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RPPR Final Report

as of 20-Mar-2023

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STEM Degrees:

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Major Goals: This is final report for Army Award: W911NF-20-C- 0032. Our focus is to develop RSFQ circuit technology into a low-energy, computer-oriented technology that can eventually be commercialized. RSFQ is a technology based on niobium-based, Josephson junctions (JJs). RSFQ circuits can operate at very high speeds (even above 100 GHz). To operate RSFQ circuits requires significant cooling because RSFQ circuits operate at 4K. Confusion around whether or when we should deliver our final report resulted from several issues. As of late May, 2022, we had not received notification that our third year of contract support would be forthcoming; this, of course, put a tremendous strain on our ability to retain our workers as we were asking them to commit to work on our project without being able to tell them that we had funding for our third year.

In early June, 2022, we asked whether funding had materialized; it had not. So, we informed our Army COR of our decision to stop work until the situation could be normalized; such a normalization did not occur, and we were informed in March, 2023, that our final report was due.

This contract examined the application of self- timing to superconducting circuits. Self-timing is important to superconducting circuits because light travels at only about 100 microns per picosecond. 100 microns is about the width of four human hairs side by side and a picosecond is about the switching time for superconducting logic gates. Early in our work we decided to focus on the RSFQ family of circuits for their simplicity in spite of their slightly larger energy consumption. We believe that additional energy saving available from ERSFQ circuits is practical future item to pursue.

Accomplishments: Over the last two years, several accomplishments stand out, and they will provide lasting value: standardized units, a top view schematic notation, use of area attachment, automatic estimation of wire inductance, and a circuit-level simulator for RSFQ circuits.

We have written a series of memos describing the RSFQ circuits we have explored. These memos might form the basis for a textbook on RSFQ circuits. What remains to do is to gain enough experience with actual layouts to provide guidance on not only how to design and simulate circuits, but also a practical guide about how to layout working circuits. How must RSFQ circuits be connected to ground? How must return paths for bias current reduce the inductance of power supplies?

We need a systematic investigation of working and failing designs to understand the relative importance of moats, ground return paths, and stray coupling. The layout of superconducting circuits remains an art. We seek to turn it

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into a science.

We wrote the VWSIM Builder Manual (included with this report), which describes the mathematics of both voltage-based and phase-based circuit models for all the components typically used to implement JJ-based circuits. This document guided the development of the VWSIM simulator, and it serves as a reference for the mathematics used to model JJ-based circuits. With this manual, one can build a simulator for JJ-based circuits.

Useful design units deserve simple names. We know precisely what a Liter is, but the normalized unit of current for superconductive circuits, 125 microAmps, has been in use for over 30 years without a name. We assigned the name Lick to this normalized unit of current, 125 microAmps. and chose Lk as its abbreviation. We assigned the name oHenry to the normalized unit of Inductance, $2.632e-12$ Henries, and assigned the abbreviation oH to it. We devised the top-view notation reported in FHI 2021-is11. The top-view notation provides an icon to represent grounded JJs with suitable bypass resistors. The top-view abstraction offers a more compact schematic representation of complex circuits such as the D2Latch and the 2x2 rendezvous. Because the Electric CAD tool permits definition of icons, we were able immediately to simulate circuits represented in the new top-view notation. We explored and put to use the layout concept of an area attachment; area attachment provides useful assistance for layout. By using Electric, one may attach a wire to a sub circuit either at a particular point, along a defined line, or anywhere inside a designated area. A wire attached to any part of a metal rectangle associated with a JJ can remain attached when moved to any other position. The ability to make such adjustment is a critical layout aid to fit parts together into a compact layout.

We devised and put into practice an estimation method for wire inductance. The method is fast enough to see inductance values change as one modifies a layout. This technique has proven of great value for adjusting the layout of the TAP Controller.

We have developed a simulator for RSFQ circuits, as well as other electric circuits, that will be distributed as a part of the ACL2 books (library) repository. This repository is updated with each release of ACL2 and the code in the repository is maintained to work with future releases of the ACL2 system. Using the ACL2 logic to define our VWSIM simulator provides a general-purpose environment to generate test inputs and to filter and check outputs. In addition, ACL2 was used to prove many mathematical observations about the VWSIM source code – these observations can be used subsequently to prove properties about RSFQ circuit models.

We have collected a series of memos describing many of the circuits we have explored. These memos could form the basis for a textbook on RSFQ circuits. What remains to do is to gain enough experience with actual circuits and their layouts to provide guidance on not only how to design and simulate circuits, but it could serve as a practical guide about how to layout working circuits. How must RSFQ circuits be connected to ground? How must return paths for bias current reduce the inductance of power supplies? We need a systematic investigation of working and failing designs to understand the relative importance of moats, ground return paths, and stray coupling. The layout of superconducting circuits remains an art. Using our tools and what we have learned, we seek to turn RSFQ design into a science.

Training Opportunities: Nothing to Report

Results Dissemination: Nothing to Report

Honors and Awards: Nothing to Report

Protocol Activity Status:

Technology Transfer: Nothing to Report

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Person Months Worked: 2.00

Project Contribution:

National Academy Member: N

Funding Support:

Participant Type: Co PD/PI

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Participant Type: Staff Scientist (doctoral level)

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Project Contribution:

National Academy Member: N

Funding Support:

Participant Type: Technician

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National Academy Member: N

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National Academy Member: N

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National Academy Member: N

Funding Support:

Participant Type: Consultant
Participant: Navenn Katam
Person Months Worked: 1.00
Project Contribution:
National Academy Member: N

Funding Support:

Partners

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I certify that the information in the report is complete and accurate:

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Signature Date: 3/17/23 4:50PM

DEVELOPMENT AND ANALYSIS OF LINK-AND-JOINT-BASED, RSFQ LOGIC SYSTEMS

ForrestHunt, Inc.

Final Report

Submitted to: Army Research Office

Award number: W911NF-20-C-0032

Project Title: Development and Analysis of Link-and-Joint-Based,
RSFQ Logic Systems

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Organization: ForrestHunt, Inc.

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Report Frequency: Final -- a Single, Comprehensive Report

Signature:



DEVELOPMENT AND ANALYSIS OF LINK-AND-JOINT-BASED, RSFQ LOGIC SYSTEMS

Warren A. Hunt, Jr. and Ivan E. Sutherland
ForrestHunt, Inc.

Abstract. This is our final report for Army Award number W911NF-20-C-0032. Our focus is to develop RSFQ-based circuit technology into a computer-focused design and manufacturing method that can eventually be commercialized. This report summarizes our investigation toward our making RSFQ-based systems practical.

1 Accomplishments

This is final report for Army Award: W911NF-20-C-0032. Our focus is to develop RSFQ circuit technology into a low-energy, computer-oriented technology that can eventually be commercialized. RSFQ is a technology based on niobium-based, Josephson junctions (JJs). RSFQ circuits can operate at very high speeds (even above 100 GHz). To operate RSFQ circuits requires significant cooling because RSFQ circuits operate at 4K.

Confusion around whether or when we should deliver our final report resulted from several issues. As of late May, 2022, we had not received notification that our third year of contract support would be forthcoming; this, of course, put a tremendous strain on our ability to retain our workers as we were asking them to commit to work on our project without being able to tell them that we had funding for our third year.

We reached the end of our second year of support without knowing whether the expected third year of funding would actually materialize. Also, during our monthly status report given in May, 2022, we received significant criticism concerning about

whether our approach would eventually yield satisfactory results. Given our lack of confidence in our sponsor funding our third year and the lack (expressed at our last meeting) of sponsor confidence in our approach, we choose to pause our work at the end of May, 2022. In early June, 2022, we asked whether funding had materialized; it had not. So, we informed our Army COR of our decision to stop work until the situation could be normalized; such a normalization did not occur, and we were informed in March, 2023, that our final report was due.

This contract examined the application of self-timing to superconducting circuits. Self-timing is important to superconducting circuits because light travels at only about 100 microns per picosecond. 100 microns is about the width of four human hairs side by side and a picosecond is about the switching time for superconducting logic gates. Early in our work we decided to focus on the RSFQ family of circuits for their simplicity in spite of their slightly larger energy consumption. We believe that additional energy saving available from ERSFQ circuits is practical future item to pursue.

2 Our Approach and Results

This final report is a largely a summary of our monthly reports (June, 2020 to May 2022). We thank the Laboratory for Physical Sciences and the US Army for their support of our work. We have learned a great deal, and we have a series of memos written during our effort that, when combined and edited, would make the basis of a good textbook about this technology area.

Our objective for this contract was to learn enough about the RSFQ family to design self-timed circuits. We recorded our understanding so as to share that understanding with others. We regularly reported our work in memos, typically of 4-to-6 pages each. Al-

though we have worked in this area of technology for a total of only four years (two of which were supported by this contract), we have accumulated over 200 such short memos. We continue to refine our memos with the idea that selected versions of these memos may become chapters in a textbook about RSFQ circuits. Even though our effort has completed, that plan proceeds.

Initially, we used a paper by Bunyk, Likharev, and Zinoviev as a guide to learning about RSFQ. We decided to focus on RSFQ circuits knowing that they could, later on, easily be adapted to ERSFQ format to save power. We also, wisely as it turned out, decided to break logic functions into blocks where each of which starts with an inductor and ends with a grounded JJ driver. By breaking logic chains into blocks that start with an inductor and end with a grounded junction, all grounded JJ circuit modules produce identical output pulses regardless of the size of their junctions. Their identical output pulses make modules completely interchangeable. Starting logic blocks with an inductor provides for fan-out; one output can drive more than one input provided the output is strong enough to drive the collection of inputs.

During the period of this contract, we achieved several important results: development of a simulator for Josephson Junction (JJ) based circuit models, designed and taped out two chips (which were built by MIT-LL), the use of the Link-and-Joint circuit paradigm for JJ-based circuits, RSFQ transformer-based circuits suitable for transferring circuit values to JJ-based circuits operating with separate voltage supplies, and other innovations reported previously.

Early in our effort, we attempted to use public-domain simulators capable of simulating circuits that include Josephson Junctions. However, during the first few months of use, we were several times baffled by the results we received. In each case, we spent some weeks failing to understanding the results

we received, and later realizing that the primary (and widely used) simulator we had been using was incorrect in various subtle ways. This lack of a reliable simulator was of great concern because we couldn't rely on the results we were receiving. After chasing down several bugs, which we passed along to the simulator authors, we set out to write our own simulator in a formal logic; writing in logic forced us to be very precise in coding our mathematics-based implementation.

At first, we developed a very simple, but easy to validate simulator. As a guide for our work, we wrote a 70-page report (included with this submission) that describes the mathematics required to model Josephson Junction based circuit models. Our initial simulator provided very accurate results, but with low simulation performance; this initial simulator became our reference simulator, and using this simulator, we discovered many additional bugs in other publicly-available simulators. The development of our simulator forced us to understand the subtle mathematics required to model Josephson-Junction-based circuit models.

After appreciating the value of a simulator that we can extend and monitor its operation, we re-wrote our initial simulator with data structures that provided performance comparable to existing simulators. During this effort, we extended the monitoring and stimulus tools so we could perform simulations under program control and to state requirements that could be checked automatically by the simulation process. For instance, we could perform simulations under multiple initial conditions and even with time-varying component values and time-varying input signals. In addition, our simulator can monitor and confirm computational requirements without the need to graph and check results by *eye*. This capability allowed us to perform thousands (or even millions) of simulations and check for correct circuit operation under program control; such an ap-

proach permits much faster circuit design iteration. Our VWSIM simulator will be released with the next public release (expected Q3 2023) of the ACL2 system.

Although it took us nearly a year to understand the concept of phase, we captured that understanding in the memo FHI# 2021-is18v3: M.C. Escher and a Turn of Phase., by Marly Roncken and Ivan Sutherland, 27 March 2021. The notion that phase is a cyclic rather than a monotonic potential is both difficult to grasp and very important. In many cases the cyclic nature of phase eliminates the time that might otherwise be wasted for reset.

We have refined our schematic capture and layout tool, Electric in several ways. First, a “technology” file called “Josephson” connects Electric to Lincoln Laboratory’s SFQ5ee chip manufacturing process. Second, we connected Electric to VWIM and to JoSIM, an open-source simulation tool for superconducting circuits. Third, Electric can now estimate the inductance of conductors in layout and report such estimates in real time as a user changes a layout. Fourth, under user control, Electric can modify the width of an entire conductor to reach a desired target inductance. We have used Electric extensively for both schematic capture and layout. Electric is an open-source tool available on the web from Static Free Software.

We taped out a simple test chip on 3 June 2021 that the MIT Lincoln Laboratory built and tested. Our first chip worked just well enough to suggest improvements to include in our second test chip. We taped out our second test chip, FHI#2022-10, in May of 2022. Our second chip includes a TAP controller comprising about 400 Josephson junctions and occupying an on-chip area of only 580×280 micrometers, less than 1% of the area available on a typical chip. The IEEE standard chip test method, widely used in industry, requires such an on-chip TAPcontroller. Our TAPcontroller will bring IEEE standard

test methods to superconducting circuits. Our TAP controller chip has been built and awaits test at the MIT Lincoln Laboratory. The Lincoln Laboratory plans to make our TAP controller available on future chips they build. Our TAPcontroller, is the subject of a paper published in the Journal of Super Conductivity.

We devised an entirely new notation for RSFQ circuits that we call the *Top View* notation. Our memo on this subject, FHI# 2021-is11v6, is included with this report. Because our top view schematic notation is more compact than older notations, the top view notation allows a user to visualize the operation of more complex circuits. The top view notation will be available in any text that we may publish about superconducting electronics.

Marly Roncken devised an improved and completely general description of her communication and storage channels called *Links*. Her memo on the subject is FHI# 2020-is11v5, Symmetric Links by Marly Roncken and Ivan Sutherland, 26 March 2020. Those ideas are scheduled to appear in publications relevant to Asynchronous design. Roncken adapted her key circuit for testing self-timed circuits, called MrGO, to RSFQ technology. The RSFQ version of MrGO combines three parts: a non-destructive read out (NDRO) latch, an ordinary RSFQ D2latch, and an ordinary RSFQ coincidence junction. We thank Drs. Roncken for inspiring this design and pushing the circuit to completion. The circuit is reported via the Mayo Clinic contract that supports Roncken.

In July of 2021, we began to study the transformer work of Vasili Semenov, see reference. Vasili kindly sent us layouts of his design and described some subtle points not described in his published paper. We plan to extend this work and have some ideas that may result in circuits less sensitive to the manufacturing flaws that have limited Vasili’s experiments. We plan a follow-on effort so we may continue with

this effort.

We explored two different versions of the 2x2 rendezvous circuit. The 2x2 rendezvous circuit will be at the heart of the arithmetic calculations for a future pipelined array multiplier. The two versions each limit the input current to their central square of eight JJs, but differ in the form of current limitation used. One method, easier to understand, uses an inductor of carefully chosen size to limit current. The other uses a carefully chosen bias resistor to limit current. Both forms work, but we don't yet understand why one is faster than the other.

Test Site 2:

Our second test site, shown in In Figure 1, includes the following experiments. Pad frame (PF) connections exclude ground unless ground connections are being provided for measurement. Internal post (IP) connection counts similarly exclude ground connections.

- North-edge heater patterned after Vasili Semenov's design (1 PF) [6].
- A DC-to-RSFQ to RSFQ-to-DC converter (7 PF)
- NE-corner, 6-bit shift register (5 PF)
- SW-corner, 6-bit shift register (13 IP connections)
- SE-corner two-input, merge circuit (8 IP connections)
- Two collections of resistor calibration tests (56 IP connections)
- Three different DC-to-RSFQ to RSFQ-to-DC converters (3 * 8 IP connections)

- Test-Access Port Controller (14 PF, 4 connections are for power)

In Figure 2 we show the layout of our DC-to-RSFQ converter; this converter is followed by our RSFQ-to-DC converter. This diagram also includes connections to power and ground. Labels within the diagram identify the constituent pieces.

The layout of our TAP Controller appears in Figure 3. Again, the embedded labels help identify the primary parts of our design.

3 Project Accomplishments

Over the last two years, several accomplishments stand out, and they will provide lasting value: standardized units, a *top view* schematic notation, use of *area attachment*, automatic estimation of wire inductance, and a circuit-level simulator for RSFQ circuits.

We have written a series of memos describing the RSFQ circuits we have explored. These memos might form the basis for a text-book on RSFQ circuits. What remains to do is to gain enough experience with actual layouts to provide guidance on not only how to design and simulate circuits, but also a practical guide about how to layout working circuits. How must RSFQ circuits be connected to ground? How must return paths for bias current reduce the inductance of power supplies?

We need a systematic investigation of working and failing designs to understand the relative importance of moats, ground return paths, and stray coupling. The layout of superconducting circuits remains an art. We seek to turn it into a science.

