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AN IMPROVED ELECTRONIC ANALOG COMPUTING CIRCUIT

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ABSTRACT

An improved electronic analog computing circuit, employing only two vacuum tubes and characterized by both simplicity and accuracy of computation, is described and analyzed. The theoretical error is calculated for integration, differentiation, and constant multiplication with step-function and sine-wave forms of excitation. The results of a cursory experiment, made to evaluate the performance of the improved circuit functioning as a differentiator, agree with the theoretical results. The details of the circuit analysis are included in an appendix.

PROBLEM STATUS

This is an interim report on one phase of this problem; work is continuing.

AUTHORIZATION

NRL Problem No. P10-01R

AN IMPROVED ELECTRONIC ANALOG COMPUTING CIRCUIT

INTRODUCTION

Applications of electronic circuits in analog computing techniques have been described extensively in the recent literature.[1,2,3,4] In the present state of the art certain basic circuits for performing the most common mathematical operations, such as: addition, integration, and multiplication by a constant, have become more or less standardized. In each of these circuits accuracy and reliability are attained at the expense of simplicity. The improved circuit described in this paper is characterized by simplicity as well as excellent performance.

In an article entitled, "Designing Industrial Controllers by Analog,"[5] George A. Philbrick presents a computing circuit wherein a high degree of accuracy is sacrificed for simplicity of design and certain other advantages. The improved circuit is similar to the Philbrick design.

PHILOSOPHY OF THE IMPROVED CIRCUIT

In describing electronic circuits for integrating and differentiating voltage-input functions,[6] it is customary to begin with the basic resistance-capacitance networks and then show how the application of the feed-back amplifier reduces the inherent error to negligible proportions. In contrast to this approach, suppose the ideal realization of these mathematical operations by electrical circuits is considered.

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[1] I. A. Greenwood, Jr., J. V. Holdam, Jr. and D. Macrae, Jr., Electronic Instruments, M.I.T. Radiation Laboratory Series, Vol. 21, McGraw-Hill, 1948.

[2] F. J. Murray, The Theory of Mathematical Machines, King's Crown Press, 1948.

[3] Seymour Frost, Compact Analog Computer, Electronics, 21, 116-122, July, 1948.

[4] Granino A. Korn, Elements of D. C. Analog Computers, Electronics, 21, 122-127, April, 1948.

[5] G. A. Philbrick, Designing Industrial Controllers by Analog, Electronics, 21, 108-111, June, 1948.

[6] Granino A. Korn, op. cit.

The instantaneous voltage drop, $e_c(t)$, across a capacitor as a function of the instantaneous current, $i_c(t)$, flowing is

$$e_c = \frac{1}{c}\int i_c dt + e_{co}, \tag{1}$$

where

e_{co} = initial voltage drop across the capacitor representing the initial condition of integration.

The equivalent electrical circuit for equation (1) consists of a lossless capacitor fed by a constant current generator as shown in Figure 1. Theoretically, then, if the voltage-input function can be converted into a proportional current function, perfect integration by analog can be realized. It is important to note that this generator must have zero internal shunt admittance.

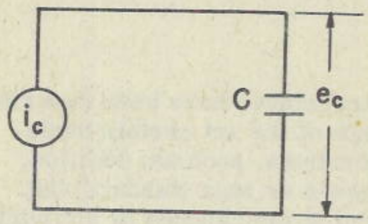

Figure 1

For differentiation the inverse procedure is considered. The instantaneous current, $i_c(t)$, flowing in a capacitor as a function of the instantaneous voltage applied, $e_c(t)$, is

$$i_c = C\,\frac{de_c}{dt}. \tag{2}$$

The equivalent electrical circuit for equation (2) consists of a lossless capacitor fed by a constant-voltage generator as shown in Figure 2. Thus, if the capacitor current can be converted into a proportional voltage without disturbing the circuit, perfect differentiation by analog can be realized. It is important to note that this generator must have zero internal series impedance.

