

PROPAGATION OF ELECTROMAGNETIC WAVES THROUGH A STRATIFIED MEDIUM PART 1 GENERAL ANALYSIS

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ABSTRACT

A general analysis of the propagation of electromagnetic waves through a stratified medium is developed from Maxwell's equations. The medium is assumed to be divided into m plane-parallel layers, the first and m th of which are semi-infinite. The electromagnetic properties of the m layers are arbitrary and dissimilar and, in general, the thicknesses of the $(m-2)$ finite layers are also different. A steady-state monochromatic plane wave, generated in the m th layer and having either its electric or magnetic vector normal to the plane of incidence, is assumed to be incident at any angle upon the $(m-1)$ th layer. Formulas for the resulting reflection and transmission are derived.

In Part II, to be published, the methods and results of the general analysis will be used to obtain detailed reflection and transmission characteristics of a variety of multi-layer systems which are of interest in radio and optics.

PROBLEM STATUS

The work reported here was done several years ago and has been referred to in the literature; it is presented here in response to requests made during the past few years for copies of the work. Work on the problem continues.

AUTHORIZATION

NRL Problem No. R11-01R (NR 511-010)

the layers is uniform and isotropic, but the electromagnetic properties of the m layers are arbitrary and in general dissimilar, as are also the thicknesses of the $(m-2)$ finite layers. A steady-state monochromatic plane wave generated in the m th layer is assumed to be incident at any angle upon the $(m-1)$ th layer. Because the layers are dissimilar, a reflected wave is also present in the m th layer. Similarly, a transmitted wave and a reflected wave are present in each of the finite layers. In the first, or final, layer only a transmitted wave is present. Because the layer extends to infinity, no reflection occurs.

The essential problem is to provide a quantitative description of the propagation of these waves in the layers of the stratified medium, and in particular to determine the reflection and transmission at the interface between m th and $(m-1)$ th layers, and the transmission through the final interface. These quantities must be expressed in terms of the thicknesses and electromagnetic properties of the m layers and the state of polarization and angle of incidence of the wave which is incident upon the $(m-1)$ th layer.

THEORY

The relations between \mathbf{H} and \mathbf{E} , the complex vector magnetic and electric amplitudes at any point in a layer, are expressed by the Maxwell equations

$$\begin{aligned} \text{curl } \mathbf{H} &= (PA) \mathbf{E} \\ -\text{curl } \mathbf{E} &= (P/A) \mathbf{H} \end{aligned} \quad (1)^*$$

Both \mathbf{H} and \mathbf{E} vary as $\exp(i\omega t)$, where $i = (-1)^{1/2}$, ω is the angular frequency of the steady-state source, and t is the time. The propagation constant, P , is proportional to the complex index of refraction, or to the square root of the product of complex dielectric constant and complex permeability. The characteristic admittance, A , for which there is no corresponding optical term, is equal to the square root of the ratio of complex dielectric constant to complex permeability. These parameters are related to the electromagnetic properties and the angular frequency in the following ways:

$$\begin{aligned} P &= (i\omega/c_0) [(\epsilon - i\epsilon') (\mu - i\mu')]^{1/2} = [(\sigma + i\omega\epsilon\epsilon_0) (\rho + i\omega\mu\mu_0)]^{1/2} \\ A &= A_0 [(\epsilon - i\epsilon') / (\mu - i\mu')]^{1/2} = [(\sigma + i\omega\epsilon\epsilon_0) / (\rho + i\omega\mu\mu_0)]^{1/2} \end{aligned} \quad (2)$$

In equations (2), ϵ and ϵ' are the real and imaginary components of the complex relative dielectric constant, μ and μ' are the real and imaginary parts of the complex relative magnetic permeability, ϵ_0 and μ_0 are the electric and magnetic properties of free space, σ is the electric conductivity, and ρ is the magnetic resistivity, a term corresponding to one introduced by Heaviside⁵ to make the curl relations more symmetrical. The units used are those of the MKS system, in which the velocity of wave propagation in free space is $c_0 = (\mu_0\epsilon_0)^{-1/2} \approx 3 \times 10^8$ meters per second, and the characteristic admittance of free space is $A_0 = (\epsilon_0/\mu_0)^{1/2} \approx 2.65 \times 10^{-3}$ mho.

*Equations of a general nature are numbered in ordinary type. Corresponding equations which apply only to a specific problem are numbered in italicized type.

⁵Heaviside, O., "Electrical Papers," Copley, 1925. See vol. I, p. 449, and vol. II, p. 378.

The electromagnetic properties σ and ρ , or the corresponding properties ϵ' and μ' , which are responsible for energy loss or absorption in a layer, are always positive or zero. The properties ϵ and μ are usually, but not necessarily, greater than or equal to unity. In radio and optical problems, ρ and μ' are usually zero. Since the parameters P and A are square roots of complex quantities, a convention must be adopted with respect to choice of sign for these quantities. The convention used here is the customary one—that the real part of P , and therefore also the real part of A , is always positive. This choice of sign results in the interpretation of $\exp(-Px)$ as the symbolic representation of a wave which decreases in amplitude along the positive x -direction, and correspondingly in the interpretation of $\exp(Px)$ as a wave which decreases in amplitude along the negative x -direction.

