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14. ABSTRACT					
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RPPR Final Report

as of 22-Aug-2022

Agency Code: 21XD

Proposal Number: 71518MS

Agreement Number: W911NF-18-1-0031

INVESTIGATOR(S):

Name: Ph.D. Guoliang Huang

Email: huangg@missouri.edu

Phone Number: 5738825915

Principal: Y

Organization: **University of Missouri - Columbia**

Address: 115 Business Loop 70W, Columbia, MO 652110001

Country: USA

DUNS Number: 153890272

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Report Date: 31-Oct-2022

Date Received: 22-Aug-2022

Final Report for Period Beginning 15-Dec-2017 and Ending 31-Jul-2022

Title: Degenerate Torque Materials for Perfect Cloaking in Elastic Media

Begin Performance Period: 15-Dec-2017

End Performance Period: 31-Jul-2022

Report Term: 0-Other

Submitted By: Ph.D. Guoliang Huang

Email: huangg@missouri.edu

Phone: (573) 882-5915

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STEM Degrees: 4

STEM Participants: 3

Major Goals: The central object of this proposal is the study of cloaking materials within the framework of a new class of elastic materials termed “Degenerate Torque Materials” (DTMs) and characterized as materials admitting a set of collapse mechanisms and torque-induced non-symmetric stress states. The research will establish a theoretical framework for the modelling of DTMs and will tackle the inverse problem of design of specific DTMs targeting particular macroscopic constitutive parameters, typical of those required by elastic cloaks. The necessity in elastic cloaking of a material with engineered collapse mechanisms and torque distributions, as well as modern advances in additive manufacturing technology, all encourage the use of truss- or lattice-based design paradigms. The research will accordingly explore paths towards the design and fabrication as well as dynamic modeling and homogenization of specific degenerate mechanical trusses and lattices admitting collapse mechanisms and exhibiting aptitude for the support of torque densities and non-symmetric stresses.

Accomplishments: Elastic cloaking has been a longstanding open problem in the area of transformation-based wave control. While theory and applications progressed smoothly in transformation optics and acoustics, the multitude of coupled propagated modes (e.g., shear and pressure) in elasticity introduced an unparalleled complication. Indeed, it turns out that the application of the transformation method necessitates the use of so-far lacking materials that exhibit asymmetric stresses as well as specific stressless deformation modes. In first two years of this funded project, we obtained the first lattice-based metamaterials that fulfil the requirements of the transformation method. An invisibility cloak in fully coupled 2D elasticity was then designed by spatially grading these metamaterials. To impose the required torques, we either (i) ground the masses using rotational springs, in which case the design is equally functional in all frequency ranges; or (ii) embed within each mass a rotational resonator, in which case the design is optimally functional at the resonance frequency. Then, in order to accommodate the necessary zero modes, the springs are connected to masses through hinges and spatially distributed so as to allow exactly one mode of stressless deformations. Once the local architecture of the lattice has been chosen, the geometry, orientation and constitutive parameters are graded in space so as to target the elasticity tensors $a=a(x)$ within the transformed solid. However, it is of interest to develop a new polar materials to minimize the number of unit cells within the cloak, or the transformed domain more generally. Accordingly, an optimal tradeoff between feasibility and performance is to be found. To extend application of polar material, our second goal is to develop inverse design theory “discrete transformation theory” to realize elastic wave cloaking and waveguiding under the general transformation. Third, due to the presence of collapse mechanisms and of stress asymmetry, the dynamics of polar materials transcends the standard framework of waves and vibrations in elastic solids. Therefore, such a topic is worthy of investigation independently of cloaking and other applications related to the transformation method. In the project, we have built on these results and expanded our research efforts through analytical, numerical and experimental approaches in three directions.

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Total 19 highly impacted journal papers have been published in top journals including Physical Review Letters and Journal of Mechanics of Physics and Solids. One patent has been filed (see attachment). Two postdoctoral fellows under this project have successfully secured faculty position in research-oriented universities.

Training Opportunities: Three graduate students and two postdoctoral fellows have been working under this project to get training on analytical, numerical and experimental skills. Two postdoctoral fellows have successfully secured faculty position in US research-oriented universities. Three graduate students have now graduated, one graduate student has found a faculty position in overseas university and the other two found postdoctoral positions at Duke University and University of Missouri. The PI have designed projects in the senior capstone course by integrating the topics related to the proposed research. These projects aim to inspire the students' interest and curiosity in emerging functional materials and improve their problem-solving abilities. The hands-on skills and fundamental knowledge that the students obtain have improved their prospects for acquiring future positions in academia or industry. Besides, the PI have been actively participating in the Engineering Summer Camp Program, Pre-Engineering Bridge Program, and Mizzou Youth Experience Program by giving seminars and tech-demos to K-12 students to promote the results in this research.

Results Dissemination: The PI's Physical Review Letters paper, entitled "Physical Realization of Elastic Cloaking with a Polar Material", was reported by more than 52 media across the world, including Science Daily, Phys.org and La Nación. Based on the MU News Bureau, these stories have potentially reached 51 million people, and were shared on social media over 1,400 times. Total 19 highly impacted journal papers have been published in top journals including Physical Review Letters and Journal of Mechanics of Physics and Solids. One patent has been filed in 2021.

Honors and Awards:

1. VSET Fellow Award, 2020.
2. Senior Faculty Research Award, College of Engineering, University of Missouri, 2018.
3. Fellow of International Association of Advanced Materials (IAAMs), 2020.
4. Vebleo Fellow, 2020.
5. Member, EU Academy of Sciences (EUAS), 2019.

Protocol Activity Status:

Technology Transfer: UM Invention Disclosure No. 20UMC080 entitled "Flexible Material to Protect Buildings"

PARTICIPANTS:

Participant Type: PD/PI

Participant: Guoliang Huang

Person Months Worked: 10.00

Project Contribution:

National Academy Member: N

Funding Support:

Participant Type: Postdoctoral (scholar, fellow or other postdoctoral position)

Participant: Hussein Nassar

Person Months Worked: 12.00

Project Contribution:

National Academy Member: N

Funding Support:

Participant Type: Postdoctoral (scholar, fellow or other postdoctoral position)

Participant: Yangyang Chen

Person Months Worked: 6.00

Project Contribution:

National Academy Member: N

Funding Support:

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Participant Type: Graduate Student (research assistant)
Participant: Xianchen Xu
Person Months Worked: 15.00
Funding Support:
Project Contribution:
National Academy Member: N

Participant Type: Graduate Student (research assistant)
Participant: Qian Wu
Person Months Worked: 15.00
Funding Support:
Project Contribution:
National Academy Member: N

Participant Type: Undergraduate Student
Participant: Raymond Waidmann
Person Months Worked: 6.00
Funding Support:
Project Contribution:
National Academy Member: N

Participant Type: Graduate Student (research assistant)
Participant: Hui Chen
Person Months Worked: 12.00
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ARTICLES:

Publication Type: Journal Article Peer Reviewed: Y **Publication Status:** 1-Published

Journal: Journal of the Mechanics and Physics of Solids

Publication Identifier Type: DOI **Publication Identifier:**

Volume: 117 **Issue:** **First Page #:** 22

Date Submitted: 10/5/18 12:00AM

Date Published: 4/1/18 10:00AM

Publication Location:

Article Title: A study of topological effects in 1D and 2D mechanical lattices

Authors: Hui Chen, Hussein Nassar, Guoliang Huang

Keywords: Kagome lattice; Topological mechanics; Edge states; Stoneley waves

Abstract: Topological insulators are new phases of matter whose properties are derived from a number of qualitative yet robust topological invariants rather than specific geometric features or constitutive parameters. Their salient feature is that they conduct localized waves along edges and interfaces with negligible scattering and losses induced by the presence of specific varieties of defects compatible with their topological class. The paper investigates two lattice-based topological insulators in one and two spatial dimensions. In 1D, relevant background on topological invariants, how they arise in a classical mechanical context and how their existence influences the dynamic behavior within bandgaps, is provided in a simple analytical framework. In 2D, we investigate Kagome lattices based on an asymptotic continuum model.

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Volume: Issue: 98

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Date Submitted: 10/5/18 12:00AM

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Publication Location:

Article Title: Elastic quantum spin-Hall effect in Kagome lattice

Authors: Hui Chen, Hussein Nassar, Andrew Norris, Gengkai Hu, Guoliang Huang

Keywords: Elastic wave; quantum hall; Kagome lattice

Abstract: A mechanical analog to the quantum spin Hall insulator (QSHI) is implemented into a simple mass-spring kagome lattice. The transition from the trivial state to the topological one is described by an invariant Chern number function of a contrast parameter. The band diagram and helical edge states characteristic of QSHI are obtained by a combination of numerical and analytical methods. In particular, these states are shown to be Stoneley wave solutions to a set of asymptotic continuous motion equations. Last, scatterless propagation of polarized topological edge waves around sharp corners is demonstrated and robustness is assessed through a parametric study.

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Publication Type: Journal Article Peer Reviewed: Y **Publication Status:** 1-Published

Journal: Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences

Publication Identifier Type: DOI

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Volume: 474 Issue:

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Date Submitted: 8/21/19 12:00AM

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Publication Location:

Article Title: A degenerate polar lattice for cloaking in full 2D elastodynamics and statics

Authors: Hussein Nassar, Yangyang Chen, Guoliang Huang

Keywords: Degenerate lattice; cloaking; elastodynamics and statics

Abstract: A lattice design of a cloak for full 2D elasticity is suggested when the background continuum is isotropic. The lattice is polar in the sense that it elastically resists rotations; and is degenerate meaning it admits a stressless collapse mechanism. These characteristics are enabled through the use of appropriately distributed restoring torques in conjunction with hinge-like spring-mass contacts. Thus, the lattice is proven to exhibit a rank-3 elasticity tensor lacking minor symmetry. Accordingly, it rigorously adheres to the form-invariance requirements of the transformation method under the Brun-Guenneau-Movchan gauge. The cloak is numerically tested in statics and in dynamics under pressure and shear incident waves and shows satisfactory performance. Finally, a theoretical generalization extends the design to 3D and to arbitrarily anisotropic backgrounds so as to enable cloaking as well as other transformation-based static and dynamic field manipulation in these cases.

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Journal: Journal of Mechanics and Physics of Solids

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Volume: 129

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Date Submitted: 8/21/19 12:00AM

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Publication Location:

Article Title: Isotropic polar solids for conformal transformation elasticity and cloaking

Authors: Hussein Nassar, Yangyang Chen and Guoliang Huang

Keywords: Cloaking, minor symmetry, chirality, polarization, hexachiral lattice

Abstract: The transformation method provides designs for elastic invisibility cloaks and other wave steering devices. In general nonetheless, it requires materials where stress symmetry is broken. We call polar a solid where Hook's law applies without enforcing stress symmetry, that is without suppressing body torques. Isotropic polar solids in two and three space dimensions are characterized and their material parameters identified and analyzed. Most importantly, we show that stress asymmetry permits for 2D polar solids to be chiral while remaining isotropic. Chirality is demonstrated to manifest in statics, as a coupling between rotation and change in area, as well as in dynamics, as a biasing tilt modifying the polarizations of plane harmonic waves. Finally, designs of 2D chiral isotropic polar solids are suggested in the form of grounded hexachiral lattices. Then, predictions based on polar elasticity are shown to accurately describe the behavior of these lattices in the homogenization limit.

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Journal: Physical Review Applied

Publication Identifier Type: DOI

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Volume: 11

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Date Submitted: 8/21/19 12:00AM

Date Published: 5/21/19 3:37PM

Publication Location:

Article Title: Mechanical Quantum Hall Effect in Time-Modulated Elastic Materials

Authors: H. Chen, L. Yao, H. Nassar, and G.L. Huang

Keywords: Time-modulated materials; Mechanical Quantum Hall effects

Abstract: We investigate a mechanical wave analogue of Thouless pumping and the quantum Hall effect (QHE) in one- (1D) and two-dimensional (2D) periodically time-modulated materials, respectively. In one dimension, wave propagation in the time-modulated system is characterized based on the adiabatic theorem, and topologically protected one-way edge modes are numerically demonstrated by the principle of bulk-edge correspondence. In two dimensions, a time-modulation scheme of a hexagonal lattice is suggested and polarized edge states characteristics of QHE are put into evidence by the plane wave expansion (PWE) method. We numerically demonstrate the existence of topologically protected one-way edge states immune to scattering by sharp corners, defects, randomly disordered modulation phases, and dissipation effects.

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Volume: 144

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Publication Location:

Article Title: Microtwist elasticity: A continuum approach to zero modes and topological polarization in Kagome lattices

Authors: Hussein Nassar, Hui Chen, Guoliang Huang

Keywords: Zero modes, Topological polarization, Kagome lattices, Isostatic lattices, Microtwist continuum, Effective medium theory, Mechanics of generalized continua

Abstract: The topologically polarized isostatic lattices discovered by Kane and Lubensky (2014, Nat. Phys. 10, 39–45) challenged the standard effective medium theories used in the modeling of many truss-based materials and metamaterials. As a matter of fact, these exhibit Parity (P) asymmetric distributions of zero modes that induce a P-asymmetric elastic behavior, both of which cannot be reproduced within Cauchy elasticity. Here, we propose a new effective medium theory baptized "microtwist elasticity" capable of rendering polarization effects on a macroscopic scale. The theory is valid for trusses on the brink of a polarized-unpolarized phase transition in which case they necessarily exhibit more periodic zero modes than they have dimensions. By mapping each periodic zero mode to a macroscopic degree of freedom, the microtwist theory ends up being a kinematically enriched theory.

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Publication Type: Journal Article

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Journal: Physical Review Applied

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Date Submitted: 8/17/20 12:00AM

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Publication Location:

Article Title: In-Plane Second-Order Topologically Protected States in Elastic Kagome Lattices

Authors: Qian Wu, Hui Chen, Xiaopeng Li, Guoliang Huang

Keywords: Second-order topological insulators, In-plane elastic dynamics, Quantized Wannier centers, Corner states

Abstract: Second-order topological insulators, which exhibit capability of hosting topologically protected zero-dimensional corner states distinct from the well-studied topological edge states, unveil a horizon beyond the conventional bulk-edge correspondence. Motivated by recent experimental observation of Wannier-type second-order corner states in acoustic structures, we investigate numerically and demonstrate experimentally the in-plane edge and corner states in a mechanical kagome lattice. By manipulating simply lattice geometry and quantized characterization, we exploit that the emerging corner states are topologically robust against disorders. We further present a second-order topological insulator with multi-interfaces such that the mechanical energy can be localized in multiple locations, which provides the possibility of practical application in energy harvesting devices.

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Publication Location:

Article Title: Physical Realization of Elastic Cloaking with a Polar Material

Authors: Xianchen Xu, Chen Wang, Wan Shou, Zongliang Du, Yangyang Chen, Beichen Li, Wojciech Matusik, N

Keywords: Cloaking, Polar materials, Transformation elasticity

Abstract: Cloaking against elastic disturbances has been demonstrated using several designs and gauges. None, however, tolerate the coexistence of normal and shear stresses due to a shortage of physical realization of transformation-invariant elastic materials. Here, we overcome this limitation to design and fabricate a new class of polar materials with a distribution of body torque that exhibits asymmetric stresses. A static cloak for full two-dimensional elasticity is thus constructed based on the transformation method. The proposed cloak is made of a functionally graded multilayered lattice embedded in an isotropic continuum background. While one layer is tailored to produce a target elastic behavior, the other layers impose a set of kinematic constraints equivalent to a distribution of body torque that breaks the stress symmetry. Experimental testing under static compressive and shear loads demonstrates encouraging cloaking performance in good agreement with our theoretical prediction.

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Publication Location:

Article Title: Polar Metamaterials: A New Outlook on Resonance for Cloaking Applications

Authors: H. Nassar, Y.Y. Chen, G.L. Huang

Keywords: Cloaking, Rotationally resonant metamaterials, Homogenization, Transformation method

Abstract: Rotationally resonant metamaterials are leveraged to answer a longstanding question regarding the existence of transformation-invariant elastic materials and the ad hoc possibility of transformation-based passive cloaking in full plane elastodynamics. Combined with tailored lattice geometries, rotational resonance is found to induce a polar and chiral behavior, that is, a behavior lacking stress and mirror symmetries, respectively. The central, and simple, idea is that a population of rotating resonators can exert a density of body torques strong enough to modify the balance of angular momentum on which hang these symmetries. The obtained polar metamaterials are used as building blocks of a cloaking device. Numerical tests show satisfactory cloaking performance under pressure and shear probing waves, further coupled through a free boundary.

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Publication Location:

Article Title: Optimal quantum valley Hall insulators by rationally engineering Berry curvature and band structure

Authors: Zongliang Du, Hui Chen, Guoliang Huang

Keywords: Optimal valley Hall insulators, Systematic design, Explicit geometry description, Band structure and Berry curvature engineering

Abstract: We propose the concept of "optimal quantum valley Hall insulators" and formulate it to an optimization formulation. In this new design paradigm, mechanical valley Hall insulators with simple geometries, large relative operating bandwidths and highly localized interface modes can be obtained in a systematic way. Compared to existing classical work based trail-and-error method via breaking symmetry, advantages of the new strategy are: (1) it simultaneously confines the Berry curvature and tailors the band structure to achieve robustness and optimality; (2) it adopts an explicit structural optimization method to describe and optimize the unit cells with a few design variables, which paves a rational way to explore a relatively large design space efficiently; (3) valley Hall insulators with quite different geometries and a better performance can be produced directly through an optimization process without introducing and manually tuning the unit cell with Dirac cone feature.

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Publication Location:

Article Title: Realization of active metamaterials with odd micropolar elasticity

Authors: Yangyang Chen, Xiaopeng Li, Colin Scheibner, Vincenzo Vitelli, Guoliang Huang

Keywords: Active metamaterials, wave control

Abstract: Materials made from active, living, or robotic components can display emergent properties arising from local sensing and computation. Here, we realize a freestanding active metabeam with piezoelectric elements and electronic feed-forward control that gives rise to an odd micropolar elasticity absent in energy-conserving media. The non-reciprocal odd modulus enables bending and shearing cycles that convert electrical energy into mechanical work, and vice versa. The sign of this elastic modulus is linked to a non-Hermitian topological index that determines the localization of vibrational modes to sample boundaries. At finite frequency, we can also tune the phase angle of the active modulus to produce a direction-dependent bending modulus and control non-Hermitian vibrational properties. Our continuum approach, built on symmetries and conservation laws, could be exploited to design others systems such as synthetic biofilaments and membranes with feed-forward control loops.

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Publication Location:

Article Title: Discrete transformation elasticity: An approach to design lattice-based polar metamaterials

Authors: Yangyang Chen, Hussein Nassar, Guoliang Huang

Keywords: discrete transform theory, carpet cloaking

Abstract: The transformation method is a powerful tool providing the constitutive parameters necessary for arbitrary geometric transformations of solution fields. These constitutive parameters, in elasticity, describe a constitutive law that, unlike conventional Hooke's law, is polar and chiral and that no known solids exhibit. This raises the question of whether polar and chiral elastic solids can be designed from the bottom up as architected lattice-based materials; this design task is a major challenge in the field of transformation elasticity. The present study aims to provide a theoretically justified design methodology based on a discrete transformation method. The key idea is to let the gradient of the geometric transformation operate not only on the elastic properties but on the underlying lattice-based architecture of the solids. As an outstanding application, we leverage the proposed design paradigm to construct a polar lattice metamaterial subsequently used for elastic carpet cloaking

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Volume: 12

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Article Title: Creating synthetic spaces for higher-order topological sound transport

Authors: Hui Chen, Hongkuan Zhang, Qian Wu, Yu Huang, Huy Nguyen, Emil Prodan, Xiaoming Zhou, Guoliang

Keywords: Synthetic space, topological sound transport

Abstract: Modern technological advances allow for the study of systems with additional synthetic dimensions. Higher-order topological insulators in topological states of matters have been pursued in lower physical dimensions by exploiting synthetic dimensions with phase transitions. While synthetic dimensions can be rendered in the photonics and cold atomic gases, little to no work has been succeeded in acoustics because acoustic wave-guides cannot be weakly coupled in a continuous fashion. Here, we formulate the theoretical principles and manufacture acoustic crystals composed of arrays of acoustic cavities strongly coupled through modulated channels to evidence one-dimensional (1D) and two-dimensional (2D) dynamic topological pumpings. In particular, the higher-order topological edge-bulk-edge and corner-bulk-corner transport are physically illustrated in finite-sized acoustic structures.

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Publication Location:

Article Title: Microtwist homogenization of three-dimensional Pyrochlore lattices on zero modes and mechanical polarization

Authors: Rongyu Xia, Hussein Nassar, Hui Chen, Zheng Li, Guoliang Huang

Keywords: Microtwist continuum, zero modes

Abstract: The mechanical Pyrochlore lattice was experimentally tested to demonstrate an intrinsically polar behavior of the material, which is soft on one side and hard on the opposite side (Bilal et al., 2017). The topological polarization in Pyrochlore lattices begs for developing a new effective medium theory because conventional Cauchy effective theories cannot predict the polarization phenomenon. In this study, we develop a 3D microtwist effective theory of Pyrochlore lattices to capture the P-asymmetric zero modes by which polarization emerges or fades on a macroscopic scale. By mapping three periodic zero modes to three macroscopic degrees of freedom, the 3D microtwist theory ends up being a kinematically enriched theory. The 3D microtwist elasticity is formulated by using two-scale asymptotic approach and its constitutive and balance equations are derived for a fairly generic isostatic lattice.

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Publication Location:

Article Title: Non-reciprocal Rayleigh wave propagation in space–time modulated surface

Authors: Qian Wu, Hui Chen, Hussein Nassar, Guoliang Huang

Keywords: space-time modulation, Rayleigh waves

Abstract: Research on non-reciprocal propagation of waves is of great significance in the field of photonic and phononic crystals for realizing flexible one-way propagation devices with potential engineering applications. Here, non-reciprocal Rayleigh waves are investigated in a continuous two-dimensional (2D) semi-infinite medium bound with an array of space–time modulated spring–mass oscillators. The involved modulation is a wave-like perturbation of the surface of the continuous medium that breaks time-reversal symmetry and reciprocity. To characterize the propagation of Rayleigh waves in such a complex 2D medium with continuous and discrete interface, an analytical study is performed to obtain dispersion-engineered bandgaps by adopting the asymptotic method and coupled mode theory, which is also validated by numerical simulation.

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Authors: Qian Wu; Guoliang Huang

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Abstract: Mode conversion and anomalous wave polarization between shear and longitudinal waves have been mainly explored in anisotropic solids. However, the use of elastic anisotropy inevitably introduces fundamental limitation in operative directions, which is not desirable for applications. In this study, we propose to use two-dimensional (2D) isotropic polar media that exhibit polarity and chirality moduli to achieve the above behaviors in an omnidirectional way, thereby realizing omnidirectional wave polarization manipulation. First, we formulate analytically and numerically a nearly perfect wave mode conversion, enabled by the chirality modulus, through a polar slab. Then, we emphasize that the polarity modulus causes the omnidirectional anomalous wave polarization which features faster shear modes. Two supplementary examples of the anomalous wave polarization, including singly polarized transmission and anomalous wave refraction, are presented.

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Authors: Jiayi Chen, Yangyang Chen, Xianchen Xu, Weijian Zhou, Guoliang Huang

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Abstract: Mode conversion and anomalous wave polarization between shear and longitudinal waves have been mainly explored in anisotropic solids. However, the use of elastic anisotropy inevitably introduces fundamental limitation in operative directions, which is not desirable for applications. In this study, we propose to use two-dimensional (2D) isotropic polar media that exhibit polarity and chirality moduli to achieve the above behaviors in an omnidirectional way, thereby realizing omnidirectional wave polarization manipulation. First, we formulate analytically and numerically a nearly perfect wave mode conversion, enabled by the chirality modulus, through a polar slab. Then, we emphasize that the polarity modulus causes the omnidirectional anomalous wave polarization which features faster shear modes. Two supplementary examples of the anomalous wave polarization, including singly polarized transmission and anomalous wave refraction, are presented.

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Authors: Hui Chen, Shaoyun Wang, Xiaopeng Li, Guoliang Huang

Keywords: Microtwist continuum Zero modes Parity asymmetry Polarized behavior Hinge-based metamaterials

Abstract: Mode conversion and anomalous wave polarization between shear and longitudinal waves have been mainly explored in anisotropic solids. However, the use of elastic anisotropy inevitably introduces fundamental limitation in operative directions, which is not desirable for applications. In this study, we propose to use two-dimensional (2D) isotropic polar media that exhibit polarity and chirality moduli to achieve the above behaviors in an omnidirectional way, thereby realizing omnidirectional wave polarization manipulation. First, we formulate analytically and numerically a nearly perfect wave mode conversion, enabled by the chirality modulus, through a polar slab. Then, we emphasize that the polarity modulus causes the omnidirectional anomalous wave polarization which features faster shear modes. Two supplementary examples of the anomalous wave polarization, including singly polarized transmission and anomalous wave refraction, are presented.

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Article Title: Physical Observation of a Robust Acoustic Pumping in Waveguides with Dynamic Boundary

Authors: Xianchen Xu, Qian Wu, Hui Chen, Hussein Nassar, Yangyang Chen, Andrew Norris, Michael R. Haberman

Keywords: Acoustic pumping; topological waveguide

Abstract: Research on breaking time-reversal symmetry to realize one-way wave propagation is a growing area in photonic and phononic crystals and metamaterials. In this Letter, we present physical realization of an acoustic waveguide with spatiotemporally modulated boundary conditions to realize nonreciprocal transport and acoustic topological pumping. The modulated waveguide inspired by a water wheel consists of a helical tube rotating around a slotted tube at a controllable speed. The rotation of the helical tube creates moving boundary conditions for the exposed waveguide sections at a constant speed. We experimentally demonstrate acoustic nonreciprocity and topologically robust bulk-edge correspondences for this system, which is in good agreement with analytical and numerical predictions. The nonreciprocal waveguide is a one-dimensional analog to the two-dimensional quantum Hall effect for acoustic circulators and is characterized by a robust integer-valued Chern number.

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Authors: H. Nassar, Y.Y. Chen, G.L. Huang

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Authors: H. Nassar, Y.Y. Chen, G.L. Huang
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Paper Title: Lattice cloaks for in-place elasticity: design and simulation
Authors: H. Nassar, Y.Y. Chen, G.L. Huang
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Authors: H. Hussein, Y.Y. Chen, G.L. Huang
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Partners

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I certify that the information in the report is complete and accurate:
Signature: Guoliang Huang
Signature Date: 8/22/22 12:58PM

1. Introduction

Cloaking for various wave phenomena is an issue of fundamental importance both practical and theoretical. Being an extreme example of wave steering, the feasibility of cloaking readily implies that a significant number of wave control devices are realizable with direct applications in stealth technology, imaging and energy harvesting (Chen et al., 2010; Pendry et al., 2000, 2006; Milton et al., 2006). The transformation method (Figure 1) provided the ultimate tool for designing cloaking devices for phenomena governed by equations that are form-invariant under a curvilinear change of coordinates such as the conductivity equation (Kohn et al., 2008, 2010; Greenleaf et al., 2003a,b), the acoustic wave equation (Norris, 2008) and Maxwell's equations of electromagnetism (Pendry et al., 2000; Leonhardt, 2006). Unlike in these domains where research has advanced to experimental and manufacturing stages, little to no work succeeded in probing the problem of cloaking for elastic waves; the major obstacle being that the corresponding equations are not form-invariant (Norris and Shuvalov, 2011). Form-invariance is in its essence a problem in material design. As a matter of fact, the form of the equations of elastic waves is a direct consequence of Hook's law. A curvilinear change of coordinates then transforms Hook's law into a different constitutive law that no known elastic solid satisfies. The ultimate purpose of the proposed research is to model, design and fabricate materials that will fill in this "behavioral gap". The suggested "cloaking materials" will have a generalized form-invariant constitutive behavior, making them excellent candidates for the design of elastic wave control devices in general, and of elastic cloaks in particular.

2. Objective

The central object of this proposal is the study of cloaking materials within the framework of a new class of elastic materials termed "Degenerate Torque Materials" and characterized as materials admitting a set of collapse mechanisms and torque-induced non-symmetric stress states. The concept of DTMs sets forth the basis of a markedly distinct approach towards cloaking for elasticity. The proposed research aims to theoretically justify the fundamental need for DTMs, by critically revisiting the transformation method. The research will establish a theoretical framework for the modelling of DTMs and will tackle the inverse problem of design of specific DTMs targeting particular macroscopic constitutive parameters, typical of those required by elastic cloaks. The necessity in elastic cloaking of a material with engineered collapse mechanisms and torque distributions, as well as modern advances in additive manufacturing, all encourage the use of truss- or lattice-based design paradigms. The research will accordingly explore paths towards the design and fabrication as well as dynamic modeling and homogenization of specific degenerate mechanical trusses and lattices admitting collapse mechanisms and exhibiting aptitude for the support of body torque densities and non-symmetric stresses, such as, but not restricted to, chiral isostatic lattices.

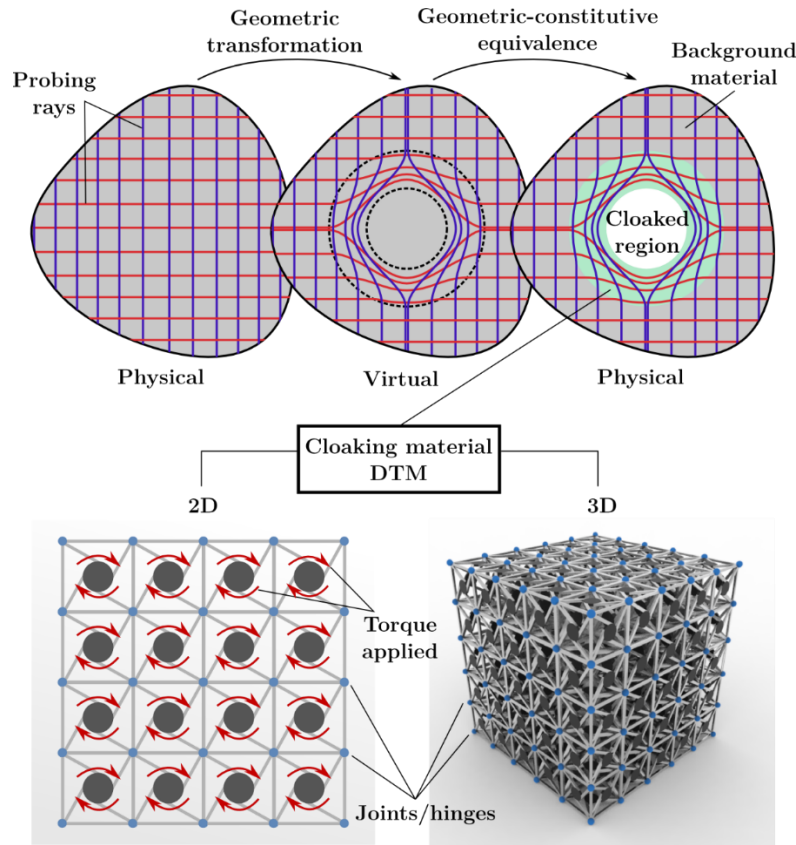


Figure 1.1 Scheme of the transformation method and of Degenerate Torque cloaking materials in 2D and 3D.

3. Approach

The proposed research is planned according to five tasks.

Task 1 is to propose and characterize a novel class of elastic materials, DTMs, that (i) possess a set of collapse mechanisms and (ii) admit torque-induced non-symmetric stress states. The need for DTMs is justified by a novel analysis of the transformation methods extracting some of its key features that have never been reported before.

Task 2 is to design **on a mesoscopic scale** lattice-based materials with targeted collapse mechanisms and to explore various ways to induce within strong body torque distributions by passive (e.g., resonance-based) and active (e.g., magnetism-based) means.

Task 3 is to model the effective elastic behavior of the lattice-based materials designed in Task 2 on the macroscopic scale leveraging **a consistent and accurate homogenization method**. The purpose is to build a reliable and systematic connection bridging the mesoscopic and macroscopic descriptions of DTMs.

Task 4 is to apply acquired knowledge of lattice-based DTMs to the **design and numerical testing of an elastic cloak**. Other elastic wave phenomena taking place in DTMs will be investigated as well. Focus will be on unique wave steering capabilities such as immunity to back scattering and robustness against defects.

Task 5 is to **fabricate and test experimentally** DTMs and DTM-based elastic cloaks and to assess their performance across various geometries and frequency bands.

4. A degenerate polar lattice for cloaking in full two-dimensional elastodynamics and statics

4.1 Introduction

An elastic cloak is a coating that can be applied to an arbitrary inclusion to make it indistinguishable from the background medium. The coated inclusion becomes undetectable by elastographic techniques, be they static or dynamic. To determine the material the coating is made of, one applies a curvilinear change of coordinates given by a singular geometric transformation that (i) blows-up a point of the background into the inclusion; and (ii) does not disturb the background beyond a horizon outlining the boundary of the cloak; see, e.g. Fig. 4.1 and other examples by Hu et al. (2009). The constitutive parameters of the cloaking material can then be read from the equations of elasticity written in the new coordinates. This design method, known as the transformation method, was first discovered for conductivity by Greenleaf *et al.* (2003a, 2003b), then for geometric optics by Leonhardt (2006) and for electromagnetism by Pendry *et al.* (2006) and later applied to acoustics by others (Cummer and Schurig, 2007; Chen and Chan, 2007; Norris, 2015).

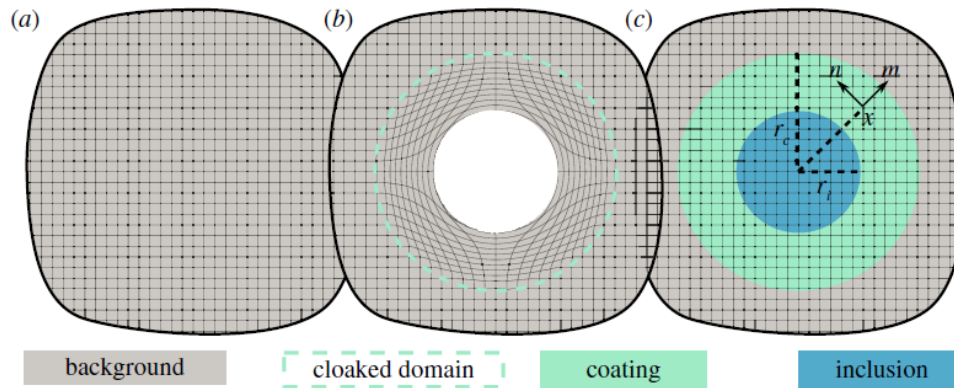


Figure 4.1 The transformation method: (a) background medium described by Cartesian coordinates $\mathbf{X}$; (b) background medium described by curvilinear coordinates $\mathbf{x} = \boldsymbol{\phi}(\mathbf{X})$; (c) Reinterpreting $\mathbf{x}$ as Cartesian coordinates makes appear a gradient of constitutive properties (in green). Here, the illustrated transformation blows-up a point from (a) into a disc (in white) on (b) without disturbing the region falling beyond some horizon (in dashed lines) and is suitable for cloaking purposes. The disc, having a scattering cross section equal to that of a point, can be filled with any inclusion without disturbing the fields beyond the horizon.

Among these, the work of Norris and Parnell (2012) stands out. It is the only one to adhere to the transformation method in its full form rigorously. In comparison, the other contributions, one way or another, discarded altogether the tensorial character of elasticity in favour of a scalar theory. While this attitude is natural in some cases, as in bending of plates or in acoustics, it is hardly

justifiable in other cases. Unfortunately, Norris and Parnell (2012) revealed that their approach requires the blow-up ratio, the one transforming Fig. 4.1a into 4.1b, to be smaller than two. This means that their prestressed hyperelastic materials can only substitute the cloak near its outer boundary. At the inner boundary, their approach fails as the blow-up ratio nears infinity. In practical terms, their design reduces scattering of in-plane waves without achieving perfect cloaking.

The main purpose of the paper is to suggest a lattice cloak for full two-dimensional elastodynamics and statics. By a scrutiny of the transformation method, we demonstrate that such a lattice is necessarily polar in that it elastically resists rotations; and degenerate in that it admits a stressless collapse mechanism. Both requirements are then satisfied through the use of appropriately distributed torques and hinge-like spring-mass contacts. Thus, the designed lattice is proven to exhibit a rank-3 elasticity tensor lacking the minor symmetries in a way that rigorously fulfills the form-invariance requirements of the BGM gauge over the complete domain of the cloak. A stability analysis shows that the design is valid when the background is isotropic with a Lamé pair (λ, μ) satisfying $\mu \leq \lambda$. Last, the cloak's performance is assessed numerically and found to be satisfactory under exhaustive static/dynamic and pressure/shear loadings.

As a theoretical generalization, we construct lattice materials which render the set of elastic materials invariant by arbitrary transformations under the BGM gauge so as to enable cloaking as well as other static and dynamic field manipulation techniques in three dimensions and in the presence of a possibly anisotropic background medium. Essential to this endeavour is an adaptation of Milton and Cherkaev's (1995) pentamode materials. Thus, by mixing appropriately modified pentamode materials submitted to distributed torques, we show that all non-negative elasticity tensors with major symmetry and with or without the minor symmetries can be constructed. We stress that Guevara Vasquez et al. (2012) similarly observed that transformation elasticity under the BGM gauge is possible in lattices featuring 'torque springs' with non-central forces; our insight is equivalent to theirs.

4.2 Cloaking in full two-dimensional elasticity

Hereafter, we briefly recall how the elasticity equations transform in the BGM gauge and analyse the nature of the elasticity tensors that emerge as an outcome. A degenerate polar lattice whose behavior can match that of a perfect two-dimensional cloak is introduced and its effective elasticity tensor calculated. Once the relevant geometric and constitutive parameters have been identified, the cloak's performance is assessed based on an exhaustive set of numerical tests in static and dynamic regimes.

4.2.1 Transformation elasticity

We begin by analyzing the constitutive behavior of the transformed/coating material in the BGM gauge. Let

$$L(\nabla \mathbf{U}, \dot{\mathbf{U}}) = \frac{\nabla \mathbf{U} \cdot \mathbf{C} \nabla \mathbf{U} - \dot{\mathbf{U}} \cdot \mathbf{R} \dot{\mathbf{U}}}{2}, \quad (4.1)$$

be the Lagrangian density of an elastic domain $\{X\}$ of elasticity tensor $\mathbf{C}$ and mass density R written as a function of displacement gradient $\nabla \mathbf{U}$ and velocity $\dot{\mathbf{U}}$. Then, in the transformed domain $\{x = \phi(X)\}$, the Lagrangian density reads

$$\ell(\nabla \mathbf{u}, \dot{\mathbf{u}}) = \frac{1}{J} L(\nabla \mathbf{u} \mathbf{F}, \dot{\mathbf{u}}) = \frac{\nabla \mathbf{u} \cdot \mathbf{c} \nabla \mathbf{u} - \dot{\mathbf{u}} \cdot \rho \dot{\mathbf{u}}}{2}, \quad (4.2)$$

where $\mathbf{F} = \partial \mathbf{x} / \partial \mathbf{X}$ is the transformation gradient, J its determinant and the BGM gauge enforcing the equality between original and transformed displacements, namely $\mathbf{u}(\mathbf{x}) = \mathbf{U}(\mathbf{X})$, has been adopted. The transformed elasticity tensor $\mathbf{c}$ and mass density ρ can be extracted and read

$$c_{ijkl} = \frac{F_{jm} F_{ln} C_{imkn}}{J}, \quad \rho = \frac{R}{J}. \quad (4.3)$$

Therein, it can be readily checked that when $\mathbf{C}$ has the minor symmetries, $\mathbf{c}$ does not. That is, $C_{ijkl} = C_{jikl} = C_{ijlk}$ does not imply a similar relation for $\mathbf{c}$. Consequently, the stress $\boldsymbol{\sigma}$ that reigns in $\{x\}$ in the presence of a displacement gradient $\mathbf{e} = \nabla \mathbf{u}$ is given by

$$\boldsymbol{\sigma} = \mathbf{c} \mathbf{e} \quad \text{or} \quad \sigma_{ij} = c_{ijkl} e_{kl} \quad (4.4)$$

and is asymmetric in general: $\sigma_{ij} \neq \sigma_{ji}$.

For our cloaking purposes, we take $\{x\}$ to be a two-dimensional elastic medium composed of three regions (Fig. 4.1c). The central region $\{\|\mathbf{x}\| < r_i\}$ is an inclusion with arbitrary elasticity tensor $\mathbf{c}(\mathbf{x})$. The outer region $\{\|\mathbf{x}\| > r_c\}$ is the background medium, assumed isotropic with a Lamé pair (λ, μ) and mass density R . The remaining ring $\{r_i < \|\mathbf{x}\| < r_c\}$ is an axisymmetric coating of thickness $r_c - r_i$ fulfilling the function of a cloak. Its constitutive relation is determined thanks to Eq. (4.3) once ϕ has been specified to the blow-up transformation of Fig. 4.1. Following Brun et al. (2009), that relation reads

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \\ \sigma_{21} \end{bmatrix} = \begin{bmatrix} (2\mu + \lambda)f & \lambda_0 & 0 & 0 \\ \lambda & \frac{(2\mu + \lambda)}{f} & 0 & 0 \\ 0 & 0 & \frac{\mu}{f} & \mu \\ 0 & 0 & \mu & \mu f \end{bmatrix} \begin{bmatrix} e_{11} \\ e_{22} \\ e_{12} \\ e_{21} \end{bmatrix}, \quad f = \frac{\|\mathbf{x}\| - r_i}{\|\mathbf{x}\|}, \quad (4.5)$$

where components have been calculated in the normalized polar basis (m, n) with $m = \mathbf{x} / \|\mathbf{x}\|$. Finally, the mass density of the cloak is

$$\rho = \frac{r_c^2}{(r_c - r_i)^2} fR. \quad (4.6)$$

The major obstacle facing elastic cloaking is to design elastic materials whose behavior is described by the two foregoing relations (4.5) and (4.6).

4.2.2 Design heuristics: why degenerate polar lattices are necessary

There are three insights that can guide our efforts in designing cloaking materials. These are presented and discussed next.

The first insight is that a material where the minor symmetries are broken is necessarily a polar material, i.e. a material that elastically resists rotations. At the same time, a polar material is a material submitted to a distribution of restoring body torques. To see that, let c be an elasticity tensor lacking the minor symmetries, e.g. with $c_{1212} \neq c_{1221}$. Then let $\mathbf{u}$ describe an infinitesimal plane rotation of angle θ . Accordingly, the displacement gradient e is skewsymmetric with $e_{11} = e_{22} = 0$ and $e_{21} = -e_{12} = \theta$. In that case, the shear stresses $\sigma_{12} = (c_{1221} - c_{1212})\theta$ and $\sigma_{21} = (c_{2121} - c_{2112})\theta$ are non-zero meaning that the material is indeed polar. Furthermore, the difference between the shear stresses $\sigma_{12} - \sigma_{21} = (2c_{1221} - c_{1212} - c_{2121})\theta$ is also non-zero and directly quantifies the restoring body torques that emerge as a response to a rotation by an angle θ .

The second insight is rank conservation: seen as linear maps, C and c have the same rank $d(d+1)/2$ in dimension d . Therefore, they have an equal number of zero modes, namely $d(d-1)/2$. That is, for each tensor E such that $CE = 0$, there exists a tensor $e = EF^{-1}$ such that $ce = 0$. If the original material is a classical one, then the only zero modes it admits are rotations meaning that all the E_s are skewsymmetric. But then, owing to the multiplication by F^{-1} , e is no longer skewsymmetric. That is, the transformed material admits zero modes that are not rotations; it is degenerate. Therefore, the cloak's material must exhibit a high volume fractions of void, as in foams or lattice materials, so as to stretch or shear under zero stress. In the present case with $d = 2$, c is of rank 3 and admits a unique zero mode. This can be checked through relation (4.5). As a matter of fact, $e_{zm} = f \mathbf{m} \otimes \mathbf{n} - \mathbf{n} \otimes \mathbf{m}$ satisfies $ce_{zm} = 0$.

The third and last insight is particular to isotropic backgrounds. It consists in observing that the principal directions of F yield planes of mirror symmetry.

In conclusion, with an isotropic background in two dimensions, it is necessary that our lattice design of an elastic cloak (i) feature distributed restoring body torques, (ii) collapse effortlessly under e_{zm} and (iii) be rectangular so as to exhibit two orthogonal planes of symmetry.

4.2.3 A degenerate polar lattice cloak

Following the guidelines described above, we fill the coating ring $\{r_i < \|\mathbf{x}\| < r_c\}$ with an inhomogeneous lattice as shown in Fig. 4.2a. A magnified local view of the lattice is depicted in Fig. 4.2b. The lattice is rectangular with a unit cell containing two diagonal springs of constant α

and one vertical spring of constant β . Furthermore, a unit cell contains a single mass m submitted to a restoring torque per radian per unit cell area κ as illustrated in Fig. 4.2c. Finally, all contacts are assumed to be hinge-like.

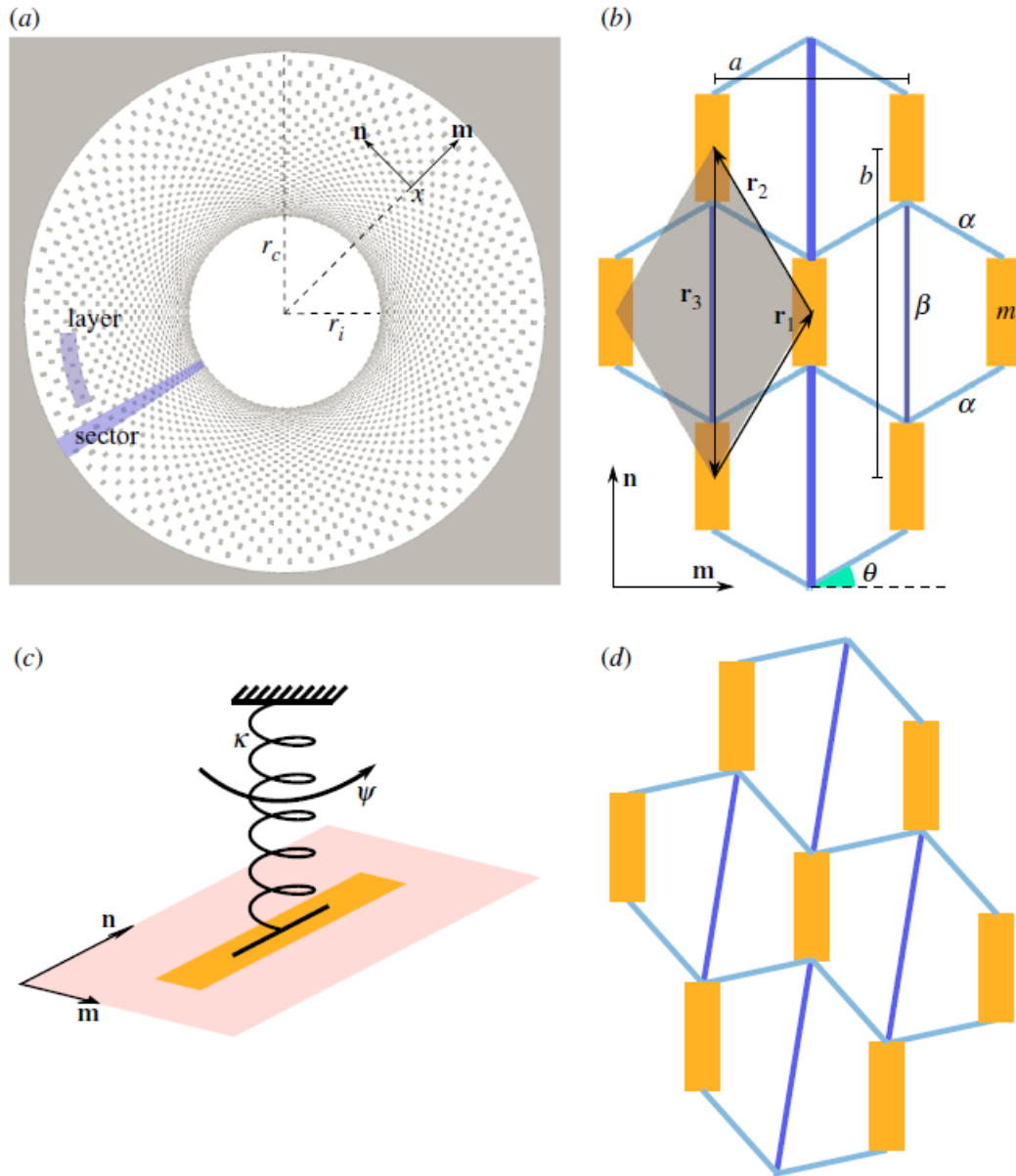


Figure 4.2 A two-dimensional lattice elastic cloak. (a) Cloak geometry: the inclusion is a void coated by a lattice and embedded in an isotropic background; the bonds of the lattice are not depicted. (b) Local lattice geometry at position : rectangles are rigid masses submitted to a restoring torque; edges are springs; all contacts are hinge-like. (c) A mechanism by which restoring torques can be applied. (d) The configuration of the lattice under the action of the zero mode $e_{z\ m}$.

Now we focus on the local elastic response of the lattice in the vicinity of position x . Call ϵ the elastic energy per unit cell area under a uniform displacement gradient e of the lattice and a uniform twisting of the masses by an angle ψ . Energy ϵ is composed of three terms

$$\epsilon = \epsilon_\alpha + \epsilon_\beta + \epsilon_\kappa \quad (4.7)$$

corresponding to the energy stored in the springs of constant α , the spring of constant β and in the grounded torsion spring of constant κ , respectively. These admit the expressions

$$\left. \begin{aligned} \epsilon_\alpha &= \frac{\alpha}{2v} \left[(\mathbf{er}_1 + 2d\psi \mathbf{m}) \cdot \mathbf{m}_\theta \right]^2 + \frac{\alpha}{2v} \left[(\mathbf{er}_2 + 2d\psi \mathbf{m}) \cdot \mathbf{m}_\theta \right]^2 \\ \epsilon_\beta &= \frac{\beta}{2v} (\mathbf{er}_3 \cdot \mathbf{n})^2, \quad \epsilon_\kappa = \frac{\kappa}{2} \psi^2 \end{aligned} \right\} \quad (4.8)$$

where $v = ab/2$ is the unit cell area, $d = (bc - as)/(4c)$ is half the length of a mass, $\mathbf{r}_1 = a\mathbf{m}/2 + b\mathbf{n}/2$, $\mathbf{r}_2 = -a\mathbf{m}/2 + b\mathbf{n}/2$ and $\mathbf{r}_3 = -b\mathbf{n}$ are lattice vectors, and where $\mathbf{m}_\theta = c\mathbf{m} + s\mathbf{n}$ and $\mathbf{m}'_\theta = -c\mathbf{m} + s\mathbf{n}$ are unit vectors with $c \equiv \cos \theta$ and $s \equiv \sin \theta$.