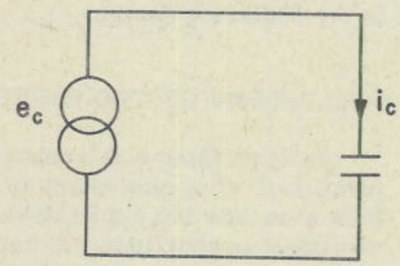

Figure 2

The simplest type of electronic circuit approximating these generator specifications is the "cathode follower" from which either a constant voltage or a constant current can be supplied, depending upon the manner in which the load is connected.[7] The equivalent form of such a circuit is shown in Figure 3.

$e_i(t)$ = input voltage
$e_k(t)$ = cathode voltage
$e_o(t)$ = output voltage
$i_p(t)$ = plate current
$i(t)$ = load or capacitor current
C = computing capacitor
R = computing capacitor
S = switch for interchanging computing element connections
(1) integration (2) differentiation
R_p = dynamic plate resistance of the vacuum tube.

---

[7] I. A. Greenwood, Jr., J. V. Holdam, Jr. and D. Macrae, op. cit.

The inherent degeneration of the cathode follower justifies the approximation,

$$e_i \approx e_k . \quad (3)$$

Therefore, with the capacitor connected to the cathode of the tube, a constant voltage is applied to the capacitor and the resulting current is not only constant but also proportional to the derivative of the input voltage according to equation (2). The output voltage is provided by the drop across R, or

$$e_o = -Ri = -RC\frac{de_k}{dt} \approx -RC\frac{de_i}{dt}. \quad (4)$$

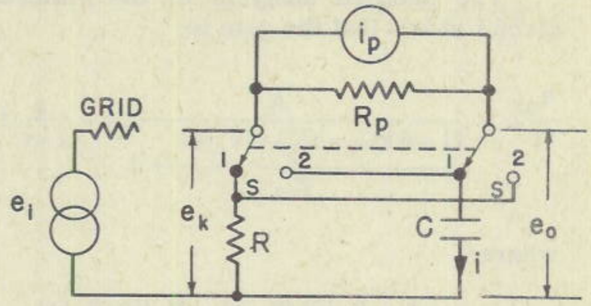

Figure 3

With resistor R connected to the cathode, current i is proportional to the input voltage, and the capacitor is effectively fed by a constant-current generator. According to equation (1), then,

$$e_o = -\frac{1}{RC}\int Ridt + e_{co} \approx -\frac{1}{RC}\int e_i dt + e_{co}. \quad (5)$$

The essential difference between the circuit of Figure 3 and the corresponding passive circuits is that, in the former case, the input voltage generator is not required to deliver power by virtue of the vacuum-tube action.

The circuit of Figure 3 becomes a constant multiplier if the capacitor is replaced by another resistor, R_o, since

$$i = -\frac{e_o}{R_o} = \frac{e_k}{R} \approx \frac{e_i}{R}, \quad (6)$$

where: R is connected to the cathode. Therefore, the multiplying constant is

$$\frac{e_o}{e_i} \approx -\frac{R_o}{R}. \quad (7)$$

The equivalent circuit of Figure 3 can be realized by the physical circuits described in the following paragraphs. The effectiveness of each is determined by the accuracy with which the condition of equation (3) is realized.

GENERAL ANALYSIS OF THE PHYSICAL CIRCUITS

The circuit presented by Philbrick, and illustrated in Figure 4 is equivalent to that of Figure 3 except for the addition of resistor R_b, which is connected in shunt with the current generator. The arrangement of the batteries and the proper adjustment of this resistor establish the conditions for linear operation of the tube and zero output with no input.

The detailed analysis of the Philbrick circuit shows that the gain is:

$$\frac{e_o}{e_i} = - \frac{A}{\frac{(1+A)(G_b+G_p)+Y_1(p)}{g_m} + 1} = -\frac{A}{1+\epsilon}, \quad (8)$$

where:

$A = Z_2(p)/Z_1(p) = Y_1(p)/Y_2(p)$

$Z_2(p)$ = generalized output impedance

$Z_1(p)$ = generalized cathode impedance

$R_b = 1/G_b$

$R_p = 1/G_p$

g_m = tube transconductance

ϵ = error term.

