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AN ELECTRONIC METHOD FOR APPROXIMATING THE FREQUENCY SPECTRA OF TRANSIENT FUNCTIONS

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ABSTRACT

It is shown that any method for obtaining the spectra (i.e., plots of component amplitude versus frequency) of periodic functions can be used, in a manner described, to determine approximately the spectra of a broad class of transient functions. The problem is thus simplified in that only an instrument for obtaining the spectra of periodic functions need be found. A wave analyzer is then considered as such an instrument.

PROBLEM STATUS

This is an interim report on a portion of the problem; work is continuing.

AUTHORIZATION

NRL Problem No. F03-07R

AN ELECTRONIC METHOD FOR APPROXIMATING
THE FREQUENCY SPECTRA OF TRANSIENT FUNCTIONS

INTRODUCTION

Experimental investigations of physical phenomena frequently result in data which are plots of various dynamical quantities versus time. Often it is of theoretical and practical importance to represent these curves as a superposition of sinusodial components of various frequencies, each component being characterized by an amplitude and a phase angle. A graph of component amplitude versus frequency constitutes a frequency spectrum.

While frequency spectra can in theory be obtained analytically by means of Fourier's series (or Fourier's integral) expansions, complexity of the data may make analytical procedures impracticable. In this report, the question of evolving an electronic method for obtaining frequency spectra is discussed.

It is shown that any method for obtaining the spectra of periodic functions can be used (in a manner to be described) to determine approximately the spectra of a broad class of transient functions. The problem is thus simplified in that only an instrument for obtaining the spectra of periodic functions need be found. A wave analyzer is then considered as such a device.

THE NATURE OF THE SPECTRA OF PERIODIC AND OF TRANSIENT FUNCTIONS

Consider a periodic function p(t), of period τ, such that shown in Figure 1. By means of the Fourier series expansion, p(t) can be represented as a superposition of sinusoidal components:

$$p(t) = \sum_{\nu = -\infty}^{\infty} a_\nu e^{i \nu \omega_0 t} , \qquad (1)$$

where

$$a_\nu = \frac{\omega_0}{2\pi} \int_0^\tau p(t)\, e^{-i \nu \omega_0 t}\, dt, \qquad (2)$$

and

$$\omega_0 = \frac{2\pi}{\tau} .$$

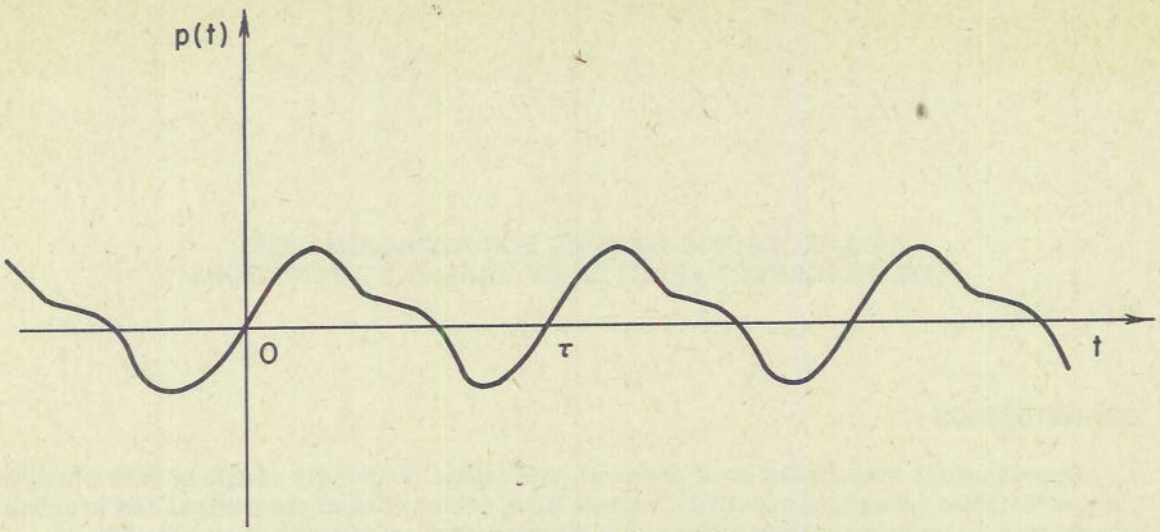

Fig. 1 - A typical periodic function, p(t)