At equilibrium, ψ is such that ϵ is at its minimum characterized by

$$\frac{\partial \epsilon}{\partial \psi} = 0, \quad (4.9)$$

or equivalently,

$$2cd\alpha (\mathbf{er}_1 \cdot \mathbf{m}_\theta + 2cd\psi) - 2cd\alpha (\mathbf{er}_2 \cdot \mathbf{m}'_\theta - 2cd\psi) + v\kappa\psi = 0 \quad (4.10)$$

This leads to a new expression of ϵ as a function of e alone and where ψ can be expressed as

$$\psi = -2cd\alpha \frac{ase_{21} + bce_{12}}{8c^2d^2\alpha + v\kappa}. \quad (4.11)$$

The lattice's Hooke's Law $\sigma = \partial \epsilon / \partial e$ follows immediately and reads

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \\ \sigma_{21} \end{bmatrix} = \begin{bmatrix} \frac{ac^2\alpha}{b} & cs\alpha & 0 & 0 \\ cs\alpha & \frac{b(\alpha s^2 + 2\beta)}{a} & 0 & 0 \\ 0 & 0 & \frac{b^2c^2\alpha\kappa}{\alpha(bc-as)^2 + ab\kappa} & \frac{abcs\alpha\kappa}{\alpha(bc-as)^2 + ab\kappa} \\ 0 & 0 & \frac{abcs\alpha\kappa}{\alpha(bc-as)^2 + ab\kappa} & \frac{a^2s^2\alpha\kappa}{\alpha(bc-as)^2 + ab\kappa} \end{bmatrix} \begin{bmatrix} e_{11} \\ e_{22} \\ e_{12} \\ e_{21} \end{bmatrix}. \quad (4.12)$$

Last, it is worth mentioning that the suggested lattice is polar as it features an asymmetric stress and is degenerate as it admits a zero mode of the form $e_{zm} = (as/bc)\mathbf{m} \otimes \mathbf{n} - \mathbf{n} \otimes \mathbf{m}$. The action of the zero mode on the lattice is exemplified in Fig. 4.2d.

4.2.4 Parameters identification

A straightforward comparison between the lattice's Hooke's Law (4.12) and that of the cloak (4.5) reveals that they match if the parameters $(\alpha, \beta, \kappa, \theta, a/b)$ are chosen to satisfy

$$f = \frac{as}{bc}, \quad \lambda = cs\alpha, \quad \mu = \frac{abcs\alpha\kappa}{\alpha(bc-as)^2 + ab\kappa}, \quad (4.13)$$

but also

$$2\frac{\mu}{\lambda} + 1 = \frac{c^2}{s^2} = \frac{s^2 + 2\beta/\alpha}{c^2} = \frac{2ab\kappa}{\alpha(bc-as)^2 + ab\kappa} + 1. \quad (4.14)$$

Solving these relations yields the local design parameters of a lattice fulfilling the function of a cloak, namely

$$\left. \begin{aligned} \kappa &= \frac{\lambda\mu}{\lambda-\mu} \frac{(1-f)^2}{f} \\ \theta &= \arctan \sqrt{\frac{\lambda}{2\mu+\lambda}} \in (0, \pi/2), \\ \alpha &= 2(\mu+\lambda) \sqrt{\frac{\lambda}{2\mu+\lambda}} \\ \beta &= \frac{(2\mu+\lambda)^2 - \lambda^2}{2\sqrt{\lambda}\sqrt{2\mu+\lambda}} > 0 \\ \frac{a}{b} &= f \sqrt{\frac{2\mu+\lambda}{\lambda}} \end{aligned} \right\} \quad (4.15)$$

As for mass, it is given by

$$m = \frac{ab}{2} \rho. \quad (4.16)$$

Note that the suggested design maintains the parameters θ, α and β uniform throughout the cloak whereas parameters $\kappa, a/b$ and m depend on position x through f . Note also that enforcing stability by rejecting torsion springs with a negative constant κ implies $\mu \leq \lambda$ and places a constraint on the isotropic backgrounds for which a lattice cloak can be designed in this manner. A generalization of our design, that would work for arbitrary anisotropic backgrounds in three dimensions, is presented in the next section.

For the example treated next, we set $\lambda = \mu$ for simplicity. Thus, the design parameters simplify into

$$\kappa \rightarrow \infty, \quad \theta = \frac{\pi}{6}, \quad \alpha = \beta = \frac{4\mu}{\sqrt{3}}, \quad \frac{a}{b} = f\sqrt{3}. \quad (4.17)$$

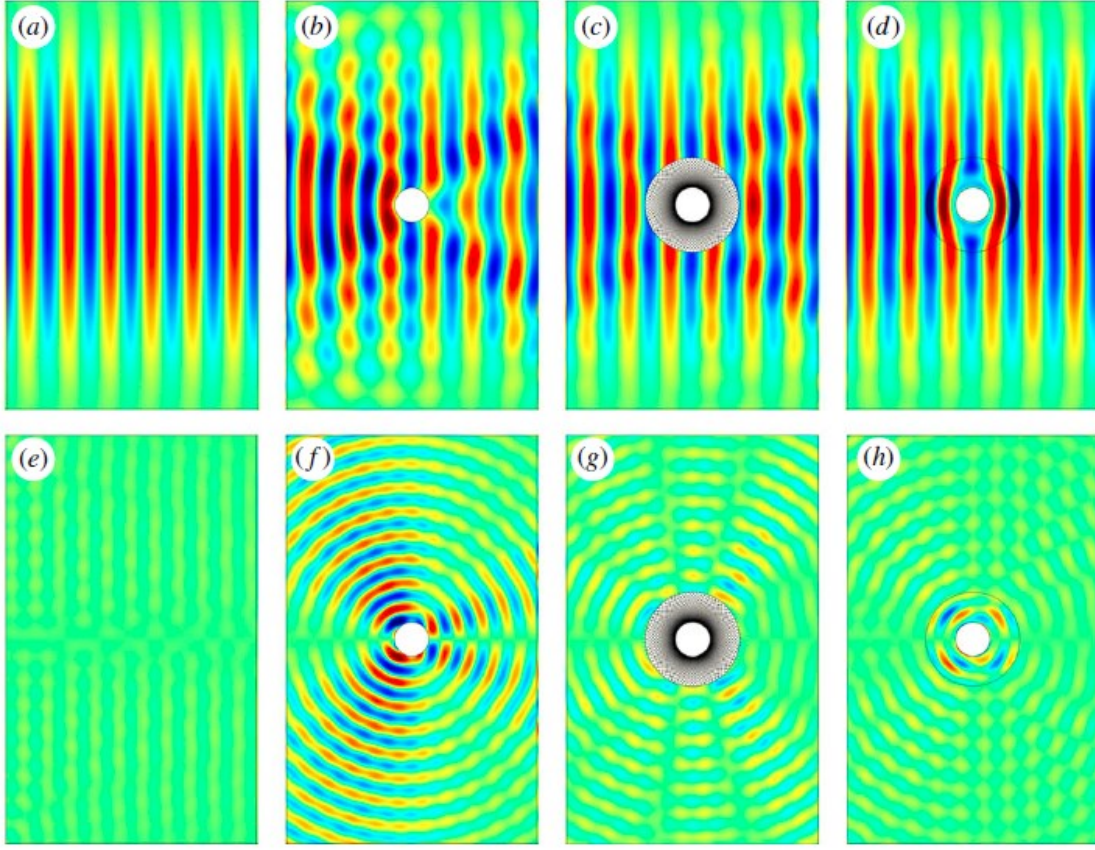


Figure 4.3 Normalized divergence (a – d) and curl (e – h) of the displacement field as colour maps for an incident 40kHz plane pressure wave. Ideal cloaking should show zero curl out of the cloak's domain: residual curl can be interpreted as an error measure. Plots (a, e) correspond to the intact background medium; (b, f) to a non-coated void; (c, g) to a lattice-coated void; and (d, h) to a continuously coated void.

In particular, κ is infinite and mass rotation is completely blocked. Then, letting N be the number of angular sectors in the coating, it comes that

$$b = \frac{2\pi \|\mathbf{x}\|}{N}, \quad a = 2\pi\sqrt{3} \frac{\|\mathbf{x}\| - r_i}{N}, \quad m = \frac{2\pi^2\sqrt{3}}{N^2} r_c^2 \left(\frac{\|\mathbf{x}\| - r_i}{r_c - r_i} \right)^2 R. \quad (4.18)$$

As for the number of layers M , it is infinite in theory so as to permit $\|\mathbf{x}\|$ to reach the inner radius r_i . In practice, M is chosen large and $\|\mathbf{x}\|$ reaches a minimum $r'_i > r_i$. Cloaking is, therefore, approximate in practice but becomes ideal in the theoretical limit $M, N \rightarrow \infty$ as $r'_i \rightarrow r_i$.

4.2.5 Cloaking simulations

In the following numerical simulations, the plane stress hypothesis was adopted. We cloak a void of radius $r'_i = 55$ mm with a coating of radius $r_c = 150$ mm embedded in a rectangular plate of 1 mm thickness and of inplane Lamé parameters $\lambda = \mu = 2.5 \times 10^6$ Pa $\times$ m and mass density $R = 2.7$ kg m⁻² (Fig. 4.2a). The lattice cloak consists of $M = 22$ layers and $N = 75$ sectors and

has a theoretical inner radius of $r_i = 50$ mm. Navier's equations in the background medium coupled to Newton's equations in the coating were solved using the finite-element method and with the help of the commercial software COMSOL Multiphysics. Each mass in the cloak's domain is modelled as a solid medium of sufficiently large modulus. Furthermore, its nodes are constrained to have equal displacements so as to block rotations. Last, springs are modelled as pairs of point forces depending on nodal displacements.

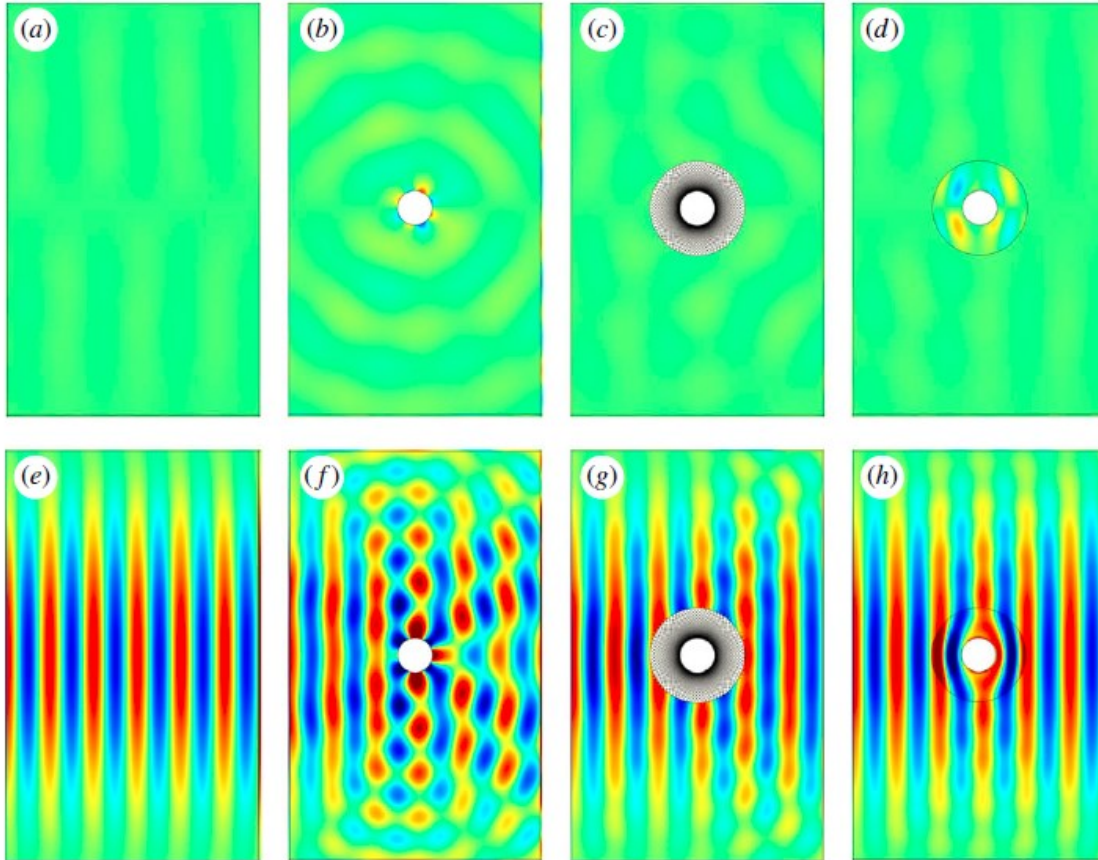


Figure 4.4 Normalized divergence ($a - d$) and curl ($e - h$) of the displacement field as colour maps for an incident 22kHz plane shear wave. Ideal cloaking should show zero divergence out of the cloak's domain: residual divergence can be interpreted as an error measure. Plots (a, e) correspond to the intact background medium; (b, f) to a non-coated void; (c, g) to a lattice-coated void; and (d, h) to a continuously coated void.

Given linearity and isotropy, it is sufficient to characterize the cloak's behavior in three configurations: under an incident plane harmonic pressure wave, under an incident plane harmonic shear wave and under uniaxial static compression. In the dynamic regime, incident waves with Gaussian profiles are emitted from the left boundary of the simulation domain and perfectly matched layers are appended to all sides so as to eliminate reflections. Note that reflections would not jeopardize the cloak's performance but constitute a nuisance for the interpretation of the results. In the static regime, uniform normal displacements are applied to the left and right boundaries while leaving the top and bottom boundaries free of normal stresses.

Results are shown in Figs. 4.3-4.5 for an incident pressure wave, incident shear wave and for unidirectional compression, respectively. For reference, we carried out the same simulations where the background medium is intact (plots *a* and *e*); where the void is non-coated (*b* and *f*); where the void is lattice-coated (*c* and *g*); and in the homogenization limit where the discrete cloak is substituted by a fictitious continuous material of the same behaviour (*d* and *h*). Therein, the real part of the divergence and curl of the displacement field is shown as a color map over continuous domains but is not available over discrete regions a priori and these are left uncolored. Most importantly, beyond the cloak's domain, whenever the divergence or the curl component is dominant over the other, the secondary component can be interpreted as an error or a scattering measure.

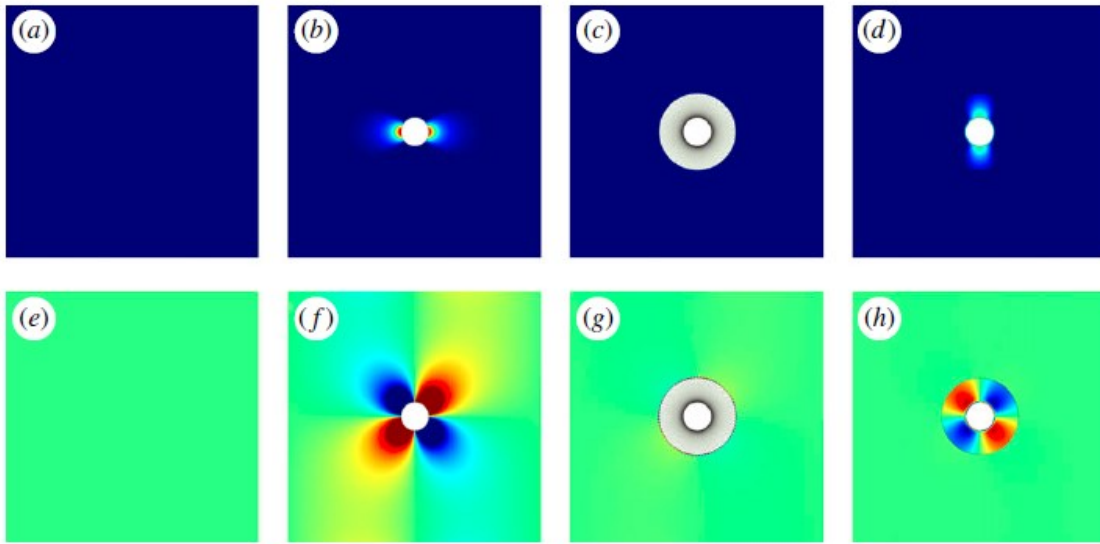


Figure 4.5 Normalized divergence (*a* – *d*) and curl (*e* – *h*) of the displacement field as colour maps under horizontal compression. Ideal cloaking should show zero curl out of the cloak's domain: residual curl can be interpreted as an error measure. Plots (*a*, *e*) correspond to the intact background medium; (*b*, *f*) to a non-coated void; (*c*, *g*) to a lattice-coated void; and (*d*, *h*) to a continuously coated void.

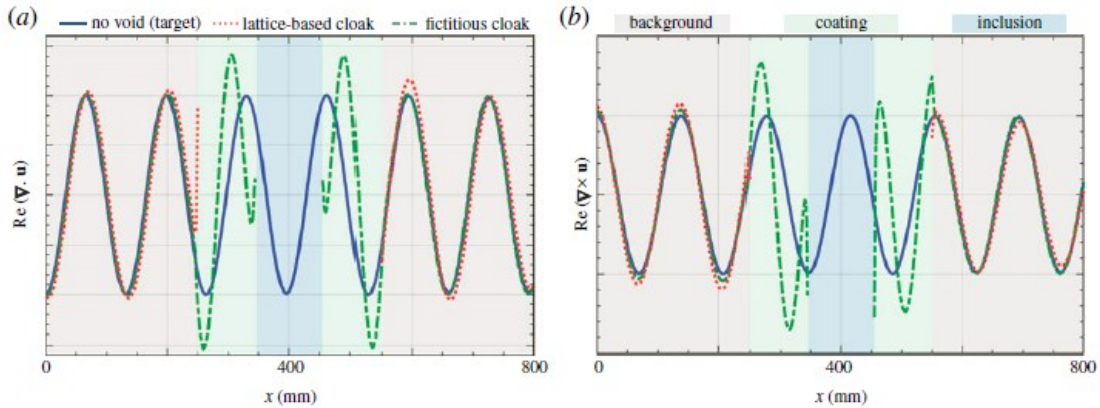


Figure 4.6 Normalized divergence (*a*) and curl (*b*) of the displacement field as a function of position x along the midspan of the simulation domain, respectively, for the configurations of figure 3*a, c, d* and for the configurations of figure 4*a, c, d*.

Comparing plots (*b, f*) to (*c, g*), respectively, it is seen that the lattice cloak manages to suppress the scattering, or strain localization, by or around the void to a high and satisfactory degree although without being perfect. The cloaking error, e.g. the residual scattering, has two sources: first, by comparing plots (*c, g*) to (*d, h*), it is seen that the lattice is not operating yet in the homogenization limit; second, by comparing plots (*a, e*) to (*d, h*), it is seen that even the homogenization limit does not show perfect cloaking due to the inner radius r'_i being slightly larger than the theoretical value r_i . The remedy to both sources of error is an increase in the number of layers and sectors N and M . This analysis is further confirmed by the fact that the performance of the cloak is visibly better in statics than in dynamics, that is when wavelengths are larger compared to the unit cell parameters.

To consolidate our results quantitatively, the fields in Figs. 4.3*a, c, d* are plotted against position x along the midspan of the simulation domain. The normalized profiles are depicted in Fig. 4.6*a*. It is seen that over the background medium, all three profiles match well overall. Remarkably, the most significant surge of error occurs at the boundary of the cloak at $x = 250$ mm but then dies exponentially fast as x moves into the bulk of the background. Consequently, at that interface, where the discrete domain of the lattice cloak meets the continuous domain of the background, evanescent fields emerge. These are similar to the evanescent fields that appear in a half-space as a result to a point force being applied to its free boundary. As M and N grow, the number of point forces applied to the background increases and the spacing between their points of application decreases. In the limit $M, N \rightarrow \infty$, the point forces converge toward a smoothly distributed load and the parasitic evanescent fields disappear as a result; only then our lattice cloak becomes perfect. Fig. 4.6*b* shows the normalized profile of the fields of Figs. 4.4*e, c, d* as a function of x . Similar observations hold in that case.

4.3 A generalization to anisotropic backgrounds in three dimensions

The lattice cloak of the previous section is specific to two-dimensional isotropic backgrounds. Furthermore, if only positive torsion spring constants κ are permitted, then the background must satisfy $\mu \leq \lambda$. Here, we prove that lattice designs of cloaks exist in three dimensions for arbitrary anisotropic backgrounds. More generally, we construct lattice materials whose behavior is forminvariant under the BGM gauge. The proof closely follows Milton and Cherkaev's (1995) construction of arbitrary elastic solids as mixtures of pentamode materials. First, the approach is motivated thanks to a spectral decomposition of the elasticity tensor. Second, we modify Milton and Cherkaev's pentamodes so as to enable access to elasticity tensors without the minor symmetries. These modified pentamodes are lattice materials that can support a unique stress state, be it symmetric or not, and are renamed 'rank-1 materials'. Finally, we conclude that mixtures of rank-1 materials can exhibit any elasticity tensor with or without the minor symmetries.

4.3.1 Spectral decomposition and definitions

Once transformed, the elasticity tensor $\mathbf{c}$ loses the minor symmetries. Fortunately, $\mathbf{c}$ still has major symmetry and remains non-negative. Hence, it can be decomposed into a sum of projectors

$$\mathbf{c} = \sum_{s=1}^{d^2} \mathbf{c}_s = \sum_{s=1}^{d^2} \mu_s \boldsymbol{\tau}_s \otimes \boldsymbol{\tau}_s, \quad (4.19)$$

where the μ_s are non-negative elastic moduli and the $\boldsymbol{\tau}_s$ are orthonormal eigentensors such that $\mathbf{c}\boldsymbol{\tau}_s = \mu_s \boldsymbol{\tau}_s$ and $\boldsymbol{\tau}_s \cdot \boldsymbol{\tau}_{s'} = \delta_{ss'}$. Note that since $\mathbf{c}$ does not obey the minor symmetries, there exists at least one asymmetric $\boldsymbol{\tau}_s$. Conversely, if all $\boldsymbol{\tau}_s$ are symmetric, then $\mathbf{c}$ has the minor symmetries. A material whose elasticity tensor is equal to one of the $\mathbf{c}_s$ is called a rank-1 material. Such a material can support a unique stress state proportional to $\boldsymbol{\tau}_s$ and collapses effortlessly under any gradient $\mathbf{e}$ that is orthogonal to $\boldsymbol{\tau}_s$. As a matter of fact, $\boldsymbol{\tau}_s \cdot \mathbf{e} = 0$ implies $\mathbf{c}_s \mathbf{e} = 0$. More generally, a rank- n material is one capable of supporting exactly n linearly independent stress states while collapsing under all gradients orthogonal to these states. Note that a material is of rank n if and only if its elasticity tensor $\mathbf{c}$ is of rank n when seen as a linear map over the space of second-order tensors.

With this definition, recall that a classical elastic solid in dimension d is of rank $d(d+1)/2$ since it supports d traction/compression states and $d(d-1)/2$ shear states while 'collapsing' under $d(d-1)/2$ rotations. As changes of coordinates preserve ranks, the possibility for a full theory of transformation elasticity under the BGM gauge hinges on designing arbitrary nonnegative elasticity tensors of rank $d(d+1)/2$ with major symmetry and with or without the minor symmetries.

Hereafter, we closely follow Milton and Cherkaev (1995) and design a material with elasticity tensor $\mathbf{c}$ in two steps. First, we build rank-1 materials whose elasticity tensors are equal to the individual $\mathbf{c}_s$. Second, we mix these building blocks into a single material whose elasticity tensor is equal to $\mathbf{c}$.

4.3.2 Design of rank-1 materials

Consider the lattice whose unit cell is depicted in Fig. 4.7. A unit cell is composed of a single rigid mass placed at the origin of the coordinate system and connected to $d+1$ vertices placed at the tips of vectors $\mathbf{r}_j$. The connections are spring-like with constants k_j and are directed along the unitary vectors $\mathbf{s}_j$; note that the $\mathbf{s}_j$ and $\mathbf{r}_j$ are not necessarily aligned. A mass originally has $d(d+1)/2$ degrees of freedom corresponding to rigid body motions. However, we assume each mass to be submitted to $d(d-1)/2$ restoring torques sufficiently strong to block rotations while leaving free the d degrees of freedom of translational motion. Accordingly, the elastic energy density within one unit cell of volume v submitted to a displacement gradient $\mathbf{e}$ is

$$\epsilon = \frac{1}{2v} \sum_{j=0}^d k_j \left[\mathbf{s}_j \cdot (\mathbf{e} \mathbf{r}_j - \boldsymbol{\delta}) \right]^2, \quad (4.20)$$

where $\mathbf{er}_j$ is the imposed displacement of vertex j and δ is the displacement of the mass. Hence, $\mathbf{s}_j \cdot (\mathbf{er}_j - \delta)$ is the elongation of spring j and the expression of ϵ follows. At equilibrium, δ is such that ϵ is equal to its minimum characterized by

$$\frac{\partial \epsilon}{\partial \delta} = 0, \quad (4.21)$$

or equivalently

$$\sum_{j=0}^d k_j \left[\mathbf{s}_j \cdot (\mathbf{er}_j - \delta) \right] \mathbf{s}_j = 0. \quad (4.22)$$

The $\mathbf{s}_j$ being $d + 1$ vectors in d dimensions, there exist scalars $x_j, j = 0 \dots d$, such that

$$\sum_{j=0}^d x_j \mathbf{s}_j = 0. \quad (4.23)$$

Comparing (4.22) and (4.23), it comes that

$$k_j \mathbf{s}_j \cdot (\mathbf{er}_j - \delta) = \gamma x_j, \quad (4.24)$$

where γ is a j -independent similarity ratio. Multiplying by x_i/k_j and summing over j then yields

$$\sum_{j=0}^d x_j \mathbf{s}_j \cdot \mathbf{er}_j = \gamma \sum_{k=0}^d \frac{x_j^2}{k_j} \quad (4.25)$$

or

$$\gamma = \frac{\boldsymbol{\tau}^{\text{lat}} \cdot \mathbf{e}}{\sum_{k=0}^d x_j^2 / k_j}; \quad \boldsymbol{\tau}^{\text{lat}} = \sum_{j=0}^d x_j \mathbf{s}_j \otimes \mathbf{r}_j. \quad (4.26)$$

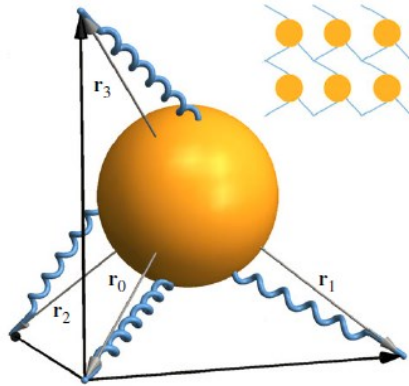


Figure 4.7 The unit cell of a rank-1 material: black arrows are lattice vectors; vectors $\mathbf{r}_j$ are in grey; vectors $\mathbf{s}_j$ (not shown) are directed along the coils; all contacts (coil-coil and coil-sphere) behave like hinges; coils behave like linear springs; spheres have a fixed absolute orientation but are free to translate. When the $\mathbf{r}_j$ are parallel to the coils, the

lattice becomes equivalent to a Milton and Cherkaev's pentamode. The inset shows a two-dimensional representation of the lattice.

Now, combining (4.20), (4.24) and (4.26), we deduce that the elastic energy in fact depends on a unique scalar, namely $\tau^{\text{lat}} \cdot \mathbf{e}$, through

$$\epsilon = \frac{(\tau^{\text{lat}} \cdot \mathbf{e})^2}{2\nu \sum_j x_j^2 / k_j}. \quad (4.27)$$

The lattice is, therefore, clearly of rank 1: for any strain $\mathbf{e}$ orthogonal to tensor τ^{lat} , the scalar product $\tau^{\text{lat}} \cdot \mathbf{e}$ vanishes and so does ϵ . Conversely, the unique stress state that the lattice is capable of supporting is given by tensor τ^{lat} . This can also be seen by deriving the expression of the effective elasticity tensor. We find

$$\mathbf{c}^{\text{lat}} = \frac{\partial^2 \epsilon}{\partial \mathbf{e} \partial \mathbf{e}} = \frac{1}{\nu \sum_j x_j^2 / k_j} \tau^{\text{lat}} \otimes \tau^{\text{lat}} \quad (4.28)$$

is a dyadic product and is of rank 1.

In conclusion of this subsection, as long as the vectors $\mathbf{r}_j$ and $\mathbf{s}_j$ are misaligned, the stress τ^{lat} will be asymmetric and cat will lack the minor symmetries. When the $\mathbf{r}_j$ and the $\mathbf{s}_j$ are aligned, the rank-1 material becomes identical to a Milton and Cherkaev's pentamode. In any case, by appropriately choosing vectors $\mathbf{r}_j$ and $\mathbf{s}_j$, it is possible to target any single eigentensor τ_s and subsequently any single elasticity tensor $\mathbf{c}_s$. We refer to Milton and Cherkaev's original paper for more details regarding this assertion.

4.3.3 Design of rank-n materials

Now take two weakly interacting interpenetrated lattices with elasticity tensors $\mathbf{c}_s$ and $\mathbf{c}_{s'}$. Applying a displacement gradient $\mathbf{e}$ will generate an energy $\mu_{s,s'} (\tau_{s,s'} \cdot \mathbf{e})^2 / 2$ in the first and second lattice, respectively. Accordingly, the elasticity tensor of the mixture is simply $\mathbf{c}_s + \mathbf{c}_{s'}$. Iterating the process and mixing up to d^2 lattices, all non-negative elasticity tensors $\mathbf{c}$ satisfying major symmetry can be generated. This concludes our proof that the set of elastic materials submitted to restoring torques is invariant by transformations under the BGM gauge.

4.4 Conclusion

In conclusion, full transformation elasticity and cloaking, in particular, have been proved possible in principle thanks to a class of degenerate polar lattices that elastically resist rotations while admitting a set of collapse mechanisms. In two dimensions, the proposed lattice exhibits a rank-3 elasticity tensor lacking the minor symmetries as required by the transformation method under the BGM gauge. The lattice cloak was numerically tested under various static and dynamic loadings and showed satisfactory performance.

Experimental demonstrations are much desired but face two challenges. The first is to devise a practical way of imposing the required torques. The second is to reduce the number of sectors and layers in the cloak as much as possible. This would contradict the homogenization limit however and an optimal trade-off is to be found. In doing so, it could be beneficial to investigate better ways to connect the background continuum to the discrete domain of the cloak and to take into account the jumps occurring at that interface so as to eliminate any parasitic evanescent fields. As for the restoring torques, they can be applied in practice thanks to grounded torsion springs. If external intervention is prohibited or deemed disadvantageous, effective restoring torques can be dynamically applied by embedding a rotational resonator within each mass. In that case, our results must be restated as they become valid at single non-zero frequencies in the harmonic regime, thus excluding statics. Future efforts tackling these issues are most welcome.

5. Isotropic polar solids for conformal transformation elasticity and cloaking

5.1 Introduction

The transformation method (Greenleaf et al., 2003a,b) is a powerful tool for the design of structures that guide waves in specific purposeful manners. Indeed, there exist transformation rules dictating what gradients of material properties are needed in order to spatially transform and rearrange the fields that reign within a domain.

In the case of elasticity and in contrast to optics and acoustics, spatial transformations are not only susceptible of changing the material symmetry of the constitutive tensors but also their index symmetries (Brun et al., 2009; Diatta and Guenneau, 2014; Milton et al., 2006; Norris and Shuvalov, 2011). Namely, the image elasticity tensor a turns out to generally violate the minor index symmetries $a_{ijkl} = a_{jikl} = a_{jilk}$. In other words, the image domain tolerates asymmetric stresses; we qualify such domains as "polar". With that in mind, it is the purpose of this paper to provide a constitutive theory of isotropic polar solids thus laying the foundations of an isotropy-preserving transformation method.

Several theories of elasticity with asymmetric stresses are possible, including the one by the Cosserat brothers (1909) also known as micropolar elasticity (Cemal Eringen, 1999; Maugin, 1998). As a matter of fact, the polar theory proposed hereafter can be seen as a version of micropolar elasticity simplified and modified to fit the requirements of the transformation method. First, starting with the full theory, microrotation is dismissed, one way or another, to obtain what the literature refers to as the couple stress theory (Hadjefandiari and Dargush, 2011; Mindlin, 1962; Mindlin and Tiersten, 1962; Toupin, 1964). Second, we neglect couple stresses but not body torque. Third, we assume that body torque is linearly dependent on deformations. Fourth, we embed torque equilibrium into Hooke's law. The outcome of this process is what we refer to as polar elasticity. In doing so, the elasticity tensor gains extra freedom as its asymmetric components no longer need to vanish. Specifically, under the assumption of isotropy, it is found that the elasticity tensor features, in addition to the usual bulk and shear moduli, one extra independent elastic constant in 3D, and two extra constants in 2D. In the latter case, the new constants quantify

to which degree stress and mirror symmetries are broken. Indeed, it turns out that 2D polar elasticity enables isotropic yet chiral behaviors.

In order to illustrate the usefulness of the suggested theory in 2D, we show that an isotropy-preserving transformation method is possible if spatial transformations are restricted to conformal ones, i.e., to transformations that locally conserve angles. The corresponding transformation rules then dictate that the image domain is composed of a gradient of isotropic polar materials (i) with the same shear modulus as the original domain; (ii) with bulk, polarity and chirality moduli depending solely on the original bulk modulus as well as on the local angle of rotation of the conformal transformation; (iii) that are degenerate, in that they admit a stressless strainful deformation mode. It should be noted that conformal transformations have been known to preserve the isotropy of optical and acoustic properties; the present work extends these efforts to the case of elasticity and resolves the complications related to the loss of the minor symmetries.

As an application, a lattice design of a conformal cloak of the "carpet" variety is proposed following ideas by Leonhardt (2006) and by Li and Pendry (2008). Carpet cloaks can render invisible objects placed along an interface, e.g., a free boundary. The cloak's design is found in two steps. In the first step, spring-mass lattices whose behavior in the homogenization limit can target that of the constitutive materials of the cloak are parametrized. Aiming for a chiral yet isotropic behavior, the adopted lattices are based on an adaptation of the well-known hexachiral lattices. In the second step, the derived transformation rules are used to solve for the lattice parameters at each position in the domain of the cloak. The lattice cloak is subsequently tested numerically under combined pressure and shear dynamic loads and at various angles of incidence with the simulations showing approximate but satisfactory cloaking performance.

5.2 Isotropic polar solids

In what follows, we investigate the linear elastic behavior of isotropic polar solids. We calculate general expressions for isotropic fourth-order elasticity tensors that exhibit the major index symmetry but not necessarily the minor ones and account for the extra independent elastic constants needed for their description. For later purposes, emphasis is given to the conditions under which the behavior is degenerate. Degeneracy and polarity are in fact main byproducts of spatial transformations but the connection to the transformation method is left for the next section. Meanwhile, in 2D, we show that polar solids are chiral in general and deduce some of the salient symptoms of chirality in statics and dynamics.

5.2.1 Governing equations

Within the context of infinitesimal deformations, we define a linear elastic polar solid as one governed by Cauchy's law of motion

$$\sigma_{ij,j} + \rho f_i = \rho \dot{v}_i \quad (5.1)$$

and whose strain energy $\psi = \psi(\mathbf{e})$ is quadratically dependent over the displacement gradient $\mathbf{e}$. Therein, $\boldsymbol{\sigma}$ denotes the stress tensor, ρ is mass density, $\mathbf{f}$ is body force and $\mathbf{v}$ is particle velocity. Hooke's law is obtained by differentiation and reads

$$\sigma_{ij} = \frac{\partial \psi}{\partial e_{ij}} = a_{ijkl} e_{kl} \quad (5.2)$$

Since $\mathbf{a}$ derives from ψ , it satisfies the major index symmetry $a_{ijkl} = a_{klij}$; but $\mathbf{a}$ does not necessarily satisfy the minor symmetries. In the same manner, ψ depends on both the symmetric and skew parts of $\mathbf{e}$. In terms of stress, this implies that the skew part $\sigma_{ij} - \sigma_{ji}$ does not necessarily vanish. As a matter of fact, Cauchy's second law of motion expresses a local version of the balance of angular momentum that reads

$$C_i = \varepsilon_{ijk} \sigma_{jk} \quad (5.3)$$

where $\mathbf{c}$ is body torque and ε_{ijk} is the permutation symbol. Thus, the solid being polar is equivalent to there being a body torque density linearly dependent over the displacement gradient:

$$c_i = \varepsilon_{ijk} a_{jklm} e_{lm} \quad (5.4)$$

Hereafter, in 3D then in 2D, we determine a general expression for isotropic majorly symmetric fourth-order elasticity tensors $\mathbf{a}$, count the maximum number of independent elastic constants they can exhibit and describe the zero modes they possibly support.

5.2.2 Dimension three

We begin by recalling that the space of second-order tensors can be decomposed into a sum of three subspaces all stable under rotations. Thus, such a decomposition is well suited for the description of an isotropic behavior. These subspaces are those of spherical tensors, symmetric deviatoric tensors and skew tensors. For the displacement gradient $\mathbf{e}$, we write

$$\mathbf{e} = \frac{t}{3} \mathbf{I} + \mathbf{W} + \tilde{\mathbf{e}} \quad (5.5)$$

where

$$t = \text{tr } \mathbf{e}, \quad \mathbf{W} = \frac{\mathbf{e} - \mathbf{e}^T}{2} \text{ and } \tilde{\mathbf{e}} = \frac{\mathbf{e} + \mathbf{e}^T}{2} - \frac{\text{tr } \mathbf{e}}{3} \mathbf{I} \quad (5.6)$$

are the relative infinitesimal change in volume, the infinitesimal rotation and the deviatoric strain, respectively. Note that the infinitesimal rotation $\mathbf{W}$, being skew, can be represented by a (pseudo)vector $\mathbf{w} = \theta \hat{\mathbf{w}}$ product of the infinitesimal rotation angle θ by the unitary axis of rotation $\hat{\mathbf{w}}$. Component-wise, that vector is

$$w_i = -\frac{1}{2} \varepsilon_{ijk} W_{jk} = -\frac{1}{2} \varepsilon_{ijk} e_{jk} \quad (5.7)$$

Accordingly, the deformation energy is a quadratic form

$$\psi(\mathbf{e}) = \psi(t, \mathbf{W}, \tilde{\mathbf{e}}) \quad (5.8)$$

Given that the behavior is isotropic, the six coupling tensors

$$\frac{\partial^2 \psi}{\partial t^2}, \quad \frac{\partial^2 \psi}{\partial \mathbf{w} \partial \mathbf{W}}, \quad \frac{\partial^2 \psi}{\partial \tilde{\mathbf{e}} \partial \tilde{\mathbf{e}}}, \quad \frac{\partial^2 \psi}{\partial t \partial \mathbf{w}}, \quad \frac{\partial^2 \psi}{\partial \mathbf{w} \partial \tilde{\mathbf{e}}} \quad \text{and} \quad \frac{\partial^2 \psi}{\partial \tilde{\mathbf{e}} \partial t} \quad (5.9)$$

should all be isotropic.

- The first tensor of the list above is a scalar and is automatically isotropic; it will be denoted κ .
- The second is a second-order tensor; being isotropic, it is proportional to the identity through a modulus denoted α .
- The third is an isotropic majorly symmetric fourth-order tensor 2μ coupling $\tilde{\mathbf{e}}$ to itself. Deviatoric strains being symmetric and traceless, tensor 2μ can further be supposed to satisfy $\mu_{ijkl} = \mu_{jikl}$ and $\mu_{ijkk} = 0$ without loss of generality. Therefore, it is deduced that 2μ is proportional to the fourth-order identity through a modulus denoted 2μ .
- The fourth coupling tensor is an isotropic vector and therefore vanishes.
- The fifth is an isotropic third-order tensor and therefore is proportional to the permutation symbol. As it couples $\mathbf{w}$ to $\tilde{\mathbf{e}}$ which is symmetric, it can be set to zero without loss of generality.
- The sixth is an isotropic second-order tensor and therefore is proportional to the identity. As it couples t to $\tilde{\mathbf{e}}$ which is traceless, it can be set to zero without loss of generality as well.

All in all, the elastic energy stored for a deformation $\mathbf{e}$ specifies into

$$\psi = \mu \tilde{\mathbf{e}} : \tilde{\mathbf{e}} + \frac{1}{2} \kappa t^2 + \frac{1}{2} \alpha \theta^2 \quad (5.10)$$

and features three independent elastic moduli. Once more, Hooke's law is obtained by differentiation of ψ with respect to $\mathbf{e}$; it reads

$$\boldsymbol{\sigma} = 2\mu \tilde{\mathbf{e}} + \kappa t \mathbf{I} + \frac{\alpha}{2} \mathbf{W} \quad (5.11)$$

and breaks into three parts

$$\boldsymbol{\sigma} = 2\mu \tilde{\mathbf{e}}, \quad p = -\kappa t \quad \text{and} \quad \mathbf{c} = -\alpha \mathbf{W} \quad (5.12)$$

where

$$\boldsymbol{\sigma} = \frac{\boldsymbol{\sigma} + \boldsymbol{\sigma}^T}{2} + p\mathbf{I}, \quad p = -\frac{\text{tr } \boldsymbol{\sigma}}{3} \quad \text{and} \quad c_i = \varepsilon_{ijk} \sigma_{jk} \quad (5.13)$$

are deviatoric stress, hydrostatic pressure and body torque, respectively. Equivalently, the elasticity tensor admits the expression

$$a_{ijkl} = \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \left(\kappa - \frac{2}{3}\mu\right)\delta_{ij}\delta_{kl} + \frac{\alpha}{4}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) \quad (5.14)$$

Therein, the existence of parameter α is characteristic of a polar solid. For $\alpha = 0$, a is minorly symmetric and the stress tensor is systematically symmetric: we say that the solid is a Cauchy's solid. For $\alpha \neq 0$, a lacks the minor symmetries and can yield asymmetric stresses: we say that the solid is polar. In the latter case, the skew part of stress directly quantifies the externally applied body torque $\mathbf{c}$ as proportional to the infinitesimal rotation vector $\mathbf{w}$. Therefore, α can be interpreted as a torsional rigidity. As for moduli κ and μ , they express the usual relationships between pressure and change in volume and between shear strains and shear stresses; they are bulk and shear moduli, respectively.

Last, from Eq. (5.10), it is seen that all three moduli, μ , κ and α , must be positive in order to enforce stability; if either vanishes, the polar solid becomes degenerate in the sense that it exhibits nonzero displacement gradients $\mathbf{e}$ with zero stress $\boldsymbol{\sigma}$. Such gradients are called zero modes and are denoted $\mathbf{e}_{zm}$. For instance, $\mu = 0$ enables five zero modes given by simple shears; $\kappa = 0$ enables a single zero mode, namely a pure dilatation; and $\alpha = 0$ enables three zero modes corresponding to infinitesimal rotations. The latter describes the standard case of a Cauchy's solid. Last, when multiple moduli vanish simultaneously, the foregoing cases can be linearly combined.

5.2.3 Dimension two

Interestingly, in 2D, isotropy tolerates an extra material parameter denoted β . As a matter of fact, in 2D, the skew part of $\mathbf{e}$ can be represented as a scalar θ . This enables a fourth material parameter coupling θ to t . Therefore, the deformation energy specifies into

$$\psi = \mu \tilde{\mathbf{e}} : \tilde{\mathbf{e}} + \frac{1}{2} \kappa t^2 + \frac{1}{2} \alpha \theta^2 + \beta t \theta \quad (5.15)$$

and the elasticity tensor into

$$a_{ijkl} = \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + (\kappa - \mu)\delta_{ij}\delta_{kl} + \frac{\alpha}{4}J_{ij}J_{kl} + \frac{\beta}{2}(\delta_{ij}J_{kl} + J_{ij}\delta_{kl}) \quad (5.16)$$

Equivalently, Hooke's law becomes

$$\boldsymbol{\sigma} = (\kappa t + \beta \theta)\mathbf{I} + 2\mu \tilde{\mathbf{e}} + \frac{\beta t + \alpha \theta}{2} \mathbf{J} \quad (5.17)$$

where $\mathbf{J}$ is the $\pi/2$ plane rotation. Most notably, hydrostatic pressure and body torque are coupled according to the constitutive relations

$$p = -\kappa t - \beta \theta \quad \text{and} \quad c = -\beta t - \alpha \theta \quad (5.18)$$

where c and θ are now given by

$$c = -\mathbf{J} : \boldsymbol{\sigma} \quad \text{and} \quad \theta = \frac{1}{2} \mathbf{J} : \mathbf{e} \quad (5.19)$$

In order to enforce stability, μ, κ and α should remain positive while the newly found coupling should remain small, or

$$\beta^2 \leq \kappa \alpha \quad (5.20)$$

more precisely. Accordingly, in addition to the cases previously discussed in 3D, isotropic polar solids in 2D can also become degenerate if

$$\beta^2 = \kappa \alpha \quad (5.21)$$

without μ, κ or α being zero. In that case, the solid admits a unique zero mode $\mathbf{e}_{zm}$ characterized by

$$\tilde{\mathbf{e}}_{zm} = \mathbf{0}, \quad \kappa t + \beta \theta = 0 \quad \text{and} \quad \beta t + \alpha \theta = 0 \quad (5.22)$$

Consequently, $\mathbf{e}_{zm}$ has only spherical and skew parts and admits the expression $\mathbf{e}_{zm} = a\mathbf{I} + b\mathbf{J}$. Substituting into (5.22), we find

$$\mathbf{e}_{zm} = \beta \mathbf{I} + 2\kappa \mathbf{J} \quad (5.23)$$

Remarkably, 2D isotropic polar solids turn out to be able to support zero modes that are function of their constitutive parameters (e.g., β and κ). As a result, a gradient of polar materials can support a gradient of zero modes. By contrast, in 3D, all supported zero modes are parameter-independent and therefore cannot be continuously varied over space.

5.2.4. Chirality

Recall that elasticity, being described by an even-order tensor, is automatically inversion-invariant. In 3D, inversion is a reflection meaning that 3D elasticity is automatically nonchiral. In 2D, inversion is not a reflection implying that 2D elasticity can be chiral in principle. Minor symmetry however, combined with isotropy (or any nontrivial symmetry for that matter, see He and Zheng, 1996), does preclude chirality. By contrast, polar solids, as they lack minor symmetry, can be chiral while remaining isotropic. Here, we highlight chirality-related effects in their simplest static manifestation.

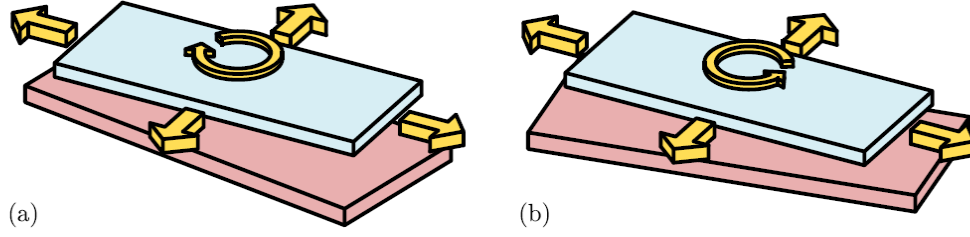


Figure 5.1 A sketch of the chiral coupling between infinitesimal rotation and change in area in (a) ambidextral and (b) dextral polar media: top and bottom correspond to reference and current configurations respectively. The arrows illustrate motion, not efforts.

For that purpose, consider the uniform deformation of a 2D isotropic polar solid under a purely hydrostatic stress of pressure p , i.e., with $c = 0$. A relationship between the relative infinitesimal change in area t and the infinitesimal rotation θ can be found and reads

$$\theta = -\frac{\beta}{\alpha} t \quad (5.24)$$

Accordingly, an increase in area ($t > 0$) is accompanied by a counterclockwise rotation if $\beta < 0$, by a clockwise rotation if $\beta > 0$, and by no rotation if $\beta = 0$, all in the absence of body torque. Thus, isotropic polar solids are chiral when $\beta \neq 0$ and, in that case, are of two types: solids with $\beta < 0$ will be referred to as "dextral" whereas solids with $\beta > 0$ will be referred to as "ambidextral"; see Fig. 5.1.

A similar conclusion holds when the solid deforms uniformly under a net torque with zero hydrostatic pressure. Note however that the value of the coupling between rotation and change in area is now different in general since $p = 0$ implies

$$\theta = -\frac{\kappa}{\beta} t \quad (5.25)$$

Finally, in the critical regime of a degenerate solid, $p = 0$ and $c = 0$ occur simultaneously so that the solid deforms freely under no external efforts. In that case, as the solid changes its area, it rotates either clockwise or anticlockwise depending on its handedness while under zero stress. This stretch-and-rotate motion describes then the zero mode $\mathbf{e}_{zm}$ calculated previously in Eq. (5.23).

5.2.5 Propagation of plane harmonic waves

We pursue here our investigation of 2D isotropic polar solids with emphasis on chiral effects but in the dynamic regime this time. The results obtained here for the most part can be readily generalized, or rather adapted, to 3D by setting $\beta = 0$.

