \left(\rho^{\text{fR}}(a + g) \delta \hat{k} + \hat{k}(a + g) \delta \rho^{\text{fR}} + \hat{k} \rho^{\text{fR}} \delta a \right) A dX \\ = & \int_0^{X=H} \frac{\partial r}{\partial X} \rho^{\text{fR}} \hat{k} \left((a + g) \frac{n^s}{J} \left[\frac{3}{n^f} - \frac{2n^f}{1 - (n^f)^2} \right] (\beta \Delta t^2) \frac{\partial(\delta a)}{\partial X} \right. \\ & \left. + (a + g) \frac{1}{K_f^\eta} (\beta \Delta t^2) \delta \ddot{p}_f + \delta a \right) A dX \end{aligned} \quad (\text{B-22})$$

For poroelasticity, Eq. B-22 becomes

$$\begin{aligned} \delta \mathcal{H}_4^{\text{INT}} = & \int_0^{X=H} \frac{\partial r}{\partial X} \rho^{\text{fR}} \hat{k} \left(g \frac{n^s}{J} \left[\frac{3}{n^f} - \frac{2n^f}{1 - (n^f)^2} \right] (\gamma \Delta t) \frac{\partial(\delta v)}{\partial X} \right. \\ & \left. + g \frac{1}{K_f^\eta} (\gamma \Delta t) \delta \dot{p}_f \right) A dX \end{aligned} \quad (\text{B-23})$$

The next linearization is given by

$$\begin{aligned}
\delta\mathcal{H}^{\text{stab}} &= \int_0^{X=H} \frac{\partial r}{\partial X} \alpha^{\text{stab}} \delta \left(F_{11}^{-1} \frac{\partial \dot{p}_f}{\partial X} \right) A dX \\
&= \int_0^{X=H} \frac{\partial r}{\partial X} \alpha^{\text{stab}} \left(\frac{\partial \dot{p}_f}{\partial X} \delta(F_{11}^{-1}) + F_{11}^{-1} \frac{\partial \delta \dot{p}_f}{\partial X} \right) A dX \\
&= \int_0^{X=H} \frac{\partial r}{\partial X} \alpha^{\text{stab}} F_{11}^{-1} \left((\gamma \Delta t) \frac{\partial \delta \ddot{p}_f}{\partial X} - \frac{\partial \dot{p}_f}{\partial X} F_{11}^{-1} (\beta \Delta t^2) \frac{\partial \delta a}{\partial X} \right) A dX \quad (\text{B-24})
\end{aligned}$$

For poroelasticity, Eq. B-24 becomes

$$\delta\mathcal{H}^{\text{stab}} = \int_0^{X=H} \frac{\partial r}{\partial X} \alpha^{\text{stab}} F_{11}^{-1} \left(\frac{\partial \delta \dot{p}_f}{\partial X} - \frac{\partial \dot{p}_f}{\partial X} F_{11}^{-1} (\gamma \Delta t) \frac{\partial \delta v}{\partial X} \right) A dX \quad (\text{B-25})$$

B.3 Linearization of the $(\mathbf{u}-\mathbf{u}_f-p_f)$ Formulation Variational Equations

Refer to Section 5.1.5 of Irwin et al.¹ The first sought linearization initiates as follows:

$$\begin{aligned}
\delta\mathcal{G}_1^{\text{INT}} &= \int_0^{X=H} w \delta (\rho_0^s a + \rho_0^f a_f) A dX \\
&= \int_0^{X=H} w (\rho_0^s \delta a + a_f \delta \rho_0^f + \rho_0^f \delta a_f) A dX \\
&= \int_0^{X=H} w \left(\rho_0^s \delta a + a_f (\beta \Delta t^2) \left[\rho^{\text{fR}} \frac{\partial (\delta a)}{\partial X} + \frac{J n^f \rho^{\text{fR}}}{K_f^\eta} \delta \ddot{p}_f \right] \right. \\
&\quad \left. + \rho_0^f \delta a_f \right) A dX \quad (\text{B-26})
\end{aligned}$$

The linearization $\delta\mathcal{G}_2^{\text{INT}}$ remain unchanged from elastodynamics and elastostatics for the $(\mathbf{u}-\mathbf{u}_f-p_f)$ formulation.

The next linearization is obtained via

$$\begin{aligned}
\delta\mathcal{I}_1^{\text{INT}} &= \int_0^{X=H} q\delta(\rho_0^f a_f) A dX = \int_0^{X=H} q(a_f\delta\rho_0^f + \rho_0^f\delta a_f) A dX \\
&= \int_0^{X=H} q\left(a_f(\beta\Delta t^2)\left[\rho^{\text{fR}}\frac{\partial(\delta a)}{\partial X} + \frac{Jn^f\rho^{\text{fR}}}{K_f^\eta}\delta\ddot{p}_f\right] + \rho_0^f\delta a_f\right) A dX \quad (\text{B-27})
\end{aligned}$$

The linearization of the next term proceeds as follows:

$$\begin{aligned}
\delta\mathcal{I}_2^{\text{INT}} &= \int_0^{X=H} q\delta\left(n^f\frac{\partial p_f}{\partial X}\right) A dX = \int_0^{X=H} q\left(\frac{\partial p_f}{\partial X}\delta n^f + n^f\frac{\partial(\delta p_f)}{\partial X}\right) A dX \\
&= \int_0^{X=H} q\left(\frac{n^s}{J}\frac{\partial p_f}{\partial X}(\beta\Delta t^2)\frac{\partial(\delta a)}{\partial X} + n^f(\beta\Delta t^2)\frac{\partial(\delta\ddot{p}_f)}{\partial X}\right) A dX \quad (\text{B-28})
\end{aligned}$$

The next linearization is then

$$\begin{aligned}
\delta\mathcal{I}_3^{\text{INT}} &= \int_0^{X=H} q\delta\left(\frac{J(n^f)^2}{\hat{k}}\tilde{v}_f\right) A dX \\
&= \int_0^{X=H} q\left(\frac{(n^f)^2}{\hat{k}}\tilde{v}_f\delta(J) + \frac{2Jn^f}{\hat{k}}\tilde{v}_f\delta(n^f) \right. \\
&\quad \left. + J(n^f)^2\tilde{v}_f\delta(\hat{k}^{-1}) + \frac{J(n^f)^2}{\hat{k}}\delta(\tilde{v}_f)\right) A dX \\
&= \int_0^{X=H} q\left(\frac{(n^f)^2}{\hat{k}}\tilde{v}_f(\beta\Delta t^2)\frac{\partial(\delta a)}{\partial X} + \frac{2Jn^f\tilde{v}_f}{\hat{k}}\frac{n^s}{J}(\beta\Delta t^2)\frac{\partial(\delta a)}{\partial X} \right. \\
&\quad \left. + J(n^f)^2\tilde{v}_f(-\hat{k}^{-2})\frac{\hat{k}n^s}{J}\left[\frac{3}{n^f} - \frac{2n^f}{1-(n^f)^2}\right](\beta\Delta t^2)\frac{\partial(\delta a)}{\partial X} \right. \\
&\quad \left. + \frac{J(n^f)^2}{\hat{k}}(\gamma\Delta t)[\delta a_f - \delta a]\right) A dX \quad (\text{B-29})
\end{aligned}$$

Rearranging like terms,

$$\begin{aligned} \delta \mathcal{I}_3^{\text{INT}} = \int_0^{X=H} q \left(\left(1 + \frac{2n^s}{n^f} - n^s \left[\frac{3}{n^f} - \frac{2n^f}{1 - (n^f)^2} \right] \right) \frac{(n^f)^2 \tilde{v}_f}{\hat{k}} (\beta \Delta t^2) \frac{\partial \delta a}{\partial X} \right. \\ \left. + \frac{J(n^f)^2}{\hat{k}} (\gamma \Delta t) (\delta a_f - \delta a) \right) A dX \quad (\text{B-30}) \end{aligned}$$

The next requisite linearization proceeds as

$$\begin{aligned} \delta \mathcal{I}_4^{\text{INT}} &= \int_0^{X=H} q \delta (\rho_0^f g) A dX = \int_0^{X=H} q \delta (\rho_0^f) g A dX \\ &= \int_0^{X=H} q \left((\beta \Delta t^2) \left[\rho^{\text{fR}} \frac{\partial (\delta a)}{\partial X} + \frac{J n^f \rho^{\text{fR}}}{K_f^\eta} \delta \ddot{p}_f \right] \right) g A dX \quad (\text{B-31}) \end{aligned}$$

The subsequent linearization for the $(\mathbf{u}-\mathbf{u}_f-p_f)$ formulation is similar to that derived in Eq. B-17 for the $(\mathbf{u}-p_f)$ formulation, except that we now account for a variation on the pore fluid acceleration rather than the mixture acceleration when the Darcy velocity is expanded:

$$\begin{aligned} \delta \mathcal{H}_2^{\text{INT}} &= \int_0^{X=H} r \delta \left(\frac{1}{K_f^\eta} \frac{\partial p_f}{\partial X} n^f \tilde{v}_f \right) A dX \\ &= \int_0^{X=H} r \left(n^f \tilde{v}_f \frac{\partial (\delta p_f)}{\partial X} + \frac{\partial p_f}{\partial X} \delta (n^f \tilde{v}_f) \right) \frac{A}{K_f^\eta} dX \\ &= \int_0^{X=H} r \left(n^f \tilde{v}_f (\beta \Delta t^2) \frac{\partial (\delta \ddot{p}_f)}{\partial X} + \frac{\partial p_f}{\partial X} \left[\underbrace{- \left(\frac{\partial p_f}{\partial X} + \rho^{\text{fR}} (a_f + g) \right) \delta \hat{k}}_{\text{Term 1}} \right. \right. \\ &\quad \left. \left. - \hat{k} \left(\frac{\partial (\delta p_f)}{\partial X} + (a_f + g) \delta \rho^{\text{fR}} + \rho^{\text{fR}} \delta a_f \right) \right] \right) \frac{A}{K_f^\eta} dX \quad (\text{B-32}) \end{aligned}$$

Term 2

$$\begin{aligned}
\mathbf{Term\ 1} &= -\left(\frac{\partial p_f}{\partial X} + \rho^{\text{fR}}(a_f + g)\right) \frac{\hat{k} n^s}{J} \left(\frac{3}{n^f} - \frac{2n^f}{1 - (n^f)^2}\right) (\beta \Delta t^2) \frac{\partial(\delta a)}{\partial X} \\
&= \frac{n^s}{J} \left(\frac{3}{n^f} - \frac{2n^f}{1 - (n^f)^2}\right) (n^f \tilde{v}_f) (\beta \Delta t^2) \frac{\partial(\delta a)}{\partial X} \\
\mathbf{Term\ 2} &= -\hat{k} \left((\beta \Delta t^2) \frac{\partial(\delta \ddot{p}_f)}{\partial X} + (a_f + g) \frac{\rho^{\text{fR}}}{K_f^\eta} (\beta \Delta t^2) \delta \ddot{p}_f + \rho^{\text{fR}} \delta a_f \right)
\end{aligned}$$

Combining terms leaves us with

$$\begin{aligned}
\delta \mathcal{H}_2^{\text{INT}} &= \int_0^{X=H} r \left(n^f \tilde{v}_f (\beta \Delta t^2) \frac{\partial(\delta \ddot{p}_f)}{\partial X} + \frac{\partial p_f}{\partial X} \left[\frac{n^s}{J} \left(\frac{3}{n^f} - \frac{2n^f}{1 - (n^f)^2} \right) \times \right. \right. \\
&\quad \left. \left. (n^f \tilde{v}_f) (\beta \Delta t^2) \frac{\partial(\delta a)}{\partial X} - \hat{k} \left((\beta \Delta t^2) \frac{\partial(\delta \ddot{p}_f)}{\partial X} + (a + g) \frac{\rho^{\text{fR}}}{K_f^\eta} \times \right. \right. \right. \\
&\quad \left. \left. \left. (\beta \Delta t^2) \delta \ddot{p}_f + \rho^{\text{fR}} \delta a \right) \right] \right) \frac{A}{K_f^\eta} dX \quad (\text{B-33})
\end{aligned}$$

Rearranging like terms, we arrive at

$$\begin{aligned}
\delta \mathcal{H}_2^{\text{INT}} &= \int_0^{X=H} r \left(\left[(n^f \tilde{v}_f) - \hat{k} \frac{\partial p_f}{\partial X} \right] (\beta \Delta t^2) \frac{\partial(\delta \ddot{p}_f)}{\partial X} - \hat{k} \frac{\partial p_f}{\partial X} (a_f + g) \frac{\rho^{\text{fR}}}{K_f^\eta} \times \right. \\
&\quad \left. (\beta \Delta t^2) \delta \ddot{p}_f + \frac{\partial p_f}{\partial X} \frac{n^s}{J} \left[\frac{3}{n^f} - \frac{2n^f}{1 - (n^f)^2} \right] (n^f \tilde{v}_f) (\beta \Delta t^2) \frac{\partial(\delta a)}{\partial X} \right. \\
&\quad \left. - \frac{\partial p_f}{\partial X} \hat{k} \rho^{\text{fR}} \delta a_f \right) \frac{A}{K_f^\eta} dX \quad (\text{B-34})
\end{aligned}$$

The final necessary linearization for the three-field case proceeds as follows:

$$\begin{aligned}
\delta\mathcal{H}_4^{\text{INT}} &= \int_0^{X=H} \frac{\partial r}{\partial X} \delta(\hat{k}\rho^{\text{fR}}(a_{\text{f}} + g)) A dX \\
&= \int_0^{X=H} \frac{\partial r}{\partial X} (\rho^{\text{fR}}(a_{\text{f}} + g)\delta(\hat{k}) + \hat{k}(a_{\text{f}} + g)\delta(\rho^{\text{fR}}) \\
&\quad + \hat{k}\rho^{\text{fR}}\delta(a_{\text{f}})) A dX \\
&= \int_0^{X=H} \frac{\partial r}{\partial X} \rho^{\text{fR}} \hat{k} \left((a_{\text{f}} + g) \frac{n^{\text{s}}}{J} \left[\frac{3}{n^{\text{f}}} - \frac{2n^{\text{f}}}{1 - (n^{\text{f}})^2} \right] (\beta\Delta t^2) \frac{\partial(\delta a)}{\partial X} \right. \\
&\quad \left. + (a_{\text{f}} + g) \frac{1}{K_{\text{f}}^{\eta}} (\beta\Delta t^2) \delta\ddot{p}_{\text{f}} + \delta a_{\text{f}} \right) A dX \tag{B-35}
\end{aligned}$$

Appendix C. Derivation of the Finite-Element (FE) Equations

C.1 Implicit Integration

C.1.1 Elastodynamics and Elasticity

Numerical ingredients such as consistent tangents for the FE formulation in Section 5 of Irwin et al.¹ are derived in this Appendix.

For elastodynamics, we first apply the FE discretization as follows:

$$\delta \mathcal{G}_1^{\text{INT}} = \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{s,e}} \underbrace{\{\mathbf{c}^e\}^T}_{1 \times n_{\text{dof}}^{s,e}} \cdot \left(\int_{-1}^1 \underbrace{\{\mathbf{N}^{e,u}\}^T}_{n_{\text{dof}}^{s,e} \times 1} \rho_0^{h,e} \underbrace{\{\mathbf{N}^{e,u}\}}_{1 \times n_{\text{dof}}^{s,e}} \underbrace{\{\delta \ddot{\mathbf{d}}^e\}}_{n_{\text{dof}}^{s,e} \times 1} A j^e d\xi \right) \quad (\text{C-1})$$

Then, pulling out the variation $\delta \ddot{\mathbf{d}}^e$ leaves us with

$$\delta \mathcal{G}_1^{\text{INT}} = \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{s,e}} \underbrace{\{\mathbf{c}^e\}^T}_{1 \times n_{\text{dof}}^{s,e}} \cdot \underbrace{[\mathbf{k}_{u,u}^{\mathcal{G}_1^{\text{INT}},e}]}_{n_{\text{dof}}^{s,e} \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\{\delta \ddot{\mathbf{d}}^e\}}_{n_{\text{dof}}^{s,e} \times 1} \quad (\text{C-2})$$

where we recover

$$\underbrace{[\mathbf{k}_{u,u}^{\mathcal{G}_1^{\text{INT}},e}]}_{n_{\text{dof}}^{s,e} \times n_{\text{dof}}^{s,e}} = \int_{-1}^1 \rho_0^{h,e} \underbrace{\{\mathbf{N}^{e,u}\}^T}_{n_{\text{dof}}^{s,e} \times 1} \underbrace{\{\mathbf{N}^{e,u}\}}_{1 \times n_{\text{dof}}^{s,e}} A j^e d\xi \quad (\text{C-3})$$

Next we apply the FE discretization as

$$\delta \mathcal{G}_2^{\text{INT}} = \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{s,e}} \underbrace{\{\mathbf{c}^e\}^T}_{1 \times n_{\text{dof}}^{s,e}} \cdot \left(\int_{-1}^1 \underbrace{\{\mathbf{B}^{e,u}\}^T}_{n_{\text{dof}}^{s,e} \times 1} [\mu + (\lambda - \lambda \ln [J^{h,e}] + \mu) (F_{11}^{h,e})^{-2}] \times \right. \\ \left. [\beta \Delta t^2] \underbrace{\{\mathbf{B}^{e,u}\}}_{1 \times n_{\text{dof}}^{s,e}} \underbrace{\{\delta \ddot{\mathbf{d}}^e\}}_{n_{\text{dof}}^{s,e} \times 1} A j^e d\xi \right) \quad (\text{C-4})$$

¹Irwin Z, Regueiro R, Clayton J. A large deformation multiphase continuum mechanics model for shock loading of lung parenchyma. Part II: Numerical methods. DEVCOM Army Research Laboratory (US); 2023. Report No.: ARL-TR-9687.

