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77 Massachusetts Avenue
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14. ABSTRACT

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RPPR Final Report
as of 16-May-2020

Agency Code:

Proposal Number: 70674MAYIP **Agreement Number: W911NF-17-1-0068**

INVESTIGATOR(S):

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DUNS Number: 001425594 EIN: 042103594

Report Date: 01-Apr-2020 Date Received: 15-May-2020
Final Report for Period Beginning 02-Jan-2017 and Ending 01-Jan-2020
Title: Smooth Modeling of Flows on Graphs
Begin Performance Period: 02-Jan-2017 **End Performance Period:** 01-Jan-2020
Report Term: 0-Other
Submitted By: Justin Solomon Email: jsolomon@mit.edu
Phone: (703) 623-4762
Distribution Statement: 1-Approved for public release; distribution is unlimited.

STEM Degrees: 4 **STEM Participants:** 34

Major Goals: The high-level goal of this project is to bridge the gap between theoretical developments in the field of optimal transport---designed to understand flows along smooth domains---and the analysis of signals over graphs. In contrast to existing combinatorial algorithms and models for flows on graphs, our approach is inspired by continuous ideas from differential equations, functional analysis, and geometry. In the end, we propose fundamentally different constructions from well-known graph algorithms that more directly link smooth (continuously-varying in time and space) and discrete interpretations of flows.

While mathematically-oriented research is subject to adjustment as conjectures are verified or disproven and as models are tested against collected data, the goals laid out in the original proposal for this project are as follows:

* Modeling with quadratic transportation costs (year 1)
* Adaptation of fluid based models for transport (year 1)
* Discretization and numerical optimization (years 1-2)
* Application to interpolation (year 2)
* Application to data sequences and extrapolation (years 2-3)
* Extension to unbalanced transport and gradient flows (year 3)
* Application in computational disciplines (year 3)

In addition to these research-oriented tasks, as a grant for the Army Young Investigator Award, a secondary goal is to support the inception of the new MIT Geometric Data Processing group led by Prof. Justin Solomon, and to open lines of communication and feedback between the group and various research/engineering organizations supported by the Army.

Accounting for minor adjustments expected from speculative mathematical research, the goals outlined above were largely achieved (100% completion) and successful. Work funded from this grant has fundamentally advanced the theory and practice of computational optimal transport and brought new insights into its connections with graph theory, large-scale optimization, statistics, and applications. Moreover, we are pleased to continue related work through future engagements with Army Research.

Accomplishments: We refer readers to the annual reports associated with this proposal for details about the many activities we carried out during the period of support.

Speaking broadly, the research supported from this Army Young Investigator Award was overwhelmingly successful. Note that this report is accompanied by 33 publications produced by the team during the period of support, as well as multiple theses/dissertations. Many of these publications are already well-cited, and they have built the foundation for research in our group.

The main goal in the proposed research was to bring insight from smooth differential equation-style flow problems to discrete problems on graphs and networks. In the process, we have considered development of practical algorithms for optimal transport on domains typically considered in the computer science and operations communities, primarily graphs and discretized geometric domains. Below, we highlight some of the many outcomes of this work including pointers into the bibliographic references provided in this report.

On the mathematical side, our group has provided mathematical theory and numerical methods for flow-based optimal transport, including theoretical results showing that these new approaches to network flow problems preserve basic properties of the smooth analog (see e.g. [Lavenant et al. 2018]). We have advanced understanding of how optimal transport computation interacts with noise, adversarial perturbation, and sampling (see e.g. [Claici et al. 2018; Frogner et al. 2019] among many others), as well as extensions for specially-structured instances that appear in applications (see e.g. [Monteiller et al. 2019; Yurochkin et al. 2019] among others). We also have advanced fundamental understanding of transport problems on graphs [Essid et al. 2018; Abrishami et al. 2019].

A remarkable side effect of our research---and progress in applied optimal transport broadly---is that the algorithms for optimal transport have advanced to a point where they can be used in a variety of applications. Our research reflects this trend, with practical implications for many different application areas. To name a few, our work has led to state-of-the-art methods for tasks in data sparsification [Claici et al. 2019], distributionally robust learning [Frogner et al. 2019], Bayesian inference [Monteiller et al. 2019], document retrieval [Yurochkin et al. 2019], audio effects [Henderson and Solomon 2019], vector field design [Solomon and Vaxman 2019], representation learning [Frogner et al. 2019], interpolation [Lavenant et al. 2018], and regularized graph matching [Essid and Solomon 2018].