Having cloaking applications, as well as related wave phenomena in mind, it is of interest to inspect how harmonic plane waves propagate through polar solids. We let

$$\mathbf{u} = \mathbf{u}_0 e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)} \quad (5.26)$$

be a candidate solution of the motion equation of wavenumber $\mathbf{k}$ and angular frequency ω . The corresponding eigenvalue problem can then be put in the form

$$\Gamma \mathbf{u}_0 = \frac{\omega^2}{k^2} \mathbf{u}_0, \quad [\Gamma] = \frac{1}{\rho} \begin{bmatrix} \mu + \kappa & \beta/2 \\ \beta/2 & \mu + \alpha/4 \end{bmatrix} \quad (5.27)$$

where tensor Γ has been decomposed in the canonical basis $(\mathbf{k}, \mathbf{Jk})$. Matrix $[\Gamma]$ being nondiagonal signifies that, for $\beta \neq 0$, purely longitudinal and purely transverse modes do not exist. This is another symptom of chirality. Indeed, it is expected that the coupling between dilation and rotation precludes purely longitudinal modes and, then necessarily, purely transverse ones as well. The supported modes still have mutually orthogonal polarizations denoted $\mathbf{u}_\pm$ and admitting the expressions

$$[\mathbf{u}_+] = \begin{bmatrix} \cos \gamma \\ -\sin \gamma \end{bmatrix} \quad \text{and} \quad [\mathbf{u}_-] = \begin{bmatrix} \sin \gamma \\ \cos \gamma \end{bmatrix}, \quad -\pi/2 < \gamma \leq \pi/2 \quad (5.28)$$

The corresponding eigenvalues are then

$$c_\pm^2 = \frac{\mu}{\rho} + \frac{1}{2\rho} (\kappa + \alpha/4) \pm \frac{1}{2\rho} \sqrt{(\kappa - \alpha/4)^2 + \beta^2} \quad (5.29)$$

and are equal to the propagation velocities $c_\pm$ squared of the respective modes. It will also be useful to introduce the centered normalized wave velocities

$$\bar{c}_\pm = \sqrt{\frac{\rho c_\pm^2 - \mu}{\kappa + \alpha/4}} \quad (5.30)$$

As for angle γ , it is characterized by

$$\cos(2\gamma) = \frac{\kappa - \alpha/4}{\sqrt{(\kappa - \alpha/4)^2 + \beta^2}} \quad \text{and} \quad \sin(2\gamma) = -\frac{\beta}{\sqrt{(\kappa - \alpha/4)^2 + \beta^2}} \quad (5.31)$$

The range of possible polarizations is illustrated on Fig. 5.2. Therein, the x -axis corresponds to parameter $(\kappa - \alpha/4)/(\kappa + \alpha/4)$ and qualitatively measures to which degree stress symmetry is broken, i.e., how far the medium is from a Cauchy's medium (point C). As for the y -axis, it corresponds to parameter $\beta/(\kappa + \alpha/4)$ and measures the amplitude and handedness of chiral effects. Enforcing stability implies that only pairs of parameters falling within the big circle of equation $\beta^2 = \kappa\alpha$ are within reach. For each pair of parameters, a cross represents the orthogonal polarizations of the $\mathbf{u}_+$ and $\mathbf{u}_-$ modes oriented with respect to wavenumber $\mathbf{k}$ supposed everywhere horizontal and pointing towards the right. The length of the longer mark is proportional to $\bar{c}_+$, that of the shorter one to $\bar{c}_-$. Thus, starting at point C and going deeper into quadrant (I), the faster mode $\mathbf{u}_+$ (resp., the slower mode $\mathbf{u}_-$) becomes less and less longitudinal (resp., transverse). As we reach the y -axis, modes $\mathbf{u}_\pm$ are at $\pm\pi/4$ from $\mathbf{k}$. As we go deeper into the second quadrant, the faster (resp., slower) mode becomes more and more transverse (resp., longitudinal). As we reach

the other side of the x -axis, i.e., $\kappa < \alpha/4$ and $\beta = 0$, the faster (resp., slower) mode becomes purely transverse (resp., longitudinal). Last, at the origin, $\beta = 0, \kappa = \alpha/4$, all polarizations are possible, a fact symbolized on the plot with a small circle. In that case, all modes propagate at the same centered normalized velocity $\bar{c}_{\pm} = 1/\sqrt{2}$.

5.2.6 Example: Grounded hexachiral lattices

Isotropic degenerate polar solids, their mechanics and the potentially useful phenomena they host, all remain a theoretical exercise so long as we have not demonstrated that they can actually be designed. It is the purpose of this subsection to provide designs of lattice materials whose behavior in the homogenization limit is that of an isotropic degenerate polar solid. Aiming for a chiral yet isotropic outcome, the hexachiral spring-mass lattice of Fig. 3a emerges as a natural candidate. Its unit cell contains a single mass of value m and radius b and three springs of constant k and directions $\mathbf{n}_i$ that are tangent to the masses' outer circumferences. Last, call $\mathbf{r}_i$ the three elementary lattice vectors and call γ the (oriented) angle between $\mathbf{n}_i$ and $\mathbf{r}_i$.

Loosely speaking, a polar solid is one that resists rotations. Polar solids are therefore necessarily grounded. Unlike common elastic foundations that resist displacements, polar solids require a foundation that resists displacement gradients, e.g., rotations, without hindering the displacements themselves. Thus, we connect each mass to a parallel rigid plane through a torsional spring of constant $\eta = 12bb'k$ where b' is a positive radius equivalent to a torsional rigidity; see Fig. 5.3b. The spring being vertical, small inplane displacements of the masses remain unaware of the ground. In contrast, mass rotation is penalized by a restoring torque $-\eta\phi$ proportional to the rotation angle ϕ .

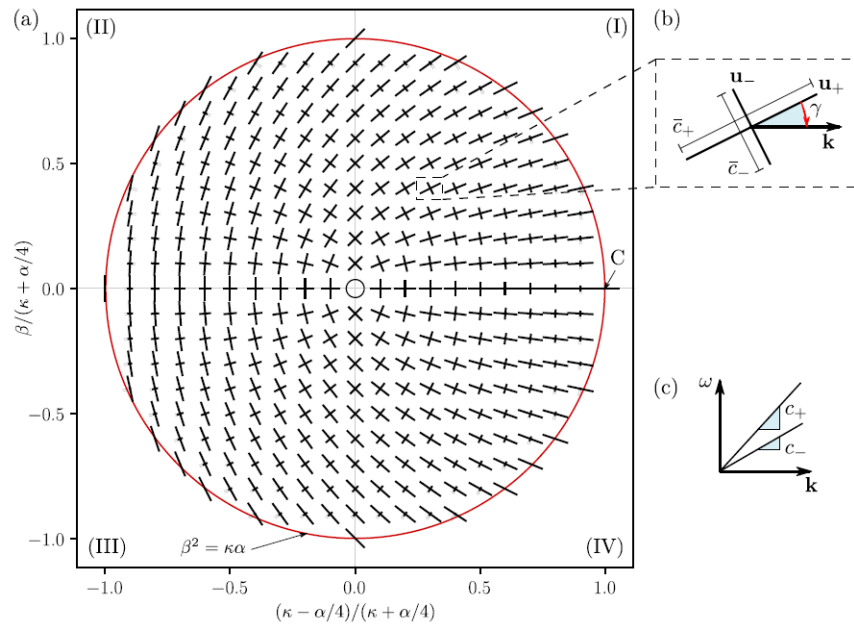


Figure 5.2 Plane harmonic waves in a polar isotropic solid. (a) Polarization field in the $((\kappa - \alpha/4)/(\kappa + \alpha/4), \beta/(\kappa - \alpha/4))$ -space: the big circle bounds the stable region $\beta^2 \leq \kappa\alpha$; point C corresponds to a Cauchy's medium. (b) Magnified view: at each pair of parameters, a cross represents the orthogonal polarizations of the $\mathbf{u}_+$ and $\mathbf{u}_-$ modes; the lengths of the marks are proportional to the normalized wave velocities $\bar{c}_\pm$. (c) A typical dispersion curve.

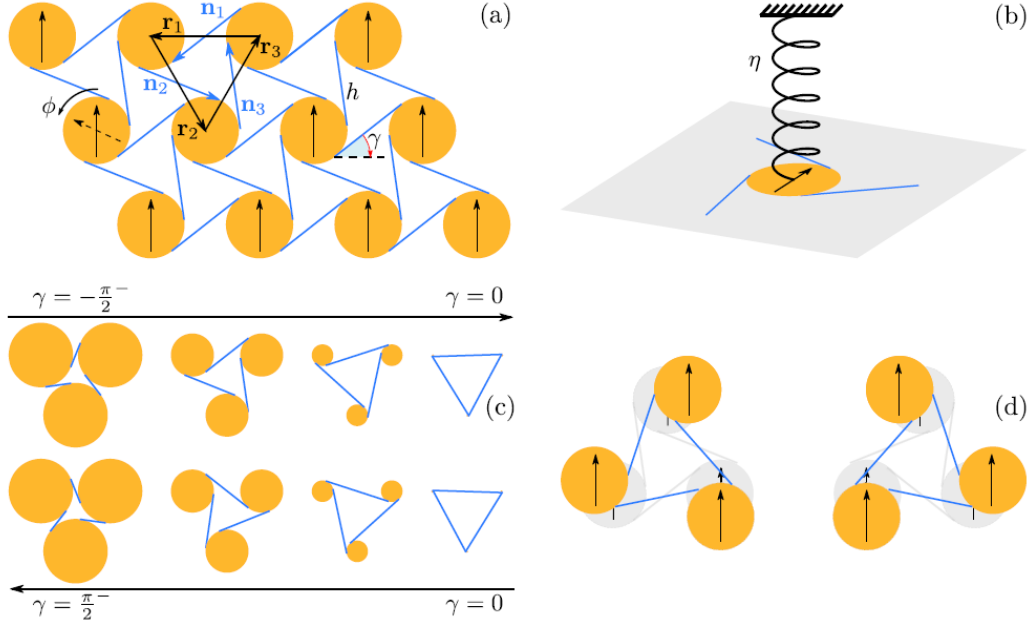


Figure 5.3 Grounded hexachiral lattices as isotropic polar solids. (a) An annotated example with $\gamma < 0$; the arrows serve as a reference to the masses' initial orientation. (b) Grounding mechanism: An out-of-plane torsional spring is attached to each mass thus providing a restoring torque acting against mass rotation without hindering displacements. (c) Unit cell variations parametrized with $\gamma \in (-\pi/2, \pi/2)$. (d) The action of the zero mode on the configuration of a unit cell for an ambidextral (left) and a dextral (right) solid: reference state is in the background, the deformed one in the foreground.

Hence, the elastic energy of the lattice per unit cell area a for a displacement gradient $\mathbf{e}$ is given by

$$\psi(\mathbf{e}) = \min_{\phi} E(\mathbf{e}, \phi) \quad \text{with} \quad E(\mathbf{e}, \phi) = \frac{k}{2a} \sum_i (\langle \mathbf{e} \mathbf{r}_i, \mathbf{n}_i \rangle - 2b\phi)^2 + \frac{\eta}{2a} \phi^2 \quad (5.32)$$

At the minimum, the rotation angle satisfies the zero derivative condition

$$\frac{\partial E}{\partial \phi} = 0, \quad \text{i.e.,} \quad -2bk \sum_i (\langle \mathbf{e} \mathbf{r}_i, \mathbf{n}_i \rangle - 2b\phi) + \eta\phi = 0 \quad (5.33)$$

and is given by

$$\phi = \mathbf{s} : \mathbf{e} \quad \text{Where} \quad \mathbf{s} = \frac{1}{6(b+b')} \sum_i \mathbf{n}_i \otimes \mathbf{r}_i = \frac{r}{4(b+b')} \mathbf{R}^T \quad (5.34)$$

and $\mathbf{R}$ is the plane rotation of angle γ . Consequently, the elastic energy density can be recast into

$$\psi(\mathbf{e}) = \frac{k}{2a} \sum_i (\langle \mathbf{e} \mathbf{r}_i, \mathbf{n}_i \rangle - 2b\mathbf{s} : \mathbf{e})^2 + \frac{\eta}{2a} (\mathbf{s} : \mathbf{e})^2 \quad (5.35)$$

Thus, the elasticity tensor of the lattice is

$$\mathbf{a} = \frac{\partial^2 \psi}{\partial \mathbf{e} \partial \mathbf{e}} = \frac{k}{a} \sum_i (\mathbf{n}_i \otimes \mathbf{r}_i - 2b\mathbf{s}) \otimes (\mathbf{n}_i \otimes \mathbf{r}_i - 2b\mathbf{s}) + \frac{\eta}{a} \mathbf{s} \otimes \mathbf{s} \quad (5.36)$$

and, skipping a few index manipulations, can be expressed by

$$a_{pqrs} = \frac{k\sqrt{3}}{4} \left(\frac{b' - b}{b' + b} R_{qp} R_{sr} + R_{sp} R_{qr} + \delta_{pr} \delta_{qs} \right) \quad (5.37)$$

Therefrom, $\mathbf{a}$ can be readily verified to be isotropic and the material parameters of the effective polar solid can be extracted. They read

$$\mu = \frac{k\sqrt{3}}{4}, \quad \kappa = \frac{kb'\sqrt{3}}{2(b+b')} \cos^2 \gamma, \quad \alpha = \frac{2kb'\sqrt{3}}{b+b'} \sin^2 \gamma \quad \text{and} \quad \beta = -\frac{kb'\sqrt{3}}{b+b'} \cos \gamma \sin \gamma \quad (5.38)$$

In particular, relation (5.21) holds systematically meaning that the effective polar solid is degenerate. The corresponding zero mode has been illustrated on Fig. 5.3d.

Formulae (5.38) are perhaps more useful inverted so as to yield the lattice parameters necessary for targeting the elastic parameters of a given degenerate isotropic polar solid. We find

$$k = \frac{4\mu}{\sqrt{3}}, \quad \frac{b'}{b} = \frac{\kappa + \frac{\alpha}{4}}{2\mu - \kappa - \frac{\alpha}{4}} \quad \text{and} \quad \gamma = \text{sgn } \beta \arctan \sqrt{\frac{\alpha/4}{\kappa}} \quad (5.39)$$

Add to that the expression of the mass density of the masses given by

$$\frac{m}{\pi b^2} = \frac{2\sqrt{3}\rho}{\pi} \left(1 + \frac{\kappa}{\alpha/4} \right) \quad (5.40)$$

where ρ is the mass density of the target solid.

It is noteworthy that the proposed lattices remain stable in the homogenization limit even when b' , or equivalently η , is negative as long as $1 + b/b'$ is positive. If however the stability of the lattice's constitutive elements, taken separately, is desired as well, then b' must be positive. In conclusion, under the element-wise stability requirement, the proposed grounded hexachiral lattices can target all degenerate isotropic polar solids satisfying

$$2\mu \geq \kappa + \frac{\alpha}{4} \quad (5.41)$$

The salient static and dynamic characteristics of these lattices, in the homogenization limit, can of course be deduced from the general study of the previous subsections. More importantly, in any application (e.g., cloaking) where such polar solids are needed, the suggested lattices can be used but only so in a sufficiently low frequency regime.

5.3 Conformal transformation theory

In this section, we briefly recall the general transformation method and then specify it to conformal transformations. We prove that conformal transformations map a 2D isotropic elastic solid into a gradient of isotropic degenerate polar materials. This is done by exhibiting the transformation rules that relate the original and image constitutive tensors. It should be noted that 3D conformal transformations are limited to translations, rotations, reflections, scalings and inversions. By contrast, in 2D, all analytic functions of a complex variable are conformal. This implies that the use of conformal transformations for field manipulation purposes while avoiding anisotropic constitutive materials is far more potent in 2D than in 3D. Thus, from now on, we restrict our presentation to 2D.

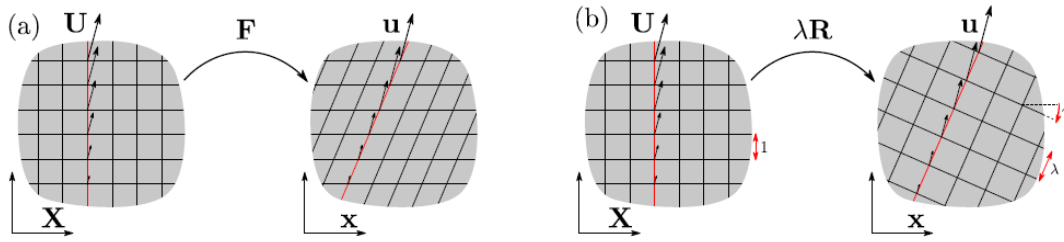


Figure 5.4 Local view of the transformation method: (a) nonconformal; (b) conformal. Note how displacements are redistributed but otherwise remain the same. This fictitious redistribution can be realized by enforcing the rules (42).

5.3.1 General transformation rules

Let $\mathbf{U}$ be the displacement field over a body $\{\mathbf{X}\}$ of mass density R and elasticity tensor $\mathbf{A}$. It is of interest to determine what fields ρ and $\mathbf{a}$ should replace R and $\mathbf{A}$ so as to redistribute the displacements $\mathbf{U}$ in a controlled and purposeful manner. Facing that inverse problem, the transformation method provides a solution. Specifically, letting $\mathbf{X} = \boldsymbol{\phi}(\mathbf{x})$ transform $\{\mathbf{X}\}$ into a new body $\{\mathbf{x}\}$, the field $\mathbf{u}(\mathbf{x}) = \mathbf{U}(\mathbf{X})$ reigns over $\{\mathbf{x}\}$ if the new properties derive from the original ones following

$$\rho = R / J \quad \text{and} \quad a_{ijkl} = F_{jp} F_{lq} A_{ipkq} / J, \quad (5.42)$$

with $F_{ij} = \partial x_i / \partial X_j$ and $J = \det \mathbf{F}$ (Norris and Shuvalov, 2011); see Fig. 5.4. The Hooke's law corresponding to the new moduli a_{ijkl} then reads

$$\sigma_{ij} = a_{ijkl} e_{kl} = F_{jp} F_{lq} A_{ipkq} e_{kl} / J \quad (5.43)$$

where $e_{kl} = \partial u_k / \partial x_l$ is the displacement gradient.

Note that we have set $\mathbf{u}(\mathbf{x}) = \mathbf{U}(\mathbf{X})$ whereby the point of application of a displacement changes but not the displacement itself. This transformation rule was first introduced by Brun et al. (2009) and is adopted here as it is capable of preserving isotropy. Other gauges, with $(\mathbf{x}) \neq \mathbf{U}(\mathbf{X})$, were suggested by Milton et al. (2006) and by Norris and Shuvalov (2011); these do not seem to preserve isotropy except in trivial cases.

5.3.2 Conformal transformation rules

Suppose now that A satisfies the major and minor index symmetries (i.e., $A_{ijkl} = A_{klij} = A_{jikl}$) and is isotropic with a shear modulus μ_0 and a bulk modulus κ_0 , namely

$$A_{ijkl} = \mu_0 (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + (\kappa_0 - \mu_0) \delta_{ij}\delta_{kl}. \quad (5.44)$$

Also, let ϕ be conformal in the sense that it preserves oriented angles. Therefore, F is shear-free and reads as the product of a pure dilatation by a plane rotation, i.e.,

$$\mathbf{F} = \lambda \mathbf{R} \quad (5.45)$$

where $\lambda = \sqrt{J}$ is a positive stretch ratio and $\mathbf{R}$ is again the plane rotation of angle γ . Employing the transformation rule (5.42), and skipping elementary index manipulations, we deduce that A takes the form (5.16) with the elasticity constants μ, κ, α and β given respectively by

$$\mu = \mu_0, \quad \kappa = \kappa_0 \cos^2 \gamma, \quad \alpha = 4\kappa_0 \sin^2 \gamma \quad \text{and} \quad \beta = -2\kappa_0 \cos \gamma \sin \gamma. \quad (5.46)$$

In particular, (i) the shear modulus is invariant; (ii) the local stretch λ is irrelevant; and (iii) the image parameters systematically satisfy

$$\alpha\kappa = 4\kappa_0^2 \cos^2 \gamma \sin^2 \gamma = \beta^2 \quad (5.47)$$

In conclusion, in 2D, we have proven that conformal transformations applied to an isotropic solid yield a gradient of isotropic denegerate polar materials.

5.3.3 Discussion

As described earlier, a conformal transformation locally resembles a rotation of angle γ followed by a dilatation of ratio λ . Thus, a plane harmonic wave traveling along a direction inclined at an angle δ away from an absolute axis in body $\{\mathbf{X}\}$ has an image that travels at angle $\delta + \gamma$ away from the same axis in $\{\mathbf{x}\}$. Given the gauge $\mathbf{u}(\mathbf{x}) = \mathbf{U}(\mathbf{X})$, the wave and its image have identical amplitudes. Accordingly, if the wave is longitudinally polarized in $\{\mathbf{X}\}$, then its image is polarized at angle $-\gamma$ with respect to the propagation direction in $\{\mathbf{x}\}$; similarly, if it is transversally polarized in $\{\mathbf{X}\}$, then its image is polarized at an angle $\pi/2 - \gamma$; see Fig. 5.4b.

Furthermore, since space in $\{\mathbf{x}\}$ is stretched by a ratio λ in comparison to $\{\mathbf{X}\}$, the image of a wave is expected to travel at lower (resp., higher) speed if $\lambda < 1$ (resp., $\lambda > 1$). Equivalently, the transformation rule for mass density,

$$\rho = R / \lambda^2, \quad (5.48)$$

implies the same decrease or increase in propagation speeds. By that logic, it is expected that pressure and shear waves have images that all propagate at the same speeds

$$c_+ = \lambda \sqrt{\frac{\kappa_0 + \mu_0}{R}} \text{ and } c_- = \lambda \sqrt{\frac{\mu_0}{R}} \quad (5.49)$$

respectively.

Note that these effects, namely bias in polarization and change in speed, can be quantitatively checked by injecting the derived expressions (5.46) into the formulae (5.29). Both derivations (and notations) are therefore consistent.

5.4 Application: Conformal carpet cloaking

Cloaking is a notable application of the general transformation method. Here, using strictly conformal transformations, we demonstrate a version of cloaking known as "carpet" cloaking by hiding an inclusion along the free boundary of an elastic half-plane. The geometry and elastic parameters of the cloak as well as of its lattice microstructure are derived and numerical simulations are carried so as to assess cloaking performance.

5.4.1 Geometry

Let $\{\mathbf{x}\}$ be an elastic half-space composed of three regions: the first is a void half-disk of radius a ; the second is a coating of that void of approximate width $2w$ and height h ; and the third is an isotropic homogeneous background of elastic constants κ_0 and μ_0 . Let ϕ be a transformation whose inverse eclipses the void of radius a while leaving the background invariant; see Fig. 5.5. Thus, by filling the coating with a gradient of materials following the transformation rules (5.42), body $\{\mathbf{x}\}$ can be seen as the image of a homogeneous isotropic elastic half-space $\{\mathbf{X}\}$ of parameters κ_0 and μ_0 . Accordingly, the fields that reign over $\{\mathbf{x}\}$ can be obtained by redistributing the fields that reign over $\{\mathbf{X}\}$. In particular, having assumed $\phi(\mathbf{X}) = \mathbf{X}$ over the background, the fields over there remain unaltered. Consequently, from the perspective of an observer located in the background, bodies $\{\mathbf{x}\}$ and $\{\mathbf{X}\}$ are indistinguishable meaning that the void of radius a has been cloaked.

Unfortunately, when ϕ is conformal, $\phi(\mathbf{X}) = \mathbf{X}$ over any region of finite extent readily implies, by the uniqueness of analytic continuation, that $\phi(\mathbf{X}) = \mathbf{X}$ everywhere thus making $\{\mathbf{x}\}$ and $\{\mathbf{X}\}$ identical and leaving no room for a void to be cloaked. To circumvent this limitation, we only enforce $\phi(\mathbf{X}) = \mathbf{X}$ in the limit $|\mathbf{X}| \rightarrow \infty$ and take

$$\mathbf{x} = \phi(\mathbf{X}) = \begin{cases} \left(\mathbf{X} + \sqrt{\mathbf{X}^2 - 4a^2} \right) / 2, & \text{for } |\mathbf{X}| \text{ finite} \\ \mathbf{X}, & \text{for } |\mathbf{X}| \rightarrow \infty \end{cases} \quad (5.50)$$

where $\mathbf{x}$ and $\mathbf{X}$ are represented by complex numbers and where the complex square root is taken to have a positive imaginary part. Fig. 5.5 illustrates the action of ϕ . Therein, it is visible that ϕ is conformal since it transforms a square grid into another square grid.

In practice, and in order to bound the cloak's domain, we adopt

$$\mathbf{x} = \phi(\mathbf{X}) = \begin{cases} \left(\mathbf{X} + \sqrt{\mathbf{X}^2 - 4a^2} \right) / 2 & \text{for } \mathbf{X} \in D \equiv (-w, w) + i(0, h) \\ \mathbf{X}, & \text{for } \mathbf{X} \notin D \end{cases} \quad (5.51)$$

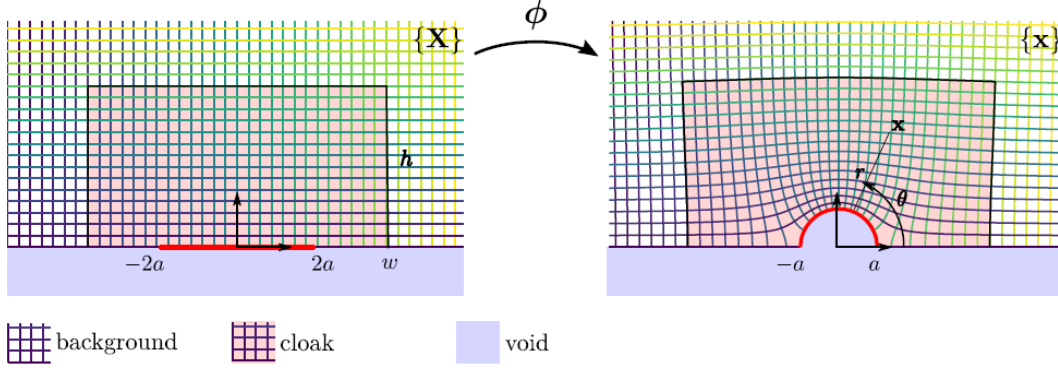


Figure 5.5 The contour lines of the Cartesian coordinates of $\mathbf{X}$ before and after transformation. Note that the real segment $[-2a, 2a]$ transforms into a semi circle of radius a thus leaving a void along the free boundary of the half-plane where objects can be hidden. The domain of the cloak is hed originally, it is a rectangle of width $2w$ and height h .

where the image of the rectangle D , i.e., $\phi(D)$, defines the cloak's domain. Accordingly, ϕ is piecewise conformal and is discontinuous along the boundary of D ; as a consequence, cloaking is expected to be approximate but improves as w and h get larger. Cloaking becomes perfect in the theoretical limit $w, h \rightarrow \infty$.

5.4.2 Elastic parameters

From expression (5.50), it is possible to derive that, within the cloak,

$$\frac{\partial \mathbf{X}}{\partial \mathbf{x}} = -\frac{a^2}{\mathbf{x}^2} + 1 \quad (5.52)$$

By extracting real and imaginary parts, the above relation can be recast into the matrix form

$$\mathbf{F}^{-1} = \left(1 - \frac{a^2}{r^2} \cos(2\theta) \right) \mathbf{I} + \frac{a^2}{r^2} \sin(2\theta) \mathbf{J} \quad (5.53)$$

with

$$\mathbf{x} = r \exp(i\theta) \quad (5.54)$$

see Fig. 5.5. Elementary algebraic manipulations then permit to reveal that $\mathbf{F}$ is the product of a pure dilatation of stretch λ by a plane rotation of angle γ respectively given by

$$\begin{aligned}\lambda &= \frac{1}{\sqrt{a^4 / r^4 + 1 - 2a^2 / r^2 \cos(2\theta)}}, \\ \cos \gamma &= \frac{1 - a^2 / r^2 \cos(2\theta)}{\sqrt{a^4 / r^4 + 1 - 2a^2 / r^2 \cos(2\theta)}} \\ \text{and } \sin \gamma &= -\frac{a^2 / r^2 \sin(2\theta)}{\sqrt{a^4 / r^4 + 1 - 2a^2 / r^2 \cos(2\theta)}}\end{aligned}\tag{5.55}$$

5.4.3 Lattice microstructure

To construct a lattice model of the cloak, we determine the way in which the polar hexachiral lattices investigated earlier need to be graded and spatially distributed to fit the local elastic parameters of the cloak. First, we discretize domain D , including its boundary, into a finite hexagonal point lattice with a sufficiently small lattice parameter δ . The points are transformed by ϕ and now constitute a discretization of the cloak's domain $\phi(D)$ with a local lattice parameter $\lambda\delta$. Then, at each point, a rigid circular mass is centered and given the radius $\delta\lambda\sin(\gamma)/2$. Last, each mass is grounded using a torsional spring of constant η and each pair of neighboring masses are connected along a common internal tangent using a linear spring of constant k ; see Fig. 5.3 for which tangent to select depending on the sign of angle γ . The various steps of this construction are outlined on Fig. 5.6.

It is important to highlight that the lattice parameters are $\mathbf{x}$ -, or equivalently (r, θ) -, dependent. Specifically, Eqs. (5.39) and (5.40) determine the lattice parameters $(k, b'/b, \gamma, b, m)$ in function of the cloak's effective parameters $(\mu, \kappa, \alpha, \beta, \rho)$; Eqs. (5.46) and (5.48) provide expressions of $(\mu, \kappa, \alpha, \beta, \rho)$ in function of the background medium constitutive parameters (μ_0, κ_0, R) and of the transformation gradient's parameters (λ, γ) ; last, Eq. (5.55) yields (λ, γ) in function of position (r, θ) . All in all, the lattice parameters read

$$\begin{aligned}k &= \frac{4\mu_0}{\sqrt{3}}, \\ \frac{b'}{b} &= \frac{\kappa_0}{2\mu_0 - \kappa_0}, \\ \gamma &= -\arctan\left(\frac{a^2 / r^2 \sin(2\theta)}{1 - a^2 / r^2 \cos(2\theta)}\right), \\ b &= \frac{\delta}{2} \frac{a^2 / r^2 \sin(2\theta)}{a^4 / r^4 + 1 - 2a^2 / r^2 \cos(2\theta)} \\ \text{and } m &= \frac{\sqrt{3}}{2} R\delta^2.\end{aligned}\tag{5.56}$$

Hereafter, for simplicity, we consider the case $2\mu_0 = \kappa_0$ for which b'/b is infinite meaning that the masses' rotational degree of freedom is completely suppressed.

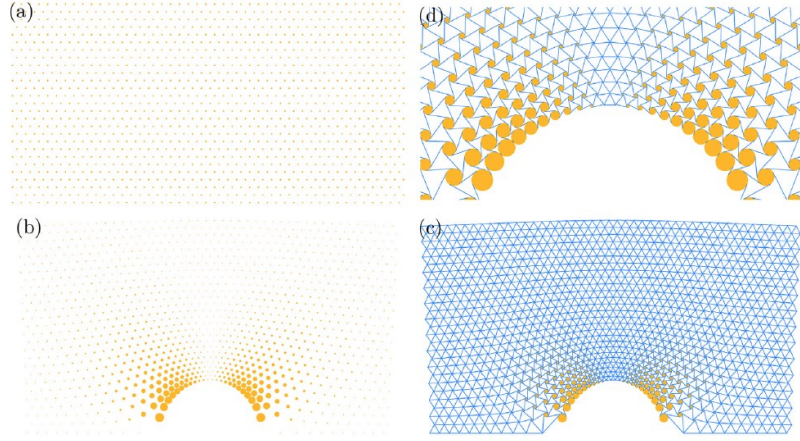


Figure 5.6 Building the lattice cloak in three steps: in step (a), the domain D is discretized; in step (b), the points are transformed by ϕ into a discretization of the cloak's domain $\phi(D)$; at each point, a mass is centered and grounded; in step (c), masses are connected along common internal tangent lines. On (d) is a magnified view of the cloak's geometry near its vertical axis of symmetry. Notice in particular the change in chirality occurring at the crossing of that axis.

Note that the element-wise stability requirement of Eq. (5.41) can be recast into a condition weighing on the elastic parameters of the background medium, namely

$$2\mu_0 \leq \kappa_0 \quad (5.57)$$

Consequently, it is only for such background media that the proposed lattice is element-wise stable. For other backgrounds, other lattices are to be found.

5.4.4 Cloaking simulations

We are now ready to perform cloaking simulations. We set the background medium to a plate made out of an Aluminum-like material with a mass density of 2700 kg/m^3 , a shear modulus of 25 GPa , a bulk modulus of 66.67 GPa and a thickness of 1 mm . Thus, the 2D plane stress equivalent parameters of the background are given by

$$R = 2.7 \text{ kg/m}^2, \quad \kappa_0 = 2\mu_0 = 50 \text{ MPa.m.}$$

The cloak's original dimensions are $2w = 300 \text{ mm}$ by $h = 130 \text{ mm}$ whereas the cloaked void has a radius $= 40 \text{ mm}$. The lattice parameter δ was chosen sufficiently small so as to enable satisfactory cloaking performance but not too small so as to avoid impractical simulation times. A good compromise was found at about $\delta = 6.67 \text{ mm}$ for a loading frequency of $f_0 = 25 \text{ kHz}$. Equivalently, the lattice cloak contains 44 to 45 unit cells in its horizontal span and 27 to 28 unit cells in its vertical span. The simulations presented next were carried at f_0 ; evidently, the cloaking performance is expected to improve for lower frequencies, or eventually in statics. Conversely, the performance is expected to gradually deteriorate for higher frequencies and as the probing wavelength approaches the lattice parameter. Simulations were performed using the finite element

method over the continuum background coupled to a weak formulation of Newton's equations over the discrete domain of the cloak.

Simulation results are presented in two series of plots tabulated on Figs. 5.7 and 5.8. In all cases, a plane wave with a Gaussian profile is generated by forcing the displacements along a black line visible in the simulation domain. Perfectly matched layers are appended to said domain so as to avoid parasitic reflections. The parameters of the simulation were varied so as to thoroughly characterize the cloak's performance: Figs. 5.7 and 5.8 respectively correspond to pressure and shear probing waves. On each Figure, the prefix "p" (resp., "s") identifies the contour plots of the normalized pressure (resp., shear) component whereas the suffixes "0", "30" and "60" correspond, in degrees, to the angle of incidence. The cloaking performance is assessed based on several references: plots in column (a) correspond to the background medium with its free boundary unaltered; in column (b), an uncoated circular void is added; in column (c), the void is coated with a cloak made out of fictitious continuous materials; last, in column (d), the void is coated with the proposed lattice cloak. Accordingly, comparing column (d) to (a) measures the total cloaking error; comparing (d) to (b) measures the amount of scattering suppressed by the lattice cloak; and, comparing (d) to (c) measures a partial cloaking error, the one introduced by the fact that the lattice is not yet operating in the homogenization limit. The remaining cloaking error comes from the fact that the cloak's domain is not infinite and can be visualized by comparing column (c) to (a).

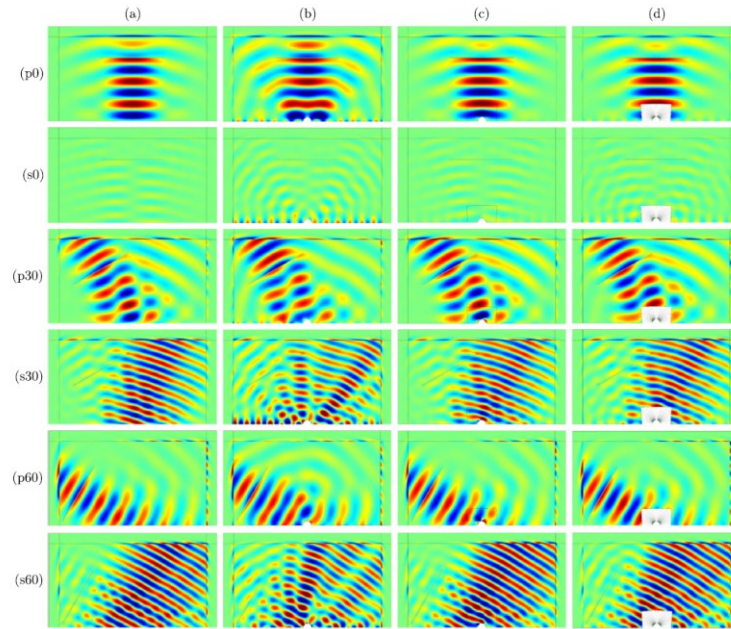


Figure 5.7 Normalized contour plots of the pressure and shear components under pressure incidence. Plots' nomenclature: "p" $\rightarrow$ pressure; "s" $\rightarrow$ shear; "0", "30" and "60" $\rightarrow$ the angle of incidence in degrees; (a) $\rightarrow$ the background medium unaltered; (b) $\rightarrow$ the background with an uncoated void; (c) $\rightarrow$ the background with a void cloaked using fictitious continuous materials; (d) $\rightarrow$ the background with a void cloaked using the proposed lattice.

The results demonstrate overall a satisfactory cloaking performance and consistently so over a range of angles of incidence while in the presence of coupled pressure and shear fields. Notably,

in addition to suppressing pressure and shear bulk scattering, the lattice cloak is capable of mitigating the generation of surface Rayleigh waves. Consider for instance the case of normal shear incidence of Fig. 5.8, plots $(p0, s0) \times (a - d)$. It is seen that the uncoated void causes a quasi-total conversion of the incident shear wave into a scattered surface Rayleigh wave. The lattice cloak, as well as its theoretical continuous limit, manage however to suppress this scattering and arrange for the incident shear wave to be totally and normally reflected, unconverted.

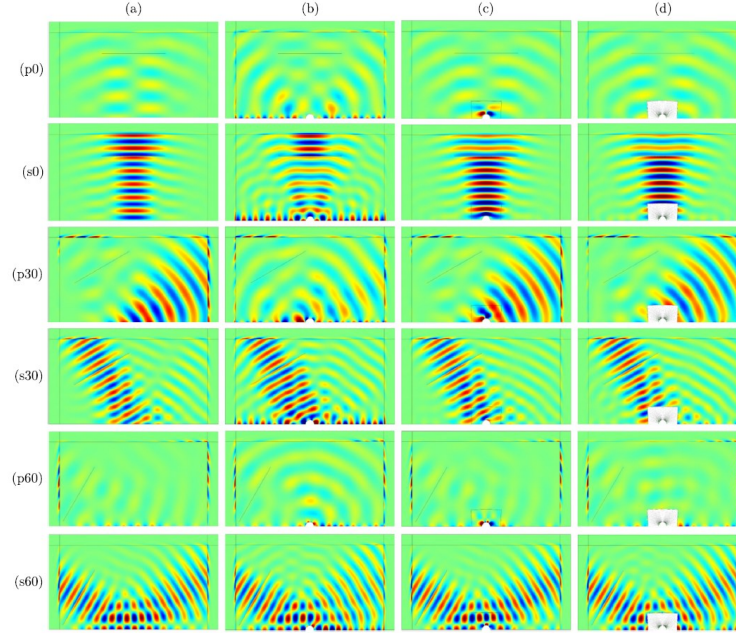


Figure 5.8 Normalized contour plots of the pressure and shear components under shear incidence. Plots' nomenclature: "p" $\rightarrow$ pressure; "s" $\rightarrow$ shear; "0", "30" and "60" $\rightarrow$ the angle of incidence in degrees; (a) $\rightarrow$ the background medium unaltered; (b) $\rightarrow$ the background with an uncoated void; (c) $\rightarrow$ the background with a void cloaked using fictitious continuous materials; (d) $\rightarrow$ the background with a void cloaked using the proposed lattice.

5.5 Conclusion

As stress symmetry breaks down, additional constitutive parameters emerge and modify Hooke's law. The resulting theory was referred to as polar elasticity. When degenerate, polar elastic solids have been proven to emerge in the context of transformation elasticity where the underlying spatial transformation is conformal. In these cases, rules relating the elastic parameters pre- and post-transformation were derived. As an application, a carpet cloak composed of a gradient of isotropic degenerate polar lattice materials was designed and numerically tested.

The adoption of conformal transformations in the present contribution was motivated by the fact that they preserve isotropy. This raises the question of whether other classes of geometric transformations preserve other classes of material symmetries. Future efforts aimed at such classifications, under different transformation gauges, should be beneficial to the understanding of elastic materials whose behavior is form-invariant and therefore to the design and fabrication of elastic invisibility cloaks and other elastic wave manipulation devices.

6. Physical realization of elastic cloaking with a polar material

6.1 Introduction

Polarity turns out to be necessary if shear and hydrostatic stresses are coupled as is typically the case in elasticity (Nassar et al., 2018). The lack of subwavelength microstructures, whose unit cell size is much smaller than operating wavelength, producing an effective polar elastic behavior has blocked the progress in the area of cloaking in solids for decades. Here, we propose a new way to fabricate and test the first of these structures for cloaking applications in two-dimensional elasticity.

It has long been thought that the Cosserats' micropolar solids are suitable for cloaking given that they naturally feature asymmetric stresses (Norris and Shuvalov, 2011). This common wisdom is misleading, however, since what truly distinguishes the Cosserat theory from the standard one by Cauchy, kinetically and kinematically, is the presence of couple stresses on one hand and of microrotations on the other hand instead of the nature of the Cauchy stress tensor. This observation has led us, within the standard theory, to revisit the principle responsible for the symmetry of stresses, namely the local balance of angular momentum. It then becomes clear that achieving cloaking in solids by breaking the stress symmetry relies on finding the structures where the balance of angular momentum is maintained differently. In what follows, we physically realize that by introducing the distribution of large body torques in the form of constraints which limit the local rotations (Nassar et al., 2018). We propose a design of a metamaterial with several layers, one to guide stresses, and the others to impose the constraints. Tests performed under static compression and shear demonstrate satisfactory cloaking performance of a semi-circular void or inclusion. We hope that the present contribution will fill the current gap which separates the transformation elasticity, in both static and dynamic regimes, from the already mature fields of transformation acoustics and optics thus enabling similar capabilities in the control and steering of general stress fields.

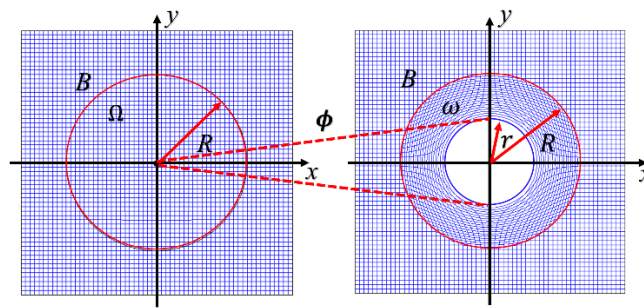


Figure 6.1 Schematic representation of the transformation method: Ω is a fictitious reference medium and ω is the physical medium; transformation ϕ maps a point of the reference into a circular area of radius r hosting the cloaked object while leaving unchanged the background beyond a distance R from the center; the remaining annulus of thickness $R - r$ hosts the cloak.

6.2 Cloaking materials

Consider the two media Ω and ω of Fig. 6.1: the former is a fictitious reference medium and the latter is the physical medium. While they match in shape and composition over the background B , they differ within a disk of radius R corresponding to a cloaked area, which is composed of a cloak of thickness $R - r$ and of a cloaked circular object of radius r . Cloaking is successful when the respective displacement fields $\mathbf{U}$ and $\mathbf{u}$ of Ω and ω are equal over the background B regardless of the applied loading or of the composition of the cloaked object. To derive the elastic properties of the cloak, we ensure that Ω and ω have elastic energy densities $\mathcal{L}(\nabla \mathbf{U}) = C_{ijkl} U_{i,j} U_{k,l} / 2$ and $\ell(\nabla \mathbf{u}) = c_{ijkl} u_{i,j} u_{k,l} / 2$ that are identical up to the change of variables $\mathbf{U}(\mathbf{X}) = \mathbf{u}(\mathbf{x})$ for $\mathbf{x} = \boldsymbol{\phi}(\mathbf{X})$ where $\boldsymbol{\phi}$ maps Ω to ω . That is $\ell(\nabla \mathbf{u}) = \mathcal{L}(\nabla \mathbf{u} \mathbf{F}) / J$ or in terms of elasticity tensors: $c_{ijkl} = J^{-1} F_{jm} F_{ln} C_{imkn}$, $\mathbf{F} = \nabla \boldsymbol{\phi}$, and $J = |\mathbf{F}|$ (Nassar et al., 2019). In what follows, the background B is assumed to be homogeneous and isotropic; the cloaked object is a void; and the cloaking transformation is radially symmetric: $\mathbf{x} = \frac{f(r)}{\|\mathbf{X}\|} \mathbf{X}$ with $f(r) = (\|\mathbf{x}\| - r) / \|\mathbf{x}\|$, where the radius r is measured from the center of the cloaked area.

Therefore, cloaking an object in a background B of elasticity tensor $\mathbf{C}$ requires materials whose elasticity tensor is $\mathbf{c}$; the problem is that no such materials are known. As a matter of fact, standard elasticity tensors satisfy three fundamental properties: (i) they are positive definite, namely if $\mathbf{E}$ is non-skew then $C_{ijkl} E_{ij} E_{kl} > 0$; (ii) they have the major symmetry $C_{ijkl} = C_{klij}$ and (iii) the minor symmetry $C_{ijkl} = C_{ijlk}$. By analyzing the constitutive behavior of the transformed/coating material in the Brun–Guenneau–Movchan (BGM) gauge [1], we found that $\mathbf{c}$ only satisfies condition (ii) of major symmetry (Nassar et al., 2018; Achaoui et al., 2020). We come to conclude that cloaking materials, were they to exist, must accommodate a compliant mechanism, i.e., a zero-energy deformation mode or “zero mode”. In the present case, the zero mode is $\mathbf{E}_{zm} = f(r) \mathbf{e}_1 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_1$ (detailed derivation in Appendix 6B). As for condition (iii), it is a consequence of Cauchy’s second law of motion stating that the skew part of the stress tensor is equal to the externally applied body torque: $t = \epsilon_{3jk} \sigma_{jk} = \sigma_{12} - \sigma_{21}$. Typically, body torque is zero, the stress tensor is symmetric and the elasticity tensor has minor symmetry. Negating the latter, we deduce that cloaking materials are necessarily a polar material, a material that elastically resists rotation, and must be subject to an externally applied body torque $t = \mu \left[\left(\frac{1}{f} - 1 \right) E_{12} - (f - 1) E_{21} \right]$ with $E_{ij} = U_{i,j}$.

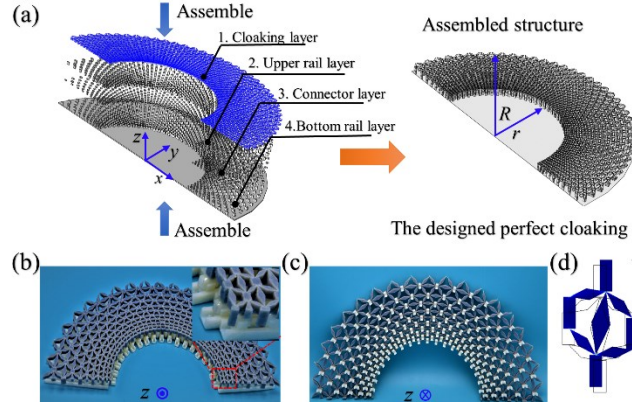


Figure 6.2 The design and fabrication of the polar-mechanical perfect cloak. (a) The cloak consists of 4 different layers, in the z -axis view, from top to bottom, which is arranged as: layer 1-cloaking lattice; layer 2-upper rail; layer 3-connector; and layer 4-bottom rail. The bottom rail layer is fixed in a rigid plate as a ground condition. The size cloaking region and cloaking lattice are $r=95$ mm and $R=200$ mm respectively. Any object can be placed inside of the hollow interior and thereby becomes ‘unfilmable’. For any given pulling direction, the cylindrical core-shell geometry exhibits a symmetry plane normal to the pushing direction and cutting through the middle of the cylinder. Thus, it is sufficient to study the half-cylinder geometry. (b) The top view of the assembled cloak. The layer 1 (cloaking lattice) has been assembled with layer 2, layer 3 and layer 4. (c) The bottom view reveals the details of the polar-mechanical perfect cloak, the blue lattice is layer 1, and the white part is layer 2. (d) Illustration of zero-mode of the unit cell of cloaking layer.

6.3 Model description

In this study, we design and fabricate a new class of cloaking (polar) materials based on the transformation method. The cloaking material is composed of a functionally graded four-layered lattice embedded in an isotropic continuum background as shown in Figs. 6.2a-c. A mechanism to make interconnection among the four layers is proposed: layer 1 (cloaking layer) works as the lattice guiding the stresses; layer 2 (upper rail layer) is the first set of rails whereas, layer 4 (bottom rail layer) is the second set of rails bound to a relatively rigid material as an effective ground; layer 3 (connector layer) connects layer 2 and layer 4 and ensures the transmission of torques between them as shown in Fig. 6.2a. In our design, each lattice site is free to move by sliding over an assembly of two orthogonal rails. By grounding the rails, the sites’ rotations are impeded. In other words, the grounded rails act as a torsional spring: they apply a torque proportional to the site’s angle of rotation. (Detailed in Appendix 6A). As a result, the distributed external torques t are properly applied to break stress symmetry. Given that, the lattice material shown in Figs. 6.2b and 6.2c exhibit an effective elasticity tensor of the same form as $\mathbf{c}$. At each location x , the thickness of the bars and the aspect ratio of the unit cell must be adjusted to fit exactly the targeted tensor $\mathbf{c}$; this is confirmed for each unit cell through a numerical homogenization procedure. Meanwhile, the thickness of the hinges is kept to a minimum to approximately reproduce the behavior of an ideal pin transmitting zero bending moment. Therefore, the zero mode of the lattice shown in Fig. 6.2d can be easily obtained by adjusting the geometrical relations among the bars as tension-compression elements (Detailed in Appendix 6B).

The layers were 3D printed and manually assembled; the material properties are listed in Table 6.1 (Brown, 2000). It is noteworthy that the Poisson's ratio of the background medium is 0.33; in this particular case, the elastic moduli of the rails, connectors, and ground become irrelevant as long as they are significantly higher than those of the background. For other Poisson's ratios, these moduli become important design parameters. To validate our design strategy, numerical simulations of a void cloaked in a two-dimensional plate under either pressure or shear loading are conducted by using a fictitious polar continuum and then by using the proposed lattice-based medium; the comparison showed very good agreement (Details in Appendix 6C). In the simulation, we also demonstrate the displacement fields in the proposed lattice structures with and without body torque and the excellent elastic cloaking performance from the polar material.

Table.6.1 Material properties of the designed cloaking

	<i>Young's modulus (GPa)</i>	<i>Density(kg/m³)</i>	<i>Poisson's ratio</i>
<i>Layer 1</i>	1.50	1170	0.33
<i>Layer (2~4)</i>	2.50	1180	0.33
<i>Host medium</i>	1.25	1270	0.33

6.4 Experiment

To test its cloaking abilities, the proposed design is bonded to a background medium through its stress guiding layer, i.e., layer 1. In the quasi-static characterization experiments shown in Fig. 6.3a, a holder holds the top side of the sample and pull onto the sample via a motorized translational stage (Bückmann et al., 2014). The loading is applied by an MTS system both for tension and shear tests. The loading speed is 0.2 mm/min. To keep the deformations in the linear region, we apply a maximum global strain of about 1%. The holder at the bottom is fixed. In the tested sample, the void is a half circle with a radius of 95 mm and the radial thickness of the lattice cloak is 200 mm attached to the host medium plate with a length of 300 mm and a width of 600 mm. Meanwhile, the backside of layer 4 is fixed to an aluminum plate with length 400 mm and width 250 mm. We optically image the entire structure from the side, that is, in a plane perpendicular to the sample, and film the deformation using a DANTEC dynamics system while varying the loading. The obtained images are analyzed using an autocorrelation software tracking individual points (red dotted line in experimental setup) and delivering a spatial resolution beyond that of the individual camera pixels. This analysis provides us with the displacement directly from the experiment with a good signal-to-noise ratio. In this manner, the background medium is loaded and displacements are measured in two cases pertaining to tension and shear. The measurements are then compared to those obtained by loading two reference samples of the background medium, one with the void and one without it, both in the absence of the cloak.