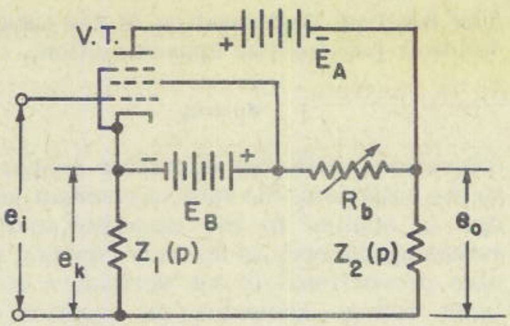

Figure 4

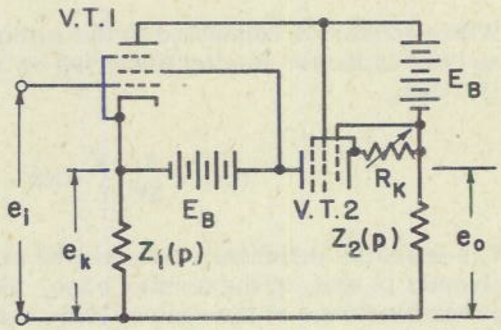

Figure 5

The error term is relatively small because the tube type is selected for the largest possible transconductance. The dominant quantity in the numerator of this term is G_b, the conductance of the balancing resistor, provided the tube is a pentode. The approximate magnitude of G_b is determined by battery voltage E_B, and the static plate current of the tube, I_p; i.e.,

$$G_b = \frac{I_p}{E_B}. \quad (9)$$

The accuracy of computation can be greatly improved by a drastic reduction in the magnitude of G_b. A scheme for doing this is shown in Figure 5, wherein the balancing resistor is replaced by another vacuum tube of the same type. Thus, the "constant current" property of the pentode provides the desired reduction in G_b and still maintains the necessary static operating conditions. The cathode resistor, R_k, provides the "fine" balancing adjustment. This resistor is so small that its presence can be neglected entirely in the analysis of the dynamic operation of the circuit.

The gain of the improved circuit, which is derived from equation (8) by simply replacing R_b with the dynamic plate resistance of the additional tube, is

$$\frac{e_o}{e_i} = - \frac{A}{\frac{(1+A)(G'_p+G_p)+Y_1(p)}{g_m} + 1} = -\frac{A}{1+\epsilon'}, \quad (10)$$

where:

G'_p = dynamic plate conductance of the additional tube,

ϵ' = error term.

ANALYSIS OF THE IMPROVED CIRCUIT

As an Integrator

The integration gain function is derived from equation (10) by substituting

$$Y_1(p) = G_1 = \frac{1}{R_1}$$

and

$$Y_2(p) = Cp, \tag{11}$$

and rearranging. Hence

$$\frac{e_o}{e_i} = - \frac{\frac{1}{R_1 Cp}}{\frac{G_1(G_p' + G_p)}{g_m Cp} + \left(\frac{G_p' + G_p + G_1}{g_m} + 1\right)} \approx - \frac{1}{R_1 Cp} . \tag{12}$$

Equation (12) is more useful in the compact form,

$$\frac{e_o}{e_i} = - \frac{\frac{1}{R_1 Cp}}{\frac{a}{p} + (1 + b)} . \tag{13}$$

The accuracy of integration depends upon the nature of the input function. As the simplest example of a nonperiodic input, the unit step function is considered:

$$\begin{cases} e_i(t) = 0 , \; t < 0 \\ e_i(t) = 1 , \; t \geqslant 0. \end{cases} \tag{14}$$

Since "p" is defined as the derivative operator, equation (13) is a differential equation, the solution of which is, after substitution of equation (14),

$$\begin{aligned} e_o &= - \frac{1}{a R_1 C}\left[1 - \exp.\left(\frac{-a}{1+b}\right)t\right] \\ &= - \frac{t}{R_1 C} \cdot \frac{1}{1+b}\left[1 - \frac{at}{(1+b)2!} + \frac{a^2 t^2}{(1+b)^2 3!} - \dots\right], \end{aligned} \tag{15}$$

in which

$$a = \frac{G_p' + G_p}{g_m R_1 C} \tag{16}$$

and

$$b = \frac{G_p' + G_p + G_1}{g_m} . \tag{17}$$

If $a = b = 0$, then the integration would be perfect and the output would increase linearly with time. Normally, b is so small that it can be neglected. Although a is also very small, it can not be neglected because it appears in the coefficient of t in the error terms. It is advantageous to make a as small as possible to make the integration time longer for a specified error.