ω_0 is, of course, the fundamental angular frequency of p(t). The amplitude of the componen of frequency $\nu\omega_0$ is

$$A_\nu = \left|a_\nu\right| = \frac{\omega_0}{2\pi}\left|\int_0^\tau p(t)\, e^{-i\nu\omega_0 t}\, dt\right| (\nu = 0, 1, 2, \ldots). \tag{3}$$

Equation (3), then, gives the frequency spectrum of p(t). The spectrum of a periodic function is thus composed of discrete points and is not a continuous curve; i.e., such a function possesses components of only those frequencies which are integral multiples of the fundamental frequency of the function (Figure 2).

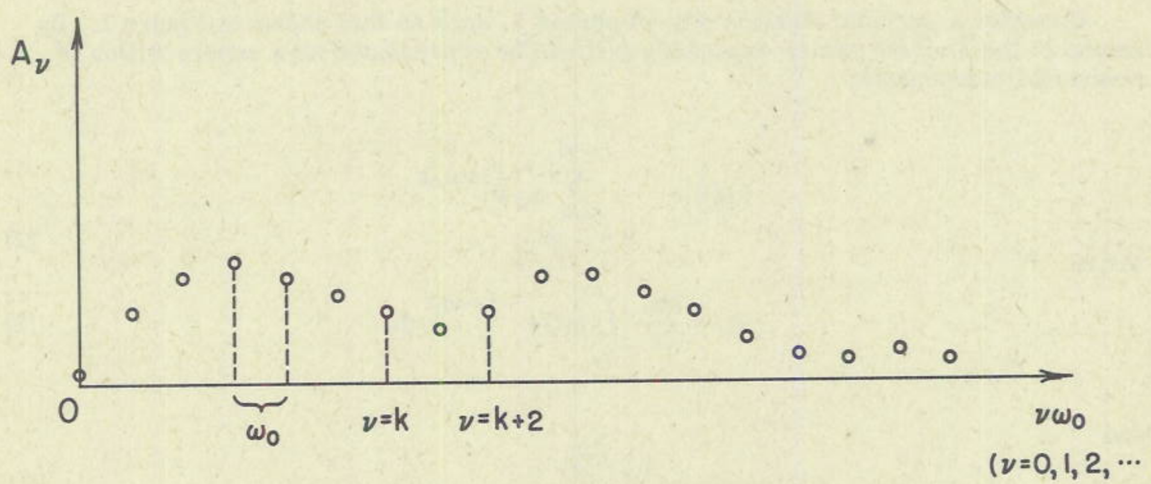

Fig. 2 - The spectrum of p(t)