We wrote the *VWSIM Builder Manual* (included with this report), which describes the mathematics

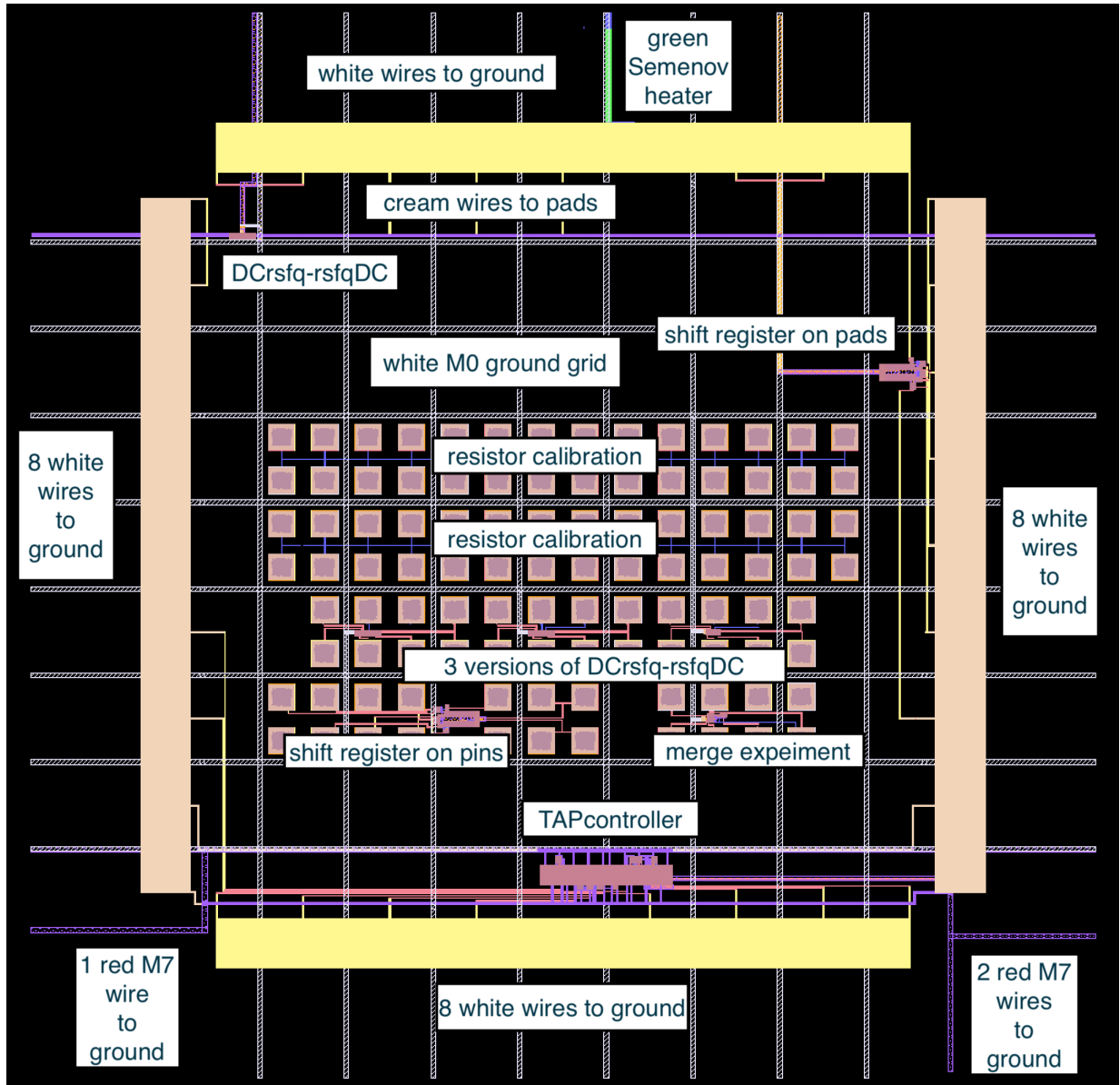


Figure 1: Pictorial diagram of our second test site. The figure omits the chip's 36 bonding pads and ground-ring. The eight wires extending out each of the four sides of the Figure connect to the bonding pads. Red and white wires connecting experiments to bonding pads are visible. The **M0** ground grid (white) connects to the ground ring between bonding pads.

A green heater resistor can be seen at top center. Wire bond *posts* appear on a grid between the white **M0** ground grid. The shift register experiment appears at the right, and the TAP Controller connects to bonding pads at the bottom of the chip (and the figure).

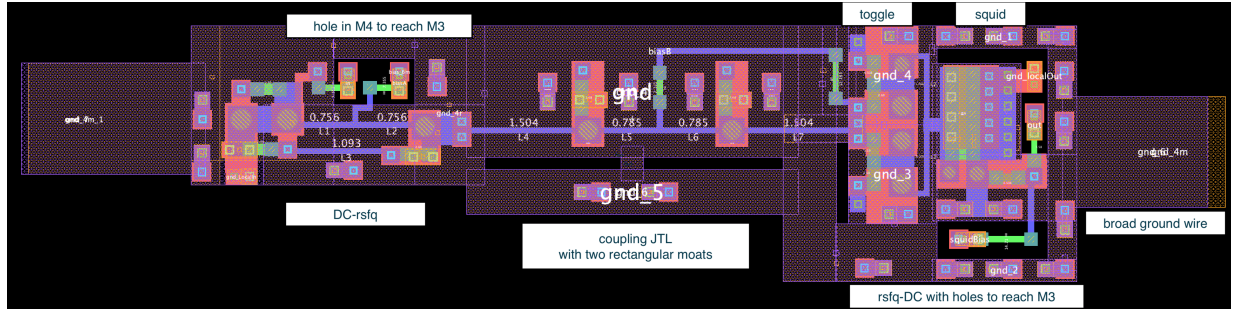


Figure 2: Layout of our DC-to-RSFQ-to-DC converter, including its IO complement. This experiment is attached to the **M0** ground grid post just to its left and the **M0** pad. It also connects to the ground ring via a broad **M7** wire extending out its right side above the **M3** wires.

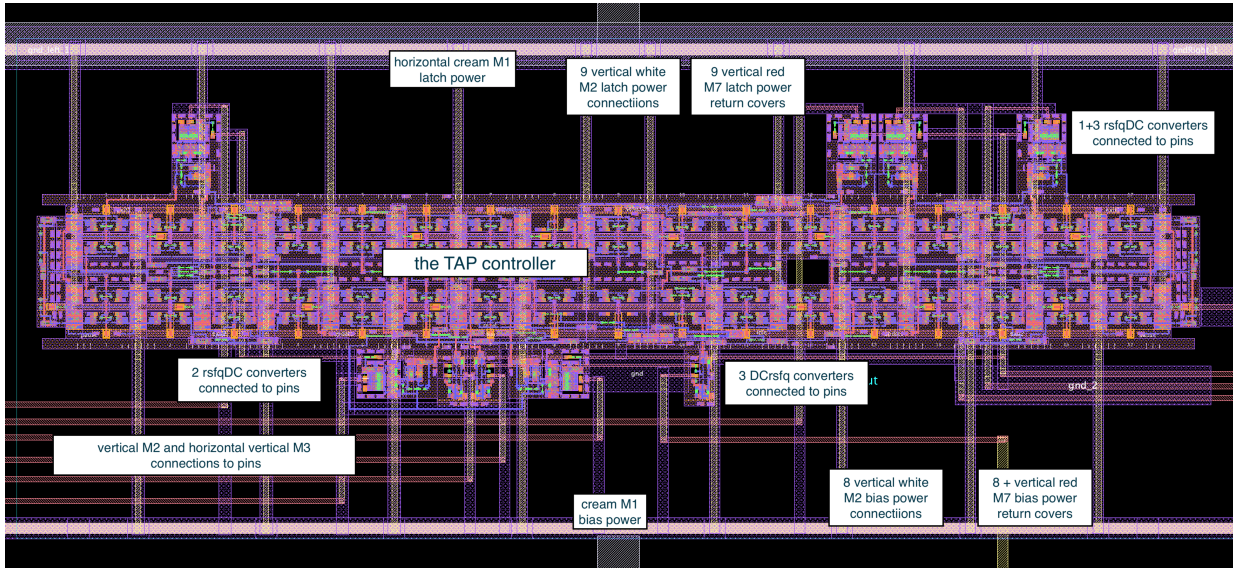


Figure 3: Layout of our TAP Controller, including its IO complement. The entire TAP Controller could fit in the space of a single bonding pad.

of both voltage-based and phase-based circuit models for all the components typically used to implement JJ-based circuits. This document guided the development of the VWSIM simulator, and it serves as a reference for the mathematics used to model JJ-based circuits. With this manual, one can build a simulator for JJ-based circuits.

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3.1 Interactions with MIT-LL Personnel

Even though our contract ended in May, 2022, we have continued to hold weekly meetings with Evan Golden, Matt Guyton, and Alex Wynn, and from time-to-time with Sergey Tolpygo and other MIT-LL personnel. Vasili Semenov has provided a great deal of useful feedback, and we still interact with him reg-

ularly.

3.2 What are the major goals of the project?

We seek to develop a methodology to specify, design, analyze, verify, build, test, validate, and measure increasingly complex RSFQ-based circuits – so as to develop the most energy efficient computational fabric possible.

SFQ signaling works at the milli-volt level. The *quiet* of the cold environment makes possible signaling with very small energy; an SFQ pulse carries less energy than is stored in the capacitance of CMOS wire. Unlike CMOS (operating at any temperature), the very low voltage of SFQ technology renders CV^2 energy nearly irrelevant.

For SFQ technology to be successful, we must deploy it where success is possible. The SFQ community has tried to design and build general-purpose microprocessors, but such devices are very poorly suited to effective use of SFQ technology. Instructions at the level of adding the contents of two registers are too primitive for an SFQ system. The instruction-set architecture of SFQ circuits must be specified at a much higher, *flow-through* level, such as a 128-point FFT primitive operation. Energy efficient SFQ systems will take advantage of very high utilization levels impossible with CMOS.

High utilization SFQ systems will be deeply pipelined and highly parallel. Memory will be distributed throughout such systems, rather than concentrated in addressable arrays. Again, consider the FFT: we can imagine a radix-8 or radix-16 FFT engine fed by a collection of routing circuits where the *state* values are distributed through the switching network. A separate memory, in the conventional sense, will be absent because storage will be distributed throughout SFQ circuits. Obviously, this

will require a complete change in architecture from contemporary microprocessors.

It appears that SFQ technology can also be deployed to stream-based search. A regular expression being applied to streaming data could be distributed through an SFQ regular-expression (circuit) recognizer. When a new search expression is offered, the search pattern will be distributed throughout the SFQ search engine hardware to enable search of 400-Gbit data streams in real time. For this kind of search, a serial, rather than an addressable, memory would be suitable – and such a memory could be constructed from long super-conducting wires, but thousands of times faster than the mercury-delay lines of old.

The energy required in SFQ circuits is very small; in fact, we pass only a single fluxon to indicate an event. Even though experimental present-day SFQ circuits are built with 25-year-old (e.g., 200 nm) technology, we believe that SFQ circuits with energy efficiency superior to CMOS can be developed, manufactured, tested, and deployed.

3.3 What was accomplished under these goals?

We have developed a simulator and analysis system for JJ-based circuit models. We have written over 50 short reports that describe various observations and things we have learned over the two-year duration of our contract. We have designed two SFQ-oriented integrated circuits, and these two designs were fabricated at MIT-LL. The first chip was tested successfully, and our second test site awaits testing. We gave two presentations at the 2022 Applied Superconductivity Conference (ASC). We have published three papers about this work, two at ASC 2022 and one at the ACL2 Workshop.

3.4 What opportunities for training and professional development has the project provided?

We involved four university-level students. Two of the students finished in the summer of 2020. UT Austin student Vivek Ramanathan has worked with us since the beginning of our project, and he continues working on our project. Portland State University Quinn Morgan student joined us in August, 2021, and he has continued to work even though our contract has ended.

Unfortunately, the funding approach used on our contract makes it very difficult for us to hire anyone to assist us in our investigation.

3.5 How have the results been disseminated to communities of interest?

We presented our VWSIM simulator paper at the 2022 ACL2 Workshop; it is available on ArXiv. We will distribute our VWSIM simulator as a part of the ACL2 community *books*. We are improving the VWSIM documentation; this documentation is built into the source code, and when a user wishes, the ACL2 documentation system can generate Emacs Info, HTML, PDF versions of our system documentation automatically.

Two abstracts we submitted to the 2022 Applied Superconductivity Conference (ASC 2022) – and both abstracts were accepted for publication! We presented our efforts this October in Honolulu, Hawaii, and two papers associated with our effort will soon appear in the Journal of Super Conductivity.

We continue to exchange information with several key members of the RSFQ community: Coenrad Fourie, Vasili Semenov and Sergey Tolpygo. Our interactions with these very experienced scientists has

accelerated as we have learned enough to be interesting to them.

3.6 What do you plan to do during the next reporting period to accomplish the goals and objectives?

Nothing. We our effort has finished.

4 Optional Categories

4.1 Products

Nothing to report.

4.2 Publications

We have published three papers about this work, two at ASC 2022 and one at the ACL2 Workshop. We gave two presentations at the 2022 Applied Superconductivity Conference (ASC).

4.3 Websites

In the near future, the VWSIM simulator will distributed with the ACL2 system; see <http://acl2.org>.

4.4 Technologies or Techniques

We have developed several techniques for circuit descriptions and for simulation that are unique; these items were described above.

4.5 Inventions, Patent Applications, and/or Licenses

All software designs developed during the course of our effort either are, or will soon be, available publicly for free download.

4.6 Other Products

Nothing to report.

Role: Engineer
Worked: 625.0 hours
Contribution: Assisted with tools
Funding: This project

5 Participants & Other Collaborating Organizations

During the entirety of our effort, we worked with MIT-LL scientists. Our points of contact are: Alex Wynn (primary), Matt Guyton, Evan Golden, and Sergey Tolpygo. We had a contract with SeeQC; we worked with Oleg Mukhanov (President), Alex Kirichenko (Senior Scientist), and Naveen Katam (Engineer).

Name: J Strother Moore
Role: Scientist
Worked: 873.0 hours
Contribution: Assist with simulator analyzer development
Funding: This project

Name: Maria Noack
Role: Advisory Engineer
Worked: 206.15 hours
Contribution: Circuit analysis
Funding: This project

5.1 What Individuals Have Worked on the Project?

The people that are working on our project include: Warren A. Hunt, Jr., Matt Kaufmann, J Strother Moore, Steve Rubin, and Ivan Sutherland; these people are all senior personnel – their resumes all appeared in our proposal. Below are the cumulative hours by contractor for our entire effort.

Name: Vivek Ramanathan
Role: Technician
Worked: 1057.43 hours
Contribution: Circuit simulator
Funding: This project

Name: Steven Rubin
Role: Scientist
Worked: 256.0 hours
Contribution: Extend Electric tool for RSFQ-logic design
Funding: This project

Name: Warren A. Hunt, Jr.
Role: Scientist, Manager
Worked: 1622.1 hours
Contribution: Develop formal-logic circuit simulator and analyzer
Funding: This project

Name: Vasili Semenov
Role: Scientist
Worked: 48.0 hours
Contribution: RSFQ Logic Expert
Funding: This project

Name: Matt Kaufmann
Role: Scientist
Worked: 702.3 hours
Contribution: Assist with simulator analyzer development
Funding: This project

Name: Ivan Sutherland
Role: Scientist
Worked: 2752.4 hours
Contribution: Study, develop RSFQ RSFQ theory, logic
Funding: This project

Name: Quinn Morgan

With the reduction in support to \$10,000 per

month, SeeQC took a position that they would either provide a report or individual labor hours. We discussed SeeQC's position with Joseph Michael (Mike) Coyle, our COR; Dr Coyle agreed with our choice to select a report with a fixed amount of average-priced hours. For the first year, here are the aggregated hours provided by SeeQC personnel.

Name: SeeQC, Navenn Katam,
Alex Kirichenko,
Oleg Mukhanov
Role: Scientists, Engineers
Worked: 771.5 aggregated hours
Contribution: Circuits, engineering
Funding: This project

We engaged Vasili Semenov for 16 hours/month starting in March, 2022, for three months. Dr. Semenov provided consulting in the area of RSFQ circuit design.

5.2 What other organizations have been involved as partners?

Gary Delp, of the Mayo Clinic Special-Purpose Processor Design Group (SPPDG), continues with work related to our work. Delp's continued involvement with our effort provides additional communication between our effort and the Laboratory of Physical Sciences. We value his assistance and council highly. In addition, the Mayo Clinic supports Professor Marly Roncken (of Portland State University) with whom we are collaborating in the areas of logic, test, and analysis.

We are in communication with Coenrad Fourie. In late April, 2022, Coenrad and Joey Delpport visited us in Portland, Oregon, for a full day of discussion.

5.3 Have other collaborators or contacts been involved?

None other than the people already mentioned in previous sections.

6 Impact

It is too early to list the impact our work might have. If we are successful, our techniques will make it easier to specify, develop, analyze, verify, manufacture, test, and deploy RSFQ-based circuits.

As we continue learning about RSFQ circuits, we anticipate that we will wish to write a book about our experiences. We are still some time away from developing such a book, but as we learn from our experiments, we are producing information suitable for publication.

Another key impact would be lowering the energy required to perform computations. If successful, this work could provide 10x faster circuits that consume less than 1% of the energy of equivalent CMOS circuits.

6.1 What is the impact on the development of the principal discipline(s) of the project?

It's too early to say.

6.2 What is the impact on other disciplines?

It's too early to say.

6.3 What is the impact on the development of human resources?

It's too early to say.

6.4 What is the impact on physical, institutional, and information resources that form infrastructure?

It's too early to say.

6.5 What is the impact on technology transfer?

It's too early to say.

6.6 What is the impact on society beyond science and technology?

It's too early to say.