According to equation (15), the time constant of the integrator is

$$T_c = \frac{1+b}{a} \approx \frac{1}{a} = \frac{g_m R_1 C}{G_p' + G_p}. \tag{18}$$

Equation (18) indicates that: (1) replacing the balancing resistor by another tube increases the integrating time constant by at least an order of magnitude, and (2) for the same integrating components, the time constant of the improved circuit is several thousand times greater than that for a passive circuit.

Since an arbitrary input function, $e_i(t)$, can be resolved into sinusoidal components, the operation of the circuit for this type of input can be analyzed on the basis of one such component, say,

$$e_i(t) = E_i \cos \omega t. \tag{19}$$

In this case, the output is the steady-state solution of equation (13),

$$e_o = -\frac{E_i}{R_1 C} \frac{\sin(\omega t - \Psi)}{\sqrt{a^2 + \omega^2 (1+b)^2}}, \tag{20}$$

where

$$\Psi = \arc\tan \frac{a}{\omega(1+b)}.$$

Since a and b are very small, the magnitude of the output is approximately

$$\left|e_o\right| = \frac{E_i}{\omega R_1 C} \cdot \frac{1}{(1+b)} \left[1 - \frac{a^2}{2(1+b)^2 \omega^2}\right], \tag{21}$$

and the phase angle error of the output is approximately

$$\Psi = \frac{a}{\omega(1+b)} \approx \frac{a}{\omega} = \frac{1}{\omega T_c}. \tag{22}$$

For a prescribed maximum allowable error, equations (21) and (22) show that the lower frequency limit is determined by the circuit time constant, hence, the advantage of the increased time constant of the improved circuit.

When integration is performed with an electronic circuit, the initial condition is established by appropriately charging the integrating capacitor at $t = 0$. This charge is dissipated exponentially according to the circuit time constant (provided, of course, the integrator is not overloaded). Since this dissipation must be negligible for the duration of a given input function, the necessity for a very long time constant is again indicated.

As a Differentiator

The differentiation gain function is derived from equation (10) by substituting

$$Y_1(p) = C_p, \quad Y_2(p) = G_2 = \frac{1}{R_2}, \tag{23}$$

and rearranging;

$$\frac{e_o}{e_i} = -\frac{R_2 Cp}{\left(1 + \frac{G_p' + G_p}{g_m}\right) + \frac{G_p' + G_p + G_2}{G_2 g_m} Cp} . \tag{24}$$

Equation (24) is more useful in the form

$$\frac{e_o}{e_i} = -\frac{R_2 C_p}{(1 + m) + np} . \tag{25}$$

The accuracy of differentiation depends upon the nature of the input function. As the simplest example of a nonperiodic input, the linear time function is considered

$$\begin{cases} e_i(t) = 0, \ t < 0, \\ e_i(t) = t, \ t \geqslant 0. \end{cases} \tag{26}$$

The solution of equation (25), subject to the conditions of equation (26), is

$$e_o = -\frac{R_2 C}{1 + m}\left[1 - \exp. \frac{-(1 + m)}{n} t\right], \tag{27}$$

in which,

$$m = \frac{G_p + G_p'}{g_m} , \tag{28}$$

and

$$n = \frac{\left(G_p + G_p' + G_2\right)}{g_m} R_2 C. \tag{29}$$

According to equation (27), the time constant is

$$T_c = \frac{n}{1 + m} \approx n = \frac{\left(G_p + G_p' + G_2\right)}{g_m} R_2 C. \tag{30}$$

Equation (30) indicates that: (1) replacing the balancing resistor by another tube decreases the differentiation time constant, the extent of which depends upon the relative magnitude of G_2; (2) for the same differentiation components, the time constant of the improved circuit may be several hundred times smaller than that for a passive circuit; and (3) the decay of the exponential error term in equation (27) is much more rapid with the smaller time constant.

