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AN UNSTRUCTURED MESH TRANSFORMATION FDTD METHOD FOR THE TM MODE EQUATIONS

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14. ABSTRACT
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An Unstructured Mesh Transformation FDTD Method for the TM Mode Equations

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***Abstract*—We introduce a new finite-difference time-domain (FDTD) method for solving the TM (transverse magnetic) mode reduction of Maxwell's equations with complicated material interfaces. The method uses an unstructured quadrilateral mesh intended to conform to the complicated geometry of material interfaces. Using a quadrilateral to unit-square local coordinate transformation, we remap all the field components from the physical grid to a computational grid consisting of unit-squares. On the computational grid of unit-squares we update the fields in time using the usual FDTD field updates. The local coordinate remap, however, changes a scalar material on the physical space to an anisotropic material on computational space; we handle the mixed components of the anisotropic material by interpolating the non-colocated field components. Finally, we resolve material parameter jumps across interfaces using a harmonic-mean. By testing our method on the TM$_{0,1}$ mode in partially filled infinite cylindrical cavity, we verify that the method converges, is second order accurate, and maintains stability.**

I. Introduction

The FDTD method [1] is one the most popular computational methods used for solving electromagnetic problems. Rightfully so, it has many desirable properties such as explicit time-stepping and simple parallelization. However, FDTD does suffer from key problem dependent issues; the most pertinent of which is staircasing of boundary conditions and material interfaces. The main tools to address staircasing in FDTD are methods which conform the grid to perfect electric conductor (PEC) boundary conditions [2] and handle interfaces with sub-cell averaging techniques [3].

Nonetheless, these fixes skirt the crux of the issue that structured rectangular grids do not conform to complicated geometry. To address it directly, researchers have developed several unstructured grid approaches which use FDTD on conformal grids produced by meshing tools. These include unstructured triangle-polygonal meshes [4] and non-orthogonal mixed-polyhedral unstructured meshes [5]; all of which perform the FDTD update on the physical grid (as opposed to a computational grid).

Contrasting with prior unstructured conforming mesh approaches, our approach borrows a technique from the finite-element community [6] in which a physical grid element is mapped to the unit-square where time updates will take place. Then we transform back to the physical grid to measure fields.

For this conference paper we describe Maxwell's equations under a coordinate transformation, introduce the unit-square to quadrilateral transformation (called the Piola transformation that's equivalent to transformation optics), describe our TM mode equation FDTD method on an unstructured quadrilateral grid, and verify our method. We finalize by summarizing the results in a conclusion.

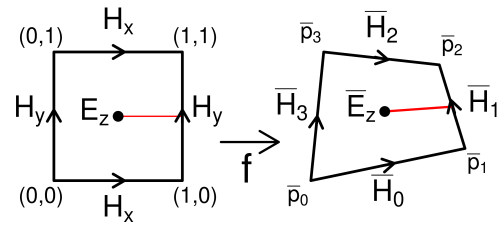

Fig. 1. Depiction of the unit-square to quadrilateral bilinear transformation used in unstructured mesh FDTD method. Equivalent Faraday law integration paths for physical and computational cell are shown in red.

II. Maxwell's Equations Under a Coordinate Transformation

Maxwell's equations

$$\partial_t \mathbf{D} = \nabla \times \mathbf{H}, \quad \partial_t \mathbf{B} = -\nabla \times \mathbf{E} \tag{1}$$

with the constitutive relations $\mathbf{H} = \mu^{-1}\mathbf{B}$ and $\mathbf{E} = \epsilon^{-1}\mathbf{D}$ are invariant under coordinate transformations. That is, under a map $f : \mathbb{R}^3 \to \mathbb{R}^3$ with Jacobian $\mathcal{J} = [\partial_x f, \partial_y f, \partial_z f]$, the coordinate transformed fields $\overline{\mathbf{E}} = \mathcal{J}^{-T}\mathbf{E}$ and $\overline{\mathbf{H}} = \mathcal{J}^{-T}\mathbf{H}$ satisfy Maxwell's equations with the transformed constitutive parameters $\overline{\mu} = \frac{\mathcal{J}\mu\mathcal{J}^T}{\det\mathcal{J}}$ and $\overline{\epsilon} = \frac{\mathcal{J}\epsilon\mathcal{J}^T}{\det\mathcal{J}}$. Note that the tranformation may make $\overline{\mu}$ and $\overline{\epsilon}$ anisotropic.