A secondary goal of this Young Investigator project was to provide support for establishing Prof. Solomon's research group at MIT, the Geometric Data Processing (GDP) group in the MIT Computer Science and Artificial Intelligence Laboratory (CSAIL). This goal has been accomplished unequivocally. At the time of submitting this proposal in 2017, our group had two graduate students and one short-term postdoctoral associate. Currently, we have expanded to four postdoctoral associates, eight PhD students, two MEng students, and many interns and undergraduate researchers. While of course the finances needed to support a group of this scale come from many sources, a large part of the team is dedicated to pursuing a research program linked to results and ideas generated by work supported by this grant.

Moving forward, we are pleased to recently have received notification that a proposal titled "Geometric Approaches to Near-Optimization" has been accepted for funding through the ARO Probability and Statistics program. While this proposal is shorter term, we are excited to continue our research into geometry, probability, and optimal transport. We also remain eager to expand our interaction with the ARO through other projects and will gladly submit proposals or updates on our going work at any point.

RPPR Final Report
as of 16-May-2020

Training Opportunities: The Geometric Data Processing (GDP) group is intensively involved in programs intended to train future generations of researchers in geometry and related fields at all levels spanning from introductory to advanced.

Locally, the group currently houses four postdoctoral research associates, three of whom are in the final stages of their time at MIT. Two of these postdoctoral associates have received offers for faculty positions (Washington State University mathematics department; Boston University computer science department), and the third is pursuing industrial research positions in machine learning and autonomous driving. In addition to mentorship from Prof. Solomon, these postdoctoral associates have participated in mentorship and advising programs hosted by the MIT Computer Science and Artificial Intelligence Laboratory (CSAIL), including seminars/workshops on the job search and giving talks.

Prof. Solomon mentors a large number of undergraduate students through MIT's UROP and SuperUROP programs; these undergraduate research programs support semester- and year-long research, respectively. Undergraduates hosted in the Geometric Data Processing group are afforded the same consideration as PhD students, including weekly meetings with Prof. Solomon and/or GDP student mentors. In total, the group has hosted over 20 undergraduate researchers since 2016, including 10 year-long intensive SuperUROPs. Note these students also receive research advising from the Electrical Engineering and Computer Science (EECS) department and MIT Writing Center.

Each summer, the group hosts a high school student researcher through the Research Science Institute program. The summer student in 2019 worked on open problems in graph theory, extending complexity-theoretic results about sampling and enumerating graph partitions to certain graphs of bounded treewidth. The team has volunteered to host another RSI student in summer 2020 (remotely, given current quarantine considerations).

The group also has taken on a number of long-term research visitors, who receive mentorship at MIT and also interact with the students in the group. A sampling of our visitors include several students from the École Normale Supérieure (ENS) program in France focusing on optimal transport theory, a visiting PhD student from the University of Wisconsin mathematics department focusing on sampling problems in graph partitioning, a visiting PhD student from UT Austin focusing on numerical algorithms on discretized surfaces, and a visiting PhD student from RWTH Aachen focusing on surface correspondence problems.

Prof. Solomon has also incorporated results from the research supported by this grant into his professional education course titled "Modeling and Optimization for Machine Learning" (MIT Professional Education), co-taught annually with Prof. Suvrit Sra.

Finally, Prof. Solomon and staff regularly attend and contribute to a broad variety of research conferences and gatherings, including the International Conference on Machine Learning (ICML), NeurIPS, the Symposium on Geometry processing, ACM SIGGRAPH, and several events supported by the Society for Industrial and Applied Mathematics (SIAM). They also have organized workshops, conferences, and summer-long undergraduate research programs, as detailed in past reports for this grant.

RPPR Final Report
as of 16-May-2020

Results Dissemination: Research supported by this ARO grant has been disseminated to the technical and general-interest communities through a variety of avenues. ARO support is acknowledged in all printed research reports and related products.

Of course, the major means of dissemination in our discipline is through publication. Readers are invited to see the official list of research products accompanying this report for a full list.

A unique aspect of the research by Prof. Solomon and students/staff in the MIT Geometric Data Processing Group is that it appears in an interdisciplinary set of venues. Publications reported in this project appeared in several top-tier venues including:

* ACM Transactions on Graphics/SIGGRAPH (computer graphics)
* AMS Short Courses (mathematics)
* Computer Graphics Forum (computer graphics)
* Computer Vision and Pattern Recognition (computer vision)
* European Journal of Applied Mathematics (applied mathematics)
* International Conference on Computer Vision (vision)
* International Conference on Digital Audio Effects (audio)
* International Conference on Machine Learning (machine learning)
* NeurIPS (machine learning)
* SIAM Journal on Applied Algebra and Geometry (applied mathematics)
* SIAM Journal on Scientific Computing (numerical mathematics)
* Snapshots of Modern Mathematics from Oberwolfach (mathematics)

Beyond publications, Prof. Solomon and staff have given invited talks at a variety of institutions. Please refer to annual project reports for comprehensive lists of the many relevant invited talks.