6.7 What dollar amount of the award's budget is being spent in foreign country(ies)?

None.

7 Changes/Problems

At the end of our second year of an anticipated three-year effort, we were not provided with additional funds; actually, we asked to end our effort. We told our Contracting Officer, Dr. Mike Coyle (Army Research Office) that we will not continue with our effort.

7.1 Changes in approach and reasons for change

Even though our support from LPS has ended, our technical efforts have continued. The primary reason we stopped our LPS-supported effort was the hostility demonstrated toward us as people. For the 40+ years that we have been working as a contractors, we have never experienced such outright scorn. The last six months of our relationship with LPS was the worst work experience that we have experienced in our careers.

7.2 Actual or anticipated problems or delays and actions or plans to resolve them

There are delays that are effecting our productivity: long wait for feedback from the test design we submitted last August and the very late arrival (either after the start of our work year or a day before our work year) of our funding. This problem persisted, and for our third year, the funding did not arrive until well after the third contract year had begun. It is impossible to keep a team together with such an uncertain funding model.

In summary, the Government has provided our first two years of funding either one business day after our work start date or one business day prior to the date we are expected to start work.

7.3 Changes that have a significant impact on expenditures

We operated with a Firm, Fixed-Price (FFP) contract, and we use the funding provided as our contract stipulated.

7.4 Significant changes in use or care of human subjects, vertebrate animals, and/or biohazards

Nothing to Report.

7.5 Change of primary performance site location from that originally proposed

Nothing to Report.

8 Special Reporting Requirements

Nothing to Report.

9 Budgetary Information

Nothing to Report; all budgetary information was submitted previously.

FORRESTHUNT, Inc.

Superconductivity Project

Subject: M.C. Escher and a Turn of Phase
Date: 27 March 2021
From: Marly Roncken and Ivan Sutherland
FHI#: 2021-is18v3

References:

M.C Escher: Klimmen en dalen(Ascending and Descending), Litho, 353x285mm. March 1960

PURPOSE

This memo addresses the meaning of phase in Josephson junctions. Phase is a little like M.C. Escher's famous staircase, there's an up direction and a down direction but continuing ascent or descent brings one back to the starting point. This memo attempts to aid understanding of superconducting circuits by offering a path towards intuition about phase.

WALKABOUT IN MANHATTAN

Manhattan is renowned for its North-South avenues and East-West streets. A pedestrian in Manhattan can keep track of three things: a street number, an avenue letter, and the direction she faces. Let us contemplate the fundamental difference in these three variables. Whereas street and avenue identifiers in Manhattan seem ordered, one can walk in only four directions. Every four left turns brings one back to the very same facing direction: North, South, East or West. In contrast, every four blocks walked brings one to an entirely different street or avenue, although a pedestrian lost in Manhattan may think Manhattan's streets repetitive and unending.

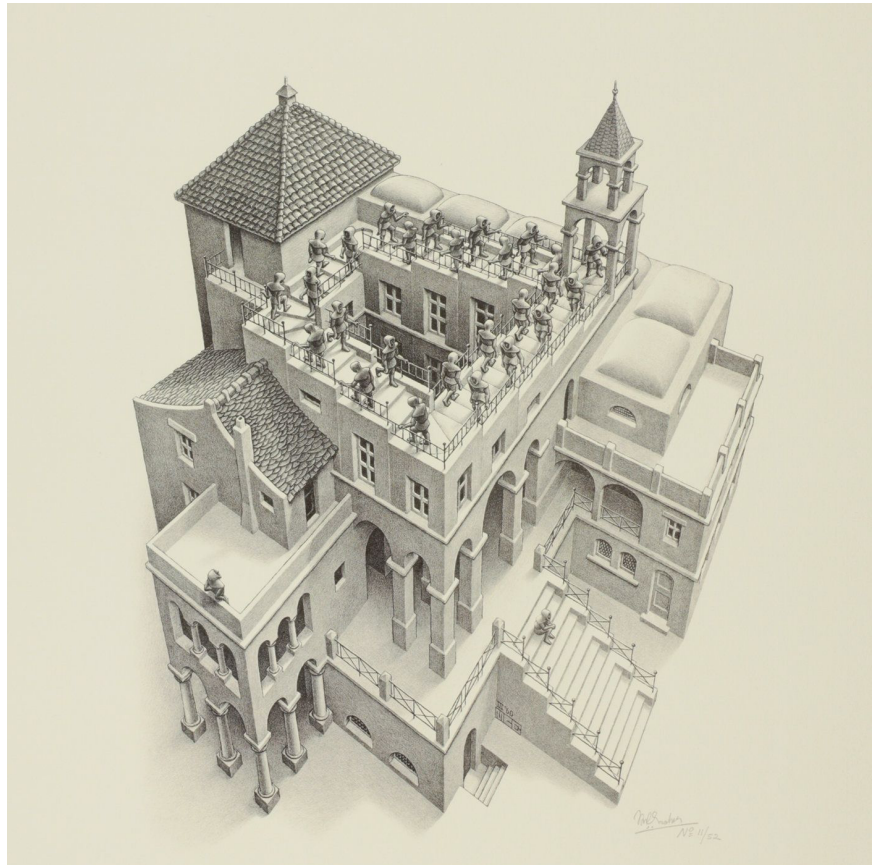
The phase we use in superconducting circuits is much like the direction one faces in Manhattan. Perhaps a better view is M.C. Escher's famous never-ending staircase, because Escher's staircase has distinct "up" and "down" directions that North, South, East and West lack. Phase is a potential, but unlike monotonic potentials such as voltage or altitude, phase is periodic. Escher's staircase attracts immediate attention because it makes altitude periodic, but altitude is a potential we think of as monotonic.

We find it useful to think of phase as "advancing" rather than "rising." Two nodes in a superconducting circuit can have different phase, one more advanced than the other. Current tends to flow from the more advanced phase to the less advanced

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phase, although the periodic nature of phase may give ambiguity to which is the more or less advanced.

The current through an inductor depends linearly on the difference in phase of its two ends. The current through a Josephson junction, however, depends not on the difference in phase, but on the mathematical sine function of the difference in phase. For small phase differences, the current through a Josephson junction is almost linear with phase difference. But for large differences, the current may be positive, negative, or even zero. This is behavior as remarkable as Escher's staircase.



Imagine walking your dog on Escher's staircase. The dog's leash, like an inductor, must get longer the further forward he runs. He might even catch up to you after running a full turn around the staircase. The leash "remembers" the path he took. It has to stretch further the more times your dog circumnavigates the stairs. In contrast, the dog's affection forgets how far he has run; he is just as affectionate after meeting you on his next trip around as he was at the start.

The sinusoidal relationship between phase and current in a Josephson junction means that JJs have no "memory" of full turns of phase. JJs are sensitive only to the fractional part of phase difference, measured in turns, and not to the integer number of turns. Like a ballet dancer who spins, the number of spins is immaterial. Only the final facing direction matters. Inductors, like the dog's leash, remember turns and like a dog's leash, might tangle with the dancer's feet.

RADIAN MEASURE

Physicists and Mathematicians like to measure angles in radians. Surveyors, navigators and grade school children measure angles in degrees. For superconducting electronics, we prefer to measure phase in turns to avoid dividing by 360 or dividing by π . Measuring phase in turns makes Josephson junctions insensitive to the integer part of turns of phase. JJs respond only to the fractional part of their phase drop measured

in turns. The pun “turns of phase” that Microsoft’s spelling corrector thinks should be turns of phrase, is a delightful side effect of measuring phase in turns.

One can teach radian measure to children by using radian measure when cutting cakes or pies or any circular desert. A child quickly learns that a full radian of cake is almost 1/6 of the whole thing. A full radian is a slice almost an equilateral triangle whose circular edge is the same length as the two radial edges cut straight by a knife. One full radian is rather more than a reasonable serving. A child soon learns to ask for about half a radian to get a reasonable slice.¹

A current of 70.7% of its critical current advances the phase drop across a JJ by 0.125 turns = 1/8 of a turn. Why? Because $\sin(0.125 \text{ turns}) = 1/\sqrt{2} = 0.707 = 70.7\%$. A current near 100% of its critical current advances the phase drop across a JJ by slightly less than one quarter of a turn. Why? Because $\sin(0.25 \text{ turns}) = \sin(90^\circ) = 1$.

What phase drop across a Josephson junction does a current of 101% of its critical current induce? Find the angle, x , such that $\sin(x) = 1.01$. There isn’t such an angle, and therein lies the non-linearity of the Josephson junction. Too much current results in a sudden “leap” between turns of phase as the circuit seeks a new stable operating point.

As its phase drop approaches one quarter of a turn, 0.25, a JJ begins to do strange things. Well-designed circuits drive JJs rapidly to beyond 0.25 turns, the top value in any one integer number of turns, so as to advance their sinusoidal behavior cleanly into the next full turn of phase.

Josephson junctions are useful as circuit elements precisely because of their cyclic relationship between current and phase. That this behavior seems strange is only because we are accustomed to monotonic rather than cyclic behavior of potentials like gravity, voltage, wealth, or mass. Common experience with periodic potentials is limited to sound, time², and alternating current circuits. What makes Escher’s drawing so striking is precisely its perspective giving a periodic form to height, a potential familiar to all as monotonic.

¹ This is exactly how Ivan learned radian measure.

² Only our measurement rather than time itself is periodic. Is Spring before or after Fall? Is Monday before or after Friday?

FORRESTHUNT, Inc.

Superconductivity Project

Subject: A Top View of RSFQ Schematics
Date: 5 November 2022
From: Ivan Sutherland
FHI#: 2021-is11v6

References:

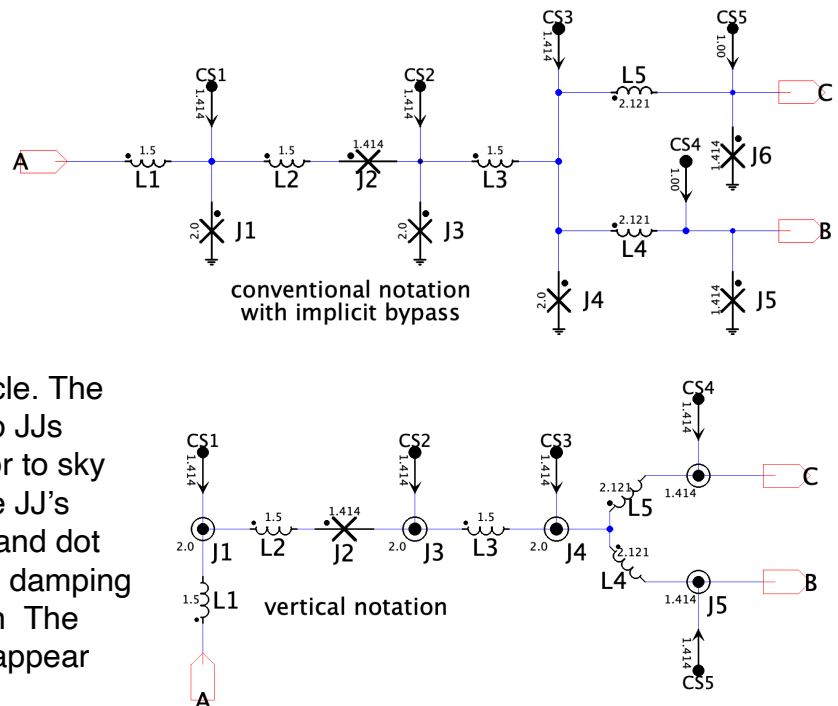
Rubin, Steven, Electric User's Manual, version 9.07, 1 November 2016

PURPOSE

This memo outlines a notation that simplifies RSFQ schematics. The notation, implemented in our Electric CAD tool, not only enables us to represent more complex schematic diagrams, but also helps us to understand how they work. This memo was first drafted on 24 February 2021. We have used the notation ever since.

THE NOTATION

A conventional schematic has “vertical” Josephson junctions, like J1 and J3, and “horizontal” JJs, like J2. Many JJ symbols implicitly include appropriate damping resistors, as do these. Our “top view” notation represents each grounded “vertical” JJs as a dot inside a circle. The circle dot notion applies equally to JJs connected to ground plane, M4, or to sky plane, M7. The dot represents the JJ’s ungrounded, terminal. The circle and dot symbol implicitly includes the JJ’s damping resistor and its ground connection. The JJ’s name and its critical current appear nearby. The dot “connects” to the



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ungrounded terminal of the JJ. We draw inductors using the usual curlicue notation.

This notation not only saves space in schematic's but also offers greater flexibility for implicit meaning. For example, the notation allows us to bend the input terminal, A, down, as seen in the lower figure, perhaps to imply that an input comes from "South of" this schematic. We are also free to use diagonal inductor symbols like L4 and L5 to represent current splits. The new notation looks down on our circuit from above, just as a city map looks down on the streets of a city. The appendix has several examples of schematics in this top view notation.

The top view notation also aids understanding by offering the third dimension, in and out of the page, to aid thoughts about phase. Whenever a junction fires, regardless of its size, it advances in phase by about one full turn. The top view notation lets us think about each such phase advance as if the junction symbol jumped up out of the page. One can think of such a jump as being a fixed distance: a centimeter, an inch, the thickness of a favorite book, or perhaps to the next floor in a building. JJs that have advanced different numbers of turns rest in equally-spaced layers over the page. These discrete layers reveal the quantized nature of flux.

Current in inductors and JJs flows "down" in phase from more advanced JJs (further above the page) to less advanced JJs. Current flows away from elevated JJs towards less elevated JJs and thence to ground. Given equal difference in phase, or elevation above the page, the amount of current flow is in inverse proportion to the inductance of each path; lots of current flows in low inductance paths, less current flows in higher inductance paths. Every superconducting grounded JJ is a potential "sink hole" for current. Thus, current from an elevated JJ usually flows through inductors into adjacent JJs and thence to ground.

One can think of each current source as an open bathroom tap that allows water to flow down-slope and eventually into the sink hole of some grounded JJ. But, if such a grounded JJ receives too much flow, it fires instead of draining, rising up by one "floor" to retard or even reverse horizontal current flow. However, any bias current source feeding that JJ continues to deliver current, just as a bathroom tap in an upper floor bathroom delivers water just as well as a bathroom tap on the main floor.

Even before firing, JJs carrying small currents elevate slightly. How much each JJ elevates when carrying a small current depends on the size of the grounded JJ "drain". Small JJs behave like big inductors and resist flow whereas big JJs behave like small inductors, providing "storm water size" drain action. All JJs rise about 1/8 of a turn (partway up a flight of stairs between floors) for their "normal" 70.7% bias current. If an adjacent JJ rises by a full turn, the inductor between them will see a "slope" of 7/8 of a turn. With that much slope, the connecting inductor is a much better carrier than the upper drain, and will carry nearly all of the bias current of the higher JJ to the lower one instead. This extra current is more than enough to overburden the lower JJ and cause it to fire.

A JTL is like a string of dominos that tumble in turn, except that the JJs rise in turn, each rising to the next floor above to overload the next JJ's ability to drain current to ground. When the next JJ fires, it joins its predecessor on the floor above, so that in the end the row of dominos is reset again one floor (one turn of phase) above exactly where it was before. Would that resetting strings of dominos were so easy.

UNITS

We use the usual normalized units. However, we have chosen to give them names. We call the normalized unit of current, 125 micro Amps, one *Lick*. Both the critical current of JJs and the source current of bias sources are calibrated in Licks. When appropriate, we abbreviate Lick as Lk with upper case L and lowercase k.

We use a dark arrow to represent a bias current source. Its parameter, in Licks, usually appears nearby, as seen in the figures. For simulation the arrow is replaced by a resistor whose resistance is inversely proportional to the current source value.

We use the usual normalized units of inductance, $2.632\text{e-}12$ Henry. However, we have chosen to name oHenry (after the American humorist) as the name of the normalized unit of inductance. One oHenry is $2.632\text{e-}12$ Henry. When appropriate, we abbreviate oHenry as oH, lower case o and upper case H. The notation $\text{oH}=\langle\text{numeric value}\rangle$ describes the value of an inductance.

By the way, O. Henry was American rather than Irish. The “O” is the first letter of his given name that he used as part of his nom de plum. Passing one Lick through one oHenry produces a phase drop of one radian, and so these normalized units still involve π . It might have been more rational to pick a value 2π larger for the oHenry so as to measure phase angles in turns rather than in radians.

ASYNCHRONOUS RESEARCH CENTER

Portland State University

Subject: Symmetric Links
Date: March 26, 2020
From: Marly Roncken and Ivan Sutherland
ARC#: 2020-is11v5

References:

Marly Roncken, Swetha Mettala Gilla, Hoon Park, Navaneeth Jamadagni, Chris Cowan, and Ivan Sutherland. *Naturalized Communication and Testing*. IEEE International Symposium on Asynchronous Circuits and Systems (ASYNC), pages 77–84, 2015 [Ronc2015]

HISTORY

Marly has worked out a new Link and Joint structure using bi-directional Links. To capture the form of her new ideas, Ivan is upgrading our Java simulator to conform to and document Marly's new ideas. The Java simulator is intended as much as a way for Ivan to record Marly's structural ideas as it is intended to provide useful simulations. This memo records some of the new features in the Java code.

PURPOSE

This memo describes the new structure for our Java simulation of the LinkJoint model. The new model makes Links strictly symmetric, with a new name SymLink for Symmetric Link. Each SymLink has an unordered Set of two HalfLinks that provide storage at each end of the SymLink. SymLink and Joint methods reflect Marly's notions of the activities possible to them. This memo also records some other insights I have had about the simulator.

