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TRANSFORMATION OF CRYSTALLOGRAPHIC ELASTIC AND PIEZOELECTRIC COEFFICIENTS

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ABSTRACT

Transformation equations for obtaining the elastic and piezoelectric coefficients of a crystal in a rectangular coordinate system rotated with respect to the crystallographic axes are given. The case of a general rotation about the three principal axes, and the special case of a rotation about a single axis, are treated.

PROBLEM STATUS

This is a final report on a small portion of the larger problem; work on the larger problem still continues.

AUTHORIZATION

NRL Problem No. S01-09R.

TRANSFORMATION OF CRYSTALLOGRAPHIC ELASTIC AND PIEZOELECTRIC COEFFICIENTS

INTRODUCTION

For most applications, crystal plates and bars assume orientations that are rotated with respect to the crystallographic axes. Hence it becomes expedient to derive formulae by which the elastic and piezoelectric coefficients in the transformed system are related to those in the original one.

Voigt[1] and Cady[2] have given expressions for several elastic and piezoelectric coefficients for various rotations. The relationships presented in this article comprise a complete list of expressions for general rotations and for particular rotations about each of three principal crystallographic axes.

We shall consider our original crystallographic system as a right-handed system specified by the axes X, Y, and Z. The transformed system after a rotation about any or all of the axes will be the new system X', Y', and Z'. Direction cosines relating the two are given by the following table.

	X'	Y'	Z'
X	α_1	β_1	γ_1
Y	α_2	β_2	γ_2
Z	α_3	β_3	γ_3

The angle of rotation is taken as positive when it appears counterclockwise to an observer looking back toward the origin from the positive end of the axis of rotation.

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[1] W. Voigt, Lehrbuch der Kristallphysik, Teubner, Leipzig, pp. 590-596, 838, and 840 (1910).

[2] W. G. Cady, Piezoelectricity, McGraw-Hill, pp. 70-79 and 194-196 (1946).

ELASTIC RELATIONS

The stress matrix T will be written as

$$T = \left\{ X_x ,\ Y_y ,\ Z_z ,\ Y_z ,\ Z_x ,\ X_y \right\}$$

The first three terms of the column matrix denote the normal stress components; the latter three, the shearing stress components. In like manner, the strain matrix S is

$$S = \left\{ x_x ,\ y_y ,\ z_z ,\ y_z ,\ z_x ,\ x_y \right\}$$

where the first three are normal strains and the latter shear strains.

Since the strain in an elastic body is proportional to the stress,

$$S = s\ T$$

where s, the matrix of elastic compliance coefficients is written as

$$s = \left[s_{hj} \right] = \begin{bmatrix} s_{11} & s_{12} & s_{13} & s_{14} & s_{15} & s_{16} \\ s_{21} & s_{22} & s_{23} & s_{24} & s_{25} & s_{26} \\ s_{31} & s_{32} & s_{33} & s_{34} & s_{35} & s_{36} \\ s_{41} & s_{42} & s_{43} & s_{44} & s_{45} & s_{46} \\ s_{51} & s_{52} & s_{53} & s_{54} & s_{55} & s_{56} \\ s_{61} & s_{62} & s_{63} & s_{64} & s_{65} & s_{66} \end{bmatrix}$$

Here $s_{hj} = s_{jh}$ yielding a total of 21 compliance coefficients. Also

$$T = c\ S \qquad (1)$$

where c is the matrix of elastic stiffness coefficients, and can be written as

$$c = \left[c_{hj} \right] = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{41} & c_{42} & c_{43} & c_{44} & c_{45} & c_{46} \\ c_{51} & c_{52} & c_{53} & c_{54} & c_{55} & c_{56} \\ c_{61} & c_{62} & c_{63} & c_{64} & c_{65} & c_{66} \end{bmatrix}$$

Analogously, $c_{hj} = c_{jh}$ giving 21 stiffness coefficients.[3]

Relating T and S in the new axial system to the old one, we have

$$T' = A\,T \quad \text{or} \quad T = A^{-1}\,T' \tag{2a}$$

and

$$S' = A_t^{-1}\,S \quad \text{or} \quad S = A_t\,S' \tag{2b}$$

where A and A^{-1} are given as follows:

$$A = \begin{bmatrix} \alpha_1^2 & \alpha_2^2 & \alpha_3^2 & 2\alpha_2\alpha_3 & 2\alpha_3\alpha_1 & 2\alpha_1\alpha_2 \\ \beta_1^2 & \beta_2^2 & \beta_3^2 & 2\beta_2\beta_3 & 2\beta_3\beta_1 & 2\beta_1\beta_2 \\ \gamma_1^2 & \gamma_2^2 & \gamma_3^2 & 2\gamma_2\gamma_3 & 2\gamma_3\gamma_1 & 2\gamma_1\gamma_2 \\ \beta_1\gamma_1 & \beta_2\gamma_2 & \beta_3\gamma_3 & \beta_2\gamma_3+\beta_3\gamma_2 & \beta_3\gamma_1+\beta_1\gamma_3 & \beta_1\gamma_2+\beta_2\gamma_1 \\ \gamma_1\alpha_1 & \gamma_2\alpha_2 & \gamma_3\alpha_3 & \gamma_2\alpha_3+\gamma_3\alpha_2 & \gamma_3\alpha_1+\gamma_1\alpha_3 & \gamma_1\alpha_2+\gamma_2\alpha_1 \\ \alpha_1\beta_1 & \alpha_2\beta_2 & \alpha_3\beta_3 & \alpha_2\beta_3+\alpha_3\beta_2 & \alpha_3\beta_1+\alpha_1\beta_3 & \alpha_1\beta_2+\alpha_2\beta_1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \alpha_1^2 & \beta_1^2 & \gamma_1^2 & 2\beta_1\gamma_1 & 2\gamma_1\alpha_1 & 2\alpha_1\beta_1 \\ \alpha_2^2 & \beta_2^2 & \gamma_2^2 & 2\beta_2\gamma_2 & 2\gamma_2\alpha_2 & 2\alpha_2\beta_2 \\ \alpha_3^2 & \beta_3^2 & \gamma_3^2 & 2\beta_3\gamma_3 & 2\gamma_3\alpha_3 & 2\alpha_3\beta_3 \\ \alpha_2\alpha_3 & \beta_2\beta_3 & \gamma_2\gamma_3 & \beta_2\gamma_3+\beta_3\gamma_2 & \gamma_2\alpha_3+\gamma_3\alpha_2 & \alpha_2\beta_3+\alpha_3\beta_2 \\ \alpha_3\alpha_1 & \beta_3\beta_1 & \gamma_3\gamma_1 & \beta_3\gamma_1+\beta_1\gamma_3 & \gamma_3\alpha_1+\gamma_1\alpha_3 & \alpha_3\beta_1+\alpha_1\beta_3 \\ \alpha_1\alpha_2 & \beta_1\beta_2 & \gamma_1\gamma_3 & \beta_1\gamma_2+\beta_2\gamma_1 & \gamma_1\alpha_2+\gamma_2\alpha_1 & \alpha_1\beta_2+\alpha_2\beta_1 \end{bmatrix}$$

A_t and A_t^{-1} are written by transposing A and A^{-1} respectively. A matrix is transposed by interchanging the rows and columns.

