SOME THEORETICAL CONSIDERATIONS IN THE DESIGN OF DIRECTIONAL-NULL-TYPE ANTENNAS

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ABSTRACT

The essential characteristics of directional-null-type antenna patterns as produced by the superposition of two primary antenna patterns phased 180 degrees apart are derived mathematically for the purpose of determining the design parameters and estimating the effect of their variation. Both omni-directional (in one plane) and directional types of primary antenna elements are considered. Equations are derived for the rate of change of field strength with azimuth angle near the central null and for the angular position of the first maximum. The effects of squinting the beams by introducing an angular displacement between the directions of maximum gain are considered as well as the effects of mutual impedance and inequalities of current amplitude, phase, and primary antenna patterns.

PROBLEM STATUS

This analysis developed as a by-product of work on problem S1234. Work is continuing.

SOME THEORETICAL CONSIDERATIONS IN THE DESIGN OF DIRECTIONAL-NULL-TYPE ANTENNAS

INTRODUCTION

In the production of an antenna pattern containing a sharp null in the forward direction, two antenna beams may be generated which are phased 180 degrees apart with the superposition of these two beams producing a sharp null. There are two general methods of arranging the angular position of the beams which, for convenience in reference, may be called Squinting and Separation.

By "Squinting" is meant that the two beams are turned outward with respect to each other so that the maxima of the two beams occur at slightly different values of the azimuth angle θ. The null then appears in the region of overlap of the two beams.

By "Separation" is meant that the two beams are pointed in the same angular direction but that the apparent sources of the two beams are separated by a considerable distance in terms of wavelengths. For example, two antennas set side by side, each antenna producing a pattern having a maximum in the same direction, would produce a null pattern in that same direction providing the antennas were fed 180 degrees out of phase with each other.

It is proposed in this paper to investigate further the characteristics of the patterns produced by these two general methods and to deduce the effect of varying the design parameters on such quantities as sharpness of the null and position of the maxima. Only the pattern in one principal plane with linear polarization in free space will be considered in this paper.

SEPARATION OF DIPOLES

First, the simple case of two vertical half-wave dipoles phased 180 degrees apart and separated a distance d as in Figure 1 will be considered.

At some large distance r, in the horizontal plane off the center line at an angle θ, the signal strength due to dipole B will be given by the following equation:

$$E_B = \frac{KP^{\frac{1}{2}}}{r} e^{j\omega(t - \frac{r}{c})} \qquad (1)$$

where K is a constant dependent on the choice of units. Similarly, the signal strength due to dipole A will be given by the equation

$$E_A = \frac{KP^{\frac{1}{2}}}{r+a} e^{j\omega(t - \frac{r+a}{c})} \qquad (2)$$

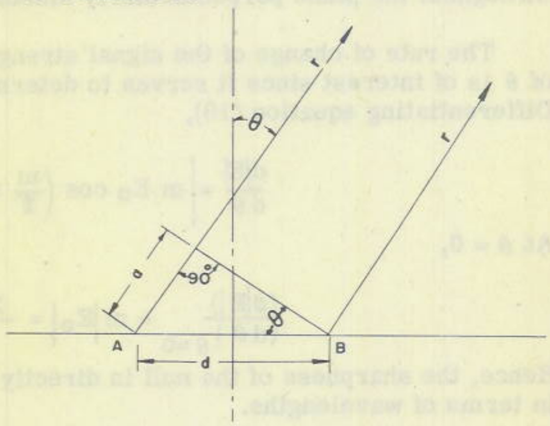

Fig. 1 - Separated Dipoles

assuming equal power is fed to each antenna. From Figure 1,

$$a = d \sin\theta \quad (3)$$

Since the dipoles are phased 180 degrees apart, the resultant signal strength is the difference of E_B and E_A or

$$E = E_B - E_A = \frac{KP^{\frac{1}{2}}}{r} e^{j\omega(t - \frac{r}{c})} \left(1 - e^{-j\omega\frac{a}{c}}\right) \quad (4)$$

since, for a sufficiently distant point, (r + a) may be set equal to r in the amplitude term but not in the phase term.

Combining equations (3) and (4), and expressing the magnitude of E in trigonometric form,

$$\left|E\right| = \left| E_0 \left\{ \left[1 - \cos\left(\frac{\omega d \sin\theta}{c}\right)\right]^2 + \left[\sin\left(\frac{\omega d \sin\theta}{c}\right)\right]^2 \right\}^{\frac{1}{2}} \right| \quad (5)$$

which reduces to

$$\left|E\right| = \left|E_0 \left[2 - 2\cos(m \sin\theta)\right]^{\frac{1}{2}}\right| \quad (6)$$

with

$$\left|E_0\right| \equiv \left|\frac{KP^{\frac{1}{2}}}{r} e^{j\omega(t - \frac{r}{c})}\right| \quad (7)$$

and

$$m \equiv 2\pi d/\lambda = \omega d/c \quad (8)$$

The trigonometric identity,

$$1 - \cos x = 2 \sin^2 x/2 \quad (9)$$

further simplifies the equation for E to

$$\left|E\right| = \left|2E_0\right| \sin\left(\frac{m}{2} \sin\theta\right)\Big| . \quad (10)$$

It is seen from equation (10) that E is zero when θ is zero and hence, a null exists throughout the plane perpendicularly bisecting the plane of the two dipoles.

The rate of change of the signal strength with change of azimuth angle near the origin of θ is of interest since it serves to determine the width of the null for strong signals. Differentiating equation (10),

$$\frac{d|E|}{d\theta} = \left| m E_0 \cos\left(\frac{m}{2} \sin\theta\right) \cos\theta \right| . \quad (11)$$

At $\theta = 0$,

$$\left(\frac{d|E|}{d\theta}\right)_{\theta=0} = m\left|E_0\right| = \frac{2\pi d}{\lambda}\left|E_0\right| . \quad (12)$$

Hence, the sharpness of the null is directly proportional to the spacing of the dipoles in terms of wavelengths.

The maximum angular width of the null, given by twice the angle θ_m representing the position of the first maximum of the null pattern, is also of interest. This angle θ_m is found by setting equation (11) equal to zero.

$$mE_0 \cos\theta \cos\left(\frac{m}{2} \sin\theta\right) = 0. \tag{13}$$

The roots, from inspection, are

$$\theta_1 = (2n+1)\frac{\pi}{2}, \qquad n = 0, 1, 2, 3, \ldots \tag{14}$$

and

$$\theta_2 = \arcsin\frac{(2n+1)\pi}{m}, \quad n = 0, 1, 2, \ldots \tag{15}$$

The root θ_1, is not immediately of interest since its smallest value is 90 degrees. The root θ_2 gives maximum points of the pattern as may be seen by inspection of equation (10). The lowest root, n = 0, gives

$$\theta_m = \arcsin\frac{\pi}{m} \tag{16}$$

which, for small angles, is approximately

$$\theta_m \doteq \frac{\pi}{m} = \frac{\lambda}{2d} \tag{17}$$

A plot of θ_m from equation (16) versus the separation in wavelength is given in Figure 2. From equations (6) and (15), it may be seen that for these values of azimuth angle the signal strength is equal to $2E_0$ and, hence, all of the maxima of the null pattern have the same amplitude. The minima of the pattern result when equation (10) equals zero,

or

$$\frac{m}{2}\sin\theta_3 = n'\pi, \qquad n' = 0, 1, 2, \ldots \tag{18}$$

giving

$$\theta_3 = \frac{2n'\pi}{m} = \frac{n'\lambda}{d} \tag{19}$$

or

$$\theta_3 = \frac{2n'}{2n+1}\,\theta_2 \tag{20}$$

For the values of n equal to zero (corresponding to the first maximum) and n equal to one (corresponding to the first minimum after the central null), θ_3 equals $2\,\theta_2$. Thus, the first few maxima and minima alternate at almost equal intervals, the minima producing zero field strength and the maxima $2E_0$. The pattern is symmetrical about $\theta = 0$, of course. It is interesting to note that, if the two dipoles had been connected in phase, the positions of maxima and minima would have been exactly interchanged.

DIPOLES IN FRONT OF A PLANE REFLECTOR

The effect of a plane reflector behind the dipoles may be treated by the method of images in which the reflector is replaced by the images of the dipoles, 180 degrees out of phase, and placed as far behind the reflector as the dipoles are in front of it. Referring to Figure 3, the field strength may be written down as the sum of contributions from the

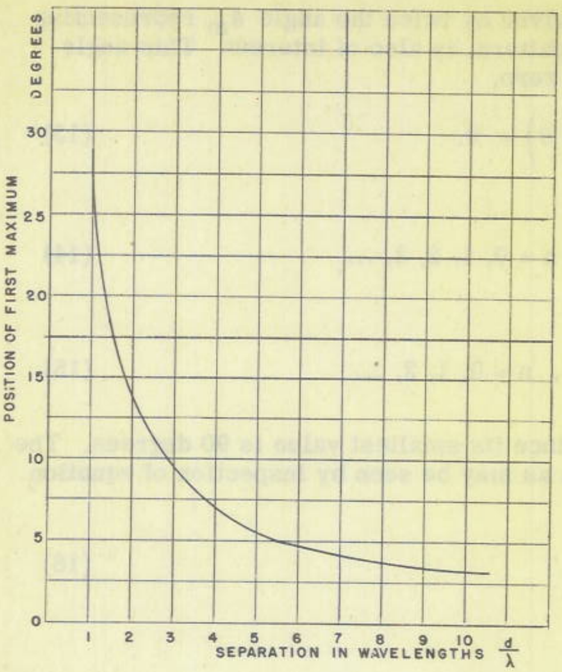

Fig. 2 - Position of First Maximum for Separated Dipoles

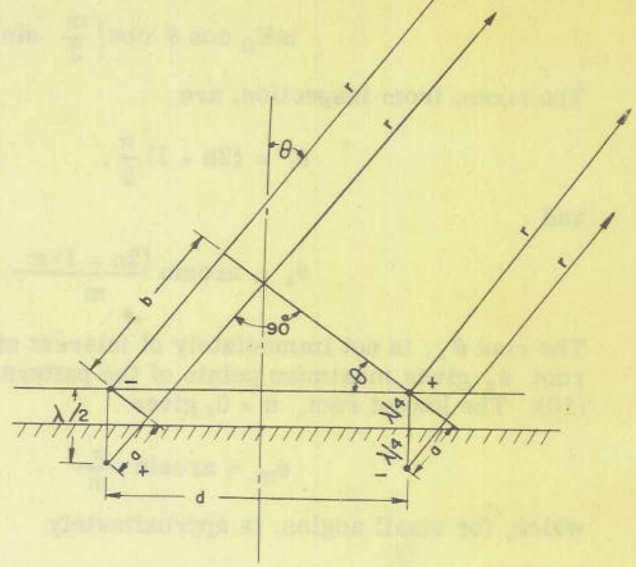

Fig. 3 - Separated Dipoles in Front of a Reflector

dipoles and their images. Thus

$$E = E_1 e^{j\omega(t - \frac{r}{c})} - E_1 e^{j\omega(t - \frac{r}{c} - \frac{a}{c})} - E_1 e^{j\omega(t - \frac{r}{c} - \frac{b}{c})} + E_1 e^{j\omega(t - \frac{r}{c} - \frac{b}{c} - \frac{a}{c})}$$