COMMANDS and MULTI-THREADING

The Commands in the simulator are a means to keep track of multiple threads of computation. Each Command identifies a method that will start up at some later time. When I started using Commands I gave the double purpose of topological pointers. Thus Components coupled to each other by exchanging commands. Data in the Command class allowed each to identify the other indirectly. When one wished to activate the other it merely executed the already-existing command. Using the Comomand as a topological pointer complicates the code.

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I plan, going forward, to create Commands only when they must be enqueued. Thus, a Link will ask a Joint for its Command only when the Link wishes to “awake” the Joint at a later time. This “just in time” request for a Command will replace the much earlier request now made during construction of the network. Just in time requests for Coammands will reduce the number of instances of Command now stored as topological pointers.

This change will appear in all versions of the code later than 26 March 2020.

JOINTS and ACTIONS

A key question is the relationship between Joints and Actions. The simulator has heretofore treated the Joint as the primary topological entity. Each Joint is connected to several Links but avoids specifying how it will use those Links. Each Joint also has several actions that it might do. Each Joint also has a list of its Actions.

Only a Joint’s Actions know how they will use the Links attached to the Joint. Thus, one might treat Actions rather than Joints as the topological entity. In that view each Action would keep a list of the Links that it uses, possibly separating its List into sub-lists by intended use. Input Links, Output Links, indexing Links, etc. Such a separation by purpose seems at odds with Marly’s new thinking.

The new thinking unifies input and output Links and uses the notion of “turn” rather than FULL or EMPTY to say when a Link is available to an Action for the Action either to read from or to write into the Link. An action can happen when the subset of Links that it uses have the proper turn. Surely, we must keep a sub-list of Links by Action if only to speed the check that an Action could happen.

In fact, the existing simulator keeps an individual LinkHandle for each of the Links that it uses. Its guard process accesses these specifically one by one rather than by iterating through a List to find out if the Action is possible. When the Action is first attached to the Joint, the attaching code specifies the index positions in the Joint’s table of LinkHandles. As Java creates the Action code it copies the particular LinkHandles thus specified into particular named storage cells in the Action code. Thus, the existing simulator has the redundant storage of which I’ve been writing.

During network construction one might just as well specify the particular Links to use by name rather than by index number. The construction code would look up the Link by name in the Joint’s table of attached Links and then copy the reference into the particular place in the code that uses it. The name lookup would happen exactly once, when building the network. It seems wise to change this “cacheing” of Link references as little as possible.

The Joint vs Action question also has implication for wakeup. When a HalfLink gets a turn, whom should it wake? Should it wake an entire Joint which must then search through its actions to find one that is ready to execute. Or should the

HalfLink wake only the actions that use that HalfLink? In real hardware, concurrency makes moot this question; the question is relevant only for simulation speed. Thus, we must err in the direction of clarity, interpreting Marly's ideas independent of their impact on software efficiency.

The relationship between Joints and Actions makes no difference in principle; either view will produce identical simulation results, and quite similar code. The difference is a conceptual one. Does a Joint offer use of its Links to its actions, or do the Actions collectively provide the Joint access to the union of their Links. The issue becomes important only because we must specify the particular Links associated with particular Actions as we establish the Link-Joint network.

Past versions of the simulator addressed this issue poorly. It made types of Joints for each specific attached Action. Thus, in the existing simulator a FIFO Joint is a generic Joint with a copy Action attached which copies from the "first" Link of its Joint into the "second." Specific integers in the Action code identify the position in the Joint's array of Links to use.

My choice to cache, in each Action, the specific references to the Links that it uses takes a mixed view. At the cost of redundancy, each Joint will retain a list of all its Links without specifying purpose. In addition, each of its Actions will keep a redundant record, in a form convenient only to the Action of the Links it must access for whatever purpose. The Action's purpose will remain embedded in the Action code as myR and myW calls. The redundant storage of topology will speed the code that guards each Action.

The generic Joint, as an entity, may eventually vanish, giving way to a new Joint type that is merely the union of all its Actions. After all, Joints matter only for checking the correctness of the network as a whole, and for printing an exhibit of the network for human interpretation. The redundant representation costs additional storage space, but provides flexibility.

TOPOLOGY

Each Joint connects topologically to any number of HalfLinks. HalfLinks associate in symmetric pairs called SymLinks. Each SymLink has exactly two HalfLinks which must be different instances. Loopback Links will have two different HalfLinks associated with the same Joint. The two HalfLinks of a SymLink are called "mates". The SymLink keeps a record of which mate's "turn" it is. Only one HalfLink at a time has the "turn". A HalfLink can find out if the turn belongs to it, i.e. whether or not it can take its turn.

The HalfLinks provide for the SymLink's storage, giving each SymLink double the storage capacity available in earlier versions of Link. The items stored are TimedValues. Each TimedValue carries a Value (which may be any structure or may be null) and the Time at which that value (or null value) was established. The real-time

sequence of events is carried exclusively by the Time stamps associated with each Value. We must provide a method to find out when and how this SymLink last changed and when new data will be available to the end about to “take a turn.”

Joints and HalfLinks exchange Commands for their topological connection and use those Commands to pass the action from one to the other. These Commands are the ONLLY topological connections between SymLinks and Joints. Of course, each SymLink knows its HalfLinks and each HalfLink knows its SymLink and its mate. Each SymLink has a Set of two rather than a List of two HalfLinks to avoid ordering the HalfLinks.

A private class of Joint Command, called a WakeUpCommand, serves as the topological pointer to the Joint for the HalfLinks to use. Each Joint gives its WakeUpCommand to the HalfLinks attached to it. The SymLink uses this WakeUpCommand to initiate Joint actions after the action of the SymLink is done, i.e. after any communication delay caused by the physical length of the SymLink.

A private class of LinkCommand, called LinkHandle, serves as the topological pointer to HalfLinks used by Joints. Each HalfLink gives its LinkHandle to the Joint to which it is attached so that the Joint may initiate HalfLink action and thus SymLink action when Joint action is complete. The LinkHandle serves also as the topological pointer to the HalfLink. Joints keep lists of the LinkHandles of their adjacent HalfLinks. The LinkHandle provides the means for a Joint to signal that it is done with its turn.

The two HalfLinks in each SymLink alternate turns. Thus, a Joint interacting with a HalfLink can be assured of stable data during the turn of a nearby HalfLink. When it is done reading or writing the SynLink’s data, the Joint signals the end of its HalfLink’s turn by enqueueing that command.

CONSTRUCTORS

The SymLink constructor requires two HalfLinks that it conjoins. The constructor installs proper pointers into each HalfLink making them “mates” by pointing to each other and to their SymLink.

The HalfLink constructor requires a Joint to which the new HalfLink is connected. The constructor adds the new HalfLink to the Joint’s list of HalfLinks which is actually a list of LinkHandles. The constructor also add’s the Joint’s WakeUpCommand to the new HalfLink.

PROGRAMATIC INTERACTION

Joint’s access to the data in HalfLinks depends on whose turn it is. Accessing data in a HalfLink may involve getting the data from the HalfLink’s mate to behave as if the access was to the SymLink rather than to the HalfLink. When a HalfLink adjacent to it gets a turn, the Joint may write into its nearby storage using a method called

HalfLink.myW(value), a stand in for SymLink .myW(value). myW will return true if the write was successful, false otherwise if the myW call was out of turn.

Likewise, a Joint may also read from the storage at the other end of the SymLink with a method called, HalfLink.myR(). The myR method will return null if the request was out of turn. Otherwise it will return the appropriate TimedValue.

The Joint may also tell if this HalfLink has the turn using HalfLink.myTurn() which returns a Boolean. The Joint can indicate that its turn has come to an end by using the myHandle Command of the HalfLink. A call to HalfLink.myHandle indicates that the Joint has terminated its turn. And allows the SymLink to do its part of the simulation.

QUESTIONS FOR MARLY

Who is the parent of a HalfLink? Its Joint, or its SymLink. There's redundancy here. HalfLinks can be constructed adjacent to or independent from Joints. They are later combined into SymLinks. Answer: parentage is orthogonal to topology; we will make the parents of stuff be a group rather than a local component. The parent value will thus avoid becoming a topological feature. The target of a Command provides the topological connection.

Do HalfLinks need names? Probably yes, but maybe their name could be based on their topology. For example, HalfLinks could carry the name of their Joint and an index of where they are in its List of connections.

The HalfLinks of a SymLink take turns alternately. Is it a Joint's "turn" when ALL of its HalfLinks get a turn? No, but a Joint can read or write only SymLinks whose near end has the "turn."

VWSIM Builder's Manual

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Abstract

The **VWSIM** simulator is designed to specify and simulate Rapid Single Flux Quantum (RSFQ) circuits. The purpose of this document is to present and explain the underlying mathematics implemented by the **VWSIM** simulator to represent circuit primitive devices.

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1 Introduction

The **VWSIM** simulator is designed to specify and simulate Rapid, Single-Flux Quantum (RSFQ) circuits. The purpose of this document is to present and explain the equations used by **VWSIM** to represent circuit primitive devices: resistors, inductors, capacitors, Josephson junctions, mutual inductance, transmission lines, and input sources. Most of the equations were derived and provided by Delport [1]. This allowed us to implement **VWSIM** in ACL2 with similar functionality to that provided by JoSIM. In this document, the derivation of each of the equations (later called stamps) is mostly a detailed reiteration of those produced by Delport [1]. We also provide a fully articulated description of the Josephson junction model.

The circuit analysis approach used by the JoSIM simulator is based on the Modified Nodal Analysis (MNA) introduced by Ho, Ruehli, and Brennan [3]. By solving repeatedly a set of equations governing a circuit model of interest, this approach can be used to provide a forward simulation procedure. To initialize this procedure, one places appropriate values into the equation $Ax = b$, and solves the system of equations.

Stamps are operational approximations for primitive devices that we use to initialize the construction of the A matrix and b vector mentioned. If two nodes are connected together by one of these devices, we place a component's *stamp* in the matrix to relate the nodes to each other. The details of constructing this matrix are detailed later in this report. For now, we concentrate on the stamps.

Like JoSIM, **VWSIM** provides two simulation modes: voltage-based and phase-based. In a voltage-based simulation, the simulator records the voltages of the nodes and the currents through primitive devices with one exception: for JJs, the phase is stored. In a phase-based simulation, the simulator stores the phases of the nodes and the currents through the primitive devices, again with one exception: for JJs, the voltage is stored.

To assure ourselves that **VWSIM** implements the component requirements faithfully, we derive the discrete-time-step equations suitable for implementation with computer code. We have implemented **VWSIM** using the ACL2 logic [2] to help assure ourselves that our implementation is sound.

This document is concerned primarily with assuring ourselves that the discrete-time-step equations are based on the fundamental mathematics that govern RSFQ components. The first section derives the stamps **VWSIM** uses for a voltage-based simulation, and the second section derives those for a phase-based simulation.

2 The Voltage-based Analysis Stamps

For each primitive component used in RSFQ circuit designs, we provide a continuous (mathematical) model, and from that model, then derive a discrete-time-step model that we used in our VWSIM simulator.

2.1 Resistor

The voltage across the resistor is defined as

$$v(t) = Ri(t) \quad (1)$$

and written as a function of the current time step

$$V_n = RI_n \quad (2a)$$

$$\frac{1}{R}V_n = I_n \quad (2b)$$

Substituting the voltage as the difference in potential between the resistor's positive and negative nodes, we get

$$\frac{1}{R}V_n^+ - \frac{1}{R}V_n^- = I_n \quad (3)$$

$$\frac{1}{R}V_n^+ - \frac{1}{R}V_n^- - I_n = 0 \quad (4)$$

Let $G = \frac{1}{R}$.

Then, the resulting stamp for a resistor is

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ G & -G & -1 \end{bmatrix} \begin{bmatrix} V_n^+ \\ V_n^- \\ I_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (5)$$

Suppose V_n^- is the reference node ($= V_0$). We can eliminate the corresponding row and column, and the simplified stamp is

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ G & -G & -1 \end{bmatrix} \begin{bmatrix} V_n^+ \\ V_n^- \\ I_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ G & -1 \end{bmatrix} \begin{bmatrix} V_n^+ \\ I_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (6)$$

The resistor stamp provided above can be simplified. Since the resistance is known and the voltage is calculated, by equation (1), the I_n current can be determined from these values. Thus, the current through the resistor is a known value and its row and column in the stamp are redundant.

The simplified 2x2 resistor stamp would be

$$\begin{bmatrix} G & -G \\ -G & G \end{bmatrix} \begin{bmatrix} V_n^+ \\ V_n^- \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (7)$$

2.2 Capacitor

The current through a capacitor is described by

$$i(t) = C \frac{dv(t)}{dt} \quad (8)$$

and written as a function of the current time step

$$I_n = C \frac{dV_n}{dt} \quad (9a)$$

Using h_n to denote the time step size (the difference in time between the n and $n-1$ time steps), we can rewrite the equation above with the trapezoidal integration method

$$I_n = C \left[\frac{2}{h_n} (V_n - V_{n-1}) - \left(\frac{dV}{dt} \right)_{n-1} \right] \quad (9b)$$

and simplifying

$$I_n = \frac{2C}{h_n} (V_n - V_{n-1}) - C \left(\frac{dV}{dt} \right)_{n-1} \quad (9c)$$

$$I_n = \frac{2C}{h_n}V_n - \frac{2C}{h_n}V_{n-1} - C \left(\frac{dV}{dt} \right)_{n-1} \quad (9d)$$

From equation (8), we know the voltage derivative is related to the current at the previous time step. Substituting, we get

$$I_n = \frac{2C}{h_n}V_n - \frac{2C}{h_n}V_{n-1} - C \left(\frac{I_{n-1}}{C} \right) \quad (9e)$$

$$I_n = \frac{2C}{h_n}V_n - \frac{2C}{h_n}V_{n-1} - I_{n-1} \quad (9f)$$

Multiplying both sides by $\frac{h_n}{2C}$

$$\frac{h_n}{2C}I_n = V_n - V_{n-1} - \frac{h_n}{2C}I_{n-1} \quad (9g)$$

Rearranging the equation

$$V_n - \frac{h_n}{2C}I_n = V_{n-1} + \frac{h_n}{2C}I_{n-1} \quad (9h)$$

Substituting the voltage as the difference in potential between the capacitors's positive and negative nodes

$$V_n^+ - V_n^- - \frac{h_n}{2C}I_n = V_{n-1}^+ - V_{n-1}^- + \frac{h_n}{2C}I_{n-1} \quad (10)$$

Using equation (10), we can write the capacitor stamp as

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & -1 & -\frac{h_n}{2C} \end{bmatrix} \begin{bmatrix} V_n^+ \\ V_n^- \\ I_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ V_{n-1}^+ - V_{n-1}^- + \frac{h_n}{2C}I_{n-1} \end{bmatrix} \quad (11)$$

Suppose V_n^- is ground. We eliminate the corresponding row and column, and the simplified stamp is

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & -1 & -\frac{h_n}{2C} \end{bmatrix} \begin{bmatrix} V_n^+ \\ V_n^- \\ I_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ V_{n-1}^+ - V_{n-1}^- + \frac{h_n}{2C}I_{n-1} \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & -\frac{h_n}{2C} \end{bmatrix} \begin{bmatrix} V_n^+ \\ I_n \end{bmatrix} = \begin{bmatrix} 0 \\ V_{n-1}^+ + \frac{h_n}{2C}I_{n-1} \end{bmatrix} \quad (12)$$

2.3 Inductor

The voltage across an inductor is described by

$$v(t) = L \frac{di(t)}{dt} \quad (13)$$

and written as a function of the current time step

$$V_n = L \left(\frac{dI}{dt} \right)_n \quad (14a)$$

Rewriting with the trapezoidal integration method

$$V_n = L \left[\frac{2}{h_n} (I_n - I_{n-1}) - \left(\frac{dI}{dt} \right)_{n-1} \right] \quad (14b)$$

and simplifying

$$V_n = \frac{2L}{h_n} I_n - \frac{2L}{h_n} I_{n-1} - L \left(\frac{dI}{dt} \right)_{n-1} \quad (14c)$$

Knowing that $V_{n-1} = L \left(\frac{dI}{dt} \right)_{n-1}$

$$V_n = \frac{2L}{h_n} I_n - \frac{2L}{h_n} I_{n-1} - V_{n-1} \quad (14d)$$

Substituting the voltage as the difference in potential between the inductors's positive and negative nodes

$$V_n^+ - V_n^- = \frac{2L}{h_n} I_n - \frac{2L}{h_n} I_{n-1} - (V_{n-1}^+ - V_{n-1}^-) \quad (14e)$$

Rearranging the equation

$$V_n^+ - V_n^- - \frac{2L}{h_n} I_n = -\frac{2L}{h_n} I_{n-1} - (V_{n-1}^+ - V_{n-1}^-) \quad (15)$$

Using equation (15), we can write the inductor stamp as

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & -1 & -\frac{2L}{h_n} \end{bmatrix} \begin{bmatrix} V_n^+ \\ V_n^- \\ I_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\frac{2L}{h_n} I_{n-1} - (V_{n-1}^+ - V_{n-1}^-) \end{bmatrix} \quad (16)$$

Suppose V_n^- is ground. We eliminate the corresponding row and column, and the simplified stamp is

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & -1 & -\frac{2L}{h_n} \end{bmatrix} \begin{bmatrix} V_n^+ \\ V_n^- \\ I_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\frac{2L}{h_n} I_{n-1} - (V_{n-1}^+ - V_{n-1}^-) \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & -\frac{2L}{h_n} \end{bmatrix} \begin{bmatrix} V_n^+ \\ I_n \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{2L}{h_n} I_{n-1} - V_{n-1}^+ \end{bmatrix} \quad (17)$$

2.4 Resistively and Capacitively Shunted Josephson Junction (RCSJ)

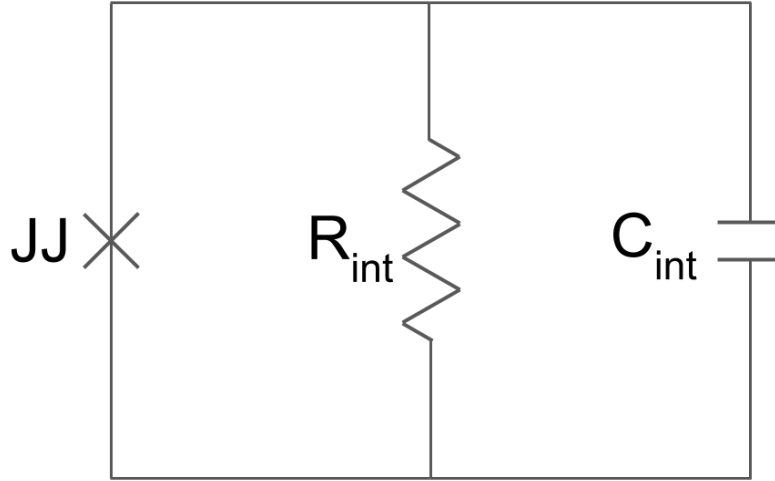


Figure 1: RCSJ model of a Josephson Junction

The equations that describe the RCSJ model are given by

$$i(t) = I_c \sin \phi(t) + C \frac{dv(t)}{dt} + \frac{1}{R} v(t) \quad (18)$$

$$\dot{\phi}(t) = \frac{2e}{\hbar} v(t) \quad (19)$$

where I_c the junction critical current, ϕ the junction phase, C the capacitance of the internal capacitor C_{int} , R the resistance of the nonlinear (voltage-dependent) internal resistor R_{int} , e the electron charge, and $\hbar$ Planck's constant.