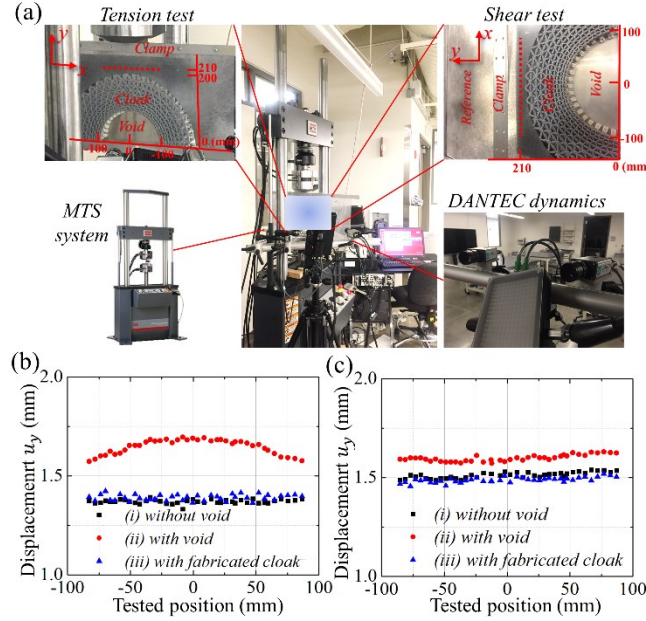


Figure 6.3 (a) The experimental setup of both tension and shear static test of the elastic cloak. The sample consists of lattice cloak, background material, void, and clamp. The tension and shear loading are supplied by the MTS system. The displacement in dotted red line due to the strain is measured by DANTEC dynamics system (The width of the dotted red line is from -100 mm to 100 mm in the x -axis and the distance from the measured line to the bottom of the plate is 210mm in the y -axis); (b) Measured displacement fields with and without the cloaking device under a static pressure field applied to the top boundary and fixed boundary conditions from below: (i) without void (black squares), (ii) with void (red dots), (iii) with fabricated cloak (blue triangles). (c) Measured displacement fields with and without the cloaking device under a static shear field applied to the center of the plate and fixed boundary conditions on the other side: (i) without void (black squares), (ii) with void (red dots), (iii) with fabricated cloak (blue triangles).

The measured displacements are depicted in Figs. 6.3b and 6.3c for the tension test and the shear test, respectively. Both plots show satisfactory cloaking performance. In particular, the lattice cloak succeeds in suppressing the localization expected and observed in the vicinity of the uncloaked void and restores the uniform displacement profile observed in the reference sample. In other words, the cloak reinforces the void without disturbing the fields in the background. Comparison of numerical (Appendix 6C) and experimental results is satisfactory as well: the minor differences observed could be attributed to the variability of the material properties produced by the 3D printing of both the lattice cloak and the host medium.

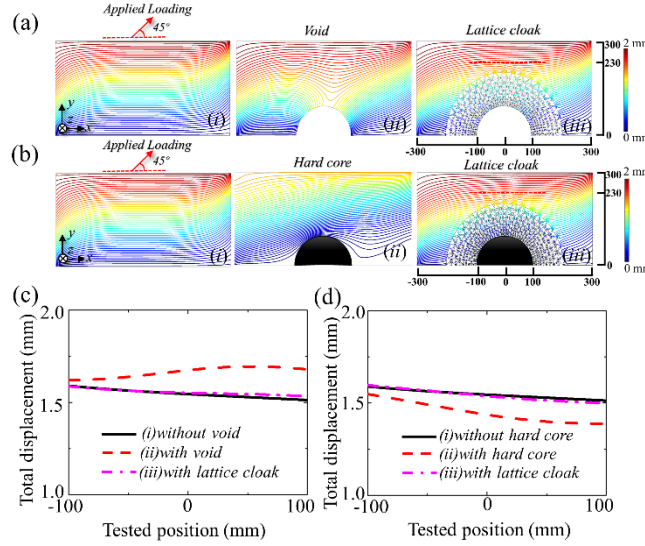


Figure 6.4 The total displacement field under a general mechanical loading (red arrow) at the top boundary of the test plate has been shown: (a) (i) without void, (ii) with void, (iii) with lattice cloak; (b) (i) without hard core, (ii) with hard core, (iii) with lattice cloak. The total displacement at dotted line (see iv, from -100 mm to 100 mm in the x-axis, at 230 mm in the y-axis) for plate with void and hard core have been shown in (c) and (d) respectively.

Coated inclusions that do not disturb background fields obtained under loading at infinity are known as “neutral inclusions” (Chen et al., 2019). Typically, however, the properties of the coating need to be changed in function of the loading and of the core’s properties. By contrast, the proposed cloak is universal. On one hand, the provided experimental results, along with the isotropy and linearity assumptions, show that the tested loading can be combined and rotated so as to guarantee similar cloaking performance for any static loading. As an example, a mechanical cloaking is simulated in the presence of a localized force applied at a 45° angle to the top boundary with satisfactory results (Fig. 6.4a). On the other hand, based on the transformation method, it is possible to predict similar cloaking performance for any core properties (void, elastic or rigid). Indeed, the void’s region is equivalent to a vanishingly small region, ideally a point, in the original domain. Therefore, filling the void with different material amounts to changing the elastic properties of a small region of the original medium; such a perturbation has negligible effects on any static equilibrium. Here too, numerical simulations validate this hypothesis: cloaking a hard core (aluminum) instead of a void, using the same lattice, produces the same fields in the background as those present in the absence of both inclusions (Figs. 6.4b-6.4d).

6.5 Conclusions

In this study, we report on the design and fabrication of a static cloak shielding against combined pressure and shear stress fields using lattice-based polar materials that exhibit asymmetric stresses. We experimentally and numerically investigate the characteristics of the proposed cloak and find very good cloaking performance under both tension and shear loadings. The cloak is further universal in that it is able to hide an inclusion of arbitrary composition from external loadings of arbitrary orientation. Our results here, restricted to statics, will help make cloaking against stress

waves and dynamic loadings experimentally accessible in the near future using similar polar materials-based architectures.

Appendix 6A Geometry of the polar-mechanical perfect cloak

The lattice cloak is 3D printed by Stratasys Objet Connex 260 using two base materials *Vero White Plus* and *Tango Black Plus*. The 3D structural models in Mph format were built in COMSOL Multiphysics and then converted into STL file for printing. For the top lattice layer, due to the fabrication limitation, as shown in Fig. 6A1, halves were printed separately using RGD8530 (a combination of two base materials) and glued together with the same materials under UV curing, while the components of rail layers and connector layer were printed with pure Vero White Plus individually. All the supporting material was cleaned carefully by waterjet.

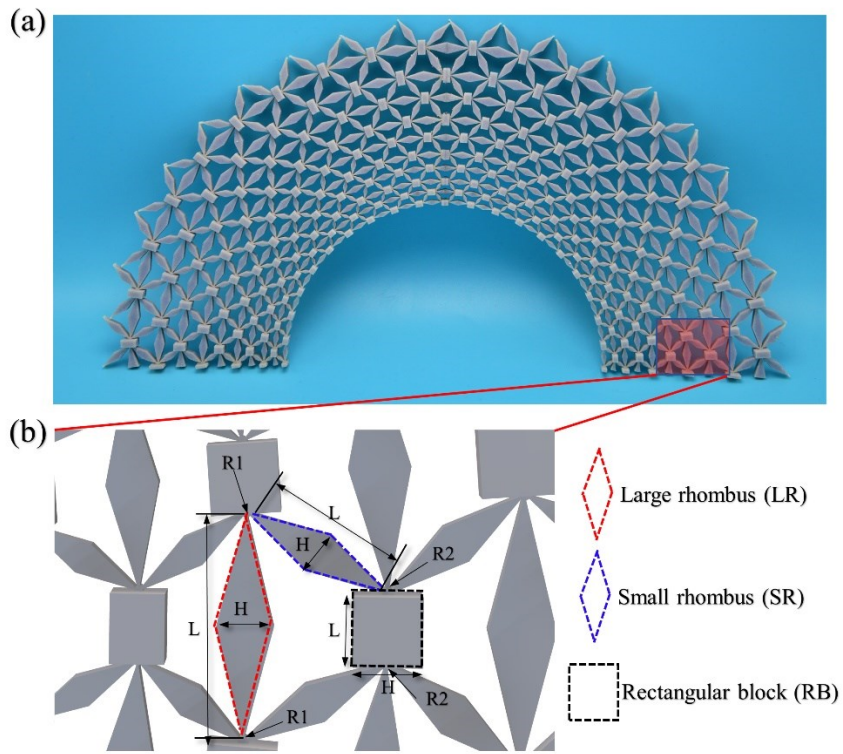


Figure 6A1 (a) The glued top lattice layer composed of 200 unit cells. (b) An enlarged drawing of the portion framed in (a). The red dashed rhombus named as the large rhombus (LR), the blue dashed rhombus named as small rhombus (SR), and the black rectangle named as rectangular block. The joint at the top band bottom of LR is semicircle with radius $R1$ and the joint at left and right of SR is also semicircle with radius $R1$ and $R2$, respectively. More geometry details are presented in Table 6A1.

Table 6A1 Geometry parameters of lattice cloak in each layer.

Title	L (mm)	H (mm)	T (mm)	R1 (mm)	R2 (mm)
LR1	24.9	6.2	10	0.5	0.5
LR2	22.4	5.6	10	0.47	0.47

LR3	20.4	5.1	10	0.45	0.45
LR4	18.5	4.6	10	0.41	0.41
LR5	16.9	4.2	10	0.39	0.39
LR6	15.4	3.8	10	0.36	0.36
LR7	14.4	3.5	10	0.33	0.33
LR8	12.9	3.2	10	0.30	0.30
LR9	11.9	3.0	10	0.27	0.27
LR10	11.0	2.8	10	0.25	0.25
SR1	19.5	4.9	10	0.5	0.5
SR2	18.8	4.7	10	0.47	0.5
SR3	15.4	3.9	10	0.45	0.47
SR4	14.8	3.7	10	0.41	0.45
SR5	12.1	3.0	10	0.39	0.41
SR6	11.6	2.9	10	0.36	0.39
SR7	9.5	2.4	10	0.33	0.36
SR8	9.2	2.3	10	0.30	0.33
SR9	7.5	1.9	10	0.27	0.30
SR10	7.2	1.8	10	0.25	0.27
SM1	3.9	8.0	14	0.5	0.5
SM2	3.9	8.0	14	0.47	0.47
SM3	3.9	6.3	14	0.45	0.45
SM4	3.9	6.3	14	0.41	0.41
SM5	3.9	5.0	14	0.39	0.39
SM6	3.9	5.0	14	0.36	0.36
SM7	3.9	3.9	14	0.33	0.33
SM8	3.9	3.9	14	0.30	0.30
SM9	3.9	3.1	14	0.27	0.27
SM10	3.9	3.1	14	0.25	0.25

In each unit cell, the guide system consists of three parts as shown in Fig. 6A2a. The bottom rail is fixed on the ground, and the top rail is fixed with the top lattice layer. By means of connecting the top and bottom guiders by a connector, the top lattice layer can only move along x -direction and y -direction while rotation is prohibited. Moreover, lubricating oil is filled between the rails to reduce friction. The geometry details of the guide system are presented in Figs. 6A2b-d.

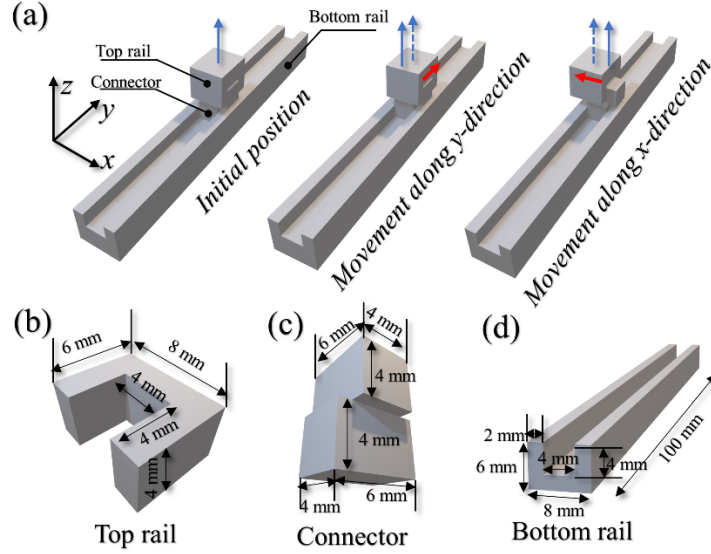


Figure 6A2 (a) The control mechanism of the guide system. From the left to right, the initial position, movement along y -direction and movement along x -direction respectively. Unit cell geometry of connects layer. The geometry of the guide system: from left to right: (b) the top rail; (c) the middle connector; (d) the bottom rail.

Appendix 6B Transformation method and homogenization of the polar elastic metamaterial

The main purpose of the paper is to suggest a lattice cloak for the 2D static cloaking test. By using the transformation of elasticity equations in the BGM gauge and analyzing the nature of the elasticity tensors that emerge as an outcome, a degenerate polar lattice whose behavior matches the elasticity tensor of 2D ideal cloak can be reached. Therein, let's consider the constitutive behavior of the harmonic elastodynamic wave in an elastic domain $\{X\}$, and let

$$\nabla \cdot \boldsymbol{\sigma}' = -\omega^2 \rho' \mathbf{U} \text{ and } \boldsymbol{\sigma}' = \mathbf{C} : \nabla \mathbf{U} \quad (6B1)$$

where ω , ρ' and $\mathbf{C}$ are the frequency, mass density, and elasticity tensor in the original domain $\{X\}$ respectively. Thus, the Lagrangian density of the elastic domain $\{X\}$ can be read as

$$\mathcal{L}(\nabla \mathbf{U}, \dot{\mathbf{U}}) = \frac{\nabla \mathbf{U} : \mathbf{C} : \nabla \mathbf{U} - \dot{\mathbf{U}} \cdot \rho' \dot{\mathbf{U}}}{2} \quad (6B2)$$

Under the mapping $\{X\} \rightarrow \{\mathbf{x} = \phi(X)\}$ and $\{\mathbf{U}(X) \rightarrow \mathbf{u}(\mathbf{x})\}$, the Lagrangian density reads as

$$\ell(\nabla \mathbf{u}, \dot{\mathbf{u}}) = \frac{\nabla \mathbf{u} : \mathbf{c} : \nabla \mathbf{u} - \dot{\mathbf{u}} \cdot \rho \dot{\mathbf{u}}}{2} = \frac{1}{J} \mathcal{L}(\nabla \mathbf{u} \mathbf{F}, \dot{\mathbf{u}}) \quad (6B3)$$

where $\mathbf{F} = \frac{\partial \mathbf{x}}{\partial X}$ is the transformation gradient and J is the corresponding determinant. The transformed elasticity tensor $\mathbf{c}$ and mass density ρ can be expressed as

$$c_{ijkl} = J^{-1} F_{jm} F_{ln} C_{imkn}, \rho = \rho' / J \quad (6B4)$$

Here, when $\mathbf{C}$ has the minor symmetry and $\mathbf{c}$ does not, that is, $C_{ijkl} = C_{jikl} = C_{ijlk}$ and $c_{ijkl} \neq c_{jikl} \neq c_{ijlk}$, the stress $\boldsymbol{\sigma}$ in the domain $\{\mathbf{x}\}$ can be expressed as

$$\boldsymbol{\sigma} = \mathbf{c} : \mathbf{e}, \mathbf{e} = \nabla \mathbf{u} \text{ with } \sigma_{ij} \neq \sigma_{ji} \quad (6B5)$$

In our cloaking design, as we motioned in paper and shown in Fig. 1, the coating region $\{r < ||\mathbf{x}|| < R\}$ is fulfilling the function of a cloak and its elasticity property is determined by equation (10B4). Following Brun et al. (2009), that relation reads

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \\ \sigma_{21} \end{bmatrix} = \begin{bmatrix} (2\mu + \lambda)f & \lambda & 0 & 0 \\ \lambda & (2\mu + \lambda)/f & 0 & 0 \\ 0 & 0 & \mu/f & \mu \\ 0 & 0 & \mu & \mu f \end{bmatrix} \begin{bmatrix} e_{11} \\ e_{22} \\ e_{12} \\ e_{21} \end{bmatrix}, f = \frac{||\mathbf{x}|| - r}{||\mathbf{x}||} \quad (6B6)$$

where the matrix represents the elasticity tensor in Voigt form. The null-space of the matrix is one-dimensional with the basis $(0, 0, f, -1)^T$, which is corresponding to the zero-mode $\mathbf{E}_{zm} = f \mathbf{e}_1 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_1$.

In the design of polar medium, the new zero-mode is necessary because the infinitesimal rotation, which is the zero-mode of Cauchy's media, is constrained by the body torque. The required zero-mode can be realized by adjusting the geometrical relations among the bars. Figure 10B1 shows the unit cell of the lattice. Under the strain $\mathbf{E}_{zm}$, the displacement of the point at $\mathbf{m} = |\mathbf{m}| \cos(\beta) \mathbf{e}_1 + |\mathbf{m}| \sin(\beta) \mathbf{e}_2$ is $\mathbf{u} = \mathbf{E}_{zm} \cdot \mathbf{m} = f |\mathbf{m}| \sin(\beta) \mathbf{e}_1 - |\mathbf{m}| \cos(\beta) \mathbf{e}_2$. The upper-right end of the top bar, which is represented by $\mathbf{n} = |\mathbf{n}| \cos(\alpha) \mathbf{e}_1 + |\mathbf{n}| \sin(\alpha) \mathbf{e}_2$, undergoes the same displacement as the point at $\mathbf{m}$, since the rotation of the block is locked. The zero-mode motion only occurs when the top bar bears no extension or compression, which means $\mathbf{n} \cdot \mathbf{u} = f |\mathbf{m}| |\mathbf{n}| \cos(\alpha) \sin(\beta) - |\mathbf{m}| |\mathbf{n}| \sin(\alpha) \cos(\beta) = 0$. Therefore, the required zero-mode can be achieved by keeping $f = \tan(\alpha) / \tan(\beta)$.

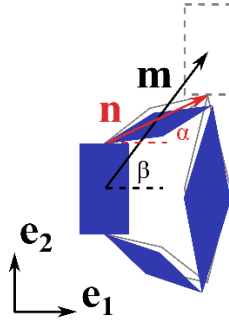


Figure 6B1 Determination of the zero-mode of the cloak lattice

In the numerical simulations of the paper, the plane stress hypothesis was adopted. The mass densities of the materials are not important here since the cloaking test is in the static case. In particular, a void of radius $r = 95$ mm is to be cloaked, with a coating of radius $R = 200$ mm, embedded in a rectangular plate of 1 mm thickness and of in-plane Lamé parameters $\mu = \lambda = 0.41 \times 10^6 \text{ Pa} \cdot \text{m}$ and mass density. The coating layer thickness is 1 mm with Young's modulus $E = 30 \times 10^6 \text{ Pa} \cdot \text{m}$. And lattice cloak consists of 10 layers and 40 sectors and has a theoretical inner radius of $r = 80$ mm. Due to the polar assumption and the control mechanism in Fig. 6A2, the homogenized property of the proposed lattice can be described using the following equation:

$$\sigma_{ij} = c_{ijkl} e_{kl} \quad (6B7)$$

where σ_{ij} is the macroscopic or homogenized stress components and $e_{ij} = \frac{\partial u_i}{\partial x_j}$ is the macroscopic or homogenized strain components, u_i is the displacement along the i -th direction and c_{ijkl} are the components of the homogenized elasticity tensor.

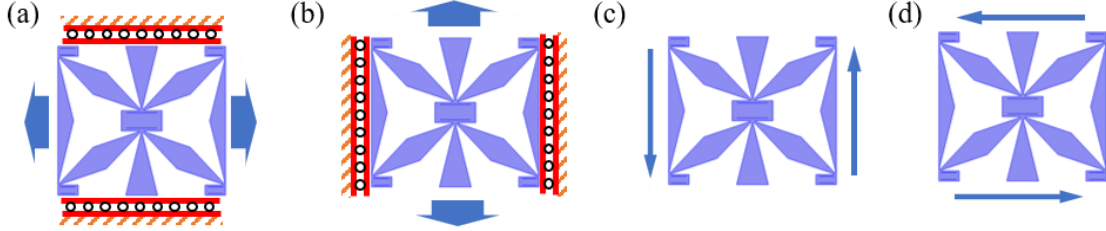


Figure 6B2 Homogenization of the polar elastic metamaterials. Uniform displacement fields applied (a) for extension along horizontal direction; (b) for extension along vertical direction; (c) for shear along vertical direction; (d) for shear along the horizontal direction.

In order to get the homogenized elasticity tensor, a unit cell of the polar elastic metamaterial is loaded with respect to different uniform displacement fields as shown in Fig. 6B2 via COMSOL-Multiphysics. In particular, as illustrated by Fig. 6B2a, a uniform horizontal displacement field is at the left and right boundaries of the unit cell while keeping the top and bottom boundary fixed vertically. The corresponding reaction forces at the left and right boundaries are collected to obtain the components of the elasticity effective tensor with respect to x -direction. Similarly, other components are determined by applying the uniform displacement boundaries illustrated by Fig. 6B2b-d. As a comparison, the theoretical values and the numerical values of the homogenized effective properties are plotted in Fig. 6B35, in which a very good agreement is observed. Note that, as we mentioned in equation S6, due to the $c_{iijj} = c_{ijji}$, ($i \neq j$), only one of these equal parameters, c_{1122} , is shown in Fig. 6B3.

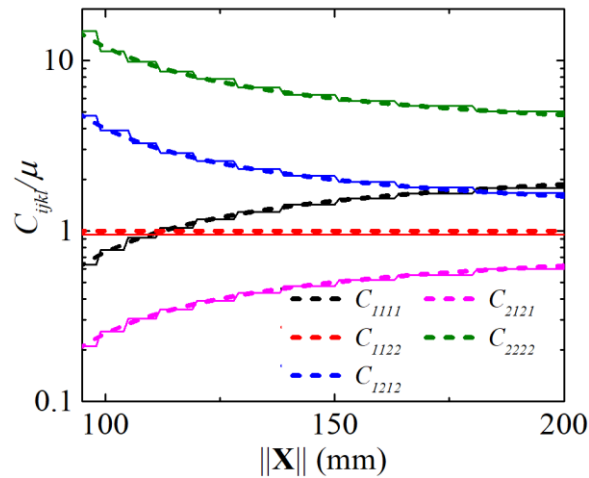


Figure 6B3 Homogenization of polar elastic metamaterials. The dotted lines and the solid lines show the theoretical homogenized properties and the numerical homogenized properties respectively.

Appendix 6C Simulation of the polar-mechanical perfect cloak

We performed finite element analysis (FEA) using the COMSOL Multiphysics to simulate the performance of different cloaks under uniform tension and shear loading, which can be further adopted to design and optimize the proposed polar-mechanical perfect cloak.

To be specific, for the tension loading case, plates without void, with void, with the ideal cloak and with the proposed polar-mechanical perfect cloak are simulated in parallel (c.f. Fig. 10C1a). Similar to the experimental setup, for the plates, all the displacement components of the bottom and the components in the horizontal plane of the top are fixed, while uniform vertical displacement along y -direction is applied on the top and the left and right boundaries are free. The displacement u_y of two horizontal lines dashed in red at distances of 210 mm and 230 mm from the bottom boundary, i.e., shown in Fig. 10C1(iv), is considered. The corresponding simulated results are presented in Figs. 10C1b and c respectively. Obviously, the red dashed lines in Figs. 10C1 b and c corresponding to results of the plate with the void is much different from the others. Although not as perfect cloaking, the deviation between the result of the proposed design and the ideal response is not very large. In addition, due to the local effect, the deviation at the distance of 210 mm is larger than that at the distance of 230 mm.

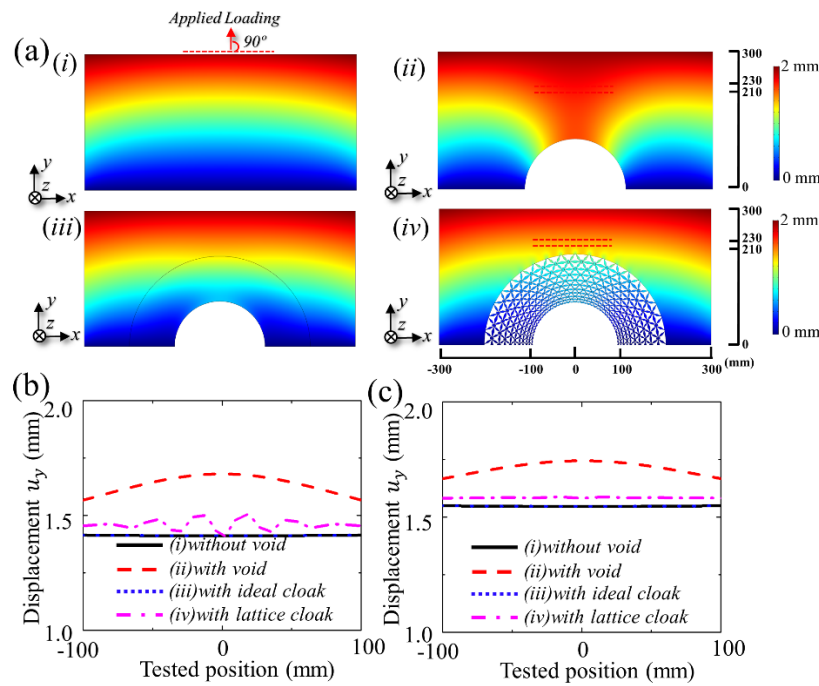


Figure 6C1 The simulated displacement magnitude field under a uniform tension loading (red arrow at (i)) at the top boundary of the test plates: (a) (i) without void, (ii) with void, (iii) with the ideal cloak, (iv) with the proposed cloak. The displacement field along y -direction of the dotted lines of those plates (see (iv), from -100 mm to 100 mm along the x -axis) at distances of 210 mm and 230 mm from the bottom shown in (b) and (c) respectively.

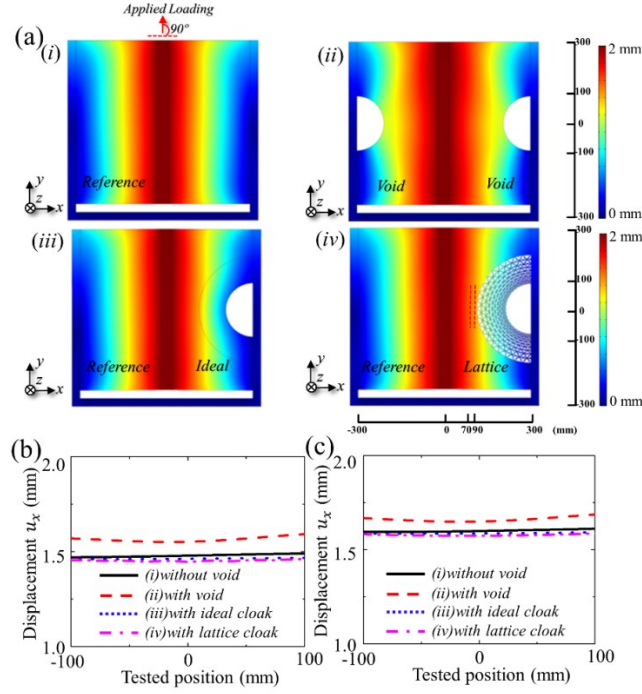


Figure 6C2 The simulated displacement magnitude field under a shear load (red arrow at (i)) of the plates: (a) (i) without void, (ii) with void, (iii) with the ideal cloak, (iv) with the proposed cloak. The displacement field along y -direction at the dotted red lines shown in (b) and (c) respectively.

Different with the tension case, a holder frame carrying both the reference plate and the objective sample is modeled as shown in Fig. 6C2a. The displacement at lines at distances of 210 mm and 230 mm from the right boundary of the sample is studied. The corresponding displacement field along y -direction of all plates is extracted and shown in Figs. 6C2b and c respectively. The response of the plate with the proposed cloak shows a good agreement with the result of the plate without defect. The FEM simulation under both tension and shear loading show satisfactory performance qualitatively with experimental results (See paper Fig. 6.3) Meanwhile, without body torque transmitted in the proposed lattice, the cloak cannot block the void any more as shown in Fig. 6C3.

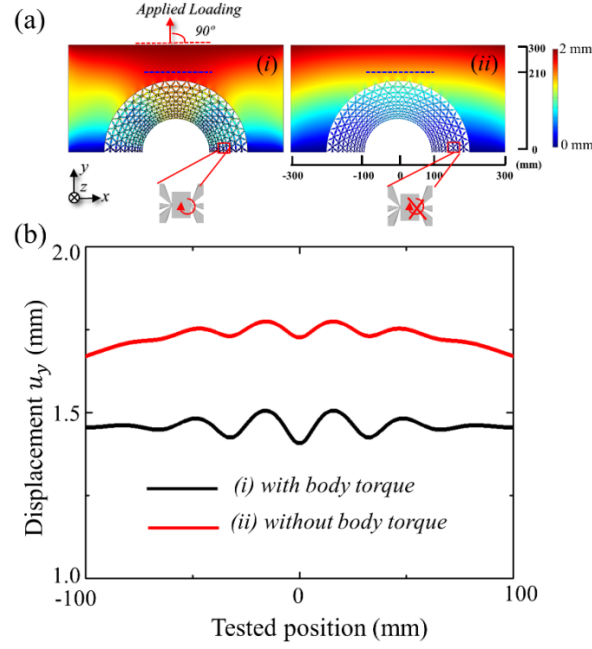


Figure 6C3 (a) The simulated displacement field along y-direction under a tensile load at the top boundary of plates containing the proposed clock: (i) with body torque transmitted, (ii) without body torque transmitted, respectively. (b) The displacement field along y-direction at the dotted blue lines of the two cases in Fig. 6C2a.

7. Polar metamaterials: a new outlook on resonance for cloaking applications

7.1 Introduction

The peculiar behavior of locally resonant metamaterials has long been understood within Cauchy's model of elasticity albeit with constitutive properties that are frequency dependent and possibly negative (Ma and Sheng, 2016). However, as we show here, resonance has other profound effects on wave motion that escape the paradigm of frequency-dependent effective parameters. In particular, we demonstrate how rotational resonance modifies not only the values of the effective moduli but the very form of the elasticity tensor as well.

The conventional form of the elasticity tensor is strongly constrained by a number of symmetries (Landau and Lifshitz, 1970). The minor index symmetries ($a_{ijkl} = a_{jikl} = a_{ijlk}$) are equivalent to the symmetry of Cauchy's stress tensor; they stem from the balance of angular momentum and reduce the number of independent coefficients in a two-dimensional elasticity tensor $\mathbf{a} = a_{ijkl}$ from 16 to 9. The major index symmetry ($a_{ijkl} = a_{klij}$) derives from the existence of a strain energy and further reduces that number to 6. Then isotropy, understood as invariance under proper rotations, specifies the form of $\mathbf{a}$ into

$$a_{ijkl} = \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + (\kappa - \mu)\delta_{ij}\delta_{kl}, \quad (7.1)$$

leaving only two independent coefficients, the bulk and shear moduli κ and μ . Notably, without asking for it, $\mathbf{a}$ has mirror symmetry (Forte and Vianello, 1996). Rotational resonance shakes this

state of affairs: it generates torques strong enough to modify the balance of angular momentum thus dissolving stress symmetry and, with it, mirror symmetry. The elasticity tensor then gains two resonance-induced degrees of freedom, one quantifying chiral effects due to the loss of mirror symmetry and one quantifying polar effects due to the loss of stress symmetry.

Accordingly, resonance effectively widens the design space of elastic materials to include polar metamaterials with dynamically broken stress symmetry. In the material, resonance-induced chirality and polarity are distinct from those usually observed due to size-dependent effects in micropolar elasticity. Thus, the polar metamaterials introduced here are permitted to tune the shape of the elastic behavior in the deep subwavelength limit. As an application, polar metamaterials in the form of resonant hexachiral lattices are shown to be suitable for the realization of elastic cloaks. We numerically demonstrate how, at the resonance frequency, a spatial gradient of these lattices can guide coupled pressure and shear waves around a cloaked region with minimal scattering. The theory underpinning the proposed cloak has been described in earlier work (Nassar et al., 2019) and substantiated leveraging grounded torsional springs for the production of the necessary torques (Nassar et al., 2018). Here, thanks to rotational resonance, we bypass the need for active intervention, through grounding or other infrastructure, and provide the first "no-strings-attached" portable and passive metamaterial-based solution, which could be easily implemented in practical applications.

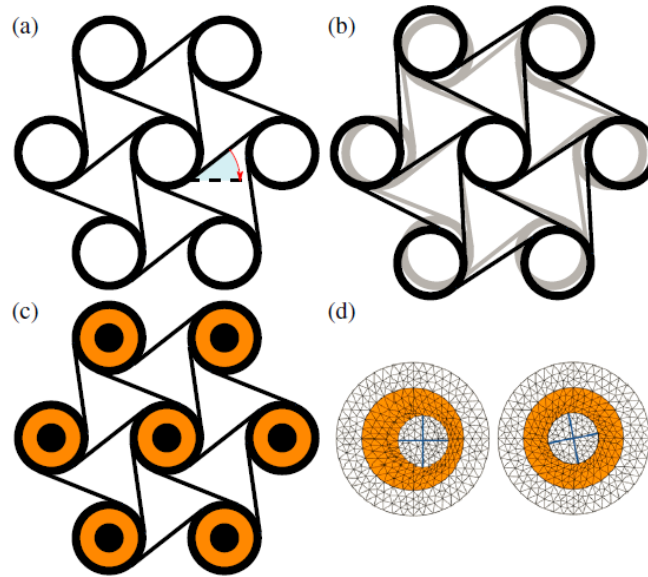


Figure 7.1 Hexachiral lattices. (a) Nonresonant lattice: black circles are rigid masses and black segments are massless elastic rods; contact points are perfect hinges. The highlighted angle is the twisting angle γ . (b) Isotropic zero mode: the lattice expands (black) from its reference configuration (gray) without stretching any rods. (c) Polar metamaterial: black disks are embedded resonators; colored coating is elastic and massless. (d) Resonance modes under imposed outer displacement: translational (ω_t) to the left and rotational (ω_r) to the right.

7.2 Model description

Consider the chiral lattice sketched on Fig. 7.1a for starters. Each of its unit cells hosts a rigid mass capable of translation and rotation in the plane and connected through hinges to a set of massless elastic rods. We are interested in the behavior of the lattice in the homogenization limit; i.e., when the unit cell size is infinitely small compared to the typical propagated wavelength. In that regime, the lattice displays an elastic behavior with elasticity tensor $\mathbf{a}$ given by

$$a_{ijkl} = \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} - \delta_{ij} \delta_{kl}), \quad (7.2)$$

where

$$\mu = k\sqrt{3} / 4 \quad (7.3)$$

is shear modulus, k being the spring constant of the rods. The bulk modulus on the other hand is zero because the lattice can expand equally in all directions without stretching any rods [Fig. 7.1b]. More relevant to our purposes is the following conundrum: while the lattice is chiral for any twisting angle $\gamma \neq 0$, tensor $\mathbf{a}$ is systematically mirror symmetric. This is due to the conspiracy of space dimensionality, the index symmetries and isotropy, as portrayed in the introduction. Together, these constraints leave no room in $\mathbf{a}$ to express a chiral behavior.

Next, let us embed within each mass a rigid core coated with a soft light material [Fig. 7.1c]. The obtained mass-in-mass resonator has two resonance frequencies ω_t and ω_r corresponding to translational and rotational oscillation modes and depicted in Fig. 7.1d. The translational motion of the core modifies the balance of linear momentum. It generates a frequency-dependent body force which can be absorbed into the expression of effective mass density

$$\rho = \frac{M}{A} + \frac{m/A}{1 - \omega^2 / \omega_t^2}, \quad (7.4)$$

where M and m are the outer and inner masses, respectively, A is the unit cell area, and ω is angular frequency. Otherwise, the translational motion of the inner mass has no influence on $\mathbf{a}$. By contrast, the rotational motion of the inner mass modifies the balance of angular momentum. As mass M rotates through an angle Ψ , it perceives a restoring torque equal to $-\eta\Psi$ where

$$\eta = -\omega^2 \frac{i}{1 - \omega^2 / \omega_r^2} \quad (7.5)$$

is an effective torsion stiffness and i is the moment of inertia of the inner mass. It is important to remain consistent with the premises of homogenization here by maintaining that ω is infinitely small when compared to the typical cutoff frequency $\omega_c \propto \sqrt{k/M}$. Accordingly, the torques $-\eta\Psi \propto \omega^2 i$ can be neglected except near the rotational resonance where the small denominator $1 - \omega^2 / \omega_r^2$ conveniently tames the small numerator $\omega^2 i$. This is assumed henceforth; specifically, we work in the asymptotic regime $\omega = \omega_r + \delta\omega \ll \omega_c$ where $\delta\omega / \omega_c \propto \omega_r^3 / \omega_c^3$ and the torsional stiffness

$$\eta = \frac{\omega_r^3}{2\delta\omega} i = \frac{\omega_r^3 / \omega_c^3}{2\delta\omega / \omega_c} \omega_c^2 i \quad (7.6)$$

is of leading order, which depends little on the geometry of the resonators. With that in mind, and by Cauchy's second law of motion, the stress tensor σ satisfies

$$\sigma_{12} - \sigma_{21} = -\frac{\eta\Psi}{A} \quad (7.7)$$

that is: the asymmetric part of stress is equal to the externally applied torque density. What is more is that $\mathbf{a}$ then no longer needs to enforce the minor symmetries, meaning that, while remaining isotropic, it can in principle express chirality.

Direct calculations confirm these predictions. We find

$$a_{ijkl} = \mu \left(\frac{\eta - 12b^2k}{\eta + 12b^2k} R_{ji} R_{lk} + R_{li} R_{jk} + \delta_{ik} \delta_{jl} \right) \quad (7.8)$$

where b is the radius of M and $\mathbf{R}$ is the plane rotation of angle γ . Since all plane rotations commute, it is straightforward to check that $\mathbf{a}$ is isotropic. Furthermore,

$$a_{1112} = -a_{1121} = \frac{2\eta\mu}{\eta + 12b^2k} \sin \gamma \cos \gamma \neq 0 \quad (7.9)$$

shows at once that $\mathbf{a}$ is both chiral and without minor symmetries so long as $\gamma \neq 0$. Rather than inspect $\mathbf{a}$ however, it is far more insightful to write the stress-strain relationship, say decomposed into three parts, deviatoric, hydrostatic, and skew, as in

$$\sigma = 2\mu\tilde{\epsilon}, \quad p = -\kappa t - \beta\theta, \quad c = -\beta t - \alpha\theta \quad (7.10)$$

see Figs. 7.2a-7.2c. Above, (i) μ is a shear modulus that transforms a deviatoric strain $\tilde{\epsilon}$ into a deviatoric stress $\tilde{\sigma}$, (ii) κ is a bulk modulus that transforms an infinitesimal change in area t into a hydrostatic pressure p , (iii) α is a polarity modulus that transforms an infinitesimal rotation θ into a body torque c , and (iv) β is a chiral coupling that transforms a change in area into a body torque and an infinitesimal rotation into a hydrostatic pressure. In fact, the polar metamaterial we present here is an extension of standard Cauchy elasticity that encompasses chiral and polar effects without any extra degrees of freedom or extra measures of stress.

While μ remains as it was, the other constitutive parameters have the expressions

$$\kappa = \kappa_o \cos^2 \gamma, \quad \alpha = 4\kappa_o \sin^2 \gamma, \quad \beta = -2\kappa_o \cos \gamma \sin \gamma, \quad (7.11)$$

with

$$\kappa_o = \frac{\eta}{\eta + 12b^2k} \frac{k\sqrt{3}}{2}. \quad (7.12)$$

From Eqs. (7.11) and (7.12), it can be concluded that (i) the polar metamaterials feature size-independent chirality and polarity, and (ii) stress goes back to being symmetric under long wavelength limit by setting the dynamic parameter η to zero.

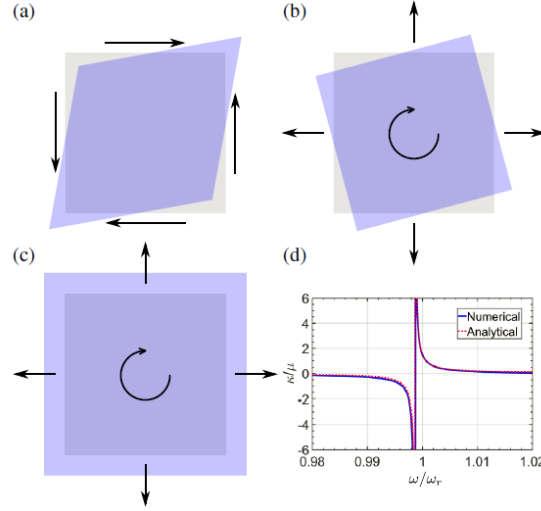


Figure 7.2 Effective constitutive law (10): response under (a) a pure shear, (b) a rigid rotation, and (c) a pure dilation; deformations are illustrated on the geometry; arrows represent forces and torques. (d) Typical profile of the effective dynamic bulk modulus in the vicinity of ω_r .

Variations on hexachiral lattices have been previously modeled as metamaterials with singly or doubly negative properties (Liu et al., 2011a; Zhu et al., 2014), as micropolar media (Liu et al., 2011b), or as strain-gradient media (Frenzel et al., 2017; Auffray et al., 2010) based on different homogenization schemes. The present asymptotic analysis shows that the proposed resonant hexachiral lattices are best understood, in the immediate vicinity of rotational resonance, as polar metamaterials, i.e., metamaterials with broken stress symmetry. The analysis is validated numerically by loading a single unit cell and plotting its response against frequency. The results for κ are shown in Fig. 7.2d; the other parameters have identical profiles up to a scaling factor. It is noteworthy that, near resonance, κ changes sign suggesting that, for suitably chosen ω_t and ω_r , a negative refraction band can be obtained. While this is true, it holds not because κ becomes negative per se but because κ_o does. Indeed, the phase velocities squared are equal to μ/ρ and an analysis based on dispersion bands alone would give access to μ , κ_o , and ρ , at best, but cannot reveal any of the access to μ , κ_o , and ρ , at best, but cannot reveal any of the broken symmetries. In particular, the fact that a is isotropic and has broken minor symmetries makes the underlying and has broken minor symmetries makes the underlying using conformal transformations (Nassar et al., 2019). This is pursued next.

A conformal transformation $\mathbf{x} = \phi(\mathbf{X})$ preserves angles (Rosi and Auffray, 2016; Norris et al., 2012). Its gradient is therefore equal to a dilation of factor λ composed with a rotation $\mathbf{R}$. Starting with an original medium $\{\mathbf{X}\}$ of shear modulus μ_o , bulk modulus κ_o and mass density ρ_o , composing a stretch

$$\lambda = \sqrt{\rho_o / \rho}, \quad (7.13)$$

with a rotation of angle γ , we retrieve an image medium $\{\mathbf{x}\}$ satisfying equations (7.4) and (7.10) exactly. Then, by the principles of transformation-based cloaking, domains $\{\mathbf{X}\}$ and $\{\mathbf{x}\}$ will be indistinguishable (Xu and Chen, 2015; Leonhardt, 2006). For our demonstration, the original domain is a free half space. The adopted transformation then creates a "cave" along the free boundary where things can hide without being detected by probing waves [Figs. 7.3a and 7.3b]. Clearly, the transformation is not uniform. As $\lambda = \lambda(\mathbf{x})$ and $\gamma = \gamma(\mathbf{x})$ change from one point $\mathbf{x}$ to another, the underlying lattice will need to be graded in space [Fig. 7.3c]. This can be done once a small uniform discretization step δ of the original domain has been selected. Then, the lattice parameter near point $\mathbf{x}$ is simply $\delta\lambda(\mathbf{x})$. This allows to calculate the outer radius $b(\mathbf{x}) = \delta\lambda(\mathbf{x})\sin[\gamma(\mathbf{x})]/2$ of mass M and the unit cell area $A(\mathbf{x}) = \delta^2\lambda^2(\mathbf{x})\sqrt{3}/2$. The inner radius of M is arbitrarily set to $2b/3$ and the radius of the core is set to $b/3$ so that the coat's thickness is $b/3$ as well.

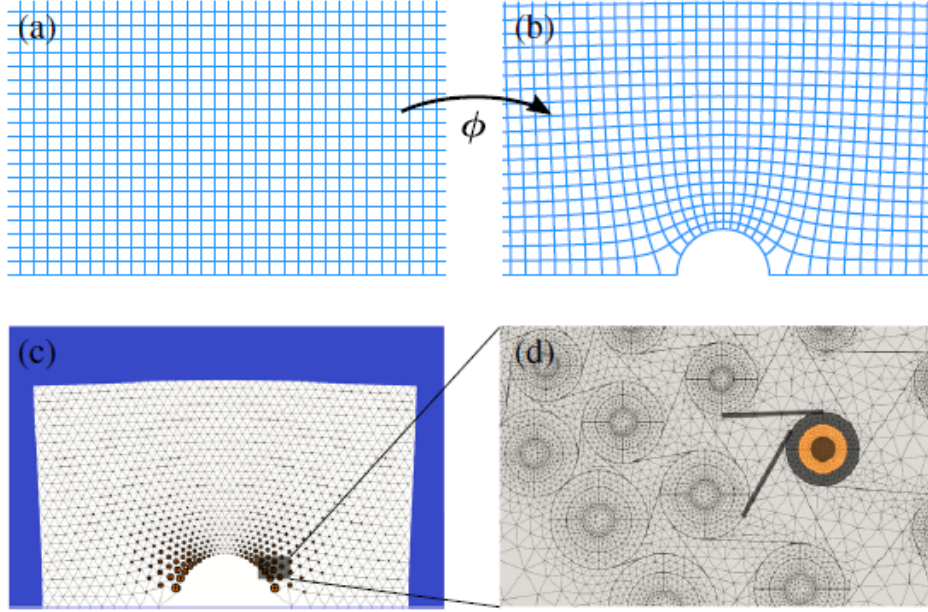


Figure 7.3 Cloak's geometry. (a) Original domain: bottom side is free; other sides extend to infinity. Lines can be thought of as rays, coordinate axes or a discretization grid. (b) Image (transformed) domain: ϕ opens a cave along the free boundary; rays are bent around the cave and back into their original paths as if the cave was not there. (c) Image domain with the cloak's microstructure resolved: the lattice is graded so as to fit the required properties at each position and is truncated beyond a given thickness. (d) Mesh used in the finite element analysis: a close up of the region framed in (c); a few structural elements are highlighted to facilitate their identification.

The above scheme determines unambiguously the geometry of the cloaking lattice. As for its constitutive parameters, they need to solve Eqs. (7.3), (7.12), and (7.13). For simplicity, we select a background medium such that $\kappa_o = 2\mu_o$ in which case it is possible to work at the exact rotational resonance frequency $\omega = \omega_r$. Thus, η is infinite and the spring constant of the lattice

can be determined easily to be $k = 2\kappa_o/\sqrt{3}$; in particular, it is uniform throughout the cloak. It is also possible to keep uniform the Lamé parameters of the core's coat; then, the inner mass $m = m(\mathbf{x})$ is graded so as to produce the same ω_r for all resonators. Last, $M = M(\mathbf{x})$ is calculated by solving for M in (7.13).

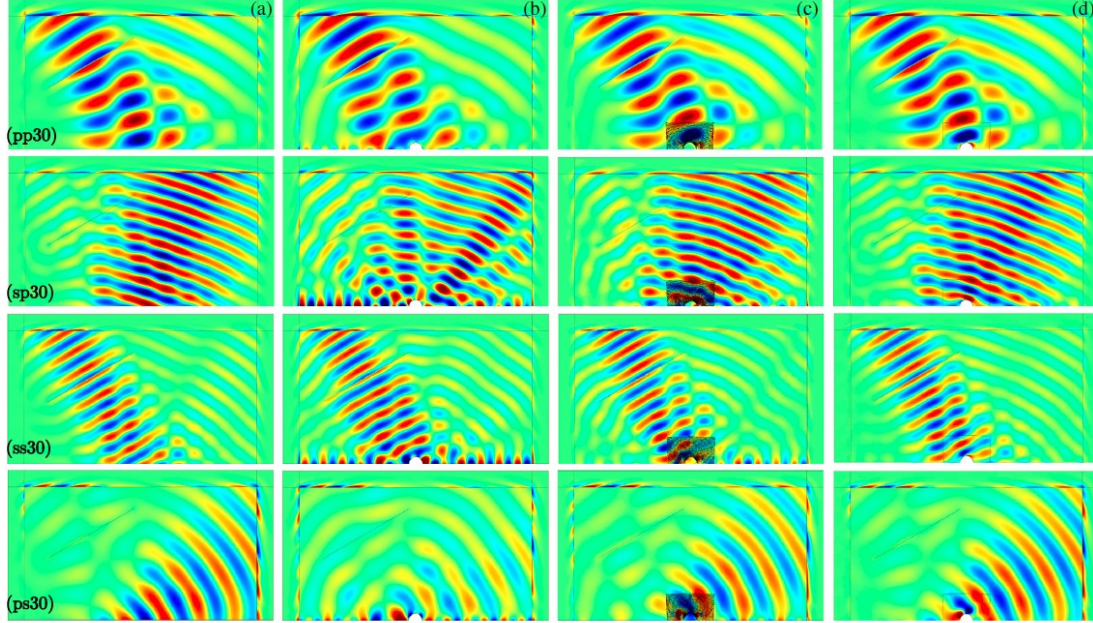


Figure 7.4 Cloaking simulations under 30° incidence: row (ij30), where i and j refer to pressure "p" or shear "s", depicts the component of type i under an incident wave of type j. Columns (a)-(d) correspond respectively to the original domain unaltered, the original domain with an uncloaked void, the void cloaked with a discrete lattice and the void cloaked with a fictitious continuous material. The numerical parameters are $\omega_r = 2\pi \times 25\text{kHz}$; $\rho_o = 2.7 \text{ kg/m}^3$; $\mu_o = 25\text{MPa}$; the cave is semicircular with a 40 mm radius; the cloak is approximately $280 \text{ mm} \times 155 \text{ mm}$ and contains about 44×26 unit cells.