At the interface between two adjoining layers, say j and k , the same laws of electromagnetic induction that are expressed by equation (1) require equality on the two sides of the interface of (a) the tangential components of both \mathbf{H} and \mathbf{E} , and (b) the normal components of both magnetic and electric current densities.

$$\left\{ \begin{array}{l} \mathbf{H}_j = \mathbf{H}_k \\ \mathbf{E}_j = \mathbf{E}_k \end{array} \right\} \text{tangential components} \quad (3.1)$$

$$\left\{ \begin{array}{l} (P/A)_j \mathbf{H}_j = (P/A)_k \mathbf{H}_k \\ (PA)_j \mathbf{E}_j = (PA)_k \mathbf{E}_k \end{array} \right\} \text{normal components} \quad (3.2)$$

The boundary planes of the layers may be assumed parallel to the plane formed by the y - and z -axes of a right-handed Cartesian system of coordinates, so that the normal to these planes is the x -axis. The y - and z -axes must be perpendicular to each other and to the x -axis, but otherwise their actual positions are arbitrary. Accordingly they are directed so as to make the normal to the incident plane wave lie in the x - z plane. The latter is thus the plane of incidence, since it contains the wave normal and the normal to the boundary planes of the layers. The angle of incidence is the angle between the two normals (see Figure 1).

The incident plane wave in the m th layer is assumed to be linearly polarized, or plane-polarized in optical terms, with either its electric or magnetic vector perpendicular to the plane of incidence. If neither vector satisfies the latter requirement, each of the vectors may be decomposed into two auxiliary vectors, one in the plane of incidence and the other perpendicular to that plane. The auxiliary vectors may then be combined so as to form two superposed linearly polarized plane waves, one of which has its electric vector perpendicular to the plane of incidence, and the other having its magnetic vector perpendicular to that plane. A solution of the more general situation thus may be obtained by superposing solutions of the two special cases to be analyzed. Furthermore, results based on the assumption of an incident linearly polarized wave are also fundamental to the case of an elliptically polarized wave, since the latter may be regarded as consisting of two linearly polarized waves, properly superposed.

Wave equations for \mathbf{H} or \mathbf{E} , applicable to any layer, are obtained by eliminating \mathbf{E} or \mathbf{H} , respectively, between the equations (1). The equations are

$$\begin{aligned} \nabla^2 \mathbf{H} &= P^2 \mathbf{H} \\ \nabla^2 \mathbf{E} &= P^2 \mathbf{E} \end{aligned} \quad (4)$$

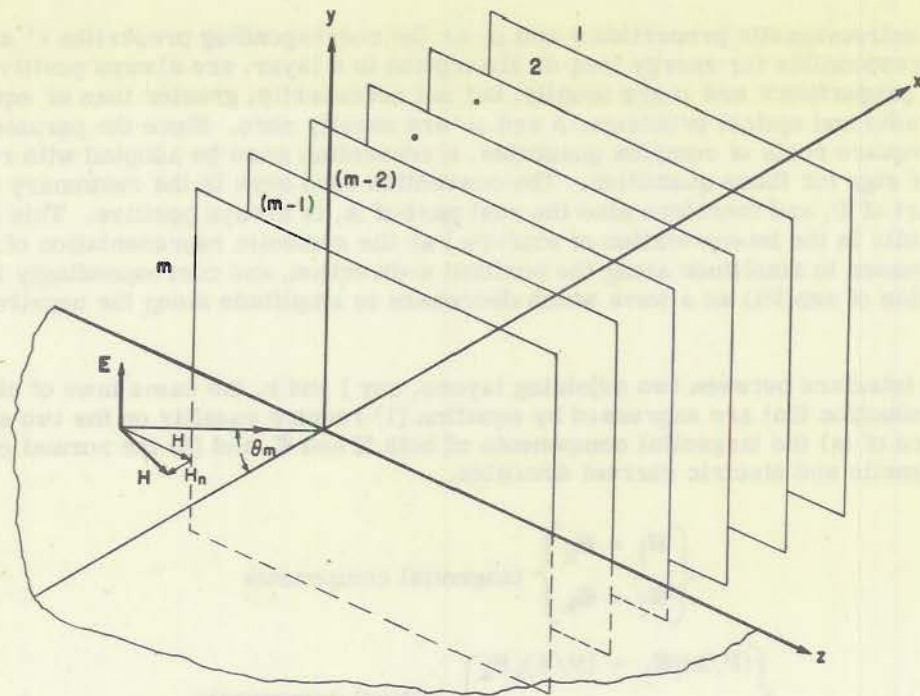


Figure 1 - The boundary planes of the m layers are parallel to the y - z plane. The normal to the incident plane wave, making an angle θ_m with the x -axis, is in the x - z plane, which is the plane of incidence. The situation depicted here corresponds to the case of transverse electric waves, for which the electric vector is perpendicular to the plane of incidence and the magnetic vector has both normal and tangential components.