If we write equation (18) in terms of the current time step and use the trapezoidal integration method, we find

$$I_n = I_c \sin \phi_n + C \left[\frac{2}{h_n} (V_n - V_{n-1}) - \dot{V}_{n-1} \right] + \frac{1}{R} V_n \quad (20a)$$

$$I_n = I_c \sin \phi_n + \frac{2C}{h_n} V_n - \frac{2C}{h_n} V_{n-1} - C \dot{V}_{n-1} + \frac{1}{R} V_n \quad (20b)$$

$$\frac{2C}{h_n} V_n + \frac{1}{R} V_n - I_n = -I_c \sin \phi_n + \frac{2C}{h_n} V_{n-1} + C \dot{V}_{n-1} \quad (20c)$$

$$\left(\frac{2C}{h_n} + \frac{1}{R} \right) V_n - I_n = -I_c \sin \phi_n + \frac{2C}{h_n} V_{n-1} + C \dot{V}_{n-1} \quad (20d)$$

When written with the junction voltage as the difference between the two node voltages

$$\left(\frac{2C}{h_n} + \frac{1}{R} \right) (V_n^+ - V_n^-) - I_n = -I_c \sin \phi_n + \frac{2C}{h_n} (V_{n-1}^+ - V_{n-1}^-) + C (\dot{V}_{n-1}^+ - \dot{V}_{n-1}^-) \quad (21)$$

Handling equation (19), the phase equation, similarly

$$\dot{\phi}(t) = \frac{2e}{\hbar} v(t) \quad (22a)$$

$$\dot{\phi}_n = \frac{2e}{\hbar} V_n \quad (22b)$$

$$\frac{2}{h_n} (\phi_n - \phi_{n-1}) - \dot{\phi}_{n-1} = \frac{2e}{\hbar} V_n \quad (22c)$$

$$\phi_n - \phi_{n-1} = \frac{h_n}{2} \frac{2e}{\hbar} V_n + \frac{h_n}{2} \dot{\phi}_{n-1} \quad (22d)$$

$$-\frac{h_n}{2} \frac{2e}{\hbar} V_n + \phi_n = \phi_{n-1} + \frac{h_n}{2} \dot{\phi}_{n-1} \quad (22e)$$

Substituting equation (22b) for the $(n - 1)^{th}$ time step

$$-\frac{h_n}{2} \frac{2e}{\hbar} V_n + \phi_n = \phi_{n-1} + \frac{h_n}{2} \frac{2e}{\hbar} V_{n-1} \quad (22f)$$

When written with the junction voltage as the difference between the two node voltages

$$-\frac{h_n}{2} \frac{2e}{\hbar} (V_n^+ - V_n^-) + \phi_n = \phi_{n-1} + \frac{h_n}{2} \frac{2e}{\hbar} (V_{n-1}^+ - V_{n-1}^-) \quad (23)$$

If we combine equations (21) and (23) into a single matrix, we get

$$\begin{bmatrix} \frac{1}{R} + \frac{2C}{h_n} & -\frac{1}{R} - \frac{2C}{h_n} & 0 \\ -\frac{1}{R} - \frac{2C}{h_n} & \frac{1}{R} + \frac{2C}{h_n} & 0 \\ -\frac{h_n}{2} \frac{2e}{\hbar} & \frac{h_n}{2} \frac{2e}{\hbar} & 1 \end{bmatrix} \begin{bmatrix} V_n^+ \\ V_n^- \\ \phi_n \end{bmatrix} = \begin{bmatrix} I_s \\ -I_s \\ \phi_{n-1} + \frac{h_n}{2} \frac{2e}{\hbar} (V_{n-1}^+ - V_{n-1}^-) \end{bmatrix} \quad (24)$$

where I_s is defined as

$$I_s = -I_c \sin \phi_n^g + \frac{2C}{h_n} (V_{n-1}^+ - V_{n-1}^-) + C(\dot{V}_{n-1}^+ - \dot{V}_{n-1}^-) \quad (25)$$

Notice that the RHS of equation (21), i.e. I_s , requires that we already know the value of ϕ_n . But we are trying to solve for ϕ_n ! So, we will have to approximate (guess) this value.

We call this the phase guess ϕ_n^g .

From equation (22f), we know

$$\begin{aligned} \phi_n^g &= \phi_{n-1} + \frac{h_n}{2} \frac{2e}{\hbar} V_{n-1} + \frac{h_n}{2} \frac{2e}{\hbar} V_n \\ &= \phi_{n-1} + \frac{h_n}{2} \frac{2e}{\hbar} (V_{n-1} + V_n) \end{aligned} \quad (26)$$

V_n is now in the equation above, which will also need to be guessed. Delpont defines the voltage guess, v_n^g , as

$$v_n^g = V_{n-1} + h_n \dot{V}_{n-1} \quad (27)$$

Why does Delpont choose to do a finite difference approximation (Euler's method) here instead of a second-order calculation? We discuss this question in Appendix B.

This voltage guess, v_n^g , requires past time-steps of voltage (and phase) guesses for each JJ being modeled. At the beginning of simulation, we have no history to perform the calculation presented in Delpont's dissertation; we discuss this further below. Delpont writes:

The only caveat in guessing the next voltage is when the previous value differs quite significantly within a time step (large $\frac{d}{dt}$) the difference between the phase guess and the previous phase dt value becomes extremely large causing unwanted behaviour. This occurs especially when initial conditions are zero and the previous two voltage values are therefore zero. We combat this by keeping the phase guess zero until around the third or fourth time step.

These three sentences left us a bit confused. We choose to read these three sentences as:

A caveat in guessing the current voltage occurs when the immediately previous voltage differs significantly (large $\frac{d}{dt}$) from the expected voltage; thus, a big change in v_n^g will have a big effect on the phase calculation (see equation 28). This effect is dramatic at simulation start, because we have no voltage values from previous steps. We may choose to mitigate this erroneous behavior in the future by choosing a phase guess of zero until sufficient history is available.

Currently, VWSIM simply calculates and uses the phase guess unaltered.

Finally, the phase guess is defined as below with v_n^g from equation (27):

$$\phi_n^g = \phi_{n-1} + \frac{h_n}{2} \frac{2e}{\hbar} (V_{n-1} + v_n^g) \quad (28)$$

Unfortunately, Delpont does not describe how to calculate $\dot{V}_{n-1}$. We suggest approximating the derivative at time-step k , after V_k has been calculated, using trapezoidal integration

$$\dot{V}_k = \frac{2}{h_k} (V_k - V_{k-1}) - \dot{V}_{k-1} \quad (29)$$

and storing $\dot{V}_k$ for every time-step k . Thus, when we are trying to guess ϕ_n^g , $\dot{V}_{n-1}$ will be available from the previous time-step.

We now summarize the JJ stamp as

$$\begin{bmatrix} \frac{1}{R} + \frac{2C}{h_n} & -\frac{1}{R} - \frac{2C}{h_n} & 0 \\ -\frac{1}{R} - \frac{2C}{h_n} & \frac{1}{R} + \frac{2C}{h_n} & 0 \\ -\frac{h_n}{2} \frac{2e}{\hbar} & \frac{h_n}{2} \frac{2e}{\hbar} & 1 \end{bmatrix} \begin{bmatrix} V_n^+ \\ V_n^- \\ \phi_n \end{bmatrix} = \begin{bmatrix} I_s \\ -I_s \\ \phi_{n-1} + \frac{h_n}{2} \frac{2e}{\hbar} (V_{n-1}^+ - V_{n-1}^-) \end{bmatrix} \quad (30)$$

with I_s defined as

$$I_s = -I_c \sin \phi_n^g + \frac{2C}{h_n} (V_{n-1}^+ - V_{n-1}^-) + C(\dot{V}_{n-1}^+ - \dot{V}_{n-1}^-) \quad (31)$$

where

$$\phi_n^g = \phi_{n-1} + \frac{h_n}{2} \frac{2e}{\hbar} (V_{n-1} + v_n^g) \quad (32)$$

$$v_n^g = V_{n-1}^+ - V_{n-1}^- + h_n(\dot{V}_{n-1}^+ - \dot{V}_{n-1}^-) \quad (33)$$

Note: We will start off the simulation with all unknown voltage and voltage derivative values as zero.

The stamp above is not complete. The junction conductance and the behavior of the JJ close to and above the critical current have neither been described nor modeled. The next few sections remedy this.

2.4.1 Rtype=0

Rtype=0 quite simply means there is no conductance present from the resistance branch in equation (30), which means infinite resistance. In order to implement this, we could simply leave the value of R to be R_{sg} , the subgap resistance. (Note: Delpont uses R_0 for the subgap resistance.) The subgap resistance is used because in practice it is impossible to reach absolute 0, and thus there will always be some resistance. By default, JoSIM uses $R_{sg} = 30 \Omega$.

We are more interested in the applications of the Rtype=1 model, and therefore end this discussion here.

2.4.2 Rtype=1

In the standard junction model, it is assumed that the internal junction damping by a non-linear (voltage-dependent) junction resistor (R_{int}) can be replaced in the first approximation by a linear resistor, R_{sg} in the subgap region ($V < V_{gap}$) and by the junction normal resistor, R_n above the gap voltage ($V > V_{gap}$) [8]. Since R_{sg} and R_n typically have different conductance values, a discontinuity occurs in the model. JoSIM (which borrows from JSIM and

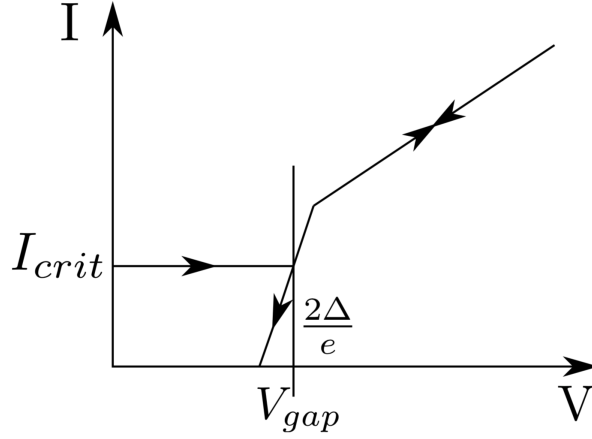


Figure 2: IV Curve of a JJ from Delport's Dissertation

WRSpice) attempts to remedy this by a linear interpolation that bridges the two regions. This bridge is referred to as the transition region.

The implementation of this piece-wise linear conductance surrounding the gap voltage is known as Rtype=1.

2.4.2.1 The JSIM Method (JoSIM does NOT use this)

JSIM defines an area that specifies 2 regions wherein the conductance transitions between the **subgap conductance** and **normal conductance**.

These regions are $V_{gap} + \Delta V$ and $V_{gap} + 2\Delta V$ where ΔV is the gap transition voltage and is set to 0.1 mV by default.

As the junction current increases and approaches the critical current, the voltage guess (v^0) is monitored. If the voltage for the next step is guessed to cross the gap voltage, the junction conductance in the LHS (equation (30)) is altered to

$$G_T = \frac{\frac{V_{gap} + \Delta V}{R_n} - \frac{V_{gap}}{R_{sg}}}{\Delta V} \quad (34)$$

where G_T is the transition conductance.

Additionally, an additional current value (I_T) is added to the RHS I_s to compensate for this transition period and can be specified as

$$I_T = V_{gap} \left(\frac{1}{R_{sg}} - G_T \right) \quad (35)$$

while $V_{gap} < V^g \leq V_{gap} + \Delta V$

$$I_T = V^g \left(\frac{1}{R_n} - G_T \right) \quad (36)$$

while $V_{gap} + \Delta V < V^g \leq V_{gap} + 2\Delta V$

Appendix F includes a derivation of the equations above.

JoSIM does not use this method, so we end this discussion here. JoSIM's approach is closer to WRspice's method, which we now discuss.

2.4.2.2 The JoSIM Method (which borrows from WRspice)

The modifications provided in JSIM are insufficient to model JJ behavior. Delport mentions in his dissertation that JSIM does not take into account the ratio of critical current to the quasiparticle step height parameter; this parameter allows the junction behaviour above the subgap region to be refined.

The transition region is defined as $V_{gap} - \frac{1}{2}\Delta V$ to $V_{gap} + \frac{1}{2}\Delta V$, where the transition conductance is

$$G_T = \frac{I_c}{I_{c_{fact}} \cdot \Delta V} \quad (37)$$

where $I_{c_{fact}}$ is the ratio of critical to gap current with a default value of $\frac{\pi}{4}$.

The voltage guess (v^g) is monitored. If the junction voltage enters the region surrounding the gap voltage ($V_{gap} - \frac{1}{2}\Delta V$ to $V_{gap} + \frac{1}{2}\Delta V$), the junction conductance in the A matrix is changed to equation (37) and a transition current is added to the RHS.

Important note: Delport's dissertation mentions that this transition current is the same as equation (35). This is incorrect. After confirming with Joey Delport, since the lower-bound of the transition region is now $V_{gap} - \frac{1}{2}\Delta V$, we must instead define the transition current as

$$I_T = \left(V_{gap} - \frac{1}{2}\Delta V \right) \left(\frac{1}{R_{sg}} - G_T \right) \quad (38)$$

Finally, when the junction voltage reaches a value above this region, the transitional current is given a constant value as defined in equation (41).

To summarize dealing with transition currents,

$$I_T = 0 \quad (39)$$

while $V^g < V_{gap} - \frac{1}{2}\Delta V$

$$I_T = V_{gap} \left(\frac{1}{R_{sg}} - G_T \right) \quad (40)$$

while $V_{gap} - \frac{1}{2}\Delta V \leq V^g < V_{gap} + \frac{1}{2}\Delta V$

$$I_T = \frac{I_c}{I_{c_{fact}}} + \left[\frac{1}{R_{sg}} \times \left(V_{gap} - \frac{1}{2}\Delta V \right) \right] - \left[\frac{1}{R_n} \times \left(V_{gap} + \frac{1}{2}\Delta V \right) \right] \quad (41)$$

while $V^g \geq V_{gap} + \frac{1}{2}\Delta V$

Note: WRspice includes more quasiparticle resistance models such as a analytic exponentially-derived approximation (Rtype=2) and a fifth order polynomial expansion model (Rtype=3), these have yet to be investigated for implementation in JoSIM.