Then, pulling out the variation $\delta \ddot{\mathbf{d}}^e$ leaves us with

$$\delta \mathcal{G}_1^{\text{INT}} = \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{s,e}} \underbrace{\{\mathbf{c}^e\}^T}_{n_{\text{dof}}^{s,e} \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\left[\mathbf{k}_{u,u}^{\mathcal{G}_2^{\text{INT}},e} \right]}_{n_{\text{dof}}^{s,e} \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\{\delta \ddot{\mathbf{d}}^e\}}_{n_{\text{dof}}^{s,e} \times 1} \quad (\text{C-5})$$

where we recover

$$\underbrace{\left[\mathbf{k}_{u,u}^{\mathcal{G}_2^{\text{INT}},e} \right]}_{n_{\text{dof}}^{s,e} \times n_{\text{dof}}^{s,e}} = \int_{-1}^1 \left(\mu + [\lambda - \lambda \ln(J^{h^e}) + \mu] (F_{11}^{h^e})^{-2} (\beta \Delta t^2) \right) \times \quad (\text{C-6})$$

$$\underbrace{\{\mathbf{B}^{e,u}\}^T}_{n_{\text{dof}}^{s,e} \times 1} \underbrace{\{\mathbf{B}^{e,u}\}}_{1 \times n_{\text{dof}}^{s,e}} A j^e d\xi \quad (\text{C-7})$$

Now, to derive the tangent for $\delta \mathcal{G}_2^{\text{INT}}$ for elasticity, apply the FE discretization as follows:

$$\delta \mathcal{G}_2^{\text{INT}} = \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{s,e}} \underbrace{\{\mathbf{c}^e\}^T}_{n_{\text{dof}}^{s,e} \times 1} \cdot \left(\int_{-1}^1 \underbrace{\{\mathbf{B}^{e,u}\}^T}_{n_{\text{dof}}^{s,e} \times 1} [\mu + (\lambda - \lambda \ln[J^{h^e}] + \mu) (F_{11}^{h^e})^{-2}] \times \right. \\ \left. [\gamma \Delta t] \underbrace{\{\mathbf{B}^{e,u}\}}_{1 \times n_{\text{dof}}^{s,e}} \underbrace{\{\delta \dot{\mathbf{d}}^e\}}_{n_{\text{dof}}^{s,e} \times 1} A j^e d\xi \right) \quad (\text{C-8})$$

Then, pulling out the variation $\delta \dot{\mathbf{d}}^e$ leaves us with

$$\delta \mathcal{G}_2^{\text{INT}} = \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{s,e}} \underbrace{\{\mathbf{c}^e\}^T}_{n_{\text{dof}}^{s,e} \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\left[\mathbf{k}_{u,u}^{\mathcal{G}_2^{\text{INT}},e} \right]}_{n_{\text{dof}}^{s,e} \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\{\delta \dot{\mathbf{d}}^e\}}_{n_{\text{dof}}^{s,e} \times 1} \quad (\text{C-9})$$

where

$$\underbrace{\left[\mathbf{k}_{u,u}^{\mathcal{G}_2^{\text{INT}},e} \right]}_{n_{\text{dof}}^{s,e} \times n_{\text{dof}}^{s,e}} = \int_{-1}^1 \left(\mu + [\lambda - \lambda \ln(J^{h^e}) + \mu] (F_{11}^{h^e})^{-2} (\gamma \Delta t) \right) \times \\ \underbrace{\{\mathbf{B}^{e,u}\}^T}_{n_{\text{dof}}^{s,e} \times 1} \underbrace{\{\mathbf{B}^{e,u}\}}_{1 \times n_{\text{dof}}^{s,e}} A j^e d\xi \quad (\text{C-10})$$

To derive the tangent for $\delta\mathcal{G}_2^{\text{INT}}$ when viscous damping is considered, we start from Eq. B-3 of Appendix B, and we then apply the FE discretization as follows:

$$\begin{aligned}
\delta\mathcal{G}_2^{\text{INT}} = & \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{s,e}} \underbrace{\{\mathbf{c}^e\}^T}_{n_{\text{dof}}^{s,e} \times 1} \cdot \left(\int_{-1}^1 [F_{11}^{h^e}]^{-2} \left[(\lambda + 2\mu) \times \right. \right. \\
& \left. \left(\nu_0 [\gamma \Delta t] - 2\nu_0 [F_{11}^{h^e}]^{-1} \frac{\partial v^{h^e}}{\partial X} [\beta \Delta t^2] \right) \right. \\
& \left. - \ln(F_{11}^{h^e}) \left(2\nu_0 [\gamma \Delta t] - \left[4\nu_0 (F_{11}^{h^e})^{-1} \frac{\partial v^{h^e}}{\partial X} - \lambda \right] [\beta \Delta t^2] \right) \right. \\
& \left. + \left([\lambda + \mu + \mu (F_{11}^{h^e})^2] - 2\nu_0 (F_{11}^{h^e})^{-1} \frac{\partial v^{h^e}}{\partial X} \right) (\beta \Delta t^2) \right] \\
& \left. \underbrace{\{\mathbf{B}^{e,u}\}^T}_{n_{\text{dof}}^{s,e} \times 1} \underbrace{\{\mathbf{B}^{e,u}\}}_{1 \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\{\delta \ddot{\mathbf{d}}^e\}}_{n_{\text{dof}}^{s,e} \times 1} A j^e d\xi \right) \quad (\text{C-11})
\end{aligned}$$

Then, pulling out the variation $\delta \ddot{\mathbf{d}}^e$ leaves us with

$$\delta\mathcal{G}_2^{\text{INT}} = \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{s,e}} \underbrace{\{\mathbf{c}^e\}^T}_{n_{\text{dof}}^{s,e} \times 1} \cdot \underbrace{[\mathbf{k}_{u,u}^{\mathcal{G}_2^{\text{INT}},e}]}_{n_{\text{dof}}^{s,e} \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\{\delta \ddot{\mathbf{d}}^e\}}_{n_{\text{dof}}^{s,e} \times 1} \quad (\text{C-12})$$

where

$$\begin{aligned}
\underbrace{[\mathbf{k}_{u,u}^{\mathcal{G}_2^{\text{INT}},e}]}_{n_{\text{dof}}^{s,e} \times n_{\text{dof}}^{s,e}} = & \int_{-1}^1 (F_{11}^{h^e})^{-2} \left([\lambda + 2\mu] \left[\nu_0 (\gamma \Delta t) - 2\nu_0 (F_{11}^{h^e})^{-1} \frac{\partial v^{h^e}}{\partial X} (\beta \Delta t^2) \right] \right. \\
& \left. - \ln(F_{11}^{h^e}) \left[2\nu_0 (\gamma \Delta t) - \left(4\nu_0 (F_{11}^{h^e})^{-1} \frac{\partial v^{h^e}}{\partial X} - \lambda \right) (\beta \Delta t^2) \right] \right. \\
& \left. + \left[(\lambda + \mu + \mu (F_{11}^{h^e})^2) - 2\nu_0 (F_{11}^{h^e})^{-1} \frac{\partial v^{h^e}}{\partial X} \right] (\beta \Delta t^2) \right) \times \\
& \underbrace{\{\mathbf{B}^{e,u}\}^T}_{n_{\text{dof}}^{s,e} \times 1} \underbrace{\{\mathbf{B}^{e,u}\}}_{1 \times n_{\text{dof}}^{s,e}} A j^e d\xi \quad (\text{C-13})
\end{aligned}$$

To derive the tangent for $\delta\mathcal{G}_2^{\text{INT}}$ when viscous damping and elasticity are considered,

start from Eq. B-5 of Appendix B and apply the FE discretization as follows:

$$\begin{aligned}
\delta \mathcal{G}_2^{\text{INT}} = & \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{s,e}} \underbrace{\left\{ \mathbf{c}^e \right\}^T}_{n_{\text{dof}}^{s,e} \times 1} \cdot \left(\int_{-1}^1 [F_{11}^{h^e}]^{-2} \left[(\lambda + 2\mu) \left(\nu_0 - 2\nu_0 [F_{11}^{h^e}]^{-1} \frac{\partial v^{h^e}}{\partial X} [\gamma \Delta t] \right) \right. \right. \\
& - \ln(F_{11}^{h^e}) \left(2\nu_0 - \left[4\nu_0 (F_{11}^{h^e})^{-1} \frac{\partial v^{h^e}}{\partial X} - \lambda \right] [\gamma \Delta t] \right) \\
& \left. \left. + \left([\lambda + \mu + \mu (F_{11}^{h^e})^2] - 2\nu_0 (F_{11}^{h^e})^{-1} \frac{\partial v^{h^e}}{\partial X} \right) (\gamma \Delta t) \right] \right. \\
& \left. \underbrace{\left\{ \mathbf{B}^{e,u} \right\}^T}_{n_{\text{dof}}^{s,e} \times 1} \underbrace{\left\{ \mathbf{B}^{e,u} \right\}}_{1 \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\left\{ \delta \dot{\mathbf{d}}^e \right\}}_{n_{\text{dof}}^{s,e} \times 1} A_j^e d\xi \right) \quad (\text{C-14})
\end{aligned}$$

Then, pulling out the variation $\delta \dot{\mathbf{d}}^e$ leaves us with

$$\delta \mathcal{G}_2^{\text{INT}} = \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{s,e}} \underbrace{\left\{ \mathbf{c}^e \right\}^T}_{n_{\text{dof}}^{s,e} \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\left[\mathbf{k}_{u,u}^{\mathcal{G}_2^{\text{INT}},e} \right]}_{n_{\text{dof}}^{s,e} \times 1} \cdot \underbrace{\left\{ \delta \dot{\mathbf{d}}^e \right\}}_{n_{\text{dof}}^{s,e} \times 1} \quad (\text{C-15})$$

where

$$\begin{aligned}
\underbrace{\left[\mathbf{k}_{u,u}^{\mathcal{G}_2^{\text{INT}},e} \right]}_{n_{\text{dof}}^{s,e} \times n_{\text{dof}}^{s,e}} = & \int_{-1}^1 (F_{11}^{h^e})^{-2} \left([\lambda + 2\mu] \left[\nu_0 - 2\nu_0 (F_{11}^{h^e})^{-1} \frac{\partial v^{h^e}}{\partial X} (\gamma \Delta t) \right] \right. \\
& - \ln(F_{11}^{h^e}) \left[2\nu_0 - \left(4\nu_0 (F_{11}^{h^e})^{-1} \frac{\partial v^{h^e}}{\partial X} - \lambda \right) (\gamma \Delta t) \right] \\
& \left. + \left[(\lambda + \mu + \mu (F_{11}^{h^e})^2) - 2\nu_0 (F_{11}^{h^e})^{-1} \frac{\partial v^{h^e}}{\partial X} \right] (\gamma \Delta t) \right) \times \\
& \underbrace{\left\{ \mathbf{B}^{e,u} \right\}^T}_{n_{\text{dof}}^{s,e} \times 1} \underbrace{\left\{ \mathbf{B}^{e,u} \right\}}_{1 \times n_{\text{dof}}^{s,e}} A_j^e d\xi \quad (\text{C-16})
\end{aligned}$$

To derive the tangent for $\delta \mathcal{G}_2^{\text{INT}}$ when shock viscosity is considered, start from Eq. B-

10 of Appendix B, and apply the FE discretization as follows:

$$\begin{aligned}
\delta\mathcal{G}_2^{\text{INT}} = & \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{s,e}} \underbrace{\{\mathbf{c}^e\}^T}_{n_{\text{dof}}^{s,e} \times 1} \cdot \left(\int_{-1}^1 \left[(\mu + [\lambda - \lambda \ln(J^{h^e}) + \mu](F_{11}^{h^e})^{-2})(\beta\Delta t^2) \right. \right. \\
& - \rho_0^{h^e} h_0 \left(C_0 h_0 (F_{11}^{h^e})^{-2} \frac{\partial v^{h^e}}{\partial X} \left[(1 - 2(F_{11}^{h^e})^{-1}) \frac{\partial v^{h^e}}{\partial X} (\beta\Delta t^2) + 2(\gamma\Delta t) \right] \right. \\
& \left. \left. - C_1 (F_{11}^{h^e})^{-1} c^{h^e} \left[\frac{3}{2} (F_{11}^{h^e})^{-1} \frac{\partial v^{h^e}}{\partial X} (\beta\Delta t^2) - (\gamma\Delta t) \right] \right] \right) \\
& \underbrace{\{\mathbf{B}^{e,u}\}^T}_{n_{\text{dof}}^{s,e} \times 1} \underbrace{\{\mathbf{B}^{e,u}\}}_{1 \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\{\delta\ddot{\mathbf{d}}^e\}}_{n_{\text{dof}}^{s,e} \times 1} A j^e d\xi \Big)
\end{aligned} \tag{C-17}$$

Then, pulling out the variation $\delta\ddot{\mathbf{d}}^e$ leaves us with

$$\delta\mathcal{G}_2^{\text{INT}} = \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{s,e}} \underbrace{\{\mathbf{c}^e\}^T}_{n_{\text{dof}}^{s,e} \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\left[\mathbf{k}_{u,u}^{\mathcal{G}_2^{\text{INT}},e} \right]}_{n_{\text{dof}}^{s,e} \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\{\delta\ddot{\mathbf{d}}^e\}}_{n_{\text{dof}}^{s,e} \times 1} \tag{C-18}$$

where

$$\begin{aligned}
\underbrace{\left[\mathbf{k}_{u,u}^{\mathcal{G}_2^{\text{INT}},e} \right]}_{n_{\text{dof}}^{s,e} \times n_{\text{dof}}^{s,e}} = & \int_{-1}^1 \left([\mu + (\lambda - \lambda \ln(J^{h^e}) + \mu)(F_{11}^{h^e})^{-2}](\beta\Delta t^2) \right. \\
& - \rho_0^{h^e} h_0 \left[C_0 h_0 (F_{11}^{h^e})^{-2} \frac{\partial v^{h^e}}{\partial X} \left[(1 - 2(F_{11}^{h^e})^{-1}) \frac{\partial v^{h^e}}{\partial X} (\beta\Delta t^2) + 2(\gamma\Delta t) \right] \right. \\
& \left. \left. - C_1 (F_{11}^{h^e})^{-1} c^{h^e} \left(\frac{3}{2} (F_{11}^{h^e})^{-1} \frac{\partial v^{h^e}}{\partial X} (\beta\Delta t^2) - (\gamma\Delta t) \right) \right] \right) \underbrace{\{\mathbf{B}^{e,u}\}^T}_{n_{\text{dof}}^{s,e} \times 1} \underbrace{\{\mathbf{B}^{e,u}\}}_{1 \times n_{\text{dof}}^{s,e}} A j^e d\xi
\end{aligned} \tag{C-19}$$

C.1.2 Poroelastodynamics (u - p_f) Formulation

As stated by Irwin et al.¹, the FE formulations for the tangents relating to the linearized terms $\delta\mathcal{G}_1^{\text{INT}}$ and $\delta\mathcal{G}_2^{\text{INT}}$ remain unchanged from elastodynamics and elasticity; refer to the equations in the preceding section. To derive the tangent for $\delta\mathcal{G}_3^{\text{INT}}$,

we start by applying the FE discretization as follows:

$$\delta\mathcal{G}_3^{\text{INT}} = \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{s,e}} \underbrace{\{\mathbf{c}^e\}^T}_{1 \times n_{\text{dof}}^{s,e}} \cdot \left(- \int_{-1}^1 \underbrace{\{\mathbf{B}^{e,u}\}^T}_{n_{\text{dof}}^{s,e} \times 1} [\beta \Delta t^2] \underbrace{\{\mathbf{N}^{e,p}\}}_{1 \times n_{\text{dof}}^{p,e}} \underbrace{\{\delta\ddot{\boldsymbol{\theta}}^e\}}_{n_{\text{dof}}^{p,e} \times 1} A_j^e d\xi \right) \quad (\text{C-20})$$