Because of the assumed geometry, equations (4) contain derivatives with respect to x and z , but not with respect to y , so that the corresponding solutions for \mathbf{H} and \mathbf{E} are independent of y . For the case in which the incident electric vector is perpendicular to the plane of incidence, there is only one component of \mathbf{E} , namely \mathbf{E}_y , which is tangential to the layer boundaries; and two components of \mathbf{H} , a tangential one, \mathbf{H}_z , and a normal one, \mathbf{H}_x . For the case in which the incident magnetic vector is perpendicular to the plane of incidence, there is only one component of \mathbf{H} , namely \mathbf{H}_y , which is tangential; and two components of \mathbf{E} , a tangential one, \mathbf{E}_z , and a normal one, \mathbf{E}_x . It should be noted that in the first case both tangential components, \mathbf{E}_y and \mathbf{H}_z , are positive, whereas in the second case one of the tangential components, \mathbf{E}_z , is intrinsically negative. In what follows, subscripts for the tangential components are omitted, and the normal components are denoted by the subscript n .

Electric Vector Perpendicular to Plane of Incidence

When the electric vector is perpendicular to the plane of incidence, generic solutions for \mathbf{E} , \mathbf{H} , and \mathbf{H}_n , applicable to any layer, are

$$\mathbf{E} = [\mathbf{C} \exp(x_0 - x)p + \mathbf{D} \exp(x - x_0)p] \exp(-qz) \quad (5.1e)^*$$

*Numbered equations which carry the additional letter e apply only to the case for which \mathbf{E} is perpendicular to the plane of incidence. Those which carry the letter h apply only to the case for which \mathbf{H} is perpendicular to the plane of incidence. Equations which are identified by a number only, apply to both cases.

$$H = (pA/P) [C \exp(x_0-x)p - D \exp(x-x_0)p] \exp(-qz) \quad (5.2e)$$

$$H_n = (qA/P) [C \exp(x_0-x)p + D \exp(x-x_0)p] \exp(-qz) \quad (5.3e)$$

in which

$$\begin{aligned} p &= P \cos \theta \\ q &= P \sin \theta \end{aligned} \quad (6)$$

That equations (5.1e) - (5.3e) are indeed solutions may be verified by carrying out the differentiations indicated by equations (4). In equations (5.1e) - (5.3e) C and D are constants, and x_0 is an origin in each layer from which the distance x is measured. For the m th and $(m-1)$ th layers x_0 is assumed to be the intersection of the x -axis with the boundary between m th and $(m-1)$ th layers. For the $(m-2)$ th layer x_0 is assumed to be the intersection of the x -axis with the boundary between $(m-1)$ th and $(m-2)$ th layers, and so on for each of the other layers. The factor $\exp(-qz)$ in equations (5.1e) - (5.3e) is, in general, complex; it therefore represents a combined amplitude and phase variation in the z -direction. The quantities p and q , the real parts of which are to be taken as positive, are propagation constants for the wave components along x - and z -directions, respectively; they are related geometrically to a length s along the incident wave normal, and to its x - and z -components, by the expression $px + qz = Ps$.

If the requirements of continuity, expressed by equations (3.1) and (3.2), are to be satisfied for all values of z at each of the interfaces between adjoining layers, the parameter q must have the same value for all layers. Therefore

$$q = P_k \sin \theta_k, \quad k = 1, 2, 3, \dots, m \quad (7)$$

In any layer, say the k th, the angle θ in equations (6) and (7) is the angle of refraction of the transmitted ray at the boundary between the $(k+1)$ th and k th layers; θ is also, therefore, the angle of incidence of the transmitted ray and $-\theta$ is the angle of reflection of the reflected ray at the boundary between k th and $(k-1)$ th layers. This angle does not appear explicitly in equations (5.1e) - (5.3e) because it is contained in the parameters p and q , given by equations (6). The connection between θ_k and the known angle of incidence, θ_m , in the m th layer is obtained from equation (7), by means of which an expression for p_k is also obtained.

$$p_k = P_k [1 - (P_m/P_k)^2 \sin^2 \theta_m]^{1/2} \quad (8)$$

The equiamplitude and equiphase planes of the waves described by equations (5.1e) - (5.3e) are defined by setting real and imaginary parts of the exponents equal to constants. This leads to an angle between equiamplitude planes and the z -axis which is given by

$$\tan \theta' = \operatorname{Re}(q)/\operatorname{Re}(p) \quad (9.1)$$

and to an angle between equiphase planes and the z -axis which is given by

$$\tan \theta'' = \operatorname{Im}(q)/\operatorname{Im}(p) \quad (9.2)$$

Here, Re indicates real part of and Im indicates imaginary part of. It should be noted that θ' is also the angle between the normal to the equiamplitude planes and the normal to the boundary planes of the layers; similarly, θ'' is also the angle between the normal to the equiphase planes and the normal to the boundary planes of the layers. When θ is real, equations (9.1), (9.2), and (6) show that $\theta' = \theta'' = \theta$. This means that the equiamplitude and