---

[3] The term "elastic compliance coefficient" used to designate the symbol s_{hj} and the term "elastic stiffness coefficient" used to designate the term c_{hj} follow the terminology used by Cady, op. cit., p. 49.

From (1) and (2)

$$T' = A\,c\,A_t S' \tag{3}$$

but

$$T' = c'\,S' \tag{4}$$

whence

$$c' = A\,c\,A_t \tag{5}$$

In like manner, the matrix of compliance coefficients in the transformed system will be related to s by

$$s' = A_t^{-1}\,s\,A^{-1} \tag{6}$$

PIEZOELECTRIC RELATIONS

When a crystal is subjected to a mechanical stress in the absence of an electric field, a polarization is set up according to the equation

$$P = d\,T \tag{7}$$

where P is the polarization matrix, T the stress matrix, and d the matrix of the piezoelectric strain coefficients. (See footnote 4, p. 5.) More explicitly

$$P = \left\{P_x,\ P_y,\ P_z\right\}$$

and

$$d = \left[d_{hk}\right] = \begin{bmatrix} d_{11} & d_{12} & d_{13} & d_{14} & d_{15} & d_{16} \\ d_{21} & d_{22} & d_{23} & d_{24} & d_{25} & d_{26} \\ d_{31} & d_{32} & d_{33} & d_{34} & d_{35} & d_{36} \end{bmatrix}$$

where $d_{hk} \neq d_{kh}$.

A rotation of axes will yield a transformed polarization P' related to P by

$$P' = a\,P \tag{8}$$

where

$$a = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix},$$

α, β, and γ denoting the direction cosines as before. The stress components of the original unprimed system will be related to those of the transformed or primed system by (2a).

Combining (7), (8), and (2a) we have

$$P' = a\, d\, A^{-1}\, T' \tag{9}$$

But

$$P' = d'\, T' \tag{10}$$

whence

$$d' = a\, d\, A^{-1} \tag{11}$$

In like manner, we may obtain an expression for e, the matrix of piezoelectric stress coefficients.[4] Polarization and strain are related according to

$$P = e\, S. \tag{12}$$

We write

$$e = \left[e_{hk}\right] = \begin{bmatrix} e_{11} & e_{12} & e_{13} & e_{14} & e_{15} & e_{16} \\ e_{21} & e_{22} & e_{23} & e_{24} & e_{25} & e_{26} \\ e_{31} & e_{32} & e_{33} & e_{34} & e_{35} & e_{36} \end{bmatrix}$$

where $e_{hk} \neq e_{kh}$. The strain components of the original system will be related to those of the transformed system by (2b). Combining (2b), (8) and (12), we have

$$P' = a\, e\, A_t\, S' \tag{13}$$

But

$$P' = e'\, S' \tag{14}$$

so that

$$e' = a\, e\, A_t \tag{15}$$

ELASTIC TRANSFORMATION

Rotation About All Three Axes

The generalized expressions for the transformed elastic compliances are given as follows:

4 The term "piezoelectric stress coefficient" for the symbol e_{hk} and the term "piezoelectric strain coefficient" for the symbol d_{hk} follow the terminology suggested by Cady, op. cit., p. 183.

$$s'_{11} = \alpha_1^4 s_{11} + \cdot\cdot + \alpha_1^2 \alpha_2^2 (2s_{12} + s_{66}) + \cdot\cdot$$
$$+ 2\alpha_1^2 \alpha_2 \alpha_3 (s_{14} + s_{56}) + \cdot\cdot$$
$$+ 2\alpha_1^3 (\alpha_2 s_{16} + \alpha_3 s_{15}) + \cdot\cdot$$

$$s'_{44} = 4\beta_1^2 \gamma_1^2 s_{11} + \cdot\cdot + (\beta_2 \gamma_3 + \beta_3 \gamma_2)^2 s_{44} + \cdot\cdot$$
$$+ 8\beta_1 \beta_2 \gamma_1 \gamma_2 s_{12} + \cdot\cdot$$
$$+ 4\beta_1 \gamma_1 \Big[(\beta_2 \gamma_3 + \beta_3 \gamma_2) s_{14} + (\beta_1 \gamma_3 + \beta_3 \gamma_1) s_{15}$$
$$+ (\beta_1 \gamma_2 + \beta_2 \gamma_1) s_{16} \Big] + \cdot\cdot$$
$$+ 2(\beta_1 \gamma_3 + \beta_3 \gamma_1)(\beta_1 \gamma_2 + \beta_2 \gamma_1) s_{56} + \cdot\cdot$$

$$s'_{12} = \alpha_1^2 \beta_1^2 s_{11} + \cdot\cdot + \alpha_2 \alpha_3 \beta_2 \beta_3 s_{44} + \cdot\cdot$$
$$+ (\alpha_1^2 \beta_2^2 + \alpha_2^2 \beta_1^2) s_{12} + \cdot\cdot + (\alpha_1^2 \beta_2 \beta_3 + \beta_1^2 \alpha_2 \alpha_3) s_{14} + \cdot\cdot$$
$$+ \alpha_1 \beta_1 \Big[(\alpha_1 \beta_3 + \alpha_3 \beta_1) s_{15} + (\alpha_1 \beta_2 + \alpha_2 \beta_1) s_{16}$$
$$+ (\alpha_2 \beta_3 + \alpha_3 \beta_2) s_{56} \Big] + \cdot\cdot$$

$$s'_{14} = 2\alpha_1^2 \beta_1 \gamma_1 s_{11} + \cdot\cdot + \alpha_2 \alpha_3 (\beta_2 \gamma_3 + \beta_3 \gamma_2) s_{44} + \cdot\cdot$$
$$+ 2(\alpha_1^2 \beta_2 \gamma_2 + \alpha_2^2 \beta_1 \gamma_1) s_{12} + \cdot\cdot$$
$$+ \Big[\alpha_1^2 (\beta_2 \gamma_3 + \beta_3 \gamma_2) + 2\alpha_2 \alpha_3 \beta_1 \gamma_1 \Big] s_{14} + \cdot\cdot$$
$$+ \Big[\alpha_1^2 (\beta_1 \gamma_3 + \beta_3 \gamma_1) + 2\alpha_1 \alpha_3 \beta_1 \gamma_1 \Big] s_{15} + \cdot\cdot$$
$$+ \Big[\alpha_1^2 (\beta_1 \gamma_2 + \beta_2 \gamma_1) + 2\alpha_1 \alpha_2 \beta_1 \gamma_1 \Big] s_{16} + \cdot\cdot$$
$$+ \Big[\alpha_1 \alpha_2 (\beta_1 \gamma_3 + \beta_3 \gamma_1) + \alpha_1 \alpha_3 (\beta_1 \gamma_2 + \beta_2 \gamma_1) \Big] s_{56} + \cdot\cdot$$

$$s_{15} = 2\alpha_1^3 \gamma_1 s_{11} + \cdot\cdot + \alpha_2 \alpha_3 (\alpha_2 \gamma_3 + \alpha_3 \gamma_2) s_{44} + \cdot\cdot$$