Then, pulling out the variation $\delta\ddot{\boldsymbol{\theta}}^e$ leaves us with

$$\delta\mathcal{G}_3^{\text{INT}} = \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{s,e}} \underbrace{\{\mathbf{c}^e\}^T}_{1 \times n_{\text{dof}}^{s,e}} \cdot \underbrace{[\mathbf{k}_{u,p_f}^{\mathcal{G}_3^{\text{INT}},e}]}_{n_{\text{dof}}^{s,e} \times n_{\text{dof}}^{p,e}} \cdot \underbrace{\{\delta\ddot{\boldsymbol{\theta}}^e\}}_{n_{\text{dof}}^{p,e} \times 1} \quad (\text{C-21})$$

where we recover

$$\underbrace{[\mathbf{k}_{u,p_f}^{\mathcal{G}_3^{\text{INT}},e}]}_{n_{\text{dof}}^{s,e} \times n_{\text{dof}}^{p,e}} = - \int_{-1}^1 (\beta \Delta t^2) \underbrace{\{\mathbf{B}^{e,u}\}^T}_{n_{\text{dof}}^{s,e} \times 1} \underbrace{\{\mathbf{N}^{e,p}\}}_{1 \times n_{\text{dof}}^{p,e}} A_j^e d\xi \quad (\text{C-22})$$

To derive the tangent for $\delta\mathcal{G}_3^{\text{INT}}$ when poroelasticity is considered, we start from Eq. B-12 of Appendix B and apply the FE discretization as follows:

$$\delta\mathcal{G}_3^{\text{INT}} = \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{s,e}} \underbrace{\{\mathbf{c}^e\}^T}_{1 \times n_{\text{dof}}^{s,e}} \cdot \left(- \int_{-1}^1 \underbrace{\{\mathbf{B}^{e,u}\}^T}_{n_{\text{dof}}^{s,e} \times 1} [\gamma \Delta t] \underbrace{\{\mathbf{N}^{e,p}\}}_{1 \times n_{\text{dof}}^{p,e}} \underbrace{\{\delta\dot{\boldsymbol{\theta}}^e\}}_{n_{\text{dof}}^{p,e} \times 1} A_j^e d\xi \right) \quad (\text{C-23})$$

Then, pulling out the variation $\delta\dot{\boldsymbol{\theta}}^e$ leaves us with

$$\delta\mathcal{G}_3^{\text{INT}} = \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{s,e}} \underbrace{\{\mathbf{c}^e\}^T}_{1 \times n_{\text{dof}}^{s,e}} \cdot \underbrace{[\mathbf{k}_{u,p_f}^{\mathcal{G}_3^{\text{INT}},e}]}_{n_{\text{dof}}^{s,e} \times n_{\text{dof}}^{p,e}} \cdot \underbrace{\{\delta\dot{\boldsymbol{\theta}}^e\}}_{n_{\text{dof}}^{p,e} \times 1} \quad (\text{C-24})$$

where

$$\underbrace{[\mathbf{k}_{u,p_f}^{\mathcal{G}_3^{\text{INT}},e}]}_{n_{\text{dof}}^{s,e} \times n_{\text{dof}}^{p,e}} = - \int_{-1}^1 (\gamma \Delta t) \underbrace{\{\mathbf{B}^{e,u}\}^T}_{n_{\text{dof}}^{s,e} \times 1} \underbrace{\{\mathbf{N}^{e,p}\}}_{1 \times n_{\text{dof}}^{p,e}} A_j^e d\xi \quad (\text{C-25})$$

To derive the tangents for $\delta\mathcal{H}_1^{\text{INT}}$, apply the FE discretization as follows:

$$\delta\mathcal{H}_1^{\text{INT}} = \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{p,e}} \underbrace{\{\alpha^e\}^T}_{1 \times n_{\text{dof}}^{p,e}} \cdot \left(\int_{-1}^1 \underbrace{\{N^{e,p}\}^T}_{n_{\text{dof}}^{p,e} \times 1} \left[\left(\frac{\dot{p}_f^{h^e}}{K_f^\eta} [\beta \Delta t^2] + [\gamma \Delta t] \right) \times \right. \right. \\ \left. \left. \underbrace{\{B^{e,u}\}}_{1 \times n_{\text{dof}}^{s,e}} \underbrace{\{\delta \ddot{\mathbf{d}}^e\}}_{n_{\text{dof}}^{s,e} \times 1} + \frac{J^{h^e} n^{f,h^e}}{K_f^\eta} [\gamma \Delta t] \underbrace{\{N^{e,p}\}}_{1 \times n_{\text{dof}}^{p,e}} \underbrace{\{\delta \ddot{\theta}^e\}}_{n_{\text{dof}}^{p,e} \times 1} \right] A_j^e d\xi \right) \quad (\text{C-26})$$

Then, pulling out the variations $\delta \ddot{\mathbf{d}}^e$ and $\delta \ddot{\theta}^e$ leaves us with

$$\delta\mathcal{H}_1^{\text{INT}} = \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{p,e}} \underbrace{\{\alpha^e\}^T}_{1 \times n_{\text{dof}}^{p,e}} \cdot \left(\underbrace{\left[\mathbf{k}_{p_f,u}^{\mathcal{H}_1^{\text{INT}},e} \right]}_{n_{\text{dof}}^{p,e} \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\{\delta \ddot{\mathbf{d}}^e\}}_{n_{\text{dof}}^{s,e} \times 1} + \underbrace{\left[\mathbf{k}_{p_f,p_f}^{\mathcal{H}_1^{\text{INT}},e} \right]}_{n_{\text{dof}}^{p,e} \times n_{\text{dof}}^{p,e}} \cdot \underbrace{\{\delta \ddot{\theta}^e\}}_{n_{\text{dof}}^{p,e} \times 1} \right) \quad (\text{C-27})$$

where we recover

$$\underbrace{\left[\mathbf{k}_{p_f,u}^{\mathcal{H}_1^{\text{INT}},e} \right]}_{n_{\text{dof}}^{p,e} \times n_{\text{dof}}^{s,e}} = \int_{-1}^1 \left(\frac{\dot{p}_f^{h^e}}{K_f^\eta} (\beta \Delta t^2) + (\gamma \Delta t) \right) \underbrace{\{N^{e,p}\}^T}_{n_{\text{dof}}^{p,e} \times 1} \underbrace{\{B^{e,u}\}}_{1 \times n_{\text{dof}}^{s,e}} A_j^e d\xi \quad (\text{C-28})$$

$$\underbrace{\left[\mathbf{k}_{p_f,p_f}^{\mathcal{H}_1^{\text{INT}},e} \right]}_{n_{\text{dof}}^{p,e} \times n_{\text{dof}}^{p,e}} = \int_{-1}^1 \frac{J^{h^e} n^{f,h^e}}{K_f^\eta} (\gamma \Delta t) \underbrace{\{N^{e,p}\}^T}_{n_{\text{dof}}^{p,e} \times 1} \underbrace{\{N^{e,p}\}}_{1 \times n_{\text{dof}}^{p,e}} A_j^e d\xi \quad (\text{C-29})$$

To derive the tangents for $\delta\mathcal{H}_1^{\text{INT}}$ when poroelasticity is considered, start from Eq. B-14 of Appendix B and apply the FE discretization as follows:

$$\delta\mathcal{H}_1^{\text{INT}} = \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{p,e}} \underbrace{\{\alpha^e\}^T}_{1 \times n_{\text{dof}}^{p,e}} \cdot \left(\int_{-1}^1 \underbrace{\{N^{e,p}\}^T}_{n_{\text{dof}}^{p,e} \times 1} \left[\left(\frac{\dot{p}_f^{h^e}}{K_f^\eta} [\gamma \Delta t] + 1 \right) \times \right. \right. \\ \left. \left. \underbrace{\{B^{e,u}\}}_{1 \times n_{\text{dof}}^{s,e}} \underbrace{\{\delta \dot{\mathbf{d}}^e\}}_{n_{\text{dof}}^{s,e} \times 1} + \frac{J^{h^e} n^{f,h^e}}{K_f^\eta} \underbrace{\{N^{e,p}\}}_{1 \times n_{\text{dof}}^{p,e}} \underbrace{\{\delta \dot{\theta}^e\}}_{n_{\text{dof}}^{p,e} \times 1} \right] A_j^e d\xi \right) \quad (\text{C-30})$$

Then, pulling out the variations $\delta \dot{\mathbf{d}}^e$ & $\delta \dot{\boldsymbol{\theta}}^e$ leaves us with

$$\delta \mathcal{H}_1^{\text{INT}} = \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{p,e}} \underbrace{\{\boldsymbol{\alpha}^e\}^T}_{1 \times n_{\text{dof}}^{p,e}} \cdot \left(\underbrace{\left[\mathbf{k}_{p_f, u}^{\mathcal{H}_1^{\text{INT}}, e} \right]}_{n_{\text{dof}}^{p,e} \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\{\delta \dot{\mathbf{d}}^e\}}_{n_{\text{dof}}^{s,e} \times 1} + \underbrace{\left[\mathbf{k}_{p_f, p_f}^{\mathcal{H}_1^{\text{INT}}, e} \right]}_{n_{\text{dof}}^{p,e} \times n_{\text{dof}}^{p,e}} \cdot \underbrace{\{\delta \dot{\boldsymbol{\theta}}^e\}}_{n_{\text{dof}}^{p,e} \times 1} \right) \quad (\text{C-31})$$

where

$$\underbrace{\left[\mathbf{k}_{p_f, u}^{\mathcal{H}_1^{\text{INT}}, e} \right]}_{n_{\text{dof}}^{p,e} \times n_{\text{dof}}^{s,e}} = \int_{-1}^1 \left(\frac{\dot{p}_f^{h^e}}{K_f^\eta} (\gamma \Delta t) + 1 \right) \underbrace{\{\mathbf{N}^{e,p}\}^T}_{n_{\text{dof}}^{p,e} \times 1} \underbrace{\{\mathbf{B}^{e,u}\}}_{1 \times n_{\text{dof}}^{s,e}} A j^e d\xi \quad (\text{C-32})$$

$$\underbrace{\left[\mathbf{k}_{p_f, p_f}^{\mathcal{H}_1^{\text{INT}}, e} \right]}_{n_{\text{dof}}^{p,e} \times n_{\text{dof}}^{p,e}} = \int_{-1}^1 \frac{J^{h^e} n^{f,h^e}}{K_f^\eta} \underbrace{\{\mathbf{N}^{e,p}\}^T}_{n_{\text{dof}}^{p,e} \times 1} \underbrace{\{\mathbf{N}^{e,p}\}}_{1 \times n_{\text{dof}}^{p,e}} A j^e d\xi \quad (\text{C-33})$$

To derive the tangents for $\delta \mathcal{H}_2^{\text{INT}}$, we apply the FE discretization as follows:

$$\begin{aligned} \delta \mathcal{H}_2^{\text{INT}} = & \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{p,e}} \underbrace{\{\boldsymbol{\alpha}^e\}^T}_{1 \times n_{\text{dof}}^{p,e}} \cdot \left(\int_{-1}^1 \underbrace{\{\mathbf{N}^{e,p}\}^T}_{n_{\text{dof}}^{p,e} \times 1} \left[\left([n^f \tilde{v}_f]^{h^e} - \hat{k}^{h^e} \frac{\partial p_f^{h^e}}{\partial X} (F_{11}^{h^e})^{-1} \right) \right. \right. \\ & \times (\beta \Delta t^2) \underbrace{\{\mathbf{B}^{e,p}\}}_{1 \times n_{\text{dof}}^{p,e}} \underbrace{\{\delta \ddot{\boldsymbol{\theta}}^e\}}_{1 \times n_{\text{dof}}^{p,e}} - \hat{k}^{h^e} \frac{\partial p_f^{h^e}}{\partial X} (a^{h^e} + g) \frac{\rho^{\text{fR}, h^e}}{K_f^\eta} (\beta \Delta t^2) \underbrace{\{\mathbf{N}^{e,p}\}}_{1 \times n_{\text{dof}}^{p,e}} \underbrace{\{\delta \ddot{\boldsymbol{\theta}}^e\}}_{n_{\text{dof}}^{p,e} \times 1} \\ & + \frac{\partial p_f^{h^e}}{\partial X} \left(\frac{n^{s,h^e}}{J^{h^e}} \left[\frac{3}{n^{f,h^e}} - \frac{2n^{f,h^e}}{1 - (n^{f,h^e})^2} \right] (n^f \tilde{v}_f)^{h^e} + \hat{k}^{h^e} \frac{\partial p_f^{h^e}}{\partial X} (F_{11}^{h^e})^{-2} \right) \\ & \left. \left. (\beta \Delta t^2) \underbrace{\{\mathbf{B}^{e,u}\}}_{1 \times n_{\text{dof}}^{s,e}} \underbrace{\{\delta \ddot{\mathbf{d}}^e\}}_{n_{\text{dof}}^{s,e} \times 1} - \frac{\partial p_f^{h^e}}{\partial X} \hat{k}^{h^e} \rho^{\text{fR}, h^e} \underbrace{\{\mathbf{N}^{e,u}\}}_{1 \times n_{\text{dof}}^{s,e}} \underbrace{\{\delta \ddot{\mathbf{d}}^e\}}_{n_{\text{dof}}^{s,e} \times 1} \right] \frac{A}{K_f^\eta} j^e d\xi \right) \quad (\text{C-34}) \end{aligned}$$

Then, pulling out the variations $\delta \ddot{\mathbf{d}}^e$ and $\delta \ddot{\boldsymbol{\theta}}^e$ leaves us with

$$\delta \mathcal{H}_2^{\text{INT}} = \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{p,e}} \underbrace{\{\boldsymbol{\alpha}^e\}^T}_{1 \times n_{\text{dof}}^{p,e}} \cdot \left(\underbrace{\left[\mathbf{k}_{p_f, u}^{\mathcal{H}_2^{\text{INT}}, e} \right]}_{n_{\text{dof}}^{p,e} \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\{\delta \ddot{\mathbf{d}}^e\}}_{n_{\text{dof}}^{s,e} \times 1} + \underbrace{\left[\mathbf{k}_{p_f, p_f}^{\mathcal{H}_2^{\text{INT}}, e} \right]}_{n_{\text{dof}}^{p,e} \times n_{\text{dof}}^{p,e}} \cdot \underbrace{\{\delta \ddot{\boldsymbol{\theta}}^e\}}_{n_{\text{dof}}^{p,e} \times 1} \right) \quad (\text{C-35})$$

whereby we recover the following tangents:

$$\begin{aligned}
\underbrace{\left[\mathbf{k}_{p_f, u}^{\mathcal{H}_2^{\text{INT}}, e} \right]}_{n_{\text{dof}}^{p, e} \times n_{\text{dof}}^{s, e}} &= \int_{-1}^1 \underbrace{\left\{ \mathbf{N}^{e, p} \right\}^T}_{n_{\text{dof}}^{p, e} \times 1} \left(\left[\frac{n^{s, h^e}}{J^{h^e}} \left(\frac{3}{n^{f, h^e}} - \frac{2n^{f, h^e}}{1 - (n^{f, h^e})^2} \right) (n^f \tilde{v}_f)^{h^e} \right. \right. \\
&\quad \left. \left. + \hat{k}^{h^e} \frac{\partial p_f^{h^e}}{\partial X} (F_{11}^{h^e})^{-2} \right] (\beta \Delta t^2) \underbrace{\left\{ \mathbf{B}^{e, u} \right\}}_{1 \times n_{\text{dof}}^{s, e}} \right. \\
&\quad \left. - \hat{k}^{h^e} \rho^{\text{fR}, h^e} \underbrace{\left\{ \mathbf{N}^{e, u} \right\}}_{1 \times n_{\text{dof}}^{s, e}} \right) \frac{\partial p_f^{h^e}}{\partial X} \frac{1}{K_f^\eta} A j^e d\xi \quad (\text{C-36})
\end{aligned}$$

$$\begin{aligned}
\underbrace{\left[\mathbf{k}_{p_f, p_f}^{\mathcal{H}_2^{\text{INT}}, e} \right]}_{n_{\text{dof}}^{p, e} \times n_{\text{dof}}^{p, e}} &= \int_{-1}^1 \underbrace{\left\{ \mathbf{N}^{e, p} \right\}^T}_{n_{\text{dof}}^{p, e} \times 1} \left(\left[(n^f \tilde{v}_f)^{h^e} - \hat{k}^{h^e} \frac{\partial p_f^{h^e}}{\partial X} (F_{11}^{h^e})^{-1} \right] \underbrace{\left\{ \mathbf{B}^{e, p} \right\}}_{1 \times n_{\text{dof}}^{p, e}} \right. \\
&\quad \left. - \hat{k}^{h^e} \frac{\partial p_f^{h^e}}{\partial X} (a^{h^e} + g) \frac{\rho^{\text{fR}, h^e}}{K_f^\eta} \underbrace{\left\{ \mathbf{N}^{e, p} \right\}}_{1 \times n_{\text{dof}}^{p, e}} \right) \frac{1}{K_f^\eta} (\beta \Delta t^2) A j^e d\xi \quad (\text{C-37})
\end{aligned}$$