$$= E_1 e^{j\omega(t - \frac{r}{c})} \left[1 - e^{-j\omega\frac{a}{c}} - e^{-j\omega\frac{b}{c}} + e^{-(j\frac{\omega}{c})(b + a)}\right]$$

$$E = E_0 \left[1 - e^{-j\omega\frac{a}{c}}\right]\left[1 - e^{-j\omega\frac{b}{c}}\right] \qquad (21)$$

$$E = E_0\left[1 - \cos\frac{\omega a}{c} + j \sin\frac{\omega a}{c}\right]\left[1 - \cos\frac{\omega b}{c} + j \sin\frac{\omega b}{c}\right]$$

$$|E| = |E_0| \left[2 - 2\cos\frac{\omega a}{c}\right]^{\frac{1}{2}}\left[2 - 2\cos\frac{\omega b}{c}\right]^{\frac{1}{2}}$$

$$= 4|E_0| \sin\frac{\omega a}{2c} \sin\frac{\omega b}{2c}$$

$$= 4|E_0| \sin\left(\frac{\pi}{2}\cos\theta\right) \sin\left(\frac{\pi d}{\lambda}\sin\theta\right) \qquad (22)$$

in which the spacing from the reflector has been chosen as a quarter-wavelength. The field strength then may be represented by the same term as previously obtained for the free dipoles in equation (10) multiplied by $\sin\left(\frac{\pi}{2}\cos\theta\right)$ which is the pattern of each dipole in front of a reflector.

The slope is found as before by taking the derivative of equation (22), obtaining

$$\frac{dE}{d\theta} = 4\left|E_0\right|\left[\sin\left(\frac{\pi}{2}\cos\theta\right)\cos\left(\frac{m}{2}\sin\theta\right)\frac{m}{2}\cos\theta - \sin\left(\frac{m}{2}\sin\theta\right)\cos\left(\frac{\pi}{2}\cos\theta\right)\frac{\pi}{2}\sin\theta\right]. \quad (23)$$

At $\theta = 0$, $\sin\theta = 0$ and

$\frac{dE}{d\theta} = m\left|E_0\right| = \frac{2\pi d}{\lambda}\left|E_0\right|$ as before. The maxima occur, however, when

$\frac{dE}{d\theta} = 0$, giving

$$\frac{m}{\pi}\tan\left(\frac{\pi}{2}\cos\theta\right)\cot\left(\frac{m}{2}\sin\theta\right) = \tan\theta. \quad (24)$$

This equation may be solved for the positions of the first maxima as in Figure 4 in which only small values of d have been used. For separations greater than a wavelength, the positions are given with adequate accuracy by the same equation (16) as for the dipoles without a reflector. Note, however, that for half-wave spacing the maximum is at 45 degrees (.785 radians) instead of 90 degrees. The minima are the same as before except that a zero always occurs at 90 degrees due to the first term of (22).

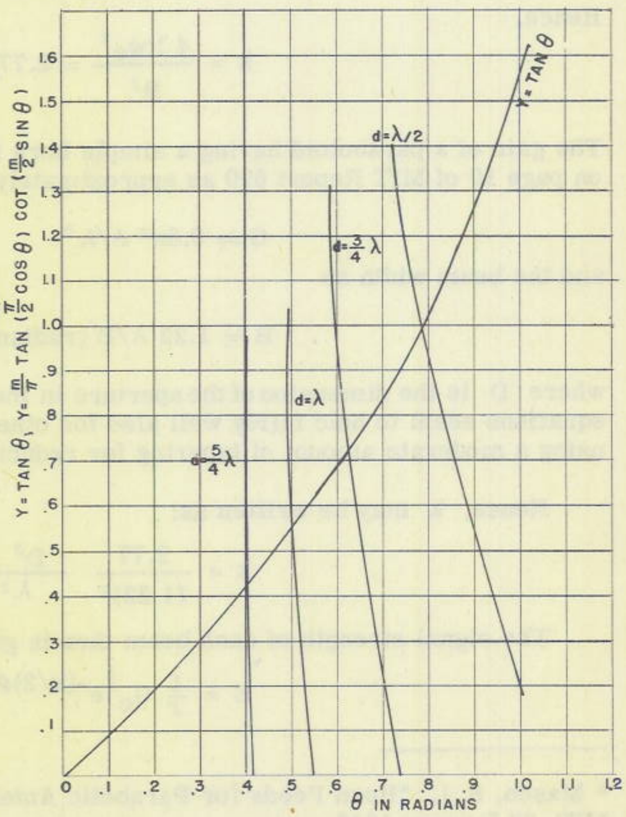

Fig. 4 - Graphical Solution of

$$\tan\theta_m = \left(\frac{m}{\pi}\tan\frac{\pi}{2}\cos\theta\right)\cot\left(\frac{m}{2}\sin\theta\right)$$

SEPARATION OF NARROW BEAMS

It will be assumed here that the shape of the main lobe in each beam is given by

$$E \propto e^{-(k/2)\theta^2} \tag{25}$$

This form, in addition to being simple and relatively easy to handle mathematically, has some claim to generality since many types of actual narrow-beam antenna patterns approximate this form over at least the region of maximum gain. From equation (2)of MTI Radiation Laboratory Report 690.*

$$G \doteq G_O C^{-k\theta^2}, \tag{26}$$

where G_O is the maximum absolute power gain and C and k are arbitrary constants. With uniform illumination and plane phase across the aperture, the maximum absolute gain is calculated to be

$$G_O \doteq 4\pi A/\lambda^2 \tag{27}$$

where A is the aperture area. Choosing C = e for convenience in computation, k is then determined by the beam width at the half-power points B. At $G = \frac{1}{2} G_O$, $\theta = (B/2)$ radians,

$$\tfrac{1}{2} = e^{-k(B/2)^2}$$

$$\log_e \tfrac{1}{2} = -\log_e 2 = -k(B/2)^2 \tag{28}$$

Hence,

$$k = \frac{4 \log_e{}^2}{B^2} = 2.77/B^2 . \tag{29}$$

The gain of a paraboloid having a simple horn feed and using tapered illumination is given on page 10 of MIT Report 690 as approximately

$$G \doteq 0.6\pi^2 A/\lambda^2$$

and the beam width as

$$B \doteq 1.22 \lambda/D \text{ (radians)} \tag{30}$$

where D is the dimension of the aperture in the plane in which B is measured. These equations seem to hold fairly well also for other types of simple narrow-beam antennas using a moderate amount of tapering for reduction of side lobes.

Hence, k may be written as:

$$k = \frac{2.77}{(1.22)^2} \frac{D^2}{\lambda^2} = 1.86 D^2/\lambda^2 . \tag{31}$$

The signal strength of each beam then is given by the equation

$$E = \frac{f}{r} G_O^{\frac{1}{2}} e^{-(k/2)\theta^2} e^{j\omega(t - \frac{r}{c})} \tag{32}$$

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* Mason, S. J., "Horn Feeds for Parabolic Antennas," Report 690, Radiation Laboratory, MIT, 22 January 1946.

where f is a function of the r-f power supplied to the antenna forming the beam.

If, as in the case of the two dipoles, the apparent sources of the two beams of radiation, phased 180 degrees apart, are separated by a distance of d, the resulting field strength will be the difference

$E_B - E_A$, or

$$E = \frac{f}{r} G_o^{\frac{1}{2}} e^{-(k/2)\theta^2}\left[e^{j\omega(t - \frac{r}{c})} - e^{j\omega(t - \frac{r+a}{c})}\right] \quad (33)$$

or

$$E = E_o e^{-(k/2)\theta^2}\left[1 - e^{-(j\omega d/c)\sin\theta}\right] \quad (34)$$

where

$$E_o = \frac{f}{r} G_o^{\frac{1}{2}} e^{-j\omega(t - \frac{r}{c})} . \quad (35)$$

Manipulating this equation as in the case of the dipoles, equation (34) reduces to

$$\left|E\right| = \left|E_o\, 2\, e^{-(k/2)\theta^2} \sin\left(\frac{m}{2}\sin\theta\right)\right|. \quad (36)$$

This equation is similar to the corresponding one for the dipoles (equation 10) except for the presence of the factor $e^{-(k/2)\theta^2}$ corresponding to the pattern of each primary antenna.

Taking the derivative with respect to θ,

$$\frac{d|E|}{d\theta} = \left|E_o\, 2e^{-(k/2)\theta^2}\left\{-k\theta \sin\left(\frac{m}{2}\sin\theta\right) + \frac{m}{2}\cos\left(\frac{m}{2}\sin\theta\right)\cos\theta\right\}\right| \quad (37)$$

At $\theta = 0$,

$$\frac{d|E|}{d\theta} = \left|E_o\, m\right| = \left|E_o\right|\left|\frac{2\pi d}{\lambda}\right| . \quad (38)$$

It is interesting to note that the slope at the origin, as in the case of the separated dipoles, is directly proportional to the separation in wavelengths. However, in this case of narrow beams, the slope also depends on the aperture size since it is proportional to E_o which, from equation (35), is proportional to the square root of the power gain of the antennas forming the two component beams.

In order to find the positions of the maxima, equation (37) is set equal to zero, giving the equation

$$\theta_m = \frac{m}{2k} \cos\theta_m \cot\left(\frac{m}{2}\sin\theta_m\right) \quad (39)$$

This equation may be solved graphically as in Figure 5 giving the results of Figure 6. The position angle of the first maximum of the null pattern becomes rapidly smaller as the separation is increased as long as the aperture dimension is small. If larger apertures are used, the position angle varies but slowly with the separation. For large separation, the position angle is practically independent of the size of aperture. The explanation of this effect is that, as the separation increases, the cotangent-like curves of Figure 5 become progressively steeper until in the limit they intersect the $y = 2k\theta$ curves at essentially the same points as those at which the cotangent-like curves go through their first zeros. Hence, the limit approached by the curves of Figure 6 with increasing separation is $\theta_m = \lambda/2d$ as in the case of separated dipoles.