The group disseminates its work to the broader community through several means. They founded and host the annual New England Symposium on Graphics, open to the broad area computer graphics research community. Prof. Solomon is organizing several upcoming workshops and conferences and plans to host a broad, large-scale conference on applied optimal transport in the coming year.

Material from the ongoing research also is incorporated into Prof. Solomon's course ("Shape Analysis") in spring 2019; all materials for this course are available for free online, including lecture videos, exercises, and course notes under development.

RPPR Final Report
as of 16-May-2020

Honors and Awards: Research and staff supported by this grant have received several honors and awards throughout the period of support. These include the following:

* DAFx best student paper award (MEng student Trevor Henderson)
* DOE Computational Science Graduate Fellowship (PhD student Paul Zhang)
* Hertz Graduate Fellowship (PhD student David Palmer)
* Mathworks Fellowship (PhD student Paul Zhang)
* MIT First-Year Graduate Fellowships (PhD students David Palmer, Yue Wang, Paul Zhang, Lingxiao Li)
* NSF "Vizzie Challenge" Experts' Choice Winner (PhD student Dmitriy Smirnov)
* NSF Graduate Research Fellowship (PhD student Mazdak Abulnaga)
* Siebel Fellowship (PhD students Mazdak Abulnaga, Yu Wang)
* NVidia research Fellowship (PhD student Yue Wang)
* Pogosyants UROP Award (undergraduate Douglas Stryker)

Prof. Solomon has received broad recognition of his work, including the following:

* Amazon Research Award in artificial intelligence
* Forbes 30 Under 30: Science
* Inaugural member, ACM Future of Computing Academy
* Junior Bose Award for Excellence in Teaching
* Prof. Amar G. Bose Research Fellowship

The team's research has been highlighted by many articles in the MIT News, including the following:

* Bose Grants for 2017 reward bold and unconventional research visions (December 14, 2017)
http://news.mit.edu/2017/bose-grants-reward-bold-and-unconventional-research-visions-1214
* Design tool reveals a product's many possible performance tradeoffs (August 16, 2018)
https://www.csail.mit.edu/news/design-tool-reveals-products-many-possible-performance-tradeoffs
* Smoothing out sketches' rough edges (September 11, 2018)
https://www.csail.mit.edu/news/smoothing-out-sketches-rough-edges
* Using math to blend musical notes seamlessly (September 27, 2019)
https://www.csail.mit.edu/news/using-math-blend-musical-notes-seamlessly
* Better fetal health - by building a map of the placenta (October 20, 2019)
https://www.csail.mit.edu/news/better-fetal-health-building-map-placenta
* Deep learning with point clouds (October 29, 2019)
https://www.csail.mit.edu/news/deep-learning-point-clouds
* Finding a good read among billions of choices (December 20, 2019)
https://www.csail.mit.edu/news/finding-good-read-among-billions-choices

Protocol Activity Status:

Technology Transfer: Nothing to Report

PARTICIPANTS:

Participant Type: Postdoctoral (scholar, fellow or other postdoctoral position)
Participant: Mikhail Bessmeltsev
Person Months Worked: 15.00 **Funding Support:**
Project Contribution:
International Collaboration:
International Travel:
National Academy Member: N
Other Collaborators:

Participant Type: Postdoctoral (scholar, fellow or other postdoctoral position)
Participant: Edward Chien
Person Months Worked: 6.00 **Funding Support:**
Project Contribution:

International Collaboration:
International Travel:
National Academy Member: N
Other Collaborators:

Participant Type: PD/PI
Participant: Justin Solomon
Person Months Worked: 15.00 **Funding Support:**
Project Contribution:
International Collaboration:
International Travel:
National Academy Member: N
Other Collaborators:

Participant Type: Postdoctoral (scholar, fellow or other postdoctoral position)
Participant: Daryl DeFord
Person Months Worked: 10.00 **Funding Support:**
Project Contribution:
International Collaboration:
International Travel:
National Academy Member: N
Other Collaborators:

Participant Type: Graduate Student (research assistant)
Participant: Sebastian Claici
Person Months Worked: 15.00 **Funding Support:**
Project Contribution:
International Collaboration:
International Travel:
National Academy Member: N
Other Collaborators:

Participant Type: Graduate Student (research assistant)
Participant: Nilai Sarda
Person Months Worked: 9.00 **Funding Support:**
Project Contribution:
International Collaboration:
International Travel:
National Academy Member: N
Other Collaborators:

Participant Type: Other (specify)
Participant: Pierre Monteiller
Person Months Worked: 4.00 **Funding Support:**
Project Contribution:
International Collaboration:
International Travel:
National Academy Member: N
Other Collaborators:

Participant Type: Undergraduate Student
Participant: Yiping Lu
Person Months Worked: 3.00 **Funding Support:**