To derive the tangents for $\delta \mathcal{H}_2^{\text{INT}}$ when poroelasticity is considered, start from Eq. B-18 of Appendix B and apply the FE discretization as follows:

$$\begin{aligned}
\delta \mathcal{H}_2^{\text{INT}} &= \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{p, e}} \underbrace{\left\{ \boldsymbol{\alpha}^e \right\}^T}_{n_{\text{dof}}^{p, e} \times 1} \cdot \left(\int_{-1}^1 \underbrace{\left\{ \mathbf{N}^{e, p} \right\}^T}_{n_{\text{dof}}^{p, e} \times 1} \left[\left([n^f \tilde{v}_f]^{h^e} - \hat{k}^{h^e} \frac{\partial p_f^{h^e}}{\partial X} (F_{11}^{h^e})^{-1} \right) \right. \right. \\
&\quad \times (\gamma \Delta t) \underbrace{\left\{ \mathbf{B}^{e, p} \right\}}_{1 \times n_{\text{dof}}^{p, e}} \underbrace{\left\{ \delta \dot{\boldsymbol{\theta}}^e \right\}}_{n_{\text{dof}}^{p, e} \times 1} - \hat{k}^{h^e} \frac{\partial p_f^{h^e}}{\partial X} g \frac{\rho^{\text{fR}, h^e}}{K_f^\eta} (\gamma \Delta t) \underbrace{\left\{ \mathbf{N}^{e, p} \right\}}_{1 \times n_{\text{dof}}^{p, e}} \underbrace{\left\{ \delta \dot{\boldsymbol{\theta}}^e \right\}}_{n_{\text{dof}}^{p, e} \times 1} \\
&\quad + \frac{\partial p_f^{h^e}}{\partial X} \left(\frac{n^{s, h^e}}{J^{h^e}} \left[\frac{3}{n^{f, h^e}} - \frac{2n^{f, h^e}}{1 - (n^{f, h^e})^2} \right] (n^f \tilde{v}_f)^{h^e} + \hat{k}^{h^e} \frac{\partial p_f^{h^e}}{\partial X} (F_{11}^{h^e})^{-2} \right) \\
&\quad \left. \times (\gamma \Delta t) \underbrace{\left\{ \mathbf{B}^{e, u} \right\}}_{1 \times n_{\text{dof}}^{s, e}} \underbrace{\left\{ \delta \dot{\mathbf{d}}^e \right\}}_{n_{\text{dof}}^{s, e} \times 1} \right] \frac{A}{K_f^\eta} j^e d\xi \quad (\text{C-38})
\end{aligned}$$

Then, pulling out the variations $\delta \dot{\mathbf{d}}^e$ & $\delta \dot{\boldsymbol{\theta}}^e$ leaves us with

$$\delta \mathcal{H}_2^{\text{INT}} = \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{p,e}} \underbrace{\{\boldsymbol{\alpha}^e\}^T}_{n_{\text{dof}}^{p,e} \times n_{\text{dof}}^{s,e}} \cdot \left(\underbrace{\left[\mathbf{k}_{p_f, u}^{\mathcal{H}_2^{\text{INT}}, e} \right]}_{n_{\text{dof}}^{p,e} \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\{\delta \dot{\mathbf{d}}^e\}}_{n_{\text{dof}}^{s,e} \times 1} + \underbrace{\left[\mathbf{k}_{p_f, p_f}^{\mathcal{H}_2^{\text{INT}}, e} \right]}_{n_{\text{dof}}^{p,e} \times n_{\text{dof}}^{p,e}} \cdot \underbrace{\{\delta \dot{\boldsymbol{\theta}}^e\}}_{n_{\text{dof}}^{p,e} \times 1} \right) \quad (\text{C-39})$$

where

$$\begin{aligned} \underbrace{\left[\mathbf{k}_{p_f, u}^{\mathcal{H}_2^{\text{INT}}, e} \right]}_{n_{\text{dof}}^{p,e} \times n_{\text{dof}}^{s,e}} &= \int_{-1}^1 \underbrace{\{\mathbf{N}^{e,p}\}^T}_{n_{\text{dof}}^{p,e} \times 1} \left(\left[\frac{n^{s,h^e}}{J^{h^e}} \left(\frac{3}{n^{f,h^e}} - \frac{2n^{f,h^e}}{1 - (n^{f,h^e})^2} \right) (n^f \tilde{v}_f)^{h^e} \right. \right. \\ &\quad \left. \left. + \hat{k}^{h^e} \frac{\partial p_f^{h^e}}{\partial X} (F_{11}^{h^e})^{-2} \right] (\gamma \Delta t) \underbrace{\{\mathbf{B}^{e,u}\}}_{1 \times n_{\text{dof}}^{s,e}} \right) \frac{\partial p_f^{h^e}}{\partial X} \frac{1}{K_f^\eta} A j^e d\xi \quad (\text{C-40}) \end{aligned}$$

$$\begin{aligned} \underbrace{\left[\mathbf{k}_{p_f, p_f}^{\mathcal{H}_2^{\text{INT}}, e} \right]}_{n_{\text{dof}}^{p,e} \times n_{\text{dof}}^{p,e}} &= \int_{-1}^1 \underbrace{\{\mathbf{N}^{e,p}\}^T}_{n_{\text{dof}}^{p,e} \times 1} \left(\left[(n^f \tilde{v}_f)^{h^e} - \hat{k}^{h^e} \frac{\partial p_f^{h^e}}{\partial X} (F_{11}^{h^e})^{-1} \right] \underbrace{\{\mathbf{B}^{e,p}\}}_{1 \times n_{\text{dof}}^{p,e}} \right. \\ &\quad \left. - \hat{k}^{h^e} \frac{\partial p_f^{h^e}}{\partial X} g \frac{\rho^{\text{fR}, h^e}}{K_f^\eta} \underbrace{\{\mathbf{N}^{e,p}\}}_{1 \times n_{\text{dof}}^{p,e}} \right) \frac{1}{K_f^\eta} (\gamma \Delta t) A j^e d\xi \quad (\text{C-41}) \end{aligned}$$

To derive the tangents for $\delta \mathcal{H}_3^{\text{INT}}$, we apply the FE discretization as follows:

$$\begin{aligned} \delta \mathcal{H}_3^{\text{INT}} &= \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{p,e}} \underbrace{\{\boldsymbol{\alpha}^e\}^T}_{n_{\text{dof}}^{p,e} \times n_{\text{dof}}^{s,e}} \cdot \left(\int_{-1}^1 \underbrace{\{\mathbf{B}^{e,p}\}^T}_{n_{\text{dof}}^{p,e} \times 1} \left[\left(n^{s,h^e} \left[\frac{3}{n^{f,h^e}} - \frac{2n^{f,h^e}}{1 - (n^{f,h^e})^2} \right] - 1 \right) \times \right. \right. \\ &\quad \left. \left. \hat{k}^{h^e} \frac{\partial p_f^{h^e}}{\partial X} (F_{11}^{h^e})^{-2} (\beta \Delta t^2) \underbrace{\{\mathbf{B}^{e,u}\}}_{1 \times n_{\text{dof}}^{s,e}} \underbrace{\{\delta \ddot{\mathbf{d}}^e\}}_{n_{\text{dof}}^{s,e} \times 1} \right. \right. \\ &\quad \left. \left. + \hat{k}^{h^e} (F_{11}^{h^e})^{-1} (\beta \Delta t^2) \underbrace{\{\mathbf{B}^{e,p}\}}_{1 \times n_{\text{dof}}^{p,e}} \underbrace{\{\delta \ddot{\boldsymbol{\theta}}^e\}}_{n_{\text{dof}}^{p,e} \times 1} \right] A j^e d\xi \right) \quad (\text{C-42}) \end{aligned}$$

Then, pulling out the variations $\delta \ddot{\mathbf{d}}^e$ and $\delta \ddot{\boldsymbol{\theta}}^e$ leaves us with

$$\delta \mathcal{H}_3^{\text{INT}} = \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{p,e}} \underbrace{\left\{ \boldsymbol{\alpha}^e \right\}^T}_{1 \times n_{\text{dof}}^{p,e}} \cdot \left(\underbrace{\left[\mathbf{k}_{p_f, u}^{\mathcal{H}_3^{\text{INT}}, e} \right]}_{n_{\text{dof}}^{p,e} \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\left\{ \delta \ddot{\mathbf{d}}^e \right\}}_{n_{\text{dof}}^{s,e} \times 1} + \underbrace{\left[\mathbf{k}_{p_f, p_f}^{\mathcal{H}_3^{\text{INT}}, e} \right]}_{n_{\text{dof}}^{p,e} \times n_{\text{dof}}^{p,e}} \cdot \underbrace{\left\{ \delta \ddot{\boldsymbol{\theta}}^e \right\}}_{n_{\text{dof}}^{p,e} \times 1} \right) \quad (\text{C-43})$$

whereby we recover

$$\underbrace{\left[\mathbf{k}_{p_f, u}^{\mathcal{H}_3^{\text{INT}}, e} \right]}_{n_{\text{dof}}^{p,e} \times n_{\text{dof}}^{s,e}} = \int_{-1}^1 \hat{k}^{h^e} \frac{\partial p_f^{h^e}}{\partial X} (F_{11}^{h^e})^{-2} \left(n^{s, h^e} \left[\frac{3}{n^{f, h^e}} - \frac{2n^{f, h^e}}{1 - (n^{f, h^e})^2} \right] - 1 \right) \times \\ (\beta \Delta t^2) \underbrace{\left\{ \mathbf{B}^{e, p} \right\}^T}_{n_{\text{dof}}^{p,e} \times 1} \underbrace{\left\{ \mathbf{B}^{e, u} \right\}}_{1 \times n_{\text{dof}}^{s,e}} A j^e d\xi \quad (\text{C-44})$$

$$\underbrace{\left[\mathbf{k}_{p_f, p_f}^{\mathcal{H}_3^{\text{INT}}, e} \right]}_{n_{\text{dof}}^{p,e} \times n_{\text{dof}}^{p,e}} = \int_{-1}^1 \hat{k}^{h^e} (F_{11}^{h^e})^{-1} (\beta \Delta t^2) \underbrace{\left\{ \mathbf{B}^{e, p} \right\}^T}_{n_{\text{dof}}^{p,e} \times 1} \underbrace{\left\{ \mathbf{B}^{e, p} \right\}}_{1 \times n_{\text{dof}}^{p,e}} A j^e d\xi \quad (\text{C-45})$$

To derive the tangents for $\delta \mathcal{H}_3^{\text{INT}}$ when poroelasticity is considered, start from Eq. B-21 of Appendix B and apply the FE discretization as follows:

$$\delta \mathcal{H}_3^{\text{INT}} = \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{p,e}} \underbrace{\left\{ \boldsymbol{\alpha}^e \right\}^T}_{1 \times n_{\text{dof}}^{p,e}} \cdot \left(\int_{-1}^1 \underbrace{\left\{ \mathbf{B}^{e, p} \right\}^T}_{n_{\text{dof}}^{p,e} \times 1} \left[\left(n^{s, h^e} \left[\frac{3}{n^{f, h^e}} - \frac{2n^{f, h^e}}{1 - (n^{f, h^e})^2} \right] - 1 \right) \times \right. \right. \\ \left. \left. \hat{k}^{h^e} \frac{\partial p_f^{h^e}}{\partial X} (F_{11}^{h^e})^{-2} (\gamma \Delta t) \underbrace{\left\{ \mathbf{B}^{e, u} \right\}}_{1 \times n_{\text{dof}}^{s,e}} \underbrace{\left\{ \delta \dot{\mathbf{d}}^e \right\}}_{n_{\text{dof}}^{s,e} \times 1} \right. \right. \\ \left. \left. + \hat{k}^{h^e} (F_{11}^{h^e})^{-1} (\gamma \Delta t) \underbrace{\left\{ \mathbf{B}^{e, p} \right\}}_{1 \times n_{\text{dof}}^{p,e}} \underbrace{\left\{ \delta \dot{\boldsymbol{\theta}}^e \right\}}_{n_{\text{dof}}^{p,e} \times 1} \right] A j^e d\xi \right) \quad (\text{C-46})$$

Then, pulling out the variations $\delta \dot{\mathbf{d}}^e$ and $\delta \dot{\boldsymbol{\theta}}^e$ leaves us with

$$\delta \mathcal{H}_3^{\text{INT}} = \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{p,e}} \underbrace{\left\{ \boldsymbol{\alpha}^e \right\}^T}_{1 \times n_{\text{dof}}^{p,e}} \cdot \left(\underbrace{\left[\mathbf{k}_{p_f, u}^{\mathcal{H}_3^{\text{INT}}, e} \right]}_{n_{\text{dof}}^{p,e} \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\left\{ \delta \dot{\mathbf{d}}^e \right\}}_{n_{\text{dof}}^{s,e} \times 1} + \underbrace{\left[\mathbf{k}_{p_f, p_f}^{\mathcal{H}_3^{\text{INT}}, e} \right]}_{n_{\text{dof}}^{p,e} \times n_{\text{dof}}^{p,e}} \cdot \underbrace{\left\{ \delta \dot{\boldsymbol{\theta}}^e \right\}}_{n_{\text{dof}}^{p,e} \times 1} \right) \quad (\text{C-47})$$

where

$$\underbrace{\left[\mathbf{k}_{p_f, u}^{\mathcal{H}_3^{\text{INT}, e}} \right]}_{n_{\text{dof}}^{p, e} \times n_{\text{dof}}^{s, e}} = \int_{-1}^1 \hat{k}^{h^e} \frac{\partial p_f^{h^e}}{\partial X} (F_{11}^{h^e})^{-2} \left(n^{s, h^e} \left[\frac{3}{n^{f, h^e}} - \frac{2n^{f, h^e}}{1 - (n^{f, h^e})^2} \right] - 1 \right) \times$$

$$(\gamma \Delta t) \underbrace{\left\{ \mathbf{B}^{e, p} \right\}^T}_{n_{\text{dof}}^{p, e} \times 1} \underbrace{\left\{ \mathbf{B}^{e, u} \right\}}_{1 \times n_{\text{dof}}^{s, e}} A j^e d\xi \quad (\text{C-48})$$

$$\underbrace{\left[\mathbf{k}_{p_f, p_f}^{\mathcal{H}_3^{\text{INT}, e}} \right]}_{n_{\text{dof}}^{p, e} \times n_{\text{dof}}^{p, e}} = \int_{-1}^1 \hat{k}^{h^e} (F_{11}^{h^e})^{-1} (\gamma \Delta t) \underbrace{\left\{ \mathbf{B}^{e, p} \right\}^T}_{n_{\text{dof}}^{p, e} \times 1} \underbrace{\left\{ \mathbf{B}^{e, p} \right\}}_{1 \times n_{\text{dof}}^{p, e}} A j^e d\xi \quad (\text{C-49})$$

To derive the tangents for $\delta \mathcal{H}_4^{\text{INT}}$, we start by applying the FE discretization as follows:

$$\delta \mathcal{H}_4^{\text{INT}} = \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{p, e}} \underbrace{\left\{ \boldsymbol{\alpha}^e \right\}^T}_{1 \times n_{\text{dof}}^{p, e}} \cdot \left(\int_{-1}^1 \underbrace{\left\{ \mathbf{B}^{e, p} \right\}^T}_{n_{\text{dof}}^{p, e} \times 1} \rho^{\text{fR}, h^e} \hat{k}^{h^e} \left[(a^{h^e} + g) \frac{n^{s, h^e}}{J^{h^e}} \right. \right.$$

$$\times \left(\frac{3}{n^{f, h^e}} - \frac{2n^{f, h^e}}{1 - (n^{f, h^e})^2} \right) (\beta \Delta t^2) \underbrace{\left\{ \mathbf{B}^{e, u} \right\}}_{1 \times n_{\text{dof}}^{s, e}} \underbrace{\left\{ \delta \ddot{\mathbf{d}}^e \right\}}_{n_{\text{dof}}^{s, e} \times 1}$$

$$+ (a^{h^e} + g) \frac{1}{K_f^\eta} (\beta \Delta t^2) \underbrace{\left\{ \mathbf{N}^{e, p} \right\}}_{1 \times n_{\text{dof}}^{p, e}} \underbrace{\left\{ \delta \ddot{\boldsymbol{\theta}}^e \right\}}_{n_{\text{dof}}^{p, e} \times 1}$$

$$\left. \left. + \underbrace{\left\{ \mathbf{N}^{e, u} \right\}}_{1 \times n_{\text{dof}}^{s, e}} \underbrace{\left\{ \delta \ddot{\mathbf{d}}^e \right\}}_{n_{\text{dof}}^{s, e} \times 1} \right] A j^e d\xi \right) \quad (\text{C-50})$$