7.3 Simulation

We simulated the cloaking of the cave against probing pressure and shear waves at various angles of incidence using the finite element solver COMSOL Multiphysics. For convenience, the whole domain was meshed, including the background medium, the lattice, the empty space between the masses, and the cloaked cave [Fig. 7.3d]. Empty spaces were assigned a massless compliant material whereas the rigid inner and outer masses were assigned a stiff material. The action of the elastic rods is taken into account in the weak form of the equations by integration along the straight lines in the meshes that connect the various neighboring masses. Besides the free boundary, perfectly matched layers are appended to the other three boundaries to suppress parasitic reflections.

The results are summarized on Fig. 7.4 for pressure and shear probing waves coupled to one another through the free boundary and incident at 30° . While not perfect, the designed resonant lattice shows satisfactory cloaking performance as it suppresses pressure-shear [plot (sp30-b) vs

(sp30-c)], shear-pressure [plot (ps30-b) vs (ps30-c)], and Rayleigh [plots (sp30-b) vs (sp30-c) and (ss30-b) vs (ss30 - c)] scattering. As a general observation, it seems that the shear component carries the dominant part of the cloaking error, in the first quadrant of plots (sp30 - c) and (ss30 - c) most notably. That is because the cloak is based on the behavior of the lattice in the homogenization limit and, at equal frequencies, shear waves propagate at almost half the wavelength of pressure waves. Thus, reducing the cloaking error further requires using a smaller discretization step δ and a larger numerical effort while the design remains essentially the same.

7.4 Conclusions

In conclusion, transformation-based cloaking for mechanical waves has progressed in a number of important cases mostly pertaining to acoustic waves in fluids (Cummer and Schurig, 2007), to flexural waves in plates (Farhat et al., 2009) and to scalar fields more generally (Bückmann et al., 2014). Progress in full elasticity has been slower, impeded by the absence of materials whose behavior is invariant under curvilinear changes of coordinates (Bückmann et al., 2015). Such materials can be designed nonetheless as lattices of subwavelength structural element. In that process, resonance plays a role that goes beyond reflection and refraction and unlocks atypical stress-strain relationships. Here, rotational resonance permitted to break stress and mirror symmetries and granted the elasticity tensor two extra degrees of freedom, polarity and chirality, rich enough to enable cloaking. In this way, the present work established the first theoretical and numerical evidence of resonance-based cloaking. In this way, the present work established the first theoretical and numerical evidence of resonance-based cloaking in full plane elasticity thanks to the proposed polar metamaterials. Finally, it is worth noting that a major drawback common to all resonance-based solutions, cloaking related or not, is their dependence over frequency. Overcoming this limitation without involving extrinsic interactions would constitute a significant step towards applications.

8. Creating synthetic spaces for higher-order topological sound transport

8.1 Introduction

Hall physics can be theoretically generalized to higher dimensions¹⁸ and experimental realizations of the effect in four-dimensional (4D) were recently reported with atomic and photonic systems as well as electric circuits (Lohse et al., 2018). These experimental works point to two different strategies for emulating higher space dimensions in physical space. The first approach (Wang et al., 2020) uses the higher dimensional lattice coordinates and orbital indices only as labels for resonators rendered in physical space, to connect them based on a standard higher dimensional model (Prodan and Schulz-Baldes, 2016). The second approach relies on synthetic dimensions generated via space modulations (Lohse et al., 2018), where the phases of the modulated structures can be used as adiabatic parameters that augment the physical space (Kraus et al., 2012). In essence, these phase variations can be treated as an additional global degree of freedom, usually called the phason, which lives on a torus. A third strategy is to weakly couple modulated waveguides, whose effective dynamics are described by a Schrödinger-like equation, with the coordinate along the

waveguides playing the role of time. This strategy has been experimentally implemented to produce topological pumping with photonic and elastic degrees of freedom (Zilberberg et al., 2018; Kraus et al., 2012; Lustig et al., 2019). The last strategy mentioned above is not feasible with sound because acoustic wave-channels cannot be weakly coupled in a continuous fashion along with the guides and, to our knowledge, it has been never implemented with acoustic degrees of freedom. In this work, we present a distinct strategy that functions in the opposite regime, where the waveguides are replaced by chains of coupled discrete resonators and strong couplings and modulations are also established in the transversal direction. In fact, the strategy we are proposing here can be better described as horizontal acoustic crystals carrying different phason values that are stacked and coupled with each other. By slowly varying the phason along the stacking direction, we demonstrate here that, with such an approach, we can explore any continuous orbit inside the phason space, and even control the speed along this path to control the shape of the pumped pattern. As a result, we can render these abstract trajectories, occurring in the synthetic dimensions, on the physical dimension along the stackings. In turn, this enables us to control the propagation of the acoustic modes in space as well as the temporal phases of the signals. In contrast to the waveguide experiments, where the samples had to be sliced for data acquisition, our designs enable a “non-demolition” measurement procedure that does not disrupt or alter the wave propagation. With the control over the phason, we demonstrate edge-to-edge topological pumping of sound in 1D-modulated acoustic crystals, as well as edge-to-edge and corner-to-corner topological pumping in 2D modulated acoustic crystals. The higher-order topological corner-to-corner modes in our system are principally different from conventional realizations. We demonstrate these types of pumping processes that the topological sound transport is robust against random fluctuations in the resonator couplings. We also demonstrate that the pumping along a given orbit in the phason space occurs only in specific space directions. We delineate the generated 2D and 4D quantum Hall systems by calculating first and second Chern numbers. We also discuss various ways in which we can control these pumping processes and, moreover, we discuss topological mode steerings in 2D modulated acoustic crystals that are entirely specific to the 4D IQHE physics. We believe that our work breaks ground for engineering applications, where the couplings in an acoustic crystal can be programmed for selective and robust point-to-point distribution of acoustic signals.

8.2 Physical rendering of synthetic spaces

We describe here the mechanism behind the physical rendering of a synthetic space, and we start by describing the general setting. It consists of a generic acoustic structure of discrete resonators such that each resonator has an address $(\mathbf{n}, m)$, with $\mathbf{n}$ being a horizontal label and m a vertical one. The couplings in the horizontal plane occur through channels whose modulated widths are specified by a phason ϕ , which lives on a phason space such as the d_s -torus. Here, d_s represents the dimension of the synthetic space. The couplings in the vertical direction are established through uniform channels. Now, we assume an infinitely long system in the z -direction and consider any curve $\phi(z)$ in the phason space with bounded first derivative $\phi'(z)$, which we parametrize by the continuous z -coordinate. We use this curve to specify the phason values for each horizontal layer,

specifically, $\phi_m = \boldsymbol{\phi}(\epsilon z_m)$, where $z_m = m a_z$ is the physical coordinate of the m th layer. The parameter ϵ is small and is there to ensure that the variations of the phason from one layer to another are small. Compared with the wave-guide setting, the major differences are the discrete character of the z -coordinate and the strong coupling in the horizontal direction. The resulting dynamical matrix D_ϕ governing the collective resonant modes depends on the chosen path $\boldsymbol{\phi}(z)$ and, while D_ϕ is not periodic in the z -direction, it displays the following covariance property

$$T_v^\dagger D_\phi T_v = D_{\phi \circ \tau}, \quad \tau(z) = z + \epsilon a_z, \quad (8.1)$$

where T_v is the vertical translation by a_z . If $Q_{n,m}$ is the acoustic resonant mode supported by the (n, m) -cavity, then the pressure field of the resonant collective modes can always be sought in the form $\int d k_z \Psi_{k_z}(\mathbf{r}; \boldsymbol{\phi})$, with

$$\Psi_{k_z}(\mathbf{r}; \boldsymbol{\phi}) = \sum_{n,m} e^{i k_z m} \varphi_{n,m}(\boldsymbol{\phi}, k_z) Q_{n,m}(\mathbf{r}) \quad (8.2)$$

and the covariance property (8.1) requires

$$\varphi_{n,m+1}(\boldsymbol{\phi}, k_z) = \varphi_{n,m}(\boldsymbol{\phi} \circ \tau, k_z) \quad (8.3)$$

Since $\boldsymbol{\phi}(z)$ is a smooth path, then, $\boldsymbol{\phi} \circ \tau \approx \boldsymbol{\phi} + \epsilon a_z \boldsymbol{\phi}'$ and, as such, $\varphi_{n,m+1}(\boldsymbol{\phi}, k_z) \approx \varphi_{n,m}(\boldsymbol{\phi}, k_z)$ to 0th order in ϵ . In these conditions, the vertical dispersion of the φ -coefficients can be ignored and the $\varphi_{n,m}$ coefficients for a fixed m become the eigen-modes of the reduced Hamiltonian

$$H_{k_z}(\boldsymbol{\phi}_m) = \sum_{\mathbf{n}} v(k_z)^2 |\mathbf{n}\rangle \langle \mathbf{n}| + \sum_{\langle \mathbf{n}, \mathbf{n}' \rangle} \kappa_{\mathbf{n}, \mathbf{n}'}(\boldsymbol{\phi}_m) |\mathbf{n}\rangle \langle \mathbf{n}'|, \quad (8.4)$$

where the last sum goes over the neighboring cavities and $\kappa_{n,n'}$ are the horizontal couplings, determined entirely by the phason value $\boldsymbol{\phi}_m = \boldsymbol{\phi}(\epsilon z_m)$. Also, $v(k_z)$ is the dispersion of the decoupled vertical channels and, if $\epsilon(\boldsymbol{\phi}_m)$ is the eigenvalue of the φ -mode, then the value of k_z at layer m is determined by the relation $f^2 - v(k_z)^2 = \epsilon(\boldsymbol{\phi}_m)$. The conclusion is that, by examining the horizontal spatial profiles of the collective resonant modes, one layer at a time, we can visualize the states of the Hamiltonian $H(\boldsymbol{\phi})$ along arbitrary paths inside the phason space.

So far we have established that the coefficients $\varphi_{n,m}$ for fixed m are eigen-modes of Hamiltonian (6.4), but what are the weight and the phase of these modes in Eq. (8.2)? An expansion in the first order in ϵ reveals that the answer is supplied by the equation

$$i\epsilon \Gamma(\boldsymbol{\phi}) \partial_\phi |\varphi(\boldsymbol{\phi})\rangle = [H(\boldsymbol{\phi}) - \epsilon(\boldsymbol{\phi})] |\varphi(\boldsymbol{\phi})\rangle \quad (8.5)$$

which governs the evolution of the φ -coefficients along the $\boldsymbol{\phi}$ trajectory. The function $\Gamma(\boldsymbol{\phi})$ is determined entirely by the vertical mode dispersion. After a proper change of variable, Eq. (8.5) becomes the classic equation of adiabatic evolution, hence the amplitudes of the modes along the

z -direction are all equal but a non-trivial phase $e^{i\alpha(z)}$ does develop along the stacking direction. It can be computed as the Berry phase of the Wilczek-Zee connection along the path $\phi(z)$ (Wilczek and Zee, 1984). The important conclusion is that we not only can control the spatial profiles of the modes but also their phases. The latter has been already proposed as vehicles for certain forms of information processing with classical metamaterials (Barlas and Prodan, 2020). Last, it is worth mentioning that if the acoustic crystal is finite in the z -direction, the collective resonant modes are given by linear superpositions of the k_z solutions of Eq. (8.5) and they occur at quantized values of the wavenumbers for which the top and bottom boundary conditions are simultaneously satisfied. Since the effective Hamiltonian from Eq. (8.4) is independent of the sign of the wavenumber, the main conclusion regarding the horizontal spatial profiles of the modes holds without modifications.

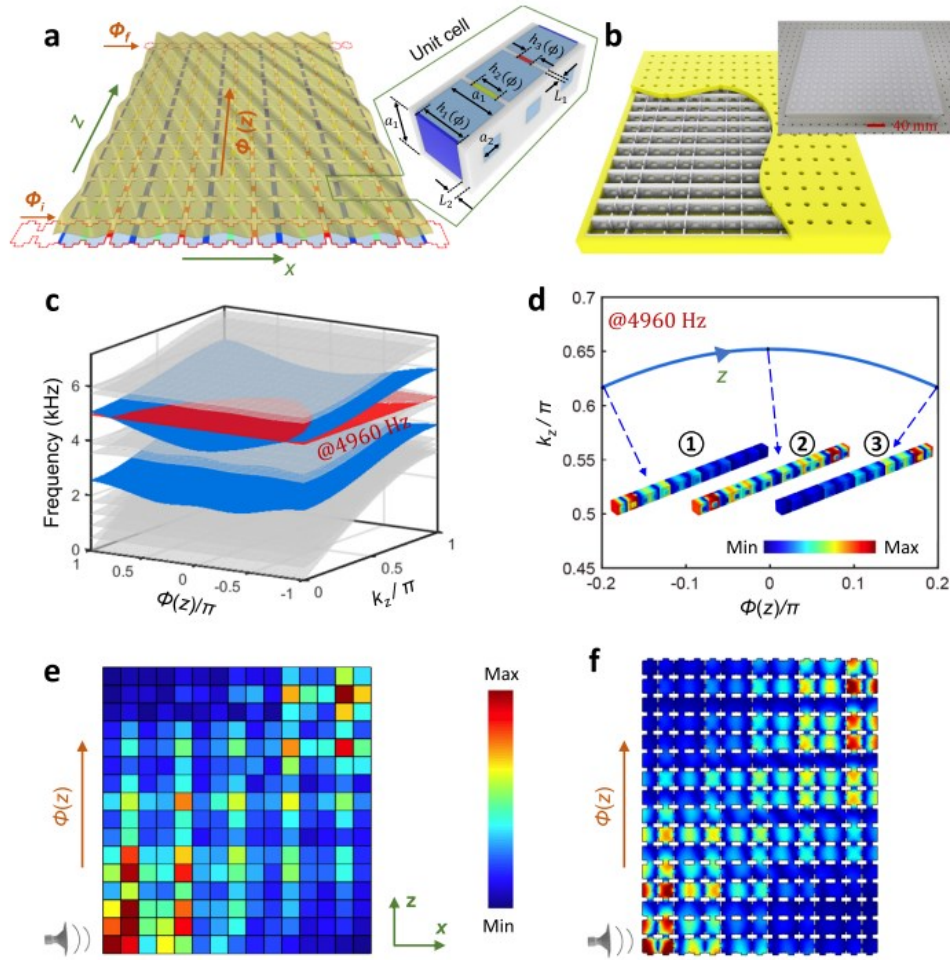


Figure 8.1 2D channel-modulated acoustic crystal and its dispersion property. (a) Schematic of the 2D channel-modulated acoustic crystal and its unit-cell. (b) Photograph of the printed 2D channel-modulated sample. (c) Dispersion diagram of a supercell composed of 15 coupled cavities, terminated by a hard-wall boundary along the x -direction, and a periodic boundary condition along the z -direction. The topological pumping of edge states is represented by blue surfaces, whereas the bulk bands are indicated by gray surfaces. (d) Wave number k_z as a function of the pumping parameter for frequency $f = 4960$ Hz, change of the pumping parameter ϕ from -0.2π to 0.2π along the z -direction. The color scale corresponds to the pressure amplitude. (e,f) Experimental (e) and numerical (f)

demonstration of topological edge pumping in the 2D crystal with the frequency at $f = 4960$ Hz. The color scale corresponds to the pressure amplitude.

8.3 1D topological pumping

Figures 8.1(a) and 8.1(b) show a planar array of acoustic cavities coupled horizontally and vertically through channels. Each cavity has an address $(n, m) \in \mathbb{Z}^2$ and the thickness of the horizontal channel connecting (n, m) and $(n + 1, m)$ resonators is modulated according to the protocol $h_{nm}^x = h_0[1 + \delta \cos(b_n \setminus b_{\text{mod}3} + \phi_m)]$, where h_0 is the average thickness of the horizontal channels, δ is the modulation amplitude, and b_j 's are free parameters. The values of phason for each layer are set by $\phi(z) = \phi_i + (\phi_f - \phi_i) \frac{z}{L_z}$, where $\phi_i = -0.2\pi$, $\phi_f = 0.2\pi$ and $L_z = 16a_z$. This results in a variation $\Delta\phi = 0.026\pi$ from one layer to another, hence within the adiabatic conditions.

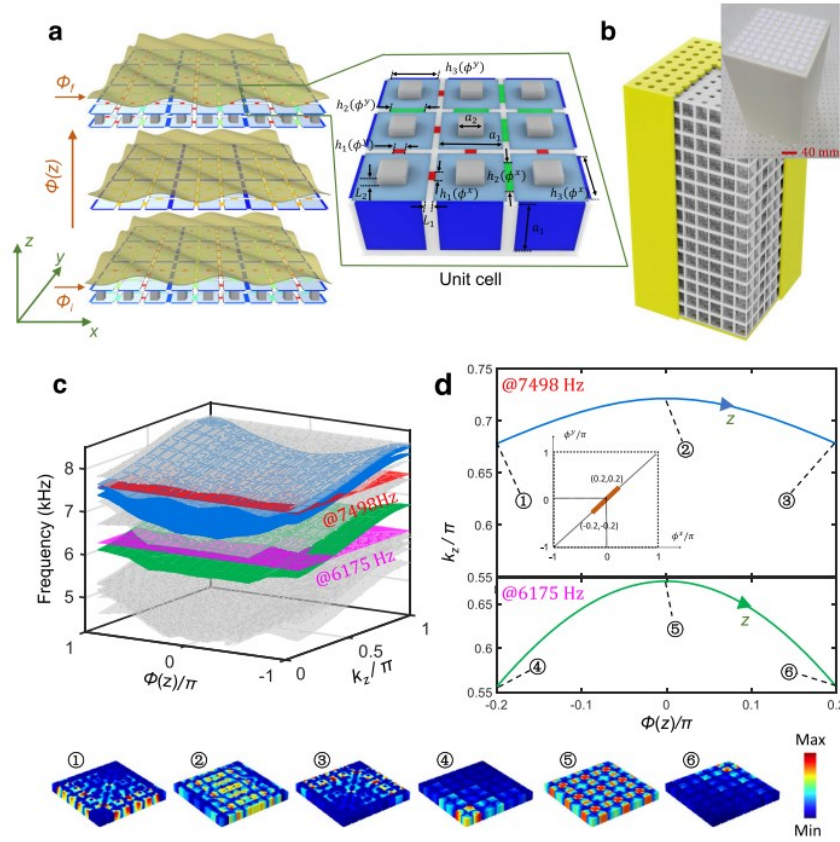


Figure 8.2 3D channel-modulated acoustic crystal and its dispersion property. (a) Schematic of the 3D channel-modulated acoustic crystal and its unit cell. (b) Photograph of the printed 3D channel-modulated sample. (c) Dispersion diagram of a supercell composed of 9×9 coupled cavities, terminated by a hard wall boundary along x and y directions, and a periodic boundary condition along the z -direction. Bulk modes are shown in gray, topological pumping of edge and corner modes are in blue and green, respectively. (d) Wave number k_z as a function of the pumping parameter for fixed frequencies $f = 7498$ Hz (upper panel) and $f = 6175$ Hz (lower panel), corresponding to the cross-section illustrated by the red and purple planes in c. The inset in the upper panel shows the diagonal orbit

with the phason value from $(-0.2\pi, -0.2\pi)$ to $(0.2\pi, 0.2\pi)$. The insets below (c) and (d) show representative mode shapes for each type of mode. The color scale corresponds to the pressure amplitude.

By design, when $b_j = (j - 1) \frac{2\pi}{3}$, the effective Hamiltonian $H_{k_z}(\phi)$ in Eq. (8.4) is just the 1D Aubry-André-Harper model associated with the 2D Hofstadter model at magnetic flux $\pi/3$, with ϕ playing the role of a quasi-momentum. Figure 8.1(c) shows the resonant spectrum of $H_{k_z}(\phi)$ as function of ϕ and k_z . The computation is carried out with COMSOL Multiphysics on the domain of a finite horizontal stack with k_z -twisted Bloch boundary conditions in the vertical direction. The bulk and the boundary spectra are shown with distinctive colors and, as expected from the Hofstadter butterfly (Hofstadter, 1976), two bulk spectral gaps are observed. Also from the Hofstadter butterfly, one can read that the lower and upper band gaps carry first Chern numbers $C_1 = \{-1, 1\}$. This is confirmed by direct calculations of the Chern numbers for an entire phase diagram computed for various values of b_j 's. At last, as expected from the bulk-boundary correspondence for the 2D IQHE, topological edge modes are observed in the spectrum reported in Fig. 8.1(c) (see the blue sheets). At fixed k_z , there are precisely one chiral edge band per edge and the slopes of these bands are consistent with the values of the Chern numbers.

We now focus on the spatial profiles of the modes, as excited at frequency $f = 4960$ Hz, indicated by the red horizontal sheet in Fig. 8.1(c). It is chosen to intersect the dispersion surface of the edge modes such that we can visualize a topological pumping of sound. All modes along the curve resulted from the intersection of the $f = 4960$ Hz plane and the dispersion surfaces will be excited. As argued in the previous section, ϕ is resolved by the z -coordinate, hence this pumping curve, shown again in Fig. 8.1(d), can be parametrized by the physical coordinate along the stacking, $(\phi(z), k_z(z))$. In other words, the spectral data from Fig. 8.1(d) have been rendered in the physical dimension, for us to observe. We now can understand the spatial profiles of the excited modes, when examined one stack at a time. Along the horizontal stack at a given coordinate z , one should observe the mode of $H(\phi(z))$ at energy $f^2 - v(k_z(z))^2$. The evolution of this mode along the pumping curve is shown in Fig. 1d, which confirms that sound is indeed pumped from one edge to the other. For example, the left edge state denoted in the inset (1) is selected as an initial state with a negative pumping value, which remains localized on the left boundary with the adiabatic increase of the pumping parameter and the corresponding wavenumber. When the pumping parameter approaches $\phi = 0$, the left edge state becomes the bulk state as inset (2) depicted. As the pumping parameter increases further, the bulk state is then transformed into an edge state (3) localized at the right side. Then our prediction is that sound is transported from one side to the opposite side of the structure and this topological sound steering can be witnessed by walking along vertical coordinate. Experimental observation and confirmation of the adiabatic pumping parameter increases further, the bulk state is then transformed into an edge state (3) localized at the right side. Then our prediction is that sound is transported from one side to the opposite side of the structure and this topological sound steering can be witnessed by walking along vertical coordinate. Experimental observation and confirmation of the adiabatic pumping via topologically protected boundary states are reported in Fig. 8.1(e). Here, the sound is injected into the bottom-

left corner of the structure and the pressure field is mapped by a microphone for each site. As one can see, the pressure distribution indeed renders the topological pumping process in the physical dimensions. The minor difference between experiment and simulation may be attributed to manufacturing deviations from connecting adiabaticity and perfect coupling to edge states. Let us also point out that there is a substantial region where the wave has a bulk character and where dissipation mostly occurs. This region can be reduced by optimizing the function $\phi(z)$ from a linear to a tangent hyperbolic profile.

8.4 2D topological pumping

We now investigate the three-dimensional (3D) acoustic structure shown in Figs.8.2(a) and 8.2(b), engineered to have a phason $\phi = (\phi^x, \phi^y)$ living on 2-torus. In this case, each cavity has an address $(n, m) \in Z^3$ and the thicknesses of the horizontal connecting channels in the $\alpha = x, y$ directions are modulated according to the protocol $h_{n,m}^\alpha = h_0[1 + \delta \cos(b_{n\alpha}^\alpha \bmod 3 + \phi_m^\alpha)]$, while the vertical connecting channels are uniform. By design, the effective horizontal Hamiltonian (8.4) is just a sum of two copies of the Hamiltonian from 1D topological pumping, $H(\phi) = H(\phi^x) \otimes I + I \otimes H(\phi^y)$, which is known to host 4D QHE physics (Zilberberg, 2018).

The system can be pumped along an orbit inside the phason space by using the strategy described in Section "Physical rendering of synthetic spaces". The particular crystal is designed to pump along the diagonal orbit $\phi(z) = (\phi(z), \phi(z))$, with $\phi(z) = \phi_i + (\phi_f - \phi_i) \frac{z}{L_z}$, where $L_z = 15a_z$. Figure 8.2(c) shows the dispersion surfaces of the resulting effective Hamiltonian (8.4) as a function of the pumping parameter ϕ and k_z . The computation, which is carried as for the 2D structure, reveals two bulk gaps and, in addition, two spectral sheets highlighted in blue for which the modes are localized along two edges, as well as one sheet highlighted in green for which the modes are localized at the corners. Fig. 8.2(d) (upper panel) shows the pumping curves resulted from the intersection of the dispersion diagram with the plane at frequency $f = 7498$ Hz, with the latter highlighted in red in Fig. 8.2(c). In contrast to the 1D topological pumping, there is more than one such pumping curve. Similarly, Fig. 8d (lower panel) shows the pumping curve resulted from the intersection with the plane at frequency $f = 6175$ Hz, with the latter highlighted in purple in Fig. 8.2(c). All these pumping curves are parametrized by the z coordinate and, as in the 1D case, we can predict that, if one examines the horizontal stack at coordinate z , one should observe the eigen-modes of $H(\phi(z))$ at energy $f^2 - v(k_z(z))^2$. Samples of these modes are rendered in the inset of Fig. 8.2(d) and, as one can see, both pumping curves that we engineered are very special. Indeed, as one pumps along the blue contours, the mode is pumped from one pair of edges to the opposite pairs of edges, whereas if one pumps along the green curve the mode is pumped from one corner to the opposite corner. Both pumping processes proceed through a bulk delocalization transition.

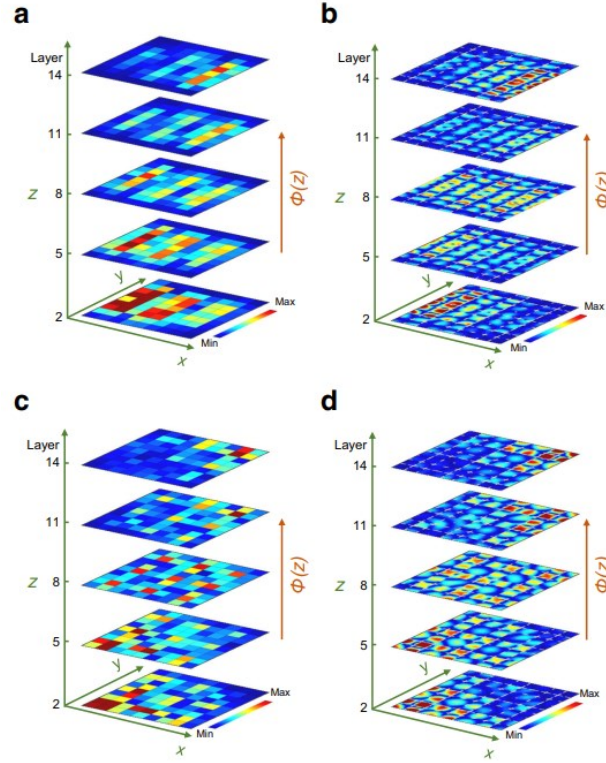


Figure 8.3 Topological pumpings in the 3D channel-modulated acoustic crystal. (a, b) Experimental (a) and numerical (b) observation of edge-bulk-edge pumping. The sound is injected at the bottom-left edge with the frequency at $f = 7498$ Hz. (c, d) Experimental (c) and numerical (d) observation of corner-bulk-corner pumping. The sound is injected at the bottom-left corner with the frequency at $f = 6175$ Hz. Note that the pumping parameter $\phi(z)$ for each horizontal layer of these samples is evenly distributed from -0.2π to 0.2π along the z -direction. The color scale corresponds to the pressure amplitude

In Fig. 8.3(a), we demonstrate experimentally that the predicted edge-bulk-edge pumping is indeed rendered in the physical space of the 3D structure. The acoustic wave is excited by a sound speaker along the bottom edge at $f = 7498$ Hz and the pressure distribution is measured by the microphone, layer by layer along the stacking direction. As seen in Fig. 8.3(a), the pressure field does evolve from the left to the right edge as one walks along the stacking direction. It is interesting to note that the edge states that we excite have the same energies as bulk states, which are also seen as being excited in the very bottom layer. However, as the bulk states extend throughout the horizontal layer, they experience increased dissipation and, as such, fade away along the stacking direction. On the other hand, the topologically pumped states are long-propagated, which further demonstrates the advantages of the topological steering of sound. A numerical simulation is also conducted to validate our experimental observation.

In Fig. 8.3(c), we demonstrate experimentally that also the predicted corner-bulk-corner pumping can be rendered in the physical space of our 3D structure. Indeed, with the source placed at a bottom corner and with the frequency adjusted at $f = 6175$ Hz, one sees the measured pressure field evolving towards the opposite corner as one walk along the staking direction. This pumping is resolved much better than in the previous case, because the source couples less effectively to

the bulk modes, hence the latter are not excited in this setup. Also, due to the reduced dimensionality of the mode, the dissipation is weaker, and the pressure field can be sustained longer along the stacking direction. Numerical simulations for the entire 3D structure are reported in Fig. 8.3(d) and they validate our experimental findings.

The emergence of the chiral boundary spectrum that makes possible the topological pumpings observed in Fig. 8.3 is owing to the second Chern number of the gaps, which is the strong topological invariant in 4D. As it is the case for any strong topological invariant, topological boundary spectrum emerges regardless of how the boundary of the crystal is cut, provided the available quasi momenta are properly sampled. In our case, the phason plays the role of synthetic momenta and this implies that a particular space direction. To demonstrate that the pumping processes are indeed highly directional, we simulate the acoustic processes are indeed highly directional, we simulate the acoustic characteristics of the 3D structure for different phason orbits and with the source placed at different space locations and encountered the following scenarios: (1) propagation along a facet; (2) propagation along an edge; (3) pumping in the x but not in the y direction; (4) pumping in the y but not in the x direction; (5) the y direction; (4) pumping in the y but not in the x direction; second diagonal $x = -y$. (6) pumping along the second diagonal $x = -y$ but not along the first diagonal. We also demonstrate what we call an "antagonistic" effect, which manifests as follows: If edge-to-edge or corner-to-corner pumping is observed with the source placed on one edge or corner, respectively, then the pumping or any propagation is completely absent if the source is moved to the opposite edge or corner. In this respect, the structure acts like a perfect "transistor" because the pumping can be turned on and off by 180° rotations. In fact, by combining all the effects listed above, our 3D structure can be transformed into a multifunctional and programmable acoustic device for sound transport and distribution. We stress that our structure does not possess any crystalline symmetry so that the bulk polarizations are not quantized. Therefore, the higher-order topological corner modes in our system are fundamentally different from previous realizations based on quantized quadrupole polarization (Serra-Garcia et al., 2018) or quantized Wannier centers (Xue et al., 2019).

Topological phases with non-zero second Chern numbers display intriguing wave transport characteristics, such as robustness against impurities or defects. It is of interest to quantify the extent of the topological protection in such conditions. To evaluate that, a 3D hollow structure is constructed by removing nine cavities at the center of the system and the topological edge-bulk-edge and corner-bulk-corner pumpings in the defected structure are experimentally measured, as shown in Fig. 8.4(a) and 8.4(c), respectively. When comparing performance with the system without defects (Fig. 8.3(a) and 8.3(c)), both edge and corner modes can be smoothly pumped despite the presence of defects on a relatively large scale. This confirms that the topological pumping is immune against back reflections from defects or discontinuity. The experimental observation is also verified by the numerical simulation based on the exact geometry (Fig. 8.4(b) and 8.3(d)).

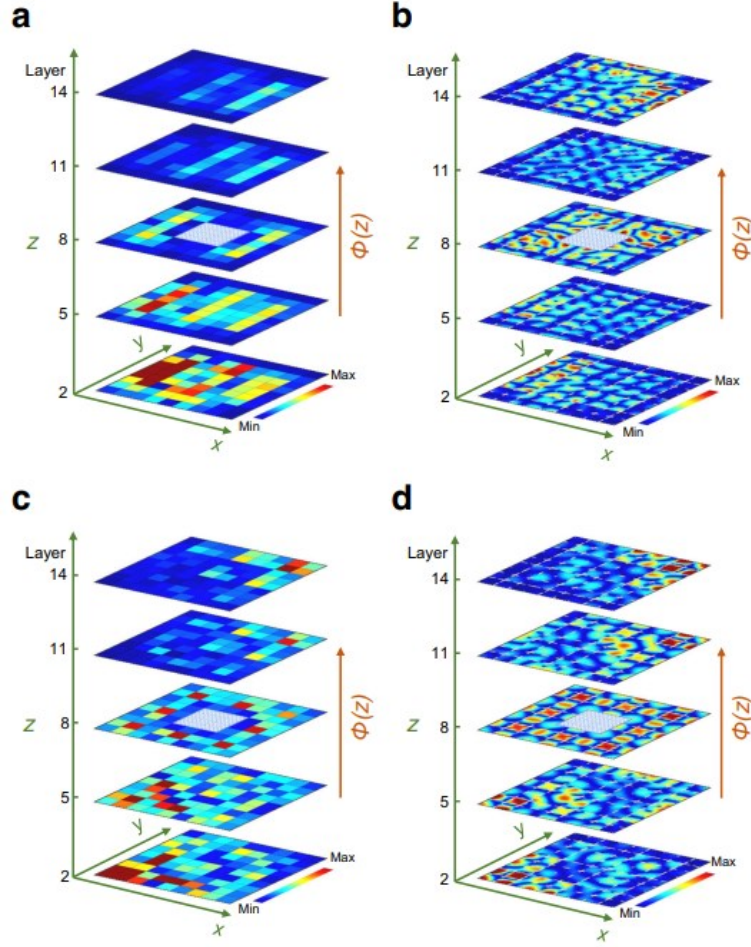


Figure 8.4 Robust topological pumpings in the 3D hollow acoustic crystal. (a, b) Experimental (a) and numerical (b) observation of edge-bulk-edge pumping. The sound is injected at the bottom-left edge with the frequency at $f = 7498$ Hz. (c, d) Experimental (c) and numerical (d) observation of corner-bulk-corner pumping. The sound is injected at the bottom-left corner with the frequency at $f = 6175$ Hz. Note that the pumping parameter $\phi(z)$ for each horizontal layer of these samples is evenly distributed from -0.2π to 0.2π along the z -direction and the hollow area is indicated by the blue grid plane where nine cavities are removed. The color scale corresponds to the pressure amplitude.

8.5 Discussion

We demonstrate a robust strategy to explore the global degrees of freedom of modulated wave media. While we have exemplified here only simple orbits inside the phason spaces, the method has no limitations on the geometry and topology of these orbits. For example, in the present study on the 2D pumping, we explored the fundamental loops x and y of the 2-torus as well as the diagonal orbit, which is topologically equivalent to the combination $C_x + C_y$. The 2-torus, however, supports an infinite number of topologically distinct paths, which can be in principle explored with the methods demonstrated in this work. It remains to be seen if the phases of the sound signals can be resolved as predicted in Section "Physical rendering of synthetic spaces", in which case the steering of the modes in both space and time domains could be controlled with the same device. In our opinion, the phason engineering may exhibit the possibility to implement the topological split-flow device, such as the topological beam splitter. In Fig. 8.5(a), we create an acoustic beam splitter to engineer two-way beam splitting. In the current design, we consider a 2D system with

different phason orbits in four quadrants: in the first quadrant, the phason value is linearly distributed from -0.2π to 0.2π along the z -direction; in the second quadrant, the phason value is linearly distributed from 0.2π to -0.2π along the z -direction; in the third quadrant, the phason value is held constant with $\phi = 0.2\pi$ along the z -direction; in the fourth quadrant, the phason value is held constant with $\phi = -0.2\pi$ along the z -direction. The input point is located at the bottom edge between the third and fourth quadrants. As shown in Fig. 8.5(b), the sound stays confined to the interface until it arrives at the junction of four quadrants. Then, it splits into the first and second quadrants, and eventually reaches the two-end sites. Thanks to topological protection, the propagation is immune against back reflection from discontinuity. As such, our design, based on phason engineering and topological pumping, provides an avenue for the application of acoustic beam splitters. In addition, by replacing the waveguides with discrete coupled resonators, one now can engineer the dispersion with respect to k_z quasi-momentum. This will involve modulations along the vertical direction, and this opens a dimension in the design space, which is yet to be explored.

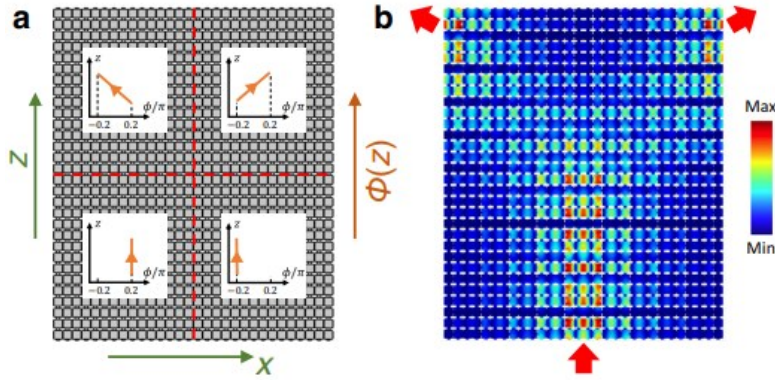


Figure 8.5 Topologically protected acoustic beam splitter. (a) Schematic of the phason engineering in the 2D acoustic system. (b) The pressure field distribution of the beam splitter. The sound is injected at the center of the bottom edge with the frequency at $f=4960$ Hz. The color scale corresponds to the pressure amplitude.

8.6 Conclusions

In conclusion, we have evidenced the topological sound transport in modulated acoustic crystals through edge-to-edge topological and corner-to-corner topological pumpings associated with the 2D and 4D quantum Hall effects by the physical rendering of synthetic spaces. These observations imply that the system is characterized by a non-zero Chern number and therefore the topological pumping is immune to bulk scattering and exhibits strong protection against design imperfections. The modulated acoustic crystals with synthetic spaces offer a platform and route for efficient acoustic topological mode transport by engineering desired patterns on a phason-torus, and the higher dimensional quantum Hall effect may provide surface acoustic phenomena in the finite structure. The phason space augments the physical space and this opens a door to higher-dimensional physics in acoustics and mechanics. Although we focused on the be generalized to other degrees of freedom, such as additional frequency dimensions can also be harnessed for the frequency modulation. Going forward, it will be important to develop matter in synthetic dimensions is very general and the extension matter in synthetic dimensions is very general and the extension of this approach to other complex orbits is much awaited. At last, regime, the number of stacks in our experimental set-ups

is appreciable and, whereas this is perfectly fine for demonstration purposes, it will be an obstacle for practical applications. Recently, Fedorova et al. (2020) showed that the non-adiabatic effects can be compensated using modulated dissipative channels and that such a strategy can be used to achieve quantized topological pumping with fast-driven cycles. It will be interesting to explore if this strategy can be deployed for our acoustic crystals to reduce the number of stacks needed for the topological pumping of sound.

9. Discrete transformation elasticity: An approach to design lattice-based polar metamaterials

9.1 Introduction

The theory of composites, broadly defined, continues to inform design procedures for artificial materials (Milton, 2002). An important thematic problem in that context is how to design materials with target elastic properties out of a limited number of elastic phases, with specific volume fractions, and using certain fabrication processes, e.g., mixing, layering, or 3D printing. The advent of metamaterials has presented the theory of composites with new challenges where the target properties are unusual in the sense that they break certain conventional symmetries of the constitutive law, e.g., the minor symmetries of Hooke's law. Most relevant to our purposes is the way in which such properties arise in the context of the transformation method where they allow to space-warp solution fields.

The desired elasticity tensors $\mathbf{c}$ in transformation elasticity are function of two parameters: (i) the elasticity tensor $\mathbf{C}$ of a standard reference medium, i.e., before applying any geometric transformations; and, (ii) the gradient F of a space-warping transformation ϕ which maps the reference medium to a physical space (Norris and Shuvalov, 2011). In a recent breakthrough, elastic cloaks have been designed by developing polar elastic lattices convenient for two particular classes of transformations, (i) that of conformal transformations ϕ where F is shear-free (i.e., proportional to a rotation) (Nassar et al., 2019; Nassar et al., 2020); and (ii) that of radially symmetric ϕ where F is rotation-free (Nassar et al., 2018; Zhang et al., 2020). By contrast, the present design paradigm holds for arbitrary transformation gradients F combining nonuniform stretch, shear, and rotation and permits to realize anisotropic materials with polar and chiral properties. The key idea is to let F operate, not only on the elastic properties to change them from $\mathbf{C}$ to $\mathbf{c}$, but on the underlying architectures themselves as well. In other words, instead of looking for architectures with the unusual tensor $\mathbf{c}$, we first target the architectures of the background medium with the conventional tensor $\mathbf{C}$. The found architectures are then transformed by F into a lattice material which automatically exhibits the sought-after tensor $\mathbf{c}$. The transformation rules according to which F operates on a lattice material to generate another lattice material constitute what we refer to as "Discrete Transformation Elasticity". A theoretical treatment of discrete transformation elasticity was first proposed by Guevara Vasquez et al. but did not lead to feasible designs (Vasquez et al., 2013); another relevant contribution, is the one by Bückmann et al. (2015), where direct lattice transformations have been suggested but ignored the polar character of the required tensors. Here, we complete these efforts in the framework of polar elasticity and propose

feasible designs based on rigorous derivations taking into account the full tensorial character of the equations of elasticity that can be anisotropic, polar, and chiral simultaneously. As an outstanding application, we leverage the proposed design paradigm to construct a polar lattice metamaterial for the observation of elastic carpet cloaking. Numerical simulations are then conducted and show excellent cloaking performance under shear and pressure, static and dynamic loads.

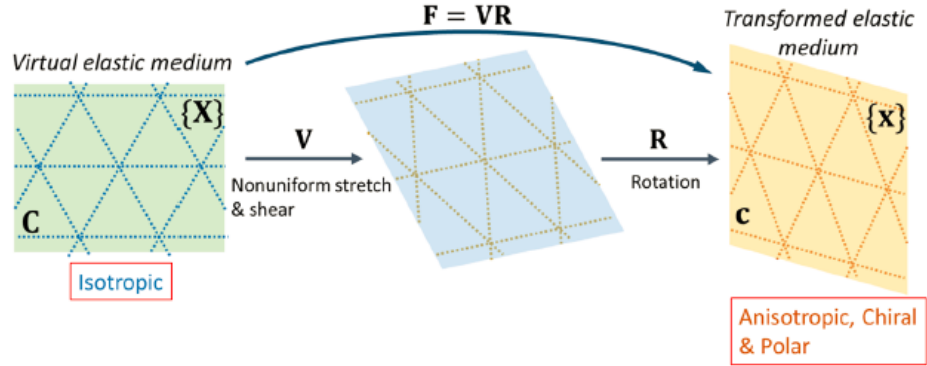


Figure 9.1. Schematic representation of continuum transformation elasticity. A continuum elastic medium with the elastic tensor $\mathbf{C}$ occupies the virtual space $\{\mathbf{X}\}$. A transformed elastic medium with the elastic tensor $\mathbf{c}$ occupies the physical space $\{\mathbf{x}\}$. The transformation gradient $\mathbf{F}$ comprises nonuniform stretch, shear, and rotation. When the virtual elastic medium is isotropic, the resulted transformed elastic medium is anisotropic, chiral, and polar.

9.2 Continuum and discrete transformation elasticity

Consider a general mapping $\mathbf{x} = \phi(\mathbf{X})$ that transforms a virtual (original) domain $\{\mathbf{X}\}$ into a physical (transformed) domain $\{\mathbf{x}\}$ (see Fig. 9.1). The transformation ϕ can encompass any combinations of stretch, shear, and rotation. Thus, the transformation gradient $\mathbf{F}$ will generally decompose into $\mathbf{F} = \mathbf{V}\mathbf{R}$ where $\mathbf{R}$ is orthogonal (rotation tensor) and $\mathbf{V}$ is symmetric positive definite (stretch tensor), as shown in Fig. 9.1. The virtual and physical domains are occupied by elastic media which, under an external load, are displaced by fields $\mathbf{U}(\mathbf{X})$ and $\mathbf{u}(\mathbf{x})$, respectively. We are interested in determining the constitutive properties of $\{\mathbf{x}\}$, or even its microstructure, that let the displacement field be warped according to the same transformation ϕ , that is, $\mathbf{U}(\mathbf{X}) = \mathbf{u}(\mathbf{x}) = \mathbf{u}(\phi(\mathbf{X}))$. To do so, it is particularly insightful to interpret the fields $\mathbf{u}(\mathbf{x})$ and $\mathbf{U}(\mathbf{X})$ as two different, but equivalent, sets of generalized Lagrangian coordinates, namely such that

$$\int_{\{\mathbf{x}\}} L(\nabla \mathbf{U}) d\mathbf{X} = \int_{\{\mathbf{x}\}} \ell(\nabla \mathbf{u}) d\mathbf{x} \quad (9.1)$$

where L and ℓ are the strain energy densities over $\{\mathbf{X}\}$ and $\{\mathbf{x}\}$, respectively. The change of coordinates formula, together with the chain rule, then yields $\ell(\nabla \mathbf{u}) = L(\nabla \mathbf{U})/J$ with $\mathbf{F} = d\mathbf{x}/d\mathbf{X}$ and $J = \det \mathbf{F}$ being the transformation gradient and its determinant. Note that Eq. (1) is valid for any sets of coordinate transformations $\{\mathbf{X}\} \rightarrow \{\mathbf{x}\}$. Accordingly, it is possible to warp the displacement field from $\mathbf{U}(\mathbf{X})$ to $\mathbf{u}(\mathbf{x})$ as long as $\{\mathbf{x}\}$ is composed of materials with the prescribed

strain energy ℓ . In terms of the constitutive properties, when the original domain $\{\mathbf{X}\}$ has an elasticity tensor $\mathbf{C}$, the transformed domain $\{\mathbf{x}\}$ has an elasticity tensor $\mathbf{c}$ with $c_{ijkl} = F_{jm}F_{ln}C_{imkn}/J$. A close inspection of the foregoing relation shows that the transformed elasticity tensor $\mathbf{c}$ is unconventional in at least three regards: (i) it is polar in the sense that it lacks the minor symmetry; (ii) it is degenerate in the sense that it admits a number of zero modes; (iii) it is chiral in the sense that it lacks mirror symmetry (in 2D). Materials with such unusual properties are unavailable and lattice-based designs, in few particular cases, have only recently been found. This state of affairs has significantly impeded progress in transformation elasticity in comparison to its optics or acoustics counterpart.

Here, we solve the material design problem by fully embracing a discrete lattice-based transformation (see Fig. 9.2). Thus, we discretize L and ℓ as if they represented the strain energy densities of two lattices. We suppose then that the background medium is made out of a periodic lattice, or at least that it has the same elasticity tensor as such a lattice. The lattice is made of a set of massless springs connecting mass nodes (Fig. 9.2). Hence, we let

$$L(\nabla \mathbf{U}) = (1/2) \sum_{m,n,p} K^p \left\langle \nabla \mathbf{U}(\mathbf{X}_{m,n}^p - \mathbf{X}_{m,n}), \mathbf{S}^p \right\rangle^2 / A_0 \quad (9.2)$$

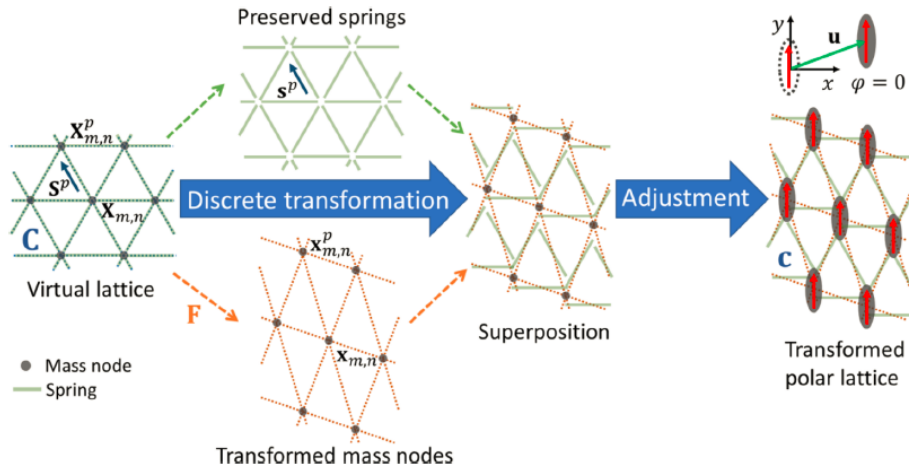


Figure 9.2 Schematic representation of discrete transformation elasticity. A virtual lattice with springs connected to mass nodes is characterized by effective modulus $\mathbf{C}$. During the discrete transformation, mass nodes are transformed to new locations following $\mathbf{F}$, while springs maintains their original directions. The transformed lattice is initially misaligned and does not contact. The final transformed polar lattice is constructed by adjusting masses and springs to ensure contact and, at the same time, suppressing rotations of the masses.

be the strain energy per unit cell area A_0 of a set of springs (index p) of constants K_p and of direction $\mathbf{S}^p = \frac{\mathbf{X}_{mn}^p - \mathbf{X}_{m,n}}{\|\mathbf{X}_{mn}^p - \mathbf{X}_{m,n}\|}$ connecting node $\mathbf{X}_{m,n}$ of a given unit cell (index m, n) to its neighboring nodes $\mathbf{X}_{mn}^p$. Then, per the above identity relating L and ℓ , domain $\{\mathbf{x}\}$ will have the same behavior as a lattice whose strain energy is

$$\ell(\nabla \mathbf{u}) = \left(\frac{1}{2} \right) \sum_{m,n,p} k^p \left\langle \nabla \mathbf{u}(\mathbf{x}_{m,n}^p - \mathbf{x}_{m,n}), \mathbf{s}^p \right\rangle^2 / a_0 \$, \quad (9.3)$$

with $(\mathbf{x}_{m,n}^p - \mathbf{x}_{m,n}) = \mathbf{F}(\mathbf{X}_{m,n}^p - \mathbf{X}_{m,n})$, $a_0 = A_0 J$, $k^p = K^p$ and $\mathbf{s}^p = \mathbf{S}^p$. In other words, the lattice of $\{\mathbf{x}\}$ can be easily deduced from that of $\{\mathbf{X}\}$ by (i) applying the transformation gradient $\mathbf{F}$ to the nodes of the lattice (Fig. 9.2) while (ii) leaving the springs constants and directions, as they were (Fig. 9.2). In particular, note that the lattice vectors anchored at the nodes, such as $\mathbf{X}_{m,n}^p - \mathbf{X}_{m,n}$, and the spring orientation $\mathbf{S}^p$ do not transform in the same fashion; indeed, $s^p \neq \frac{x_{mn}^p - x_{m,n}}{|x_{mn}^p - x_{m,n}|}$.