equiphase planes coincide, forming a uniform plane wave which makes an angle θ with the z-axis. In the special case of normal incidence, for which $\theta_m = 0$, this is always true, regardless of the properties of the layers. On the other hand, when θ_m is not zero, equation (7) indicates that even for lossless layers, for which P is imaginary, θ_k may no longer be real. For such situations θ'_k is not equal to θ''_k , which means that equiamplitude and equiphase planes in the k th layer are not coincident. The wave is now non-uniform, or hybrid.⁶ This is the type of wave which is set up, for example, when total reflection occurs at the boundary between two lossless materials; although incident and reflected waves are uniform, the refracted wave in this case is non-uniform. Equations (6) and (7) also indicate that when a uniform wave is transmitted at non-normal incidence from a lossless layer to a lossy layer, a hybrid wave is always set up in the lossy layer.

It will now be apparent that the plane waves in the different layers—including the m th, which contains the source—need not be uniform in general. When the waves are non-uniform θ is complex, and the phrase "angle of incidence" must be replaced by more explicit information with respect to the angles between the z-axis and the equiamplitude and equiphase planes.

Since the factor $\exp(-qz)$ is the same for all layers and common to all terms of equations (5.1e) - (5.3e), it may be absorbed in the coefficients C and D . Using the subscripts t and r to denote transmitted and reflected amplitudes, respectively, the tangential components of \mathbf{E} and \mathbf{H} given by equations (5.1e) and (5.2e) for each layer thus may be rewritten as

$$\mathbf{E} = \mathbf{E}_t \exp(x_0 - x)p + \mathbf{E}_r \exp(x - x_0)p \quad (10.1)$$

$$\mathbf{H} = \mathbf{H}_t \exp(x_0 - x)p - \mathbf{H}_r \exp(x - x_0)p \quad (10.2)$$

The relations between transmitted-wave amplitudes \mathbf{E}_t and \mathbf{H}_t , and reflected-wave amplitudes \mathbf{E}_r and \mathbf{H}_r , in equations (10.1) and (10.2), are obtained by comparison of these equations with equations (5.1e) and (5.2e), and making use of equations (6). These relations are

$$\begin{aligned} \mathbf{H}_t &= A' \mathbf{E}_t \\ \mathbf{H}_r &= A' \mathbf{E}_r \end{aligned} \quad (11e)$$

$$A' = pA/P = A \cos \theta$$

The first terms of equations (10.1) and (10.2) represent the electric and magnetic components of a plane wave transmitted along the positive x-direction in each layer, and the second terms represent the electric and magnetic components of a plane wave reflected along the negative x-direction in each layer except the first, in which there is no reflected wave. Plane waves of this type, for which the electric vector is perpendicular to the direction of propagation and the magnetic vector has a component along the direction of propagation, are designated as transverse electric waves. In equations (11e), A' , the real part of which is to be taken as positive, is the characteristic admittance for the wave components propagated along the x-direction; it is related to $A'' = (qA/P) = A \sin \theta$, the characteristic admittance for the wave components propagated along the z-direction, by the expression $A'^2 + A''^2 = A^2$.

Magnetic Vector Perpendicular to Plane of Incidence

When the magnetic vector is perpendicular to the plane of incidence, generic solutions for \mathbf{E} , \mathbf{H} , and \mathbf{E}_n , applicable to any layer, are

$$\mathbf{E} = [C \exp(x_0 - x)p + D \exp(x - x_0)p] \exp(-qz) \quad (5.1h)$$

⁶Fry, T. C., "Plane waves of light," *JOSA*, 15, 137-161, 1927; 16, 1-25, 1928; 22, 307-332, 1932.

$$H = (PA/p) [C \exp(x_0-x)p - D \exp(x-x_0)p] \exp(-qz) \quad (5.2h)$$

$$E_n = (q/p) [C \exp(x_0-x)p + D \exp(x-x_0)p] \exp(-qz) \quad (5.3h)$$

in which p and q are given by equations (6). That equations (5.1h) - (5.3h) are indeed solutions may be verified readily by testing them in the wave equations (4).

If the requirements of continuity, expressed by equations (3.1) and (3.2), are to be satisfied for all values of z at each of the interfaces between adjacent layers, the parameter q must again have the same value for all layers. This means that equations (7) and (8) of the preceding case are applicable here also.

The equiamplitude and equiphase planes of the waves described by equations (5.1h) - (5.3h) are determined by the same exponents as are present in equations (5.1e) - (5.3e). Consequently equation (9.1), which gives the angle between equiamplitude planes and the z -axis; and equation (9.2), which gives the angle between equiphase planes and the z -axis, are also valid here.