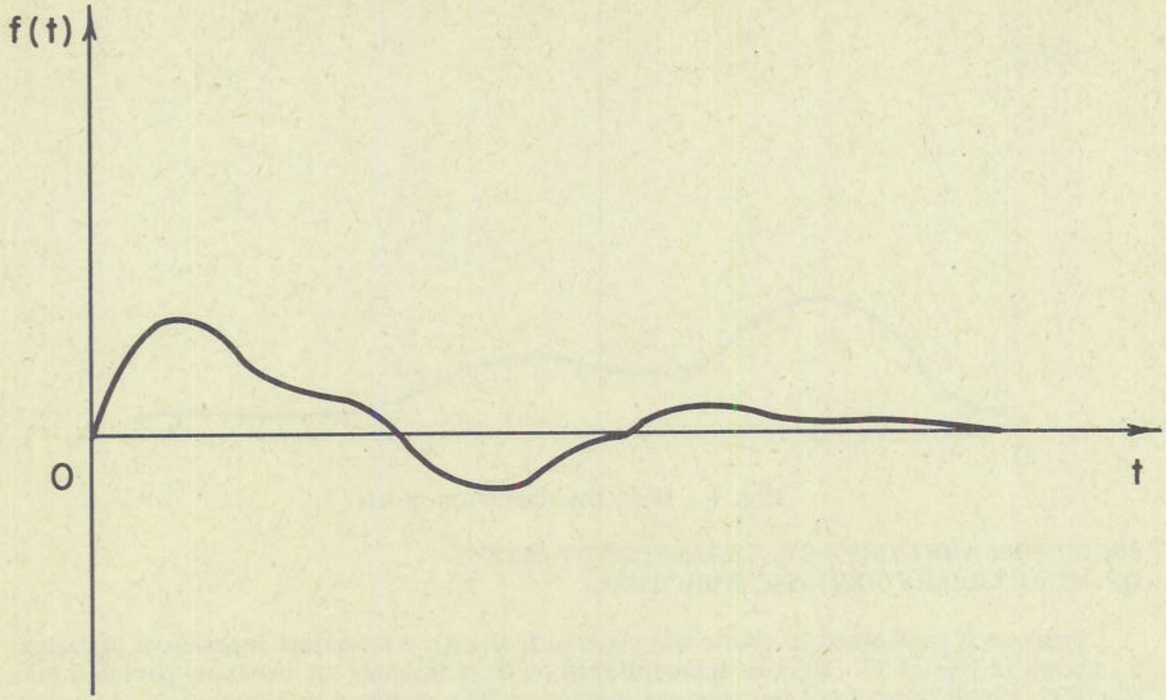

Fig. 3 - A transient function, f(t)

Consider next the transient function, f(t), shown in Figure 3. This function may be expressed as a superposition of sinusoidal components by means of the Fourier integral expansion:

$$f(t) = \int_{-\infty}^{\infty} g(\omega)\, e^{i\omega t}\, d\omega, \tag{4}$$

where

$$g(\omega) = \frac{1}{2\pi}\int_{0}^{\infty} f(t)\, e^{-i\omega t}\, dt. \tag{5}$$

The amplitude of the component of frequency ω is $|g(\omega)|d\omega$. This is an infinitesimal quantity for all ω's. Since in practice only the relative amplitudes of components of various frequencies are of interest, it will be convenient when dealing with transient functions to use the word spectrum to denote any curve proportional to that of the component amplitudes versus frequency. Accordingly the spectrum of f(t) is given by

$$B(\omega) = k'\left|g(\omega)\right| = k\left|\int_{0}^{\infty} f(t)\, e^{-i\omega t}\, dt\right|, \tag{6}$$

where k is any constant. $B(\omega)$ is shown in Figure 4.

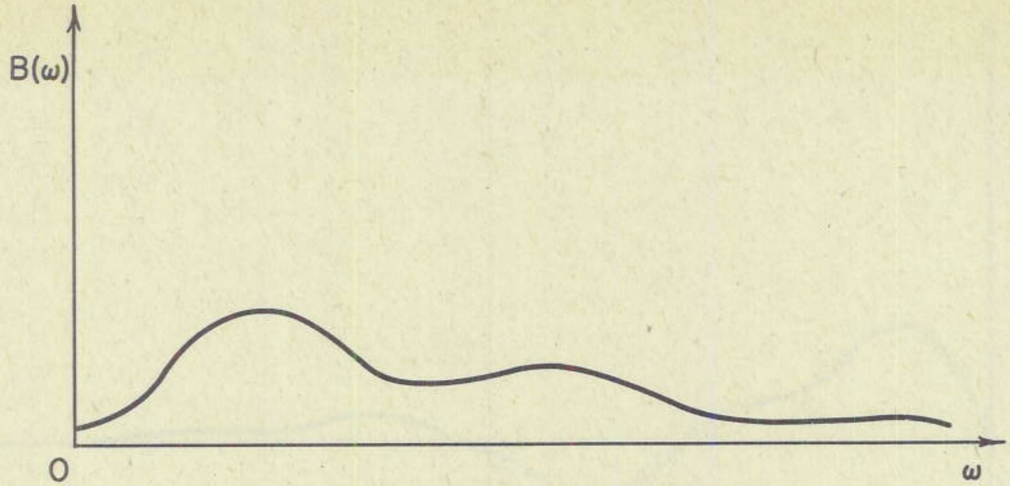

Fig. 4 - $B(\omega)$, the spectrum of $f(t)$

SPECTRUM ANALYSIS OF A TRANSIENT BY MEANS OF AN AUXILIARY PERIODIC FUNCTION

Suppose it is desired to obtain the spectrum of $F(t)$, a transient function of duration t_1, shown in Figure 5.[1] This is accomplished by first forming an auxiliary periodic function $P(t)$, which consists of the nonzero portion of $F(t)$ repeated at intervals T of time - see Figure 6. If the spectrum of $P(t)$ is then found by any method whatever, the points

