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NAVAL RESEARCH LABORATORY

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NRL REPORT NO. R-

MISSILE GUIDANCE BY JET IMPULSE

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Approved by:

Dr. R. M. Page, Superintendent, Radio Division III

Problem No. 36R05-20 November 7, 1947

NAVAL RESEARCH LABORATORY

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CONTENTS

ABSTRACT

The feasibility of guiding a spinning missile by means of the precession produced by sideways explosive impulses, properly timed and directed, is analyzed dynamically in terms of precessional forces, aerodynamic overturning moment, and stability. The rate of turn is found to be too slow for target speeds anticipated.

An alternative is proposed of guiding by means of intermittent thrusts at appropriate points along the helicoidal path of a spinning jet-assisted missile.

PROBLEM STATUS

The whole problem of the guidance of gun-launched missiles is in only the incipient stage. The present report analyzes one very small aspect of the problem.

AUTHORIZATION

The investigation of basic control of high-velocity, high-acceleration missiles was originated by Mr. Allen H. Schooley, Associate Superintendant of Radio Division III, as Laboratory Problem 36R05-20.

MISSILE GUIDANCE BY JET IMPULSE

GENERAL CONSIDERATIONS

In projectile motion, viscosity effects resulting in skin friction, nose- and tail-drag (the latter is practically nonexistent during jet propulsion), air compressibility, creation of shock waves in supersonic flow, turbulence, and, last but not least, stability - all enter into the composite result. In anticipating new designs, these component factors must be evaluated so as to determine which factors are significant and which negligible, insofar as the proposed purpose is concerned.

Above 50 feet per second, Reynolds' viscosity effects are unimportant. Consequently, surface roughness has little effect in supersonic flow. And, inasmuch as scale effects are usually associated with frictional effects present only in the boundary layer, wind-tunnel model studies are quite adequate for the study of a proposed missile.

Above a Mach number M = 0.7, air compressibility becomes of rapidly increasing importance, particularly critical in the neighborhood of the speed of sound. In fact, at about 900 feet per second, the yaw of a projectile actually tends to increase, due to the additional energy involved in producing the shock wave.

The exact dynamical equations of the motion of a projectile have been given by Nielsen and Synge (see Appendix II), derived from two basic principles: 1) that the projectile has an axis of symmetry, and 2) that the motion of the projectile is invariant with respect to any shift of the mass center along the axis of symmetry. The latter principle marks a genuine advance in projectile theory, bringing into clear relief the "obvious" fact that the atmospheric medium is ignorant of and oblivious to the actual mass distribution within the projectile and the concomitant precise location of the center of gravity. Incidentally, the vanishing of Magnus effects is invariant with respect to translation of the center of gravity along the axis of symmetry. A detailed solution of the Nielsen and Synge equations has not been effected, but would be of considerable interest and usefulness.

Inasmuch as the factors affecting missile stability and performance are of such a nature that they vary from moment to moment, it is the missile behavior over short intervals of space and time that is most significant. Thus it is that the solution of the basic problems of supersonic flight involves the forces concomitant with shock waves. The dynamics of the missile are largely determined by the aerodynamics of the shock waves, air flow, and air pressures at various points over the surface of the missile.

Aerodynamic forces acting on a missile in motion are determined entirely by the shape or geometric configuration of the missile. At subsonic velocities, the solution of the second-order differential equations of the flow is of an elliptic type, with boundary values determined by the inherent continuity, as in the usual potential theory. But, with the supersonic velocities of genuine and immediate interest, the situation is much simpler, inasmuch as discontinuities can and do occur along lines known as the "characteristic" lines associated with any abrupt corner or vertex of the missile. These characteristics separate regions of essentially different types of flow and actually represent abrupt changes in the flow velocity. The complete solution of the differential equations cannot be had in closed form but a solution can be carried out to any required accuracy in a step-by-step approximation. As long as boundary-layer flow is not involved, the method of characteristics is capable of clarifying considerably the problems of projectile resistance and motion.

Even in the case of axial symmetry, however, the solution of the exact equations (Appendix II) involves considerable difficulties. However, a theoretical solution of the problem would be of real interest in connection with the problems of stability and (for the spinning projectile) precession.