For the TM mode equations ($\partial_z = 0$, $z = 0$ and $H_z = E_x = E_y = 0$) consisting only of field components (H_x, H_y, E_z), the transformed fields and constitutive parameters become

$$\begin{gathered}\overline{\mathbf{H}} = \mathcal{J}^{-T}\mathbf{H}, \quad \overline{E}_z = E_z \\ \overline{\mu} = \frac{\mathcal{J}\mu\mathcal{J}^T}{\det\mathcal{J}}, \quad \overline{\epsilon} = \frac{\epsilon}{\det\mathcal{J}}.\end{gathered} \tag{2}$$

where $\mathcal{J}$ is now a 2×2 matrix. Using this we may apply the inverse of the Piola transformation to remap every cell (each of which has its own local transformation) from an unstructured quadrilateral mesh to a computational grid consisting only of unit-squares. The bilinear transformation together with Piola transformation, depicted in Fig. 1, is given by

$$\mathbf{f}(x,y) = \overline{\mathbf{p}}_0 + (\overline{\mathbf{p}}_1 - \overline{\mathbf{p}}_0)x + (\overline{\mathbf{p}}_3 - \overline{\mathbf{p}}_0)y + \overline{\mathbf{m}}xy \tag{3}$$

$$\mathcal{J}_{(x,y)} = [\overline{\mathbf{p}}_1 - \overline{\mathbf{p}}_0 + \overline{\mathbf{m}}y, \overline{\mathbf{p}}_3 - \overline{\mathbf{p}}_0 + \overline{\mathbf{m}}x], \tag{4}$$

where $\overline{\mathbf{m}} = \overline{\mathbf{p}}_2 - \overline{\mathbf{p}}_1 - (\overline{\mathbf{p}}_3 - \overline{\mathbf{p}}_0)$. Once on the computational grid of unit-squares, we apply an FDTD update, the details of which are described in the next section.

III. TM Mode FDTD Method on an Unstructured Quadrilateral Grid

Our TM Mode FDTD method places $\overline{E}_z$ at quadrilateral midpoint (average of all corners) and places $\overline{\mathbf{H}}$ tangential components at edge midpoints (continuity of $\overline{\mathbf{H}}_{\text{tangent}}$); see Fig. 1 for a depiction of field component locations. In order to consistently orient the edges of the quadrilateral grid (so that we may compute the circulation of H fields), we use the algorithm described in [7]. Then we map the fields and material parameters from the physical grid to the computational grid using the inverse of the transform in (2) with Jacobian (4): $\mathbf{H} = \mathcal{J}^T\overline{\mathbf{H}}$, $E_z = \overline{E}_z$, $\mu = (\det \mathcal{J})\mathcal{J}^{-1}\overline{\mu}\mathcal{J}^{-T}$ and $\epsilon = \overline{\epsilon}\det \mathcal{J}$. On the unit-square grid we update the fields with

$$D_z|^{i+1} = D_z|^i + \Delta t \sum_{i=0}^{3} \sigma_i H|_i^{i+1/2} \tag{5}$$

$$B|^{i+3/2} = B|^{i+1/2} - \Delta t(E_z|_1^{i+1} - E_z|_0^{i+1}) \tag{6}$$

$$E_z = \epsilon^{-1} D_z, \quad H = \mu_{\text{eff}}^{-1} B \tag{7}$$

where σ_i is the orientation the square's edge and the index 1 refers to the square on the positive side of the edge's orientation; 0 means the negative side. The effective permeability is given by the harmonic average $\mu_{\text{eff}}^{-1} = \frac{1}{2}(\mu_0^{-1} + \mu_1^{-1})$. In order to handle the anisotropic part of μ_{eff}^{-1} we interpolate non-colocated fields to their required location. For example, to get $H_y = \mu_{yx}^{-1} B_x + \mu_{yy}^{-1} B_y$ we interpolate B_x to the location of B_y. Once the time-updates are finished we switch back the to physical space using the forward transform in (2).