Expressions for the tangential components of \mathbf{E} and \mathbf{H} for each layer, written in terms of transmitted and reflected amplitudes, are given by equations (10.1) and (10.2), which are also valid here provided the relations between transmitted-wave amplitudes E_t and H_t , and reflected-wave amplitudes E_r and H_r , are defined by

$$\begin{aligned} H_t &= A'E_t \\ H_r &= A'E_r \\ A' &= PA/p = A/\cos \theta \end{aligned} \quad (11h)$$

In equations (11h), A' , the real part of which is to be taken as positive, is the characteristic admittance for the wave components propagated along the x -direction; it is related to $A'' = (PA/q) = A/\sin \theta$, the characteristic admittance for the wave components propagated along the z -direction, by the expression $(1/A')^2 + (1/A'')^2 = (1/A)^2$.

Plane waves of the type described by equations (10.1), (10.2), and (11h), for which the magnetic vector is perpendicular to the direction of propagation and the electric vector has a component along the direction of propagation, are designated as transverse magnetic waves.

The essential difference in these results for the case in which \mathbf{E} is perpendicular to the plane of incidence, and for the case in which \mathbf{H} is perpendicular to the plane of incidence, is exhibited by the last of equations (11e) and (11h). In the first case $A' = A \cos \theta$, whereas in the second case $A' = A/\cos \theta$. At normal incidence, for which $\theta = 0$, the two cases become indistinguishable, as should be expected. For both cases the phase velocity, or the velocity for the harmonic steady-state, is given by

$$v' = \omega/\text{Im}(p) \quad (12.1)$$

for the wave components propagated along the x -direction, and by

$$v'' = \omega/\text{Im}(q) \quad (12.2)$$

for the wave components propagated along the z -direction.

The Input Admittance

The values of E and H at one boundary of any of the $(m-2)$ finite layers may be related to the values of E and H at the other boundary in terms of the thickness and electromagnetic properties of the layer.

At the first boundary, from which x is measured, $(x-x_0) = 0$, so that equations (10.1) and (10.2) become

$$\begin{aligned} E &= E_1 = E_t + E_r \\ H &= H_1 = A' E_t - A' E_r \end{aligned} \quad (13.1)$$

At the second boundary $(x-x_0) = d$, the layer thickness, so that equations (10.1) and (10.2) become

$$\begin{aligned} E &= E_2 = E_t \exp(-pd) + E_r \exp(pd) \\ H &= H_2 = A' E_t \exp(-pd) - A' E_r \exp(pd) \end{aligned} \quad (13.2)$$

The amplitudes E_t and E_r may be eliminated between equations (13.1) and (13.2) to provide the desired relations between E_1 , H_1 and E_2 , H_2 . The results are

$$\begin{aligned} E_1 &= E_2 \cosh pd + (H_2/A') \sinh pd \\ H_1 &= E_2 A' \sinh pd + H_2 \cosh pd \end{aligned} \quad (14)$$

At a boundary between two adjoining layers, say k and j , the tangential components of both E and H are equal, according to equation (3.1). Consequently if equation (14) is applied to the k th layer, $(E_2)_k$ and $(H_2)_k$ may be replaced by $(E_1)_j$, and $(H_1)_j$, respectively.

$$\begin{aligned} (E_1)_k &= (E_1)_j \cosh (pd)_k + [(H_1)_j/A'_k] \sinh (pd)_k \\ (H_1)_k &= (E_1)_j A'_k \sinh (pd)_k + (H_1)_j \cosh (pd)_k \end{aligned} \quad (15)$$

The ratio $(H_1/E_1)_k = Y_k$ is termed the input admittance of the k th layer. It has the dimensions of mho (ampere/volt), or reciprocal ohm. It is the value of H in amperes per meter associated with unit value of E , expressed in volts per meter, at the first boundary of the k th layer. By means of equation (15), Y_k may be written in terms of the input admittance, Y_j , of the j th layer.

$$Y_k = A'_k [Y_j + A'_k \tanh(pd)_k] / [A'_k + Y_j \tanh(pd)_k] \quad (16)$$

Equation (16) is a formula for deriving Y_k from Y_j and the properties A' and pd of the k th layer. Furthermore, since the subscripts k, j refer to any two adjoining layers, the input admittance of the $(m-1)$ th layer may be derived from the input admittance of the first layer and the properties of the $(m-2)$ finite layers by repeated use of equation (16). In detail, this process of iteration goes as follows:

The input admittance of the first layer is simply $Y_1 = A'_1$. This result is obtained by making use of the fact that there is no reflected wave in the first layer, so that $(H_1/E_1)_1 = Y_1 = A'_1$, from equation (13.1). The input admittance of the second layer is given by equation (16) if j is replaced by 1, and k is replaced by 2. Similarly, the input admittance of the third layer is given by equation (16) if j is replaced by 2, and k is replaced by 3. This process is repeated until j takes on the value $(m-2)$ and k takes on the value $(m-1)$, whereupon an explicit expression for Y_{m-1} is obtained.