Then, pulling out the variations $\delta \ddot{\mathbf{d}}^e$ and $\delta \ddot{\boldsymbol{\theta}}^e$ leaves us with

$$\delta \mathcal{H}_4^{\text{INT}} = \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{p, e}} \underbrace{\left\{ \boldsymbol{\alpha}^e \right\}^T}_{1 \times n_{\text{dof}}^{p, e}} \cdot \left(\underbrace{\left[\mathbf{k}_{p_f, u}^{\mathcal{H}_4^{\text{INT}, e}} \right]}_{n_{\text{dof}}^{p, e} \times n_{\text{dof}}^{s, e}} \cdot \underbrace{\left\{ \delta \ddot{\mathbf{d}}^e \right\}}_{n_{\text{dof}}^{s, e} \times 1} + \underbrace{\left[\mathbf{k}_{p_f, p_f}^{\mathcal{H}_4^{\text{INT}, e}} \right]}_{n_{\text{dof}}^{p, e} \times n_{\text{dof}}^{p, e}} \cdot \underbrace{\left\{ \delta \ddot{\boldsymbol{\theta}}^e \right\}}_{n_{\text{dof}}^{p, e} \times 1} \right) \quad (\text{C-51})$$

where we recover

$$\underbrace{\left[\mathbf{k}_{p_f, u}^{\mathcal{H}_4^{\text{INT}, e}} \right]}_{n_{\text{dof}}^{p, e} \times n_{\text{dof}}^{s, e}} = \int_{-1}^1 \underbrace{\left\{ \mathbf{B}^{e, p} \right\}^T}_{n_{\text{dof}}^{p, e} \times 1} \left(\left[\frac{3}{n^{f, h^e}} - \frac{2n^{f, h^e}}{1 - (n^{f, h^e})^2} \right] (a^{h^e} + g) \frac{n^{s, h^e}}{J^{h^e}} \times \right. \\ \left. (\beta \Delta t^2) \underbrace{\left\{ \mathbf{B}^{e, u} \right\}}_{1 \times n_{\text{dof}}^{s, e}} + \underbrace{\left\{ \mathbf{N}^{e, u} \right\}}_{1 \times n_{\text{dof}}^{s, e}} \right) \hat{k}^{h^e} \rho^{\text{fR}, h^e} A j^e d\xi \quad (\text{C-52})$$

$$\underbrace{\left[\mathbf{k}_{p_f, p_f}^{\mathcal{H}_4^{\text{INT}, e}} \right]}_{n_{\text{dof}}^{p, e} \times n_{\text{dof}}^{p, e}} = \int_{-1}^1 \hat{k}^{h^e} (a^{h^e} + g) \frac{\rho^{\text{fR}, h^e}}{K_f^\eta} (\beta \Delta t^2) \underbrace{\left\{ \mathbf{B}^{e, p} \right\}^T}_{n_{\text{dof}}^{p, e} \times 1} \underbrace{\left\{ \mathbf{N}^{e, p} \right\}}_{1 \times n_{\text{dof}}^{p, e}} A j^e d\xi \quad (\text{C-53})$$

To derive the tangents for $\delta \mathcal{H}_4^{\text{INT}}$ when poroelasticity is considered, start from Eq. B-23 of Appendix B and apply the FE discretization as follows:

$$\delta \mathcal{H}_4^{\text{INT}} = \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{p, e}} \underbrace{\left\{ \alpha^e \right\}^T}_{1 \times n_{\text{dof}}^{p, e}} \cdot \left(\int_{-1}^1 \underbrace{\left\{ \mathbf{B}^{e, p} \right\}^T}_{n_{\text{dof}}^{p, e} \times 1} \rho^{\text{fR}, h^e} \hat{k}^{h^e} \left[g \frac{n^{s, h^e}}{J^{h^e}} \times \right. \right. \\ \left. \left(\frac{3}{n^{f, h^e}} - \frac{2n^{f, h^e}}{1 - (n^{f, h^e})^2} \right) (\gamma \Delta t) \underbrace{\left\{ \mathbf{B}^{e, u} \right\}}_{1 \times n_{\text{dof}}^{s, e}} \underbrace{\left\{ \delta \dot{\mathbf{d}}^e \right\}}_{n_{\text{dof}}^{s, e} \times 1} \right. \\ \left. \left. + g \frac{1}{K_f^\eta} (\gamma \Delta t) \underbrace{\left\{ \mathbf{N}^{e, p} \right\}}_{1 \times n_{\text{dof}}^{p, e}} \underbrace{\left\{ \delta \dot{\boldsymbol{\theta}}^e \right\}}_{n_{\text{dof}}^{p, e} \times 1} \right] A j^e d\xi \right) \quad (\text{C-54})$$

Then, pulling out the variations $\delta \dot{\mathbf{d}}^e$ and $\delta \dot{\boldsymbol{\theta}}^e$ leaves us with

$$\delta \mathcal{H}_4^{\text{INT}} = \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{p, e}} \underbrace{\left\{ \alpha^e \right\}^T}_{1 \times n_{\text{dof}}^{p, e}} \cdot \left(\underbrace{\left[\mathbf{k}_{p_f, u}^{\mathcal{H}_4^{\text{INT}, e}} \right]}_{n_{\text{dof}}^{p, e} \times n_{\text{dof}}^{s, e}} \cdot \underbrace{\left\{ \delta \dot{\mathbf{d}}^e \right\}}_{n_{\text{dof}}^{s, e} \times 1} + \underbrace{\left[\mathbf{k}_{p_f, p_f}^{\mathcal{H}_4^{\text{INT}, e}} \right]}_{n_{\text{dof}}^{p, e} \times n_{\text{dof}}^{p, e}} \cdot \underbrace{\left\{ \delta \dot{\boldsymbol{\theta}}^e \right\}}_{n_{\text{dof}}^{p, e} \times 1} \right) \quad (\text{C-55})$$

where

$$\underbrace{\left[\mathbf{k}_{p_f, u}^{\mathcal{H}_{4, e}^{\text{INT}}} \right]}_{n_{\text{dof}}^{p, e} \times n_{\text{dof}}^{s, e}} = \int_{-1}^1 \underbrace{\left\{ \mathbf{B}^{e, p} \right\}^T}_{n_{\text{dof}}^{p, e} \times 1} \left(\left[\frac{3}{n_{\text{f}, h^e}} - \frac{2n_{\text{f}, h^e}}{1 - (n_{\text{f}, h^e})^2} \right] g \frac{n^{s, h^e}}{J^{h^e}} \times \right. \\ \left. (\gamma \Delta t) \underbrace{\left\{ \mathbf{B}^{e, u} \right\}}_{1 \times n_{\text{dof}}^{s, e}} \right) \hat{k}^{h^e} \rho^{\text{fR}, h^e} A j^e d\xi \quad (\text{C-56})$$

$$\underbrace{\left[\mathbf{k}_{p_f, p_f}^{\mathcal{H}_{4, e}^{\text{INT}}} \right]}_{n_{\text{dof}}^{p, e} \times n_{\text{dof}}^{p, e}} = \int_{-1}^1 \hat{k}^{h^e} g \frac{\rho^{\text{fR}, h^e}}{K_{\text{f}}^\eta} (\gamma \Delta t) \underbrace{\left\{ \mathbf{B}^{e, p} \right\}^T}_{n_{\text{dof}}^{p, e} \times 1} \underbrace{\left\{ \mathbf{N}^{e, p} \right\}}_{1 \times n_{\text{dof}}^{p, e}} A j^e d\xi \quad (\text{C-57})$$

To derive the tangents for $\delta \mathcal{H}^{\text{stab}}$, start from Eq. B-24 of Appendix B and apply the FE discretization as follows:

$$\delta \mathcal{H}^{\text{stab}} = \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{p, e}} \underbrace{\left\{ \boldsymbol{\alpha}^e \right\}^T}_{1 \times n_{\text{dof}}^{p, e}} \cdot \left(\int_{-1}^1 \underbrace{\left\{ \mathbf{B}^{e, p} \right\}^T}_{n_{\text{dof}}^{p, e} \times 1} \alpha^{\text{stab}} (F_{11}^{h^e})^{-1} \left[(\gamma \Delta t) \underbrace{\left\{ \mathbf{B}^{e, p} \right\}}_{1 \times n_{\text{dof}}^{p, e}} \cdot \underbrace{\left\{ \delta \ddot{\boldsymbol{\theta}}^e \right\}}_{n_{\text{dof}}^{p, e} \times 1} \right. \right. \\ \left. \left. - \frac{\partial \dot{p}_{\text{f}}^{h^e}}{\partial X} (F_{11}^{h^e})^{-1} (\beta \Delta t^2) \underbrace{\left\{ \mathbf{B}^{e, u} \right\}}_{1 \times n_{\text{dof}}^{s, e}} \cdot \underbrace{\left\{ \delta \ddot{\mathbf{d}}^e \right\}}_{n_{\text{dof}}^{s, e} \times 1} \right] A j^e d\xi \right) \quad (\text{C-58})$$

Then, pulling out the variations $\delta \ddot{\mathbf{d}}^e$ and $\delta \ddot{\boldsymbol{\theta}}^e$ leaves us with

$$\delta \mathcal{H}^{\text{stab}} = \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{p, e}} \underbrace{\left\{ \boldsymbol{\alpha}^e \right\}^T}_{1 \times n_{\text{dof}}^{p, e}} \cdot \left(\underbrace{\left[\mathbf{k}_{p_f, u}^{\mathcal{H}^{\text{stab}, e}} \right]}_{n_{\text{dof}}^{p, e} \times n_{\text{dof}}^{s, e}} \cdot \underbrace{\left\{ \delta \ddot{\mathbf{d}}^e \right\}}_{n_{\text{dof}}^{s, e} \times 1} + \underbrace{\left[\mathbf{k}_{p_f, p_f}^{\mathcal{H}^{\text{stab}, e}} \right]}_{n_{\text{dof}}^{p, e} \times n_{\text{dof}}^{p, e}} \cdot \underbrace{\left\{ \delta \ddot{\boldsymbol{\theta}}^e \right\}}_{n_{\text{dof}}^{p, e} \times 1} \right) \quad (\text{C-59})$$

where we recover

$$\underbrace{\left[\mathbf{k}_{p_f, u}^{\mathcal{H}^{\text{stab}, e}} \right]}_{n_{\text{dof}}^{p, e} \times n_{\text{dof}}^{s, e}} = - \int_{-1}^1 \alpha^{\text{stab}} (F_{11}^{h^e})^{-2} \frac{\partial \dot{p}_{\text{f}}^{h^e}}{\partial X} (\beta \Delta t^2) \underbrace{\left\{ \mathbf{B}^{e, p} \right\}^T}_{n_{\text{dof}}^{p, e} \times 1} \underbrace{\left\{ \mathbf{B}^{e, u} \right\}}_{1 \times n_{\text{dof}}^{s, e}} A j^e d\xi \quad (\text{C-60})$$

$$\underbrace{\left[\mathbf{k}_{p_f, p_f}^{\mathcal{H}^{\text{stab}, e}} \right]}_{n_{\text{dof}}^{p, e} \times n_{\text{dof}}^{p, e}} = \int_{-1}^1 \alpha^{\text{stab}} (F_{11}^{h^e})^{-1} (\gamma \Delta t) \underbrace{\left\{ \mathbf{B}^{e, p} \right\}^T}_{n_{\text{dof}}^{p, e} \times 1} \underbrace{\left\{ \mathbf{B}^{e, p} \right\}}_{1 \times n_{\text{dof}}^{p, e}} A j^e d\xi \quad (\text{C-61})$$

To derive the tangents for $\delta\mathcal{H}^{\text{stab}}$ when poroelasticity is considered, start from Eq. B-25 and apply the FE discretization as follows:

$$\delta\mathcal{H}^{\text{stab}} = \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{p,e}} \underbrace{\left\{ \boldsymbol{\alpha}^e \right\}^T}_{1 \times n_{\text{dof}}^{p,e}} \cdot \left(\int_{-1}^1 \underbrace{\left\{ \mathbf{B}^{e,p} \right\}^T}_{n_{\text{dof}}^{p,e} \times 1} \alpha^{\text{stab}}(F_{11}^{h^e})^{-1} \left[\underbrace{\left\{ \mathbf{B}^{e,p} \right\}}_{1 \times n_{\text{dof}}^{p,e}} \cdot \underbrace{\left\{ \delta \dot{\boldsymbol{\theta}}^e \right\}}_{n_{\text{dof}}^{p,e} \times 1} \right. \right. \\ \left. \left. - \frac{\partial \dot{p}_f^{h^e}}{\partial X} (F_{11}^{h^e})^{-1} (\gamma \Delta t) \underbrace{\left\{ \mathbf{B}^{e,u} \right\}}_{1 \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\left\{ \delta \dot{\mathbf{d}}^e \right\}}_{n_{\text{dof}}^{s,e} \times 1} \right] A j^e d\xi \right) \quad (\text{C-62})$$

Then, pulling out the variations $\delta \dot{\mathbf{d}}^e$ and $\delta \dot{\boldsymbol{\theta}}^e$ leaves us with

$$\delta\mathcal{H}^{\text{stab}} = \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{p,e}} \underbrace{\left\{ \boldsymbol{\alpha}^e \right\}^T}_{1 \times n_{\text{dof}}^{p,e}} \cdot \left(\underbrace{\left[\mathbf{k}_{p_f,u}^{\mathcal{H}^{\text{stab},e}} \right]}_{n_{\text{dof}}^{p,e} \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\left\{ \delta \dot{\mathbf{d}}^e \right\}}_{n_{\text{dof}}^{s,e} \times 1} + \underbrace{\left[\mathbf{k}_{p_f,p_f}^{\mathcal{H}^{\text{stab},e}} \right]}_{n_{\text{dof}}^{p,e} \times n_{\text{dof}}^{p,e}} \cdot \underbrace{\left\{ \delta \dot{\boldsymbol{\theta}}^e \right\}}_{n_{\text{dof}}^{p,e} \times 1} \right) \quad (\text{C-63})$$

where

$$\underbrace{\left[\mathbf{k}_{p_f,u}^{\mathcal{H}^{\text{stab},e}} \right]}_{n_{\text{dof}}^{p,e} \times n_{\text{dof}}^{s,e}} = - \int_{-1}^1 \alpha^{\text{stab}}(F_{11}^{h^e})^{-2} \frac{\partial \dot{p}_f^{h^e}}{\partial X} (\gamma \Delta t) \underbrace{\left\{ \mathbf{B}^{e,p} \right\}^T}_{n_{\text{dof}}^{p,e} \times 1} \underbrace{\left\{ \mathbf{B}^{e,u} \right\}}_{1 \times n_{\text{dof}}^{s,e}} A j^e d\xi \quad (\text{C-64})$$

$$\underbrace{\left[\mathbf{k}_{p_f,p_f}^{\mathcal{H}^{\text{stab},e}} \right]}_{n_{\text{dof}}^{p,e} \times n_{\text{dof}}^{p,e}} = \int_{-1}^1 \alpha^{\text{stab}}(F_{11}^{h^e})^{-1} \underbrace{\left\{ \mathbf{B}^{e,p} \right\}^T}_{n_{\text{dof}}^{p,e} \times 1} \underbrace{\left\{ \mathbf{B}^{e,p} \right\}}_{1 \times n_{\text{dof}}^{p,e}} A j^e d\xi \quad (\text{C-65})$$

C.1.3 Poroelastodynamics (u - u_f - p_f) Formulation

To derive the tangents for $\delta\mathcal{G}_1^{\text{INT}}$, we start by applying the FE discretization as follows:

$$\begin{aligned} \delta\mathcal{G}_1^{\text{INT}} = & \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{s,e}} \underbrace{\{\mathbf{c}^e\}^T}_{n_{\text{dof}}^{s,e} \times 1} \cdot \left(\int_{-1}^1 \underbrace{\{\mathbf{N}^{e,u}\}^T}_{n_{\text{dof}}^{s,e} \times 1} \left[\underbrace{\rho_0^{s,h^e} \{\mathbf{N}^{e,u}\}}_{1 \times n_{\text{dof}}^{s,e}} \underbrace{\{\ddot{\mathbf{d}}^e\}}_{n_{\text{dof}}^{s,e} \times 1} \right. \right. \\ & + a_f^{h^e} (\beta \Delta t^2) \left(\underbrace{\rho^{\text{fR},h^e} \{\mathbf{B}^{e,u}\}}_{1 \times n_{\text{dof}}^{s,e}} \underbrace{\{\ddot{\mathbf{d}}^e\}}_{n_{\text{dof}}^{s,e} \times 1} + \frac{J^{h^e} n^{f,h^e} \rho^{\text{fR},h^e}}{K_f^\eta} \underbrace{\{\mathbf{N}^{e,p}\}}_{1 \times n_{\text{dof}}^{p,e}} \underbrace{\{\ddot{\boldsymbol{\theta}}^e\}}_{n_{\text{dof}}^{p,e} \times 1} \right. \\ & \left. \left. \left. + \rho_0^{f,h^e} \underbrace{\{\mathbf{N}^{e,u_f}\}}_{1 \times n_{\text{dof}}^{f,e}} \underbrace{\{\ddot{\mathbf{d}}_f^e\}}_{n_{\text{dof}}^{f,e} \times 1} \right] A_j^e d\xi \right) \end{aligned} \quad (\text{C-66})$$