Project Contribution:
International Collaboration:
International Travel:
National Academy Member: N
Other Collaborators:

Participant Type: Other (specify)
Participant: Hugo Lavenant
Person Months Worked: 3.00 **Funding Support:**
Project Contribution:
International Collaboration:
International Travel:
National Academy Member: N
Other Collaborators:

Participant Type: Undergraduate Student
Participant: Trevor Henderson
Person Months Worked: 4.00 **Funding Support:**
Project Contribution:
International Collaboration:
International Travel:
National Academy Member: N
Other Collaborators:

Participant Type: High School Student
Participant: Qianqia Zhang
Person Months Worked: 2.00 **Funding Support:**
Project Contribution:
International Collaboration:
International Travel:
National Academy Member: N
Other Collaborators:

Participant Type: High School Student
Participant: Simon Lam
Person Months Worked: 2.00 **Funding Support:**
Project Contribution:
International Collaboration:
International Travel:
National Academy Member: N
Other Collaborators:

Participant Type: High School Student
Participant: Ignasi Segura Vicente
Person Months Worked: 2.00 **Funding Support:**
Project Contribution:
International Collaboration:
International Travel:
National Academy Member: N
Other Collaborators:

ARTICLES:

Publication Type: Journal Article Peer Reviewed: Y **Publication Status:** 1-Published
Journal: European Journal of Applied Mathematics
Publication Identifier Type: Publication Identifier:
Volume: Issue: First Page #:
Date Submitted: 1/31/18 12:00AM Date Published:
Publication Location:
Article Title: Quantum optimal transport for tensor field processing
Authors: Gabriel Peyre, Lenaic Chizat, Francois-Xavier Vialard, Justin Solomon
Keywords: Optimal transport, tensors
Abstract: This article introduces a new notion of optimal transport (OT) between tensor fields, which are measures whose values are positive semidefinite (PSD) matrices. This "quantum" formulation of OT (Q-OT) corresponds to a relaxed version of the classical Kantorovich transport problem, where the fidelity between the input PSD-valued measures is captured using the geometry of the Von-Neumann quantum entropy. We propose a quantum-entropic regularization of the resulting convex optimization problem, which can be solved efficiently using an iterative scaling algorithm. This method is a generalization of the celebrated Sinkhorn algorithm to the quantum setting of PSD matrices. We extend this formulation and the quantum Sinkhorn algorithm to compute barycenters within a collection of input tensor fields. We illustrate the usefulness of the proposed approach on applications to procedural noise generation, anisotropic meshing, diffusion tensor imaging and spectral texture synthesis.
Distribution Statement: 3-Distribution authorized to U.S. Government Agencies and their contractors
Acknowledged Federal Support: **Y**

Publication Type: Journal Article Peer Reviewed: Y **Publication Status:** 1-Published
Journal: SIAM Journal on Scientific Computing
Publication Identifier Type: ISSN Publication Identifier: 1095-7197
Volume: 40 Issue: 4 First Page #: A1961
Date Submitted: 8/8/18 12:00AM Date Published: 7/23/18 4:00AM
Publication Location:
Article Title: Quadratically-regularized optimal transport on graphs
Authors: Montacer Essid, Justin Solomon
Keywords: Optimal transport, graphs
Abstract: Optimal transportation provides a means of lifting distances between points on a geometric domain to distances between signals over the domain, expressed as probability distributions. On a graph, transportation problems can be used to express challenging tasks involving matching supply to demand with minimal shipment expense; in discrete language, these become minimum-cost network flow problems. Regularization typically is needed to ensure uniqueness for the linear ground distance case and to improve optimization convergence; state-of-the-art techniques employ entropic regularization on the transportation matrix. In this paper, we explore a quadratic alternative to entropic regularization for transport over a graph. We theoretically analyze the behavior of quadratically-regularized graph transport, characterizing how regularization affects the structure of flows in the regime of small but nonzero regularization. We further exploit elegant second-order structure in the dual of this prob
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Volume: 36 Issue: 5 First Page #: 37
Date Submitted: 1/31/18 12:00AM Date Published: 8/1/17 12:00PM
Publication Location:
Article Title: Isometry-Aware Preconditioning for Mesh Parameterization
Authors: Sebastian Claici, Mikhail Bessmeltsev, Scott Schaefer, Justin Solomon
Keywords: Optimization, parameterization, manifold
Abstract: This paper presents a new preconditioning technique for large-scale geometric optimization problems, inspired by applications in mesh parameterization. Our positive (semi-)definite preconditioner acts on the gradients of optimization problems whose variables are positions of the vertices of a triangle mesh in R2 or of a tetrahedral mesh in R3, converting localized distortion gradients into the velocity of a globally near-rigid motion via a linear solve. We pose our preconditioning tool in terms of the Killing energy of a deformation field and provide new efficient formulas for constructing Killing operators on triangle and tetrahedral meshes. We demonstrate that our method is competitive with state-of-the-art algorithms for locally injective parameterization using a variety of optimization objectives and show applications to two- and three-dimensional mesh deformation.
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Acknowledged Federal Support: **Y**