Considering a component of an arbitrary input function,

$$e_i(t) = E_i \cos \omega t, \tag{19}$$

the steady-state solution of equation (25) is

$$e_o = \frac{E_i R_2 \omega C \sin(\omega t - \psi)}{\sqrt{(1+m)^2 + \omega^2 n^2}}, \tag{31}$$

where

$$\psi = \arc\tan \frac{\omega n}{1+m}.$$

Since m and n are very small, the magnitude of the output is approximately

$$|e_o| = \frac{E_i R_2 \omega C}{(1+m)} \left[1 - \frac{\omega^2 n^2}{2(1+m)^2}\right], \tag{32}$$

and the phase angle error of the output is approximately

$$\psi = \frac{\omega n}{1+m} = \omega T_C \approx \omega n. \tag{33}$$

For a prescribed maximum error, the upper frequency limit is determined by the circuit time constant.

As a Constant Multiplying Amplifier

The gain function as an amplifier is derived from equation (10) by substituting

$$\begin{aligned} Y_1(p) &= G_1 = \frac{1}{R_1}, \\ Y_2(p) &= G_2 = \frac{1}{R_2}, \end{aligned} \tag{34}$$

and rearranging;

$$\frac{e_o}{e_i} = -\frac{\frac{G_1}{G_2}}{\frac{(G_1 + G_2)(G_p' + G_p) + G_1 G_2}{G_2 g_m} + 1} \approx -\frac{G_1}{G_2} = -\frac{R_2}{R_1}. \tag{35}$$

The error in multiplication can be minimized by keeping the multiplication constant relatively small (say, less than 10) and the computing resistances as large as possible, consistent with stray capacitance and leakage considerations.

Neither the Philbrick nor the improved circuit can be employed as a summing amplifier because there is only one isolated input. However, passive resistive networks[8] in conjunction with these circuits are satisfactory for summing several inputs. By using a common output resistance, it is also possible to sum the outputs of several circuits connected as amplifiers.

IMPEDANCE CONSIDERATIONS

Whenever computing circuits are interconnected, the reaction of each circuit on all the others must be carefully considered. In circuits of the improved type, the input impedance is essentially infinite, so that the input of the one will not "load" the other in a simple tandem connection. However, when a resistive summing network intervenes, loading does occur, and the gain of the loaded circuit must be corrected accordingly.

The effect of output loading on the improved circuit can be determined from the internal impedance at the output terminals. Calculated on the admittance basis, this is

$$Y_g = Y_2 + \frac{Y_1(G_p + G_p')}{G_p + G_p' + Y_1 + g_m} \approx Y_2 + \frac{Y_1(G_p + G_p')}{g_m} \; . \qquad (36)$$

The second term of the right-hand member is the internal admittance exclusive of the output admittance, Y_2, and is very small compared to Y_2. The output of the circuit, therefore, has the characteristics of a constant-current generator, and any loading introduces error in computation unless suitable corrections are applied. This difficulty can be circumvented by employing another circuit as a one-to-one isolation amplifier with the load connected to the cathode. The equivalent shunt admittance at the cathode terminals is

$$Y_k = Y_1 + \frac{Y_2(G_p + G_p' + g_m)}{Y_2 + G_p + G_p'} \approx Y_1 + g_m . \qquad (37)$$

Since the second term of the right-hand member is large compared to the normal cathode admittance, Y_1, the output at the cathode terminals is essentially constant with loading. In this application the improved circuit becomes a d-c cathode follower.

EXPERIMENTAL RESULTS

The work described in this paper was initiated primarily to develop a simple and reliable electronic differentiating circuit. When it was determined that the computation accuracy of the Philbrick was not adequate, the improved circuit was developed; the superiority of the latter demonstrated in the foregoing analyses has been experimentally verified for differentiation.

The experiments consisted of:

(1) Measurement of gain magnitude as a function of frequency for sinusoidal voltage excitation.