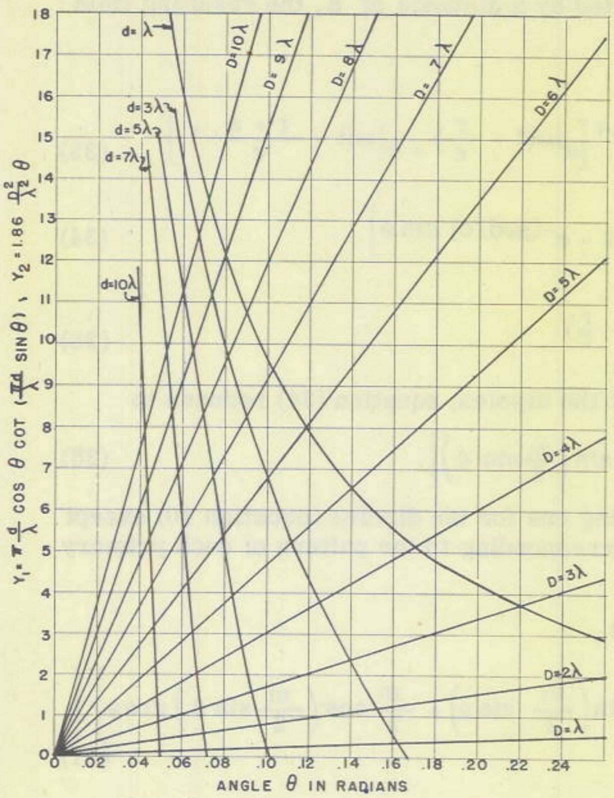

Fig. 5 - Graphical Solution of $\theta_m = \frac{m}{2k} \cos \theta_m \cot \frac{m}{2} \sin\theta$

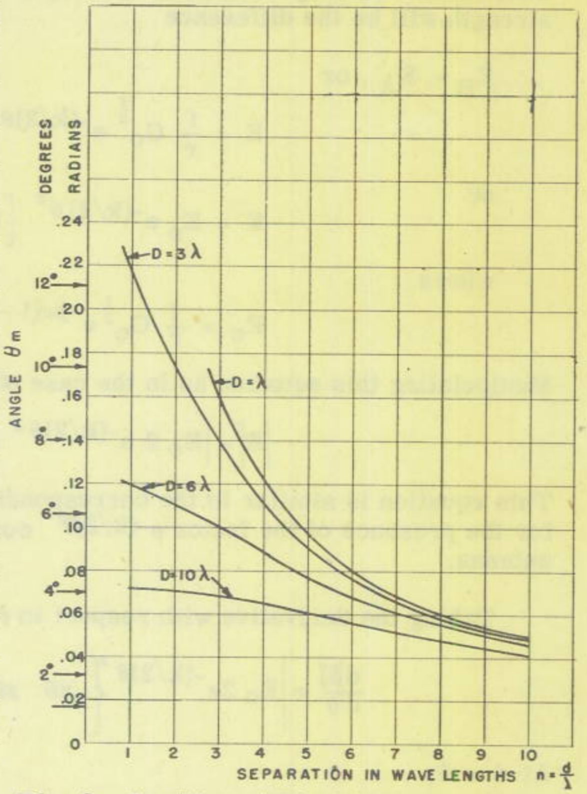

Fig. 6 - Position of First Maximum for Separated Narrow Beams

The amplitude of the first maxima is given by equation (36) and the graphical results of Figure 6. The curves of Figure 7 show the dependence of the amplitude of the first maxima on the aperture width and the separation, plotted in decibels relative to $2E_o$ which would be the peak amplitude if the two beams were combined in phase instead of 180 degrees out of phase. Although not indicated by the curves of Figure 7, increasing the gain of the antennas producing the component beams by increasing either principal plane dimension increases the absolute amplitude of the first maximum since E_o is increased. Hence, for example, if the principal plane dimension perpendicular to the plane in which E is measured is held constant and D/λ varied, then the curves of Figure 7 would be modified by adding to the ordinates of each curve the quantity $10 \log_{10} D/\lambda$. If the two principal antenna dimensions were made equal, then $20 \log_{10} D/\lambda$ should be added to each curve, the ordinates then representing gain in the direction of the pattern maximum relative to the gain of the $D/\lambda = 1$ curve.

The positions of the first few minima of the pattern are given by the same equation (18) as for the separated dipoles since, in equation (36), the factor $e^{-(k/2)\theta^2}$ decreases monotonically. However, only the main lobe of each beam was considered in writing equations (25) and (32). Therefore, the amplitudes and positions of side lobes farther from the center than the angle at which $e^{-(k/2)\theta^2}$ becomes small (less than one-tenth full amplitude, perhaps), as calculated from the equations presented here, must be considered unreliable.

SQUINTING OF NARROW BEAMS

Considering now the method of obtaining a null pattern by the method of "Squinting", as defined previously, the angle through which each beam is turned from the perpendicular will be called α and the separation of the sources of the two beams will be considered vanishingly small. Under these conditions, the equation for the signal strength resulting from the super-position of the two beams phased 180 degrees apart becomes

$$E = \frac{f}{r} G_0^{\frac{1}{2}} e^{j\omega(t - \frac{r}{c})} \left[e^{-(k/2)(\theta - \alpha)^2} - e^{-(k/2)(\theta + \alpha)^2} \right]$$

or

$$|E| = \left| E_0 \left[e^{-(k/2)(\theta - \alpha)^2} - e^{-(k/2)(\theta + \alpha)^2} \right] \right| \tag{40}$$

where $|E_0|$ is defined the same as in the case of separation of narrow beams.

At $\theta = 0$, $|E| = 0$, so that here also a null is produced in the perpendicular plane.

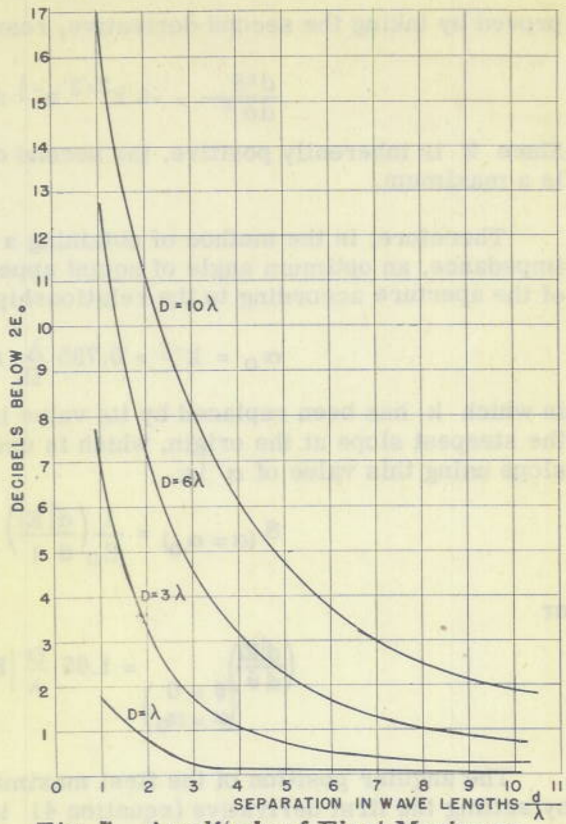

Fig. 7 - Amplitude of First Maximum for Separated Narrow Beams

$$\frac{d|E|}{d\theta} = \left| E_0 \left[-k(\theta - \alpha) e^{-(k/2)(\theta - \alpha)^2} + k(\theta + \alpha) e^{-(k/2)(\theta + \alpha)^2} \right] \right| . \tag{41}$$

At $\theta = 0$,

$$\left(\frac{d|E|}{d\theta}\right)_{\theta = 0} = \left| E_0 \left[2 k\alpha e^{-(k/2)\alpha^2} \right] \right| . \tag{42}$$

Since equation (42) may have a maximum slope for some particular value of α, let

$$S = \frac{1}{|E_0|} \left(\frac{dE}{d\theta}\right)_{\theta = 0} = 2k\alpha e^{-(k/2)\alpha^2} \tag{43}$$

and take the derivative of S with respect to α,

$$\frac{dS}{d\alpha} = 2k\, e^{-(k/2)\alpha^2} - 2k^2\alpha^2 e^{-(k/2)\alpha^2} \tag{44}$$

Setting (44) equal to zero,

$$\alpha_0^2 = \frac{1}{k}, \qquad \alpha_0 = \pm k^{-\frac{1}{2}} . \tag{45}$$

Only the positive root of (45) has physical significance and the other roots $\alpha = \pm \infty$ are trivial. That equation (45) gives the value of α which makes S a maximum may be proved by taking the second derivative, resulting in:

$$\frac{d^2S}{d\alpha^2} = -4\,k^{3/2}\,e^{-\frac{1}{2}} \text{ at } \alpha_o = k^{-\frac{1}{2}}. \tag{46}$$

Since k is inherently positive, the second derivative is negative and, hence, this point is a maximum.