$$+ 2\alpha_1 \alpha_2(\alpha_1 \gamma_2 + \alpha_2 \gamma_1) s_{12} + \cdot\cdot$$

$$+ \left[\alpha_1^2(\alpha_2 \gamma_3 + \alpha_3 \gamma_2) + 2\alpha_1 \alpha_2 \alpha_3 \gamma_1\right] s_{14} + \cdot\cdot$$

$$+ \alpha_1^2\left[(\alpha_1 \gamma_3 + 3\alpha_3 \gamma_1) s_{15} + (\alpha_1 \gamma_2 + 3\alpha_2 \gamma_1) s_{16}\right] + \cdot\cdot$$

$$+ \left[\alpha_1 \alpha_2(\alpha_1 \gamma_3 + \alpha_3 \gamma_1) + \alpha_1 \alpha_3(\alpha_1 \gamma_2 + \alpha_2 \gamma_1)\right] s_{56} + \cdot\cdot$$

$$s'_{16} = 2\alpha_1^3 \beta_1 s_{11} + \cdot\cdot + \alpha_2 \alpha_3(\alpha_2 \beta_3 + \alpha_3 \beta_2) s_{44} + \cdot\cdot$$

$$+ 2\alpha_1 \alpha_2(\alpha_1 \beta_2 + \alpha_2 \beta_1) s_{12} + \cdot\cdot$$

$$+ \left[\alpha_1^2(\alpha_2 \beta_3 + \alpha_3 \beta_2) + 2\alpha_1 \alpha_2 \alpha_3 \beta_1\right] s_{14} + \cdot\cdot$$

$$+ \alpha_1^2\left[(\alpha_1 \beta_3 + 3\alpha_3 \beta_1) s_{15} + (\alpha_1 \beta_2 + 3\alpha_2 \beta_1) s_{16}\right] + \cdot\cdot$$

$$+ \left[\alpha_1 \alpha_3(\alpha_1 \beta_2 + \alpha_2 \beta_1) + \alpha_1 \alpha_2(\alpha_1 \beta_3 + \alpha_3 \beta_1)\right] s_{56} + \cdot\cdot$$

$$s'_{56} = 4\alpha_1^2 \beta_1 \gamma_1 s_{11} + \cdot\cdot + (\alpha_2 \gamma_3 + \alpha_3 \gamma_2)(\alpha_2 \beta_3 + \alpha_3 \beta_2) s_{44} + \cdot\cdot$$

$$+ 4\alpha_1 \alpha_2(\beta_1 \gamma_2 + \beta_2 \gamma_1) s_{12} + \cdot\cdot$$

$$+ 2\left[\alpha_1 \beta_1(\alpha_2 \gamma_3 + \alpha_3 \gamma_2) + \alpha_1 \gamma_1 (\alpha_2 \beta_3 + \alpha_3 \beta_2)\right] s_{14} + \cdot\cdot$$

$$+ 2\left[\alpha_1 \beta_1 (\alpha_1 \gamma_3 + \alpha_3 \gamma_1) + \alpha_1 \gamma_1(\alpha_1 \beta_3 + \alpha_3 \beta_1)\right] s_{15} + \cdot\cdot$$

$$+ 2\left[\alpha_1 \beta_1(\alpha_1 \gamma_2 + \alpha_2 \gamma_1) + \alpha_1 \gamma_1(\alpha_1 \beta_2 + \alpha_2 \beta_1)\right] s_{16} + \cdot\cdot$$

$$+ \left[(\alpha_1 \gamma_3 + \alpha_3 \gamma_1)(\alpha_1 \beta_2 + \alpha_2 \beta_1) + (\alpha_1 \gamma_2 + \alpha_2 \gamma_1)(\alpha_1 \beta_3 + \alpha_3 \beta_1)\right] s_{56} + \cdot\cdot$$

Each missing term denoted by a dot is obtained from the term just preceding by raising the subscripts of the compliance coefficients and of the direction cosines by one step, i.e, 2, 3, 1, 5, 6, 4 are written in place of 1, 2, 3, 4, 5, 6.

Employing this rule, we may, for example, write s'_{12} in complete form, as follows:

$$
\begin{aligned}
s'_{12} = {} & \alpha_1^2 \beta_1^2 s_{11} + \alpha_2^2 \beta_2^2 s_{22} + \alpha_3^2 \beta_3^2 s_{33} + \alpha_2 \alpha_3 \beta_2 \beta_3 s_{44} \\
& + \alpha_1 \alpha_3 \beta_1 \beta_3 s_{55} + \alpha_1 \alpha_2 \beta_1 \beta_2 s_{66} + (\alpha_1^2 \beta_2^2 + \alpha_2^2 \beta_1^2) s_{12} \\
& + (\alpha_2^2 \beta_3^2 + \alpha_3^2 \beta_2^2) s_{23} + (\alpha_3^2 \beta_1^2 + \alpha_1^2 \beta_3^2) s_{13} \\
& + (\alpha_1^2 \beta_2 \beta_3 + \beta_1^2 \alpha_2 \alpha_3) s_{14} + (\alpha_2^2 \beta_1 \beta_3 + \beta_2^2 \alpha_1 \alpha_3) s_{25} \\
& + (\alpha_3^2 \beta_1 \beta_2 + \beta_3^2 \alpha_1 \alpha_2) s_{36} + \alpha_1 \beta_1 \Big[(\alpha_1 \beta_3 + \alpha_3 \beta_1) s_{15} \\
& + (\alpha_1 \beta_2 + \alpha_2 \beta_1) s_{16} + (\alpha_2 \beta_3 + \alpha_3 \beta_2) s_{56} \Big] \\
& + \alpha_2 \beta_2 \Big[(\alpha_2 \beta_1 + \alpha_1 \beta_2) s_{26} + (\alpha_2 \beta_3 + \alpha_3 \beta_2) s_{24} + (\alpha_3 \beta_1 + \alpha_1 \beta_3) s_{46} \Big] \\
& + \alpha_3 \beta_3 \Big[(\alpha_3 \beta_2 + \alpha_2 \beta_3) s_{34} + (\alpha_3 \beta_1 + \alpha_1 \beta_3) s_{35} \\
& + (\alpha_1 \beta_2 + \alpha_2 \beta_1) s_{45} \Big]
\end{aligned}
$$

A cyclical permutation of the direction cosines and of the subscripts of the primed compliance coefficients according to the table below will yield equations for all 21 transformed constants.