It may be mentioned, in passing, that equation (16) represents a conformal transformation of the complex variable Y_j to the complex variable Y_k , in terms of the complex variables A'_k and $(pd)_k$. A graphical chart constructed in accordance with this conformal transformation may be used to effect a considerable reduction in the labor of computation which would be required in any given numerical case.⁷

Reflection and Transmission Coefficients

The reflection and transmission coefficients may be defined in terms of the tangential components of \mathbf{E} and \mathbf{H} , since only these components are involved in energy flow across the layer boundaries.

The amplitude reflection coefficient is defined as the ratio of reflected to incident electric or magnetic amplitudes in the m th layer, at the interface between m th and $(m-1)$ th layers. A formula for this coefficient, in terms of the input admittance Y_{m-1} , may be derived in the following way: The intersection of the x -axis with the boundary between m th and $(m-1)$ th layers is the origin from which x is measured in the m th and $(m-1)$ th layers. Therefore, from equation (13.1), the values of \mathbf{E} and \mathbf{H} in the m th layer, at the boundary, are $(E_t + E_r)_m$ and $A'_m(E_t - E_r)_m$, respectively. In accordance with equation (3.1) these amplitudes are equal to the corresponding amplitudes $(E_1)_{m-1}$ and $(E_1)_{m-1} Y_{m-1}$ in the $(m-1)$ th layer on the other side of the same boundary.

$$\begin{aligned} (E_t + E_r)_m &= (E_1)_{m-1} \\ A'_m (E_t - E_r)_m &= (E_1)_{m-1} Y_{m-1} \end{aligned} \quad (17)$$

By eliminating $(E_1)_{m-1}$ between the pair (17), a formula for the amplitude reflection coefficient is obtained.

$$r = (E_r/E_t)_m = (H_r/H_t)_m = (A'_m - Y_{m-1})/(A'_m + Y_{m-1}) \quad (18)$$

The electric amplitude transmission coefficient is defined as the ratio of electric amplitude in the $(m-1)$ th layer, at the boundary between m th and $(m-1)$ th layers, to the incident electric amplitude in the m th layer, at the same boundary. The magnetic amplitude transmission coefficient is defined as the ratio of corresponding magnetic amplitudes. Formulas for these coefficients are obtained from equations (17) and (18).

$$\begin{aligned} t &= (E_1)_{m-1}/(E_t)_m = 1 + r = 2 A'_m/(A'_m + Y_{m-1}) \\ t' &= (H_1)_{m-1}/(H_t)_m = 1 - r = 2 Y_{m-1}/(A'_m + Y_{m-1}) = t(Y_{m-1}/A'_m) \end{aligned} \quad (19)$$

⁷ There is an extensive literature on this subject. Some useful references are: Kennelly, A. E., "Chart Atlas of Complex Hyperbolic and Circular Functions," Cambridge, Harvard University Press, 1924. Rybner, J., "Nomograms of Complex Hyperbolic Functions," Copenhagen, J. Gjellerups Forlag, 1947. Cafferata, H., "The calculation of input, or sending-end impedance of feeders and cables terminated by complex loads," Marconi Rev., 6, (No. 64), 12-19, Jan.-Feb. 1937; 6, (No. 67), 21-39, Sept.-Dec., 1937. The last reference describes a chart by means of which an input impedance may be obtained from a terminal impedance and from the characteristic impedance and propagation constant of the transmission system. Impedance is the reciprocal of admittance, but since the chart is expressed in terms of impedance ratios, it may be used directly and without reciprocating admittances to obtain a numerical solution of equation (16). A limited quantity of charts of this type is available at Naval Research Laboratory. They may be obtained by requesting Charts H. O. Misc. 9999, 9999a, 9999b, 9999b-1 from the Technical Information Division, Naval Research Laboratory, Washington 25, D. C.

The over-all electric amplitude transmission coefficient is defined as the ratio of electric amplitude in the first layer, at the boundary between first and second layers, to the incident electric amplitude in the m th layer, at the boundary between m th and $(m-1)$ th layers. The over-all magnetic amplitude transmission coefficient is defined as the ratio of corresponding magnetic amplitudes. Formulas for these coefficients may be derived as follows: By means of equations (14) and (3.1) the ratios of electric and magnetic amplitudes at the two boundaries of the k th layer are expressed in terms of the properties of that layer and the input admittance, $Y_j = (H_1/E_1)_j$, of the adjoining layer.