Therefore, in the method of obtaining a null pattern by squinting, neglecting mutual impedance, an optimum angle of squint appears to exist which is dependent on the size of the aperture according to the relationship:

$$\alpha_o = k^{-\frac{1}{2}} = 0.735\,\frac{\lambda}{D} \text{ radians,} \tag{47}$$

in which k has been replaced by its value in equation (31). This optimum angle produces the steepest slope at the origin, which is usually the characteristic of major interest. The slope using this value of α is:

$$S_{(\alpha=\alpha_o)} = \frac{1}{E_o}\left(\frac{d|E|}{d\,\theta}\right)_{\substack{\alpha=\alpha_o \\ \theta=0}} = 2\,k^{\frac{1}{2}}\,e^{-\frac{1}{2}} \tag{48}$$

or

$$\left(\frac{d|E|}{d\,\theta}\right)_{\left(\substack{\theta=0 \\ \alpha=\alpha_o}\right)} = 1.65\,\frac{D}{\lambda}\,|E_o|\,. \tag{49}$$

The angular position of the first maximum in the squinting method is found as before by setting the first derivative (equation 41 in this case) equal to zero, giving

$$k(\theta-\alpha)\,e^{-(k/2)(\theta-\alpha)^2} = k(\theta+\alpha)\,e^{-(k/2)(\theta+\alpha)^2} \tag{50}$$

or

$$\frac{\theta-\alpha}{\theta+\alpha} = e^{-2k\theta\alpha}\,. \tag{51}$$

Using the optimum value for $\alpha(k^{-\frac{1}{2}})$,

$$\frac{k^{\frac{1}{2}}\theta-1}{k^{\frac{1}{2}}\theta+1} = e^{-2k^{\frac{1}{2}}\theta}, \tag{52}$$

or, expressing in terms of α_o,

$$\frac{\theta_m-\alpha_o}{\theta_m+\alpha_o} = e^{-2\theta/\alpha_o}\,. \tag{53}$$

This may be rearranged to give

$$\theta_m = \alpha_o\left[\frac{1+e^{-2\theta/\alpha_o}}{1-e^{-2\theta/\alpha_o}}\right], \tag{54}$$

or

$$\frac{\theta_m}{\alpha_o} = \coth\frac{\theta}{\alpha_o}\,. \tag{55}$$

If $X = \theta_m/\alpha_o$, then

$$X = \coth X \tag{56}$$

giving

$$X = \frac{\theta_m}{\alpha_o} = 1.20, \tag{57}$$

or

$$\theta_m = 1.20\alpha_o = 1.20\ k^{-\frac{1}{2}} = 0.882\ \lambda/D \text{ radians} = 50.5\lambda/D \text{ degrees.} \tag{58}$$

This curve is plotted versus D/λ on Figure 8.

That the angular position corresponding to $\theta_m = 1.20\alpha_o$ is a maximum of the null pattern may be proven by taking the second derivative of E with respect to θ which, after substituting equation (52), reduces to

$$\frac{d^2|E|}{d\theta^2} = -k\ E_o \left[(1 + 2k^{\frac{1}{2}}\theta)\ e^{-(k/2)(\theta+\alpha)^2} + e^{-(k/2)(\theta-\alpha)^2}\right]. \tag{59}$$

Since all the terms inside the brackets are positive, the second derivative is negative and the point is a maximum.

To find the amplitude of the first maximum when the angle of squint has been chosen so as to maximize the slope at the origin, equation (40) may be rewritten as

$$|E| = \left|E_o\ e^{-(k/2)(\theta^2+\alpha_o^2)}\left[e^{k\theta\alpha_o} - e^{-k\theta\alpha_o}\right]\right| \tag{60}$$

which reduces to

$$|E| = \left|E_o\ 2e^{-\frac{1}{2}}\ e^{-(k/2)\theta^2}\ \sinh k^{\frac{1}{2}}\theta\right|. \tag{61}$$

At the maximum, $\theta_m = 1.20\ k^{-\frac{1}{2}}$,

$$|E| = \left|E_o\ 2e^{-(\frac{1}{2}+0.72)}\sinh 1.20\right| = 0.90\left|E_o\right|, \tag{62}$$

a result which is independent of the aperture dimensions as long as the squinting angle is optimum. However, it must not be forgotten that E_o depends on the antenna gain as previously shown and mutual impedance has been neglected.

Equation (61) has no zero other than the central null, the function approaching zero asymptotically. In practice, however, we may expect additional lobes to appear due to their presence in the primary beams.

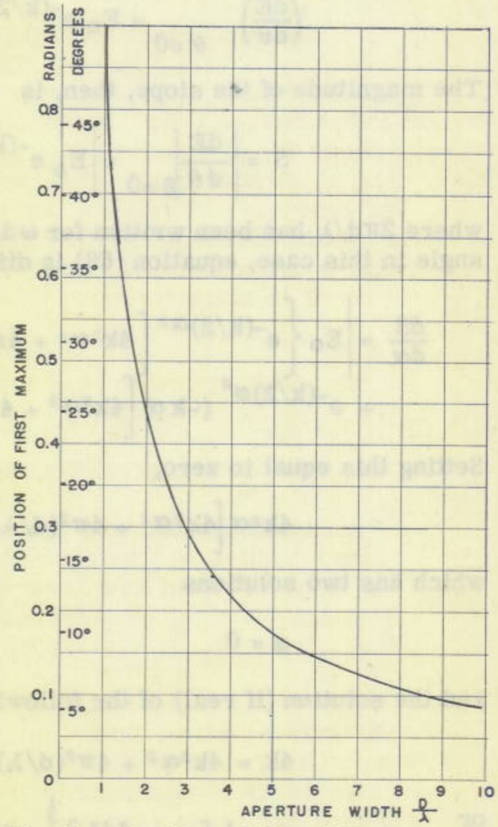

Fig. 8 - Position of First Maximum for Squinting of Narrow Beams

SQUINTING AND SEPARATION

Since it is conceivable that improved results might be obtained if both squinting and separation were used, the combined case will be investigated in this section. The two beams will be turned through an angle with respect to each other and the apparent sources of the beams will be separated. Using the same nomenclature as before,

$$E = \frac{f}{r} G_0^{\frac{1}{2}} e^{j\omega(t - \frac{r}{c})}\left[e^{-(k/2)(\theta - \alpha)^2} - e^{-(k/2)(\theta + \alpha)^2 - j\omega(d/c)\sin\theta}\right]$$

$$= E_0\left[e^{-(k/2)(\theta - \alpha)^2} - e^{-(k/2)(\theta + \alpha)^2 - j\omega(d/c)\sin\theta}\right] \tag{63}$$

which again is zero at $\theta = 0$. The slope at the origin is readily obtained if the equation is left in the complex form, yielding

$$\frac{dE}{d\theta} = E_0\left\{-k(\theta - \alpha)\, e^{-(k/2)(\theta - \alpha)^2} - e^{-(k/2)(\theta + \alpha)^2 - j\omega(d/c)\sin\theta}\left[-k(\theta + \alpha) - j\omega(d/c)\cos\theta\right]\right\} \tag{64}$$

$$\left(\frac{dE}{d\theta}\right)_{\theta = 0} = E_0\, e^{-(k/2)\alpha^2}\left[2k\alpha + j\omega(d/c)\right]. \tag{65}$$

The magnitude of the slope, then, is

$$S = \left|\frac{dE}{d\theta}\right|_{\theta = 0} = \left|E_0\, e^{-(k/2)\alpha^2}\left[4k^2\alpha^2 + 4\pi^2(d/\lambda)^2\right]^{\frac{1}{2}}\right| \tag{66}$$

where $2\pi d/\lambda$ has been written for $\omega d/c$. To determine if there is an optimum squinting angle in this case, equation (66) is differentiated with respect to α, obtaining

$$\frac{dS}{d\alpha} = \left|E_0\left\{e^{-(k/2)\alpha^2}\left[4k^2\alpha^2 + 4\pi^2(d/\lambda)^2\right]^{-\frac{1}{2}}(4k^2\alpha) + e^{-(k/2)\alpha^2}(-k\alpha)\left[4k^2\alpha^2 + 4\pi^2(d/\lambda)^2\right]^{\frac{1}{2}}\right\}\right| \tag{67}$$

Setting this equal to zero,

$$4k^2\alpha\left[4k^2\alpha^2 + 4\pi^2(d/\lambda)^2\right]^{-\frac{1}{2}} = k\alpha\left[4k^2\alpha^2 + 4\pi^2(d/\lambda)^2\right]^{\frac{1}{2}} \tag{68}$$

which has two solutions

$$\alpha = 0 \tag{69}$$

and the solution (if real) of the following equation

$$4k = 4k^2\alpha^2 + 4\pi^2(d/\lambda)^2 \tag{70}$$

or

$$\alpha = \frac{1}{k}\left[k - \frac{\pi^2 d^2}{\lambda^2}\right]^{\frac{1}{2}} = \frac{0.735}{d/\lambda}\left[1 - 5.32(d/D)^2\right]^{\frac{1}{2}}. \tag{71}$$

Equation (71) has a real solution if

$$k \geq \pi^2 d^2/\lambda^2 \tag{72}$$

or, replacing k by its equivalent from equation (31),

$$1.86 \frac{D^2}{\lambda^2} \geq \pi^2 \frac{d^2}{\lambda^2}; \qquad d \leq 0.434\ D. \tag{73}$$

To investigate further the meaning of equations (69) and (71), the second derivative of the slope with respect to α is taken, obtaining

$$\frac{1}{|E_0|}\frac{d^2S}{d\alpha^2} = \frac{4k^3\alpha^2 e^{-(k/2)\alpha^2}}{4k^2\alpha^2 + 4\pi^2(d/\lambda)^2} \cdot \left\{ \frac{-4k}{[4k^2\alpha^2 + 4\pi^2(d/\lambda)^2]^{\frac{1}{2}}} - \left[4k^2\alpha^2 + 4\pi^2(d/\lambda)^2\right]^{\frac{1}{2}} \right\}$$

$$+ \left\{ \frac{4k}{[4k^2\alpha^2 + 4\pi(d/\lambda)^2]^{\frac{1}{2}}} - \left[4k^2\alpha^2 + 4\pi^2(d/\lambda)^2\right]^{\frac{1}{2}} \right\} \left[ke^{-(k/2)\alpha^2} - k^2\alpha^3 e^{-(k/2)\alpha^2}\right]. \tag{74}$$

When equation (70) is satisfied, the second term of the above equation vanishes and the first term is negative showing that the slope is a maximum. When α is zero, the first term vanishes and the second term becomes

$$\frac{1}{|E_0|}\left(\frac{d^2S}{d\alpha^2}\right)_{\alpha=0} = k\left[\frac{2k\lambda}{\pi d} - \frac{2\pi d}{\lambda}\right] = \frac{2k\lambda}{\pi d}\left[k - \frac{\pi^2 d^2}{\lambda^2}\right]. \tag{75}$$

This point ($\alpha = 0$) is a maximum, then, when $k \leq \frac{\pi^2 d^2}{\lambda^2}$ or $d \geq 0.434$ D but, when $K > \frac{\pi^2 d^2}{\lambda^2}$ or $d < 0.434$ D, the slope is a maximum when

$$\alpha = \frac{1}{k}\left(k - \pi^2 d^2/\lambda^2\right)^{\frac{1}{2}}.$$

If two arrays are set side by side to produce the null pattern, the separation d cannot be less than the width of either of the arrays D so that equation (73) cannot be satisfied. In this case, the optimum squinting angle is zero, reducing the problem to one of separation only which has already been considered. However, if two sources are used to feed a reflector, the spacing of the feeds may be small in which case equation (73) applies and the value of the slope is found by combining equations (66) and (70), giving

$$S = |E_0|\, 2k^{\frac{1}{2}} e^{-(k/2)\alpha^2} = |E_0|\, 2k^{\frac{1}{2}} e^{-\frac{1}{2} - (\pi^2 d^2/2\lambda^2 k)} \tag{76}$$

which reduces for d = 0 to the value $E_0\ 2k^{\frac{1}{2}} e^{-\frac{1}{2}}$ previously obtained for squinting only and, for $\alpha = 0$, to $E_0\,(2\pi d/\lambda)$, as in the case of separation only, when use is made of equation (70).