INITIAL SUGGESTIONS

That the direction of flight of a rotating projectile might be changed by a sideways explosive impulse was initially suggested as a possibility. These explosive thrusts could be quite naturally timed by a lobing radar which scanned the forward field as the projectile rotated. In this way, the projectile could be guided towards an objective. The explosive thrusts would be applied at some point as far from the center of gravity of the projectile as convenient - possibly both fore and aft of the center of gravity, if greater turning moment were needed than could readily be supplied by a single explosive source.

With a rotating projectile, the result of such an explosive impulse would be a momentary presession in a direction at right angles to the thrust and the axis of spin or, if the thrust couple were of sufficient magnitude, in a resultant direction somewhere between the directions of thrust and precession.

The question arose: By means of intermittent impulses, controlled by a lobing radar, could the course of a spinning projectile be changed, and how rapidly? Possibly, it could almost be made to "turn corners"! Further, if the projectile were so turned in its flight, would it track — i. e. would its length be effectively tangent to the trajectory pursued by its center of gravity — or would it proceed cocked to one side of its path?

ANALYSIS OF PROBLEM

The solution of such a problem as the one proposed involves several considerations of a dynamic and aerodynamic nature:

1) a measure of the precessional couple;
2) a measure of the aerodynamic overturning moment; and
3) a measure of the resulting stability of flight.

These do not all lend themselves, at the present status of ballistic theory, to theoretical solution. A complete solution would involve supersonic aerodynamics, including the theory of shock waves and supersonic flow, which would clarify the present questions of stability and precession. However, a fairly good approximation can be made on the basis of present dynamics for precession and stability. A good estimate of the aerodynamic overturning moment can be had from wind-tunnel tests of shell models.

Precession. — The fundamental theorem of gyrostatics states that: If a body which has an angular momentum AN about an axis Ox of rotation be under the action of an impressed impulsive torque K about a perpendicular axis Oy, then the angular momentum will be rotated about the mutually perpendicular axis Oz with a resulting angular velocity Ω satisfying the relation

$$K = AN\Omega,$$

where

K = torque couple applied ($\text{lb-ft}^2/\text{sec}^2$)
A = axial moment of inertia (lb-ft^2)
N = axial rate of spin (radians/sec)
Ω = angular rate of precession (radians/sec)

The relation here stated shows that, for a given projectile design (A = constant), fired with a predetermined spin (N = constant), the precession rate can be increased by increasing the applied couple. However, an increased precession rate can also be had from a fixed torque by decreasing the spin rate.

Aerodynamic Overturning Moment. — The aerodynamic overturning moment of a rotating projectile about its center of gravity is defined to be

$$M_o = \frac{1}{2} C_m \rho v^2 (\pi d^2/4) d$$

where

M_o = overturning moment about center of gravity (lb-ft)
ρ = air density (slugs/ft^3)
v = velocity of projectile (ft/sec)
d = diameter of projectile (ft)
C_m = aerodynamic moment coefficient, depending on shape, attitude, and Mach number
+ "restoring" if center of gravity is ahead of center of pressure
- "overturning" if center of pressure is ahead of center of gravity

The aerodynamic moment coefficient C_m depends on the shape, attitude, and velocity of the projectile and is evaluated empirically from wind-tunnel tests.

Stability. — Stability demands that any off-path disturbance shall not be increased by the dynamic and aerodynamic energy of the system but, rather, tend to be damped out or, at most, remain constant. A dynamic solution of the problem of stability can be had from the theory of gyrostatics.

The motion of a spinning shell is identical with that of a spinning top such that:

1) they both have the same rate of axial spin and the same axial moment of inertia,
2) the transverse moment of inertia of the top about its support point is the same as the transverse moment of inertia of the shell about its center of gravity, and
3) the moment of gravity about the point of the top is the same as the moment of the force system on the shell about its center of gravity.

Motion of the shell can be studied most conveniently with respect to a moving coordinate frame whose origin coincides with the center of gravity of the shell and whose principal axis is tangent to the trajectory.