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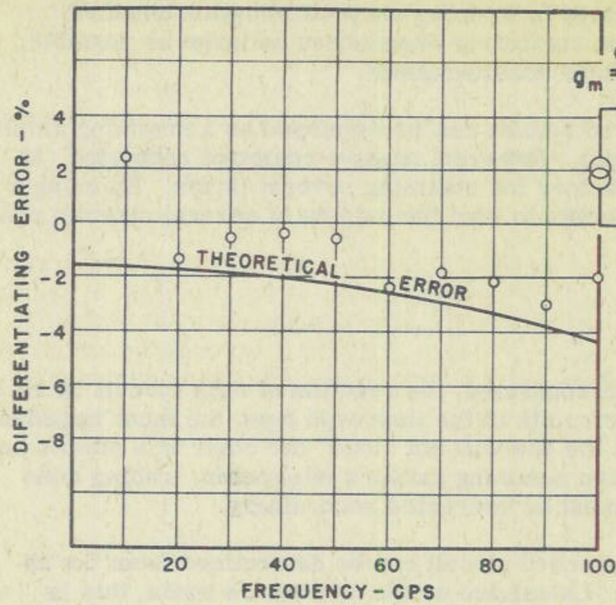
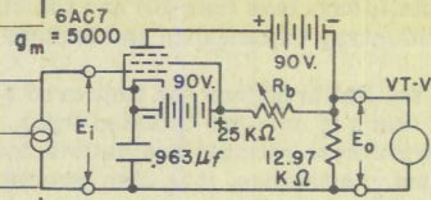
[8] F. J. Murray, op. cit.





$$E_o = \frac{E_i R_2 \omega C}{(1+m)} \left[1 - \frac{\omega^2 n^2}{2(1+m)^2}\right] \quad (32)$$

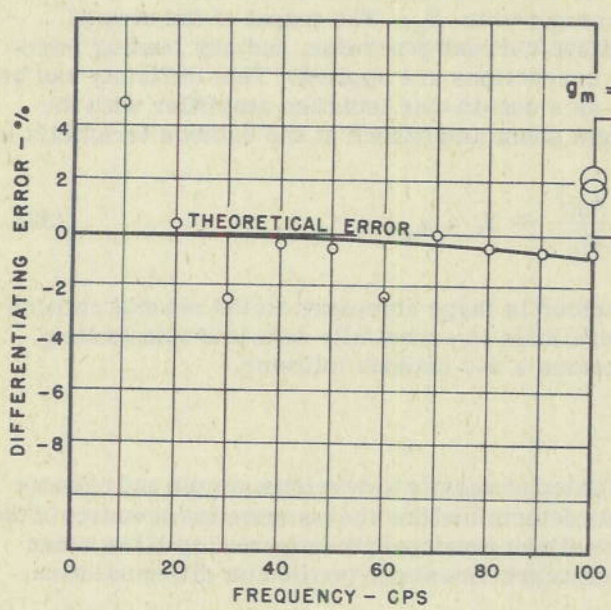
$$m = \frac{G_p + G_b}{g_m} \quad (28)$$

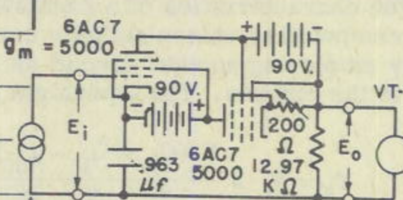
$$n = \frac{G_p + G_b + G_2}{g_m} R_2 C \quad (29)$$

$$\%\ \text{Error} = \frac{1}{(1+m)}\left[1 - \frac{\omega^2 n^2}{2(1+m)^2}\right] 100 \text{ (Theor.)}$$

$$\%\ \text{Error} = \frac{(E_o/E_i)\ \text{meas.}}{R_2 \omega C}\ 100 \text{ (Exper.)}$$

Fig. 6A - Philbrick Differentiator Performance
Gain Magnitude Error





$$E_o = \frac{E_i R_2 \omega C}{(1+m)} \left[1 - \frac{\omega^2 n^2}{2(1+m)^2}\right] \quad (32)$$