Hence, it may be concluded that when it is possible to obtain a separation of sources greater than 0.434 times the aperture dimension, the squinting angle should be zero for maximum slope at the origin, but that, when the separation is restricted to less than that amount, there exists an optimum angle of squint given by equation (71). This result will be modified, however, by the effect of mutual impedance to be considered subsequently.

In order to find the position of the first maximum, the magnitude of the field strength is found from equation (63).

$$|E| = \left| E_0\, 2^{\frac{1}{2}} e^{-(k/2)(\theta^2 + \alpha^2)} \left[\cosh 2k\theta\alpha - \cos(m \sin\theta)\right]^{\frac{1}{2}} \right|. \tag{77}$$

The derivative with respect to θ is

$$\frac{d|E|}{d\theta} = \left| E_0 \, 2^{\frac{1}{2}} \, e^{-(k/2)(\theta^2 + \alpha^2)} \left\{ \frac{k\alpha \sinh(2k\theta\alpha) + (1/2)m\cos\theta \sin(m\sin\theta)}{[\cosh(2k\theta\alpha) - \cos(m\sin\theta)]^{\frac{1}{2}}} - k\,\theta\,[\cosh(2k\theta\alpha) - \cos(m\sin\theta)]^{\frac{1}{2}} \right\} \right| \tag{78}$$

Setting this equal to 0, there follows for the position of the maximum

$$\theta_m = \frac{m\cos\theta \sin(m\sin\theta) + 2k\alpha\sinh(2k\theta\alpha)}{2k[\cosh(2k\theta\alpha) - \cos(m\sin\theta)]}, \tag{79}$$

which, for $\alpha = 0$, reduces to the equation (39) previously found for separation only and, for $d = 0$, to $\alpha/\tanh(k\theta\alpha)$. When $\alpha = k^{-\frac{1}{2}}$, this reduces to equation (55) for squinting only.

In the intermediate region, for d greater than zero but less than 0.434 D, a smooth transition of the position of the first maximum occurs from its value for squinting only to that for separation only. For angles less than about 25 degrees, $\cos\theta$ is equal to 1.0 with an error less than 10 percent and $\sin\theta = \theta$ with an even smaller error. Hence, equation (79) may be closely approximated by

$$\theta_m = \frac{m}{2k}\left[\frac{\sin m\theta + \frac{2k\alpha}{m}\sinh 2k\alpha\theta}{\cosh 2k\alpha\theta - \cos m\theta}\right] \tag{80}$$

Using equations (71) and (31),

$$2k\alpha = \frac{2\pi d}{\lambda}\left[0.188\,\frac{D^2}{d^2} - 1\right]^{\frac{1}{2}} \tag{81}$$

so that

$$Z = m\theta_m = \frac{2\pi^2 d^2}{1.86D^2}\left[\frac{\sin Z + b\sinh bZ}{\cosh bZ - \cos Z}\right], \tag{82}$$

in which

$$b = \left[0.188\,\frac{D^2}{d^2} - 1\right]^{\frac{1}{2}}. \tag{83}$$

This equation for Z depends only on the ratio of d/D. Its graphical solution is shown in Figure 9 and a plot of the resulting values of Z versus d/D is given in Figure 10. From this curve, the position of the maximum may be quickly obtained for any set of values d and D for which the ratio d/D is less than 0.434. It is assumed, of course, that the optimum angle of squint is used as given by equation (71). The amplitude of the first maximum may be found by substitution in equation (77) of the value for θ_m found from Figure 9. For ratios of d/D greater than or equal to 0.434, the optimum angle of squint is zero and equation (80) reduces to

$$Z = 10.64\,\frac{d^2}{D^2}\cot Z/2 \tag{84}$$

or

$$U = \frac{Z}{2} = 5.32 \frac{d^2}{D^2} \cot U . \qquad (85)$$

From this equation and a table of cotangents, an approximate solution for the position of the first maximum in the case of separation may be readily found without referring to Figure 6 or for values of d and D not covered there.

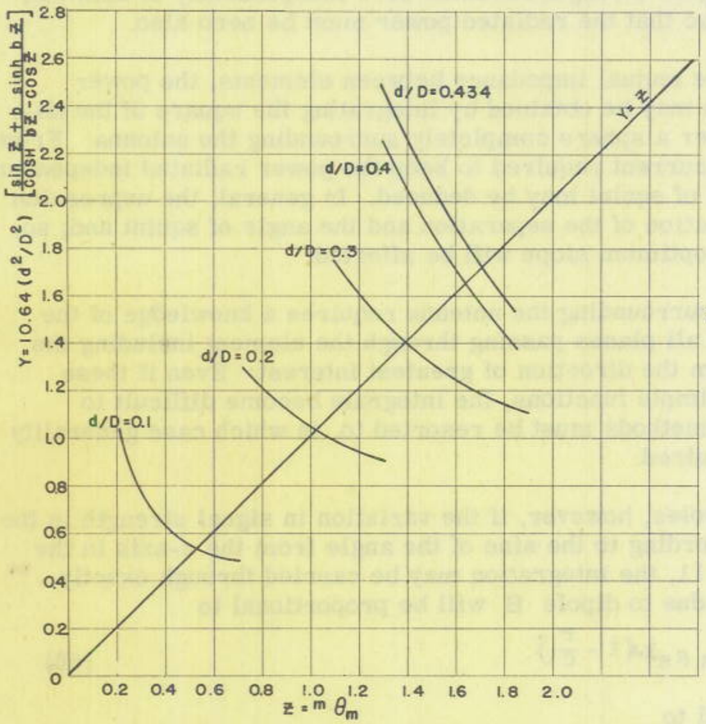

Fig. 9 - Graphical Solution of

$$Z = 10.64(d^2/D^2)\left[\frac{\sin Z + b \sinh b Z}{\cosh b Z - \cos Z}\right]$$

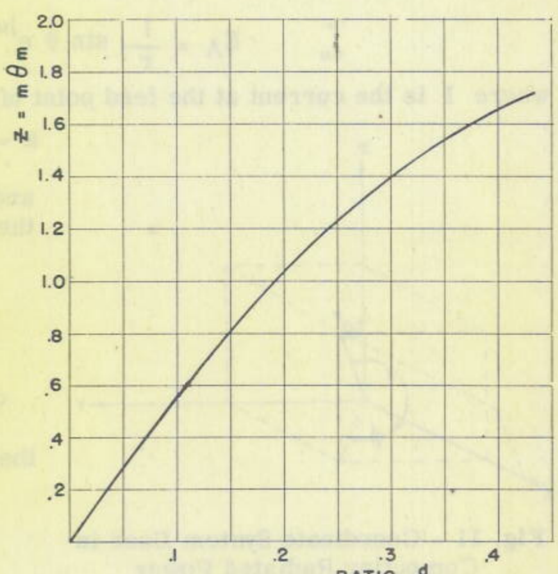

Fig. 10 - Position of First Maximum for Both Squinting and Separation with $d < 0.434D$

EFFECT OF MUTUAL IMPEDANCES

The foregoing analysis has ignored the effect of mutual impedance between the antenna elements. In effect, a current has been fed to each antenna element which has been held constant as the spacing or angle of squint is varied. The total power radiated, however, does not necessarily remain constant during this process as can readily be seen by considering the limiting case in which both the separation and angle of squint are made to approach zero. In this case, the signal strength becomes zero independently of azimuth, as can be seen from equation (63), so that the radiated power must be zero also.

In order to take into account the mutual impedance between elements, the power radiated by the antenna combination may be obtained by integrating the square of the absolute value of the field strength over a sphere completely surrounding the antenna. From this result the variation in antenna current required to keep the power radiated independent of variations in separation or angle of squint may be deduced. In general, the expression for the power radiated will be a function of the separation and the angle of squint and, so, the derivation of the conditions for optimum slope will be affected.

The integration over a sphere surrounding the antenna requires a knowledge of the pattern for each antenna element in all planes passing through the element including the pattern behavior at wide angles from the direction of greatest interest. Even if these patterns are assumed in terms of simple functions, the integrals become difficult to evaluate analytically and graphical methods must be resorted to, in which case generality may be lost or else great labor required.

In the case of the separated dipoles, however, if the variation in signal strength in the vertical plane is assumed to be according to the sine of the angle from the z-axis in the coordinate system shown in Figure 11, the integration may be carried through exactly. The signal strength at the point P due to dipole B will be proportional to

$$E_B = \frac{I}{r} \sin\theta\, e^{j\omega(t - \frac{r}{c})} \qquad (86)$$

and that due to dipole A proportional to

$$E_A = \frac{I}{r} \sin\theta\, e^{j\omega[t - (\frac{r+a}{c})]} \quad , \qquad (87)$$

where I is the current at the feed point of the dipoles,

$$a = d \cos\psi \ , \qquad (88)$$

and ψ is the angle between the radius vector of the point P and the x-axis. From Figure 11,

$$\psi = \cos^{-1}\frac{x}{r} \ , \qquad (89)$$

$$x = r\sin\theta\cos\phi \ , \qquad (90)$$

$$\cos\psi = \frac{x}{r} = \sin\theta\cos\phi \ , \qquad (91)$$

therefore

$$a = d\sin\theta\cos\phi \ . \qquad (92)$$

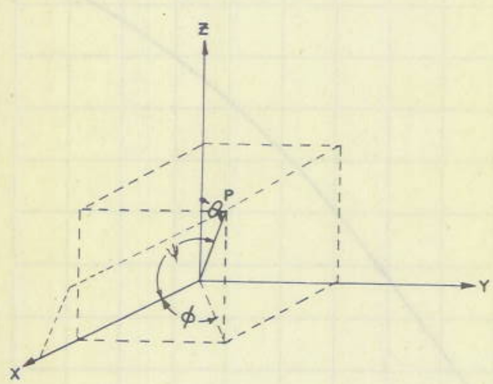

Fig. 11 - Coordinate System Used in Computing Radiated Power from Dipoles

Hence, the resultant signal strength at point P will be

$$E = \frac{I}{r} \quad e^{j\omega(t - \frac{r}{c})} \sin\theta \left[1 - e^{j\omega(d/c)\sin\theta\cos\theta}\right] \tag{93}$$

which, as in equation (6), may be written

$$|E| = I_0 \sin\theta \left[2 - 2\cos\left(\frac{2\pi d}{\lambda}\sin\theta\cos\phi\right)\right]^{\frac{1}{2}} \tag{94}$$