Publication Type: Journal Article Peer Reviewed: Y **Publication Status:** 1-Published
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Volume: 38 Issue: 1 First Page #:
Date Submitted: 3/17/19 12:00AM Date Published: 2/1/19 5:00AM
Publication Location:
Article Title: Steklov Spectral Geometry for Extrinsic Shape Analysis
Authors: Yu Wang, Mirela Ben-Chen, Iosif Polterovich, Justin Solomon
Keywords: Shape analysis, boundary element method, triangle mesh
Abstract: We propose Steklov geometry processing, an extrinsic approach to spectral geometry processing and shape analysis. Intrinsic approaches, usually based on the Laplace-Beltrami operator, cannot capture the spatial embedding of a shape up to rigid motion, while many previous extrinsic methods lack theoretical justification. Instead, we propose a systematic approach by considering the Steklov eigenvalue problem, computing the spectrum of the Dirichlet-to-Neumann operator of a surface bounding a volume. A remarkable property of this operator is that it encodes the volumetric geometry. We use the boundary element method (BEM) to discretize the operator, accelerated by hierarchical numerical schemes and preconditioning; this pipeline allows us to solve eigenvalue and linear problems on large-scale meshes despite the density of the Dirichlet-to-Neumann discretization.
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Journal: NIPS
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Volume: Issue: First Page #:
Date Submitted: 1/31/18 12:00AM Date Published: 7/1/17 8:00AM
Publication Location:
Article Title: Parallel Streaming Wasserstein Barycenters
Authors: Matthew Staib, Sebastian Claici, Justin Solomon, Stefanie Jegelka
Keywords: Optimal transport, stochastic optimization
Abstract: Efficiently aggregating data from different sources is a challenging problem, particularly when samples from each source are distributed differently. One principled way to fuse probability distributions is via the lens of optimal transport. However, computing the barycenter scales poorly and requires discretization of all input distributions and the barycenter itself. Improving on this situation, we present a scalable, communication-efficient, parallel algorithm for computing the Wasserstein barycenter of arbitrary distributions. Our algorithm can operate directly on continuous input distributions and is optimized for streaming data. Our method is even robust to nonstationary input distributions and produces a barycenter estimate that tracks the input measures over time. The algorithm is semi-discrete, needing to discretize only the barycenter estimate. We also provide the first bounds on the quality of the approximate barycenter as the discretization becomes finer.
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Publication Type: Journal Article Peer Reviewed: Y **Publication Status:** 1-Published
Journal: ACM Transactions on Graphics
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Volume: 38 Issue: 1 First Page #:
Date Submitted: 3/17/19 12:00AM Date Published: 2/1/19 5:00AM
Publication Location:
Article Title: Vectorization of Line Drawings via Polyvector Fields
Authors: Mikhail Bessmeltsev, Justin Solomon
Keywords: Vectorization, frame fields
Abstract: Image tracing is a foundational component of the workflow in graphic design, engineering, and computer animation, linking hand-drawn concept images to collections of smooth curves needed for geometry processing and editing. Even for clean line drawings, modern algorithms often fail to faithfully vectorize junctions, or points at which curves meet; this produces vector drawings with incorrect connectivity. This subtle issue undermines the practical application of vectorization tools and accounts for hesitance among artists and engineers to use automatic vectorization software. To address this issue, we propose a novel image vectorization method based on state-of-the-art mathematical algorithms for frame field processing. Our algorithm is tailored specifically to disambiguate junctions without sacrificing quality.
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Publication Type: Journal Article Peer Reviewed: Y **Publication Status:** 4-Under Review
Journal: AMS Short Course on Discrete Differential Geometry
Publication Identifier Type: Publication Identifier:
Volume: Issue: First Page #:
Date Submitted: 2/6/18 12:00AM Date Published:
Publication Location:
Article Title: Optimal Transport on Discrete Domains
Authors: Justin Solomon
Keywords: optimal transport, discretization, numerics
Abstract: Inspired by the matching of supply to demand in logistical problems, the optimal transportation (or Monge-Kantorovich) problem involves the matching of probability distributions defined over a geometric domain such as a surface or manifold. After discretization, optimal transportation becomes a large-scale linear program, which typically is infeasible to solve efficiently on triangle meshes, graphs, point clouds, and other domains encountered in graphics and machine learning. Recent breakthroughs in numerical optimal transportation enable scalability to orders-of-magnitude larger problems, solvable in a fraction of a second. In these lecture notes, we discuss advances in numerical optimal transport that leverage understanding of both discrete and smooth aspects of the problem. State-of-the-art techniques in discrete optimal transportation combine insight from partial differential equations (PDE) with convex analysis to reformulate, discretize, and optimize transportation problems.
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Acknowledged Federal Support: **Y**