$$A_\nu = \frac{\omega_o}{2\pi}\left|\int_o^T P(t)\,e^{-i\nu\omega_o t}\,dt\right|$$

will be known. Since $P(t)$ is identical to $F(t)$ for $o < t < T$, and since $F(t) = o$ for $t_1 < t < T$, it follows that the known points A_ν are

$$A_\nu = \frac{\omega_o}{2\pi}\left|\int_{o.}^{t_1} F(t)\,e^{-i\nu\omega_o t}\,dt\right| = k_1\left|\int_o^{t_1} F(t)\,e^{-i\nu\omega_o t}\,dt\right|, \qquad (7)$$

where $k_1 = \omega_o/2\pi$. According to equation (6), the desired spectrum of $F(t)$ is

$$B(\omega) = k\left|\int_o^{\infty} F(t)\,e^{-i\omega t}\,dt\right|;$$

but since $F(t) = o$ for $t > t_1$, and since k is any constant,

$$B(\omega) = k_1\left|\int_o^{t_1} F(t)\,e^{-i\omega t}\,dt\right|. \qquad (8)$$

---

[1] The present analysis is limited to functions of time which are zero for all times before a certain instant t_o, and for all times after $t_o + t_1$. Such a function can represent a physical phenomenon which decays to an amplitude below the detectability of the recording instrument (or below the random noise level) in t_1 seconds.