The total power radiated is

$$P = \int |E|^2 \, d\tau \tag{95}$$

$$= 2 I_0^2 \int_0^{\pi} \int_0^{2\pi} \left[1 - \cos(m \sin\theta\cos\phi)\right] \sin^3 \theta \, d\theta \, d\phi . \tag{96}$$

(97)

Since $1 - \cos x = 2(1 - \cos^2 \frac{x}{2})$,

$$P = 4 I_0^2 \int_0^{\pi}\int_0^{2\pi} \left[1 - \cos^2\left(\frac{m}{2}\sin\theta\cos\phi\right)\right] \sin^3\theta \, d\theta \, d\phi \tag{98}$$

and
$$P = 4 I_0^2 \times 8\pi/3 - \frac{16}{3}\pi I_0^2 \left[1 + \frac{3}{2m}\left(\sin m + \frac{\cos m}{m} - \frac{\sin m}{m^2}\right)\right] \tag{99}$$

The integral has been evaluated by reference to a paper by Schelkunoff. †

$$I_0^2 = \frac{P}{\frac{16\pi}{3}\left[1 - \frac{3}{2m}\left(\sin m + \frac{\cos m}{m} - \frac{\sin m}{m^2}\right)\right]} . \tag{100}$$

The equivalent radiation resistance of the antenna combination, then, is

$$R_r = \frac{16\pi}{3}\left[1 - \frac{3}{2m}\left(\sin m + \frac{\cos m}{m} - \frac{\sin m}{m^2}\right)\right]. \tag{101}$$

For large values of m (large separation), this reduces to a constant which must be the radiation resistance of an isolated elementary dipole (80 ohms) multiplied by 1/2 since the dipoles are effectively in parallel. For small values of m, the radiation resistance approaches zero as may be seen by differentiating and allowing m to approach zero. A plot of R_r versus d/λ is presented in Figure 12.

The field strength, then, is given by

$$E = \frac{P^{\frac{1}{2}} e^{j\omega(t - \frac{r}{c})} \sin\theta \, (1 - e^{jm\sin\theta\cos\phi})}{(40)^{\frac{1}{2}} r \left[1 - \frac{3}{2}\left(\frac{\sin m}{m} + \frac{\cos m}{m^2} - \frac{\sin m}{m^3}\right)\right]^{\frac{1}{2}}} \tag{102}$$

In the plane $\theta = 90°$, $\sin\theta = 1$, and

$$|E| = \frac{|E_0| \left[2 - 2\cos(m\cos\theta)\right]^{\frac{1}{2}}}{\left[1 - \frac{3}{2}\left(\frac{\sin m}{m} + \frac{\cos m}{m^2} - \frac{\sin m}{m^2}\right)\right]^{\frac{1}{2}}} , \tag{103}$$

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† Schelkunoff, S. A., "A General Radiation Formula," Proc. IRE, 27, 663, October 1939

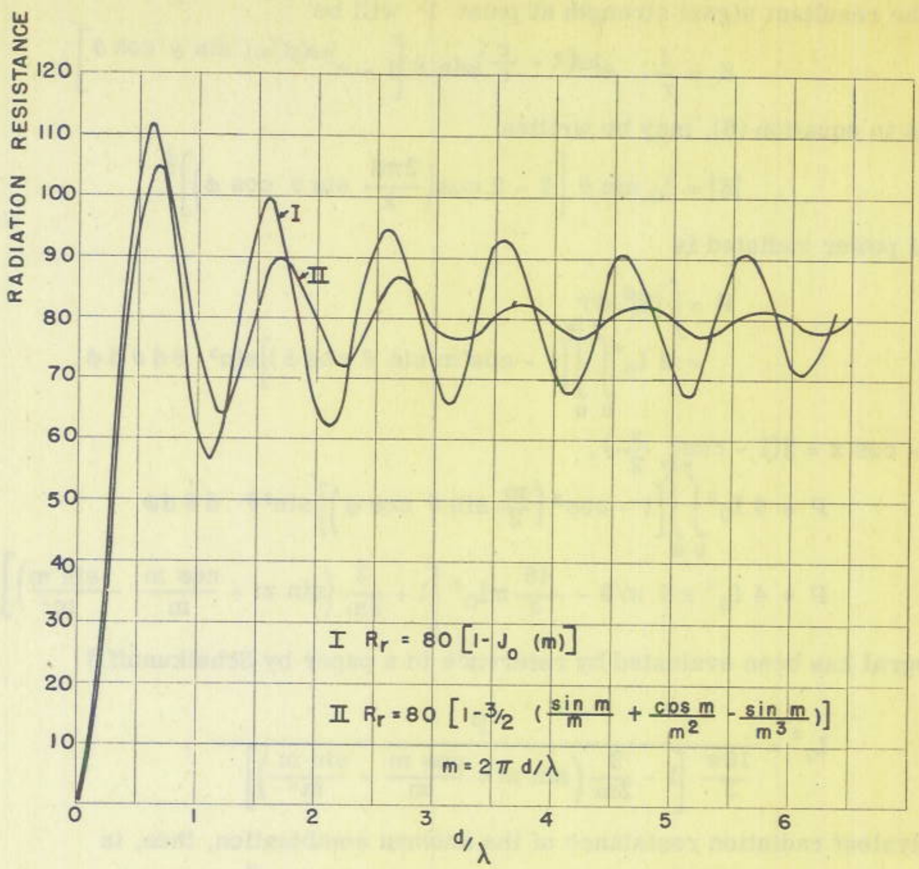

Fig. 12 - Radiation Resistance of Separated Elementary Dipoles

where

$$|E_0| = \left| \frac{P^{\frac{1}{2}} e^{j\omega(t - \frac{r}{c})}}{(40)^{\frac{1}{2}} r} \right| . \tag{104}$$

$$\frac{d|E|}{d\phi} = |E_0| \frac{m \cos\left(\frac{m}{2} \cos\phi\right) \sin\phi}{\left[1 - \frac{3}{2}\left(\frac{\sin m}{m} + \frac{\cos m}{m^2} - \frac{\sin m}{m^3}\right)\right]^{\frac{1}{2}}} \tag{105}$$

and, at $\phi = 90$,

$$\frac{d|E|}{d\phi} = \frac{m |E_0|}{1 - \frac{3}{2}\left[\left(\frac{\sin m}{m} + \frac{\cos m}{m^2} - \frac{\sin m}{m^3}\right)\right]^{\frac{1}{2}}} . \tag{106}$$

It has been shown that, for large values of m, the denominator is a constant and the slope is directly proportional to m and consequently to the distance of separation. The behavior of the slope for small values of m is shown by the plot in Figure 13.

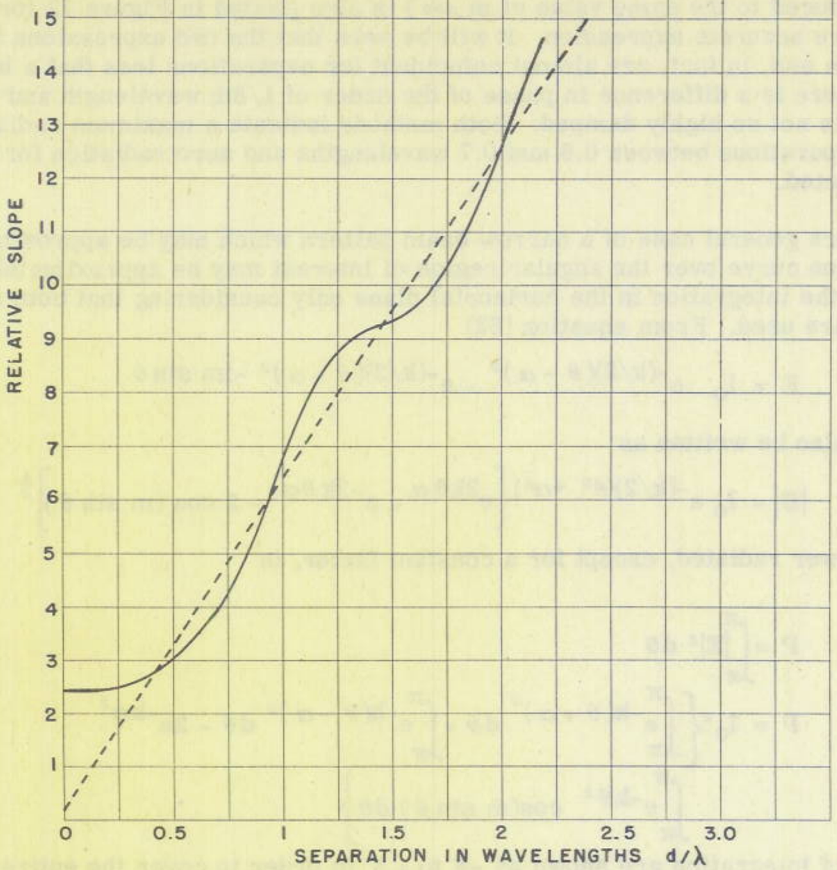

Fig. 13 - Elementary Dipoles Slope vs Separation

As would be expected, it is much easier to carry out the integration if it is assumed that the pattern is independent of the angle of elevation. The error involved in this simplifying assumption need not be very great, at least as far as the dependence of the slope of the null upon the separation is concerned. Considering the dipole case again and ignoring the variation with angle of elevation, the following expression is obtained:

$$P = 2\, I_O^2 \int_0^{2\pi} [1 - \cos(m \sin\phi)]\, d\phi \quad . \tag{107}$$

But

$$\int_0^{2\pi} \cos(m \sin\phi)\, d\phi = 2\pi J_O(m) \tag{108}$$

where $J_O(m)$ is the Bessel function of first kind and order zero. Hence,

$$P = 4\pi I_O^2\, [1 - J_O(m)] \tag{109}$$

and

$$I_O = \frac{P^{\frac{1}{2}}}{\left[4\pi[1 - J_O(m)]\right]^{\frac{1}{2}}} \quad . \tag{110}$$

The resulting variation in radiation resistance as a function of separation using this function (reduced to the same value at $m = \infty$) is also plotted in Figure 12 for comparison with the more accurate expression. It will be seen that the two expressions have the same general form and, in fact, are almost coincident for separations less that a half-wavelength. However, there is a difference in phase of the order of 1/8th wavelength and the simpler expression is not so highly damped. Both methods indicate a maximum radiation resistance for separations between 0.6 and 0.7 wavelengths and zero radiation for zero spacing as was expected.