$$(E_1/E_2)_k = \cosh(pd)_k + (Y_j/A'_k) \sinh(pd)_k \quad (20)$$

$$(H_1/H_2)_k = (A'_k/Y_j) \sinh(pd)_k + \cosh(pd)_k$$

The over-all transmission coefficients may be written as

$$\begin{aligned} t_{m,1} &= \frac{(E_1)_1}{(E_t)_m} = \frac{(E_1)_{m-1}}{(E_t)_m} \cdot \frac{(E_1)_{m-2}}{(E_1)_{m-1}} \cdot \frac{(E_1)_{m-3}}{(E_1)_{m-2}} \cdots \frac{(E_1)_1}{(E_1)_2} \\ &= \frac{(E_1)_{m-1}}{(E_t)_m} \left(\frac{E_2}{E_1} \right)_{m-1} \left(\frac{E_2}{E_1} \right)_{m-2} \cdots \left(\frac{E_2}{E_1} \right)_2 \end{aligned} \quad (21)$$

$$\begin{aligned} t'_{m,1} &= \frac{(H_1)_1}{(H_t)_m} = \frac{(H_1)_{m-1}}{(H_t)_m} \cdot \frac{(H_1)_{m-2}}{(H_1)_{m-1}} \cdot \frac{(H_1)_{m-3}}{(H_1)_{m-2}} \cdots \frac{(H_1)_1}{(H_1)_2} \\ &= \frac{(H_1)_{m-1}}{(H_t)_m} \left(\frac{H_2}{H_1} \right)_{m-1} \left(\frac{H_2}{H_1} \right)_{m-2} \cdots \left(\frac{H_2}{H_1} \right)_2 \end{aligned} \quad (22)$$

The third form of equations (21) and (22) is obtained from the second by use of equation (3.1). By use of equations (20) and (19) in the final forms of equations (21) and (22), formulas for the over-all transmission coefficients are obtained as the continued products:

$$t_{m,1} = (E_1)_1/(E_t)_m = t \cdot \prod_{k=(m-1)}^{k=2} [\cosh(pd)_k + (Y_j/A'_k) \sinh(pd)_k]^{-1} \quad (23)$$

$$\begin{aligned} t'_{m,1} = (H_1)_1/(H_t)_m &= (A'_1/A'_m) t_{m,1} = t' \cdot \prod_{k=(m-1)}^{k=2} [(A'_k/Y_j) \sinh(pd)_k \\ &\quad + \cosh(pd)_k]^{-1} \end{aligned} \quad (24)$$

Equivalent alternative formulas for these coefficients are

$$t_{m,1} = t \cdot \prod_{k=(m-1)}^{k=2} (1 + r_{kj}) \cdot [\exp(pd)_k + r_{kj} \exp(-pd)_k]^{-1} \quad (23.1)$$

$$t'_{m,1} = t' \cdot \prod_{k=(m-1)}^{k=2} (1 - r_{kj}) \cdot [\exp(pd)_k - r_{kj} \exp(-pd)_k]^{-1} \quad (24.1)$$

where

$$\Gamma_{kj} = (A'_k - Y_j)/(A'_k + Y_j) \quad (18.1)$$

The power, or rate at which energy is transmitted through unit area of the interface between m th and $(m-1)$ th layers is

$$\begin{aligned} P_T &= (1/2)\text{Re}(E_1 H_1^*)_{m-1} \\ &= (1/2)\text{Re}(E_t H_t^*)_m - (1/2)\text{Re}(E_r H_r^*)_m + (1/2)\text{Re}(E_r H_t^* - E_t H_r^*)_m \end{aligned} \quad (25)$$

where * indicates that the conjugate of the complex quantity with which it is associated is to be used. The second form of equation (25) is obtained by using equation (17). According to usual definitions, the first term on the right-hand side of the second form of equation (25) is identified as the incident power per unit area, P_t , and the second term is identified as the reflected power per unit area, P_r . The third term consists of cross products of incident and reflected amplitudes, and cannot therefore be uniquely identified with P_t or P_r . It is clear that unless the latter term is identically zero, conservation of energy, expressed in the form $P_t = P_T + P_r$, where these symbols are defined in the usual way, is not achieved. This difficulty conceivably could be circumvented by generalizing the definitions of P_t and P_r so as to include the components of the third term. Unfortunately, since these components are cross products, there is no unique way to carry out such a generalization. Consequently, except for special situations for which the third term vanishes, the concept and terminology of incident and reflected power has, strictly speaking, no real significance. In contrast, the concept and terminology of amplitude reflection may always be used.⁸ The important circumstances in which the third term of equation (25) vanishes are: the layer in which the source is located is lossless, or more generally, the layer in which the source is located is an absorbing one, but of a type which is designated as distortionless, for which the phase of A'_m is zero. Fortunately, in many important technological situations the third term is small in comparison with the other terms, so that the concept of incident and reflected power may be used with only small error. In the general case, however, the concept must be abandoned; this appears to have been overlooked in the literature.⁹

When circumstances are such that incident and reflected powers can be defined uniquely, power reflection and power transmission coefficients (or in optical terms, reflectivity and transmissivity) may be defined. These are

$$\begin{aligned} R &= P_r/P_t = |r|^2 \\ T &= P_T/P_t = 1-R = |t|^2 \text{Re}(Y_{m-1})/A'_m \end{aligned} \quad (26)$$

Similarly an over-all power transmission coefficient, or over-all transmissivity, may then be defined.

$$T_{m,1} = \text{Re}(E_1 H_1^*)_1 / \text{Re}(E_t H_t^*)_m = |t_{m,1}|^2 \text{Re}(A'_1)/A'_m \quad (27)$$

⁸ Salzberg, B., "A note on the significance of power reflection," *Amer. J. of Phys.*, 16, 444-446, 1948.