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Journal: Snapshots of Modern Mathematics from Oberwolfach
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Date Submitted: 8/8/18 12:00AM Date Published: 12/21/17 10:00AM
Publication Location:
Article Title: Computational Optimal Transport
Authors: Justin Solomon
Keywords: optimal transport, scientific computing, numerics
Abstract: Optimal transport is the mathematical discipline of matching supply to demand while minimizing shipping costs. This matching problem becomes extremely challenging as the quantity of supply and demand points increases; modern applications must cope with thousands or millions of these at a time. Here, we introduce the computational optimal transport problem and summarize recent ideas for achieving new heights in efficiency and scalability.
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Date Submitted: 8/8/18 12:00AM Date Published: 7/1/18 12:00PM
Publication Location:
Article Title: Kernel Functional Maps
Authors: Larry Wang, Anne Gehre, Michael Bronstein, Justin Solomon
Keywords: correspondence, geometry
Abstract: Functional maps provide a means of extracting correspondences between surfaces using linear-algebraic machinery. While the functional framework suggests efficient algorithms for map computation, the basic technique does not incorporate the intuition that pointwise modifications of a descriptor function (e.g. composition of a descriptor and a nonlinearity) should be preserved under the mapping; the end result is that the basic functional maps problem can be underdetermined without regularization or additional assumptions on the map. In this paper, we show how this problem can be addressed through kernelization, in which descriptors are lifted to higher-dimensional vectors or even infinite-length sequences of values. The key observation is that optimization problems for functional maps only depend on inner products between descriptors rather than descriptor values themselves. These inner products can be evaluated efficiently through use of kernel functions. In addition to deriving a kern
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Date Submitted: 8/8/18 12:00AM Date Published: 7/1/18 8:00PM
Publication Location:
Article Title: Interactive Curve Constrained Functional Maps
Authors: Anne Gehre, Michael Bronstein, Leif Kobbelt, Justin Solomon
Keywords: shape correspondence
Abstract: Functional maps have gained popularity as a versatile framework for representing intrinsic correspondence between 3D shapes using algebraic machinery. A key ingredient for this framework is the ability to find pairs of corresponding functions (typically, feature descriptors) across the shapes. This is a challenging problem on its own, and when the shapes are strongly non-isometric, nearly impossible to solve automatically. In this paper, we use feature curve correspondences to provide flexible abstractions of semantically similar parts of non-isometric shapes. We design a user interface implementing an interactive process for constructing shape correspondence, allowing the user to update the functional map at interactive rates by introducing feature curve correspondences. We add feature curve preservation constraints to the functional map framework and propose an efficient numerical method to optimize the map with immediate feedback. Experimental results show that our approach establis
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Journal: Proceedings of Machine Learning Research (Proc. ICML)
Publication Identifier Type: Publication Identifier:
Volume: 80 Issue: First Page #: 998
Date Submitted: 8/8/18 12:00AM Date Published: 7/1/18 4:00AM
Publication Location:
Article Title: Stochastic Wasserstein Barycenters
Authors: Sebastian Claici, Edward Chien, Justin Solomon
Keywords: optimal transport, barycenter, stochastic
Abstract: We present a stochastic algorithm to compute the barycenter of a set of probability distributions under the Wasserstein metric from optimal transport. Unlike previous approaches, our method extends to continuous input distributions and allows the support of the barycenter to be adjusted in each iteration. We tackle the problem without regularization, allowing us to recover a sharp output whose support is contained within the support of the true barycenter. We give exampleswhere our algorithm recovers a more meaningful barycenter than previous work. Our method is versatile and can be extended to applications such as generating super samples from a given distribution and recovering blue noise approximations.
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Date Submitted: 8/8/18 12:00AM Date Published: 8/12/18 8:00AM
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Article Title: Interactive Exploration of Design Trade-Offs
Authors: Adriana Schulz, Harrison Wang, Eitan Grinspun, Justin Solomon, Wojciech Matusik
Keywords: Pareto set
Abstract: Typical design for manufacturing applications requires simultaneous optimization of conflicting performance objectives: Design variations that improve one performance metric may decrease another performance metric. In these scenarios, there is no unique optimal design but rather a set of designs that are optimal for different trade-offs (called Pareto-optimal). In this work, we propose a novel approach to discover the Pareto front, allowing designers to navigate the landscape of compromises efficiently. Our approach is based on a first-order approximation of the Pareto front, which allows entire neighborhoods rather than individual points on the Pareto front to be captured. In addition to allowing for efficient discovery of the Pareto front and the corresponding mapping to the design space, this approach allows us to represent the entire trade-off manifold as a small collection of patches that comprise a high-quality and piecewise-smooth approximation. We illustrate how this technique