$$m = \frac{G_p + G'_p}{g_m} \quad (28)$$

$$n = \frac{G_p + G'_p + G_2}{g_m} R_2 C \quad (29)$$

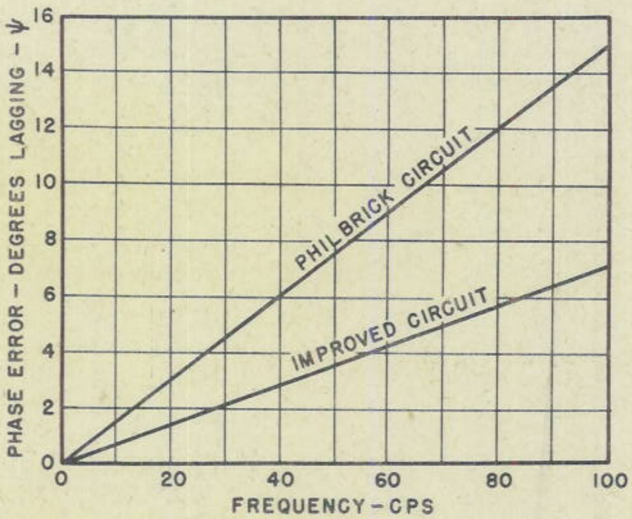
$$\%\ \text{Error} = \frac{1}{1+m}\left[1 - \frac{\omega^2 n^2}{2(1+m)^2}\right] 100 \text{ (Theor.)}$$

$$\%\ \text{Error} = \frac{(E_o/E_i)\ \text{meas.}}{R_2 \omega C}\ 100 \text{ (Exper.)}$$

Fig. 6B - Improved Differentiator Performance
Gain Magnitude Error

(2) oscillographic observation of output and phase shift for sinusoidal and square-wave excitation.

The gain magnitude measurements are shown in the graphs of Figure 6, wherein differentiation error is plotted against frequency for both circuits, and the theoretical error is also plotted for comparison. Measurements for the low-frequency end of the range are of no value because of the errors introduced by the dielectric absorption and hysteresis of the differentiating capacitor. These undesirable phenomena could have been eliminated by using a polystyrene capacitor, which was not available for the experiments. The dispersion of the points of the experimental curves is largely the result of the measurement error rather than computing error. The phase shift was observed on an oscilloscope by comparing input and output voltages with an electronic switch. The crudeness of the technique and the small deviations from the theoretical 90^0 phase shift did not permit quantitative measurements. However, the theoretical values for the experimental frequency range are plotted in the graphs of Figure 7.



$$\psi = \arc\tan \frac{\omega n}{1+m} \approx \frac{\omega n}{1+m}, \qquad (31, 33)$$

For Philbrick Circuit

$$m = \frac{G_p + G_b}{g_m} \qquad (28)$$

$$n = \frac{G_p + G_b + G_2}{g_m} R_2 C \qquad (29)$$

For Improved Circuit

$$m = \frac{G_p + G_p'}{g_m} \qquad (28)$$

$$n = \frac{G_p + G_p' + G_2}{g_m} R_2 C \qquad (29)$$

See Figure 6 for Circuits

Fig. 7 - Differentiator Performance Gain Phase Error

The plates and screens of the tubes in the experimental circuits were supplied by "B" batteries. This is not practical for a finished computing device. It will be noted that the tubes in these circuits, as well as the batteries, are floating with respect to ground. A 60-cycle power supply, in lieu of these batteries, presents some acute shielding and filtering problems, since any extraneous output due to the power supply results in computing error. It may be feasible to use, for the plates and screens as well as the filaments, a radio-frequency power supply, similar to those now incorporated in many television receivers. Because of the great difference in the power and operating frequencies, the former can be easily filtered out without introducing computing error. The output voltage of a radio-frequency oscillator can be well regulated by very simple means, and much better isolation can be attained with radio-frequency components.

* * *

APPENDIX

Analysis of the Computing Circuits

Figure 8 shows the equivalent circuit, on the nodal basis of analysis, which corresponds to the physical circuits shown in Figures 4 and 5.

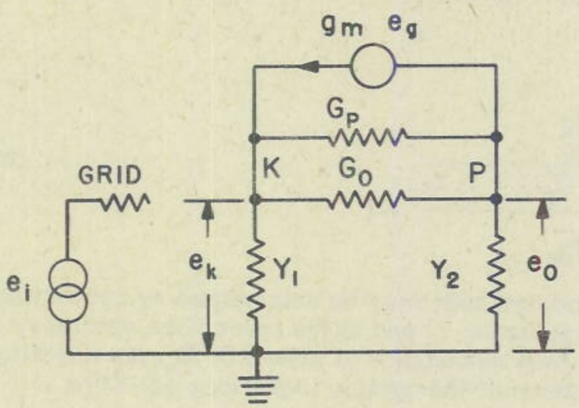


Figure 8

$G_o = G_b$ for Philbrick Circuit

$G_o = G_p'$ for Improved Circuit

$e_i = e_i(t),\ e_k = e_k(t)$

$e_o = e_o(t)$

$e_g = e_g(t)$

$Y_1 = Y_1(p),\ Y_2 = Y_2(p)$

The nodal equations which describe the circuit of Figure 8 are

$$\begin{cases} e_g g_m = e_k(Y_1 + G_o + G_p) - e_o(G_p + G_o), \\ -e_g g_m = -e_k(G_p + G_o) + e_o(Y_2 + G_o + G_p), \end{cases} \quad (38)$$