⁹ See, for example, pp. 511-513 of J. A. Stratton, "Electromagnetic Theory," New York, McGraw-Hill, 1941; and Section 3 of L. Pincherle, "Refraction of plane non-uniform electromagnetic waves between absorbing media," *Phys. Rev.*, 72, 232-235, 1947.

The total power absorbed in the layers of finite thickness, expressed as a ratio with respect to the incident power, P_t , is $(T - T_{m,1})$.

In Part II the use of the foregoing methods and results will be illustrated by applying them to the solution of several problems of interest in radio and optics.

$$P_t = P_{inc} + P_{refl} + P_{abs} \quad (22)$$

where P_{inc} indicates that the average of the complex quantity with which it is associated is to be used. The second term of equation (22) is obtained by using equation (17). According to usual definitions, the first term on the right-hand side of the second term of equation (22) is identified as the incident power per unit area, P_i , and the second term is identified as the reflected power per unit area, P_r . The third term consists of cross products of incident and reflected amplitudes, and cannot therefore be unambiguously identified with P_i or P_r . It is clear that unless the latter term is identically zero, conservation of energy, as expressed in the form $P_i - P_r = P_a$, where these symbols are defined in the usual way, is not satisfied. This difficulty conceivably could be circumvented by generalizing the definition of P_i and P_r so as to include the components of the third term. Unfortunately, since these components are cross products, there is no unique way to carry out such a generalization. Consequently, except for special situations for which the third term vanishes, the concept and terminology of incident and reflected power has, strictly speaking, no real significance. In contrast, the concept and terminology of amplitude reflection may always be used. The important circumstances in which the third term of equation (22) vanishes are: (a) the layer in which the source is located is lossless, or more generally, the layer in which the source is located is an absorbing one, but of a type which is designated as A_{10} lossless, for which the value of A_{10} is zero. Fortunately, in many important instances, first attention to the third term is small in comparison with the other terms, so that the concept of incident and reflected power may be used with only small error. In the general case, however, the concept must be abandoned; this appears to have been overlooked in the literature.

When circumstances are such that incident and reflected power can be defined unambiguously, power reflection and power transmission coefficients (or in optical terms, reflectivity and transmissivity) may be defined. These are

$$R = P_r/P_i = |r|^2 \quad (23)$$

$$T = P_t/P_i = |t|^2 \quad (24)$$

Similarly, an all-power transmission coefficient, or over-all transmissivity, may then be defined.

$$T_{all} = P_t/P_i = |t|^2 \quad (25)$$

¹ Ibid., p. 13, where the definition of these coefficients is given. A. W. Maue, *ibid.*
² *ibid.*, p. 13, where the definition of these coefficients is given. A. W. Maue, *ibid.*
³ *ibid.*, p. 13, where the definition of these coefficients is given. A. W. Maue, *ibid.*

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List of Symbols

- A** - characteristic admittance of a layer
- A₀** - characteristic admittance of free space
- A'** - characteristic admittance for tangential wave components
- A''** - characteristic admittance for normal wave components
- c₀** - velocity of propagation in free space
- d** - layer thickness
- E** - complex vector electric amplitude
- E** - tangential component of **E**
- E_n** - normal component of **E**
- E_r** - reflected component of **E**
- E_t** - incident or transmitted component of **E**
- H** - complex vector magnetic amplitude
- H** - tangential component of **H**
- H_n** - normal component of **H**
- H_r** - reflected component of **H**
- H_t** - incident or transmitted component of **H**
- i** = $(-1)^{1/2}$
- j, k, m** - subscripts which identify particular layers
- n** - index of refraction
- P** - propagation constant of a layer
- p** - propagation constant normal to layer boundaries
- q** - propagation constant parallel to layer boundaries
- R** - reflectivity

r	- amplitude reflection coefficient
s	- distance along incident wave normal
T	- transmissivity
$T_{m,1}$	- over-all transmissivity
t	- time
t	- electric amplitude transmission coefficient
t'	- magnetic amplitude transmission coefficient
$t_{m,1}$	- over-all electric amplitude transmission coefficient
$t'_{m,1}$	- over-all magnetic amplitude transmission coefficient
v	- phase velocity
x, y, z	- axes of coordinates
ϵ	- relative dielectric constant
ϵ_0	- dielectric constant of free space
ϵ'	- imaginary component of complex relative dielectric constant
μ	- relative magnetic permeability
μ_0	- permeability of free space
μ'	- imaginary component of complex relative permeability
ρ	- magnetic resistivity
σ	- electric conductivity
θ	- angle of incidence
θ'	- angle between equiamplitude planes and z-axis
θ''	- angle between equiphase planes and z-axis
ω	- angular frequency
∇^2	- Laplacian