The more general case of a narrow beam pattern which may be approximated by an error function curve over the angular region of interest may be approximately treated by performing the integration in the horizontal plane only considering that both squinting and separation are used. From equation (63)

$$E = I_0 \quad e^{-(k/2)(\theta - \alpha)^2} - e^{-(k/2)(\theta + \alpha)^2} \; -im \sin\theta \tag{111}$$

which may also be written as

$$|E| = I_0 \, e^{-(k/2)(\theta^2 + \alpha^2)} \left[e^{2k\theta\alpha} + e^{-2k\theta\alpha} - 2\cos(m \sin\theta) \right]^{\frac{1}{2}} \tag{112}$$

The total power radiated, except for a constant factor, is

$$P = \int_{-\pi}^{\pi} |E|^2 \, d\theta \tag{113}$$

$$P = I_0^2 \left\{ \int_{-\pi}^{\pi} e^{-k(\theta + \alpha)^2} d\theta + \int_{-\pi}^{\pi} e^{-k(\theta - \alpha)^2} d\theta - 2e^{-k\alpha^2} \int_{-\pi}^{\pi} e^{-k\theta^2} \cos(m \sin\theta) \, d\theta \right\} \tag{114}$$

The limits of integration are shown as $-\pi$ to $+\pi$ in order to cover the entire plane but it is advantageous to take cognizance of the concentration of energy in a narrow beam by reducing the region of integration in order to replace the $\sin\theta$ by θ in the third integral of equation (114). Since this replacement is accurate to within about 5 percent for angles up to 30 degrees, the results will be reasonably accurate for narrow-beam antennas. An estimate of the minimum aperture width for which this approximation is satisfactory may be obtained by noting that the value of k for which $e^{-k\theta^2}$ is down to about 5 percent at 30 degrees is about 11, which, from equation (31) corresponds to an aperture of about $2\frac{1}{2}$ wavelengths.

The first two integrals of equation (114) correspond to the area under the primary antenna pattern curves and, hence may be written

$$2k^{-\frac{1}{2}} \int_0^{\pi k^{\frac{1}{2}}/6} e^{-\beta_1^2} \, d\beta_1 \quad \text{and} \quad 2k^{-\frac{1}{2}} \int_0^{\pi k^{\frac{1}{2}}/6} e^{-\beta_2} \, d\beta_2 \tag{115}$$

respectively, where $\beta_1 = k^{\frac{1}{2}}(\theta + \alpha)$ and $\beta_2 = k^{\frac{1}{2}}(\theta - \alpha)$. For aperture dimensions greater than the above limit of $2\frac{1}{2}$ wavelengths, the integrals differ from $\frac{1}{2}\pi^{\frac{1}{2}}$ by less than one percent as may be seen by reference to tables of the probability integral‡

‡ "Tables of Probability Functions," Vol. I, Federal Works Agency, Work Projects Administration for the City of New York, 1941.

The third integral may be written as the real part of

$$2\int_0^{\pi/6} e^{-k\theta^2 + jm\theta}\, d\theta\ ; \qquad (116)$$

which after completing the square in the exponent, becomes

$$2\int_0^{\pi/6} e^{-[k^{\frac{1}{2}}\theta - j(m/2k^{\frac{1}{2}})]^2 - (m^2/4k)}\, d\theta\ . \qquad (117)$$

Letting $Z = k^{\frac{1}{2}}\theta - j(m/2k^{\frac{1}{2}})$,

$$2k^{-\frac{1}{2}} e^{-m^2/4k}\int_{-j(m/2k^{\frac{1}{2}})}^{\pi k^{\frac{1}{2}}/6 - j(m/2k^{\frac{1}{2}})} e^{-Z^2}\, dZ\ . \qquad (118)$$

Since the path of integration in the complex plane has the same ordinate at both limits, the integral has the same value as a real integral from 0 to $k^{\frac{1}{2}}\pi/6$ which as before is very close to $\frac{1}{2}\pi^{\frac{1}{2}}$.

The total power, therefore, is

$$P = 2\, I_o^{\,2}\, (\pi/k)^{\frac{1}{2}} \left[1 - e^{-k\alpha^2 - (m^2/4k)}\right]. \qquad (119)$$

Since each primary antenna radiates half the total power, the radiation resistance of each primary antenna is, except for a constant factor,

$$R_r = P/2\, I_o^{\,2} = (\pi/k)^{\frac{1}{2}} \left[1 - e^{-k\alpha^2 - (m^2/4k)}\right], \qquad (120)$$

and

$$I_o = 2^{-\frac{1}{2}}\, P^{\frac{1}{2}}\, (k/\pi)^{\frac{1}{4}} \left[1 - e^{-k\alpha^2 - (m^2/4k)}\right]^{-\frac{1}{2}}. \qquad (121)$$

Substituting this latter expression in equation (66) for the magnitude of the slope yields

$$\left(\frac{dE}{d\theta}\right)_{\theta=0} = \frac{2^{-\frac{1}{2}}\, P^{\frac{1}{2}}\, (k/\pi)^{\frac{1}{4}}\, e^{-k\alpha^2/2}\, (4k^2\alpha^2 + m^2)^{\frac{1}{2}}}{[1 - e^{-k\alpha^2 - (m^2/4k)}]^{\frac{1}{2}}}$$

$$= E_1 \left[\frac{4k^2\alpha^2 + m^2}{e^{k\alpha^2} - e^{-m^2/4k}}\right]^{\frac{1}{2}} \qquad (122)$$

Where $E_1 \equiv 2^{-\frac{1}{2}}\, P^{\frac{1}{2}}\, (k/\pi)^{\frac{1}{4}}$ (123)

Let

$$S \equiv \frac{1}{E_1}\left(\frac{dE}{d\theta}\right)_{\theta=0}.$$

Then

$$\frac{dS}{d\alpha} = \frac{1}{2}\left[\frac{e^{k\alpha^2} - e^{-m^2/4k}}{4k^2\alpha^2 + m^2}\right]^{\frac{1}{2}} \frac{[(4k^2\alpha^2 + m^2)(2k\alpha)\, e^{k\alpha^2} - 8k^2\alpha(e^{k\alpha^2} - e^{-m^2/4k})]}{[e^{k\alpha^2} - e^{-m^2/4k}]^2}. \qquad (124)$$

Setting this equal to zero to find the value of α which gives the maximum slope, it is obvious from inspection that α equals zero is one solution of the equation. After further simplification,

$$(4k^2\alpha^2 + m^2)\, e^{k\alpha^2} = 4k\,(e^{k\alpha^2} - e^{-m^2/4k})$$

$$4k^2\alpha^2 + m^2 = 4k(1 - e^{-k\alpha^2 - m^2/4k})$$

$$k\alpha^2 + \frac{m^2}{4k} = 1 - e^{-k\alpha^2 - m^2/4k}$$

Let

$$u = k\alpha^2 + m^2/4k\ .$$

Then

$$u = 1 - e^{-u}, \tag{125}$$

whose only root is zero, which gives imaginary values for α.

Hence, the maximum slope is obtained with zero squinting angle for all values of m as might have been seen directly from inspection of equation (122). The imaginary roots result from setting the denominator of (122) equal to zero which, of course, is impossible for real values of α or m other than zero. Likewise, inspection reveals that the slope is greater than $m\ E_1$ for small values of m since the denominator is less than one for $\alpha = 0$. A plot of the slope vs d/λ for an aperture of three wavelengths and three values of α is given in Figure 14.

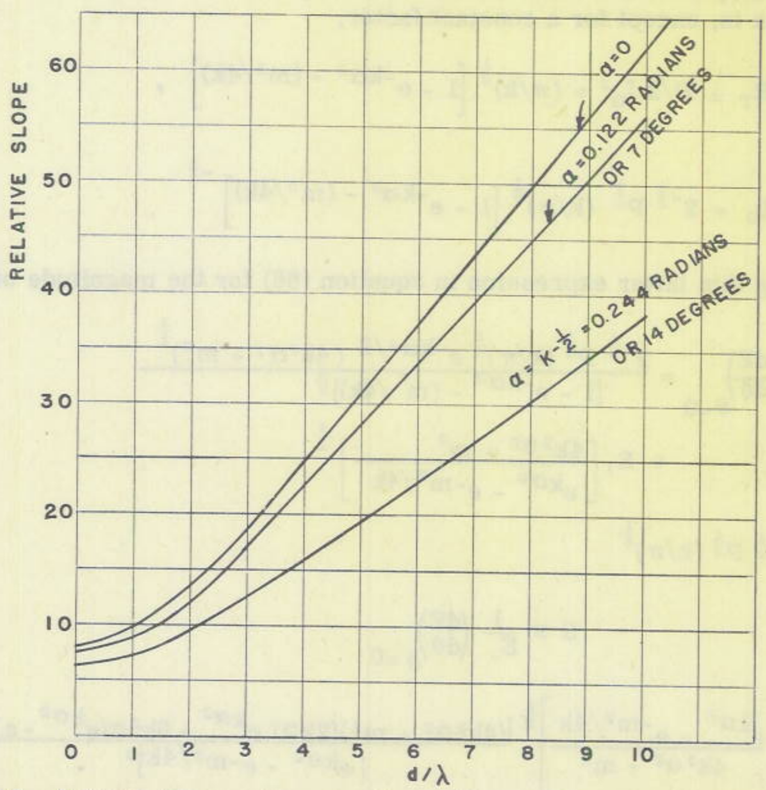

Fig. 14 - Narrow Beams Slope vs Separation with Angle of Squint as Parameter Including Effect of Mutual Impedance Aperture Width D = 3λ

Thus, an approximate treatment of the case of narrow beams, considering constant radiated power, yields the result that the greatest slope at the origin is obtained with zero squinting angle for all distances of separation in contrast with the result for constant feed current that an optimum angle of squint exists for small separations. However, the rate of change of slope with change in angle of squint is zero in the region of $\alpha = 0$ so that small squinting angles do not reduce the slope very much. In practice, it would probably be found very difficult to maintain constant radiated power as the distance of separation decreases because of the low radiation resistance and the required high antenna currents, so that the constant current solution might not be too far off after all.