Energy of Moving System. — The kinetic energy of the spinning shell is

$$K = \tfrac{1}{2} A\omega_1^2 + \tfrac{1}{2} B(\omega_2^2 + \omega_3^2)$$

By means of spherical coordinates

$$\omega_1 = \dot{\vartheta} \sin\varphi - \dot{\psi} \sin\vartheta \cos\varphi$$

$$\omega_2 = \dot{\vartheta} \cos\varphi + \dot{\psi} \sin\vartheta \sin\varphi$$

$$\omega_3 = \dot{\varphi} + \dot{\psi} \cos\vartheta$$

this can be transformed into

$$K = \tfrac{1}{2}\left[A(\dot{\vartheta}^2 + \dot{\psi}^2 \sin^2\vartheta) + B(\dot{\varphi} + \dot{\psi}\cos\vartheta)^2\right].$$

Thus the total energy of the moving system is

$$T = \tfrac{1}{2}\left[A(\dot{\vartheta}^2 + \dot{\psi}^2 \sin^2\vartheta) + B(\dot{\varphi} + \dot{\psi}\cos\vartheta)^2\right] - M\cos\vartheta.$$

It would be interesting to show the analogy of this last equation with the representation of the streamlines passing through the stagnation point for a source in a uniform stream, wherein the last term would correspond to a velocity potential and the part involving the square brackets would correspond to the flux past a cylindrical disturbing source, so that the total-energy equation could be had from either gyrostatic or aerodynamic considerations.

Stability Conditions. — By a transformation in coordinates

$$\sin\vartheta \cos\varphi = \xi$$

$$\sin\vartheta \sin\varphi = \eta,$$

which is equivalent to

$$\cot\varphi = \xi/\eta$$

$$\sin\vartheta = (\xi^2 + \eta^2)^{1/2},$$

the equation of total energy

$$T = \tfrac{1}{2} A(\dot{\varphi} + \dot{\psi}\cos\vartheta)^2 + \tfrac{1}{2} B(\dot{\vartheta}^2 + \dot{\psi}^2 \sin^2\vartheta) - M\cos\vartheta$$

can be transformed into the approximation, to within first order differentials (See Appendix I for details),

$$T = \tfrac{1}{2} An(\xi\dot{\eta} - \eta\dot{\xi}) + \tfrac{1}{2} B(\dot{\xi}^2 + \dot{\eta}^2) + \tfrac{1}{2} M(\xi^2 + \eta^2) - M.$$

Lagrange's equations

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = 0 \qquad (q_i = \xi, \eta)$$

must be satisfied. Now

$$\frac{\partial T}{\partial \dot{\xi}} = -\tfrac{1}{2} A n \eta + B \dot{\xi}, \qquad \frac{\partial T}{\partial \dot{\eta}} = \tfrac{1}{2} A n \xi + B \dot{\eta}$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\xi}} = -\tfrac{1}{2} A n \dot{\eta} + B \ddot{\xi}, \qquad \frac{d}{dt}\frac{\partial T}{\partial \dot{\eta}} = \tfrac{1}{2} A n \dot{\xi} + B \ddot{\eta}$$

$$\frac{\partial T}{\partial \xi} = \tfrac{1}{2} A n \dot{\eta} + M \xi, \qquad \frac{\partial T}{\partial \eta} = -\tfrac{1}{2} A n \dot{\xi} + M \eta$$

so that the Lagrangean conditions become

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\xi}} - \frac{\partial T}{\partial \xi} = 0 = B\ddot{\xi} - A n \dot{\eta} - M\xi$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\eta}} - \frac{\partial T}{\partial \eta} = 0 = B\ddot{\eta} + A n \dot{\xi} - M\eta .$$

If we put

$$\xi + i\eta = \zeta$$

and hence

$$i\dot{\xi} - \dot{\eta} = i\dot{\zeta}$$

the two Lagrangean conditions can be combined into the single equation

$$B(\ddot{\xi} + i\ddot{\eta}) + A n(-\dot{\eta} + i\dot{\xi}) - M(\xi + i\eta) = 0$$

or

$$B\ddot{\zeta} + A n i \dot{\zeta} - M\zeta = 0.$$

The condition that this equation have periodic solutions

$$\zeta = \exp i\alpha, \quad \bar{\zeta} = \exp(-i\alpha)$$

so that a disturbance if once small remains small, thus guaranteeing stability, is that the discriminant