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Publication Location:
Article Title: Singularity-Constrained Octahedral Fields for Hexahedral Meshing
Authors: Heng Liu, Paul Zhang, Edward Chien, Justin Solomon, David Bommes
Keywords: Hexahedral meshing, topology
Abstract: Despite high practical demand, algorithmic hexahedral meshing with guarantees on robustness and quality remains unsolved. A promising direction follows the idea of integer-grid maps, which pull back the Cartesian hexahedral grid formed by integer isoplanes from a parametric domain to a surface-conforming hexahedral mesh of the input object. Since directly optimizing for a high-quality integer-grid map is mathematically challenging, the construction is usually split into two steps: (1) generation of a surfacealigned octahedral field and (2) generation of an integer-grid map that best aligns to the octahedral field. The main robustness issue stems from the fact that smooth octahedral fields frequently exhibit singularity graphs that are not appropriate for hexahedral meshing and induce heavily degenerate integer-grid maps. The first contribution of this work is an enumeration of all local configurations that exist in hex meshes with bounded edge valence, and a generalization of the Hopf-
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Date Submitted: 8/8/18 12:00AM Date Published:
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Article Title: Wasserstein Coresets for Lipschitz Costs
Authors: Sebastian Claici, Justin Solomon
Keywords: Optimal transport, coreset
Abstract: Sparsification is becoming more and more relevant with the proliferation of huge data sets. Coresets are a principled way to construct representative weighted subsets of a data set that have matching performance with the full data set for specific problems. However, coreset language neglects the nature of the underlying data distribution, which is often continuous. In this paper, we address this oversight by introducing a notion of measure coresets that generalizes coreset language to arbitrary probability measures. Our definition reveals a surprising connection to optimal transport theory which we leverage to design a coreset for problems with Lipschitz costs. We validate our construction on support vector machine (SVM) training, k-means clustering, k-median clustering, and linear regression and show that we are competitive with previous coreset constructions.
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Date Submitted: 3/17/19 12:00AM Date Published: 11/1/18 8:00AM
Publication Location:
Article Title: OptCuts: joint optimization of surface cuts and parameterization
Authors: Minchen Li, Danny Kaufman, Vladimir Kim, Justin Solomon, Alla Sheffer
Keywords: parameterization, topology, optimization
Abstract: Low-distortion mapping of three-dimensional surfaces to the plane is a critical problem in geometry processing. The intrinsic distortion introduced by these UV mappings is highly dependent on the choice of surface cuts that form seamlines which break mapping continuity. Parameterization applications typically require UV maps with an application-specific upper bound on distortion to avoid mapping artifacts; at the same time they seek to reduce cut lengths to minimize discontinuity artifacts. We propose OptCuts, an algorithm that jointly optimizes the parameterization and cutting of a three-dimensional mesh. OptCuts starts from an arbitrary initial embedding and a user-requested distortion bound. It requires no parameter setting and automatically seeks to minimize seam lengths subject to satisfying the distortion bound of the mapping computed using these seams.
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Article Title: Dynamical optimal transport on discrete surfaces
Authors: Hugo Lavenant, Sebastian Claici, Edward Chien, Justin Solomon
Keywords: optimal transport, discrete differential geometry
Abstract: We propose a technique for interpolating between probability distributions on discrete surfaces, based on the theory of optimal transport. Unlike previous attempts that use linear programming, our method is based on a dynamical formulation of quadratic optimal transport proposed for flat domains by Benamou and Brenier [2000], adapted to discrete surfaces. Our structure-preserving construction yields a Riemannian metric on the (finite-dimensional) space of probability distributions on a discrete surface, which translates the so-called Otto calculus to discrete language. From a practical perspective, our technique provides a smooth interpolation between distributions on discrete surfaces with less diffusion than state-of-the-art algorithms involving entropic regularization. Beyond interpolation, we show how our discrete notion of optimal transport extends to other tasks, such as distribution-valued Dirichlet problems and time integration of gradient flows.
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Volume: 38 Issue: 2 First Page #: 15
Date Submitted: 10/7/19 12:00AM Date Published: 4/1/19 8:00AM
Publication Location:
Article Title: Reversible Harmonic Maps between Discrete Surfaces
Authors: Danielle Ezuz, Justin Solomon, Mirela Ben-Chen
Keywords: Triangle meshes, correspondence