Then, pulling out the variations $\delta\ddot{\mathbf{d}}^e$, $\delta\ddot{\mathbf{d}}_f^e$, and $\delta\ddot{\boldsymbol{\theta}}^e$ leaves us with

$$\begin{aligned} \delta\mathcal{G}_1^{\text{INT}} = & \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{s,e}} \underbrace{\{\mathbf{c}^e\}^T}_{n_{\text{dof}}^{s,e} \times 1} \cdot \left(\underbrace{\left[\mathbf{k}_{u,u}^{\mathcal{G}_1^{\text{INT}},e} \right]}_{n_{\text{dof}}^{s,e} \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\{\delta\ddot{\mathbf{d}}^e\}}_{n_{\text{dof}}^{s,e} \times 1} + \underbrace{\left[\mathbf{k}_{u,u_f}^{\mathcal{G}_1^{\text{INT}},e} \right]}_{n_{\text{dof}}^{s,e} \times n_{\text{dof}}^{f,e}} \cdot \underbrace{\{\delta\ddot{\mathbf{d}}_f^e\}}_{n_{\text{dof}}^{f,e} \times 1} \right. \\ & \left. + \underbrace{\left[\mathbf{k}_{u,p_f}^{\mathcal{G}_1^{\text{INT}},e} \right]}_{n_{\text{dof}}^{s,e} \times n_{\text{dof}}^{p,e}} \cdot \underbrace{\{\delta\ddot{\boldsymbol{\theta}}^e\}}_{n_{\text{dof}}^{p,e} \times 1} \right) \end{aligned} \quad (\text{C-67})$$

where we recover the following tangent matrices:

$$\underbrace{\left[\mathbf{k}_{u,u}^{\mathcal{G}_1^{\text{INT}},e} \right]}_{n_{\text{dof}}^{s,e} \times n_{\text{dof}}^{s,e}} = \int_{-1}^1 \underbrace{\{\mathbf{N}^{e,u}\}^T}_{n_{\text{dof}}^{s,e} \times 1} \left(\underbrace{\rho_0^{s,h^e} \{\mathbf{N}^{e,u}\}}_{1 \times n_{\text{dof}}^{s,e}} + a_f^{h^e} \rho^{\text{fR},h^e} (\beta \Delta t^2) \underbrace{\{\mathbf{B}^{e,u}\}}_{1 \times n_{\text{dof}}^{s,e}} \right) A_j^e d\xi \quad (\text{C-68})$$

$$\underbrace{\left[\mathbf{k}_{u,u_f}^{\mathcal{G}_1^{\text{INT}},e} \right]}_{n_{\text{dof}}^{s,e} \times n_{\text{dof}}^{f,e}} = \int_{-1}^1 \rho_0^{f,h^e} \underbrace{\{\mathbf{N}^{e,u}\}^T}_{n_{\text{dof}}^{s,e} \times 1} \underbrace{\{\mathbf{N}^{e,u_f}\}}_{1 \times n_{\text{dof}}^{f,e}} A_j^e d\xi \quad (\text{C-69})$$

$$\underbrace{\left[\mathbf{k}_{u,p_f}^{\mathcal{G}_1^{\text{INT}},e} \right]}_{n_{\text{dof}}^{s,e} \times n_{\text{dof}}^{p,e}} = \int_{-1}^1 \frac{a_f^{h^e} J^{h^e} n^{f,h^e} \rho^{\text{fR},h^e}}{K_f^\eta} (\beta \Delta t^2) \underbrace{\{\mathbf{N}^{e,u}\}^T}_{n_{\text{dof}}^{s,e} \times 1} \underbrace{\{\mathbf{N}^{e,p}\}}_{1 \times n_{\text{dof}}^{p,e}} A_j^e d\xi \quad (\text{C-70})$$

The tangents for $\delta\mathcal{G}_2^{\text{INT}}$ and $\delta\mathcal{G}_3^{\text{INT}}$ remain unchanged from elastodynamics and poroelastodynamics, respectively. Refer to Eqs. C-6, C-13, and C-22.

To derive the tangents for $\delta\mathcal{I}_1^{\text{INT}}$, we apply the FE discretization as follows:

$$\begin{aligned} \delta\mathcal{I}_1^{\text{INT}} = & \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{f,e}} \underbrace{\{\boldsymbol{\omega}^e\}^T}_{1 \times n_{\text{dof}}^{f,e}} \cdot \left(\int_{-1}^1 \underbrace{\{\mathbf{N}^{e,u_f}\}^T}_{n_{\text{dof}}^{f,e} \times 1} \left[a_f^{h^e} (\beta \Delta t^2) \left(\underbrace{\rho^{\text{fR},h^e}}_{1 \times n_{\text{dof}}^{s,e}} \underbrace{\{\mathbf{N}^{e,u}\}}_{n_{\text{dof}}^{s,e} \times 1} \underbrace{\{\ddot{\mathbf{d}}^e\}}_{n_{\text{dof}}^{s,e} \times 1} \right. \right. \right. \\ & \left. \left. \left. + \frac{J^{h^e} n^{\text{f},h^e} \rho^{\text{fR},h^e}}{K_f^\eta} \underbrace{\{\mathbf{N}^{e,p}\}}_{1 \times n_{\text{dof}}^{p,e}} \underbrace{\{\ddot{\boldsymbol{\theta}}^e\}}_{n_{\text{dof}}^{p,e} \times 1} \right) + \rho_0^{\text{f},h^e} \underbrace{\{\mathbf{N}^{e,u_f}\}}_{1 \times n_{\text{dof}}^{f,e}} \underbrace{\{\ddot{\mathbf{d}}_f^e\}}_{n_{\text{dof}}^{f,e} \times 1} \right] A j^e d\xi \right) \quad (\text{C-71}) \end{aligned}$$

Then, pulling out the variations $\delta\ddot{\mathbf{d}}^e$, $\delta\ddot{\mathbf{d}}_f^e$, and $\delta\ddot{\boldsymbol{\theta}}^e$ leaves us with

$$\begin{aligned} \delta\mathcal{I}_1^{\text{INT}} = & \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{f,e}} \underbrace{\{\boldsymbol{\omega}^e\}^T}_{1 \times n_{\text{dof}}^{f,e}} \cdot \left(\underbrace{\left[\mathbf{k}_{u_f,u}^{\mathcal{I}_1^{\text{INT}},e} \right]}_{n_{\text{dof}}^{f,e} \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\{\delta\ddot{\mathbf{d}}^e\}}_{n_{\text{dof}}^{s,e} \times 1} + \underbrace{\left[\mathbf{k}_{u_f,u_f}^{\mathcal{I}_1^{\text{INT}},e} \right]}_{n_{\text{dof}}^{f,e} \times n_{\text{dof}}^{f,e}} \cdot \underbrace{\{\delta\ddot{\mathbf{d}}_f^e\}}_{n_{\text{dof}}^{f,e} \times 1} \right. \\ & \left. + \underbrace{\left[\mathbf{k}_{u_f,p_f}^{\mathcal{I}_1^{\text{INT}},e} \right]}_{n_{\text{dof}}^{f,e} \times n_{\text{dof}}^{p,e}} \cdot \underbrace{\{\delta\ddot{\boldsymbol{\theta}}^e\}}_{n_{\text{dof}}^{p,e} \times 1} \right) \quad (\text{C-72}) \end{aligned}$$

where we recover

$$\underbrace{\left[\mathbf{k}_{u_f,u}^{\mathcal{I}_1^{\text{INT}},e} \right]}_{n_{\text{dof}}^{f,e} \times n_{\text{dof}}^{s,e}} = \int_{-1}^1 \rho^{\text{fR},h^e} a_f^{h^e} (\beta \Delta t^2) \underbrace{\{\mathbf{N}^{e,u_f}\}^T}_{n_{\text{dof}}^{f,e} \times 1} \underbrace{\{\mathbf{B}^{e,u}\}}_{1 \times n_{\text{dof}}^{s,e}} A j^e d\xi \quad (\text{C-73})$$

$$\underbrace{\left[\mathbf{k}_{u_f,u_f}^{\mathcal{I}_1^{\text{INT}},e} \right]}_{n_{\text{dof}}^{f,e} \times n_{\text{dof}}^{f,e}} = \int_{-1}^1 \rho_0^{\text{f},h^e} \underbrace{\{\mathbf{N}^{e,u_f}\}^T}_{n_{\text{dof}}^{f,e} \times 1} \underbrace{\{\mathbf{N}^{e,u_f}\}}_{1 \times n_{\text{dof}}^{f,e}} A j^e d\xi \quad (\text{C-74})$$

$$\underbrace{\left[\mathbf{k}_{u_f,p_f}^{\mathcal{I}_1^{\text{INT}},e} \right]}_{n_{\text{dof}}^{f,e} \times n_{\text{dof}}^{p,e}} = \int_{-1}^1 \frac{a_f^{h^e} \rho_0^{\text{f},h^e}}{K_f^\eta} (\beta \Delta t^2) \underbrace{\{\mathbf{N}^{e,u_f}\}^T}_{n_{\text{dof}}^{f,e} \times 1} \underbrace{\{\mathbf{N}^{e,p}\}}_{1 \times n_{\text{dof}}^{p,e}} A j^e d\xi \quad (\text{C-75})$$

To derive the tangents for $\delta\mathcal{I}_2^{\text{INT}}$, we start by applying the FE discretization as follows:

$$\delta\mathcal{I}_2^{\text{INT}} = \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{f,e}} \underbrace{\{\boldsymbol{\omega}^e\}^T}_{1 \times n_{\text{dof}}^{f,e}} \cdot \left(\int_{-1}^1 \underbrace{\{\mathbf{N}^{e,u_f}\}^T}_{n_{\text{dof}}^{f,e} \times 1} \left[\underbrace{\frac{n^{s,h^e}}{J^{h^e}} \frac{\partial p_f^{h^e}}{\partial X}}_{1 \times n_{\text{dof}}^{s,e}} \underbrace{\{\mathbf{B}^{e,u}\}}_{1 \times n_{\text{dof}}^{s,e}} \underbrace{\{\ddot{\mathbf{d}}^e\}}_{n_{\text{dof}}^{s,e} \times 1} \right. \right. \\ \left. \left. + n^{f,h^e} \underbrace{\{\mathbf{B}^{e,p}\}}_{1 \times n_{\text{dof}}^{p,e}} \underbrace{\{\ddot{\boldsymbol{\theta}}^e\}}_{n_{\text{dof}}^{p,e} \times 1} \right] (\beta \Delta t^2) A j^e d\xi \right) \quad (\text{C-76})$$

Then, pulling out the variations $\delta\ddot{\mathbf{d}}^e$ & $\delta\ddot{\boldsymbol{\theta}}^e$ leaves us with

$$\delta\mathcal{I}_2^{\text{INT}} = \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{f,e}} \underbrace{\{\boldsymbol{\omega}^e\}^T}_{1 \times n_{\text{dof}}^{f,e}} \cdot \left(\underbrace{\left[\mathbf{k}_{u_f,u}^{\mathcal{I}_2^{\text{INT}},e} \right]}_{n_{\text{dof}}^{f,e} \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\{\delta\ddot{\mathbf{d}}^e\}}_{n_{\text{dof}}^{s,e} \times 1} + \underbrace{\left[\mathbf{k}_{u_f,p_f}^{\mathcal{I}_2^{\text{INT}},e} \right]}_{n_{\text{dof}}^{f,e} \times n_{\text{dof}}^{p,e}} \cdot \underbrace{\{\delta\ddot{\boldsymbol{\theta}}^e\}}_{n_{\text{dof}}^{p,e} \times 1} \right) \quad (\text{C-77})$$

where we recover the following expressions:

$$\underbrace{\left[\mathbf{k}_{u_f,u}^{\mathcal{I}_2^{\text{INT}},e} \right]}_{n_{\text{dof}}^{f,e} \times n_{\text{dof}}^{s,e}} = \int_{-1}^1 \frac{n^{s,h^e}}{J^{h^e}} \frac{\partial p_f^{h^e}}{\partial X} (\beta \Delta t^2) \underbrace{\{\mathbf{N}^{e,u_f}\}^T}_{n_{\text{dof}}^{f,e} \times 1} \underbrace{\{\mathbf{B}^{e,u}\}}_{1 \times n_{\text{dof}}^{s,e}} A j^e d\xi \quad (\text{C-78})$$

$$\underbrace{\left[\mathbf{k}_{u_f,p_f}^{\mathcal{I}_2^{\text{INT}},e} \right]}_{n_{\text{dof}}^{f,e} \times n_{\text{dof}}^{p,e}} = \int_{-1}^1 n^{f,h^e} (\beta \Delta t^2) \underbrace{\{\mathbf{N}^{e,u_f}\}^T}_{n_{\text{dof}}^{f,e} \times 1} \underbrace{\{\mathbf{B}^{e,p}\}}_{1 \times n_{\text{dof}}^{p,e}} A j^e d\xi \quad (\text{C-79})$$

To derive the tangents for $\delta\mathcal{I}_3^{\text{INT}}$, we start by applying the FE discretization as follows:

$$\begin{aligned}
\delta\mathcal{I}_3^{\text{INT}} &= \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{f,e}} \underbrace{\left\{ \omega^e \right\}^T}_{n_{\text{dof}}^{f,e} \times 1} \\
&\times \left(\int_{-1}^1 \underbrace{\left\{ \mathbf{N}^{e,u_f} \right\}^T}_{n_{\text{dof}}^{f,e} \times 1} \left[\left(\left[1 + \frac{2n^{s,h^e}}{n^{f,h^e}} - n^{s,h^e} \left(\frac{3}{n^{f,h^e}} - \frac{2n^{f,h^e}}{1 - (n^{f,h^e})^2} \right) \right] \right. \right. \\
&\times \frac{(n^{f,h^e})^2 \tilde{v}_f^{h^e}}{\hat{k}^{h^e}} (\beta \Delta t^2) - \frac{J^{h^e} (n^{f,h^e})^2}{\hat{k}^{h^e}} (\gamma \Delta t) \left. \right) \underbrace{\left\{ \mathbf{N}^{e,u} \right\}}_{1 \times n_{\text{dof}}^{s,e}} \underbrace{\left\{ \ddot{\mathbf{d}}^e \right\}}_{n_{\text{dof}}^{s,e} \times 1} \\
&+ \frac{J^{h^e} (n^{f,h^e})^2}{\hat{k}^{h^e}} (\gamma \Delta t) \underbrace{\left\{ \mathbf{N}^{e,u_f} \right\}}_{1 \times n_{\text{dof}}^{f,e}} \underbrace{\left\{ \ddot{\mathbf{d}}_f^e \right\}}_{n_{\text{dof}}^{f,e} \times 1} \left. \right] A_j^e d\xi \Bigg)
\end{aligned} \tag{C-80}$$

Then, pulling out the variations $\delta\ddot{\mathbf{d}}^e$ and $\delta\ddot{\mathbf{d}}_f^e$ leaves us with

$$\delta\mathcal{I}_3^{\text{INT}} = \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{f,e}} \underbrace{\left\{ \omega^e \right\}^T}_{n_{\text{dof}}^{f,e} \times 1} \cdot \left(\underbrace{\left[\mathbf{k}_{u_f,u}^{\mathcal{I}_3^{\text{INT}},e} \right]}_{n_{\text{dof}}^{f,e} \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\left\{ \delta\ddot{\mathbf{d}}^e \right\}}_{n_{\text{dof}}^{s,e} \times 1} + \underbrace{\left[\mathbf{k}_{u_f,u_f}^{\mathcal{I}_3^{\text{INT}},e} \right]}_{n_{\text{dof}}^{f,e} \times n_{\text{dof}}^{f,e}} \cdot \underbrace{\left\{ \delta\ddot{\mathbf{d}}_f^e \right\}}_{n_{\text{dof}}^{f,e} \times 1} \right) \tag{C-81}$$

whereby we recover the tangents

$$\begin{aligned}
\underbrace{\left[\mathbf{k}_{u_f,u}^{\mathcal{I}_3^{\text{INT}},e} \right]}_{n_{\text{dof}}^{f,e} \times n_{\text{dof}}^{s,e}} &= \int_{-1}^1 \underbrace{\left\{ \mathbf{N}^{e,u_f} \right\}^T}_{n_{\text{dof}}^{f,e} \times 1} \left(\left[1 + \frac{2n^{s,h^e}}{n^{f,h^e}} - n^{s,h^e} \left(\frac{3}{n^{f,h^e}} - \frac{2n^{f,h^e}}{1 - (n^{f,h^e})^2} \right) \right] \times \right. \\
&\left. \frac{(n^{f,h^e})^2 \tilde{v}_f^{h^e}}{\hat{k}^{h^e}} (\beta \Delta t^2) \underbrace{\left\{ \mathbf{B}^{e,u} \right\}}_{1 \times n_{\text{dof}}^{s,e}} - \frac{J^{h^e} (n^{f,h^e})^2}{\hat{k}^{h^e}} (\gamma \Delta t) \underbrace{\left\{ \mathbf{N}^{e,u} \right\}}_{1 \times n_{\text{dof}}^{s,e}} \right) A_j^e d\xi
\end{aligned} \tag{C-82}$$

$$\underbrace{\left[\mathbf{k}_{u_f,u_f}^{\mathcal{I}_3^{\text{INT}},e} \right]}_{n_{\text{dof}}^{f,e} \times n_{\text{dof}}^{f,e}} = \int_{-1}^1 \frac{J^{h^e} (n^{f,h^e})^2}{\hat{k}^{h^e}} (\gamma \Delta t) \underbrace{\left\{ \mathbf{N}^{e,u_f} \right\}^T}_{n_{\text{dof}}^{f,e} \times 1} \underbrace{\left\{ \mathbf{N}^{e,u_f} \right\}}_{1 \times n_{\text{dof}}^{f,e}} A_j^e d\xi \tag{C-83}$$