Last, to maintain physical contact between the mass nodes and the misaligned springs, the mass nodes must assume a finite size and become rigid bodies (see the transformed polar lattice in Fig. 9.2). This has the unintended effect of introducing an extra rotational degree of freedom which must be suppressed using an external ground or other rotational resonant substructures. Together, the misalignment and the suppressed rotations, are responsible for the unconventional properties of the transformed tensor $\mathbf{c}$. It is also important here to stress that the original domain $\{\mathbf{X}\}$ need not be a lattice, and only to behave like one. In particular, all isotropic materials with a Poisson's coefficient equal to $1/3$ behave like a triangular truss and can therefore be transformed in this fashion. More generally, square, rectangular, and oblique lattices with anisotropic effective tensors $\mathbf{C}$ can be transformed in the same manner. Conversely, the availability of a lattice representation of the original domain is the only limitation weighing on the present approach.

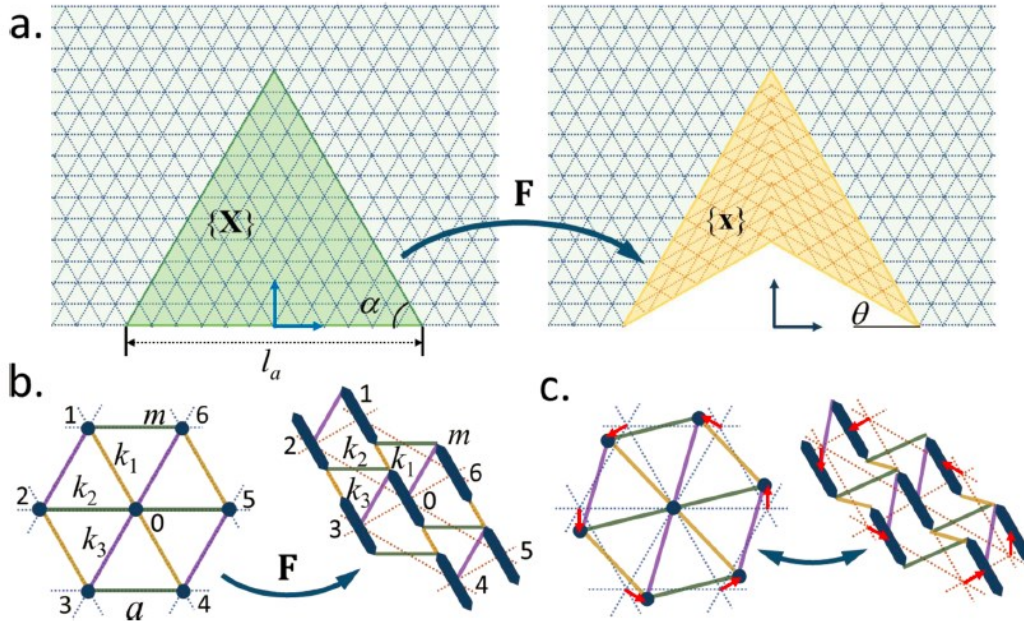


Figure 9.3 Design of the discrete mass-spring polar lattice for a carpet cloak. (a) Geometric transformation for realizing a carpet cloak concealing a triangular void; (b) The mass-spring lattice is designed using the discrete transformation.

For the design, $\alpha = 60^\circ$, and the lengths of springs, k_1, k_2, k_3 , and the rigid masses are $a \left[1 - \frac{2\sqrt{3}}{3} \tan(\theta) \right]$, $a \left[1 - \frac{2\sqrt{3}}{3} \tan(\theta) \right]$, $a \left[1 - \frac{2\sqrt{3}}{3} \tan(\theta) \right]$, and $a \left[1 - \frac{2\sqrt{3}}{3} \tan(\theta) \right]$ respectively.

$\frac{\sqrt{3}}{3} \tan(\theta) \Big], a \Big[1 - \frac{\sqrt{3}}{3} \tan(\theta) \Big]$, and $\frac{2\sqrt{3}a}{3} \tan(\theta)$, respectively; (c) Zero-mode of the transformed lattice is equivalent to the rigid rotation of the virtual lattice.

Given that the behavior post-transformation exhibits asymmetric stresses, it would be tempting to describe it in the general context of micropolar elasticity (Pasternak & Dyskin, 2018). This would be uncalled however for two main reasons: on one hand, the original medium is a classical (Cauchy) continuum and therefore should transfer no couple stresses to the cloaked domain. On the other hand, the extra degrees of freedom of a micropolar medium, namely microrotations, do not appear in the strain energy density ℓ . Accordingly, the transformed domain remains of Cauchy type, with a single stress measure, albeit an asymmetric one, and a single, again asymmetric, strain measure. The asymmetry can be explained by recalling Cauchy's second law of motion, $\sigma_{ij} - \sigma_{ji} = e_{ijk} c_k$, namely that skew stresses are acceptable as long as they are in balance with an appropriate torque density. This torque density is exactly what impedes mass rotation in our designs. Simply, rather than them being constant, the torques are proportional to the applied strain. Cauchy elasticity augmented with torques that are proportional to strain constitutes what we refer to as "polar elasticity". It is when the original medium itself is micropolar that one needs to consider how micro-rotations and couple-stresses transform, but such a generalization is out of the scope of the present paper.

9.3 A polar lattice for elastic carpet cloaking

Having introduced the continuous and discrete versions of transformation elasticity, it is time to illustrate the usefulness of the suggested paradigm in the design of polar lattices displaying nonstandard elastic properties never realized before. Our particular interest is in constructing an elastic carpet cloak with material properties that are uniform in space; this can be very advantageous from a manufacturing point of view. With that in mind, we apply a geometric transformation that linearly compresses a triangular domain in the virtual space (shaded green area) into a concave polygon in the physical domain (shaded yellow area). The transformation leaves a void along the lower boundary where an object can be concealed; the other boundaries remain unchanged (see Fig. 9.3a). Formally, the transformation ϕ reads

$$x = X, \quad y = \frac{\tan(\alpha) - \tan(\theta)}{\tan(\alpha)} Y + [l_\alpha - \text{sgn}(x)x] \tan(\theta) \quad (9.4)$$

where α, θ and l_α are two angles and a length characterizing the triangular domain and void pre- and post-transformation and are shown in Fig. 9.3a. We then obtain the deformation gradient of ϕ as

$$\mathbf{F} = \begin{bmatrix} 1 & 0 \\ -\text{sgn}(x) \tan(\theta) & 1 - \frac{\tan(\theta)}{\tan(\alpha)} \end{bmatrix} \quad (9.5)$$

The polar decomposition leads to

$$\mathbf{V} = \begin{bmatrix} p_1 \cos^2(\varphi) + p_2 \sin^2(\varphi) & (p_2 - p_1) \cos(\varphi) \sin(\varphi) \\ (p_2 - p_1) \cos(\varphi) \sin(\varphi) & p_2 \cos^2(\varphi) - p_1 \sin^2(\varphi) \end{bmatrix}, \mathbf{R} = \mathbf{V}^{-1} \mathbf{F} \quad (9.6)$$

where $p_{1,2} = \frac{1+F_{21}^2+F_{22}^2}{2} \mp \sqrt{\left(\frac{1-F_{21}^2-F_{22}^2}{2}\right)^2 + F_{21}^2}$ and $\tan(-2\varphi) = \frac{2F_{11}}{1-F_{21}^2-F_{22}^2}$. It can be clearly seen that the stretch is non-isotropic, as $V_{11} \neq V_{22}$. In addition, the transformation comprises a shear deformation, since $V_{12} = V_{21} \neq 0$. Finally, the deformed grid needs to experience a rigid rotation $R \neq I$ to realize the transformation gradient F .

Assume the material in the virtual space is isotropic with Lamé parameters (λ, μ) and mass density ρ_0 . Applying the rules of transformation elasticity (Norris & Shuvalov, 2011), the mass density and constitutive relations of the transformed material (i.e., the cloak in our context) read

$$\rho = \rho_0 A \quad (9.7)$$

$$\rho = \rho_0 A \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \\ \sigma_{21} \end{bmatrix} = \begin{bmatrix} (\lambda + 2\mu)A & \lambda & (\lambda + 2\mu)C & 0 \\ \lambda & (\lambda + 2\mu)/A + \mu BC & (\lambda + \mu)B & \mu C \\ (\lambda + 2\mu)C & (\lambda + \mu)B & \mu/A + (\lambda + 2\mu)BC & \mu \\ 0 & \mu C & \mu & \mu A \end{bmatrix} \begin{bmatrix} e_{11} \\ e_{22} \\ e_{12} \\ e_{21} \end{bmatrix} \quad (9.8)$$

with $A = \frac{\tan(\alpha)}{\tan(\alpha) - \tan(\theta)}$, $B = -\text{sgn}(x)\tan(\theta)$ and $C = AB$. Eq. (9.7) indicates that the total mass of an area in the virtual space is equal to the total mass of its transformed area in the physical space. As for the matrix in the right-hand side of Eq. (9.8), it represents the transformed elastic tensor c . Therein, note how the properties of cloak should be anisotropic, as $c_{1111} \neq c_{2222}$ and $c_{1212} \neq c_{2121}$ due to the anisotropic stretch in the transformation. Furthermore, the transformed material needs to be chiral ($c_{1211} \neq c_{2111}$ and $c_{1222} \neq c_{2122}$), and the material is not mirror symmetric since the transformation comprises a rotation. If we are able to design a "cloaking material", i.e., a medium equivalent to the transformed material, described in Eqs. (9.7) and (9.8), then we are able to demonstrate a cloaking behavior for elastic waves in the physical space.

We tackle the inverse problem of designing the cloaking material using discrete transformation elasticity. Hereafter, we specify Eq. (9.8) to the case a reference medium with a Poisson's ratio of $1/3$ ($\lambda = \mu$). Such a medium is conveniently represented by a triangular lattice with massless springs connecting massive nodes (see Fig. 7.3b; left panel). The length of the springs is denoted by a . The spring constants (k_1, k_2, k_3) and mass m of the nodes can be determined from the isotropic material properties by $k_1 = k_2 = k_3 = \frac{4\mu}{\sqrt{3}}$ and $m = \frac{\sqrt{3}\rho_0 a^2}{2}$. Performing the proposed discrete transformation, mass nodes of the virtual lattice are mapped to new locations in the transformed space following the transformation ϕ (see Points 0 – 6 in Fig. 9.3b before and after the transformation), and directions of the connecting springs remain as they were in the virtual lattice. To make masses and springs physically connected, we adjust the lengths of the springs

(keeping their constants unchanged) as well as the sizes of the rigid masses (keeping their masses unchanged); the specific shape of the rigid masses is irrelevant. While the lattice configuration can be arbitrary in principle, avoiding overlap among masses and springs will lead to simpler and more practical designs. In the adopted design, we selected elongated hexagonal rigid masses, and enforce springs k_1 and k_2 to the locations that transformed from ϕ . We first connect the rigid masses on Points 0 and 1 with the spring k_1 (see the right figure in Fig. 9.3b). By doing so, rigid masses must be in alignment with the spring k_1 to ensure periodicity. We then connect the rigid masses on Points 0 and 2 with the spring k_2 (see the right figure in Fig. 9.3b). Lengths of springs and masses are consequently determined after this procedure. Finally, the spring k_3 is adjusted into the transformed lattice by connecting Point 0 to the rigid mass on Point 6. In the design, all contacts are assumed to be hinge-like, and rotations of masses are suppressed.

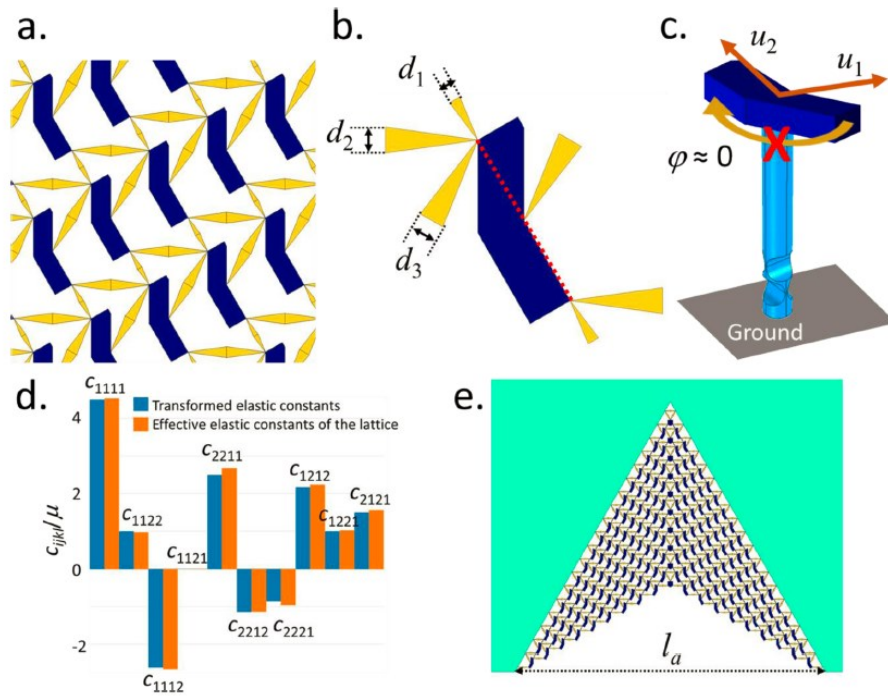


Figure 9.4 Microstructure realization of the lattice-based polar metamaterial. (a) The topology of the lattice metamaterial; (b) The unit cell of the lattice metamaterial. In the design, $a = 18$ mm, $\theta = 30^\circ$, $d_1 = 0.38$ mm, and $d_2 = d_3 = 0.84$ mm. The Young's modulus and Poisson's ratio of the soft and hard materials are 1GPa, 0.33, 100GPa, and 0.33, respectively; (c) The hard material is connected to the ground with a rod that supports near-zero rotation but nearly free in-plane translation; (d) Effective elastic constants of the lattice metamaterial in comparison with the desired transformed elastic constants; (e) A carpet cloak is constructed using the lattice metamaterial, where $l_a = 360$ mm. For the carpet cloaking, material parameters of the background material are selected as $\lambda = \mu = 9.14$ MPa and $\rho_0 = 1000$ kg/m³. To satisfy the mass requirement, the mass density of the hard material is assumed as $\rho_h = 7035$ kg/m³.

Now we focus on the homogenized elastic response of the transformed lattice. Our aim is to validate the design step. Using the volume average approach, the effective elastic constants of the transformed spring-mass system can be analytically obtained in terms of θ , k_1 , k_2 , and k_3 for $x >$

0 as

$$\begin{aligned}
\bar{c}_{1111} &= \bar{A}(16k_1 + k_2 + k_3), \bar{c}_{1122} = \bar{A}(3k_2 + \bar{B}k_3), \bar{c}_{1112} = \bar{A}(-16\tan(\theta)k_1 - \sqrt{3}k_2 + \bar{C}k_3), \\
\bar{c}_{1121} &= \sqrt{3}\bar{A}(-k_2 + k_3), \bar{c}_{2222} = \bar{A}(9k_2 + 3\bar{C}^2k_3), \bar{c}_{2212} = \sqrt{3}\bar{A}(-3k_2 + \bar{C}^2k_3), \\
\bar{c}_{2221} &= 3\bar{A}(-\sqrt{3}k_2 + \bar{C}k_3), \bar{c}_{1212} = \bar{A}(16\tan^2(\theta)k_1 + 3k_2 + \bar{C}^2k_3), \bar{c}_{1221} = \bar{A}(3k_2 + \bar{B}k_3), \\
\bar{c}_{2121} &= 3\bar{A}(k_2 + k_3)
\end{aligned} \tag{9.9}$$

where $\bar{A} = \frac{1}{\Omega[\sqrt{3}-\tan(\theta)]}$, $\bar{B} = 3 - 2\sqrt{3}\tan(\theta)$ and $\bar{C} = \sqrt{3} - 2\tan(\theta)$. Letting the spring constants be those of the virtual lattice: $k_1 = k_2 = k_3 = \frac{4\mu}{\sqrt{3}}$ and $\lambda = \mu$, the obtained effective elastic moduli in Eq. (9.9) are exactly those that appear in the transformed material parameters of Eq. (9.8); thus our predictions based on discrete transformation elasticity are valid.

It should be worth mentioning that the lattice we constructed through discrete transformation elasticity naturally admits a zero-mode $\mathbf{e}_0 = \mathbf{E}_0 \mathbf{F}^{-1}$, where $\mathbf{E}_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Physically, the rigid rotation in the virtual lattice is transformed into a zero-mode in the transformed lattice. As shown in Fig. 9.3c, the motions on Points 1-6 are identical before (when the motion is a rigid rotation) and after (when the motion is a zero-mode) the transformation, and naturally satisfy the displacement gauge $U(\mathbf{x}) = \mathbf{u}(\mathbf{x})$ adopted in the transformation.

9.4 Microstructure realization of the polar metamaterial

Informed by the transformed mass-spring lattice, we now explore ways to numerically design a polar metamaterial to confirm the extent to which a candidate metamaterial microstructure meets the requirements of a cloaking material and is ready to be fabricated. Figs. 9.4a and 9.4b illustrate the metamaterial design, where a hard material (blue area) is employed for constructing rigid masses, and a soft material (yellow area) functions as springs. The geometry of the hard material can be modified to meet the mass density requirement as long as it maintains contact with the soft material at the three points located on the red dotted. To suppress the rotational motion of each mass, we introduce a rod connecting the mass to the ground (see Fig. 9.4c). The rod near the ground is indented to significantly reduce its bending stiffness along u_1 and u_2 directions. As a result, the rotational stiffness of the rod is much greater than its bending stiffness. Therefore, rotation of the supported mass is efficiently suppressed and only in-plane translation is allowed. The conventional diamond-shaped bar made of the soft material is selected to mimic the spring (Fig. 9.4 b), which leads to negligible bending moments at its ends when the bar twists around the mass. By varying parameters d_1 , d_2 , and d_3 , we can realize different effective spring constants $k_{\text{eff}} = k_1 = k_2 = k_3$ for the transformed lattice.

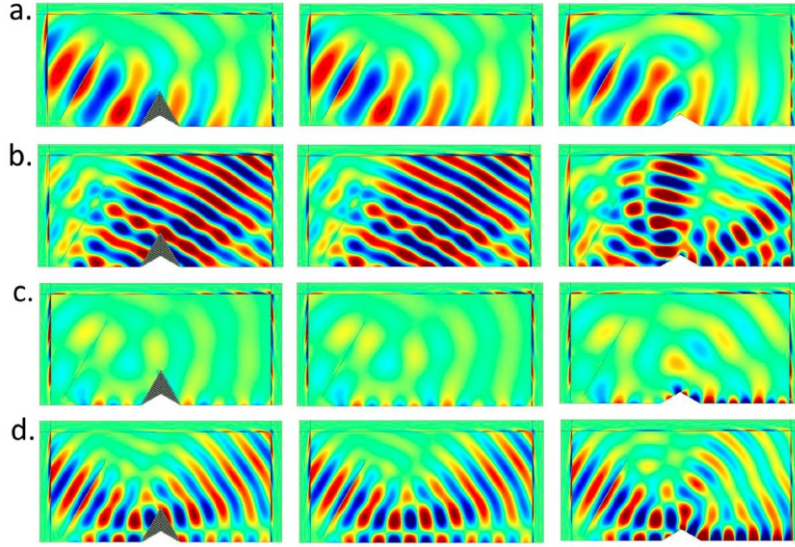


Figure 9.5 Numerical simulations of the carpet cloak with the lattice-based polar metamaterial. Figures in the first column show the results when the lattice metamaterial is embedded in the background medium for cloaking. Figures in the second column show the results when the background medium is intact. Figures in the third column show the results when the void is non-coated. (a) Divergence of the displacement field with a pressure incidence; (b) Curl of the displacement field with a pressure incidence; (c) Divergence of the displacement field with a shear incidence; (d) Curl of the displacement field with a shear incidence.

Having selected geometric and material parameters of the metamaterial, we now determine the effective elastic constants needed for the cloaking application. We numerically calculate the effective elastic constants based on the volume average approach similar to that employed analytically in (Nassar et al., 2018). In the approach, the mechanical response of the metamaterial and the effective medium is considered equivalent if the strain energy density of the former is equal to that of the latter. To obtain the effective c , ten independent numerical tests are performed for solving the ten independent components in c dictated by the major symmetry ($c_{ijkl} = c_{klij}$). In each of the tests, we apply the rotational constraint on the left bottom boundary of the mass to mimic the required grounded connection. We prescribe displacements on the boundaries of the unit cell to induce different strain states. Other boundaries are left free. During the simulations, we first individually apply the four uniform strain states: two uniaxial and two shear strains (e_{11} , e_{22} , e_{12} , or e_{21}), and then apply the six mixed strains: ($\{e_{11}, e_{22}\}$, $\{e_{11}, e_{12}\}$, $\{e_{11}, e_{21}\}$, $\{e_{22}, e_{12}\}$, $\{e_{22}, e_{21}\}$, or $\{e_{12}, e_{21}\}$). The strain energy per unit cell is numerically calculated for each of the ten cases with the plane stress hypothesis, from which the ten independent elastic constants can be retrieved. As shown in Fig. 9.4d, effective elastic constants of the lattice metamaterial (orange bars) agree very well with the transformed parameters in Eq. (9.8) (blue bars), which demonstrates the validity of the microstructure design.

9.5 Cloaking simulations

With the proposed microstructure of the lattice-based metamaterial, we are ready to construct a carpet cloak for elastic wave propagation tests (see Fig. 9.4e). Since the lattice-based metamaterial is originated from the discrete transformation elasticity, the metamaterial can automatically match the transformed physical space without any geometric alterations. This procedure, therefore, provides significant advantages in building lattice topologies for transformation elasticity. The number of unit cells in the cloak is chosen sufficiently large so as to enable satisfactory cloaking performance but not too large so as to avoid impractical simulation times. In the design, we chose 19 unit cells in the bottom row of the lattice, and tessellate them row-by-row with a total of 19 rows to create the carpet cloak. Note that the lattice metamaterial has seamless connections with the background material as shown in Fig. 9.4e. From a fabrication point of view, the total mass of the hard material can be tuned or adjusted by geometry modifications. In particular, special treatments are made for masses on the middle line of the cloak, where discontinuities emerge. We merge the two half masses into one mass and properly enlarge its area equal to the area of other masses.

In numerical simulations, Navier's equations in the background medium and in the lattice metamaterial are solved using COMSOL Multiphysics. Perfectly matched layers surrounding the background medium are adopted to suppress reflected waves from boundaries. Incident waves with Gaussian profiles in the direction of -60° are emitted to the cloak region (see Supplementary Information for other incident angles). Simulation results are shown in Fig. 9.5 for an incident pressure wave (Figs. 9.5a and 9.5b) and incident shear wave (Figs. 9.5c and 9.5 d) at 400 Hz. Figures in the first column show the results when the lattice metamaterial is embedded in the background medium for cloaking. For reference, we perform the same simulations where the background medium is intact (the second column) and the void is non-coated (the third column). The divergence and curl of the displacement field are shown in Figs. 9.5a and 9.5c and Figs. 9.5b and 9.5 d, respectively. It is clearly seen that the designed lattice metamaterial demonstrates excellent cloaking performance as it almost perfectly suppresses pressure, shear, and Rayleigh scattering due to the presence of the void. Specifically, the lattice metamaterial significantly reduce scattered shear waves from the void, which are more sensitive to defects compared with pressure waves, due to the shorter wavelength. Simulations are also conducted to study the cloaking performance of the lattice metamaterial under different incidence angles. The results still demonstrate an excellent cloaking performance.

The lattice metamaterial we designed comprises no resonance structures, implying that the design could be operated at much broader frequency regions and as long as the unit cell size is small compared to the incident wavelength (that said, there are no limitations on the size of the cloaked void). To demonstrate this, we perform numerical simulations at different frequencies. As shown in Fig. 9.6, pressure waves are excited in the direction of -30° , and we compare the cloaking performance of the metamaterial (Figs. 9.6 a and 9.6b: Divergence and curl of the displacement field) with the wave fields where the background medium is intact (Figs. 9.6c and 9.6 d: Divergence and curl of the displacement field). In the figure, the first to fourth columns show the simulation results at 200, 400, 600 , and 800 Hz, respectively. It can be clearly seen that good

cloaking performances still retain at frequencies below 600 Hz, whereas, at 800 Hz, the shear wave field produced by the metamaterial cloak slightly deviates from that reflected from a flat boundary. To improve the performance, one needs to reduce the size of the metamaterial unit cell.

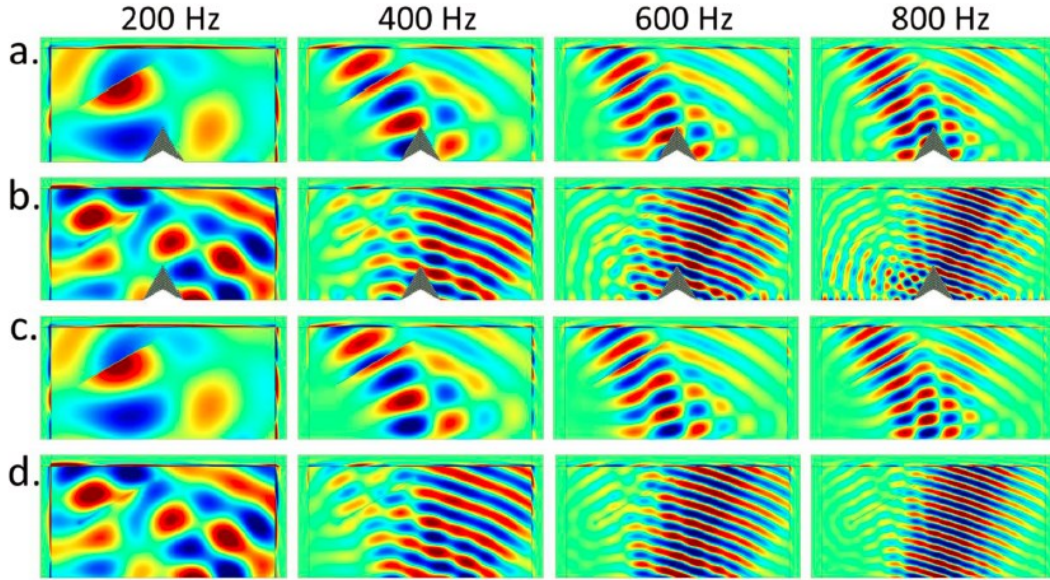


Figure 9.6 Numerical simulations of the carpet cloak with -30° incidence at different frequencies. Figures in the first to fourth columns show the results at 200, 400, 600, and 800 Hz, respectively. (a,b) Divergence (a) and curl (b) of the displacement field with a pressure incidence, where the cloak is embedded in the background medium; (c,d) Divergence (c) and curl (d) of the displacement field with a pressure incidence, where the background medium is intact.

Besides dynamic pressure and shear loads, the lattice metamaterial cloak can also hide the void from static loads. In the static numerical tests, we fix the right boundary of the background plate and prescribe displacements on its left boundary. Other boundaries are left free. The prescribed displacement is along either horizontal (Figs. 9.7a and 9.7 b) or vertical (Figs. 9.7c and 9.7 d) directions to induce elongation or shear in the background plate. Similar to Fig. 9.5, the first column in Fig. 9.7 shows the results when the lattice metamaterial cloak is embedded, and the second and third columns are for the intact background medium and the non-coated void. Again, the divergence and curl of the displacement field are shown in Figs. 9.7a and 9.7c and Figs. 9.7 b and 9.7 d, respectively. We can easily find the lattice metamaterial cloak significantly improves the pressure and shear fields when the background plate undergoes elongation. In addition, the triangular void is insensitive to the shear load, where the figures in the first to third columns are nearly identical. Finally, it is also worth mentioning that the modulus of the hard material can be reduced in real designs, whereas quantitative studies should be performed to ensure the cloaking performance.

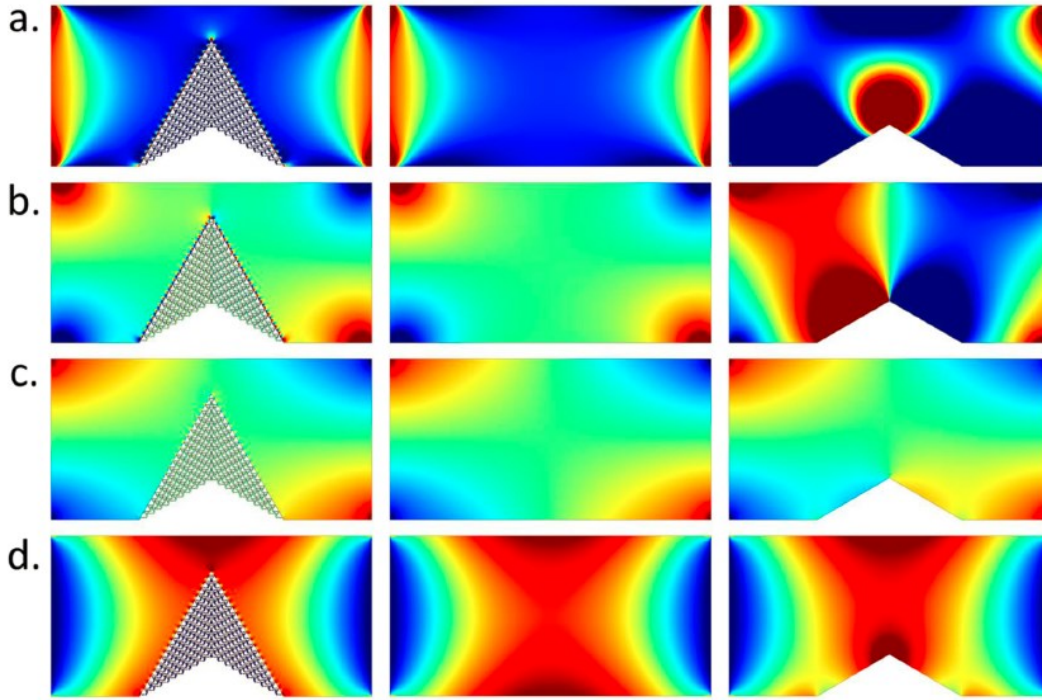


Figure 9.7 Numerical simulations of the carpet cloak with static loads. Figures in the first column show the results when the lattice metamaterial is embedded in the background medium for cloaking. Figures in the second column show the results when the background medium is intact. Figures in the third column show the results when the void is non-coated. (a) Divergence of the displacement field with an elongation along the horizontal direction; (b) Curl of the displacement field with an elongation along the horizontal direction; (c) Divergence of the displacement field with a shear along the horizontal direction; (d) Curl of the displacement field with a shear along the horizontal direction.

9.6 Conclusions

In conclusion, we report a simple approach based on discrete transformation elasticity to design lattice-based polar metamaterials that can automatically satisfy constitutive requirements from arbitrary coordinate transformations. In particular, we engineer a lattice metamaterial that exhibits desired polarity, chirality, and anisotropy of a transformed continuum. Based on this polar metamaterial, an elastic carpet cloak is constructed and demonstrated numerically for concealing a triangular void. The research establishes a theoretical framework for tackling the inverse design problems of lattice-based materials targeting transformed macroscopic constitutive tensors, not only for those required by elastic cloaks. Given the fact that transformation elasticity is a highly flexible method for exploiting new material parameters by varying coordinate transformations or materials in the virtual space, the lattice metamaterials designed based on discrete transformation elasticity could foster a wide range of control functions in dynamics and statics in general i.e. waveguiding, illusion, and shape morphing. Furthermore, the metamaterials designed based on this approach is ready to be used without geometric alterations or discontinuities. This approach, therefore, holds advantages over other methods in building lattice based topologies related to transformation elasticity. As far as the proposed approach itself goes, it generalizes to 3D in a

straightforward manner; the structure will simply extend in a third periodicity direction along which the geometry and the elastic properties are determined by the same principles that determined the in-plane properties. In 3D, fabrication becomes more challenging since a 3D system of grounded rods would be seriously impractical. In that case, the rotations could be impeded dynamically by leveraging the rotational resonance of appropriately designed embedded resonators.

10. Realization of active metamaterials with odd micropolar elasticity

10.1 Introduction

Responsive materials in both biology and engineering distinguish themselves by their ability to respond to external stimuli in tailored ways. For example, muscles contract in response to electrical signals (Caruel and Truskinovsky, 2018) and mechanocaloric solids undergo dramatic deformations in response to temperature changes (Wang et al., 2019). Unlike computers or multicellular organisms with specialized functional components, the undifferentiated physical machinery available in a distributed material presents unique challenges for sensing, information processing, and response. Yet, the range of available functionalities is fundamentally extended when the materials possess distributed, local reservoirs of energy (Needleman and Dogic, 2017). Such active materials exhibit responses not allowed by their passive, or energy conserving, counterparts.

Recent advances in metamaterial design and prototyping have utilized active components to achieve functionalities such as sensing, lasing, and cloaking (Fleury et al., 2015; Peng et al., 2014; Lin et al., 2011, frequency dependent reflectivity (Trainiti et al., 2019), unidirectional wave amplification (Brandenbourger et al., 2019) and so on. Nonetheless, all the active non-reciprocal metamaterials so-far realized exhibit either of the following fundamental limitations: the active nonreciprocal effects either vanish from the linear response in the quasistatic limit (Trainiti et al., 2019) or they require the presence of background sources of linear or angular momentum (Brandenbourger et al., 2019). As a result, their functionalities are largely restricted to finite-frequency control or fundamentally require the sample to be in contact with an additional medium that acts as a momentum sink or source. Here, we report the design, construction, and experimental demonstration of a freestanding metamaterial whose elasticity is unattainable in passive media (Fig. 10.1a–d). The metamaterial is constructed with piezoelectric elements mounted on a beam and controlled by electrical circuits. Our approach enables an asymmetric coupling between bending and shearing in micropolar solids. This results in an odd micropolar material that simultaneously breaks parity and Maxwell-Betti reciprocity. The spectrum of the metabeam exhibits a non-Hermitian topological index which results in the localization of vibrational modes at sample boundaries. We experimentally show the resulting unidirectional amplification/attenuation of waves propagating through the metamaterial. Our work sheds light on controlling non-reciprocal elasticity in artificial materials.

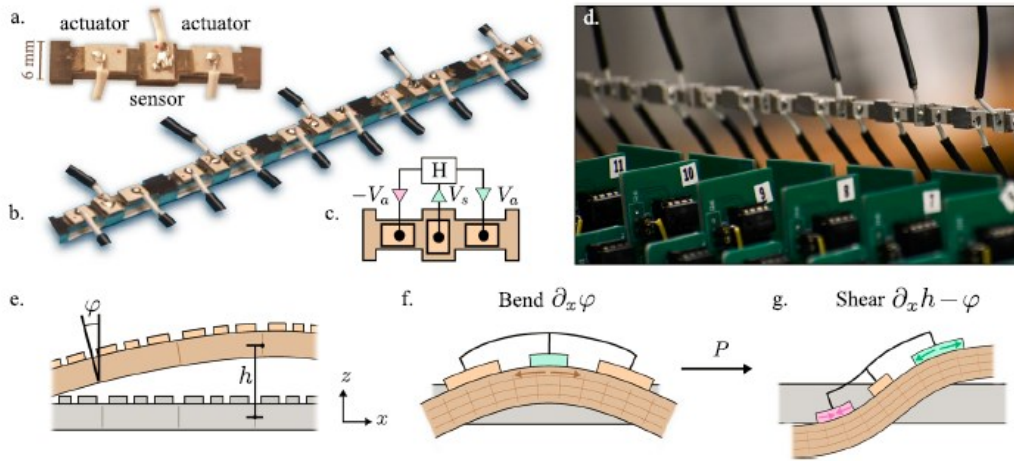


Figure 10.1 Design and mechanics of an odd micropolar metabeam. a A single unit cell featuring three piezoelectric patches mounted on a beam: one that acts as a sensor, and two that act as actuators b A segment of the full metabeam. c Each unit cell has an electronic loop. The voltage V_s induced by the central piezoelectric is fed into a transfer function $H(\omega) = V_a(\omega)/V_s(\omega)$ that sends opposing voltages V_a and $-V_a$ to the piezoelectric actuators. d A photograph of the metabeam (horizontal) with the electronic circuits in the foreground. We note that the mechanical forces from the attached wires are negligible. The wires act only as sources of energy and computation, but not of linear or angular momentum. e The motion of the metabeam can be described by two independent fields, φ and h , which parameterize the angular and vertical displacements of the metabeam. Notice that under a reflection about the z -axis, we have $\varphi \rightarrow -\varphi$ and $h \rightarrow h$. f When the beam bends, the center piezoelectric is stretched. g The antisymmetric electronic actuation then gives rise to a shearing stress proportional to the modulus P .

10.2 Design of the active metamaterial

Design of the active metamaterial. The metamaterial we design takes the form of a thick beam, whose shape is characterized by two independent degrees of freedom (Fig. 10.1e): the height $h(x)$ of the mid-plane and the angle $\varphi(x)$ of the cross section with respect to the vertical. A single unit cell in the beam is equipped with three piezoelectric patches that enable shape-sensing and response. The central patch acts as a sensor that acquires a voltage proportional to the elongation or contraction of the top surface (Fig. 10.1f). The piezoelectric patches at the front and back of the unit cell serve as mechanical actuators that elongate and contract in response to an applied voltage. A transfer function $H(\omega)$ processes the input voltage from the central patch and sends output signals to the actuating patches (Fig. 10.1c). For each unit cell, we implement $H(\omega)$ using a minimal electrical circuit (Fig. 10.1d). The electronics only couple piezoelectric patches within a single unit cell, thereby creating a control system that is both local and decentralized. The resulting system only needs to be connected to a voltage or, more generally, energy source. One practical advantage of this approach is that the voltage or power source can easily be housed inside the medium itself.

The active metamaterial we design is freestanding—it does not push or pull on an external medium. Indeed, it obeys Newton’s third law by preserving both angular and linear momentum as a traditional beam. However, the crucial difference between a traditional beam and the one we

construct is the presence of internal energy sources. We design the feedback such that when the central patch experiences elongation or compression due to bending of the beam, the electronic loop produces output voltages that are antisymmetric (Fig. 10.1c), resulting in shear stresses (Fig. 10.1g). However, the ensuring shear strain does not stretch or compress the central piezoelectric patch. Therefore, the electromechanical control loop is entirely feed-forward: bending induces shear, while shear does not induce bending.

10.3 Odd micropolar elasticity

The feedback results in an elastic response that cannot be realized without an internal source of energy. This effect, apparent in the emergent continuum equations, may be deduced solely using symmetries and conservation laws based on classical beam theory (1D micropolar elasticity). Crucial to our design is the notion of a parity inversion $\mathcal{P}$, defined here to be a mirror reflection of the beam about the z -axis that sends x to $-x$. Fig. 10.1e shows that under parity, the two independent degrees of freedom, $h(x)$ and $\varphi(x)$, transform as

$$h(x) \xrightarrow{\mathcal{P}} h(-x) \quad (10.1)$$

$$\varphi(x) \xrightarrow{\mathcal{P}} -\varphi(-x) \quad (10.2)$$

Since $\varphi(x)$ acquires a minus sign under parity, we say that $\varphi(x)$ is a micropolar degree of freedom. The equations of motion for a freestanding micropolar beam are then built out of conservation laws. Linear momentum conservation implies that

$$\rho \ddot{h} = \partial_x \sigma_{zx} \quad (10.3)$$

where σ_{zx} is the shear stress and ρ is the mass density. Moreover, angular momentum conservation implies

$$I \ddot{\varphi} = \partial_x M + \sigma_{zx} \quad (10.4)$$

where M is the bending moment and I is the cross-sectional moment of inertia. The moment M and stress σ_{zx} are themselves determined by the deformation of the beam via a set of constitutive relations. To leading order in gradients of h and φ , the internal geometry of the beam is approximated by two independent types of deformation: bending $b(x)$ (Fig. 10.1f) and shearing $s(x)$ (Fig. 10.1g), defined as

$$b(x) = \partial_x \varphi \quad (10.5)$$

$$s(x) = \partial_x h - \varphi \quad (10.6)$$

Under parity inversions, Eqs. (10.1, 10.2) imply that

$$b(x) \xrightarrow{\mathcal{P}} b(-x) \quad (10.7)$$

$$s(x) \xrightarrow{\mathcal{P}} -s(-x) \quad (10.8)$$

Assuming a linear response, one may in general write the linear constitutive relations as:

$$\begin{bmatrix} \sigma_{zx}(t) \\ M(t) \end{bmatrix} = \int_{-\infty}^{\infty} \begin{bmatrix} C_{11}(t') & C_{12}(t') \\ C_{21}(t') & C_{22}(t') \end{bmatrix} \begin{bmatrix} s(t-t') \\ b(t-t') \end{bmatrix} dt' \quad (10.9)$$

or in terms of frequency:

$$\begin{bmatrix} \sigma_{zx}(\omega) \\ M(\omega) \end{bmatrix} = \begin{bmatrix} C_{11}(\omega) & C_{12}(\omega) \\ C_{21}(\omega) & C_{22}(\omega) \end{bmatrix} \begin{bmatrix} s(\omega) \\ b(\omega) \end{bmatrix} \quad (10.10)$$

We denote the matrix operator on the right-hand side of Eq. (10.9) as $C(t)$. Causality implies that $C(t) = 0$ for $t < 0$ and reality of $C(t)$ implies $C(-\omega) = C^*(\omega)$. Direct substitution of Eq. (10.10) into Eqs. (10.3, 10.4) reveals that the equations of motion are invariant under parity if and only if C_{21} and C_{12} are zero. Hence, we say that C_{21} and C_{12} are micropolar moduli.

It is useful to parameterize the $C(\omega)$ as

$$\begin{bmatrix} C_{11}(\omega) & C_{12}(\omega) \\ C_{21}(\omega) & C_{22}(\omega) \end{bmatrix} = \begin{bmatrix} \mu(\omega) & \alpha(\omega) + \beta(\omega) \\ \alpha(\omega) - \beta(\omega) & B(\omega) \end{bmatrix} \quad (10.11)$$

where μ is the shear modulus and B is the bending modulus, and α and β are the symmetric and antisymmetric components of the micropolar moduli. If the beam lacks an internal source of energy, then the total work done by the beam on any deformation process that begins and ends in the same configuration must be less than zero: $\Delta W \leq 0$. This energy condition places the following constraints on the finite frequency linear response coefficients:

$$0 \geq \text{Im}[\mu(\omega) + B(\omega)] \quad (10.12)$$

$$0 \leq \text{Im}[\mu(\omega)] \text{Im}[B(\omega)] - \text{Im}[\alpha(\omega)]^2 - \text{Re}[\beta(\omega)]^2 \quad (10.13)$$

for all ω . Notice that the reality condition $C(-\omega) = C^*(\omega)$ implies $\text{Im}[C] = 0$ at $\omega = 0$. As a consequence, we must have $\beta(\omega \rightarrow 0) = 0$ for any passive beams.

However, our active metamaterial has an internal source of energy and thus need not obey this constraint. The feed-forward coupling between bending and shearing suggests a linear response matrix of the form:

$$\begin{bmatrix} C_{11}(\omega) & C_{12}(\omega) \\ C_{21}(\omega) & C_{22}(\omega) \end{bmatrix} = \begin{bmatrix} \mu(\omega) & P(\omega) \\ 0 & B(\omega) \end{bmatrix} \quad (10.14)$$

The electronic control loop introduces the coefficient $= 2\alpha = 2\beta$, which we refer to as the odd micropolar modulus. This modulus breaks two crucial symmetries. Since the electromechanical coupling violates parity, the active modulus P must occur in the off-diagonal entries. Moreover, P occurs only in the upper right entry because the electro-mechanical coupling is feedforward: bend

causes shear, but shear does not cause bend. As a result, the matrix C is asymmetric, indicating that the beam violates Maxwell-Betti reciprocity, even at zero frequency.

For our metabeam, we measure the moduli via COMSOL simulations in which we apply controlled displacements at finite frequency to the front and back faces of a single unit cell. By measuring the reaction forces on these faces, we determine the resulting stresses and, consequently, the moduli. We empirically find that $\mu = 1.3 \times 10^9 \text{ kg/ms}^2$ and $B = 0.112 \times 10^6 \text{ kg/s}^2$ are approximately independent of frequency, while $P(\omega) = \Pi H(\omega)$, where $H(\omega)$ is the transfer function and $\Pi = 4.7 \times 10^6 \text{ kgm/s}^2$ is a material constant. From the metabeam geometry and materials, we compute the average volumetric density $\rho = 5613 \text{ kg/m}^3$ and the average cross-sectional moment of inertia $5.9 \times 10^{-3} \text{ kg/m}$.

10.4 Energy cycles

It is useful to consider the elastic limit of Eq. (10.11) in which $C(\omega)$ is real and approximately independent of frequency. For a passive beam in this limit, the matrix C is obtained by approximating the energetic cost of deformation by a quadratic function:

$$W = \frac{\mu}{2} s^2 + \frac{B}{2} b^2 + \alpha s b \quad (10.15)$$

The work (per unit volume) done in an infinitesimal deformation of the beam is given by:

$$dW = \sigma_{zx} ds + M db \quad (10.16)$$

From Eq. (10.16), we conclude $\sigma_{zx} = \frac{\partial W}{\partial s}$ and $M = \frac{\partial W}{\partial b}$. Hence, we obtain the linear constitutive relations:

$$\begin{bmatrix} \sigma_{zx} \\ M \end{bmatrix} = \begin{bmatrix} \mu & \alpha \\ \alpha & B \end{bmatrix} \begin{bmatrix} s \\ b \end{bmatrix} \quad (10.17)$$

Since C in Eq. (10.17) is obtained via a second derivative of an energy function, it is symmetric. This is a manifestation of the Maxwell-Betti reciprocity theorem. In this case, the cumulative work done over a sequence of deformations depends only on the initial and final configurations: $\Delta W = \int dW = W_{\text{final}} - W_{\text{initial}}$. In particular, if the procedure begins and ends at the same state, we have $\Delta W = 0$.

Notice that the constitutive relation in Eq. (10.14) violates $C = C^T$. Hence, the Maxwell-Betti reciprocity theorem implies that the form of C in Eq. (10.14) does not follow from a potential energy function. To see this explicitly, consider the differential of energy given by:

$$dW = \sigma_{zx} ds + M db \quad (10.18)$$

$$= d \left(\frac{\mu}{2} s^2 + \frac{B}{2} b^2 + \frac{P}{2} s b \right) + \frac{P}{2} (b ds - s db) \quad (10.19)$$

Notice that the second term in Eq. (10.19) cannot be expressed as the differential of a potential.

By Green's theorem, integrating the work done $\oint dW$ over a closed loop in strain space yields:

$$\oint_{\partial V} dW = \oint_V P ds db = P \times \text{Area} \quad (10.20)$$

where "Area" is the signed area of the region V enclosed by the path in the space of strains. Notice that the work done per cycle can be either positive or negative depending on the orientation of the path, and is independent of the rate of the process (in the quasistatic limit). Since the crucial ingredient for such cycles is an antisymmetry in C , we refer to this form of elasticity as "odd" (i.e., antisymmetric) elasticity (Scheibner et al., 2020).

To verify that our design displays this property at low frequencies, we perform COMSOL simulations of the beam with full piezoelectric coupling. As illustrated in Fig. 10.2a, we subject a single unit cell to a four-step protocol of shearing and bending by enforcing displacement-controlled boundary conditions at the two ends of the beam. We measure the reaction forces on the control surfaces to compute the work done by the beam, plotted in Fig. 10.2b. When the deformations are performed in a clockwise direction in strain space, as shown in the top panel of Fig. 10.2b, the cumulative work done is negative once the unit cell returns to its initial configuration. Hence, energy flows from the external agent into the internal power reserves of the medium. When the cycle is reversed, so is the flow of energy. The ability to inject or extract mechanical energy through quasistatic cycles is synonymous with odd elasticity, i.e. a quasistatic stress-strain relationship that does not follow from a potential energy. In the quasistatic limit, the total work done is given by P times the area enclosed in strain space. In the Methods, we also derive energy relations for cycles at finite frequency. For those cases, the total work done is related to $|P|$, $\arg(P)$, and the area enclosed in strain space.

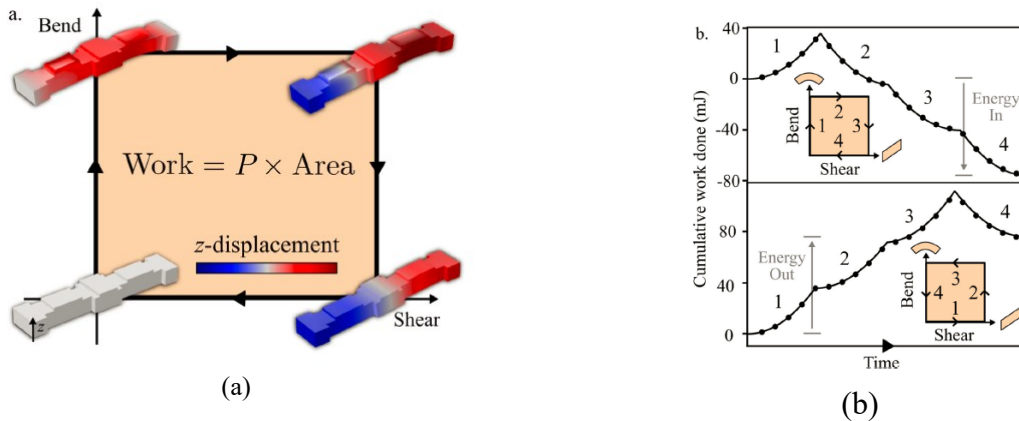


Figure 10.2 Quasistatic deformation cycles with odd micropolar elasticity. a The state of the unit cell is tracked in the space of shear and bend. When a quasistatic closed path is traced out in this space, the unit cell performs work per unit volume that is proportional to the area times the modulus P . The z -displacement is provided in arbitrary units. b We numerically compute the work done for a clockwise (top) and a counterclockwise (bottom) path. The solid lines are predictions from the continuum theory, and the black dots result from finite element simulations of the unit cell.

In the simulations, maximum amplitudes of bending and shearing are $b_{\max} \approx 10^{-1} \text{ m}^{-1}$ and $s_{\max} \approx 10^{-2}$, respectively.