$$\Delta \equiv - A^2n^2 + 4\ B\ M < 0.$$

The condition for stability is thus

$$A^2n^2 \ /\ 4\ B\ M > 0.$$

APPLICATION TO 5″/38 AA COMMON PROJECTILE

For the 5-in, 38-cal AA common projectile with axial moment of inertia

$$A = 1.4\ \text{lb-ft}^2,$$

rotating

$$N = 1306\ \text{radians per second},$$

a torque of

$$K = AN\Omega = 1.4 \times 1306 \times 0.105$$

$$K = 192\ \text{lb-ft}^2/\text{sec}^2$$

is required to produce a precessional velocity of

$$\Omega = 6^0 = 0.105\ \text{radians per second.}$$

The overturning moment for $\alpha = 6^0$ is

$$M_O = \frac{1}{2} C_m \rho v^2 (\pi d^3/4)$$

$$(-0.037)\ (0.002378)\ (2600)^2\ (\pi/4)\ (5/12)^3$$

$$-19.7\ \text{ft-lbs.}$$

For $\alpha = 4^0$, similarly, with $C_m = -0.022$,

$$M_O = -11.7\ \text{ft-lbs.}$$

The values of the aerodynamic moment coefficient C_m are taken from the German wind-tunnel evaluations* on a projectile of somewhat similar shape and, consequently, give only the general order of magnitude of forces to be expected. This overturning moment is observed to increase with yaw.

CONCLUSIONS

While some precessional guiding appears possible, the rate-of-turn limitations do

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* SANN, B., "Auftrieb-, Widerstands- und Seitenkraftmessungen an rotierenden Langgeschossen bei Unterschallgeschwindigkeit", Deutsche Luftfahrtforschung, Forschungsberichte No. 1065 (1939 May 20).

not seem to fit the requirements of maneuverability too well. The spiral motion of the projectile, maintained somewhat undamped by jet assistance, could be changed only gradually in direction as a result of precession due to intermittent impulses.

FUTURE POSSIBILITIES OF ATTACK

The natural advantages offered by jet propulsion are such that they should be put to use. Let us consider the motion of a jet-assisted shell fired from a gun at fairly high velocity. If the shell is spin-stabilized, its motion will be helicoidal, that is, in a spiral. If at one position along its spiral path it is pointed directly at the target, a burst from the jet would tend to drive it directly towards the target. However, due to the aerodynamic and precessional forces involved, the result of this thrust will be a further spiral course with an amplitude which, while momentarily enlarged, is gradually damped down by the gyrostatic force of the spinning shell. Again, in the course of the ensuing spiral motion, there will be a position where the shell is again pointed at the target, when another burst from the jet will drive it homeward, with a momentarily increased spiral but changed in direction. The momentary initial axis of each successive spiral would point very nearly targetward.

The feasibility of such a motion as described can be evaluated from the results of tests of the 75-mm shell (M4) at different initial velocities, as shown in Figure 1. The unbalance of the shell produced by the initial blast from the gun, resulting in a rapidly damped spiral motion of the shell, would be reproduced from time to time by the intermittent thrust of the controlled jet of the rocket-assisted shell. It will be observed that initial yaws of as high as 9° at 1500 feet per second initial velocity, and still 5° at 3800 feet per second can be expected within a yaw period of from 162 ft to some 1100 ft.,

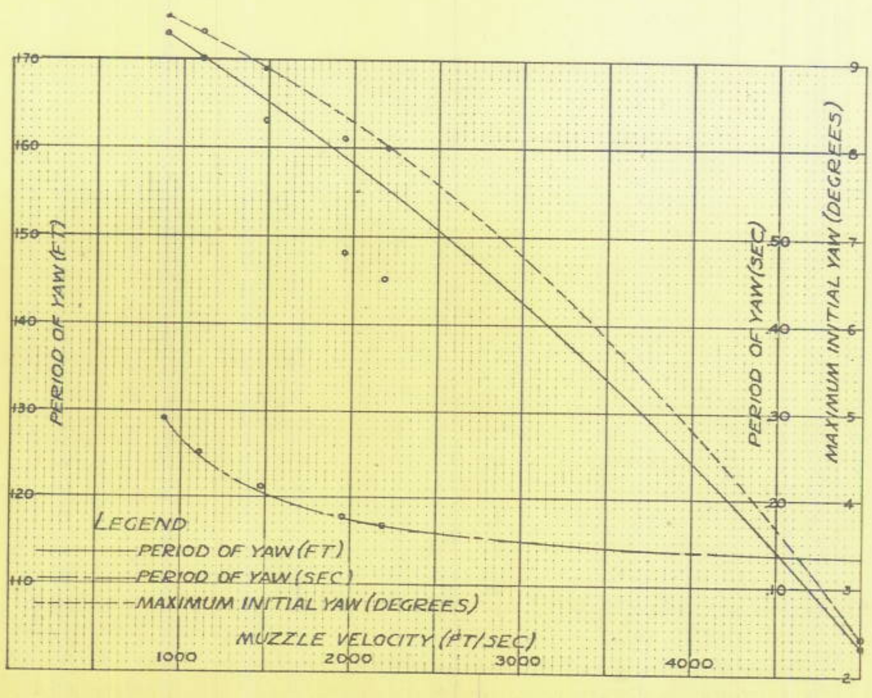

Figure 1 - Helicoidal Motion of 75mm Shell M4

correspondingly, as measured in the direction of forward motion along the axis of the spiral. An appreciably rapid change in direction could, accordingly, be accomplished.