TABLE I

SUBSCRIPTS OF PRIMED COEFFICIENTS							DIRECTION COSINES		
11	44	12	14	15	16	56	α	β	γ
22	55	23	25	26	24	46	β	γ	α
33	66	13	36	34	35	45	γ	α	β

As an illustration, the expression for s'_{13} is obtained by first writing the 21 terms of the s'_{12} equation, as was done above, and then substituting γ for α and α for β. Thus we have

$$
\begin{aligned}
s'_{13} = {} & \alpha_1^2 \gamma_1^2 s_{11} + \alpha_2^2 \gamma_2^2 s_{22} + \alpha_3^2 \gamma_3^2 s_{33} + \alpha_2 \alpha_3 \gamma_2 \gamma_3 s_{44} \\
& + \alpha_1 \alpha_3 \gamma_1 \gamma_3 s_{55} + \alpha_1 \alpha_2 \gamma_1 \gamma_2 s_{66} + (\alpha_2^2 \gamma_1^2 + \alpha_1^2 \gamma_2^2) s_{12} \\
& + (\alpha_3^2 \gamma_2^2 + \alpha_2^2 \gamma_3^2) s_{23} + (\alpha_1^2 \gamma_3^2 + \alpha_3^2 \gamma_1^2) s_{13} + (\alpha_2 \alpha_3 \gamma_1^2 + \alpha_1^2 \gamma_2 \gamma_3) s_{14} \\
& + (\alpha_1 \alpha_3 \gamma_2^2 + \alpha_2^2 \gamma_1 \gamma_3) s_{25} + (\alpha_1 \alpha_2 \gamma_3^2 + \alpha_3^2 \gamma_1 \gamma_2) s_{36}
\end{aligned}
$$

$$+\ \alpha_1\gamma_1\left[(\alpha_3\gamma_1+\alpha_1\gamma_3)s_{15}+(\alpha_2\gamma_1+\alpha_1\gamma_2)s_{16}+(\alpha_3\gamma_2+\alpha_2\gamma_3)s_{56}\right]$$

$$+\ \alpha_2\gamma_2\left[(\alpha_1\gamma_2+\alpha_2\gamma_1)s_{26}+(\alpha_3\gamma_2+\alpha_2\gamma_3)s_{24}+(\alpha_1\gamma_3+\alpha_3\gamma_1)s_{46}\right]$$

$$+\ \alpha_3\gamma_3\left[(\alpha_2\gamma_3+\alpha_3\gamma_2)s_{34}+(\alpha_1\gamma_3+\alpha_3\gamma_1)s_{35}+(\alpha_2\gamma_1+\alpha_1\gamma_2)s_{45}\right]$$

The rules following the s'_{56} expression are applied in similar manner to the transformed equations for the elastic stiffness coefficients. These expressions are given, as follows:

$$\begin{aligned} c'_{11} = {} & \alpha_1^4 c_{11}+\cdots+2\alpha_1^2\alpha_2^2(c_{12}+2c_{66})+\cdots \\ & +4\alpha_1^3(\alpha_3 c_{15}+\alpha_2 c_{16})+\cdots \\ & +4\alpha_1^2\alpha_2\alpha_3(c_{14}+2c_{56})+\cdots \end{aligned}$$

$$\begin{aligned} c'_{44} = {} & \beta_1^2\gamma_1^2 c_{11}+\cdots+2\beta_1\beta_2\gamma_1\gamma_2 c_{12}+\cdots \\ & +(\beta_2\gamma_3+\beta_3\gamma_2)^2 c_{44}+\cdots+2\beta_1\gamma_1\left[(\beta_2\gamma_3+\beta_3\gamma_2)c_{14}\right. \\ & \left.+(\beta_1\gamma_3+\beta_3\gamma_1)c_{15}+(\beta_1\gamma_2+\beta_2\gamma_1)c_{16}\right]+\cdots \\ & +2(\beta_3\gamma_1+\beta_1\gamma_3)(\beta_2\gamma_1+\beta_1\gamma_2)c_{56}\ +\cdots \end{aligned}$$

$$\begin{aligned} c'_{12} = {} & \alpha_1^2\beta_1^2 c_{11}+\cdots+4\alpha_2\alpha_3\beta_2\beta_3 c_{44}+\cdots \\ & +(\alpha_1^2\beta_2^2+\alpha_2^2\beta_1^2)c_{12}+\cdots+2(\alpha_1^2\beta_2\beta_3+\alpha_2\alpha_3\beta_1^2)c_{14}+\cdots \\ & +2\alpha_1\beta_1\left[(\alpha_1\beta_3+\alpha_3\beta_1)c_{15}+(\alpha_1\beta_2+\alpha_2\beta_1)c_{16}\right. \\ & \left.+2(\alpha_2\beta_3+\alpha_3\beta_2)c_{56}\right]+\cdots \end{aligned}$$

$$\begin{aligned} c'_{14} = {} & \alpha_1^2\beta_1\gamma_1 c_{11}+\cdots+2\alpha_2\alpha_3(\beta_2\gamma_3+\beta_3\gamma_2)c_{44}\ +\cdots \\ & +(\alpha_1^2\beta_2\gamma_2+\alpha_2^2\beta_1\gamma_1)c_{12}+\cdots+\left[\alpha_1^2(\beta_2\gamma_3+\beta_3\gamma_2)\right. \\ & \left.+2\alpha_2\alpha_3\beta_1\gamma_1\right]c_{14}+\cdots \\ & +\left[\alpha_1^2(\beta_1\gamma_3+\beta_3\gamma_1)+2\alpha_1\alpha_3\beta_1\gamma_1\right]c_{15}\ +\cdots \\ & +\left[\alpha_1^2(\beta_1\gamma_2+\beta_2\gamma_1)+2\alpha_1\alpha_2\beta_1\gamma_1\right]c_{16}\ +\cdots \\ & +2\left[\alpha_1\alpha_3(\beta_1\gamma_2+\beta_2\gamma_1)+\alpha_1\alpha_2(\beta_1\gamma_3+\beta_3\gamma_1)\right]c_{56}+\cdots \end{aligned}$$

$$c'_{15} = \alpha_1^3 \gamma_1 c_{11} + \cdots + 2\alpha_2 \alpha_3(\alpha_2 \gamma_3 + \alpha_3 \gamma_2)c_{44} + \cdots$$
$$+ \alpha_1 \alpha_2(\alpha_1 \gamma_2 + \alpha_2 \gamma_1)c_{12} + \cdots$$
$$+ \left[\alpha_1^2(\alpha_2 \gamma_3 + \alpha_3 \gamma_2) + 2\alpha_1 \alpha_2 \alpha_3 \gamma_1\right]c_{14} + \cdots$$
$$+ \alpha_1^2\left[(\alpha_1 \gamma_3 + 3\alpha_3 \gamma_1)c_{15} + (\alpha_1 \gamma_2 + 3\alpha_2 \gamma_1)c_{16}\right] + \cdots$$
$$+ 2\left[\alpha_1 \alpha_3(\alpha_1 \gamma_2 + \alpha_2 \gamma_1) + \alpha_1 \alpha_2(\alpha_1 \gamma_3 + \alpha_3 \gamma_1)\right] c_{56} + \cdots$$

$$c'_{16} = \alpha_1^3 \beta_1 c_{11} + \cdots + 2\alpha_2 \alpha_3(\alpha_3 \beta_2 + \alpha_2 \beta_3)c_{44} + \cdots$$
$$+ \alpha_1 \alpha_2(\alpha_1 \beta_2 + \alpha_2 \beta_1)c_{12} + \cdots$$
$$+ \left[\alpha_1^2(\alpha_3 \beta_2 + \alpha_2 \beta_3) + 2\alpha_1 \alpha_2 \alpha_3 \beta_1\right]c_{14} + \cdots$$
$$+ \alpha_1^2\left[(3\alpha_3 \beta_1 + \alpha_1 \beta_3)c_{15} + (3\alpha_2 \beta_1 + \alpha_1 \beta_2)c_{16}\right] + \cdots$$
$$+ 2\left[\alpha_1 \alpha_3(\alpha_2 \beta_1 + \alpha_1 \beta_2) + \alpha_1 \alpha_2(\alpha_3 \beta_1 + \alpha_1 \beta_3)\right]c_{56} + \cdots$$