10.5 Odd micropolar elastodynamics and the non-Hermitian skin effect

We now ask about the dynamic consequences of the active feed-forward control with parity violation. We first consider the elastic approximation in which $C(\omega)$ is real and frequency independent. The linearized continuum equations governing the motion of the beam are given by:

$$\rho \ddot{h} = \mu \partial_x^2 h + P \partial_x^2 \varphi - \mu \partial_x \varphi \quad (10.21)$$

$$I \ddot{\varphi} = \mu \partial_x^2 h + B \partial_x^2 \varphi + P \partial_x \varphi - \mu \varphi \quad (10.22)$$

Using Fourier transform, Eqs. (10.21, 10.22) may be cast in the form:

$$\omega \begin{bmatrix} \tilde{p}_h \\ \tilde{p}_\varphi \\ \tilde{s} \\ \tilde{b} \end{bmatrix} = \omega_1 \underbrace{\begin{bmatrix} 0 & 0 & -kl_1 & -\tilde{P}l_1k \\ 0 & 0 & i & i\tilde{P} - kl_2 \\ -kl_1 & -i & 0 & 0 \\ 0 & -kl_2 & 0 & 0 \end{bmatrix}}_{D(k)} \begin{bmatrix} \tilde{p}_h \\ \tilde{p}_\varphi \\ \tilde{s} \\ \tilde{b} \end{bmatrix} \quad (10.23)$$

$$s = \sqrt{\mu} (\partial_x h - \varphi), \quad \tilde{b} = \sqrt{B} \partial_x \varphi \quad (10.24)$$

$$p_\varphi = \sqrt{I} \dot{\varphi}, \quad \tilde{p}_h = \sqrt{\rho} \dot{h} \quad (10.25)$$

where k and ω are the wave number and frequency associated with the Bloch wave $e^{i(kx - \omega t)}$. We have introduced the notation:

Here, $\tilde{s}$ represents the shear, $\tilde{b}$ represents the bending, $\tilde{p}_\varphi$ is the angular momentum, and $\tilde{p}_h$ is the z-component of the linear momentum. This parameterization is natural since the standard inner product

$$2e = |\tilde{p}_h|^2 + |\tilde{p}_\varphi|^2 + |\tilde{s}|^2 + |\tilde{b}|^2 \quad (10.26)$$

is equal to twice the mechanical energy density e . The dynamical matrix $D(k)$ depends on four parameters: $l_1 \equiv \sqrt{I/\rho}$ is roughly the thickness of the metabeam; $l_2 \equiv \sqrt{B/\mu}$ is the distance over which shearing and bending of equal transverse deflection cost equal amounts of energy; $\omega_1 \equiv \sqrt{\mu/I}$ sets a frequency scale separating transverse flexural modes and high frequency shearing modes; finally, the parameter $\tilde{P} \equiv P/\sqrt{B\mu}$ is the normalized odd micropolar modulus. For our metabeam $l_1 \approx 10^{-3} \text{ m}$, $l_2 \approx 10^{-2} \text{ m}$, $\omega_1 \approx 10^5 \text{ Hz}$, and $\tilde{P} \lesssim 1$.

Within the continuum theory, the vibrational dynamics can be captured by solving the secular equation $\det D(k)[- \omega] = 0$, which takes the form

$$0 = \tilde{\omega}^4 - \left[1 - i \tilde{P} k l_2 + k^2 (l_1^2 + l_2^2) \right] \tilde{\omega}^2 + k^4 l_1^2 l_2^2 \quad (10.27)$$

where $\tilde{\omega} = \omega/\omega_1$. Notice that Eqs. (10.21, 10.22) are two coupled second order equations and hence permit a dispersion with four branches. For small wavenumber, the dispersion for the low frequency flexural bands is given by:

$$\omega_{\pm} = \pm \omega_1 \left[l_1 l_2 k^2 \pm i \tilde{P} l_1 l_2^2 k^3 + \mathcal{O}(l_1^2 l_2^2 k^4) \right] \quad (10.28)$$

As can be seen from Eq. (10.28), when P is nonzero and real, the periodic boundary spectrum acquires a nonzero imaginary component. The nonzero imaginary contribution arises since the active metabeam has the ability to physically introduce or remove mechanical energy. Moreover, the modulus P violates parity and hence breaks the symmetry between $k \rightarrow -k$, in contrast to the design in ref. 7 where the parity violation is induced by a term in the dispersion that is linear in k .

The simultaneous breaking of parity and energy conservation allows our active micropolar metamaterial to selectively amplify and attenuate waves based on their direction of travel. In Fig. 10.3a, we show the spectrum of the flexural mode in the right half of the complex plane for $P > 0$. The solid line is the continuum theory, valid at small $k \ll 1/\sqrt{l_1 l_2}$, and the discrete points are the results of fully coupled COMSOL simulations. In the calculations, $P = \pm 3\Pi$, and there are no free fitting parameters. (The material constants Π, ρ, B , and μ in the continuum theory are determined from independent simulations, see Odd micropolar elasticity.) As illustrated in Fig. 10.3a, for $P > 0$, $\text{Im}(\omega) > 0$ whenever $\text{Re}(d\omega/dk) > 0$ and $\text{Im}(\omega) < 0$ whenever $\text{Re}(d\omega/dk) < 0$. Physically, this means that wave packets traveling to the right are amplified, while wave packets traveling to the left are attenuated. As an illustration, Fig. 10.3b shows the inverse penetration depth κ for ω taken to lie along the positive real axis. From the continuum theory, we compute the analytical formula for κ

$$\kappa = \sqrt{\frac{\rho}{B}} \frac{P}{4\mu} \omega \quad (10.29)$$

In Fig. 10.3b, we compare Eq. (10.29) to the COMSOL simulations and a semi-analytical technique known as the transfer matrix method.

This unidirectional amplification can be understood from the point of view of the non-Hermitian skin effect. Given a complex frequency ω , we can define the following topological index:

$$v(\omega) = \frac{1}{2\pi i} \sum_{\alpha} \oint_{-\pi/L}^{\pi/L} \frac{d}{dk} \log[\omega_{\alpha}(k) - \omega] dk \quad (10.30)$$

where L is the length of a single unit cell and $\omega_{\alpha}(k)$ is the frequency of the α band. Here, we take α to run over the flexural bands. The topological index $v(\omega)$ indicates whether a system with semi-infinite boundary conditions will host a localized mode at the frequency ω . When $v(\omega) > 0$, a semi-infinite system with Homain $x \in [0, \infty)$ will host a mode localized to its left boundary at

frequency ω . Likewise, when $\nu(\omega) < 0$, a semi-infinite system with Homain $x \in (-\infty, 0]$, will host a mode localized to its right boundary.

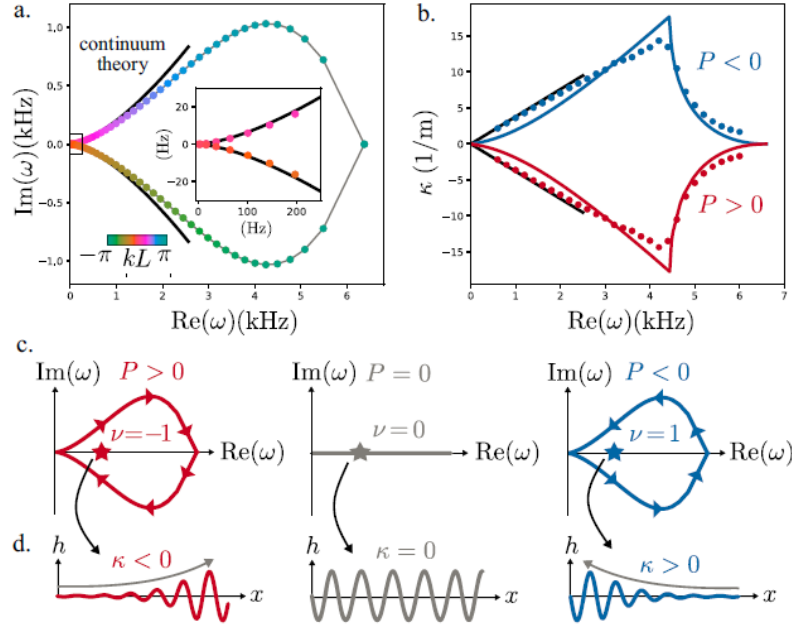


Figure 10.3 Non-Hermitian skin effect via the odd micropolar elasticity. a. The vibrational spectrum for the flexural mode of a metabeam boundary conditions and odd micropolar modulus $P = 3\pi$. The black line results from the continuum the gery plotted as a function of k in the continuum theory and in the numerics, see Fig. 12.7, respectively. The inset compares the continuum theory and simulations for small wavenumbers $\ll 1/\sqrt{I_1 I_2}$. b. The inverse penetration depth κ for real ω in a medium with open boundary conditions. The points are the κ for real ω in a medium with open boundary conditions. The points are the κ for real ω in a medium with open boundary conditions. c. The localized states are connected to a topological index $\nu(\omega)$. The periodic boundary spectrum for $P > 0$, $P = 0$, and $P < 0$ are resented schematically by the solid lines. The arrows indicate the direction of increasing k . For a given frequency ω , the winding number $\nu(\omega)$ of the periodic bectrum indicates the presence of a localized mode. d. the localization of eigenmode at the value of ω denoted by star in (c) schematically illustrated.

As shown in Fig. 10.3c, d, the sign of κ can be rationalized by examining $\nu(\omega)$ for an example frequency ω (denoted by the tar) along the real axis. For $P > 0$, the periodic boundary sectrum (red) winds once counter-clockwise around the star, and hence $\nu(\omega) < -1$. However, for $P < 0$, the localization is eversed since the direction of the contour is reversed. In the Methods, we show how to compute $\nu(\omega)$ directly from the continuum equations using a generalization of Eq. (10.30).

10.6 Pseudo-Hermitian dynamics and direction-dependent bending modulus

In the preceding section, we took $P(\omega)$ to be real and constant. However, by introducing a phase delay into the transfer unction $H(\omega)$ we can control the complex argument of P . When $\arg(P) = \pm\pi/2$, the secular Eq. (10.27) has entirely real coefficients. Hence, the periodic boundary spectrum will consist of frequencies hat come in real values or complex conjugate pairs. This additional symmetry is sometimes referred to as a generalized PT symmetry, which arises if and only if there exists an antiunitary operator that commutes with $D(k)$. We say that the PT symmetry is unbroken when the eigenvalues of $D(k)$ are entirely real, and that it is broken otherwise. In the unbroken

phase, $\mathbf{D}(k)$ is said to be pseudo-Hermitian. Pseudo-Hermiticity implies that each eigenvector of $D(k)$ will individually conserve the mechanical energy density e in Eq. (10.26). However, when two or more eigenvectors are superimposed, e can oscillate in time, though remaining centered around a constant time-averaged value.

Since pseudo-Hermiticity constrains the periodic boundary spectrum of $D(k)$ to lie along the real line, the unidirectional amplification vanishes when P is imaginary. Nonetheless, the effects of parity violation are still present. We note that Eq. (10.28) may be written in the form:

$$\omega_{\pm} = \pm \sqrt{\frac{B}{\rho}} \left(1 \pm i \frac{Pk}{\mu} \right) k^2 + \mathcal{O} \left(k^4 \frac{I^2 B^2}{\rho^2 \mu^2} \right) \quad (10.31)$$

When $\arg(P) = \pm\pi/2$, we can interpret the form of Eq. (10.28) as having a rescaled bending modulus:

$$B_{\text{eff}} = B \left(1 \mp \frac{|P|k}{2\mu} \right)^2 \quad (10.32)$$

Notice that the value of B_{eff} depends on the sign of k , and hence the effective bending modulus is direction dependent. Fig. 10.4a shows the tilting of the dispersion for $\arg(P) = \pm\pi/2$. The tilt implies that the phase and group velocities for right and left traveling waves are unequal. Numerically solved modes are shown in Fig. 10.4b. We note that the pseudo-Hermiticity endowed by $P(\omega) \propto \pm i$ must exist exclusively at finite frequency because $P(\omega)$ cannot be nonzero and imaginary at $\omega = 0$ due to the requirement that $P(-\omega) = P^*(\omega)$.

A discrete model of the odd micropolar metabeam. To gain intuition into the mechanics of the metabeam, it is useful to consider a discrete model. As shown in Fig. 10.5a, b, the i^{th} unit cell of the discrete model consists of a rod (gray) with moment of inertia J and total mass m , whose position and orientation are captured by a height h_i and an angle φ_{i-1} . The rods are connected via mass-less, rigid frames and two springs. The top spring is a Hookean spring with tension

$$T_i = k_{\mu} (h_{i-1} - h_i + L\varphi_{i-1}) \quad (10.33)$$

and a bottom spring is a torsional spring that exerts an angular tension

$$\tau_i = \kappa_B (\varphi_{i-1} - \varphi_i) \quad (10.34)$$

In Fig. 10.5c, d, we show the addition of an active element that senses the stretching of the bottom spring and actuates an additional tension in the top spring

$$T_i^a = p (\varphi_{i-1} - \varphi_i) \quad (10.35)$$

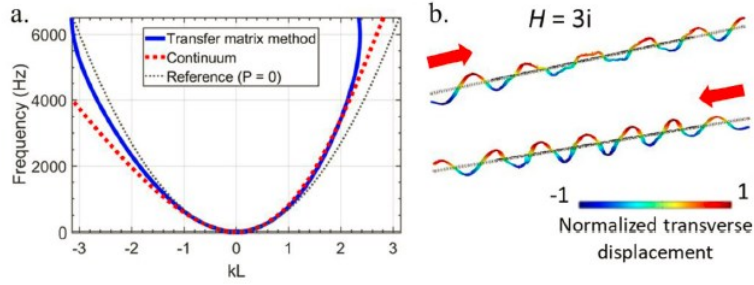


Figure 10.4 Pseudo-Hermitian dynamics. a The spectrum is shown for the metamaterial with $\arg(P) = \pi/2$. We note that the reality of the frequencies is maintained, while the modulus P breaks the $k \rightarrow -k$ symmetry. L is the unit cell length. b. Transverse displacement wave fields for the waves different directions. The left and right traveling modes are excited at equal frequencies, but have differing wavenumbers due to the odd micropolarity. The red arrows indicate the direction of travel of the wave, and H is the transfer function such that $P = \Pi H$.

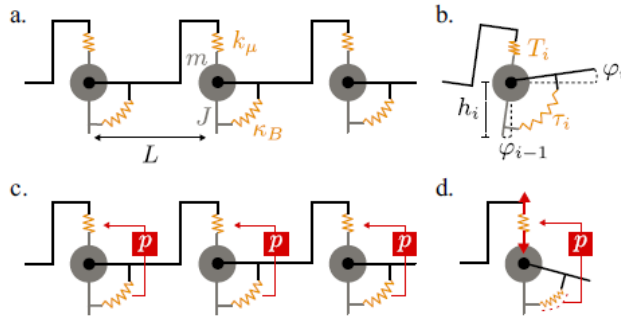


Figure 10.5 Discrete model for odd micropolar beam. a. A discrete model of a Timoshenko beam consists of a central mass m with moment of inertia J , a Hookean spring of spring constant k_μ and a torsional spring of spring constant κ , Band lattice spacing L . b. The unit cell is described by the height of the mass h_i and the angle φ_i of the black connecting rod. c, d The odd micropolar beam has an internal feedback P that senses the angle change of the torsional spring and actuates additional tension in the Hookean spring. The control loop is unidirectional: stretching or compressing the Hookean spring does not affect the torsional spring.

Summing the forces in the vertical direction yields a dynamical equation for h_i

$$m\ddot{h}_i = T_i - T_{i+1} + T_i^a - T_{i+1}^a \quad (10.36)$$

$$= k_\mu (h_{i+1} + h_{i-1} - 2h_i) + Lk_\mu (\varphi_{i-1} - \varphi_i) + p(\varphi_{i+1} + \varphi_{i-1} - 2\varphi_i) \quad (10.37)$$

Furthermore, summing the torques yields

$$J\ddot{\varphi}_i = \tau_i - \tau_{i+1} - L(T_{i+1} + T_{i+1}^a) \quad (10.38)$$

$$= \kappa_B (\varphi_{i+1} + \varphi_{i-1} - 2\varphi_i) + pL(\varphi_{i+1} - \varphi_i) + Lk_\mu (h_{i+1} - h_i - L\varphi_i) \quad (10.39)$$

Upon inspection, Eqs. (10.37) and (10.39) are precise discretizations of Eqs. (10.21) and (10.22) with $\rho = m/L$, $I = J/L$, $\mu = k_\mu L$, $B = \kappa_B L$, and $P = Lp$. Notice that the discrete model manifestly conserves linear and angular momentum since the linear and torsional springs exert equal and opposite forces and torques, respectively, on the units they connect. Even without externally applied torques, nontrivial internal angular momentum transfer occurs between the translation of its center of mass $jLm\dot{h}_j$ and the rotation of the axis $J\dot{\phi}_j$, akin to a "spin-orbit" coupling. Nonetheless, Maxwell-Betti reciprocity is violated by the asymmetry in the relationship between the linear and torsional springs: bending of the torsional spring induces a tension $T_i = p(\varphi_{i-1} - \varphi_i)$ in the linear spring, while the deformation of the linear spring $h_{i-1} - h_i + L\varphi_{i-1}$ has no response in the angular spring. This asymmetry implies that a cycle of alternating actuation and release of the linear and torsional spring is associated with a nonzero amount of work done.

Here, $\zeta = 0.48$ and $H_0 = 3$ are constants and $\omega_0 = 3\text{kHz}$ is the cutoff frequency at which $\arg(P) = -\pi/2$. We probe the vibrational dynamics of the beam by initiating waves from either the right or left side of the metamaterial via external piezoelectric elements. Fig. 10.6 b, c show the experiment at 2kHz in which waves from the right are suppressed while waves from the left are amplified.

To construct the full spectrum of the metamaterial, we perform the experiment with tone burst signals centered between 1.5 and 4kHz. The transverse velocity wave fields are measured along the medium using a laser Doppler vibrometer. We apply fast-Fourier transforms in time and in space to extract the real $k(\omega)$ and imaginary part $\kappa(\omega)$ of the wavenumber as a function of frequency for right-going (red) and left-going (blue) waves (Fig. 10.6d, e). The solid theoretical curves are produced using a semi-analytical technique known as the transfer-matrix method. The transfer-matrix method utilizes the beam geometry, known material parameters, and electronic feedback measured in simulations. No fitting parameters are used in the comparison between experiment and the transfer-matrix method curve.

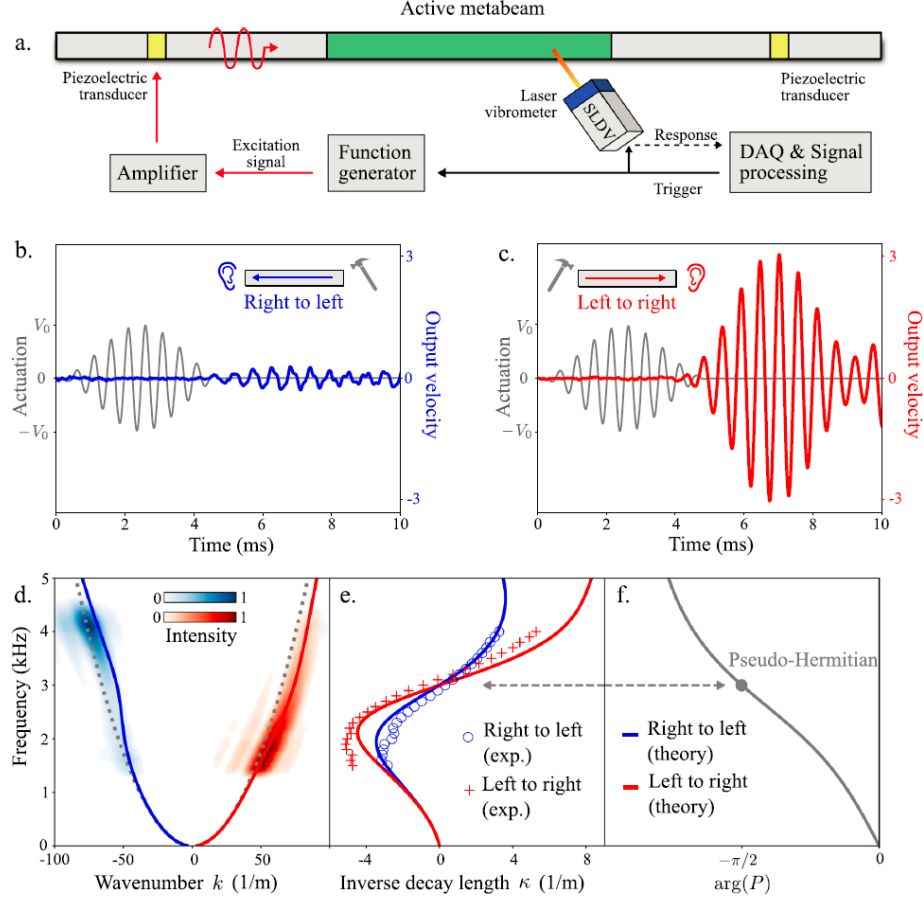


Figure 10.6 Experimental demonstration of skin modes and odd micropolar moduli. a Experimental schematic. Flexural waves are generated in the active metabeam from either the right or left side using piezoelectric actuators (yellow). A scanning laser Doppler vibrometer (SLDV) measures the transverse velocity of the surface of the active metabeam. b, c Unidirectional amplification of waves. A metamaterial consisting of 9 unit cells actuated from either the right (blue) or left (red) with a 2kHz tone burst signal (gray). The output velocity is normalized by the maximum velocity observed when the experiment is performed with no active feedback. d Observation of the non-Hermitian skin effect. Experiments are performed between 1.5kHz and 4kHz for right to left (blue) and left to right (red) traveling waves. A 2D FFT shows the intensity of the observed spectrum. The intensity is normalized by its maximum value. e The inverse decay length. In d, e the solid theoretical curves are based on the transfer matrix method. In d the gray dashed curves are theoretical predictions with no activity. f A plot of $\arg(P)$ as a function of frequency. At $\omega = \omega_0 (= 3\text{kHz})$, $\arg(P) = -\pi/2$, indicating that the system is pseudo-Hermitian and accordingly we observe $\kappa = 0$ at $\omega = \omega_0$.

Experimental demonstration. To probe the dynamic wave phenomena originating from the odd micropolar modulus P , we perform experiments in which we excite flexural waves in the metabeam using piezoelectric actuators, see Fig. 10.6a. In experiments, we implement the following electronic transfer function (cf. Fig. 10.1c):

$$H(\omega) = \frac{H_0}{(i\omega/\omega_0)^2 + 2i\zeta\omega/\omega_0 + 1} \quad (10.40)$$

As illustrated in Fig. 10.6f, the transfer function $H(\omega)$ is chosen such that $-\pi/2 < \arg(P) < 0$ when $\omega < \omega_0$. In this case, we find $\kappa < 0$ for both left- and right-propagating waves (panel e). A value of $\kappa < 0$ implies that waves propagating to the left are attenuated whereas waves propagating to the right are amplified, which is confirmed by the FFT intensity in panel d. Likewise $-\pi < \arg(P) < -\pi/2$ for $\omega > \omega_0$. In this case, we find $\kappa > 0$, indicating that waves propagating to the right are attenuated whereas waves propagating to the left are amplified. In addition, when $\omega = \omega_0$, $\arg(P) = -\pi/2$, and the waves propagating to the left and right display no attenuation or amplification. At this frequency, the effective dynamical matrix is pseudo-Hermitian, and the differences between left- and right-propagating waves reside only in the wavelengths and phase velocities. We note that, in experiments, the unidirectional amplification is limited by the maximum output voltage (± 45 V) of our electric control system. In practice, to maximize the amplification ratio, one strategy is to use a modest value of the transfer function, say $|H_0| \approx 3$, and increase the number of unit cells over which the wave is amplified.

10.7 Discussion

The metabeam presented here demonstrates active odd micropolar moduli and non-reciprocal responses absent in energy conserving media that are enabled by sensing, actuating and local computation. The minimal on-board electronics that power the active metabeam enable its multiple functions as an elastic engine, selective mode amplifier, and mechanical diode. We uncover an intrinsic relation between an odd micropolar modulus, the nonHermitian skin effect, and a corresponding topological index. Numerical and experimental results show unidirectional amplification and attenuation of waves propagating through the metamaterial. Odd micropolarity extends the range of possible couplings between conventional strains/stresses and higher-order curvatures/moments by including antisymmetry in their relationship. The electronically assisted mechanical feedback provides an appealing solution to precisely modulate odd micropolar moduli without requiring changes to the metabeam's structure, geometry, or passive moduli. Our design can be flexibly tuned through computer coding and scaled via microelectromechanical systems (MEMS). Our mechanical approach relies on a feedforward control loop, a generic concept that can exist in both metamaterial and biological contexts. The continuum theory also makes our approach especially generalizable to the mechanics in other systems. Combining the principles illustrated here with disorder, nonlinearities, and strong dissipation suggests new approaches for the control of filaments and membranes arising in biological media.

10.8 Methods

Sample fabrication. The metabeam is composed of three piezoelectric patches (STEMiNC PZT 5J: $6 \text{ mm} \times 4 \text{ mm} \times 0.55 \text{ mm}$) mounted via conductive epoxy onto a laser-cut stainless steel host beam. We achieve antisymmetric actuation without the use of an inverting voltage amplifier by mounting the two piezoelectric actuators such that their piezoelectric polarization directions are oppositely oriented.

Experimental procedures. In experiments, nine metamaterial unit cells are connected with control circuits, see Fig. 10.6a. Two piezoelectric transducers are attached on the left and right sides of the metamaterial to generate incident flexural waves. We employ ten-peak tone-burst signals with central frequencies ranging from 1.5 to 4.0kHz in step sizes of 0.1kHz. We generate and amplify incident wave signals via an arbitrary waveform generator (Tektronix AFG3022C) and a high voltage amplifier (Krohn-Hite), respectively. Transverse velocity wavefields are measured on the surface of the metamaterial by a scanning laser Doppler vibrometer (Polytec PSV-400). We note that the transfer-matrix method used to derive the theoretical curves in Fig. 12.6d, e rigorously assumes an infinite system. To experimentally approximate these conditions, we embed the active metamaterial within a larger host steel beam denoted by the gray region of Fig. 10.6a. When waves cross the boundaries from host beam to the metamaterial, the reflection at boundaries between the host beam and the metamaterial is negligible, as evidenced by our numerical and experimental results. To suppress reflected waves at the free boundaries of the host beam, we bonded two layers of clay on the host beam with sufficient lengths. This way, waves can propagate through the metamaterial with approximated infinite boundary conditions. The decay length is then obtained by calculating the wave amplitudes at different points in the metamaterial.

Finite element simulations. We calibrate the transfer matrix method and continuum equations by conducting fully three-dimensional numerical simulations of the unit cell using the commercial finite element software COMSOL Multiphysics. In all the simulations, we model the piezoelectric patches via a three-dimensional linear piezoelectric constitutive law. The central piezoelectric patch acts as a sensor whose signal is obtained by integrating the free charge over the top surface of the piezoelectric sensor. The top and bottom surfaces of the piezoelectric sensor have zero electric potential. The bottom surfaces on the piezoelectric actuators are ground, and we apply electrical potentials on the top surfaces to act as actuating voltages. The actuating voltages are related to the sensing voltages via the electronic transfer function. For the wave dispersion computations in Fig. 10.3a, Floquet periodic boundary conditions are applied on the left and right boundaries of a metamaterial unit cell. We calculate eigenfrequencies of the unit cell with different real wavenumbers. To simulate the wave propagation with open boundaries (Fig. 10.3b), a metabeam composed of 15 unit cells is placed between two external beams. Two perfectly matched layers (PMLs) are attached to both ends of the external beams in order to suppress reflected waves from the boundaries. The incident flexural wave is generated by applying a harmonic force on the boundary of the host beam. The out-of-plane displacement is measured at the left- and right-hand sides of the metabeam. The penetration depth is calculated by comparing the amplitudes of the two extracted displacements.

10.9 Energy bounds on dynamic moduli

Here we discuss Eqs. (10.12, 10.13) in the main text. For simplicity, let us collect the stresses into a vector $\mathbf{t} = (\sigma_{zx}, M)^T$ and the deformations into a vector $\mathbf{u} = (s, b)^T$. Suppose the beam is subject to a deformation procedure such that its initial and final configurations at times $t = -\infty$ and $t = \infty$ are identical. Then the total work per unit volume done by the beam is given

$$\Delta W = \int_{-\infty}^{\infty} \frac{d\mathbf{u}}{dt} \cdot \mathbf{t} dt \quad (10.41)$$

$$= -i \int_{-\infty}^{\infty} \omega_{\mathbf{u}}^{\dagger}(\omega) \cdot \mathbf{C}(\omega) \cdot \mathbf{u}(\omega) d\omega \quad (10.42)$$

$$= \int_{-\infty}^{\infty} \omega \mathbf{u}^{\dagger}(\omega) \cdot \mathbf{M}(\omega) \cdot \mathbf{u}(\omega) d\omega \quad (10.43)$$

In the final step, we have introduced the matrix $\mathbf{M}(\omega) = i[\mathbf{C}^{\dagger}(\omega) - \mathbf{C}(\omega)]$ and used the fact that $\mathbf{u}(-\omega) = \mathbf{u}^*(\omega)$ and $\mathbf{C}(-\omega) = \mathbf{C}^*(\omega)$. If the medium is passive, then we require that ΔW must be negative for all choices of $\mathbf{u}(\omega)$. Therefore, we require that the matrix $M(\omega)$ be negative semidefinite. Using the parameterization in Eq. (10.11), this requirement implies Eqs. (10.12, 10.13). See the references (Srivastava, 2015; Muhlestein et al., 2016) for related discussions in two- and three-dimensional media.

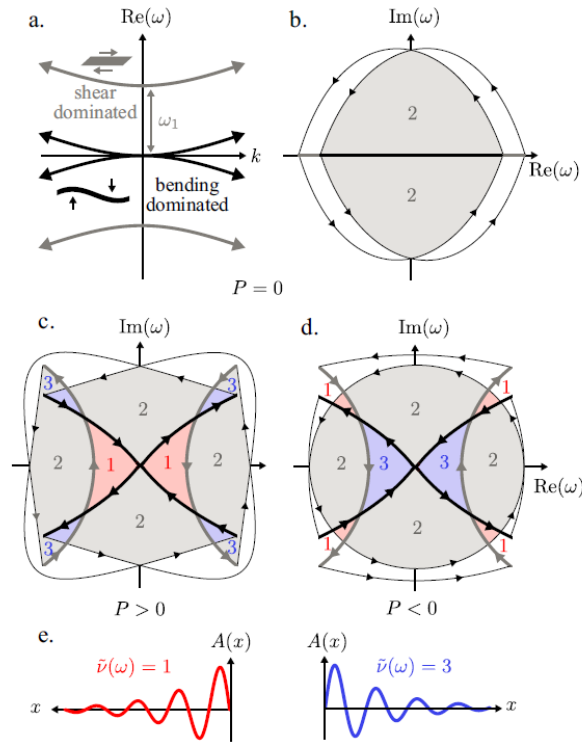


Figure 10.7 Non-Hermitian band topology via odd micropolar elasticity. a The spectrum for $P = 0$ features a pair of bending dominated bands (black) and shear dominated bands (gray) separated by a band gap ω_1 . (b-d) The spectrum is shown in the complex ω plane for $P = 0, P > 0$, and $P < 0$. The thick black lines represent the bending dominated band, while the thick gray lines represent the shear dominated bands, both with $k \in [-R, R]$ for a finite R . The thin black lines represent the analytical continuation of the spectrum for $k = \text{Re}^{i\varphi}$ for $\varphi \in [0, \pi]$. The arrows indicate the direction of increasing k . The numbers indicate the value of $\tilde{\nu}(\omega)$ for ω in the corresponding colored regions of the complex plane. This number corresponds to the number of times that the spectrum winds around a given region. e For a semi-infinite system with a free boundary, the winding number of $\tilde{\nu} = 1$ ($\tilde{\nu} = 3$) for our continuum theory indicates a mode localized to the right (left) boundary. The wave forms schematically depict the localization with $A(x)$ representing amplitude.

10.10 Cycles at finite frequency

To gain intuition on the elastodynamics of the odd micropolar metabeam, it is useful to consider the notion of a cycle at finite frequency. At a finite frequency ω , the modulus P need not be real and we may write $P = |P|e^{i\phi_P}$. In this case, both the real and imaginary parts of P will contribute to the energy extracted. For example, consider a cyclic protocol that involves bending of amplitude $|b|$ and shearing of amplitude $|s|$ and relative phase delay ϕ_B . Applying Eq. (10.43), the total energy extracted per cycle is

$$Work = -\pi |P| |s| |b| \sin(\phi_B + \phi_P) \quad (10.44)$$

Notice that Eq. (10.44) gives mechanistic insight into the amplification of the waves observed in the experiment. When P is nonzero, the eigenmodes of $D(k)$ comprising a given plane wave will generically have a phase delay between bending and shearing. Hence, after one cycle, the nonconservative stresses will have converted stored electrical energy into mechanical energy. This conversion will cause the amplitude of the eigenvector to grow in proportion to its current amplitude, for which $|s||b|$ is a proxy. Hence, the mode will be exponentially amplified, as reflected by the imaginary component of the eigenfrequencies.

10.11 Topological index in the continuum

In this section, we discuss the index $\nu(\omega)$ from the point of view of the continuum theory [54]. The spectrum as a function of k is plotted in Fig. 10.7a for $P = 0$. The spectrum is given by the roots of the secular Eq. (10.27) and contains four roots since the equations of motion are second order in time and involve two coupled fields. The spectrum consists of a pair of Goldstone modes, which for small k and $P = 0$ represents a flexural motion of the beam. Additionally, the continuum equations imply two modes separated by a band gap $\omega_1 \approx 105$ Hz. For small k and $P = 0$, these modes are dominated by a shearing motion. For the Goldstone mode, we expect a range of validity of the continuum theory at small k , since $\omega(k \rightarrow 0) = 0$. However, for the shear dominated mode, the continuum theory is not expected to self-consistently apply due to the finite gap. We numerically compute the spectrum and eigenmodes for frequencies above the experimentally relevant range using COMSOL. There we also discuss why the winding number $\nu(\omega)$ computed in the continuum is still physically relevant despite the presence of additional high frequency modes not captured by the continuum.

For a translationally invariant system, the difference between periodic and open boundary conditions is whether the differential operator that defines the equations of motion allows formal eigenvectors with complex wavenumber k . For a system with periodic boundary conditions, the spectrum consists of values of ω that solve $\det [D(k) - \omega] = 0$ for real k . However, for a system on a semi-infinite domain $x \in [0, \infty)$, we allow $\text{Im}(k) > 0$. These modes decay to the right as $\exp [(-\text{Im } k + i\text{Re } k)x]$ and therefore maintain a finite L^2 norm, which represents the mechanical energy density. To determine whether a given frequency ω is in the semi-infinite

boundary spectrum, we first must count the number of solutions to $\det [D(k) - \omega] = 0$ with $\text{Im} k > 0$. We do so by applying Cauchy's argument principle from complex analysis:

$$\nu(\omega) = \lim_{R \rightarrow \infty} \frac{1}{2\pi i} \oint_{\Gamma(R)} \frac{d}{dk} \log \det [D(k) - \omega] dk \quad (10.45)$$

where $\Gamma(R)$ is a counter clockwise curve in the complex plane given by $[-R, R]$ together with $Re^{i\varphi}$ for $\varphi \in [0, \pi]$. The winding number $\tilde{\nu}(\omega)$ itself does not directly determine the number of localized modes, since we must also consider the boundary conditions placed at $x = 0$. Suppose that γ independent homogeneous boundary conditions are placed at $x = 0$. Then the number of left localized modes will generically be given by:

$$\text{left localized modes at } \omega = \tilde{\nu}(\omega) - \gamma \quad (10.46)$$

Likewise, for a system with boundary $(0, \infty]$, the number of right localized modes is given by

$$\text{right localized modes at } \omega = d - \tilde{\nu}(\omega) - \gamma \quad (10.47)$$

where d is the degree of $\det [D(k) - \omega]$ as a polynomial in k . Whenever the right-hand side of Eqs. (10.46) or (10.47) is negative, the mode count is taken to be zero. Fig. 10.7b-d illustrates the computation of the winding number. The thick black and gray lines are the periodic boundary spectrum for $k \in [-R, R]$, and the thin black lines are the analytical continuation for $k = Re^{i\varphi}$ for $\varphi \in [-\pi, \pi]$. Colored regions are labeled by the value of $\tilde{\nu}(\omega)$. Suppose for example that the beam is given stress and moment free boundary conditions $\sigma_{zx} = 0$ and $M = 0$ at $x = 0$. In this case $\gamma = 2$, and therefore $\tilde{\nu}(\omega) = 1$ indicates the presence of a right localized mode, $\tilde{\nu}(\omega) = 3$ indicates the presence of a left localized modes, as shown in Fig. 10.7e.

10.12 Topological index from discrete models

We now discuss the topological index in Eq. (10.30), which is appropriate for discrete settings such as the discrete model and finite element simulations. Suppose the system is composed out of a unit cell of finite length L whose internal state is represented by a vector $\Psi(x)$. The components of Ψ can represent, for example, the displacement and velocities of points in a finite element mesh. Here, x is a discrete label that takes values in integer multiples of L . The Fourier transform of the equations of motion now read:

$$\omega \Psi(k) = \mathcal{D}(k) \cdot \Psi(k) \quad (10.48)$$

where $D(k)$ is the dynamical matrix for the discrete system, and k is the wave number assuming values in $[-\pi/L, \pi/L]$. For a system with periodic boundaries, the spectrum is given by solutions to $\det [D(k) - \omega] = 0$ for $k \in [-\pi/L, \pi/L]$. For a system with semi-infinite boundaries (e.g., $\in \{0, L, 2L, \dots\}$), we allow $\text{Im} k > 0$. As detailed in the Supplementary Note 1, we can invoke a similar application of the Cauchy argument principle to determine the number of eigenmodes at a given frequency ω . To do so, we write $k = -i \log \lambda$, where λ assumes values on the unit circle S in the complex plane. We can then apply Cauchy's argument principle using S as a counterclockwise contour for λ . We have:

$$v(\omega) = \frac{1}{2\pi i} \int_S \frac{d}{d\lambda} \log \det[\mathcal{D}(-i \log \lambda) - \omega] d\lambda \quad (10.49)$$

$$= \frac{1}{2\pi i} \int_{-\frac{\pi}{L}}^{\frac{\pi}{L}} \frac{d}{dk} \log \det[\mathcal{D}(\mathbf{k}) - \omega] dk \quad (10.50)$$

$$= \frac{1}{2\pi i} \sum_{\alpha} \int_{-\pi/L}^{\pi/L} \frac{d}{dk} \log[\omega_{\alpha}(k) - \omega] dk \quad (10.51)$$

where $\omega_{\alpha}(k)$ is the value of the periodic spectrum of the band α , in agreement with Eq. (10.30) from the main text. Notice that for the frequency denoted by the red star in Fig. 10.3c, we have $v(\omega) = -1$. However, for the same point in Fig. 10.7c (which lies in the red region to the right of the origin), we have $\tilde{v}(\omega) = 1$. To see the relationship between these quantities, notice that $\tilde{v}(\omega)$ in Eq. (10.45) counts the number of zeros of $f(k) = \det[D(k) - \omega]$ in the upper half plane. However, in computing v from Eq. (10.49), the interior of S contains not only the zeros of $F(\lambda) \equiv \det[D(-i \log \lambda) - \omega]$ but also a set of poles. Each of these poles physically represents a boundary condition that arises when transitioning between a Laurent operator to a Toeplitz operator. The number of poles depends on the precise discretization of the continuum equations. Hence the winding number $v(\omega)$ as given by Eq. (10.30) represents the difference between the number of zeros (candidate modes) and the number of poles (boundary conditions). Therefore, one should compare $v(\omega)$ to $\tilde{v}(\omega) - \gamma$ for an appropriate choice of γ . In practice, γ is determined by physically interpreting the boundary conditions implied by the discretization. In Supplementary Note 1, we show that the discretization used in the discrete model and the COMSOL simulations enforce displacement-free boundary conditions and hence corresponds to $\gamma = 2$. Hence $\tilde{v}(\omega) - \gamma = v(\omega) = -1$ as required for physical consistency.

10.13 Calculation of penetration depth

In Fig. 10.3b, we show the penetration depth for modes associated with positive real ω . We can compute this expression by solving $\omega_{+}(k) = \omega$ in Eq. (10.28) for complex k . In particular, substituting $k = q + i\kappa$ into Eq. (10.28) yields:

$$0 = 2k\kappa + \frac{P}{2\mu} (k^3 - 3k\kappa^2) \quad (10.52)$$

$$\omega = \sqrt{\frac{B}{\rho}} \left[\left(1 - \frac{P\kappa}{\mu} \right) (k^2 - \kappa^2) - \frac{2P}{\mu} k^2 \kappa \right] \quad (10.53)$$

Solving Eqs. (10.52, 10.53) to leading order in k , we obtain Eq. (10.29).

11. Omnidirectional wave polarization manipulation in isotropic polar solids

11.1 Introduction

In this study, the isotropic elastic polar media are leveraged to realize omnidirectional longitudinal-shear mode conversion and AWP by harnessing the polarity and chirality moduli. First, we briefly review the basic theory of the isotropic polar media. It is later found that polarity and chirality induced by the isotropy and non-vanishing asymmetric stress with broken minor symmetries play

an important role in achieving highly efficient mode conversion. Engineering the polar properties enables the isotropic polar medium to support the omnidirectional AWP which allows faster shear modes. This property subsequently leads to singly-polarized transmission and anomalous refraction. Unlike traditional schemes where elastic anisotropy is involved, all the polar media used are isotropic and allow the wave manipulation behaviors to operate in the absence of directional dependence. It is our hope that the proposed isotropic polar platform may provide additional insights into the development of modern elastic-wave-related devices.

11.2 Elastodynamics of 2D isotropic polar solids

11.2.1 Governing equations

We start with a brief review of a linear elastic polar solid governed by an equation of motion (Nassar et al., 2019)

$$\sigma_{ij,j} + \rho_p f_i = \rho_p \dot{v}_i, \quad (11.1)$$

where σ, ρ_p, f , and v denote the stress tensor, mass density, body force, and particle velocity, respectively. In general, strain energy $\psi(\mathbf{e})$ is dependent quadratically on the displacement gradient $e = u_{i,j}$, therefore, the general constitutive relation reads

$$\sigma_{ij} = \frac{\partial \psi(\mathbf{e})}{\partial e_{ij}} = a_{ijkl} e_{kl}, \quad (11.2)$$

where a should satisfy major symmetry $a_{ijkl} = a_{klij}$ but does not necessarily possess minor symmetries $a_{ijkl} \neq a_{ijlk}$. This means that the relation $\sigma_{kl} = \sigma_{lk}$ may not hold. The solid is "polar" with the existence of a body torque density linearly dependent over the displacement gradient as

$$c_i = \epsilon_{ijk} \sigma_{jk} = \epsilon_{ijk} a_{jklm} e_{lm}, \quad (11.3)$$

where c is the body torque and ϵ_{ijk} is the permutation symbol. In what follows, we will present a general expression of constitutive relations in 2D isotropic polar solid.

In 2D, a rotation angle θ is defined as the skew part of displacement gradient such that $\theta = \frac{1}{2}(u_{2,1} - u_{1,2}) = \frac{1}{2}J:e$, where J denotes the 2π rotation operation and $-J_{ij}$ is the 2D Levi-Civita notation. The deformation energy now reads

$$\psi = \mu \tilde{\mathbf{e}} : \tilde{\mathbf{e}} + \frac{1}{2} \kappa t^2 + \frac{1}{2} \alpha \theta^2 + \beta t \theta, \quad (11.4)$$

where $t = \text{tr } e$, μ is a shear modulus transforming the deviatoric strain $\tilde{\mathbf{e}}$ into the deviatoric stress $\tilde{\sigma}$, κ and α conjugated to $\frac{\partial^2 \psi}{\partial t^2}$ and $\frac{\partial^2 \psi}{\partial \theta^2}$ respectively represent a bulk modulus and a polarity modulus. The former transforms the infinitesimal change of area t into a hydrostatic pressure p , while the latter transforms the infinitesimal rotation θ into a body torque c and breaks the minor symmetry, β is induced by the minor symmetry breaking and isotropy and represents the coupling

between θ and t . With this decomposition, the strain-stress relation in terms of deviatoric, hydrostatic, and skew stresses can be expressed as,

$$\tilde{\sigma} = 2\mu\tilde{\epsilon}, \quad p = -\kappa t - \beta\theta \quad \text{and} \quad c = -\beta t - \alpha\theta, \quad (11.5)$$

which are illustrated in Fig. 11.1. In other words, hydrostatic pressure and body torque are coupled. Therein, the elastic tensor specifies into

$$a_{ijkl} = \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + (\kappa - \mu)\delta_{ij}\delta_{kl} + \frac{\alpha}{4}J_{ij}J_{kl} + \frac{\beta}{2}(\delta_{ij}J_{kl} + J_{ij}\delta_{kl}), \quad (11.6)$$

and Hooke's law becomes accordingly

$$\sigma = (\kappa t + \beta\theta)I + 2\mu\tilde{\epsilon} + \frac{\beta t + \alpha\theta}{2}J. \quad (11.7)$$

The isotropic polar solid addressed here corresponds basically to an extension of standard Cauchy elasticity with chiral and polar effects (Nassar et al., 2019). In order to enforce stability, μ, κ, α and β in Eq.(11.4) should remain positive (Nassar et al., 2019). Moreover, rewriting Eq. (11.5) into matrix form and enforcing the determinant of the stiffness matrix to be non-negative yields the stability condition reading

$$\beta^2 \leq \kappa\alpha. \quad (11.8)$$

In the extreme case of $\beta^2 = \kappa\alpha$, where no parameters are zero, the isotropic polar solid becomes degenerate, and admits a zero mode where the stress is zero with non-vanishing displacement gradient (Nassar et al., 2019).

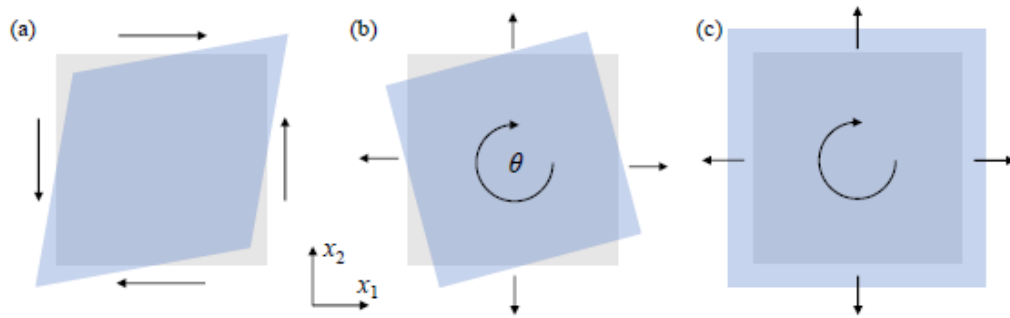


Figure 11.1 Isotropic polar solids: response under (a) a pure shear, (b) a rigid rotation, and (c) a pure dilation.

11.2.2. Wave propagation in the polar medium

We consider harmonic wave propagation in a 2D isotropic polar medium as

$$\mathbf{u} = \Psi e^{i(q \cdot \mathbf{r} - \omega t)} = (u_1, u_2)^T e^{i(q \cdot \mathbf{r} - \omega t)}, \quad (11.9)$$

where $\mathbf{q} \cdot \mathbf{r} = q_1 x_1 + q_2 x_2$ with q and $\mathbf{r}$ being the wave vector and position, respectively, Ψ and ω are, respectively, the wave amplitude and angular frequency. T represents the matrix transpose. By substituting the above equation into Eq. (11.7), we obtain the following secular equations

$$\begin{bmatrix} (\mu + \kappa)q_1^2 - \beta q_1 q_2 + \left(\mu + \frac{\alpha}{4}\right)q_2^2 & \frac{\beta}{2}(q_1^2 - q_2^2) + \left(\kappa - \frac{\alpha}{4}\right)q_1 q_2 \\ \frac{\beta}{2}(q_1^2 - q_2^2) + \left(\kappa - \frac{\alpha}{4}\right)q_1 q_2 & (\mu + \kappa)q_2^2 + \beta q_1 q_2 + \left(\mu + \frac{\alpha}{4}\right)q_1^2 \end{bmatrix} \Psi = \omega^2 \rho_p \Psi \quad (11.10)$$

where $q_1 = q \cos \theta$, $q_2 = q \sin \theta$, and θ represents the propagation angle of the considered harmonic wave. Based on Eq. (11.10) and applying $\bar{Q} = \kappa + \alpha/4$, $\mu = l\bar{Q}$, $\kappa - \alpha/4 = n\bar{Q}$ and $\beta = m\bar{Q}$, we obtain the wave phase velocities for the two orthogonal propagating modes

$$\mathbb{C}_{\pm}^2 = \frac{\bar{Q}(2l + 1 \pm \sqrt{n^2 + m^2})}{2\rho_p} \quad (11.11)$$

where the "+" / "-" sign corresponds to the faster/slower eigenmode. In particular, μ , α , κ , and β in isotropic polar solids are represented by another four independent variables m , n , l , and Q . The stability condition given by Eq.(11.8) now becomes $m^2 + n^2 \leq 1$ with $m^2 + n^2 = 1$ corresponding to the degenerate polar media. Note that we assume that the isotropic polar media used for illustrating wave manipulation functions throughout the paper are realized based on the grounding mechanism (Nassar et al., 2019; Xu et al., 2020), which causes broadband operability of the material parameters m , n , l , and $\bar{Q}$ in the continuum limit. The material parameters can be determined from the bottom-up micromechanics homogenization method by analytically formulating the unit cell strain energy under global deformation (Nassar et al., 2019).