To derive the tangents for $\delta\mathcal{I}_4^{\text{INT}}$, start by applying the FE discretization as follows:

$$\begin{aligned} \delta\mathcal{I}_4^{\text{INT}} = & \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{f,e}} \underbrace{\{\boldsymbol{\omega}^e\}^T}_{n_{\text{dof}}^{f,e} \times 1} \cdot \left(\int_{-1}^1 \underbrace{\{\mathbf{N}^{e,u_f}\}^T}_{n_{\text{dof}}^{f,e} \times 1} \left[\underbrace{\rho^{\text{fR},h^e} \{\mathbf{B}^{e,u}\}}_{1 \times n_{\text{dof}}^{s,e}} \underbrace{\{\ddot{\mathbf{d}}^e\}}_{n_{\text{dof}}^{s,e} \times 1} \right. \right. \\ & \left. \left. + \frac{J^{h^e} n^{f,h^e} \rho^{\text{fR},h^e}}{K_f^\eta} \underbrace{\{\mathbf{N}^{e,p}\}}_{1 \times n_{\text{dof}}^{p,e}} \underbrace{\{\ddot{\boldsymbol{\theta}}^e\}}_{n_{\text{dof}}^{p,e} \times 1} \right] (\beta \Delta t^2) g A j^e d\xi \right) \end{aligned} \quad (\text{C-84})$$

Then, pulling out the variations $\delta\ddot{\mathbf{d}}^e$ and $\delta\ddot{\boldsymbol{\theta}}^e$ leaves us with

$$\delta\mathcal{I}_4^{\text{INT}} = \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{f,e}} \underbrace{\{\boldsymbol{\omega}^e\}^T}_{n_{\text{dof}}^{f,e} \times 1} \cdot \left(\underbrace{\left[\mathbf{k}_{u_f,u}^{\mathcal{I}_4^{\text{INT}},e} \right]}_{n_{\text{dof}}^{f,e} \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\{\delta\ddot{\mathbf{d}}^e\}}_{n_{\text{dof}}^{s,e} \times 1} + \underbrace{\left[\mathbf{k}_{u_f,p_f}^{\mathcal{I}_4^{\text{INT}},e} \right]}_{n_{\text{dof}}^{f,e} \times n_{\text{dof}}^{p,e}} \cdot \underbrace{\{\delta\ddot{\boldsymbol{\theta}}^e\}}_{n_{\text{dof}}^{p,e} \times 1} \right) \quad (\text{C-85})$$

where we recover

$$\underbrace{\left[\mathbf{k}_{u_f,u}^{\mathcal{I}_4^{\text{INT}},e} \right]}_{n_{\text{dof}}^{f,e} \times n_{\text{dof}}^{s,e}} = \int_{-1}^1 \rho^{\text{fR},h^e} g (\beta \Delta t^2) \underbrace{\{\mathbf{N}^{e,u_f}\}^T}_{n_{\text{dof}}^{f,e} \times 1} \underbrace{\{\mathbf{B}^{e,u}\}}_{1 \times n_{\text{dof}}^{s,e}} A j^e d\xi \quad (\text{C-86})$$

$$\underbrace{\left[\mathbf{k}_{u_f,p_f}^{\mathcal{I}_4^{\text{INT}},e} \right]}_{n_{\text{dof}}^{f,e} \times n_{\text{dof}}^{p,e}} = \int_{-1}^1 \frac{J^{h^e} n^{f,h^e} \rho^{\text{fR},h^e}}{K_f^\eta} g (\beta \Delta t^2) \underbrace{\{\mathbf{N}^{e,u_f}\}^T}_{n_{\text{dof}}^{f,e} \times 1} \underbrace{\{\mathbf{N}^{e,p}\}}_{1 \times n_{\text{dof}}^{p,e}} A j^e d\xi \quad (\text{C-87})$$

The tangent for $\delta\mathcal{H}_1^{\text{INT}}$ remains unchanged from poroelastodynamics: refer to Eq. C-29.

To derive the tangents for $\delta\mathcal{H}_2^{\text{INT}}$, we start by applying the FE discretization as follows:

$$\begin{aligned}
\delta\mathcal{H}_2^{\text{INT}} = & \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{p,e}} \underbrace{\{\alpha^e\}^T}_{n_{\text{dof}}^{p,e} \times 1} \cdot \left(\int_{-1}^1 \underbrace{\{N^{e,p}\}^T}_{n_{\text{dof}}^{p,e} \times 1} \left[\left([n^f \tilde{v}_f]^{h^e} - \hat{k}^{h^e} \frac{\partial p_f^{h^e}}{\partial X} (F_{11}^{h^e})^{-1} \right) (\beta \Delta t^2) \right. \right. \\
& \times \underbrace{\{B^{e,p}\}}_{1 \times n_{\text{dof}}^{p,e}} \underbrace{\{\delta \ddot{\theta}^e\}}_{n_{\text{dof}}^{p,e} \times 1} - \hat{k}^{h^e} \frac{\partial p_f^{h^e}}{\partial X} (a_f^{h^e} + g) \frac{\rho^{\text{fR},h^e}}{K_f^\eta} (\beta \Delta t^2) \underbrace{\{N^{e,p}\}}_{1 \times n_{\text{dof}}^{p,e}} \underbrace{\{\delta \ddot{\theta}^e\}}_{n_{\text{dof}}^{p,e} \times 1} \\
& + \frac{\partial p_f^{h^e}}{\partial X} \left(\frac{n^{s,h^e}}{J^{h^e}} \left[\frac{3}{n^{f,h^e}} - \frac{2n^{f,h^e}}{1 - (n^{f,h^e})^2} \right] (n^f \tilde{v}_f)^{h^e} + \hat{k}^{h^e} \frac{\partial p_f^{h^e}}{\partial X} (F_{11}^{h^e})^{-2} \right) (\beta \Delta t^2) \\
& \left. \times \underbrace{\{B^{e,u}\}}_{1 \times n_{\text{dof}}^{s,e}} \underbrace{\{\delta \ddot{d}^e\}}_{n_{\text{dof}}^{d,e} \times 1} - \frac{\partial p_f^{h^e}}{\partial X} \hat{k}^{h^e} \rho^{\text{fR},h^e} \underbrace{\{N^{e,u_f}\}}_{1 \times n_{\text{dof}}^{f,e}} \underbrace{\{\delta \ddot{d}_f^e\}}_{n_{\text{dof}}^{f,e} \times 1} \right] \frac{A}{K_f^\eta} j^e d\xi \Bigg)
\end{aligned} \tag{C-88}$$

Then, pulling out the variations $\delta \ddot{d}^e$, $\delta \ddot{d}_f^e$, and $\delta \ddot{\theta}^e$ leaves us with

$$\begin{aligned}
\delta\mathcal{H}_2^{\text{INT}} = & \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{p,e}} \underbrace{\{\alpha^e\}^T}_{n_{\text{dof}}^{p,e} \times n_{\text{dof}}^{s,e}} \cdot \left(\underbrace{\left[\mathbf{k}_{p_f,u}^{\mathcal{H}_2^{\text{INT},e}} \right]}_{n_{\text{dof}}^{p,e} \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\{\delta \ddot{d}^e\}}_{n_{\text{dof}}^{s,e} \times 1} + \underbrace{\left[\mathbf{k}_{p_f,u}^{\mathcal{H}_2^{\text{INT},e}} \right]}_{n_{\text{dof}}^{p,e} \times n_{\text{dof}}^{f,e}} \cdot \underbrace{\{\delta \ddot{d}_f^e\}}_{n_{\text{dof}}^{f,e} \times 1} \right. \\
& \left. + \underbrace{\left[\mathbf{k}_{p_f,p_f}^{\mathcal{H}_2^{\text{INT},e}} \right]}_{n_{\text{dof}}^{p,e} \times n_{\text{dof}}^{p,e}} \cdot \underbrace{\{\delta \ddot{\theta}^e\}}_{n_{\text{dof}}^{p,e} \times 1} \right)
\end{aligned} \tag{C-89}$$

whereby we recover

$$\begin{aligned}
\underbrace{\left[\mathbf{k}_{p_f,u}^{\mathcal{H}_2^{\text{INT},e}} \right]}_{n_{\text{dof}}^{p,e} \times n_{\text{dof}}^{s,e}} = & \int_{-1}^1 \frac{\partial p_f^{h^e}}{\partial X} \left(\frac{n^{s,h^e}}{J^{h^e}} \left[\frac{3}{n^{f,h^e}} - \frac{2n^{f,h^e}}{1 - (n^{f,h^e})^2} \right] (n^f \tilde{v}_f)^{h^e} \right. \\
& \left. + \hat{k}^{h^e} \frac{\partial p_f^{h^e}}{\partial X} (F_{11}^{h^e})^{-2} \right) (\beta \Delta t^2) \underbrace{\{N^{e,p}\}^T}_{n_{\text{dof}}^{p,e} \times 1} \underbrace{\{B^{e,u}\}}_{1 \times n_{\text{dof}}^{s,e}} \frac{1}{K_f^\eta} A j^e d\xi
\end{aligned} \tag{C-90}$$

$$\underbrace{\left[\mathbf{k}_{p_f, u_f}^{\mathcal{H}_2^{\text{INT}}, e} \right]}_{n_{\text{dof}}^{p,e} \times n_{\text{dof}}^{s,e}} = - \int_{-1}^1 \hat{k}^{h^e} \rho^{\text{fR}, h^e} \frac{\partial p_f^{h^e}}{\partial X} \frac{1}{K_f^\eta} \underbrace{\left\{ \mathbf{N}^{e,p} \right\}^T}_{n_{\text{dof}}^{p,e} \times 1} \underbrace{\left\{ \mathbf{N}^{e, u_f} \right\}}_{1 \times n_{\text{dof}}^{f,e}} \quad (\text{C-91})$$

$$\begin{aligned} \underbrace{\left[\mathbf{k}_{p_f, p_f}^{\mathcal{H}_2^{\text{INT}}, e} \right]}_{n_{\text{dof}}^{p,e} \times n_{\text{dof}}^{p,e}} &= \int_{-1}^1 \underbrace{\left\{ \mathbf{N}^{e,p} \right\}^T}_{n_{\text{dof}}^{p,e} \times 1} \left(\left[\left(n^f \tilde{v}_f \right)^{h^e} - \hat{k}^{h^e} \frac{\partial p_f^{h^e}}{\partial X} (F_{11}^{h^e})^{-1} \right] \underbrace{\left\{ \mathbf{B}^{e,p} \right\}}_{1 \times n_{\text{dof}}^{p,e}} \right. \\ &\quad \left. - \hat{k}^{h^e} \frac{\partial p_f^{h^e}}{\partial X} (a_f^{h^e} + g) \frac{\rho^{\text{fR}, h^e}}{K_f^\eta} \underbrace{\left\{ \mathbf{N}^{e,p} \right\}}_{1 \times n_{\text{dof}}^{p,e}} \right) \frac{1}{K_f^\eta} (\beta \Delta t^2) A j^e d\xi \quad (\text{C-92}) \end{aligned}$$

To derive the tangents for $\delta \mathcal{H}_4^{\text{INT}}$, we start by applying the FE discretization as follows:

$$\begin{aligned} \delta \mathcal{H}_4^{\text{INT}} &= \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{p,e}} \underbrace{\left\{ \boldsymbol{\alpha}^e \right\}^T}_{n_{\text{dof}}^{p,e} \times 1} \cdot \left(\int_{-1}^1 \underbrace{\left\{ \mathbf{B}^{e,p} \right\}^T}_{n_{\text{dof}}^{p,e} \times 1} \rho^{\text{fR}, h^e} \hat{k}^{h^e} \left[(a_f^{h^e} + g) \frac{n^{s,h^e}}{J^{h^e}} \right. \right. \\ &\quad \times \left(\frac{3}{n^{f,h^e}} - \frac{2n^{f,h^e}}{1 - (n^{f,h^e})^2} \right) (\beta \Delta t^2) \underbrace{\left\{ \mathbf{B}^{e,u} \right\}}_{1 \times n_{\text{dof}}^{s,e}} \underbrace{\left\{ \delta \ddot{\mathbf{d}}^e \right\}}_{n_{\text{dof}}^{s,e} \times 1} \\ &\quad + (a_f^{h^e} + g) \frac{1}{K_f^\eta} (\beta \Delta t^2) \underbrace{\left\{ \mathbf{N}^{e,p} \right\}}_{1 \times n_{\text{dof}}^{p,e}} \underbrace{\left\{ \delta \ddot{\boldsymbol{\theta}}^e \right\}}_{n_{\text{dof}}^{p,e} \times 1} \\ &\quad \left. \left. + \underbrace{\left\{ \mathbf{N}^{e, u_f} \right\}}_{1 \times n_{\text{dof}}^{f,e}} \underbrace{\left\{ \delta \ddot{\mathbf{d}}_f^e \right\}}_{n_{\text{dof}}^{f,e} \times 1} \right] A j^e d\xi \right) \quad (\text{C-93}) \end{aligned}$$

Then, pulling out the variations $\delta \ddot{\mathbf{d}}^e$, $\delta \ddot{\mathbf{d}}_f^e$, and $\delta \ddot{\boldsymbol{\theta}}^e$ leaves us with

$$\begin{aligned} \delta \mathcal{H}_4^{\text{INT}} &= \underbrace{\mathbf{A}_e}_{1 \times n_{\text{dof}}^{p,e}} \underbrace{\left\{ \boldsymbol{\alpha}^e \right\}^T}_{n_{\text{dof}}^{p,e} \times 1} \cdot \left(\underbrace{\left[\mathbf{k}_{p_f, u}^{\mathcal{H}_4^{\text{INT}}, e} \right]}_{n_{\text{dof}}^{p,e} \times n_{\text{dof}}^{s,e}} \cdot \underbrace{\left\{ \delta \ddot{\mathbf{d}}^e \right\}}_{n_{\text{dof}}^{s,e} \times 1} + \underbrace{\left[\mathbf{k}_{p_f, u_f}^{\mathcal{H}_4^{\text{INT}}, e} \right]}_{n_{\text{dof}}^{p,e} \times n_{\text{dof}}^{f,e}} \cdot \underbrace{\left\{ \delta \ddot{\mathbf{d}}_f^e \right\}}_{n_{\text{dof}}^{f,e} \times 1} \right. \\ &\quad \left. + \underbrace{\left[\mathbf{k}_{p_f, p_f}^{\mathcal{H}_4^{\text{INT}}, e} \right]}_{n_{\text{dof}}^{p,e} \times n_{\text{dof}}^{p,e}} \cdot \underbrace{\left\{ \delta \ddot{\boldsymbol{\theta}}^e \right\}}_{n_{\text{dof}}^{p,e} \times 1} \right) \quad (\text{C-94}) \end{aligned}$$

where we finally recover the tangents

$$\underbrace{\left[\mathbf{k}_{p_f, u}^{\mathcal{H}_4^{\text{INT}}, e} \right]}_{n_{\text{dof}}^{p, e} \times n_{\text{dof}}^{s, e}} = \int_{-1}^1 \left(\left[\frac{3}{n^{f, h^e}} - \frac{2n^{f, h^e}}{1 - (n^{f, h^e})^2} \right] (a_f^{h^e} + g) \frac{n^{s, h^e}}{J^{h^e}} \right) \hat{k}^{h^e} \rho^{\text{fR}, h^e} \times \underbrace{(\beta \Delta t^2) \left\{ \mathbf{B}^{e, p} \right\}^T}_{n_{\text{dof}}^{p, e} \times 1} \underbrace{\left\{ \mathbf{B}^{e, u} \right\}}_{1 \times n_{\text{dof}}^{s, e}} A j^e d\xi \quad (\text{C-95})$$

$$\underbrace{\left[\mathbf{k}_{p_f, u_f}^{\mathcal{H}_4^{\text{INT}}, e} \right]}_{n_{\text{dof}}^{p, e} \times n_{\text{dof}}^{f, e}} = \int_{-1}^1 \hat{k}^{h^e} \rho^{\text{fR}, h^e} \underbrace{\left\{ \mathbf{B}^{e, p} \right\}^T}_{n_{\text{dof}}^{p, e} \times 1} \underbrace{\left\{ \mathbf{N}^{e, u} \right\}}_{1 \times n_{\text{dof}}^{s, e}} A j^e d\xi \quad (\text{C-96})$$

$$\underbrace{\left[\mathbf{k}_{p_f, p_f}^{\mathcal{H}_4^{\text{INT}}, e} \right]}_{n_{\text{dof}}^{p, e} \times n_{\text{dof}}^{p, e}} = \int_{-1}^1 \hat{k}^{h^e} (a_f^{h^e} + g) \frac{\rho^{\text{fR}, h^e}}{K_f^\eta} (\beta \Delta t^2) \underbrace{\left\{ \mathbf{B}^{e, p} \right\}^T}_{n_{\text{dof}}^{p, e} \times 1} \underbrace{\left\{ \mathbf{N}^{e, p} \right\}}_{1 \times n_{\text{dof}}^{p, e}} A j^e d\xi \quad (\text{C-97})$$

C.2 Explicit Integration

C.2.1 Central-Difference (CD) Scheme

In this final section, we show the derivation of the FE formulation for the $(\mathbf{u}-p_f)$ formulation using CD integration. See Section 5.3 of Irwin et al.¹ It is assumed that all time-dependent variables are evaluated at t_{n+1} unless stated otherwise. Therefore, this derivation requires manipulation of variables that have explicit dependence on solution variables.