To summarize dealing with junction conductance,

$$\frac{1}{R_{sg}} \quad (42)$$

while $V^g < V_{gap} - \frac{1}{2}\Delta V$

$$\frac{I_c}{I_{c_{fact}} \cdot \Delta V} \quad (43)$$

while $V_{gap} - \frac{1}{2}\Delta V \leq V^g < V_{gap} + \frac{1}{2}\Delta V$

$$\frac{1}{R_n} \quad (44)$$

while $V^g \geq V_{gap} + \frac{1}{2}\Delta V$

2.4.2.3 Using the Transition Current and Junction Conductance in The RCSJ Model

Transition Current

To incorporate the transition current, equation (18) will be modified to be

$$i(t) = I_c \sin \phi(t) + C \frac{dv(t)}{dt} + \frac{1}{R} v(t) + I_T \quad (45)$$

If we write the above equation in terms of the current time step and use the trapezoidal integration method, we find

$$I_n = I_c \sin \phi_n + C \left[\frac{2}{h_n} (V_n - V_{n-1}) - \dot{V}_{n-1} \right] + \frac{1}{R} V_n + I_{Tn} \quad (46a)$$

$$I_n = I_c \sin \phi_n + \frac{2C}{h_n} V_n - \frac{2C}{h_n} V_{n-1} - C \dot{V}_{n-1} + \frac{1}{R} V_n + I_{Tn} \quad (46b)$$

$$\frac{2C}{h_n} V_n + \frac{1}{R} V_n - I_n = -I_c \sin \phi_n + \frac{2C}{h_n} V_{n-1} + C \dot{V}_{n-1} - I_{Tn} \quad (46c)$$

$$\left(\frac{2C}{h_n} + \frac{1}{R} \right) V_n - I_n = -I_c \sin \phi_n + \frac{2C}{h_n} V_{n-1} + C \dot{V}_{n-1} - I_{Tn} \quad (46d)$$

When written with the junction voltage as the difference between the two node voltages

$$\left(\frac{2C}{h_n} + \frac{1}{R} \right) (V_n^+ - V_n^-) - I_n = -I_c \sin \phi_n + \frac{2C}{h_n} (V_{n-1}^+ - V_{n-1}^-) + C \dot{V}_{n-1} - I_{Tn} \quad (47)$$

Using the definition of I_s , the equation above can be rewritten as

$$\left(\frac{2C}{h_n} + \frac{1}{R} \right) (V_n^+ - V_n^-) - I_n = I_s - I_T \quad (48)$$

We avoid writing I_T as a function of the time step because its value is altered based on the voltage across the JJ.

Check Appendix D.1 for errors we discovered in the JoSIM simulator when implementing the transition current.

Junction Conductance

The model for the JJ can now be stated as

$$\left(\frac{2C}{h_n} + \frac{1}{R}\right) (V_n^+ - V_n^-) - I_n = I_s - I_T \quad (49)$$

$$-\frac{h_n}{2} \frac{2e}{\hbar} (V_n^+ - V_n^-) + \phi_n = \phi_{n-1} + \frac{h_n}{2} \frac{2e}{\hbar} (V_{n-1}^+ - V_{n-1}^-) \quad (50)$$

where $\frac{1}{R}$ is the nonlinear junction conductance. The values of the junction conductance are determined by equations (42), (43), and (44).

2.4.2.4 Finally, the JJ Stamps We Need

As seen above, the equations approximating the behavior of a Josephson Junction vary depending on the voltage across the JJ. The stamp can now be summarized as

Below the transition region ($V_n < V_{gap} - \frac{1}{2}\Delta V$)

The governing equations in this region are

$$\left(\frac{2C}{h_n} + \frac{1}{R_{sg}} \right) (V_n^+ - V_n^-) - I_n = I_s - I_T \quad (51)$$

$$-\frac{h_n}{2} \frac{2e}{\hbar} V_n + \phi_n = \phi_{n-1} + \frac{h_n}{2} \frac{2e}{\hbar} V_{n-1} \quad (52)$$

where $I_T = 0$.

Using the equations above, the stamp can be written as

$$\begin{bmatrix} \frac{1}{R_{sg}} + \frac{2C}{h_n} & -\frac{1}{R_{sg}} - \frac{2C}{h_n} & 0 \\ -\frac{1}{R_{sg}} - \frac{2C}{h_n} & \frac{1}{R_{sg}} + \frac{2C}{h_n} & 0 \\ -\frac{h_n}{2} \frac{2e}{\hbar} & \frac{h_n}{2} \frac{2e}{\hbar} & 1 \end{bmatrix} \begin{bmatrix} V_n^+ \\ V_n^- \\ \phi_n \end{bmatrix} = \begin{bmatrix} I_s \\ -I_s \\ \phi_{n-1} + \frac{h_n}{2} \frac{2e}{\hbar} (V_{n-1}^+ - V_{n-1}^-) \end{bmatrix} \quad (53)$$

In the transition region ($V_{gap} - \frac{1}{2}\Delta V \leq V_n < V_{gap} + \frac{1}{2}\Delta V$)

The governing equations in this region are

$$\left(\frac{2C}{h_n} + G_T \right) (V_n^+ - V_n^-) - I_n = I_s - I_T \quad (54)$$

$$-\frac{h_n}{2} \frac{2e}{\hbar} V_n + \phi_n = \phi_{n-1} + \frac{h_n}{2} \frac{2e}{\hbar} V_{n-1} \quad (55)$$

Using the equations above, the stamp can be written as

$$\begin{bmatrix} G_T + \frac{2C}{h_n} & -G_T - \frac{2C}{h_n} & 0 \\ -G_T - \frac{2C}{h_n} & G_T + \frac{2C}{h_n} & 0 \\ -\frac{h_n}{2} \frac{2e}{\hbar} & \frac{h_n}{2} \frac{2e}{\hbar} & 1 \end{bmatrix} \begin{bmatrix} V_n^+ \\ V_n^- \\ \phi_n \end{bmatrix} = \begin{bmatrix} I_s - I_T \\ -I_s + I_T \\ \phi_{n-1} + \frac{h_n}{2} \frac{2e}{\hbar} (V_{n-1}^+ - V_{n-1}^-) \end{bmatrix} \quad (56)$$

where

$$G_T = \frac{I_c}{I_{cfact} \cdot \Delta V} \quad (57)$$

$$I_T = (V_{gap} - \frac{1}{2}\Delta V) \left(\frac{1}{R_{sg}} - G_T \right) \quad (58)$$

$$I_{cfact} = \frac{\pi}{4} \quad (59)$$

We verify that the model is continuous at $V_n = V_{gap} - \frac{1}{2}\Delta V$ in Appendix C.1.1.

Above the transition region ($V_n \geq V_{gap} + \frac{1}{2}\Delta V$)

The governing equations in this region are

$$\left(\frac{2C}{h_n} + \frac{1}{R_n} \right) (V_n^+ - V_n^-) - I_n = I_s - I_T \quad (60)$$

$$-\frac{h_n}{2} \frac{2e}{\hbar} V_n + \phi_n = \phi_{n-1} + \frac{h_n}{2} \frac{2e}{\hbar} V_{n-1} \quad (61)$$

Using the equations above, the stamp can be written as

$$\begin{bmatrix} \frac{1}{R_n} + \frac{2C}{h_n} & -\frac{1}{R_n} - \frac{2C}{h_n} & 0 \\ -\frac{1}{R_n} - \frac{2C}{h_n} & \frac{1}{R_n} + \frac{2C}{h_n} & 0 \\ -\frac{h_n}{2} \frac{2e}{\hbar} & \frac{h_n}{2} \frac{2e}{\hbar} & 1 \end{bmatrix} \begin{bmatrix} V_n^+ \\ V_n^- \\ \phi_n \end{bmatrix} = \begin{bmatrix} I_s - I_T \\ -I_s + I_T \\ \phi_{n-1} + \frac{h_n}{2} \frac{2e}{\hbar} (V_{n-1}^+ - V_{n-1}^-) \end{bmatrix} \quad (62)$$

where

$$I_T = \frac{I_c}{I_{cfact}} + \left[\frac{1}{R_{sg}} \times \left(V_{gap} - \frac{1}{2}\Delta V \right) \right] - \left[\frac{1}{R_n} \times \left(V_{gap} + \frac{1}{2}\Delta V \right) \right] \quad (63)$$

We verify that the model is continuous at $V_n = V_{gap} + \frac{1}{2}\Delta V$ in Appendix C.1.2.

I_s is defined as

$$I_s = -I_c \sin \phi_n^g + \frac{2C}{h_n} (V_{n-1}^+ - V_{n-1}^-) + C(\dot{V}_{n-1}^+ - \dot{V}_{n-1}^-) \quad (64)$$

where

$$\phi_n^g = \phi_{n-1} + \frac{h_n}{2} \frac{2e}{h} (V_{n-1}^+ - V_{n-1}^- + v_n^g) \quad (65)$$

$$v_n^g = V_{n-1}^+ - V_{n-1}^- + h_n(\dot{V}_{n-1}^+ - \dot{V}_{n-1}^-) \quad (66)$$

2.4.2.5 Using Haberkorn's Current-Phase Relation

This section details a possible way to improve the VWSIM JJ model (presented in [1]) using the Haberkorn current-phase relation.

The current-phase relation (CPR) for a Josephson Junction is defined through the sinusoidal supercurrent equation

$$I_s = I_c \sin(\phi) \quad (67)$$

which holds true for most cases of electron tunneling between two superconducting materials. However, an investigation by Haberkorn [4] shows that this CPR can be non-sinusoidal, especially when considering ballistic tunnelling.

This non-sinusoidal effect is shown in the following equation.

$$I_s = \frac{\pi \Delta}{2eR_N} \frac{\sin \phi}{\sqrt{1 - \bar{D} \sin^2 \left(\frac{\phi}{2} \right)}} \tanh \left[\frac{\Delta}{2k_B T} \sqrt{1 - \bar{D} \sin^2 \left(\frac{\phi}{2} \right)} \right] \quad (68)$$

where

$$R_N = \frac{\pi \Delta}{2eI_c} \tanh \left(\frac{\Delta}{2k_B T} \right) \quad (69)$$

$$\Delta = \Delta_0 \sqrt{\cos \left[\frac{\pi}{2} \left(\frac{T}{T_c} \right)^2 \right]} \quad (70)$$

with

$\bar{D}$	transparency	(transmission probability of electron across the barrier)
Δ_0	$1.76k_B T_c$	(superconducting energy gap)
T_c	$9.1K$	(Critical temperature of Niobium)
T	$4.2K$	(Boiling point of liquid Helium)
k_B	$1.380649 \text{ J} \cdot \text{K}^{-1}$	(Boltzmann's constant)

Equation (68) introduces the temperature dependency through Δ , which is defined in equation (70). Implementation of this temperature dependence allows us to more accurately represent Josephson junctions dynamics through simulation. We can also alter the value of $\bar{D}$, which allows us to change the characteristics of the tunnel current. For values of $\bar{D} \ll 1$, the equation becomes the normal sinusoidal equation (i.e. equation (67)) whereas for large values of $\bar{D}$ it becomes the non-sinusoidal ballistic tunneling equation. The meaning of $\bar{D}$ is explained in Appendix E. By default, JoSIM sets $\bar{D}$ to 0, which simplifies Haberkorn's CPR (equation (68)) to the sinusoidal CPR (equation (67)).

2.5 Lossless Passive Transmission Line

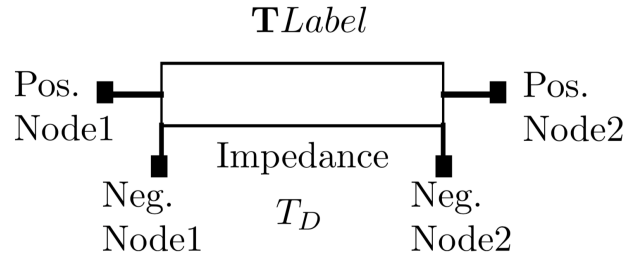


Figure 3: A Transmission Line Element from the JoSIM Manual

A Josephson Transmission Line (JTL) must be biased and, in the case of RSFQ, consumes static power. This makes them a costly component if the signal needs to propagate over a long distance. Additionally, each JTL adds roughly a 5ps delay, which also slows performance in large designs [1].

Due to these limitations, passive transmission lines (PTLs) are used in large designs. PTLs allow SFQ pulses to be transferred at near light speed with minimal loss of energy. The equations that govern this transmission are described in [1] as

$$v_1(t) - Z_0 i_1(t) = v_2(t - T_D) + Z_0 i_2(t - T_D) \quad (71)$$

$$v_2(t) - Z_0 i_2(t) = v_1(t - T_D) + Z_0 i_1(t - T_D) \quad (72)$$

The time delay in terms of the current time step can be written as

$$t - T_D \equiv n - \frac{T_D}{h_n} = n - k$$

(Note: we may choose to simulate with a variable time-step. In which case, we cannot use the above simplification, and must use $t - T_D$).

Rewriting equations (71) and (72) as a function of the time step

$$(V_1)_n - Z_0(I_1)_n = (V_2)_{n-k} + Z_0(I_2)_{n-k} \quad (73)$$

$$(V_2)_n - Z_0(I_2)_n = (V_1)_{n-k} + Z_0(I_1)_{n-k} \quad (74)$$

As a function of the voltages at both nodes, we can write these equations as

$$(V_1)_n^+ - (V_1)_n^- - Z_0(I_1)_n = (V_2)_{n-k}^+ - (V_2)_{n-k}^- + Z_0(I_2)_{n-k} \quad (75)$$

$$(V_2)_n^+ - (V_2)_n^- - Z_0(I_2)_n = (V_1)_{n-k}^+ - (V_1)_{n-k}^- + Z_0(I_1)_{n-k} \quad (76)$$

The resulting transmission line stamp is

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & -Z_0 & 0 \\ 0 & 0 & 1 & -1 & 0 & -Z_0 \end{bmatrix} \begin{bmatrix} (V_1)_n^+ \\ (V_1)_n^- \\ (V_2)_n^+ \\ (V_2)_n^- \\ (I_1)_n \\ (I_2)_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ (V_2)_{n-k}^+ - (V_2)_{n-k}^- + Z_0(I_2)_{n-k} \\ (V_1)_{n-k}^+ - (V_1)_{n-k}^- + Z_0(I_1)_{n-k} \end{bmatrix} \quad (77)$$

Similarly, writing equation 79 in terms of the current time step and using the trapezoidal integration method, we find

$$(V_2)_n = \frac{2M}{h_n}(I_1)_n - \frac{2M}{h_n}(I_1)_{n-1} - M(\dot{I}_1)_{n-1} + \frac{2L_2}{h_n}(I_2)_n - \frac{2L_2}{h_n}(I_2)_{n-1} - L_2(\dot{I}_2)_{n-1} \quad (86)$$

$$(V_2)_n = \frac{2M}{h_n}(I_1)_n - \frac{2M}{h_n}(I_1)_{n-1} + \frac{2L_2}{h_n}(I_2)_n - \frac{2L_2}{h_n}(I_2)_{n-1} - \left[M(\dot{I}_1)_{n-1} + L_2(\dot{I}_2)_{n-1} \right] \quad (87)$$

$$(V_2)_n = \frac{2M}{h_n}(I_1)_n - \frac{2M}{h_n}(I_1)_{n-1} + \frac{2L_2}{h_n}(I_2)_n - \frac{2L_2}{h_n}(I_2)_{n-1} - (V_2)_{n-1} \quad (88)$$

$$(V_2)_n - \frac{2M}{h_n}(I_1)_n - \frac{2L_2}{h_n}(I_2)_n = -\frac{2M}{h_n}(I_1)_{n-1} - \frac{2L_2}{h_n}(I_2)_{n-1} - (V_2)_{n-1} \quad (89)$$

When written with the voltages as the difference between two node voltages

$$(V_2)_n^+ - (V_2)_n^- - \frac{2M}{h_n}(I_1)_n - \frac{2L_2}{h_n}(I_2)_n = -\frac{2M}{h_n}(I_1)_{n-1} - \frac{2L_2}{h_n}(I_2)_{n-1} - (V_2)_{n-1}^+ + (V_2)_{n-1}^- \quad (90)$$

The resulting mutual inductance stamp is

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & -\frac{2L_1}{h_n} & -\frac{2M}{h_n} \\ 0 & 0 & 1 & -1 & -\frac{2M}{h_n} & -\frac{2L_2}{h_n} \end{bmatrix} \begin{bmatrix} (V_1)_n^+ \\ (V_1)_n^- \\ (V_2)_n^+ \\ (V_2)_n^- \\ (I_1)_n \\ (I_2)_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\frac{2L_1}{h_n}(I_1)_{n-1} - \frac{2M}{h_n}(I_2)_{n-1} - (V_1)_{n-1}^+ + (V_1)_{n-1}^- \\ -\frac{2M}{h_n}(I_1)_{n-1} - \frac{2L_2}{h_n}(I_2)_{n-1} - (V_2)_{n-1}^+ + (V_2)_{n-1}^- \end{bmatrix} \quad (91)$$

It is important to note that the mutual inductance stamp holds redundant information. That is, it contains the stamps of the two inductors that it is trying to couple. In **VWSIM**, since we always stamp the two inductors into the A matrix and b vector individually, the mutual inductance stamp will be

$$\begin{bmatrix} 0 & -\frac{2M}{h_n} \\ -\frac{2M}{h_n} & 0 \end{bmatrix} \begin{bmatrix} (I_1)_n \\ (I_2)_n \end{bmatrix} = \begin{bmatrix} -\frac{2M}{h_n}(I_2)_{n-1} \\ -\frac{2M}{h_n}(I_1)_{n-1} \end{bmatrix} \quad (92)$$

which adds the additional constraint of the mutual inductance.

2.6 Mutual Inductance

The mutual inductance can be described with two equations

$$v_1(t) = L_1 \dot{i}_1 + M \dot{i}_2(t) \quad (78)$$

$$v_2(t) = M \dot{i}_1(t) + L_2 \dot{i}_2(t) \quad (79)$$

where M is the mutual coupling, L_1 and L_2 are the inductances of the two inductors, i_1 and i_2 are the currents through the respective inductors.