That the presently proposed guidance path possesses the disadvantages inherent in a pursuit course should be recognized. Future position prediction of ordinary gun fire, however, has its limitations; the beam rider presents the difficulty of capture; and various navigational courses are as yet undeveloped. With high-velocity missiles, that is, high as compared with target velocity, the pursuit course disadvantage might be small enough to be overshadowed by the advantages to be gained from course corrections mid-flight against a maneuvering target.

APPENDIX I

The Total Energy Equation

The transformation of the total energy equation

$$T = \frac{1}{2} A\left(\dot{\varphi} + \dot{\psi}\cos\vartheta\right)^2 + \frac{1}{2} B\left(\dot{\vartheta}^2 + \dot{\psi}^2 \sin^2\vartheta\right) - M\cos\vartheta$$

into

$$T = \frac{1}{2} An\left(\xi\dot{\eta} - \eta\dot{\xi}\right) + \frac{1}{2} B\left(\dot{\xi}^2 + \dot{\eta}^2\right) + \frac{1}{2} M\left(\xi^2 + \eta^2\right) - M$$

follows Whittaker†. The detail, there omitted, is here supplied in order to make clear what approximations are made.

The transformation

$$\sin\vartheta\cos\varphi = \xi$$
$$\sin\vartheta\sin\varphi = \eta$$

gives, by division,

$$\cot\varphi = \xi/\eta$$

and, by squaring and adding,

$$\sin^2\vartheta = \xi^2 + \eta^2$$

Differentiation of cot φ gives

$$\csc^2\varphi\ \dot{\varphi} = \left(\xi\dot{\eta} - \eta\dot{\xi}\right)/\eta^2$$

$$\left(\xi^2 + \eta^2\right)\dot{\varphi} = \xi\dot{\eta} - \eta\dot{\xi}$$

$$\sin^2\vartheta\ \dot{\varphi} = \xi\dot{\eta} - \eta\dot{\xi}$$

† WHITTAKER, E. T., A Treatise on the Analytical Dynamics of Particles and Rigid Bodies, 4th Ed. (Cambridge, Univ. Press, 1937)

The approximation

$$1-\cos\vartheta = 1-\left[1-(\xi^2+\eta^2)\right]^{1/2} \doteq \tfrac{1}{2}(\xi^2+\eta^2)$$

easily results from the binomial expansion

$$(1-p)^{1/2} = 1-\tfrac{1}{2}p-\tfrac{1}{8}p^2-\cdots ,$$

neglecting fourth and higher powers of the variables.

The approximation

$$\dot{\vartheta}^2+\dot{\varphi}^2\sin^2\vartheta \doteq \dot{\xi}^2+\dot{\eta}^2$$

can be had in the following fashion

$$\dot{\varphi}\sin^2\vartheta = \xi\dot{\eta}-\eta\dot{\xi}$$

$$\dot{\varphi} = (\xi\dot{\eta}-\eta\dot{\xi})/(\xi^2+\eta^2)$$

$$\dot{\varphi}^2\sin^2\vartheta = (\xi\dot{\eta}-\eta\dot{\xi})^2/(\xi^2+\eta^2)$$

and

$$\vartheta = \mathrm{As}\,(\xi^2+\eta^2)^{1/2}$$

$$\dot{\vartheta} = (\xi^2+\eta^2)^{-1/2}(\xi\dot{\xi}+\eta\dot{\eta})/(1-\xi^2-\eta^2)^{1/2}$$