$$c'_{56} = \alpha_1^2 \beta_1 \gamma_1 c_{11} + \cdots + (\alpha_2 \gamma_3 + \alpha_3 \gamma_2)(\alpha_2 \beta_3 + \alpha_3 \beta_2)c_{44} + \cdots$$
$$+ \alpha_1 \alpha_2(\beta_2 \gamma_1 + \beta_1 \gamma_2)c_{12} + \cdots$$
$$+ \left[\alpha_1 \beta_1(\alpha_2 \gamma_3 + \alpha_3 \gamma_2) + \alpha_1 \gamma_1(\alpha_2 \beta_3 + \alpha_3 \beta_2)\right] c_{14} + \cdots$$
$$+ \left[\alpha_1 \beta_1(\alpha_3 \gamma_1 + \alpha_1 \gamma_3) + \alpha_1 \gamma_1(\alpha_3 \beta_1 + \alpha_1 \beta_3)\right] c_{15} + \cdots$$
$$+ \left[\alpha_1 \beta_1(\alpha_1 \gamma_2 + \alpha_2 \gamma_1) + \alpha_1 \gamma_1(\alpha_1 \beta_2 + \alpha_2 \beta_1)\right] c_{16} + \cdots$$
$$+ \left[(\alpha_1 \gamma_3 + \alpha_3 \gamma_1)(\alpha_1 \beta_2 + \alpha_2 \beta_1) + (\alpha_1 \gamma_2 + \alpha_2 \gamma_1)\right.$$
$$\left.(\alpha_1 \beta_3 + \alpha_3 \beta_1)\right] c_{56} + \cdots$$

Rotation About a Single Axis

All expressions for a rotation about a single axis are found from the general equations given above by assigning proper values to the direction cosines.

For a rotation through an angle θ about the X axis, the direction cosines reduce to:

$$\alpha_1 = 1$$
$$\alpha_2 = \alpha_3 = \beta_1 = \gamma_1 = 0$$
$$\beta_2 = \gamma_3 = \cos\theta = c$$
$$\beta_3 = -\gamma_2 = \sin\theta = s$$

Thus, the elastic compliance coefficients will become:

$$s'_{11} = s_{11}$$
$$s'_{12} = c^2 s_{12} + s^2 s_{13} + cs\, s_{14}$$
$$s'_{13} = s^2 s_{12} + c^2 s_{13} - cs\, s_{14}$$
$$s'_{14} = 2cs(s_{13} - s_{12}) + (c^2 - s^2)s_{14}$$
$$s'_{15} = cs_{15} - s\, s_{16}$$
$$s'_{16} = s\, s_{15} + c\, s_{16}$$
$$s'_{22} = c^4 s_{22} + c^2 s^2(2s_{23} + s_{44}) + s^4 s_{33} + 2cs^3 s_{34} + 2c^3 s\, s_{24}$$
$$s'_{23} = c^2 s^2(s_{22} + s_{33} - s_{44}) + (c^4 + s^4)s_{23} + cs\,(c^2 - s^2)\,(s_{34} - s_{24})$$
$$s'_{24} = -2c^3 s\, s_{22} + cs(c^2 - s^2)(2s_{23} + s_{44}) + c^2(c^2 - 3s^2)s_{24} + 2cs^3\, s_{33} + s^2(3c^2 - s^2)s_{34}$$
$$s'_{25} = c^3 s_{25} + c^2 s(s_{45} - s_{26}) + cs^2\,(s_{35} - s_{46}) - s^3 s_{36}$$
$$s'_{26} = c^3 s_{26} + c^2 s\,(s_{25} + s_{46}) + s^3 s_{35} + cs^2(s_{36} + s_{45})$$
$$s'_{33} = s^4 s_{22} + c^2 s^2(2s_{23} + s_{44}) - 2cs^3 s_{24} + c^4 s_{33} - 2c^3 s\, s_{34}$$
$$s'_{34} = -2cs^3 s_{22} - cs(c^2 - s^2)(2s_{23} + s_{44}) + s^2(3c^2 - s^2)s_{24} + 2c^3 s\, s_{33} + c^2(c^2 - 3s^2)s_{34}$$
$$s'_{35} = cs^2(s_{25} + s_{46}) - s^3 s_{26} + c^3 s_{35} - c^2 s(s_{36} + s_{45})$$
$$s'_{36} = s^3 s_{25} + cs^2(s_{26} - s_{45}) + c^2 s(s_{35} - s_{46}) + c^3 s_{36}$$
$$s_{44} = 4c^2 s^2(s_{22} + s_{33} - 2s_{23}) + (c^2 - s^2)^2\, s_{44} + 4cs(c^2 - s^2)(s_{34} - s_{24})$$

$$s'_{45} = 2c^2s\,(s_{35}-s_{25}) + 2cs^2(s_{26}-s_{36}) + (c^2-s^2)(cs_{45}-ss_{46})$$

$$s'_{46} = 2cs^2(s_{35}-s_{25}) + 2c^2s(s_{36}-s_{26}) + (c^2-s^2)(s\,s_{45}+c\,s_{46})$$

$$s'_{55} = c^2s_{55}-2cs\,s_{56}+s^2s_{66}$$

$$s'_{56} = cs\,(s_{55}-s_{66}) + (c^2-s^2)s_{56}$$

$$s'_{66} = s^2s_{55}+2cs\;\;s_{56}+c^2s_{66}$$

The elastic stiffness coefficients for a similar rotation about the X axis will become:

$$c'_{11} = c_{11}$$

$$c'_{12} = c^2c_{12}+s^2c_{13}+2cs\;c_{14}$$

$$c'_{13} = s^2c_{12}+c^2c_{13}-2cs\;c_{14}$$

$$c'_{14} = cs(c_{13}-c_{12}) + (c^2-s^2)c_{14}$$

$$c'_{15} = c\cdot c_{15}-s\;c_{16}$$

$$c'_{16} = s\;c_{15}+\;c\;c_{16}$$

$$c'_{22} = c^4c_{22}+4c^3sc_{24}+s^4c_{33}+4cs^3c_{34}+2c^2s^2(c_{23}+2c_{44})$$

$$c'_{23} = c^2s^2(c_{22}+c_{33}-4c_{44}) + (c^4+\;s^4)c_{23}+2cs(c^2-s^2)(c_{34}-c_{24})$$

$$c'_{24} = -c^3s\;c_{22}+cs(c^2-s^2)(c_{23}+2c_{44}) + cs^3c_{33}+c^2(c^2-3s^2)\;c_{24}$$
$$+\,s^2(3c^2-s^2)c_{34}$$