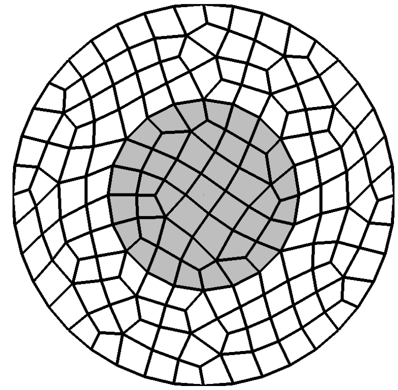

Fig. 2. Mesh of infinite cylindrical cavity with $\epsilon = 1$ and PMC boundary at $\rho = 1$. For $0 \leq \rho < 1/2$ (gray shaded area) it is filled $\mu_0 = 1$ material. For $1/2 < \rho < 1$, it is filled $\mu_1 = 2$.

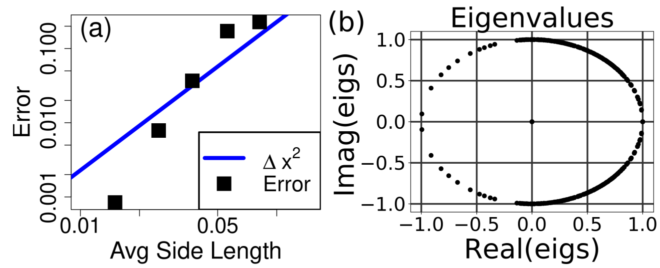

Fig. 3. Plot (a) shows convergence of the $\text{TM}_{0,1}$ solution to partially filled cylindrical cavity. The observed L^2 error for several mesh sizes are plotted along with a reference 2nd order convergence line in blue. Plot (b) shows the method's eigenvalues for the mesh in Fig. 2; all lie on the unit circle.

IV. Method Verification

We verify the method converges, is second order accurate, and is stable by testing it on a $\text{TM}_{0,1}$ mode solution in a partially filled infinite cylindrical cavity with perfect magnetic conductor (PMC) boundaries. The cavity of interest with a sample quadrilateral mesh, generated by gmsh [8], is depicted in Fig. 2. An exact solution for E_z in the cavity is given by

$$E_z^i(\rho, \phi, t) = \left[A^i J_0(\beta^i \rho) + B^i Y_0(\beta^i \rho)\right] cos(\omega t) \tag{8}$$

where J_0 and Y_0 are the 0 order Bessel functions of the 1st kind and 2nd kind, respectively. The constants A^i, B^i, β^i, $\omega = \frac{\beta^i}{\sqrt{\mu_i}}$ are determined by μ_i, the interface condition, and the PMC boundary condition [9]. By varying the mesh size and comparing the numeric solution to the exact solution in the L^2 norm at time $t = 3$, we find that the method converges with 2nd order accuracy, results are plotted in Fig. 3 (a). Furthermore, we compute the eigenvalues of our method for the mesh in Fig. 2 and plot them in Fig. 3 (b). We find they all lie on the unit circle, confirming stability for the $\text{TM}_{0,1}$ problem.

V. Conclusion

We introduced an unstructured mesh FDTD method that uses a local coordinate transformation to perform field updates on a computational grid consisting of unit-squares. We then verify it with an infinite cylindrical cavity that is filled with two different permeabilities. We find that it converges, is 2nd order accurate, and is numerically stable. For future work we will investigate adding frequency dependent material to the method and extending it to three-dimensions.

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