EFFECTS OF INEQUALITIES

In the foregoing, ideal conditions have been implicity assumed; that is, both primary antenna elements have been assumed identical in radiation resistance and pattern and fed perfectly 180 degrees out of phase. In practice, these ideal conditions may only be approached and so it is desirable to calculate the effects of small departures from equality in order to determine design tolerances. Only the method of separation is considered.

If either the current or power fed to a primary element is unequal in magnitude to that fed to the other, the result is to change the zero of the null pattern to a minimum in the same angular position whose depth depends on the amount of unbalance in current or power. This result may be shown as follows

Let
$$E = E_O\left[E_1 - E_2\, e^{-j\alpha}\right] \tag{126}$$

where E_O includes the time and distance factor as well as the primary pattern factor of each antenna. Then

$$|E| = |E_O|\left[E_1{}^2 + E_2{}^2 - 2\,E_1\,E_2\cos\alpha\right]^{\frac{1}{2}} \tag{127}$$

in which $\alpha = \frac{2\pi d}{\lambda}\sin\theta$.

In order for $|E|$ to be zero, the expression in the brackets must be zero which requires that

$$\cos\alpha = \frac{E_1{}^2 + E_2{}^2}{2\,E_1\,E_2} \quad . \tag{128}$$

If $E_1 > E_2$, assume $E_1 = E_2 + \Delta E$. Then

$$\cos\alpha = \frac{E_2{}^2 + (E_2 + \Delta E)^2}{2\,E_2\,(E_2 + \Delta E)}$$

$$= \frac{2E_2{}^2 + 2\,E_2\Delta E + \Delta E^2}{2E_2{}^2 + 2E_2\Delta E}$$

$$= 1 + \frac{\Delta E^2}{2E_1\,E_2} \quad . \tag{129}$$

Since the cosine of an angle cannot be greater than one, equation (127) cannot be zero for $E_1 > E_2$ and, since (128) is symmetrical in E_1 and E_2, (127) cannot be zero except when $E_1 = E_2$. The value of the field strength at $\theta = 0$ is given by (127) as $|E_1 - E_2| = |\Delta E|$.

For angles either side of zero, cos α becomes smaller resulting in a larger value of $|E|$ so that a minimum still exists at $\theta = 0$ but its value is given by the unbalance in antenna currents, ΔE. This unbalance in practice may arise from inequality of radiation resistance of the two primary antennas or inequality in antenna gain.

The effect of inequality in the phase of the currents to the primary antennas may be similarly found. Thus

$$E = E_0 \left[1 - e^{j(\beta - \alpha)}\right] \tag{130}$$

where β is the difference in phase and may be either positive or negative.

$$|E| = |E_0| \left[2 - 2 \cos(\beta - \alpha)\right]^{\frac{1}{2}}$$

$$= 2 |E_0| \sin\left(\frac{\beta - \alpha}{2}\right). \tag{131}$$

For this to be zero, $\beta - \alpha = 0$, and, since $\alpha = (2\pi d/\lambda) \sin\theta$,

$$\sin\theta = \frac{\lambda}{2\pi d}\,\beta\,, \tag{132}$$

which, for small differences in phase, gives

$$\theta = \frac{\lambda}{2\pi d}\,\beta\,. \tag{133}$$

Thus, the effect of a difference in phase is to shift the position of the null (which is still a zero) to a new angle proportional to the difference in phase. However, the shift in angular position is inversely proportional to the separation so that the shift may still be quite small even for quite large differences in phase. This difference in phase may arise in practice due to inequality of the reactive components of the antenna input impedances and inequality of lengths of feeder cables. Furthermore, if the phase shift β is given by an expression $2\pi L/\lambda$ where L is some dimension such as that of a feeder cable, then the position of the null, though different from zero, is independent of frequency as seen by substituting for β in equation (132) or 133).

The two primary patterns may also differ slightly in various respects which, in general, may be expected to show up as dissymmetry in the combined pattern. For simplicity, assume that both patterns have the same error function form but that the beam widths differ.

$$E = E_0 \left[e^{-(k_1/2)\theta^2} - e^{-(k_2/2)\theta^2}\, e^{-j\alpha}\right]. \tag{134}$$

At $\theta = 0$, the exponentials reduce to unity so that the field strength is still zero at the origin of θ.

$$\frac{dE}{d\theta} = E_0 \left[-k_1\,\theta\, e^{-(k_1/2)\theta^2} - k_2\,\theta\, e^{-(k_2/2)\theta^2} e^{-j\alpha} + j e^{-(k_2/2)\theta^2}\, e^{-j\alpha}\,\frac{d\alpha}{d\theta}\right]. \tag{135}$$

At $\theta = 0$,

$$\frac{dE}{d\theta} = E_0 \left[j e^{-j2\pi(d/\lambda)\sin\theta} \cdot 2\pi(d/\lambda)\cos\theta\right], \tag{136}$$

and

$$\left|\frac{dE}{d\theta}\right|_{\theta=0} = \left|E_O\right| \cdot \frac{2\pi d}{\lambda} \tag{137}$$

so that the slope at the origin is unaffected by the difference in primary patterns, providing both primary antennas have the same gain at $\theta = 0$ as was implicitly assumed in writing equation (134). If they do not have the same gain, the effect is the same as a difference in feeder power or current as previously considered.

To determine the positions of the maxima, the magnitude of E of equation (134) is written as follows

$$\left|E\right| = \left|E_O\right| \left[e^{-k_1 \theta^2} + e^{-k_2 \theta^2} - 2e^{-\frac{1}{2}(k_1 + k_2)\theta^2} \cos\alpha\right]^{\frac{1}{2}} . \tag{138}$$

$$\frac{dE}{d\theta} = \left|E_O\right| (1/2)\left[-2k_1 \theta e^{-k_1 \theta^2} - 2k_2 \theta e^{-k_2 \theta^2} + 2(k_1 + k_2)\theta e^{-(1/2)(k_1 + k_2)\theta^2} \cos\alpha \right.$$
$$\left. + 2 e^{-(1/2)(k_1 + k_2)\theta^2} \sin\alpha \frac{d\alpha}{d\theta}\right]\left[e^{-k_1 \theta^2} + e^{-k_2 \theta^2} - 2e^{-(1/2)(k_1 + k_2)\theta^2} \cos\alpha\right]^{-\frac{1}{2}}$$
$$= 0 , \tag{139}$$

and

$$k_1 \theta e^{-k_1 \theta^2} + k_2 \theta e^{-k_2 \theta^2} = (k_1 + k_2)\theta e^{-(1/2)(k_1 + k_2)\theta^2} \cos\alpha$$
$$+ e^{-(1/2)(k_1 + k_2)\theta^2} \sin\alpha \frac{d\alpha}{d\theta} . \tag{140}$$

If

$$k_1 = k_2 + \Delta k \tag{141}$$

$$(k_2 + \Delta k)\theta e^{-(k_2 + \Delta k)\theta^2} + k_2 \theta e^{-k_2 \theta^2}$$
$$= (2k_2 + \Delta k)\theta e^{-(1/2)(2k_2 + \Delta k)\theta^2} \cos\alpha + e^{-(1/2)(2k_2 + \Delta k)\theta^2} \sin\alpha \frac{d\alpha}{d\theta} \tag{142}$$

$$\theta\left[(k_2 + \Delta k) e^{-\Delta k \theta^2} + k_2\right] = (2k_2 + \Delta k)\theta e^{-\Delta k \theta^2} \cos\alpha + e^{-\Delta k \theta^2} \sin\alpha \frac{d\alpha}{d\theta} \tag{143}$$

$$\theta = \frac{\sin\alpha \frac{d\alpha}{d\theta}}{(k_2 + \Delta k) - (2k_2 + \Delta k)\cos\alpha + k_2 e^{\Delta k \theta^2}} \tag{144}$$

$$\theta = \frac{\sin\alpha \frac{d\alpha}{d\theta}}{k_1 + k_2 e^{\Delta k \theta^2} - (k_1 + k_2)\cos\alpha} . \tag{145}$$

If Δk is small, the exponential is very close to unity; so, to this order of approximation,

$$\theta_m = \frac{\sin \alpha \frac{d\alpha}{d\theta}}{(k_1 + k_2)(1 - \cos \alpha)} \tag{146}$$

$$\theta_m = \frac{m}{(k_1 + k_2)} \cos \theta_m \cot\left(\frac{m}{2} \sin \theta_m\right). \tag{147}$$

This is the same as equation (39) with $2k$ replaced by $k_1 + k_2$, and the position of the first maximum is the same as that of an antenna having a value of k equal to the average of k_1 and k_2 or, in terms of beam-widths, the angular position of the maxima is that of an equivalent antenna in which the primary elements have equal beam-widths (B) given by (from equation (29))

$$B = \sqrt{\frac{2 B_1^2 B_2^2}{B_1^2 + B_2^2}} \tag{148}$$

Another type of inequality of patterns occurs when the two primary patterns are skewed with respect to each other. This has been considered under the heading of squinting and the effect on the slope and positions of the maxima were calculated. If the direction of maximum gain of one of the primary elements has been adopted as the reference for measurement of azimuth angle, squinting of the other element with respect to this reference, of course, introduces a shift in the position of the null equal to one-half the angle of squint.

CONCLUSIONS

From the equations and curves presented here, it is possible to determine quickly the positions of the maxima and relative slopes near the origin of directional-null-type antenna patterns as a function of the distance of separation, aperture dimensions, and angle of squint of two primary antennas fed 180 degrees out of phase.

It has been shown that, at least for apertures greater than about 2-1/2 wavelengths, the optimum angle of squint for sharpest null is zero, when mutual impedance is considered.

The effect of inequality of radiated power from the two primary antennas is to reduce the depth of the null without shifting it.

The effect of departure from 180 degrees of the phase between the two primary antennas is to shift the position of the null which is still a zero. This shift is proportional to the difference in phase angle from 180 degrees and inversely proportional to the separation in wavelengths.

The effect of inequality in beamwidth of the two primary antenna patterns is to change the positions of the maxima without affecting the slope or depth of the null, providing that the antenna gains are equal.

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