The balance of momentum of the mixture is trivial to adapt to the CD scheme from the variational form of the equation. The acceleration update of the solid skeleton is readily separable from the variational equation, and the stress update does not explicitly depend on the solution variable at t_{n+1} .

Here, we begin with the balance of mass of the mixture, from which

$$\mathcal{H}_1^{\text{INT}, h} = \int_0^{X=H} r \left(\frac{J^h n^{f, h}}{K_f^\eta} p_f^h + j^h \right) A dX \quad (\text{C-98})$$

We may rewrite the above as

$$\mathcal{H}_1^{\text{INT},h} = \int_0^{X=H} r \left(\frac{J^h n^{\text{f},h}}{K_{\text{f}}^\eta} \left[\dot{p}_{\text{f},n}^h + \frac{\Delta t}{2} \ddot{p}_{\text{f},n}^h + \frac{\Delta t}{2} \ddot{p}_{\text{f}}^h \right] + \frac{\partial v_n^h}{\partial X} + \frac{\Delta t}{2} \frac{\partial a_n^h}{\partial X} + \frac{\Delta t}{2} \frac{\partial a^h}{\partial X} \right) A dX \quad (\text{C-99})$$

which we can write element-wise as

$$\begin{aligned} \mathcal{H}_1^{\text{INT},h} = & \underbrace{\mathbf{A}_e}_{n_{\text{dof}}^{\text{f},e} \times 1} \underbrace{\{\alpha\}^T}_{n_{\text{dof}}^{\text{f},e} \times 1} \cdot \left(\int_{-1}^1 \underbrace{\{N^{e,p}\}^T}_{n_{\text{dof}}^{p,e} \times 1} \left(\frac{J^h n^{\text{f},h}}{K_{\text{f}}^\eta} \left[\dot{p}_{\text{f},n}^h + \frac{\Delta t}{2} \ddot{p}_{\text{f},n}^h \right. \right. \right. \\ & \left. \left. \left. + \frac{\Delta t}{2} \underbrace{\{N^{e,p}\}}_{1 \times n_{\text{dof}}^{p,e}} \underbrace{\{\ddot{\theta}^e\}}_{n_{\text{dof}}^{p,e} \times 1} \right] + \frac{\partial v_n^h}{\partial X} + \frac{\Delta t}{2} \frac{\partial a_n^h}{\partial X} \right. \right. \\ & \left. \left. + \frac{\Delta t}{2} \underbrace{\{B^{e,u}\}}_{1 \times n_{\text{dof}}^{s,e}} \underbrace{\{\ddot{d}^e\}}_{n_{\text{dof}}^{s,e} \times 1} \right) A j^e d\xi \quad (\text{C-100}) \end{aligned}$$

Eq. C-100 can be split into a force vector, a mass matrix and a tangent coupling matrix written as follows:

$$\begin{aligned} \mathbf{f}^{\mathcal{H}_1^{\text{INT},e}} = & \int_{-1}^1 \underbrace{\{N^{e,p}\}^T}_{n_{\text{dof}}^{p,e} \times 1} \left(\frac{J^{h^e} n^{\text{f},h^e}}{K_{\text{f}}^\eta} \left[\dot{p}_{\text{f},n}^{h^e} + \frac{\Delta t}{2} \ddot{p}_{\text{f},n}^{h^e} \right] \right. \\ & \left. + \frac{\partial v_n^{h^e}}{\partial X} + \frac{\Delta t}{2} \frac{\partial a_n^{h^e}}{\partial X} \right) A j^e d\xi \quad (\text{C-101}) \end{aligned}$$

$$\underbrace{\mathbf{m}_{p_{\text{f}},p_{\text{f}}}^{\mathcal{H}_1^{\text{INT},e}}}_{n_{\text{dof}}^{p,e} \times n_{\text{dof}}^{p,e}} = \int_{-1}^1 \frac{J^{h^e} n^{\text{f},h^e}}{K_{\text{f}}^\eta} \frac{\Delta t}{2} \underbrace{\{N^{e,p}\}^T}_{n_{\text{dof}}^{p,e} \times 1} \underbrace{\{N^{e,p}\}}_{1 \times n_{\text{dof}}^{p,e}} A j^e d\xi \quad (\text{C-102})$$

$$\underbrace{\mathbf{k}_{p_{\text{f}},u}^{\mathcal{H}_1^{\text{INT},e}}}_{n_{\text{dof}}^{p,e} \times n_{\text{dof}}^{s,e}} = \int_{-1}^1 \frac{\Delta t}{2} \underbrace{\{N^{e,p}\}^T}_{n_{\text{dof}}^{p,e} \times 1} \underbrace{\{B^{e,u}\}}_{1 \times n_{\text{dof}}^{s,e}} A j^e d\xi \quad (\text{C-103})$$

Next,

$$\begin{aligned}
\mathcal{H}_2^{\text{INT},h} &= \int_0^{X=H} r \frac{1}{K_f^\eta} \frac{\partial p_{f,n}^h}{\partial X} (n^{f,h} \tilde{v}_f^h) A dX \\
&= - \int_0^{X=H} r \frac{1}{K_f^\eta} \frac{\partial p_n^h}{\partial X} \hat{k}^h \left[\frac{\partial p_{f,n}^h}{\partial X} + \rho^{\text{fR},h} (a + g) \right] A dX
\end{aligned} \tag{C-104}$$

which we can write element-wise as

$$\begin{aligned}
\mathcal{H}_2^{\text{INT},h} &= \underbrace{\mathbf{A}_e}_{n_{\text{dof}}^{p,e} \times 1} \underbrace{\{\boldsymbol{\alpha}\}^T}_{n_{\text{dof}}^{p,e} \times 1} \cdot \left(- \int_{-1}^1 \underbrace{\{\mathbf{N}^{e,p}\}^T}_{n_{\text{dof}}^{p,e} \times 1} \left[\frac{1}{K_f^\eta} \frac{\partial p_{f,n}^{h^e}}{\partial X} \hat{k}^{h^e} \left(\frac{\partial p_{f,n}^{h^e}}{\partial X} \right. \right. \right. \\
&\quad \left. \left. \left. + \rho^{\text{fR},h^e} \left[\underbrace{\{\mathbf{N}^{e,u}\}}_{1 \times n_{\text{dof}}^{s,e}} \underbrace{\{\ddot{\mathbf{d}}^e\}}_{n_{\text{dof}}^{s,e} \times 1} + g \right] \right) \right] A j^e d\xi \right)
\end{aligned} \tag{C-105}$$

Eq. C-105 can be split into a force vector and a tangent coupling matrix written as follows:

$$\mathbf{f}^{\mathcal{H}_2^{\text{INT}},e} = - \int_{-1}^1 \underbrace{\{\mathbf{N}^{e,p}\}^T}_{n_{\text{dof}}^{p,e} \times 1} \frac{1}{K_f^\eta} \frac{\partial p_{f,n}^{h^e}}{\partial X} \hat{k}^{h^e} \left(\frac{\partial p_{f,n}^{h^e}}{\partial X} - \rho^{\text{fR},h^e} g \right) A j^e d\xi \tag{C-106}$$

$$\underbrace{\mathbf{k}_{p_{f,u}}^{\mathcal{H}_2^{\text{INT}},e}}_{n_{\text{dof}}^{p,e} \times n_{\text{dof}}^{s,e}} = - \int_{-1}^1 \frac{1}{K_f^\eta} \frac{\partial p_{f,n}^{h^e}}{\partial X} \hat{k}^{h^e} \rho^{\text{fR},h^e} \underbrace{\{\mathbf{N}^{e,p}\}^T}_{n_{\text{dof}}^{p,e} \times 1} \underbrace{\{\mathbf{N}^{e,u}\}}_{1 \times n_{\text{dof}}^{s,e}} A j^e d\xi \tag{C-107}$$

Given that $\mathcal{H}_3^{\text{INT}}$ has no explicit dependence on solution variables at t_{n+1} , we omit the trivial derivation and refer the reader to Irwin et al.¹ Next,

$$\mathcal{H}_4^{\text{INT},h} = \int_0^{X=H} \frac{\partial r}{\partial X} \hat{k}^h \rho^{\text{fR},h} (a + g) A dX \tag{C-108}$$

which we can write element-wise as

$$\mathcal{H}_4^{\text{INT},h} = \underbrace{\mathbf{A}_e}_{n_{\text{dof}}^{p,e} \times 1} \underbrace{\{\alpha\}^T}_{n_{\text{dof}}^{p,e} \times 1} \cdot \left(\int_{-1}^1 \underbrace{\{\mathbf{B}^{e,p}\}^T}_{n_{\text{dof}}^{p,e} \times 1} \hat{k}^{h^e} \rho^{\text{fR},h^e} \left[\underbrace{\{\mathbf{N}^{e,u}\}}_{1 \times n_{\text{dof}}^{s,e}} \underbrace{\{\ddot{\mathbf{d}}^e\}}_{n_{\text{dof}}^{s,e} \times 1} + g \right] \times \right. \\ \left. A j^e d\xi \right) \quad (\text{C-109})$$

Eq. C-109 can be split into a force vector and a tangent coupling matrix written as follows:

$$\mathbf{f}^{\mathcal{H}_4^{\text{INT},e}} = \int_{-1}^1 \underbrace{\{\mathbf{B}^{e,p}\}^T}_{n_{\text{dof}}^{p,e} \times 1} \hat{k}^{h^e} \rho^{\text{fR},h^e} g A j^e d\xi \quad (\text{C-110})$$

$$\underbrace{\mathbf{k}_{p_f,u}^{\mathcal{H}_4^{\text{INT},e}}}_{n_{\text{dof}}^{p,e} \times n_{\text{dof}}^{s,e}} = \int_{-1}^1 \hat{k}^{h^e} \rho^{\text{fR},h^e} \underbrace{\{\mathbf{B}^{e,p}\}^T}_{n_{\text{dof}}^{p,e} \times 1} \underbrace{\{\mathbf{N}^{e,u}\}}_{1 \times n_{\text{dof}}^{s,e}} A j^e d\xi \quad (\text{C-111})$$

When pressure stabilization is enabled, we must apply the time discretization update to the $\partial \dot{p}_f / \partial X$ term:

$$\mathcal{H}^{\text{stab},h} = \int_0^{X=H} \frac{\partial r}{\partial X} \alpha^{\text{stab}}(F_{11}^h)^{-1} \frac{\partial \dot{p}_f^h}{\partial X} A dX \quad (\text{C-112})$$

In the CD scheme, we may rewrite the above as

$$\mathcal{H}^{\text{stab},h} = \int_0^{X=H} \frac{\partial r}{\partial X} \alpha^{\text{stab}}(F_{11}^h)^{-1} \frac{\partial (\dot{p}_{f,n}^h + \frac{\Delta t}{2} [\ddot{p}_{f,n} + \ddot{p}_f])}{\partial X} A dX \quad (\text{C-113})$$

which we can formulate element-wise as

$$\mathcal{H}^{\text{stab},h} = \underbrace{\mathbf{A}_e}_{n_{\text{dof}}^{p,e} \times 1} \underbrace{\{\alpha\}^T}_{n_{\text{dof}}^{p,e} \times 1} \cdot \left(\int_{-1}^1 \underbrace{\{\mathbf{B}^{e,p}\}^T}_{n_{\text{dof}}^{p,e} \times 1} \alpha^{\text{stab}}(F_{11}^{h^e})^{-1} \left[\frac{\partial \dot{p}_{f,n}^h}{\partial X} + \frac{\Delta t}{2} \frac{\partial \ddot{p}_{f,n}}{\partial X} \right. \right. \\ \left. \left. + \frac{\Delta t}{2} \underbrace{\{\mathbf{B}^{e,p}\}}_{1 \times n_{\text{dof}}^{p,e}} \cdot \underbrace{\{\ddot{\theta}^e\}}_{n_{\text{dof}}^{p,e}} \right] A j^e d\xi \right) \quad (\text{C-114})$$

Lastly, Eq. C-114 can be split into a force vector and a mass matrix as follows:

$$\underbrace{\left\{ \mathbf{f}^{\mathcal{H}^{\text{stab},e}} \right\}}_{n_{\text{dof}}^{p,e} \times 1} = \int_{-1}^1 \underbrace{\left\{ \mathbf{B}^{e,p} \right\}^T}_{n_{\text{dof}}^{p,e} \times 1} \alpha^{\text{stab}}(F_{11}^{h^e})^{-1} \left(\frac{\partial \dot{p}_{\text{f},n}^{h^e}}{\partial X} + \frac{\Delta t}{2} \frac{\partial \ddot{p}_{\text{f},n}^{h^e}}{\partial X} \right) A j^e d\xi \quad (\text{C-115})$$

$$\underbrace{\left[\mathbf{m}_{p_{\text{f}},p_{\text{f}}}^{\mathcal{H}^{\text{stab},e}} \right]}_{n_{\text{dof}}^{p,e} \times n_{\text{dof}}^{p,e}} = \int_{-1}^1 \alpha^{\text{stab}}(F_{11}^{h^e})^{-1} \frac{\Delta t}{2} \underbrace{\left\{ \mathbf{B}^{e,p} \right\}^T}_{n_{\text{dof}}^{p,e} \times 1} \underbrace{\left\{ \mathbf{B}^{e,p} \right\}}_{1 \times n_{\text{dof}}^{p,e}} A j^e d\xi \quad (\text{C-116})$$

List of Symbols, Abbreviations, and Acronyms

TERMS:

1-D	one-dimensional
3-D	three-dimensional
ARL	Army Research Laboratory
CD	central difference
CFL	Courant-Friedrichs-Lewy
DEVCOM	US Army Combat Capabilities Development Command
ECM	extra-cellular matrix
f	fluid
FE	finite element
FSI	fluid-structure interaction
s	solid
TPM	theory of porous media

MATHEMATICAL SYMBOLS:

e	internal energy per unit mass
$\mathbf{F}, F_{iJ}$	deformation gradient
p	Cauchy pressure
t	time
$\mathbf{u}, u_k$	displacement
V	volume
$\mathbf{x}, x_k$	Cartesian spatial coordinates
$\mathbf{X}, X_K$	Cartesian reference coordinates
η	entropy per unit mass
ψ	free energy per unit mass
θ	temperature

ρ	mass density
$\boldsymbol{\sigma}, \sigma_{ij}$	Cauchy stress
$\boldsymbol{v}, v_k$	velocity

1 DEFENSE TECHNICAL
(PDF) INFORMATION CTR
DTIC OCA

1 DEVCOM ARL
(PDF) FCDD RLB CI
TECH LIB

36 DEVCOM ARL
(PDF) FCDD RLA B
R BECKER
C HOPPEL
P GILLICH
A TONGE
FCDD RLA HC
J CRONE
J KNAP
FCDD RLA MB
G GAZONAS
FCDD RLA T
B LOVE
M FERMEN-COKER
S SCHOENFELD
FCDD RLA TA
S TURNAGE
C WILLIAMS
FCDD RLA TB
S ALEXANDER
R BANTON
T BAUMER
D CASEM
J CLAYTON
B FAGAN
A GOERTZ
C HAMPTON
R KARGUS
M KLEINBERGER
D KRAYTERMAN
J MCDONALD
C MEREDITH
T PLAISTED
K RAFAELS
R REGUEIRO
S SATAPATHY
T WEERASOORIYA
C WEAVER
S WOZNIAK
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