and the grid-to-cathode voltage is

$$e_g = e_i - e_k. \quad (39)$$

Substitution of equation (39) in equation (38) results in

$$\begin{cases} e_i g_m = e_k(Y_1 + G_o + G_p + g_m) - e_o(G_p + G_o), \\ -e_i g_m = -e_k(G_p + G_o + g_m) + e_o(Y_2 + G_o + G_p), \end{cases} \quad (40)$$

in which the determinant of the system is

$$\Delta = \begin{vmatrix} (Y_1 + G_o + G_p + g_m) & -(G_p + G_o) \\ -(G_p + G_o + g_m) & (Y_2 + G_o + G_p) \end{vmatrix}$$

$$= Y_1 Y_2 + Y_1(G_o + G_p) + Y_2(G_p + G_o + g_m). \quad (41)$$

Thus

$$e_o = \frac{\begin{vmatrix} (Y_1 + G_o + G_p + g_m) & e_i g_m \\ -(G_p + G_o + g_m) & -e_i g_m \end{vmatrix}}{\Delta} = \frac{-e_i g_m Y_1}{\Delta}$$

$$= \frac{-e_i g_m Y_1}{(Y_1 + Y_2)(G_o + G_p) + Y_1 Y_2 + g_m Y_2} \tag{42}$$

The formal gain function may be obtained from equation (42),

$$\frac{e_o}{e_i} = -\frac{\frac{Y_1}{Y_2}}{\frac{(Y_1 + Y_2)(G_o + G_p) + Y_1 Y_2}{Y_2 g_m} + 1} . \tag{43}$$

The internal shunt admittance at a particular node may be determined by connecting to this node an arbitrary constant current generator i, and at the same time setting e_i equal to zero. In determining the internal shunt admittance at node PO, Y_g, the conditions just stated are imposed upon the nodal equations of the system (equations 40). Then

$$\begin{cases} 0 = e_k (Y_1 + G_o + G_p + g_m) - e_o (G_p + G_o), \\ i = -e_k (G_p + G_o + g_m) + e_o (Y_2 + G_o + G_p). \end{cases} \tag{44}$$

from which, solving for e_o and the admittance i/e_o

$$e_o = \frac{\begin{vmatrix} (Y_1 + G_o + G_p + g_m) & o \\ -(G_p + G_o + g_m) & i \end{vmatrix}}{\Delta} = \frac{i(Y_1 + G_o + G_p + g_m)}{\Delta} , \tag{45}$$

and therefore

$$\frac{i}{e_o} = Y_g = \frac{\Delta}{(Y_1 + G_o + G_p + g_m)} = Y_2 + \frac{Y_1 (G_o + G_p)}{Y_1 + G_o + G_p + g_m} . \tag{46}$$

At node KO, Y_k, the internal shunt admittance is found by connecting the arbitrary current generator to this node, and again setting e_i equal to zero. Then equations (40) become

$$\begin{cases} i = e_k (Y_1 + G_o + G_p + g_m) - e_o (G_p + G_o) \\ o = -e_k (G_p + G_o + g_m) + e_o (Y_2 + G_o + G_p) \end{cases} \tag{47}$$

from which, solving for e_k and the admittance i/e_k,

$$e_k = \frac{\begin{vmatrix} i & -(G_p + G_o) \\ 0 & (Y_2 + G_o + G_p) \end{vmatrix}}{\Delta} = \frac{i\,(Y_2 + G_o + G_p)}{\Delta}, \tag{48}$$

so that

$$\frac{i}{e_k} = Y_k = \frac{\Delta}{Y_2 + G_o + G_p} = Y_1 + \frac{Y_2\,(G_p + G_o + g_m)}{(Y_2 + G_o + G_p)}. \tag{49}$$

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