Hence

$$\dot{\vartheta}^2+\dot{\varphi}^2\sin^2\vartheta = \frac{(\xi\dot{\xi}+\eta\dot{\eta})^2+(\xi\dot{\eta}-\eta\dot{\xi})^2(1-\xi^2-\eta^2)}{(\xi^2+\eta^2)(1-\xi^2-\eta^2)}$$

$$= \frac{\xi^2\dot{\xi}^2+2\xi\dot{\xi}\eta\dot{\eta}+\eta^2\dot{\eta}^2+\xi^2\dot{\eta}^2-2\xi\dot{\eta}\eta\dot{\xi}+\eta^2\dot{\xi}^2-(\xi^2+\eta^2)(\xi\dot{\eta}-\eta\dot{\xi})^2}{(\xi^2+\eta^2)(1-\xi^2-\eta^2)}$$

$$= \frac{(\xi^2+\eta^2)\left[\dot{\xi}^2+\dot{\eta}^2-(\xi\dot{\eta}-\eta\dot{\xi})^2\right]}{(\xi^2+\eta^2)(1-\xi^2-\eta^2)}$$

and thus the approximation, neglecting fourth order products,

$$\dot{\vartheta}^2+\dot{\varphi}^2\sin^2\vartheta \doteq \dot{\xi}^2+\dot{\eta}^2.$$

APPENDIX II

The Equations of Motion of a Missile

Under the assumptions of axial symmetry and of invariance with respect to translation of the center of gravity along this axis, NIELSEN and SYNGE‡ give the following conditions for the motion of a projectile:

$$\dot{\xi} + i\xi\Omega_3 - iw\eta = \xi X + \eta Y + g\cos\vartheta$$
$$\dot{\eta} + i\eta\Omega_3 - iC\omega_3\eta = \xi X' + \eta Y'$$
$$\dot{w} - u\omega_2 + v\omega_1 = F_3/m \quad - g\sin\vartheta$$
$$\omega_3 = G_3/C$$

with

$$X = P/M, \quad Y = Q/M, \quad X' = P'/A, \quad Y' = Q'/A,$$

where

- i, j, k reference triad has k along the shell axis
- m = mass of shell
- A = axial moment of inertia at O
- C = transverse moment of inertia at O
- O = mass center of shell
- ω = angular velocity of shell = $\omega_1 i + \omega_2 j + \omega_3 k$
- F = vector sum of aerodynamic forces on shell = $F_1 i + F_2 j + F_3 k$
- G = moment of aerodynamic forces about O = $G_1 i + G_2 j + G_3 k$
- F′ = weight of shell = mg cos ϑ
- q = velocity of O = ui + vj + wk
- P = $P_1 + iP_2$ = body force per unit mass acting on shell at O*
- Q = $Q_1 + iQ_2$
- ξ = u + iv
- O* = geometric centroid of solid (of uniform density)
- Ω = angular velocity of triad = $\Omega_1 i + \Omega_2 j + \Omega_3 k$

‡ NIELSEN, K. L., and SYNGE, J. L., "On the Motion of a Spinning Shell", National Research Council of Canada, reproduced in Aberdeen Proving Ground, Ballistics Research Laboratory Report #X-116 (1943 January 5).

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APPENDIX III

Characteristic Shock Waves

In order to study flow past a cone in a steady, isentropic, irrotational, non-viscous fluid, let

- x = distance from nose along symmetry axis of shell
- r = distance from shell axis
- q = flow velocity $\quad q^2 = u^2 + v^2$
- u = flow velocity in x-direction
- v = flow velocity in r-direction
- ρ = air density $= \rho(q)$
- c = local speed of sound
- η = entropy
- γ = specific heat ratio $\gamma = 1.405$ for air
- p = pressure $= p(\rho, \eta)$
- φ = potential
- ψ = stream function

Then the exact equations for three-dimensional flow with axial symmetry are:

$$\frac{\partial \varphi}{\partial x} = u = \frac{1}{\rho r}\frac{\partial \psi}{\partial r}, \qquad \frac{\partial \varphi}{\partial r} = v = -\frac{1}{\rho r}\frac{\partial \psi}{\partial r}$$

and

$$\frac{\partial}{\partial x}\left(\rho r \frac{\partial \varphi}{\partial x}\right) + \frac{\partial}{\partial r}\left(\rho r \frac{\partial \varphi}{\partial r}\right) = 0$$

Continuity Equations

$$\frac{\partial}{\partial x}\left(\frac{1}{\rho r}\frac{\partial \psi}{\partial x}\right) + \frac{\partial}{\partial r}\left(\frac{1}{\rho r}\frac{\partial \psi}{\partial r}\right) = 0$$

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