$$c'_{25} = c^3c_{25}+cs^2(c_{35}-2c_{46})-s^3c_{36}+c^2s(2c_{45}-c_{26})$$

$$c'_{26} = c^3c_{26}+s^3c_{35}+cs^2(c_{36}+2c_{45}) + c^2s(c_{25}+2c_{46})$$

$$c'_{33} = s^4c_{22}+2c^2s^2(c_{23}+2c_{44})-4\;cs^3c_{24}\;-4c^3s\;c_{34}+c^4c_{33}$$

$$c'_{34} = -cs^3c_{22}-cs(c^2-s^2)(c_{23}+2c_{44}) + c^3s\;c_{33}+s^2(3c^2-s^2)c_{24}$$
$$+\,c^2(c^2-3s^2)c_{34}$$

$$c'_{35} = cs^2(c_{25} + 2c_{46}) - s^3 c_{26} + c^3 c_{35} - c^2 s(c_{36} + 2c_{45})$$

$$c'_{36} = s^3 c_{25} + cs^2(c_{26} - 2c_{45}) + c^3 c_{36} + c^2 s(c_{35} - 2c_{46})$$

$$c'_{44} = c^2 s^2(c_{22} + c_{33} - 2c_{23}) + 2cs(c^2 - s^2)(c_{34} - c_{24}) + (c^2 - s^2)^2 c_{44}$$

$$c'_{45} = cs^2(c_{26} - c_{36}) + c^2 s(c_{35} - c_{25}) + (c^2 - s^2)(c\, c_{45} - s\, c_{46})$$

$$c'_{46} = cs^2(c_{35} - c_{25}) + c^2 s(c_{36} - c_{26}) + (c^2 - s^2)(s\, c_{45} + c\, c_{46})$$

$$c'_{55} = c^2 c_{55} - 2cs\, c_{56} + s^2 c_{66}$$

$$c'_{56} = cs(c_{55} - c_{66}) + (c^2 - s^2) c_{56}$$

$$c'_{66} = s^2 c_{55} + 2cs\, c_{56} + c^2 c_{66}$$

The above expressions may be used for a rotation about the Y or the Z axis by a cyclical permutation of the subscripts on both sides of each equation according to Table 2. The permutation is achieved by substituting the digits in the second row (rotation about Y axis), or in the third row (rotation about Z axis), for the corresponding digits in the first row.

TABLE 2

X	1	2	3	4	5	6
Y	2	3	1	5	6	4
Z	3	1	2	6	4	5

As an illustration, the s'_{34} equation will yield s'_{15} for a rotation about the Y axis, as follows:

$$s'_{15} = -2cs^3 s_{33} - cs(c^2 - s^2)(2s_{13} + s_{55}) + s^2(3c^2 - s^2)s_{35} + c^2(c^2 - 3s^2)s_{15} + 2c^3 s\, s_{11}$$

PIEZOELECTRIC TRANSFORMATION

Rotation About All Three Axes

The generalized expressions for the transformed piezoelectric strain coefficients are given, as follows:

$$d'_{11} = \alpha_1^3 d_{11} + \cdots + \alpha_1 \alpha_2^2 d_{12} + \cdots + \alpha_1 \alpha_3^2 d_{13} + \cdots$$

$$+ \alpha_1 \alpha_2 \alpha_3 d_{14} + \cdots + \alpha_1^2 \alpha_3 d_{15} + \cdots + \alpha_1^2 \alpha_2 d_{16} + \cdots$$

$$d'_{12} = \alpha_1 \beta_1^2 d_{11} + \cdots + \alpha_1 \beta_2^2 d_{12} + \cdots + \alpha_1 \beta_3^2 d_{13} + \cdots$$

$$+ \alpha_1 \beta_2 \beta_3 d_{14} + \cdots + \alpha_1 \beta_1 \beta_3 d_{15} + \cdots + \alpha_1 \beta_1 \beta_2 d_{16} + \cdots$$

$$d'_{13} = \alpha_1 \gamma_1^2 d_{11} + \cdots + \alpha_1 \gamma_2^2 d_{12} + \cdots + \alpha_1 \gamma_3^2 d_{13} + \cdots$$

$$+ \alpha_1 \gamma_2 \gamma_3 d_{14} + \cdots + \alpha_1 \gamma_1 \gamma_3 d_{15} + \cdots + \alpha_1 \gamma_1 \gamma_2 d_{16} + \cdots$$

$$d'_{14} = 2\alpha_1 \beta_1 \gamma_1 d_{11} + \cdots + 2\alpha_1 \beta_2 \gamma_2 d_{12} + \cdots + 2\alpha_1 \beta_3 \gamma_3 d_{13} + \cdots$$

$$+ \alpha_1 (\beta_2 \gamma_3 + \beta_3 \gamma_2) d_{14} + \cdots + \alpha_1 (\beta_3 \gamma_1 + \beta_1 \gamma_3) d_{15} + \cdots$$

$$+ \alpha_1 (\beta_1 \gamma_2 + \beta_2 \gamma_1) d_{16} + \cdots$$

$$d'_{15} = 2\alpha_1^2 \gamma_1 d_{11} + \cdots + 2\alpha_1 \alpha_2 \gamma_2 d_{12} + \cdots + 2\alpha_1 \alpha_3 \gamma_3 d_{13} + \cdots$$

$$+ \alpha_1 (\alpha_2 \gamma_3 + \alpha_3 \gamma_2) d_{14} + \cdots + \alpha_1 (\alpha_1 \gamma_3 + \alpha_3 \gamma_1) d_{15} + \cdots$$

$$+ \alpha_1 (\alpha_1 \gamma_2 + \alpha_2 \gamma_1) d_{16} + \cdots$$

$$d'_{16} = 2\alpha_1^2 \beta_1 d_{11} + \cdots + 2\alpha_1 \alpha_2 \beta_2 d_{12} + \cdots + 2\alpha_1 \alpha_3 \beta_3 d_{13} + \cdots$$

$$+ \alpha_1 (\alpha_2 \beta_3 + \alpha_3 \beta_2) d_{14} + \cdots + \alpha_1 (\alpha_1 \beta_3 + \alpha_3 \beta_1) d_{15} + \cdots$$

$$+ \alpha_1 (\alpha_1 \beta_2 + \alpha_2 \beta_1) d_{16} + \cdots$$

The rule following the s'_{56} expression is employed for filling in missing terms indicated by dots. A cyclical permutation of the direction cosines and subscripts of the primed piezoelectric strain coefficients according to the table below will yield equations for all 18 transformed constants.