Abstract: Information transfer between triangle meshes is of great importance in computer graphics and geometry processing. To facilitate this process, a smooth and accurate map is typically required between the two meshes. While such maps can sometimes be computed between nearly-isometric meshes, the more general case of meshes with diverse geometries remains challenging. We propose a novel approach for direct map computation between triangle meshes without mapping to an intermediate domain, which optimizes for the harmonicity and reversibility of the forward and backward maps. Our method is general both in the information it can receive as input, e.g. point landmarks, a dense map or a functional map, and in the diversity of the geometries to which it can be applied. We demonstrate that our maps exhibit lower conformal distortion than the state-of-the-art, while succeeding in correctly mapping key features of the input shapes.
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Date Submitted: 3/17/19 12:00AM Date Published: 1/11/19 5:00AM
Publication Location:
Article Title: Functional Maps on Product Manifolds
Authors: Emanuele Rodola, Zorah Laehner, Alex Bronstein, Michael Bronstein, Justin Solomon
Keywords: correspondence, functional maps
Abstract: We consider the tasks of representing, analyzing and manipulating maps between shapes. We model maps as densities over the product manifold of the input shapes; these densities can be treated as scalar functions and therefore are manipulable using the language of signal processing on manifolds. Being a manifold itself, the product space endows the set of maps with a geometry of its own, which we exploit to define map operations in the spectral domain; we also derive relationships with other existing representations (soft maps and functional maps). To apply these ideas in practice, we discretize product manifolds and their Laplace--Beltrami operators, and we introduce localized spectral analysis of the product manifold as a novel tool for map processing. Our framework applies to maps defined between and across 2D and 3D shapes without requiring special adjustment, and it can be implemented efficiently with simple operations on sparse matrices.
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Date Submitted: 10/6/19 12:00AM Date Published:
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Article Title: Learning Embeddings into Entropic Wasserstein Spaces
Authors: Charlie Frogner, Farzaneh Mirzazadeh, Justin Solomon
Keywords: optimal transport, deep learning, embedding
Abstract: Despite their prevalence, Euclidean embeddings of data are fundamentally limited in their ability to capture latent semantic structures, which need not conform to Euclidean spatial assumptions. Here we consider an alternative, which embeds data as discrete probability distributions in a Wasserstein space, endowed with an optimal transport metric. Wasserstein spaces are much larger and more flexible than Euclidean spaces, in that they can successfully embed a wider variety of metric structures. We propose to exploit this flexibility by learning an embedding that captures the semantic information in the Wasserstein distance between embedded distributions. We examine empirically the representational capacity of such learned Wasserstein embeddings, showing that they can embed a wide variety of complex metric structures with smaller distortion than an equivalent Euclidean embedding.
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Journal: ArXiv
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Volume: Issue: First Page #:
Date Submitted: 11/13/19 12:00AM Date Published:
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Article Title: Deep Sketch-Based Modeling of Man-Made Shapes
Authors: Dmitriy Smirnov, Mikhail Bessmeltsev, Justin Solomon
Keywords: deep learning, geometry
Abstract: Sketch-based modeling aims to model 3D geometry using a concise and easy to create—but extremely ambiguous—input: artist sketches. Most conventional sketch-based modeling systems target smooth shapes and, to counter the ambiguity, put manually-designed priors on the 3D shape; they also typically require clean, vectorized input. Recent approaches attempt to learn those priors from data but often produce low-quality output. Focusing on piecewise-smooth man-made shapes, we address these issues by presenting a deep learning-based system to infer a complete man-made 3D shape from a single bitmap sketch.
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Date Submitted: 10/6/19 12:00AM Date Published: 9/2/19 8:00AM
Publication Location:
Article Title: Audio Transport: A Generalized Portamento via Optimal Transport
Authors: Trevor Henderson, Justin Solomon
Keywords: audio, optimal transport
Abstract: This paper proposes a new method to interpolate between two audio signals. As an interpolation parameter is changed, the pitches in one signal slide to the pitches in the other, producing a portamento, or musical glide. The assignment of pitches in one sound to pitches in the other is accomplished by solving a 1-dimensional optimal transport problem. In addition, we introduce several techniques that preserve the audio fidelity over this highly nonlinear transformation.
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Article Title: Optimal Transport-Based Polar Interpolation of Directional Fields
Authors: Justin Solomon, Amir Vaxman
Keywords: optimal transport, directional fields, topology
Abstract: We propose an algorithm that interpolates between vector and frame fields on triangulated surfaces, designed to complement field design methods in geometry processing and simulation. Our algorithm is based on a polar construction, leveraging a conservation law from the Hopf-Poincaré theorem to match singular points using ideas from optimal transport; the remaining detail of the field is interpolated using straightforward machinery. Our model is designed with topology in mind, sliding singular points along the surface rather than having them appear and disappear, and it caters to all surface topologies, including boundary and generator loops.
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