The mutual coupling (M) is calculated as

$$M = k \sqrt{L_1 L_2} \quad (80)$$

where k is the coupling factor and $-1 \leq k \leq 1$.

If we write equation 78 in terms of the current time step and use the trapezoidal integration method, we find

$$(V_1)_n = \frac{2L_1}{h_n}(I_1)_n - \frac{2L_1}{h_n}(I_1)_{n-1} - L_1(\dot{I}_1)_{n-1} + \frac{2M}{h_n}(I_2)_n - \frac{2M}{h_n}(I_2)_{n-1} - M(\dot{I}_2)_{n-1} \quad (81)$$

$$(V_1)_n = \frac{2L_1}{h_n}(I_1)_n - \frac{2L_1}{h_n}(I_1)_{n-1} + \frac{2M}{h_n}(I_2)_n - \frac{2M}{h_n}(I_2)_{n-1} - \left[L_1(\dot{I}_1)_{n-1} + M(\dot{I}_2)_{n-1} \right] \quad (82)$$

$$(V_1)_n = \frac{2L_1}{h_n}(I_1)_n - \frac{2L_1}{h_n}(I_1)_{n-1} + \frac{2M}{h_n}(I_2)_n - \frac{2M}{h_n}(I_2)_{n-1} - (V_1)_{n-1} \quad (83)$$

$$(V_1)_n - \frac{2L_1}{h_n}(I_1)_n - \frac{2M}{h_n}(I_2)_n = -\frac{2L_1}{h_n}(I_1)_{n-1} - \frac{2M}{h_n}(I_2)_{n-1} - (V_1)_{n-1} \quad (84)$$

When written with the voltages as the difference between two node voltages

$$(V_1)_n^+ - (V_1)_n^- - \frac{2L_1}{h_n}(I_1)_n - \frac{2M}{h_n}(I_2)_n = -\frac{2L_1}{h_n}(I_1)_{n-1} - \frac{2M}{h_n}(I_2)_{n-1} - (V_1)_{n-1}^+ + (V_1)_{n-1}^- \quad (85)$$

2.7 Voltage Source

Since the voltage at every time is given, the voltage source stamp can easily be written as

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} V_n^+ \\ V_n^- \\ I_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ E_n \end{bmatrix} \quad (93)$$

2.8 Current Source

Since the current at every time is given, the current source stamp can easily be written as

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V_n^+ \\ V_n^- \\ I_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ A_n \end{bmatrix} \quad (94)$$

2.9 Phase Source

Looking ahead to (equation (102)) in the next section, we can use the voltage-phase relation to write voltage in terms of phase

$$v = \frac{\Phi_0}{2\pi} \dot{\phi} \quad (95)$$

Writing in terms of the current time step

$$V_n = \frac{\Phi_0}{2\pi} \dot{\phi}_n \quad (96)$$

Applying the trapezoidal integration method

$$V_n = \frac{\Phi_0}{2\pi} \left(\frac{2}{h_n} (\phi_n - \phi_{n-1}) - \dot{\phi}_{n-1} \right) \quad (97)$$

Using the voltage-phase relation again

$$V_n = \frac{\Phi_0}{2\pi} \left(\frac{2}{h_n} (\phi_n - \phi_{n-1}) - \frac{2\pi}{\Phi_0} V_{n-1} \right) \quad (98)$$

$$V_n = \frac{\Phi_0}{2\pi} \frac{2}{h_n} (\phi_n - \phi_{n-1}) - V_{n-1} \quad (99)$$

Using equation (99), we can write the phase source stamp as

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} V_n^+ \\ V_n^- \\ I_{\phi_n} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{\Phi_0}{2\pi} \frac{2}{h_n} (\phi_n - \phi_{n-1}) - V_{n-1} \end{bmatrix} \quad (100)$$

3 The Phase-based Analysis Stamps

The phase-voltage relation can be described as

$$\phi = \frac{2\pi}{\Phi_0} \int v dt \quad (101)$$

and taking the derivative of both sides

$$v = \frac{\Phi_0}{2\pi} \dot{\phi} \quad (102)$$

In order to perform a phase-based analysis, we will substitute equation (102) to replace the voltage terms. The derivation of this equation can be found in [6].

3.1 Resistor

The standard voltage-based equation for a resistor is

$$i = \frac{v}{R} \quad (103)$$

Substituting the phase-voltage relation (equation (102))

$$i = \frac{\Phi_0}{2\pi R} \dot{\phi} \quad (104)$$

Applying the trapezoidal integration method

$$I_n = \frac{\Phi_0}{2\pi R} \left(\frac{2}{h_n} (\phi_n - \phi_{n-1}) - \dot{\phi}_{n-1} \right) \quad (105)$$

Using equation (102) on the previous time step

$$I_n = \frac{\Phi_0}{2\pi R} \left(\frac{2}{h_n} (\phi_n - \phi_{n-1}) \right) - I_{n-1} \quad (106)$$

$$I_n = \frac{\Phi_0}{\pi h_n R} (\phi_n - \phi_{n-1}) - I_{n-1} \quad (107)$$

$$\frac{\pi h_n R}{\Phi_0} I_n = \phi_n - \phi_{n-1} - \frac{\pi h_n R}{\Phi_0} I_{n-1} \quad (108)$$

$$\phi_n - \frac{\pi h_n R}{\Phi_0} I_n = \phi_{n-1} + \frac{\pi h_n R}{\Phi_0} I_{n-1} \quad (109)$$

$$\phi_n^+ - \phi_n^- - \frac{\pi h_n R}{\Phi_0} I_n = \phi_{n-1}^+ - \phi_{n-1}^- + \frac{\pi h_n R}{\Phi_0} I_{n-1} \quad (110)$$

The resulting phase-based resistor stamp is

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & -1 & -\frac{\pi h_n R}{\Phi_0} \end{bmatrix} \begin{bmatrix} \phi_n^+ \\ \phi_n^- \\ I_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \phi_{n-1}^+ - \phi_{n-1}^- + \frac{\pi h_n R}{\Phi_0} I_{n-1} \end{bmatrix} \quad (111)$$

The resistor stamp provided above can be simplified. Since the resistance is known and the phase is calculated, by equation (105), the I_n current can be determined from these values. Thus, the current through the resistor is a known value and its row and column in the stamp are redundant.

Using equation (105), the current through the resistor can be written in terms of the phase across it.

$$I_n = \frac{\Phi_0}{2\pi R} \left(\frac{2}{h_n} (\phi_n - \phi_{n-1}) - \dot{\phi}_{n-1} \right) \quad (112a)$$

$$= \frac{\Phi_0}{2\pi R} \left(\frac{2}{h_n} (\phi_n^+ - \phi_n^- - \phi_{n-1}) - \dot{\phi}_{n-1} \right) \quad (112b)$$

$$= \frac{\Phi_0}{2\pi R} \frac{2}{h_n} \phi_n^+ - \frac{\Phi_0}{2\pi R} \frac{2}{h_n} \phi_n^- - \frac{\Phi_0}{2\pi R} \frac{2}{h_n} \phi_{n-1} - \frac{\Phi_0}{2\pi R} \dot{\phi}_{n-1} \quad (112c)$$

$$= \frac{\Phi_0}{\pi R h_n} \phi_n^+ - \frac{\Phi_0}{\pi R h_n} \phi_n^- - \frac{\Phi_0}{2\pi R} \left(\frac{2}{h_n} \phi_{n-1} + \dot{\phi}_{n-1} \right) \quad (112d)$$

$$(112e)$$

The simplified 2x2 resistor stamp is

$$\begin{bmatrix} \frac{\Phi_0}{\pi R h_n} & -\frac{\Phi_0}{\pi R h_n} \\ -\frac{\Phi_0}{\pi R h_n} & \frac{\Phi_0}{\pi R h_n} \end{bmatrix} \begin{bmatrix} \phi_n^+ \\ \phi_n^- \end{bmatrix} = \begin{bmatrix} \frac{\Phi_0}{2\pi R} \left(\frac{2}{h_n} \phi_{n-1} + \dot{\phi}_{n-1} \right) \\ -\frac{\Phi_0}{2\pi R} \left(\frac{2}{h_n} \phi_{n-1} + \dot{\phi}_{n-1} \right) \end{bmatrix} \quad (113)$$

3.2 Capacitor

The standard voltage-based equation for a capacitor is

$$i = C\dot{v} \quad (114)$$

Substituting the phase-voltage relation (equation (102))

$$I = C \frac{d}{dt} \left(\frac{\Phi_0}{2\pi} \dot{\phi} \right) \quad (115)$$

$$I = \frac{C\Phi_0}{2\pi} \ddot{\phi} \quad (116)$$

Applying the trapezoidal integration method

$$I_n = \frac{C\Phi_0}{2\pi} \left[\frac{2}{h_n} (\dot{\phi}_n - \dot{\phi}_{n-1}) - \ddot{\phi}_{n-1} \right] \quad (117)$$

Substituting equation (116)

$$I_n = \frac{C\Phi_0}{\pi h_n} (\dot{\phi}_n - \dot{\phi}_{n-1}) - I_{n-1} \quad (118)$$

Applying the trapezoidal integration method again (on $\dot{\phi}_{n-1}$)

$$I_n = \frac{C\Phi_0}{\pi h_n} \left(\frac{2}{h_n} (\phi_n - \phi_{n-1}) - \dot{\phi}_{n-1} - \dot{\phi}_{n-1} \right) - I_{n-1} \quad (119)$$

$$I_n = \frac{C\Phi_0}{\pi h_n} \left(\frac{2}{h_n} (\phi_n - \phi_{n-1}) - 2\dot{\phi}_{n-1} \right) - I_{n-1} \quad (120)$$

$$I_n = \frac{2C\Phi_0}{\pi h_n^2} \phi_n - \frac{2C\Phi_0}{\pi h_n^2} \phi_{n-1} - \frac{2C\Phi_0}{\pi h_n} \dot{\phi}_{n-1} - I_{n-1} \quad (121)$$

$$\frac{\pi h_n^2}{2C\Phi_0} I_n = \phi_n - \phi_{n-1} - h_n \dot{\phi}_{n-1} - \frac{\pi h_n^2}{2C\Phi_0} I_{n-1} \quad (122)$$

Rearranging

$$\phi_n - \frac{\pi h_n^2}{2C\Phi_0} I_n = \phi_{n-1} + h_n \dot{\phi}_{n-1} + \frac{\pi h_n^2}{2C\Phi_0} I_{n-1} \quad (123)$$

Writing the phase as the difference between the phase of the positive and negative terminals

$$\phi_n^+ - \phi_n^- - \frac{\pi h_n^2}{2C\Phi_0} I_n = \phi_{n-1}^+ - \phi_{n-1}^- + h_n (\dot{\phi}_{n-1}^+ - \dot{\phi}_{n-1}^-) + \frac{\pi h_n^2}{2C\Phi_0} I_{n-1} \quad (124)$$

The resulting phase-based capacitor stamp is

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & -1 & -\frac{\pi h_n^2}{2C\Phi_0} \end{bmatrix} \begin{bmatrix} \phi_n^+ \\ \phi_n^- \\ I_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \phi_{n-1}^+ - \phi_{n-1}^- + h_n (\dot{\phi}_{n-1}^+ - \dot{\phi}_{n-1}^-) + \frac{\pi h_n^2}{2C\Phi_0} I_{n-1} \end{bmatrix} \quad (125)$$

Remember that after each simulation cycle, each node derivative is calculated. For example,

$$\dot{\phi}_k^+ = \frac{2}{h_n} (\phi_k^+ - \phi_{k-1}^+) - \dot{\phi}_{k-1}^+ \quad (126a)$$

Therefore, the $\dot{\phi}_{n-1}^+$, $\dot{\phi}_{n-1}^-$ values will be available.

3.3 Inductor

The standard voltage-based equation for an inductor is

$$v = L\dot{i} \quad (127)$$

Substituting the phase-voltage relation (equation (102))

$$\frac{\Phi_0}{2\pi}\dot{\phi} = L\dot{i} \quad (128)$$

Assuming $\phi(0) = 0$ and $i(0) = 0$, we can integrate both sides

$$\int_0^t \frac{\Phi_0}{2\pi}\dot{\phi} = \int_0^t L\dot{i} \quad (129)$$

With respect to the current time step

$$\frac{\Phi_0}{2\pi}\phi_n = LI_n \quad (130)$$

Writing the phase as the difference between the phase of the positive and negative terminals

$$\phi_n^+ - \phi_n^- - \frac{2\pi L}{\Phi_0}I_n = 0 \quad (131)$$

The resulting phase-based inductor stamp is

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & -1 & -\frac{2\pi L}{\Phi_0} \end{bmatrix} \begin{bmatrix} \phi_n^+ \\ \phi_n^- \\ I_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (132)$$

3.4 Resistively and Capacitively Shunted Josephson Junction (RCSJ)

The equations for the JJ in phase remains the same since the phase was already calculated in the voltage-based method. So, we now calculate the phase at the positive and negative terminals (nodes) and the voltage across the JJ.

Recall equations (20d) and (22f). We can simply write the phase (ϕ) as the difference between the phase of the positive and negative nodes and leave the voltage alone. That is, we would get

$$\left(\frac{2C}{h_n} + \frac{1}{R}\right) V_n - I_n = -I_c \sin \phi_n + \frac{2C}{h_n} V_{n-1} + C\dot{V}_{n-1} \quad (133a)$$

$$-\frac{h_n}{2} \frac{2e}{\hbar} V_n + \phi_n = \phi_{n-1} + \frac{h_n}{2} \frac{2e}{\hbar} V_{n-1} \quad (134a)$$

$$-\frac{h_n}{2} \frac{2e}{\hbar} V_n + \phi_n^+ - \phi_n^- = \phi_{n-1}^+ - \phi_{n-1}^- + \frac{h_n}{2} \frac{2e}{\hbar} V_{n-1} \quad (134b)$$

The resulting phase-based Josephson Junction stamp is

$$\begin{bmatrix} 0 & 0 & \frac{1}{R} + \frac{2C}{h_n} \\ 0 & 0 & -\frac{1}{R} - \frac{2C}{h_n} \\ 1 & -1 & -\frac{h_n}{2} \frac{2e}{\hbar} \end{bmatrix} \begin{bmatrix} \phi_n^+ \\ \phi_n^- \\ V_n \end{bmatrix} = \begin{bmatrix} I_s \\ -I_s \\ \phi_{n-1}^+ - \phi_{n-1}^- + \frac{h_n}{2} \frac{2e}{\hbar} V_{n-1} \end{bmatrix} \quad (135)$$

with I_s defined as

$$I_s = -I_c \sin \phi_n^g + \frac{2C}{h_n} V_{n-1} + C\dot{V}_{n-1} \quad (136)$$

where

$$\phi_n^g = \phi_{n-1}^+ - \phi_{n-1}^- + \frac{h_n}{2} \frac{2e}{\hbar} (V_{n-1} + v_n^g) \quad (137)$$

$$v_n^g = V_{n-1} + h_n \dot{V}_{n-1} \quad (138)$$

Since the phase-based analysis JJ has the same governing equations as the voltage-based analysis JJ, we restate them below (look at Section 2.4.2.2) with a few modifications: the phase-based simulation uses the difference between the phases of the terminals and leaves the voltage terms alone.

$$\left(\frac{2C}{h_n} + \frac{1}{R}\right) V_n - I_n = I_s - I_T \quad (139)$$

$$-\frac{h_n}{2} \frac{2e}{\hbar} V_n + (\phi_n^+ - \phi_n^-) = \phi_{n-1}^+ - \phi_{n-1}^- + \frac{h_n}{2} \frac{2e}{\hbar} V_{n-1} \quad (140)$$

Below the transition region ($V_n < V_{gap} - \frac{1}{2}\Delta V$)

The governing equations in this region are

$$\left(\frac{2C}{h_n} + \frac{1}{R_{sg}}\right) V_n - I_n = I_s - I_T \quad (141)$$

$$-\frac{h_n}{2} \frac{2e}{\hbar} V_n + (\phi_n^+ - \phi_n^-) = \phi_{n-1}^+ - \phi_{n-1}^- + \frac{h_n}{2} \frac{2e}{\hbar} V_{n-1} \quad (142)$$

where $I_T = 0$.