TABLE 3

SUBSCRIPTS OF PRIMED COEFFICIENTS						DIRECTION COSINES		
11	12	13	14	15	16	α	β	γ
22	23	21	25	26	24	β	γ	α
33	31	32	36	34	35	γ	α	β

For example, in order to obtain the expression for d'_{26}, we first write the equation for d'_{15} as follows:

$$
\begin{aligned}
d'_{15} = {} & 2\alpha_1^2\gamma_1 d_{11} + 2\alpha_2^2\gamma_2 d_{22} + 2\alpha_3^2\gamma_3 d_{33} + 2\alpha_1\alpha_2\gamma_2 d_{12} \\
& + 2\alpha_2\alpha_3\gamma_3 d_{23} + 2\alpha_1\alpha_3\gamma_1 d_{31} + 2\alpha_1\alpha_3\gamma_3 d_{13} + 2\alpha_1\alpha_2\gamma_1 d_{21} \\
& + 2\alpha_3\alpha_2\gamma_2 d_{32} + \alpha_1(\alpha_2\gamma_3 + \alpha_3\gamma_2) d_{14} + \alpha_2(\alpha_3\gamma_1 + \alpha_1\gamma_3) d_{25} \\
& + \alpha_3(\alpha_1\gamma_2 + \alpha_2\gamma_1) d_{36} + \alpha_1(\alpha_1\gamma_3 + \alpha_3\gamma_1) d_{15} \\
& + \alpha_2(\alpha_2\gamma_1 + \alpha_1\gamma_2) d_{26} + \alpha_3(\alpha_3\gamma_2 + \alpha_2\gamma_3) d_{34} \\
& + \alpha_1(\alpha_1\gamma_2 + \alpha_2\gamma_1) d_{16} + \alpha_2(\alpha_2\gamma_3 + \alpha_3\gamma_2) d_{24} \\
& + \alpha_3(\alpha_3\gamma_1 + \alpha_1\gamma_3) d_{35}
\end{aligned}
$$

We then substitute β for α and α for γ to yield the expression for d'_{26} as follows:

$$
\begin{aligned}
d'_{26} = {} & 2\beta_1(\alpha_1\beta_1 d_{11} + \alpha_2\beta_2 d_{12} + \alpha_3\beta_3 d_{13}) \\
& + 2\beta_2(\alpha_1\beta_1 d_{21} + \alpha_2\beta_2 d_{22} + \alpha_3\beta_3 d_{23}) \\
& + 2\beta_3(\alpha_1\beta_1 d_{31} + \alpha_2\beta_2 d_{32} + \alpha_3\beta_3 d_{33}) \\
& + \beta_1\left[(\alpha_3\beta_2 + \alpha_2\beta_3) d_{14} + (\alpha_3\beta_1 + \alpha_1\beta_3) d_{15} + (\alpha_2\beta_1 + \alpha_1\beta_2) d_{16}\right] \\
& + \beta_2\left[(\alpha_1\beta_3 + \alpha_3\beta_1) d_{25} + (\alpha_1\beta_2 + \alpha_2\beta_1) d_{26} + (\alpha_3\beta_2 + \alpha_2\beta_3) d_{24}\right] \\
& + \beta_3\left[(\alpha_2\beta_1 + \alpha_1\beta_2) d_{36} + (\alpha_2\beta_3 + \alpha_3\beta_2) d_{34} + (\alpha_1\beta_3 + \alpha_3\beta_1) d_{35}\right]
\end{aligned}
$$

The corresponding expressions for the transformed piezoelectric stress coefficients are obtained directly from the equations for the piezoelectric strain coefficients by merely writing e_{hk} in place of d_{hk} when h=1, 2, 3 and k=1, 2, 3; and by writing $2e_{hk}$ in place of d_{hk} when h=1, 2, 3 and k = 4, 5, 6.

In accordance with this rule, we write the equation for e'_{26}, as follows:

$$
\begin{aligned}
e'_{26} = {} & \beta_1(\alpha_1\beta_1 e_{11} + \alpha_2\beta_2 e_{12} + \alpha_3\beta_3 e_{13}) \\
& + \beta_2(\alpha_1\beta_1 e_{21} + \alpha_2\beta_2 e_{22} + \alpha_3\beta_3 e_{23}) \\
& + \beta_3(\alpha_1\beta_1 e_{31} + \alpha_2\beta_2 e_{32} + \alpha_3\beta_3 e_{33}) \\
& + \beta_1\left[(\alpha_3\beta_2 + \alpha_2\beta_3) e_{14} + (\alpha_3\beta_1 + \alpha_1\beta_3) e_{15} + (\alpha_2\beta_1 + \alpha_1\beta_2) e_{16}\right]
\end{aligned}
$$

$$+ \beta_2\left[(\alpha_1\beta_3 + \alpha_3\beta_1)e_{25} + (\alpha_1\beta_2 + \alpha_2\beta_1)e_{26} + (\alpha_3\beta_2 + \alpha_2\beta_3)e_{24}\right]$$

$$+ \beta_3\left[(\alpha_2\beta_1 + \alpha_1\beta_2)e_{36} + (\alpha_2\beta_3 + \alpha_3\beta_2)e_{34} + (\alpha_1\beta_3 + \alpha_3\beta_1)e_{35}\right]$$

Rotation About a Single Axis

As before, expressions for rotation about a single axis are obtained from the general equations given above by assigning proper values to the direction cosines.

For a rotation through an angle θ about the Z axis, the direction cosines reduce to:

$$\alpha_1 = \beta_2 = \cos\theta = c$$

$$\alpha_2 = -\beta_1 = \sin\theta = s$$

$$\gamma_3 = 1$$

$$\alpha_3 = \beta_3 = \gamma_1 = \gamma_2 = 0$$

Thus, in the case of the piezoelectric strain coefficients, we have:

$$d'_{11} = c^3 d_{11} + s^3 d_{22} + cs^2(d_{12} + d_{26}) + c^2 s(d_{16} + d_{21})$$

$$d'_{12} = c^3 d_{12} + s^3 d_{21} + c^2 s(d_{22} - d_{16}) + cs^2(d_{11} - d_{26})$$

$$d'_{13} = s\, d_{23} + c\, d_{13}$$

$$d'_{14} = c^2 d_{14} - s^2 d_{25} + cs(d_{24} - d_{15})$$

$$d'_{15} = c^2 d_{15} + s^2 d_{24} + cs(d_{14} + d_{25})$$

$$d'_{16} = 2cs^2(d_{22} - d_{21}) + 2c^2 s(d_{12} - d_{11}) + c(1 - 2s^2)d_{16}$$

$$-s(1 - 2c^2)d_{26}$$

$$d'_{21} = c^3 d_{21} - s^3 d_{12} + cs^2(d_{22} - d_{16}) + c^2 s(d_{26} - d_{11})$$

$$d'_{22} = c^3 d_{22} - s^3 d_{11} - c^2 s(d_{12} + d_{26}) + cs^2(d_{21} + d_{16})$$

$$d'_{23} = c\, d_{23} - s\, d_{13}$$

$$d'_{24} = c^2 d_{24} + s^2 d_{15} - cs(d_{14} + d_{25})$$

$$d'_{25} = c^2 d_{25} - s^2 d_{14} + cs(d_{24} - d_{15})$$

$$d'_{26} = c\,(1 - 2s^2)d_{26} + s(1 - 2c^2)d_{16} + 2cs^2(d_{11} - d_{12})$$
$$+ 2c^2 s\,(d_{22} - d_{21})$$

$$d'_{31} = c^2 d_{31} + s^2 d_{32} + cs\, d_{36}$$

$$d'_{32} = s^2 d_{31} + c^2 d_{32} - cs\, d_{36}$$

$$d'_{33} = d_{33}$$

$$d'_{34} = c\, d_{34} - s\, d_{35}$$

$$d'_{35} = s\, d_{34} + c\, d_{35}$$

$$d'_{36} = (c^2 - s^2)d_{36} + 2cs\,(d_{32} - d_{31})$$

The corresponding expressions for the transformed piezoelectric stress coefficients are obtained directly from those above by writing e_{hk} for d_{hk} when h = 1, 2, 3 and k = 1, 2, 3; and $2e_{hk}$ for d_{hk} when h = 1, 2, 3 and k = 4, 5, 6.