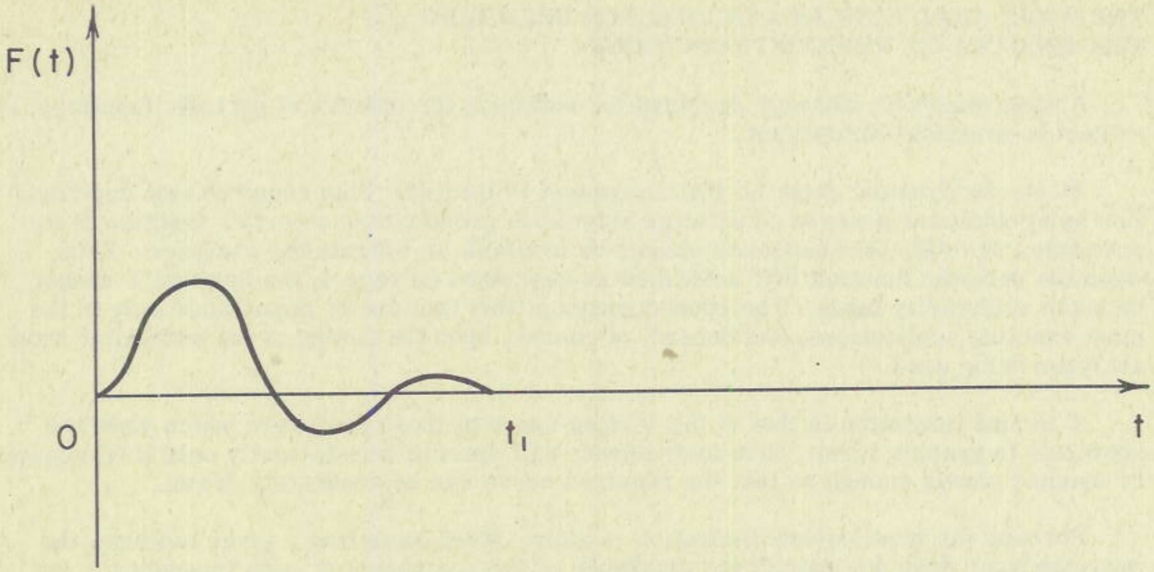

Fig. 5 - F(t), a particular transient to be analyzed

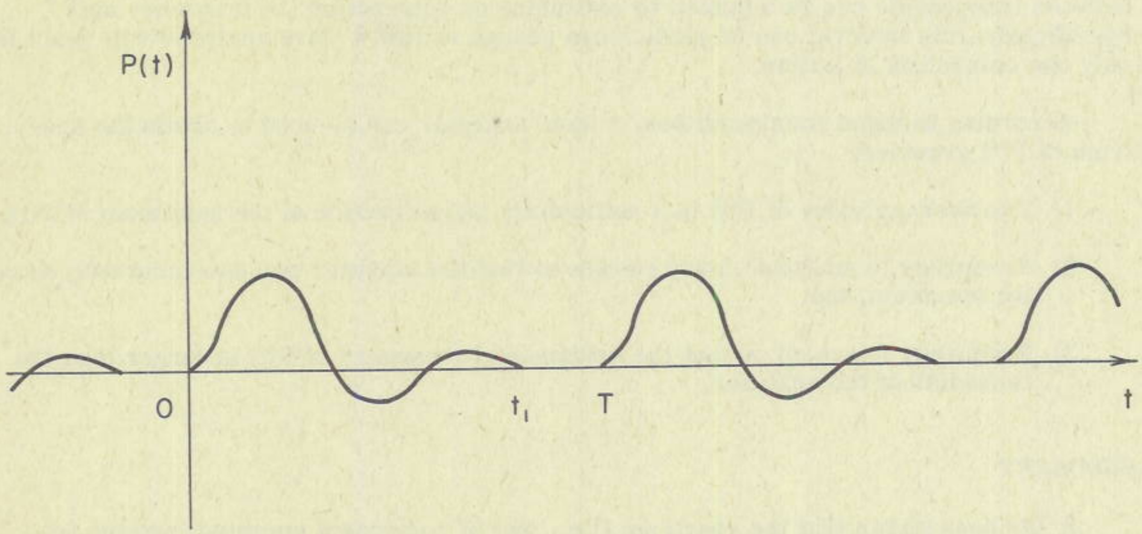

Fig. 6 - The auxiliary function, P(t)

A comparison of equations (7) and (8) shows that $A_\nu = B(\nu\omega_0)$ provided k is made equal to k_1. Thus every point A_ν of the known spectrum of P(t) is also a point $B(\nu\omega_0)$ of the desired continuous spectrum of F(t). The frequency interval between these known points is $\omega_0 = 2\pi/T$. The spectrum $B(\omega)$ of F(t) may be determined as completely as desired by making ω_0 sufficiently small, i.e., by choosing $T = 2\pi/\omega_0$ sufficiently large when forming the auxiliary function P(t). The question of determining what interval ω_0 will result in an adequate description of $B(\omega)$ is discussed in the appendix.

THE WAVE ANALYZER AS A DEVICE FOR OBTAINING THE SPECTRA OF PERIODIC FUNCTIONS

A wave analyzer, although designed for obtaining the spectra of periodic functions, is subject to practical limitations.

First, the dynamic range of the instrument is limited. This requires that any function being analyzed possess an average magnitude greater than a certain fraction of its maximum, in order that sufficient energy be available to operate the analyzer. Thus, when the periodic function P(t) is formed as described on page 4, the interval T cannot be taken arbitrarily large. The consequences of this fact are of importance only in the most exacting applications, and depend, of course, upon the design of the particular wave analyzer being used.

A second limitation is that of the writing speed of those analyzers which yield the spectrum in graphic form. Such instruments will operate satisfactorily only if frequency is scanned slowly enough so that the required curve can be accurately drawn.

Perhaps the most severe limitation is this: When analyzing a given function, the wave analyzer does not specify the amplitude of the component at each frequency ω, but instead yields an amplitude depending in a complicated manner upon all the components within a certain neighborhood of this frequency. However, since a periodic function possesses components only at integral multiples of its fundamental frequency, the interval between components can be adjusted by expanding or contracting the frequency axis.[2] Specifically, this interval can be made large enough so that a wave analyzer will react to only one component at a time.

According to these considerations, a wave analyzer can be used to obtain the spectrum of P(t) provided:

1) The average value of P(t) is a sufficiently large fraction of the maximum of P(t),

2) Frequency is scanned slowly enough so that the analyzer pen can accurately draw the spectrum, and

3) Frequency is scaled so that the fundamental frequency of P(t) is larger than the bandwidth of the analyzer.