Based on Eq.(11.10), the wave mode shapes can be determined easily as $\Psi_{\pm} = (\sin \gamma_{\pm}, \cos \gamma_{\pm})^T$, where polarization angle $\gamma_{\pm} = \arctan \frac{\pm m}{\sqrt{n^2 + m^2 \mp n}}$. Figure 11.2a illustrates the m , n -dependence of the modal polarizations (crosses) and the polarization angle γ of the faster mode with $\mathbb{C}_{+}$ (color map). Therein, n quantifies to which degree the minor symmetry is broken whereas, m measures the intensity of the induced chirality and the handedness. The cross in the plot denotes the polarization states of the orthogonal modes $\Psi_{\pm}$, with the lengths representing phase velocities $\mathbb{C}_{\pm}$. Due to the isotropy of the polar solids, a wave vector q pointing towards the right is considered without loss of generality. The stability condition in Eq. (11.8) defines all possible polarized modes; see the red circular boundary in Fig. 11.2a. First observation reveals that $m \neq 0$ leads to the existence of quasi-longitudinal (qP) and quasi-shear (qS) modes. This enables mode conversion functions between longitudinal (P) and shear (S) modes, as will be discussed later. Starting at the yellow circle (a Cauchy-elastic medium) and traveling across the first and second quadrants, the faster mode Ψ_{+} transitions from being purely longitudinal to being purely shear. In other words, when $|\gamma| > \pi/4$ or $n < 0$, qS modes propagate faster than qP modes. Specifically, S modes

outspeeding P modes can be seen when $|\gamma| = \pi/2$ is guaranteed, i.e. $m \rightarrow 0$ and $n < 0$, which leads to the occurrence of the AWP.

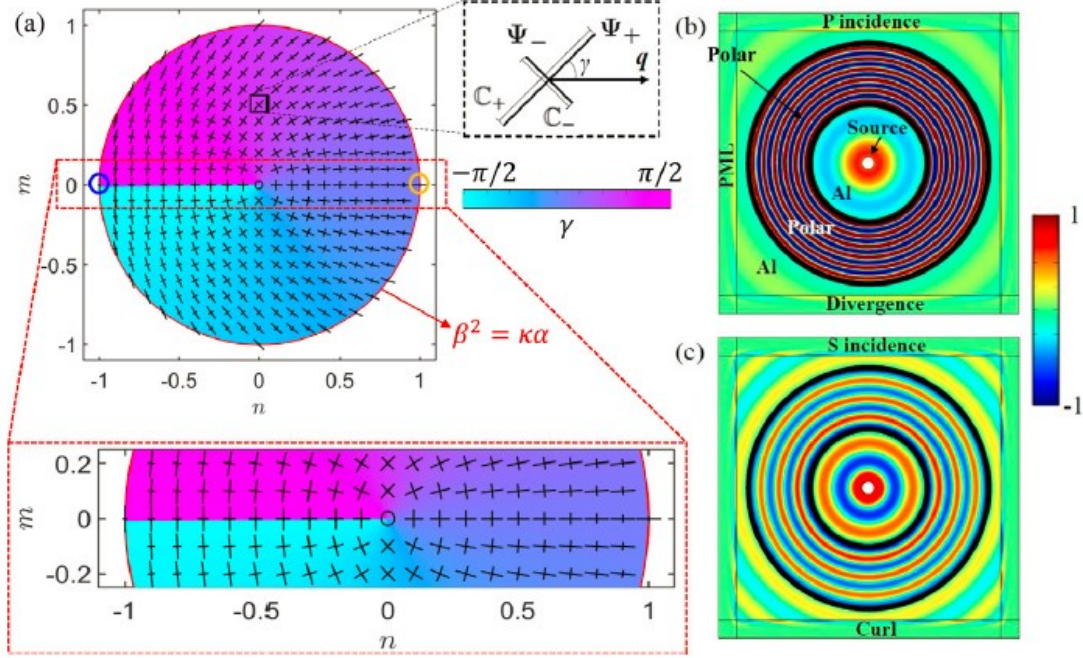


Figure 11.2 (a) Polarization field in the m, n -space. The red circle corresponding to degenerate polar media bounds the stable region defined by the stability condition Eq. (11.8). The top inset is a magnified view of the polarization state and velocity magnitudes of $\Psi_{\pm}$ at specific m and n . The bottom inset is another zoomed image for small m . The color map illustrates the m, n -dependence of polarization angle $\gamma = \arccos \sin \gamma_+$ of the faster mode Ψ_+ . The blue and yellow circles represent two extreme polarization states ($n = -1, m = 0$ and $n = 1, m = 0$, respectively). Also, the yellow circle denotes a Cauchy-elastic medium. (b-c) show (b) the normalized divergence fields under a cylindrical P-polarized incidence and (c) the normalized curl fields under a cylindrical s-polarized incidence at 53kHz. The cylindrical sources are realized with a circular boundary load at the center. The black solid curves indicate the isotropic polar slab ($c_P < c_S$, c_P and c_S being the phase velocity of P and S modes, respectively) with $m = 0.001, n = -0.999, l = 1, Q = 0.1244$ and a polar layer thickness of $0.859\lambda^{(P)}$, where $\lambda^{(P)}$ is the wavelength of P mode in the background at 53kHz. The simulation domain is surrounded by perfect matched layers (PMLs).

To numerically demonstrate the AWP that features faster S mode propagation, we conduct simulations using COMSOL for wave propagation in an isotropic polar slab sandwiched by two isotropic elastic media [see Figs. 11.2(b) and 11.2(c)]. The background medium used throughout the paper is aluminum ($\rho_0 = 2700 \text{ kg/m}^3$, $\mu_0 = 25.94 \text{ GPa}$, and $\lambda_0 = 50.35 \text{ GPa}$). Without loss of generality, the isotropic polar medium is set to be degenerate with $n = -0.999, m = 0.001, l = 1$, and $\rho_p = 5\rho_0, Q = 0.1244$, where a normalized parameter is defined as $Q = Q/(2\mu_0 + \lambda_0)$. The stiffness tensor with the additional polar constants are implemented with the help of a series of weak formulations to override the default elastic settings. The numerical results of the normalized divergence and curl fields are plotted in Figs. 11.2b and 11.2c under cylindrical P- and S-polarized incidences at 53kHz in the inner background elastic medium. The choice of this frequency is based

on the consideration that the polar slab offers large transmission for both polarized incidences at this frequency. However, even so, there still exists nonzero reflections at both interfaces due to the slight impedance mismatching ($Z_s/Z_+ = 0.4517$ and $Z_p/Z_- = 1.2627$), as can be seen from the Figs. 11.2b and 11.2c. We emphasize that under a certain polarized incidence the polar slab only supports the mode of the same polarization because the chiral effect vanishes in these cases. That said, it is reasonable to directly compare the wavelength to examine the comparison of velocity for both modes, although interference patterns from the reflection exists in the polar slab. The comparison of wavelength between the polar slab and the background shows $\lambda_p/\lambda_- = 6.3443$ for the P mode and $\lambda_s/\lambda_+ = 2.2591$ for the S mode, where $\lambda_p, \lambda_s, \lambda_+$ and λ_- represent the wavelengths of the background longitudinal and shear modes, and the faster and slower modes in the polar medium, respectively. By comparing the wavelengths inside and outside the isotropic polar slab, we can readily observe that the wavelength of the S mode is larger than that of the P mode in the polar medium with a wavelength ratio of $\lambda_-/\lambda_+ = 1.4137$, which indicates the faster S mode exhibits a phase velocity 0.4137 times larger than that of the slower P mode. Furthermore, the simulation results also demonstrate the omnidirectional nature of this phenomenon, which is significantly different from the case where the phenomenon occurs only at certain propagation angles due to anisotropy (Patil et al., 2019; Ledbetter and Kriz, 1982; Lee et al., 2017; Ahn et al., 2018)

11.3 Wave mode conversion in a finite isotropic polar slab

Different from the conventional approaches based on material anisotropy, we will study the feasibility of using an isotropic polar slab to achieve efficient wave mode conversion omnidirectionally. The mode conversion here takes place in the polar medium by harnessing the additional two parameters α and β .

11.3.1. Wave propagation from the isotropic Cauchy medium to polar medium

Recall that $\beta = m\bar{Q}$ corresponds to the measure of chirality and handedness. It couples the plane rotation angle and the change of area. With m being nonzero, the longitudinal dominant and shear dominant motions are coupled in isotropic polar media. When a certain polarized incidence strikes the interface between a Cauchy and an isotropic polar media, components of other polarization will be generated. Consider a Cauchy-polar interface shown in Fig. 11.3a. Without loss of generality, we send in a harmonic P-polarized normal incidence of unity amplitude $e^{i(qpx-\omega t)}$, where $q_p = \omega\sqrt{\rho_0/(2\mu_0 + \lambda_0)}$ denotes the wave number of P mode. The version of S-polarized normal incidence can be derived in a similar fashion. By removing the term $e^{i\omega t}$, the displacement fields on the left side of the interface including the reflected wave components can be prescribed as

$$u^{(l)} = e^{iq_p x_1} + r_{pp} e^{-iq_p x_1} \text{ and } v^{(l)} = r_{ps} e^{-iq_s x_1}, \quad (11.12)$$

where $q_S = \omega\sqrt{\rho_0/\mu_0}$ is the wave number of S mode, and r_{PP} and r_{PS} are the reflection coefficients of the reflected P and S components from the P polarized incidence, respectively. The transmitted waves can be expressed as

$$u^{(r)} = t_{PqP} \sin \gamma_+ e^{iq_+ x_1} + t_{PqS} \sin \gamma_- e^{iq_- x_1}, \quad (11.13)$$

$$v^{(r)} = t_{PqP} \cos \gamma_+ e^{iq_+ x_1} + t_{PqS} \cos \gamma_- e^{iq_- x_1}, \quad (11.14)$$

where t_{PqP} and t_{PqS} denote the transmission coefficients of the transmitted qP and qS components, $q_+ = \omega/\mathbb{C}_+$, and $q_- = \omega/\mathbb{C}_-$. Note that we have assumed throughout the paper that the isotropic polar medium is an extension of the standard Cauchy elasticity with body torques. Therefore, the continuity conditions at 2D Cauchy and polar media interface only involve two displacement continuity conditions (perpendicular to and parallel with the interface), and two Cauchy stress continuity conditions (normal and shear stresses). Based on Eqs. (11.10-11.14), we can determine stress fields at the interface reading

$$\begin{aligned} \sigma_{11}^{(l)} &= i(2\mu_0 + \lambda_0)q_p e^{iq_p x_1} - i(2\mu_0 + \lambda_0)q_p r_{PP} e^{-iq_p x_1}, \\ \sigma_{21}^{(l)} &= -i\mu_0 q_S r_{PS} e^{-iq_S x_1}, \\ \sigma_{11}^{(r)} &= i\omega\rho_p \mathbb{C}_+ t_{PqP} \sin \gamma_+ e^{iq_+ x_1} + i\omega\rho_p \mathbb{C}_- t_{PqS} \sin \gamma_- e^{iq_- x_1}, \\ \sigma_{21}^{(r)} &= i\omega\rho_p \mathbb{C}_+ t_{PqP} \cos \gamma_+ e^{iq_+ x_1} + i\omega\rho_p \mathbb{C}_- t_{PqS} \cos \gamma_- e^{iq_- x_1}. \end{aligned}$$

Therein, the displacement ($u^{(l)} = u^{(r)}$ and $v^{(l)} = v^{(r)}$) and stress continuity conditions ($\sigma_{11}^{(l)} = \sigma_{11}^{(r)}$ and $\sigma_{21}^{(l)} = \sigma_{21}^{(r)}$) at the interface lead to

$$\begin{pmatrix} -1 & 0 & \sin \gamma_+ & \sin \gamma_- \\ 0 & -1 & \cos \gamma_+ & \cos \gamma_- \\ Z_P & 0 & Z_+ \sin \gamma_+ & Z_- \sin \gamma_- \\ 0 & Z_S & Z_+ \cos \gamma_+ & Z_- \cos \gamma_- \end{pmatrix} \begin{pmatrix} r_{PP} \\ r_{PS} \\ t_{PqP} \\ t_{PqS} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ Z_P \\ 0 \end{pmatrix}, \quad (11.15)$$

where

$$Z_P = \sqrt{\rho_0(2\mu_0 + \lambda_0)}, \quad Z_S = \sqrt{\rho_0\mu_0}, \quad Z_+ = \rho_p \mathbb{C}_+, \quad \text{and} \quad Z_- = \rho_p \mathbb{C}_-.$$

Based on Eq. (11.15), the amplitudes of the reflection coefficients can be obtained as

$$r_{PP} = \frac{N_1 c_- s_+ - N_2 c_+ s_-}{N_3 c_- s_+ - N_4 c_+ s_-}, \quad (11.16)$$

$$r_{PS} = \frac{2(Z_P Z_- - Z_P Z_+) c_- c_+}{N_3 c_- s_+ - N_4 c_+ s_-}, \quad (11.17)$$

where

$$s_+ = \sin \gamma_+, s_- = \sin \gamma_-, c_+ = \cos \gamma_+, c_- = \cos \gamma_-,$$

and

$$\begin{aligned} N_1 &= Z_p Z_S + Z_p Z_- - Z_S Z_+ - Z_+ Z_- \\ N_2 &= Z_p Z_S + Z_p Z_+ - Z_S Z_- - Z_+ Z_- \\ N_3 &= Z_p Z_S + Z_p Z_- + Z_S Z_+ + Z_+ Z_- \\ N_4 &= Z_p Z_S + Z_p Z_+ + Z_S Z_- + Z_+ Z_- \end{aligned}$$

The total reflection reads

$$R_P = |r_{PP}|^2 + \frac{Z_S}{Z_P} |r_{PS}|^2. \quad (11.18)$$

Equation (11.18) can be further simplified as

$$R_P \approx \left(\frac{Z_+ (Z_P - Z_S) + Z_P Z_S - Z_+^2}{Z_+ (Z_P + Z_S) + Z_P Z_S + Z_+^2} \right)^2 \quad (11.19)$$

under the constraint of $Z_+ \approx Z_-$, which indicates that $(2l + 1)^2 \gg (m^2 + n^2)$ based on Eq. (11.11). This leads to the conclusion that $Z_+ \approx Z_P$ should be satisfied for achieving minimum reflection at the interface. Then, the optimal Q should be chosen as

$$Q = \frac{\rho_0}{\rho_p \left(l + 1/2 + \sqrt{m^2 + n^2} \right)}, \quad (11.20)$$

as long as $(2l + 1)^2 \gg (m^2 + n^2)$ is well satisfied.

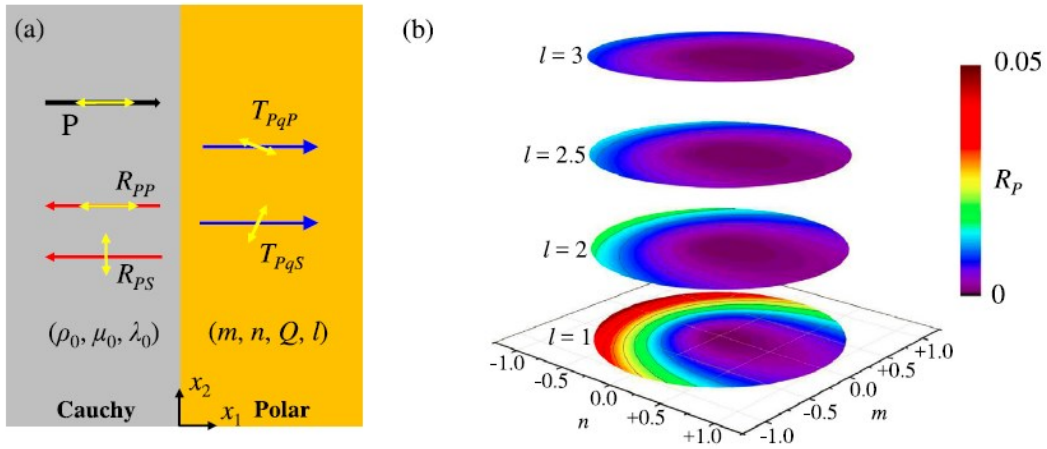


Figure 11.3 (a) Schematic of an interface defined by a Cauchy (aluminum background) and an isotropic polar media. A P-polarized incidence and two types of reflections with r_{PP} and r_{PS} are within the aluminum background. The transmitted qP and qS modes with t_{PqP} and t_{PqS} exist in the polar region. The yellow arrows denote the polarizations. (b) Calculated reflectance R_P for the P-polarized incidence at the interfaces shown in (a) in functions of m and n , with $l = 1, 2, 2.5$ and 3 .

Without loss of generality, we consider $l = 1, 2, 2.5$ and 3 , and $\rho_p = 5\rho_0$. R_p at the interface is plotted in functions of m and n in Fig. 11.3b. From 11.3b, one can easily find R_p remains overall lower than 0.05 for all the (m, n) combination. As l increases, R_p continuously decreases, due to the approximation $(2l + 1)^2 \gg (m^2 + n^2)$, $Z_+ \approx Z_-$ being gradually bettered. In addition, a closer observation of Eq.(11.16) and (11.17) reveals that as $Z_+ \approx Z_p$ performs well, $|r_{pS}|$ almost vanishes, indicating that $|r_{pP}|$ dominates in the total reflection R_p .

11.3.2 Wave propagation in the polar slab between two isotropic Cauchy media

Here, we follow the discussion in the previous subsection and consider harmonic waves propagating in a finite isotropic polar slab of thickness L , sandwiched by two Cauchy media; see Fig. 11.4a. A harmonic plane wave solution $\mathbf{u}_p = (u_1, u_2)^T$ in the sandwiched isotropic polar slab can be written as

$$\mathbf{u}_p = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \Psi_{+,1}^{(+)} & \Psi_{+,1}^{(-)} & \Psi_{-,1}^{(+)} & \Psi_{-,1}^{(-)} \\ \Psi_{+,2}^{(+)} & \Psi_{+,2}^{(-)} & \Psi_{-,2}^{(+)} & \Psi_{-,2}^{(-)} \end{bmatrix} \mathbf{P}_x^p \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix}. \quad (11.21)$$

Here, the superscripts "+" and "-" of both Ψ_+ and Ψ_- denote the propagation directions of $+x_1$ and $-x_1$, respectively. The subscripts "1" and "2" represent the first and second components of $\Psi_{\pm}$, respectively. A, B, C , and D are the displacement amplitudes, and $\mathbf{P}_x^p = \mathbf{I}(e^{iq_+x_1}, e^{-iq_+x_1}, e^{iq_-x_1}, e^{-iq_-x_1})^T$. For normal wave incidence, the field variables including the normal and shear stresses can be easily obtained at $x_1 = x$ as

$$\mathbf{F}_x^p = \begin{bmatrix} u_1 \\ u_2 \\ \sigma_{11} \\ \sigma_{12} \end{bmatrix} = \mathbf{Q}_p \mathbf{P}_x^p \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix}, \quad (11.22)$$

where $\mathbf{Q}_p$ is given in the appendix. By linking the field variables at the two interfaces, i.e. $\mathbf{F}_L^p = \mathbf{TF}_0^p$, we obtain the transfer matrix

$$\mathbf{T} = \mathbf{Q}_p \mathbf{P}_L^p \mathbf{Q}_p^{-1} (\mathbf{P}_0^p)^{-1}. \quad (11.23)$$

Similarly, the plane wave solution in the conventional isotropic media (μ_0, λ_0 and ρ_0) is written as

$$\mathbf{u}_s = A^s (1, 0)^T e^{iq_s^{(P)}x_1} + B^s (1, 0)^T e^{-iq_s^{(P)}x_1} + C^s (0, -1)^T e^{iq_s^{(S)}x_1} + D^s (0, 1)^T e^{-iq_s^{(S)}x_1} \quad (11.24)$$

where the subscript "s" denotes the isotropic background, while the superscripts " $(P)^n$ " and " $(S)^n$ " correspond to P- and S-polarized incidences. Then, the field variables at $x_1 = x$ are

$$\mathbf{F}_x^s = \begin{bmatrix} u_1^s \\ u_2^s \\ \sigma_{11}^s \\ \sigma_{12}^s \end{bmatrix} = \mathbf{Q}_s \mathbf{P}_x^s \begin{bmatrix} A_x^s \\ B_x^s \\ C_x^s \\ D_x^s \end{bmatrix}, \quad (11.25)$$

where $\mathbf{P}_x^s = \mathbf{I} \left(e^{iq_x^{(P)}x}, e^{-iq_x^{(P)}x}, e^{iq_s^{(S)}x}, e^{-iq_s^{(S)}x} \right)^T$, and $\mathbf{Q}_s$ is given in the appendix. With the continuity conditions at interfaces $x_1 = 0$ and $x_1 = L$, stating $\mathbf{F}_0^s = \mathbf{F}_0^p$ and $\mathbf{F}_L^s = \mathbf{F}_L^p$, we have

$$\begin{bmatrix} A_L^s \\ B_L^s \\ C_L^s \\ D_L^s \end{bmatrix} = (\mathbf{P}_L^s)^{-1} \mathbf{Q}_s^{-1} \mathbf{T} \mathbf{Q}_s \mathbf{P}_0^s \begin{bmatrix} A_0^s \\ B_0^s \\ C_0^s \\ D_0^s \end{bmatrix} = \mathbf{S} \begin{bmatrix} A_0^s \\ B_0^s \\ C_0^s \\ D_0^s \end{bmatrix}. \quad (11.26)$$

The transmission and reflection coefficients of each polarization for the isotropic polar slab can be determined through $\mathbf{S}$. For instance, with a P-polarized incidence from the left, the scattering parameters, t_{pp} , t_{ps} , r_{pp} , and r_{ps} can be obtained simply from

$$\begin{bmatrix} t_{pp} \\ 0 \\ t_{ps} \\ 0 \end{bmatrix} = \mathbf{S} \begin{bmatrix} 1 \\ r_{pp} \\ 0 \\ r_{ps} \end{bmatrix}. \quad (11.27)$$

In the following, we will primarily concentrate on the P-to-S conversion. From Eq. (11.27) and Fig. 11.3b, the conversion rate can be written by applying $r_{pp}, r_{ps} \approx 0$ as (Kweun et al., 2017)

$$T_{ps} = |t_{ps}|^2 \approx |S_{31}|^2 = D_s - E_s \cos(q_-L - q_+L) + G_s, \quad (11.28)$$

where

$$\begin{aligned} D_s &= \frac{c_+^2 c_-^2}{2} \frac{Y^2 (Z_S^2 Z_+^2 + Z_S^2 Z_-^2 + 2Z_+^2 Z_-^2) + X^2 (Z_+^2 + Z_-^2 + 2Z_S^2) - 8XY Z_S Z_+ Z_-}{4Z_S^2 X^2}, \\ E_s &= \frac{c_+^2 c_-^2}{2} \frac{2Y^2 (Z_S^2 Z_+ Z_- + Z_+^2 Z_-^2) + 2X^2 (Z_+ Z_- + Z_S^2) - 2XY Z_S (Z_+ + Z_-)^2}{4Z_S^2 X^2}, \\ G_s &= \frac{B_s^2 - A_s^2}{2} \cos(2q_+L) + \frac{B_s^2 - C_s^2}{2} \cos(2q_-L) - (B_s^2 + A_s C_s) \cos(q_-L + q_+L) \end{aligned}$$

with

$$\begin{aligned}
X &= \frac{\bar{Q}^2}{4}(m^2 + n^2 - 1 - 4l^2 - 4l), \\
Y &= \frac{Z_p \sqrt{-X}}{\rho_p}, \\
A_s &= -\frac{YZ_-c_+c_-}{2X} + \frac{Z_+c_+c_-}{2Z_s}, \\
B_s &= -\frac{YZ_+Z_-c_+c_-}{2Z_sX} + \frac{c_+c_-}{2}, \\
C_s &= \frac{YZ_+c_+c_-}{2X} - \frac{Z_-c_+c_-}{2Z_s}.
\end{aligned}$$

Since G_s only introduces wiggly effect due to a non-vanishing structural instability, it does not necessarily influence the maximum conversion rate much as long as the reflection remains small (Kweun et al., 2017). Based on Eq. (11.28), the conversion rate is predicted to be a sine function of frequency owing to $q_{\pm} = \omega/\mathbb{C}_{\pm}$, and the maximum conversion occurs at $(q_- - q_+)L = (2N - 1)\pi$, $N \in \mathbb{Z}$, where $\mathbb{Z}$ denotes integer numbers. This condition corresponds to the transmodal Fabry-Perot resonance (Kweun et al., 2017) and implies that the optimal conversion frequency is inversely proportional to L . Then, the maximum conversion rate approximately takes the form of $T_{PS} \approx D_s + E_s$. Because $(2l + 1)^2 \gg (m^2 + n^2)$ when l takes large values, $2\sqrt{-X} \approx Q(2l + 1)$ holds, leading to $D_s \approx E_s \geq 2c_+^2c_-^2$. To optimize the conversion rate, one should aim to maximize $c_+^2c_-^2 = m^2/(4m^2 + 4n^2)$. This indicates that $n \rightarrow 0$ leads to the optimal conversion, while m mainly determines the occurrence frequency of optimal mode conversion. It is because $\beta = m\bar{Q}$ mainly induces the coupling between the shear and longitudinal motions. Note that the mode shapes $\Psi_{\pm}$ at $n = 0$ ($\kappa = \alpha/4$) also shows the optimal conversion always takes place when the polarization vectors of qP and qS modes are ± 45 degrees with respect to the propagation direction, namely $\gamma = \pi/4$.

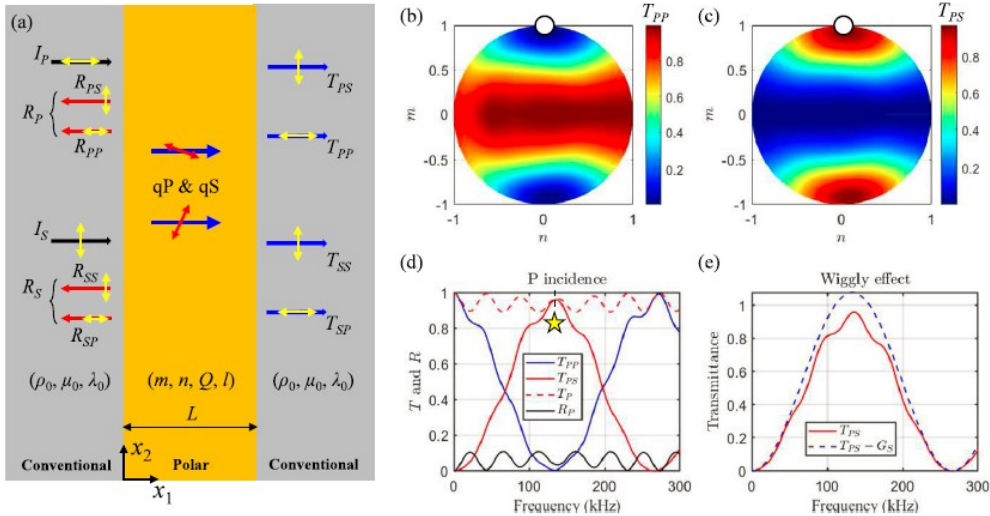


Figure 11.4 Schematic of the sandwiched system defined by two Cauchy media and an isotropic polar slab. The incident wave I_P is indicated by a black solid arrow. All the scattered fields including the intermediate (blue), transmitted and reflected components are also identified. The yellow arrows for all the field components denote the polarization states. The thickness of the polar slab is L . (b-c) Calculated transmittance of (b) P-to-P and (c) P-to-S components of the sandwiched isotropic polar slab under P-polarized incidence in functions of m and n at $l = 1$. All results in (b – c) are calculated at $f = 144.1\text{kHz}$ with the aluminum background. The polar slab thickness is chosen as $L = 0.233\lambda^{(P)}$ at 144.1kHz . A specific example related to the degenerate isotropic polar medium is taken at $m = 1$ and $n = 0$, and highlighted by the white circles. (d) Calculated frequency dependent scattering parameters of the isotropic polar slab under P-polarized incidence in the selected case in (b-c). The blue and red solid curves represents transmittance of P-to-P and P-to-S components, respectively. The red dashed and black solid curves correspond to the total transmittance and reflectance, respectively. The yellow indicates the optimal mode conversion frequency. (e) Examination of the wiggly effect for P-polarized incidence.

Figs. 11.4b and 11.4c plot the m, n - dependence of wave transmittances T_{pp} and T_{ps} at 144.1kHz . In the simulation, the parameters $\rho_p = 5\rho_0$ and $l = 1$ yielding sufficiently small reflection [Fig. 11.3b] are selected. It is found that the optimal conversion takes place around $n = 0$ and $m = 1$, which agrees well with the prediction. The conversion rate at this frequency is over 0.95 with vanishing transmission for unconverted P mode ($T_{pp} \rightarrow 0$). By selecting $m = 0.999$ and $n = 0.001$, we also plot the scattering spectra of the isotropic polar slab for P-polarized incidence in Fig. 11.4d. The frequency dependence of $T_{pp}(T_{ps})$ follows roughly a cosine(sine) function, which is consistent with Eq. (11.28). The total reflectance R_p with magnitude overall lower than 0.1 exhibits oscillation over the frequency range due to Fabry-Perot resonances. At the optimal conversion frequency, $R_p = 0.03$ is achieved. Besides, Fig. 11.4e illustrates the influence of the wiggly term G_s , which primarily induces small perturbations on the transmittance curve as long as weak reflection is well achieved, and does not influence the frequency dependence. Note that, following the same procedure, one can obtain the expression of the same form as Eq. (11.28) for a S-polarized incidence, which suggests the same optimal conversion frequencies. However, the values of D_s , E_s and G_s will be generally different (Kweun et al., 2017).

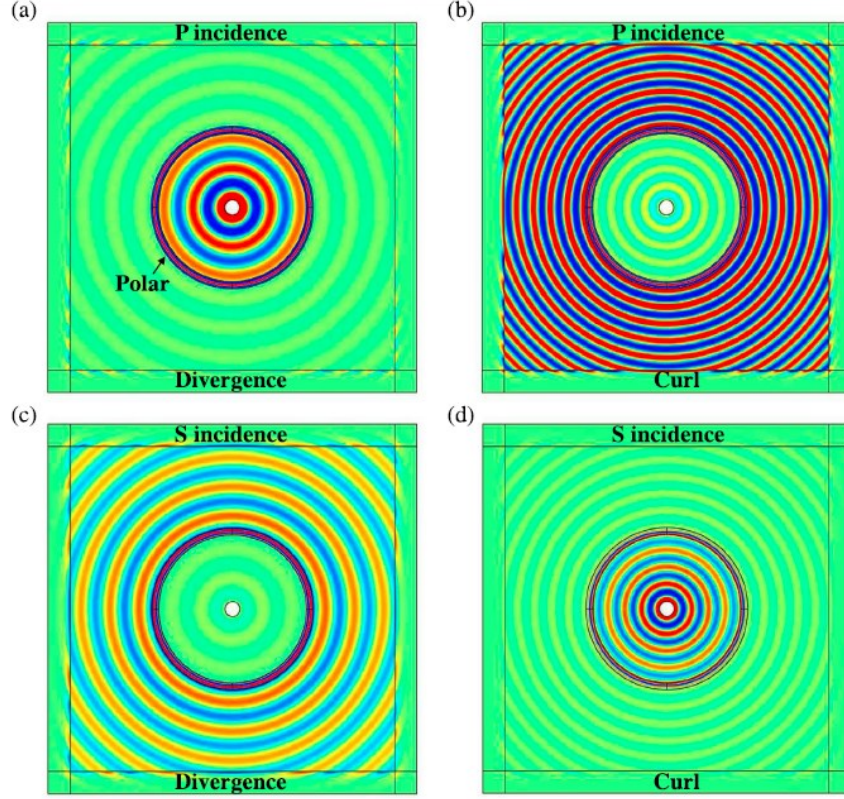


Figure 11.5 Optimal omnidirectional mode conversion within a circular isotropic polar slab of a radial thickness $L = 0.233\lambda^{(P)}$ at 144.1kHz. (a-b) show the normalized (a) divergence and (b) curl fields of displacement under the cylindrical P-polarized incidence. (c-d) give the normalized (c) divergence and (d) curl fields of displacement under the cylindrical S-polarized incidence. All the results are recorded at $f = 144.1\text{kHz}$.

To numerically verify the omnidirectional mode conversion, we utilize the simulation model used in Figs. 11.2b and 11.2c but with a different polar slab thickness. In particular, $l = 1$, $\rho_p = 5\rho_0$ and the same thickness L as that chosen in Fig. 11.4 are selected; see Fig. 11.5. At the center a cylindrical wave excitation of P wave is launched at the first optimal frequency, corresponding to the yellow pentagram in Fig. 11.4d [also see Figs. 11.5a and 11.5b]. At the optimal conversion frequency (144 kHz), the obtained divergence and curl fields of displacement indicate the occurrence of a nearly perfect omnidirectional mode conversion for P-polarized incidence. The reflection from the isotropic polar slab is heavily suppressed. In addition, we test the scattered fields for S-polarized incidence with the same configuration. As can be seen from Fig. 11.5c and 11.5d, a highly efficient S-to-P conversion is also achieved with minor reflection at the optimal conversion frequency, with an estimated conversion rate higher than 0.95.

11.4 Anomalous wave polarization in the polar medium

In addition to wave mode conversion, we point out that the polar media featuring the omnidirectional AWP can nearly perform as isotropic singlypolarized solids (SPS) where only S waves are allowed to be transmitted through omnidirectionally, without the necessity of elastic

anisotropy (Zheng et al., 2019). Also, as a straightforward demonstration of the AWP, anomalous refraction is numerically realized that an oblique incidence of P wave produces zero transmitted S wave beyond a critical angle, when a P -polarized incidence propagates from a softer normal Cauchy to a harder isotropic polar media.

11.4.1 Omnidirectional quasi-singly-polarized transmission

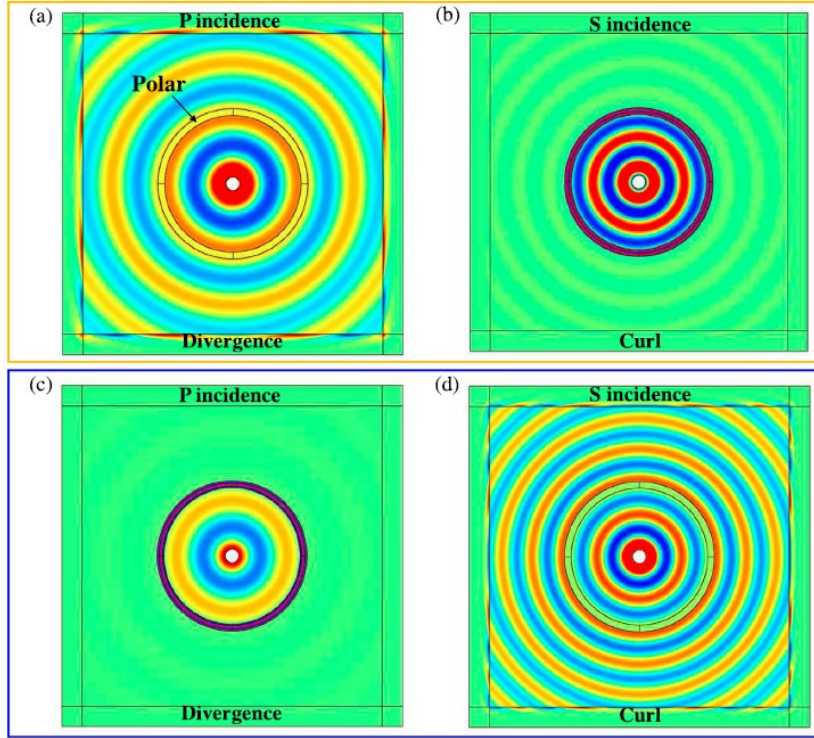


Figure 11.6 Numerical demonstration of the omnidirectional singly polarized transmission at 70kHz. The parameters of the polar media are chosen as $l = 0.001$, $\rho_p = 0.5\rho_0$, $m = 0.001$, $Q = 0.8$ and $L = 0.1134\lambda^{(P)}$ at 70kHz, (a-b) $n = 0.999$ for a longitudinal QSPS, and (c-d) $n = -0.999$ for a shear QSPS. Normalized (a,c) divergence and (b,d) curl fields under P and S incidences for both the longitudinal and shear QSPSs are illustrated, respectively.

As extreme cases, the isotropic polar media can be easily characterized as the singly polarized solids by properly harnessing the polar parameters. From Fig. 11.2a, the existence of only P and/or S modes in the isotropic polar medium requires the chirality to be vanishing with $m \rightarrow 0$. Recalling the expressions of velocities given by Eq. (11.11), we find the smaller velocity $\mathbb{C}_-$ cannot reach zero because $l > 0$. However, an alternative way to approximate the singly-polarized solids is to find the case with $\mathbb{C}_- \ll \mathbb{C}_+$. By reexamining Fig. 11.2a, the isotropic polar medium can be treated approximately as a singly-polarized solid supporting only P modes called longitudinal quasisingly-polarized solid (QSPS) by taking $n \rightarrow 1$ and supporting only S modes called shear quasi-singly-polarized solid by taking $n \rightarrow -1$. Note that the latter corresponds to an extreme case of the omnidirectional AWP where shear modes propagate faster. To numerically demonstrate these two QSPSs, we consider the term Q given by Eq. (11.20) without loss of generality. In order to meet

the requirements for both longitudinal and shear QSPSs, we set $l = 0.001$ and $\rho_p = 0.5\rho_0$, which satisfy the stability condition. We select $m = 0.001$ and $n = \pm 0.999$ with "+" and "-" corresponding to a longitudinal and a shear QSPSs, respectively. The simulation model is exactly the same as the one illustrated in Fig. 11.5. Both S- and P-polarized incidences at 70kHz are excited in the center. This frequency is chosen because it allows large transmission in the respective cases. For the longitudinal QSPS, the P-polarized incidence is nearly fully transmitted through the polar slab, whereas the S-polarized incidence is mostly reflected by the polar slab to the inner background region, causing strong curl field intensities owing to the incidence and reflection superimposed; see Figs. 11.6a and 11.6 b. Note that the physical realization of this longitudinal QSPS with our previous grounded hexagonal lattices requires zero chirality and corresponds exactly to some existing architected materials such as "pentamode lattice". In that type of lattices, the shear modulus nearly vanishes, causing longitudinal modes to drastically outspeed shear modes (Kadic et al., 2013, 2014). On the contrary, the shear QSPS forbids the transmission of P wave, but allows the transmission of S wave, as shown in Figs. 11.6c and 11.6 d. This agrees well with our previous analysis, and can be alternatively understood by impedance mismatching: the degrees of the impedance matching between the longitudinal QSPS and background aluminum for the P and S modes read $Z_+/Z_P = 1.2244$ and $Z_-/Z_S = 15.9303$, while for the shear QSPS, the two ratios become $Z_-/Z_P = 31.6254$ and $Z_+/Z_S = 0.6168$. Within the long-wavelength limit, the singly polarized transmission is in general broadband except for several Fabry-Perot resonance frequencies (Nassar et al., 2019; Xu et al., 2020). Most importantly, the singly polarized transmission phenomenon for both polarized incidences occurs omnidirectionally, vastly differing from the anisotropic cases reported previously (Zheng et al., 2019).

11.4.2 Anomalous wave refraction

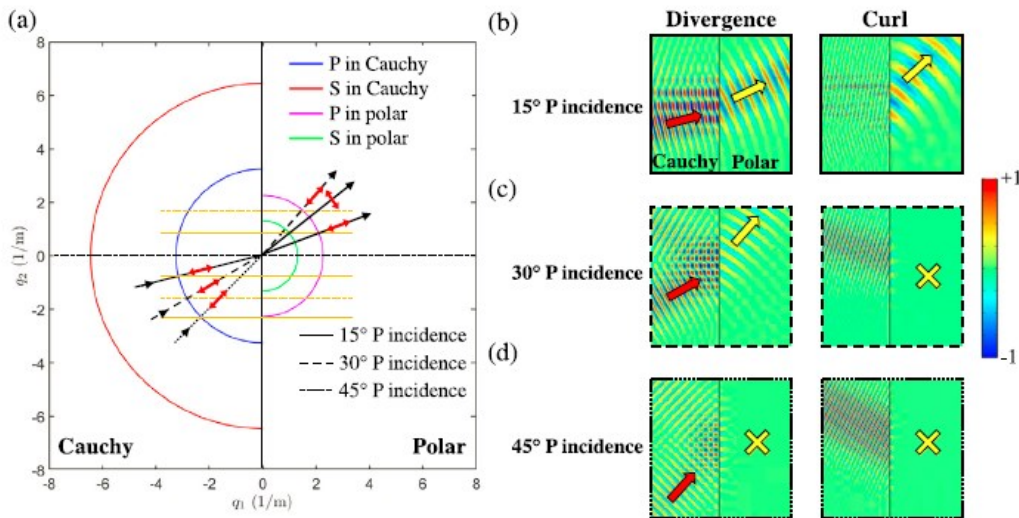


Figure 11.8 (a,b) Frequency dependent transmission and reflection of (a) P incidence and (b) S incidence by the isotropic polar slab of $L = 100$ mm, $l = 1$, $Q = 0.1224$, $\rho_p = 5\rho_0$, $m = 0.001$, and $n = -0.999$. The marked frequency (53kHz) is what we have used for the demonstration of the omnidirectional AWP that features faster shear

mode in the main text. (c,d) Frequency dependent transmission and reflection by isotropic polar slabs of (c) $L = 10$ mm, $m = 0.001$, $n = 0.999$ and (d) $L = 10$ mm, $m = 0.001$, $n = -0.999$. The marked frequency (70 kHz) is what we have used for the demonstration of the omnidirectional singly polarized transmission in the main text. In all the figures, T and R represent the transmittance and reflectance, respectively. The subscripts "P" and "S" correspond to the P- and S-polarized waves, respectively.

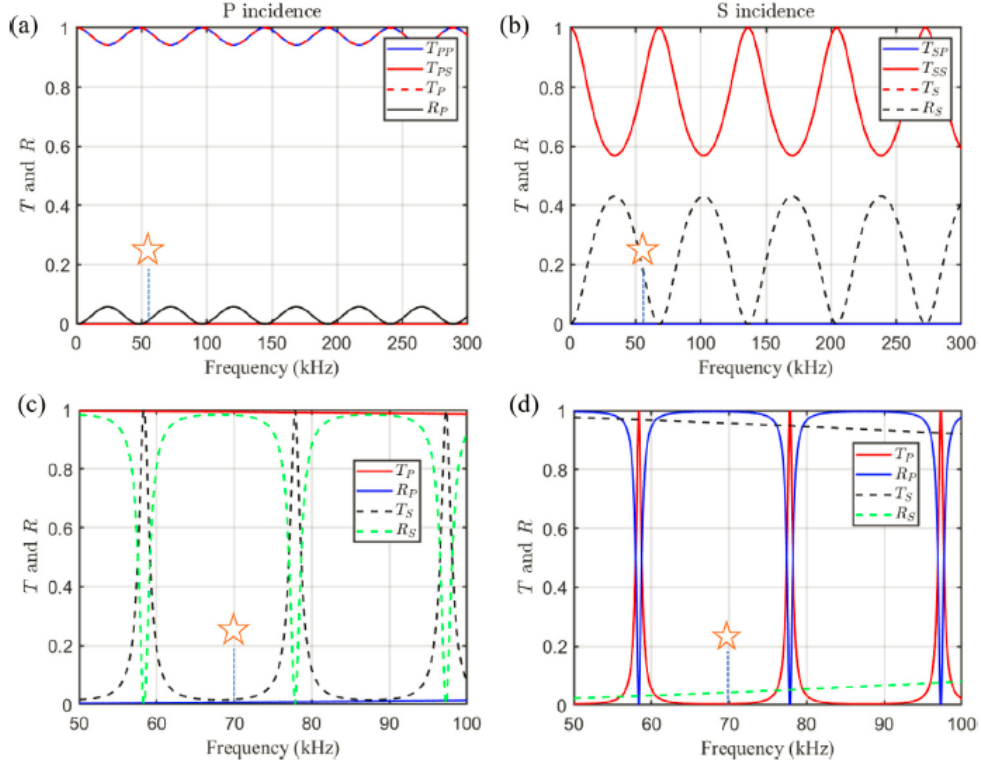


Figure 11.8 (a,b) Frequency dependent transmission and reflection of (a) P incidence and (b) S incidence by the isotropic polar slab of $L = 100$ mm, $l = 1$, $Q = 0.1224$, $\rho_p = 5\rho_0$, $m = 0.001$, and $n = -0.999$. The marked frequency (53kHz) is what we have used for the demonstration of the omnidirectional AWP that features faster shear mode in the main text. (c,d) Frequency dependent transmission and reflection by isotropic polar slabs of (c) $L = 10$ mm, $m = 0.001$, $n = 0.999$ and (d) $L = 10$ mm, $m = 0.001$, $n = -0.999$. The marked frequency (70 kHz) is what we have used for the demonstration of the omnidirectional singly polarized transmission in the main text. In all the figures, T and R represent the transmittance and reflectance, respectively. The subscripts "P" and "S" correspond to the P- and S-polarized waves, respectively.

In general, when two isotropic media supporting only normal polarizations are separated by an interface, an oblique P-polarized incidence from the soft medium striking on the interface always generates both P- and S-polarized transmissions in the hard medium. This is simply because P modes possess larger phase velocities in these media, and will be converted into their S counterparts due to transverse phase matching. Here, we take advantage of the AWP from our isotropic polar media and demonstrate the anomalous refraction where beyond a critical angle, a P-polarized incidence will not yield any S-polarized transmission from a softer medium to a harder one. This originates from the fact that the faster mode becomes S-polarized with $\mathbb{C}_+$ when $m \rightarrow 0$

and $n < 0$, as has been indicated in the previous sections. According to Snell's law, the critical angle of incidence for the occurrence of the anomalous refraction is given by

$$\theta_c \approx \sqrt{\frac{2\rho_p}{\rho_0 Q(2l+1-n)}} \quad (11.29)$$

That said, a P-polarized incidence beyond θ_c from a conventional isotropic medium will not produce S-polarized refraction. For the corresponding numerical demonstration, we construct the isotropic polar slab embedded in the isotropic Cauchy-elastic medium. The equi-frequency contours (EFCs) of the background and isotropic polar medium are presented in Fig. 11.7a. A critical angle of 23.78° for the anomalous refraction can be obtained based on Eq. (11.29). Moreover, a critical angle for the P-polarized polarization is found to be 44.2° . In this case, we consider three angles of incidence for P-polarized incidence: 15° , 30° and 45° ; see Figs. 11.7a. The corresponding simulations shown in Fig. 11.7b-d involve a Gaussian beam excitation. The isotropic polar region and the aluminum background are separated by an interface. In Fig. 11.7 b, apparently, for a small incidence angle ($15^\circ < \theta_c = 23.78^\circ$), both P- and S-polarized transmissions with finite amplitudes in the polar medium can be observed, and the propagation directions agree well with the EFC prediction. When the angle of incidence exceeds the critical angle for the anomalous refraction, i.e. $30^\circ > \theta_c$, the P-polarized incidence does not intersect with the EFC of the S mode in the polar medium [see Fig. 11.7a]. As a result, no S-polarized transmission exists in the polar area, as can be seen in Fig. 11.7c. This is the direct numerical demonstration of the anomalous refraction for P-polarized incidence in our isotropic polar system. In addition, for a P-polarized incidence of 45° exceeding the critical angle for the P-polarized refraction (44.2°), neither P- nor S-polarized transmission can be observed in the polar region [see Fig. 11.7d].

11.5 Conclusions

In the study, the isotropic polar medium with extra polarity and chirality moduli is leveraged to realize complete mode conversion and anomalous wave polarization free from directional dependence. In the proposed isotropic polar system, non-vanishing asymmetric stress with broken minor symmetries plays an important role in delivering highly efficient omnidirectional mode conversion owing to the chirality. Nearly total wave mode conversion from longitudinal to shear waves or vice versa is analytically and numerically investigated through a finite polar slab by engineering the two polar parameters. The isotropic polar solid also enables omnidirectional anomalous wave polarization which enables faster shear modes and leads to the omnidirectional singly-polarized transmission and anomalous refraction for longitudinal wave. The recent experimental realization of the elastic polar solids may push this research closer to practical applications (Xu et al., 2020). We believe the proposed wave manipulation schemes based on isotropic polar solids should be beneficial to the modern elastic mode conversion and wave steering devices.

Appendix 11A Detailed expressions in transfer matrix method

The detailed expression of Q_p in Eq. (11.22) is given below

$$Q_p = \begin{bmatrix} \Psi_{+1}^{(+)} & \Psi_{+1}^{(-)} & \Psi_{-1}^{(+)} & \Psi_{-1}^{(-)} \\ \Psi_{+2}^{(+)} & \Psi_{+2}^{(-)} & \Psi_{-2}^{(+)} & \Psi_{-2}^{(-)} \\ Q31 & Q32 & Q33 & Q34 \\ Q41 & Q42 & Q43 & Q44 \end{bmatrix}$$

with

$$\begin{aligned} Q31 &= iq_+ (a_{1111} \Psi_{+1}^{(+)} + a_{1121} \Psi_{+2}^{(+)}) \\ Q32 &= -iq_+ (a_{1111} \Psi_{+1}^{(-)} + a_{1121} \Psi_{+2}^{(-)}) \\ Q33 &= iq_- (a_{1111} \Psi_{-1}^{(+)} + a_{1121} \Psi_{-2}^{(+)}) \\ Q34 &= -iq_- (a_{1111} \Psi_{-1}^{(-)} + a_{1121} \Psi_{-2}^{(-)}) \\ Q41 &= iq_+ (a_{1211} \Psi_{+1}^{(+)} + a_{1221} \Psi_{+2}^{(+)}) \\ Q42 &= -iq_+ (a_{1211} \Psi_{+1}^{(-)} + a_{1221} \Psi_{+2}^{(-)}) \\ Q43 &= iq_- (a_{1211} \Psi_{-1}^{(+)} + a_{1221} \Psi_{-2}^{(+)}) \\ Q44 &= -iq_- (a_{1211} \Psi_{-1}^{(-)} + a_{1221} \Psi_{-2}^{(-)}) \end{aligned}$$

The detailed expressions of Q_x and P_x^z in Eq. (11.25) is given below

$$Q_x = \begin{bmatrix} c_x^{(P)} & c_x^{(P)} & s_x^{(S)} & s_x^{(S)} \\ s_x^{(P)} & -s_x^{(P)} & -c_x^{(S)} & c_x^{(S)} \\ Q31s & Q32s & 0 & 0 \\ 0 & 0 & Q43s & Q44s \end{bmatrix}$$

where all the denoted entries read

$$\begin{aligned} Q31s &= iC_{11}q_x^{(P)} \\ Q32s &= -iC_{11}q_x^{(P)} \\ Q43s &= -i\frac{1}{2}C_{66}q_x^{(S)} \\ Q44s &= -i\frac{1}{2}C_{66}q_x^{(S)} \end{aligned}$$

where $C_{11} = 2\mu_0 + \lambda_0$ and $C_{66} = \mu_0$

where $C_{11} = 2\mu_0 + \lambda_0$ and $C_{66} = \mu_0$.

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