Separate rotations about the X and Y axes may be performed by employing Table 2.

As an illustration, the d'_{16} equation will yield d'_{35} for a rotation about the Y axis, as follows:

$$d'_{35} = 2cs^2(d_{11} - d_{13}) + 2c^2 s\,(d_{31} - d_{33})$$
$$+ c(1 - 2s^2)d_{35} - s(1 - 2c^2)d_{15}$$

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* * *

$$[illegible]$$

$$[illegible]$$

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$$[illegible]$$

The corresponding expressions for the transformed piezoelectric strain coefficients are obtained directly from those above by writing d_{ik} for e_{ik} when k = 1, 2, 3 and k = 1, 2, 3; and $2d_{ik}$ for e_{ik} when k = 4, 5, 6 [illegible].

Reversed rotations about the X and Y axes may be performed as required [illegible].

As an illustration, the d_{14} equation will state d_{14}' for a rotation about the Y axis, as follows:

$$[illegible]$$

$$[illegible]$$

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* * *

APPENDIX A

I.R.E. DESIGNATION FOR ANGULAR ROTATION[5]

Employing the conventional I.R.E. designation for orientation of angles ϕ, θ, ψ, a triple rotation will yield direction cosines of the $X' Y' Z'$ axial system with respect to the XYZ axes, defined according to the accompanying matrix, as follows:

	X'	Y'	Z'
X	α_1	β_1	γ_1
Y	α_2	β_2	γ_2
Z	α_3	β_3	γ_3

where

$$\alpha_1 = \cos\phi \cos\theta \cos\psi - \sin\phi \sin\psi$$

$$\alpha_2 = \sin\phi \cos\theta \cos\psi + \cos\phi \sin\psi$$

$$\alpha_3 = -\sin\theta \cos\psi$$

$$\beta_1 = -\cos\phi \cos\theta \sin\psi - \sin\phi \cos\psi$$

$$\beta_2 = -\sin\phi \cos\theta \sin\psi + \cos\phi \cos\psi$$

$$\beta_3 = \sin\theta \sin\psi$$

$$\gamma_1 = \cos\phi \sin\theta$$

$$\gamma_2 = \sin\phi \sin\theta$$

$$\gamma_3 = \cos\theta$$

The following diagrams (Figure 1) depict a typical triple rotation in terms of the I.R.E. angles ϕ, θ and ψ.

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[5] W. G. Cady, op. cit., p. 83

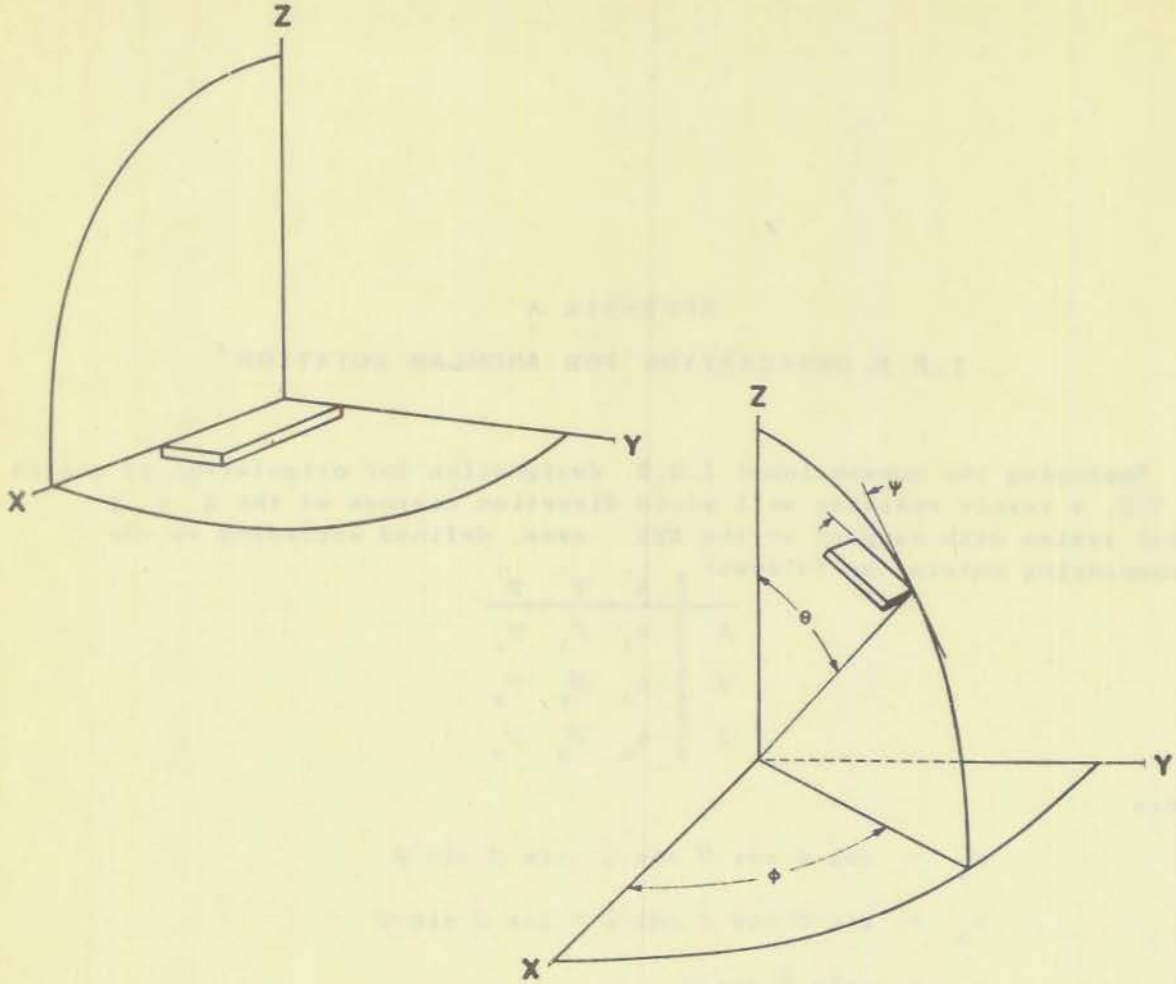

Figure 1 - I.R.E. Direction Angles for Initial Position (0,0,0) and for Final Position (ϕ, θ, ψ)

A rotation about one or two axes is readily seen to be a special case of the above. For a double rotation where $\psi = 0^0$, the direction cosines will reduce to

$\alpha_1 = \cos\phi\cos\theta$ $\qquad\beta_1 = -\sin\phi$ $\qquad\gamma_1 = \cos\phi\sin\theta$

$\alpha_2 = \sin\phi\cos\theta$ $\qquad\beta_2 = \cos\phi$ $\qquad\gamma_2 = \sin\phi\sin\theta$

$\alpha_3 = -\sin\theta$ $\qquad\beta_3 = 0$ $\qquad\gamma_3 = \cos\theta$

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