SUMMARY

It has been shown that the spectrum (i.e., plot of component amplitude versus frefrequency) of a transient function can be determined approximately by the use of any device which produces the spectra of periodic functions. This is done as follows: (a) A periodic function is formed by repetition of the nonzero portion of the transient function at intervals T of time, and (b) the spectrum of this periodic function is obtained. The points of the spectrum of the periodic function will lie on the continuous spectrum of the transient function at frequency intervals $\omega_0 = 2\pi/T$. The continuous spectrum accordingly will be determined more accurately (i.e., will be established at more closely spaced points) as the period T is taken larger.

[2]A treatment of this procedure, commonly known as "Scaling," will be found in "On Physically Similar Systems," E. Buckingham, Phys. Rev. 4: 345-376, 1914.

A wave analyzer is suitable for obtaining the spectrum of a periodic function provided:

1) The average value of the function is a sufficiently large fraction of its maximum,

2) Frequency is scanned slowly enough so that the analyzer pen can accurately draw the spectrum, and

3) Frequency is scaled so that the fundamental frequency of the function is greater than the bandwidth of the analyzer.

* * *

APPENDIX I

The Value of ω_0

ω_0 is the frequency interval between points which are known to be on the spectrum $B(\omega)$. It is desired to find a value of ω_0 which is small enough so that an adequate description of $B(\omega)$ will be insured. Now the spectrum may be expected to possess relative maxima, some of which are sharper than others. Clearly, the interval ω_0 which results in a suitable description of $B(\omega)$ near its sharpest relative maximum will suffice to describe the curve at all frequencies.

The sharpness of a relative maximum is conveniently indicated by its bandwidth $\omega_2 - \omega_1$, ω_2 and ω_1 being the half-power points of the maximum.[3] A sharp maximum thus corresponds to a narrow bandwidth, i.e., a small difference $\omega_2 - \omega_1$. If $\overline{\omega_2}$ and $\overline{\omega_1}$ are the half-power points of the sharpest relative maximum of $B(\omega)$, the completeness with which it is desired to describe $B(\omega)$ can be expressed as m, the minimum number of equally spaced frequencies between $\overline{\omega_2}$ and $\overline{\omega_1}$ at which $B(\omega)$ must be known. Accordingly ω_0 must be less than or equal to $\overline{\omega_2} - \overline{\omega_1}/m+1$. The least possible value of $\overline{\omega_2} - \overline{\omega_1}$ must yet be established in order to determine fully how small ω_0 must be.

While nothing has been assumed concerning F(t) except its time of duration t_1, it seems evident that of all functions having this duration, the one possessing the greatest concentration of sinusoidal components (and therefore the sharpest relative maximum in its spectrum) at any frequency ω is a sinusoid of frequency ω. Assuming this to be true, the smallest possible value of $\overline{\omega_2} - \overline{\omega_1}$ (for a relative maximum at the frequency ω) can be found by actually plotting the spectrum of

$$\overline{f(t)} = \begin{cases} o \text{ for } t \leq o,\ t \geq t_1 \\ \sin \omega t \text{ for } o < t < t_1 \end{cases}. \tag{9}$$

This has been done for several different frequencies and the results recorded in Table I. $\beta = 2\pi/t_1$ is, of course, the angular frequency of a sinusoid having the period t_1.

TABLE I

Width of the Sharpest Maximum Possible at Various Frequencies

Frequency	Width of Maximum
$\frac{2\pi}{t_1} = \beta$	$0.93\ \beta$
$10\ \beta$	$0.88\ \beta$
$100\ \beta$	$0.86\ \beta$
$1000\ \beta$	$0.86\ \beta$

[3] Half-power points are those frequencies at which $B(\omega) = B_{max}/\sqrt{2}$, B_{max} being the value of $B(\omega)$ at the maximum in question.

It is seen from Table I that the least possible value of $\overline{\omega_2} - \overline{\omega_1}$ is quite independent of the frequency at which the associated maximum occurs, and that a reasonably good estimate of this minimum is $2\pi/t_1$. The interval which will suitably describe the sharpest possible maximum is therefore

$$\omega_0 \leq \frac{(\overline{\omega_2} - \overline{\omega_1})_{min}}{m+1} = \frac{2\pi}{t_1(m+1)} \tag{10}$$

Accordingly, an adequate description of $B(\omega)$ will be insured if T is taken to be

$$T = \frac{2\pi}{\omega_0} \geq \frac{2\pi}{2\pi/t_1(m+1)} = t_1(m+1).$$

* * *

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