Using the equations above, the stamp can be written as

$$\begin{bmatrix} 0 & 0 & \frac{2C}{h_n} + \frac{1}{R_{sg}} \\ 0 & 0 & -\frac{2C}{h_n} - \frac{1}{R_{sg}} \\ 1 & -1 & -\frac{h_n}{2} \frac{2e}{\hbar} \end{bmatrix} \begin{bmatrix} \phi_n^+ \\ \phi_n^- \\ V_n \end{bmatrix} = \begin{bmatrix} I_s \\ -I_s \\ \phi_{n-1}^+ - \phi_{n-1}^- + \frac{h_n}{2} \frac{2e}{\hbar} V_{n-1} \end{bmatrix} \quad (143)$$

In the transition region ($V_{gap} - \frac{1}{2}\Delta V \leq V_n < V_{gap} + \frac{1}{2}\Delta V$)

The governing equations in this region are

$$\left(\frac{2C}{h_n} + G_T\right) V_n - I_n = I_s - I_T \quad (144)$$

$$-\frac{h_n}{2} \frac{2e}{\hbar} V_n + (\phi_n^+ - \phi_n^-) = \phi_{n-1}^+ - \phi_{n-1}^- + \frac{h_n}{2} \frac{2e}{\hbar} V_{n-1} \quad (145)$$

Using the equations above, the stamp can be written as

$$\begin{bmatrix} 0 & 0 & \frac{2C}{h_n} + G_T \\ 0 & 0 & -\frac{2C}{h_n} - G_T \\ 1 & -1 & -\frac{h_n}{2} \frac{2e}{\hbar} \end{bmatrix} \begin{bmatrix} \phi_n^+ \\ \phi_n^- \\ V_n \end{bmatrix} = \begin{bmatrix} I_s - I_T \\ -I_s + I_T \\ \phi_{n-1}^+ - \phi_{n-1}^- + \frac{h_n}{2} \frac{2e}{\hbar} V_{n-1} \end{bmatrix} \quad (146)$$

where

$$G_T = \frac{I_c}{I_{c \text{ fact}} \cdot \Delta V} \quad (147)$$

$$I_T = \left(V_{\text{gap}} - \frac{1}{2} \Delta V \right) \left(\frac{1}{R_{\text{sg}}} - G_T \right) \quad (148)$$

$$I_{c \text{ fact}} = \frac{\pi}{4} \quad (149)$$

We verify that the model is continuous at $V_n = V_{\text{gap}} - \frac{1}{2} \Delta V$ in Appendix C.2.1.

Above the transition region ($V_n \geq V_{\text{gap}} + \frac{1}{2} \Delta V$)

The governing equations in this region are

$$\left(\frac{2C}{h_n} + \frac{1}{R_n} \right) V_n - I_n = I_s - I_T \quad (150)$$

$$-\frac{h_n}{2} \frac{2e}{\hbar} V_n + (\phi_n^+ - \phi_n^-) = \phi_{n-1}^+ - \phi_{n-1}^- + \frac{h_n}{2} \frac{2e}{\hbar} V_{n-1} \quad (151)$$

Using the equations above, the stamp can be written as

$$\begin{bmatrix} 0 & 0 & \frac{2C}{h_n} + \frac{1}{R_n} \\ 0 & 0 & -\frac{2C}{h_n} - \frac{1}{R_n} \\ 1 & -1 & -\frac{h_n}{2} \frac{2e}{\hbar} \end{bmatrix} \begin{bmatrix} \phi_n^+ \\ \phi_n^- \\ V_n \end{bmatrix} = \begin{bmatrix} I_s - I_T \\ -I_s + I_T \\ \phi_{n-1}^+ - \phi_{n-1}^- + \frac{h_n}{2} \frac{2e}{\hbar} V_{n-1} \end{bmatrix} \quad (152)$$

where

$$I_T = \frac{I_c}{I_{c\text{fact}}} + \left[\frac{1}{R_{sg}} \times \left(V_{gap} - \frac{1}{2} \Delta V \right) \right] - \left[\frac{1}{R_n} \times \left(V_{gap} + \frac{1}{2} \Delta V \right) \right] \quad (153)$$

$$I_{c\text{fact}} = \frac{\pi}{4} \quad (154)$$

We verify that the model is continuous at $V_n = V_{gap} + \frac{1}{2} \Delta V$ in Appendix C.2.2.

I_s is defined as

$$I_s = -I_c \sin \phi_n^g + \frac{2C}{h_n} V_{n-1} + C \dot{V}_{n-1} \quad (155)$$

where

$$\phi_n^g = \phi_{n-1} + \frac{h_n 2e}{2 \hbar} (V_{n-1} + v_n^g) \quad (156)$$

$$v_n^g = V_{n-1} + h_n \dot{V}_{n-1} \quad (157)$$

3.5 Lossless Passive Transmission Line

As in the voltage-based analysis, the time delay in terms of the current time step can be written as

$$t - T_D \equiv n - \frac{T_D}{h_n} = n - k$$

and the passive lossless transmission line can be described by the voltage-based equations

$$v_1(t) - Z_0 i_1(t) = v_2(t - T_D) + Z_0 i_2(t - T_D) \quad (158)$$

$$v_2(t) - Z_0 i_2(t) = v_1(t - T_D) + Z_0 i_1(t - T_D) \quad (159)$$

Using equation (102), equations (158) and (159) can be written as

$$\frac{\Phi_0}{2\pi}\dot{\phi}_1(t) - Z_0 i_1(t) = \frac{\Phi_0}{2\pi}\dot{\phi}_2(t - T_D) + Z_0 i_2(t - T_D) \quad (160)$$

$$\frac{\Phi_0}{2\pi}\dot{\phi}_2(t) - Z_0 i_2(t) = \frac{\Phi_0}{2\pi}\dot{\phi}_1(t - T_D) + Z_0 i_1(t - T_D) \quad (161)$$

Writing in terms of the time step

$$\frac{\Phi_0}{2\pi}(\dot{\phi}_1)_n - Z_0(I_1)_n = \frac{\Phi_0}{2\pi}(\dot{\phi}_2)_{n-k} + Z_0(I_2)_{n-k} \quad (162)$$

$$\frac{\Phi_0}{2\pi}(\dot{\phi}_2)_n - Z_0(I_2)_n = \frac{\Phi_0}{2\pi}(\dot{\phi}_1)_{n-k} + Z_0(I_1)_{n-k} \quad (163)$$

Using the trapezoidal integration method

$$\frac{\Phi_0}{2\pi} \frac{2}{h_n} ((\phi_1)_n - (\phi_1)_{n-1}) - \frac{\Phi_0}{2\pi} (\dot{\phi}_1)_{n-1} - Z_0(I_1)_n = \frac{\Phi_0}{2\pi} (\dot{\phi}_2)_{n-k} + Z_0(I_2)_{n-k} \quad (164)$$

$$\frac{\Phi_0}{2\pi} \frac{2}{h_n} ((\phi_2)_n - (\phi_2)_{n-1}) - \frac{\Phi_0}{2\pi} (\dot{\phi}_2)_{n-1} - Z_0(I_2)_n = \frac{\Phi_0}{2\pi} (\dot{\phi}_1)_{n-k} + Z_0(I_1)_{n-k} \quad (165)$$

Simplifying

$$(\phi_1)_n - (\phi_1)_{n-1} - \frac{h_n}{2} (\dot{\phi}_1)_{n-1} - \frac{\pi h_n Z_0}{\Phi_0} (I_1)_n = \frac{h_n}{2} (\dot{\phi}_2)_{n-k} + \frac{\pi h_n Z_0}{\Phi_0} (I_2)_{n-k} \quad (166)$$

$$(\phi_2)_n - (\phi_2)_{n-1} - \frac{h_n}{2} (\dot{\phi}_2)_{n-1} - \frac{\pi h_n Z_0}{\Phi_0} (I_2)_n = \frac{h_n}{2} (\dot{\phi}_1)_{n-k} + \frac{\pi h_n Z_0}{\Phi_0} (I_1)_{n-k} \quad (167)$$

Rearranging the equations

$$(\phi_1)_n - \frac{\pi h_n Z_0}{\Phi_0} (I_1)_n = (\phi_1)_{n-1} + \frac{h_n}{2} [(\dot{\phi}_1)_{n-1} + (\dot{\phi}_2)_{n-k}] + \frac{\pi h_n Z_0}{\Phi_0} (I_2)_{n-k} \quad (168)$$

$$(\phi_2)_n - \frac{\pi h_n Z_0}{\Phi_0} (I_2)_n = (\phi_2)_{n-1} + \frac{h_n}{2} [(\dot{\phi}_2)_{n-1} + (\dot{\phi}_1)_{n-k}] + \frac{\pi h_n Z_0}{\Phi_0} (I_1)_{n-k} \quad (169)$$

Writing the phase as the difference between the phase of the two terminals

$$(\phi_1)_n^+ - (\phi_1)_n^- - \frac{\pi h_n Z_0}{\Phi_0} (I_1)_n = (\phi_1)_{n-1}^+ - (\phi_1)_{n-1}^- + \frac{h_n}{2} \left[(\dot{\phi}_1)_{n-1}^+ - (\dot{\phi}_1)_{n-1}^- + (\dot{\phi}_2)_{n-k}^+ - (\dot{\phi}_2)_{n-k}^- \right] + \frac{\pi h_n Z_0}{\Phi_0} (I_2)_{n-k} \quad (170)$$

$$(\phi_2)_n^+ - (\phi_2)_n^- - \frac{\pi h_n Z_0}{\Phi_0} (I_2)_n = (\phi_2)_{n-1}^+ - (\phi_2)_{n-1}^- + \frac{h_n}{2} \left[(\dot{\phi}_2)_{n-1}^+ - (\dot{\phi}_2)_{n-1}^- + (\dot{\phi}_1)_{n-k}^+ - (\dot{\phi}_1)_{n-k}^- \right] + \frac{\pi h_n Z_0}{\Phi_0} (I_1)_{n-k} \quad (171)$$

Using the equations above, the stamp can be written as

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & -\frac{\pi h_n Z_0}{\Phi_0} & 0 \\ 0 & 0 & 1 & -1 & 0 & -\frac{\pi h_n Z_0}{\Phi_0} \end{bmatrix} \begin{bmatrix} (\phi_1)_n^+ \\ (\phi_1)_n^- \\ (\phi_2)_n^+ \\ (\phi_2)_n^- \\ (I_1)_n \\ (I_2)_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ TL_1 \\ TL_2 \end{bmatrix} \quad (172)$$

where

$$TL_1 = (\phi_1)_{n-1}^+ - (\phi_1)_{n-1}^- + \frac{h_n}{2} \left[(\dot{\phi}_1)_{n-1}^+ - (\dot{\phi}_1)_{n-1}^- + (\dot{\phi}_2)_{n-k}^+ - (\dot{\phi}_2)_{n-k}^- \right] + \frac{\pi h_n Z_0}{\Phi_0} (I_2)_{n-k} \quad (173)$$

$$TL_2 = (\phi_2)_{n-1}^+ - (\phi_2)_{n-1}^- + \frac{h_n}{2} \left[(\dot{\phi}_2)_{n-1}^+ - (\dot{\phi}_2)_{n-1}^- + (\dot{\phi}_1)_{n-k}^+ - (\dot{\phi}_1)_{n-k}^- \right] + \frac{\pi h_n Z_0}{\Phi_0} (I_1)_{n-k} \quad (174)$$

3.6 Mutual Inductance

The voltage-based equations for a mutual inductance is

$$v_1(t) = L_1 \dot{i}_1 + M \dot{i}_2(t) \quad (175)$$

$$v_2(t) = M \dot{i}_1(t) + L_2 \dot{i}_2(t) \quad (176)$$

Substituting the phase-voltage relation (equation (102))

$$\frac{\Phi_0}{2\pi} \dot{\phi}_1(t) = L_1 \dot{i}_1 + M \dot{i}_2(t) \quad (177)$$

$$\frac{\Phi_0}{2\pi} \dot{\phi}_2(t) = M \dot{i}_1(t) + L_2 \dot{i}_2(t) \quad (178)$$

Integrating both sides of both equations and assuming $\phi(0) = 0$, $i_1(0) = 0$, and $i_2(0) = 0$

$$\frac{\Phi_0}{2\pi} \phi_1(t) = L_1 i_1 + M i_2(t) \quad (179)$$

$$\frac{\Phi_0}{2\pi} \phi_2(t) = M i_1(t) + L_2 i_2(t) \quad (180)$$

Writing in terms of the time step and the difference between the phase of the two terminals

$$\frac{\Phi_0}{2\pi} (\phi_1)_n^+ - \frac{\Phi_0}{2\pi} (\phi_1)_n^- = L_1 (I_1)_n + M (I_2)_n \quad (181)$$

$$\frac{\Phi_0}{2\pi} (\phi_2)_n^+ - \frac{\Phi_0}{2\pi} (\phi_2)_n^- = M (I_1)_n + L_2 (I_2)_n \quad (182)$$

Using the equation above, the stamp can be written as

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & -\frac{2\pi L_1}{\Phi_0} & -\frac{2\pi M}{\Phi_0} \\ 0 & 0 & 1 & -1 & -\frac{2\pi M}{\Phi_0} & -\frac{2\pi L_2}{\Phi_0} \end{bmatrix} \begin{bmatrix} (\phi_1)_n^+ \\ (\phi_1)_n^- \\ (\phi_2)_n^+ \\ (\phi_2)_n^- \\ (I_1)_n \\ (I_2)_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (183)$$

It is important to note that the mutual inductance stamp holds redundant information. That is, it contains the stamps of the two inductors that it is trying to couple. In **VWSIM**, since we always stamp the two inductors into the A matrix and b vector individually, the mutual inductance stamp will be

$$\begin{bmatrix} 0 & -\frac{2\pi M}{\Phi_0} \\ -\frac{2\pi M}{\Phi_0} & 0 \end{bmatrix} \begin{bmatrix} (I_1)_n \\ (I_2)_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (184)$$

which adds the additional constraint of the mutual inductance.

3.7 Voltage Source

$$v = \frac{\Phi_0}{2\pi} \dot{\phi} \quad (185)$$

Applying the trapezoidal integration method

$$V_n = \frac{\Phi_0}{2\pi} \left[\frac{2}{h_n} (\phi_n - \phi_{n-1}) - \dot{\phi}_{n-1} \right] \quad (186)$$

$$\frac{\pi h_n}{\Phi_0} V_n = \phi_n - \phi_{n-1} - \frac{h_n}{2} \dot{\phi}_{n-1} \quad (187)$$

Substituting equation (185)

$$\frac{\pi h_n}{\Phi_0} V_n = \phi_n - \phi_{n-1} - \frac{\pi h_n}{\Phi_0} V_{n-1} \quad (188)$$

$$\phi_n = \phi_{n-1} + \frac{\pi h_n}{\Phi_0} (V_{n-1} + V_n) \quad (189)$$

Writing the phase as the difference between the phase of the positive and negative terminals

$$\phi_n^+ - \phi_n^- = \phi_{n-1}^+ - \phi_{n-1}^- + \frac{\pi h_n}{\Phi_0}(V_{n-1} + V_n) \quad (190)$$

The resulting phase-based voltage source stamp is

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} \phi_n^+ \\ \phi_n^- \\ I_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \phi_{n-1}^+ - \phi_{n-1}^- + \frac{\pi h_n}{\Phi_0}(V_{n-1} + V_n) \end{bmatrix} \quad (191)$$

We identify an equivalent stamp to use in VWSIM. Rearranging equation (187)

$$\phi_n = \phi_{n-1} + \frac{h_n}{2} \dot{\phi}_{n-1} + \frac{\pi h_n}{\Phi_0} V_n \quad (192)$$

Writing as difference between positive and negative terminals

$$\phi_n^+ - \phi_n^- = \phi_{n-1}^+ - \phi_{n-1}^- + \frac{h_n}{2}(\dot{\phi}_{n-1}^+ - \dot{\phi}_{n-1}^-) + \frac{\pi h_n}{\Phi_0} V_n \quad (193)$$

The equivalent stamp is

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} \phi_n^+ \\ \phi_n^- \\ I_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \phi_{n-1}^+ - \phi_{n-1}^- + \frac{h_n}{2}(\dot{\phi}_{n-1}^+ - \dot{\phi}_{n-1}^-) + \frac{\pi h_n}{\Phi_0} V_n \end{bmatrix} \quad (194)$$

3.8 Phase Source

A phase source in a phased-based analysis can simply be described as

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} \phi_n^+ \\ \phi_n^- \\ I_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \phi_n \end{bmatrix} \quad (195)$$

3.9 Current Source

A current source in a phase-based analysis is similar to the voltage-based representation (independent of voltage/phase)

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \phi_n^+ \\ \phi_n^- \\ I_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ A \end{bmatrix} \quad (196)$$

4 A Brief List of The Voltage-based Stamps

Here, for convenience, we present the primitive device stamps used in VWSIM.

Please note:

The stamps below should NOT be taken literally. That is, we “sum” these stamps into our final matrix that represents the circuit equations. For example, looking at the resistor stamp below, performing the dot product of the first row and the column vector yields $I_G = 0$. This is NOT true in general. This is an incomplete story. All this says is that I_G is being added to the KCL of node V_n^+ . For more information on this, read [3].

For simplicity of constructing our system of equations, we include the branch current for all types of circuit elements. Therefore, the currents going through each of the elements are added as variables in our system of equations (as seen below).

Voltage Source

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} V_n^+ \\ V_n^- \\ I_{E_n} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ E_n \end{bmatrix} \quad (197)$$

Current Source

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V_n^+ \\ V_n^- \\ I_{A_n} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ A_n \end{bmatrix} \quad (198)$$

Phase Source

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} V_n^+ \\ V_n^- \\ I_{\phi_n} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{\Phi_0}{2\pi} \frac{2}{h_n} (\phi_n - \phi_{n-1}) - V_{n-1} \end{bmatrix} \quad (199)$$

Resistor

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ G & -G & -1 \end{bmatrix} \begin{bmatrix} V_n^+ \\ V_n^- \\ I_{G_n} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (200)$$

Capacitor

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & -1 & -\frac{h_n}{2C} \end{bmatrix} \begin{bmatrix} V_n^+ \\ V_n^- \\ I_{C_n} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ V_{n-1}^+ - V_{n-1}^- + \frac{h_n}{2C} I_{C_{n-1}} \end{bmatrix} \quad (201)$$

Inductor

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & -1 & -\frac{2L}{h_n} \end{bmatrix} \begin{bmatrix} V_n^+ \\ V_n^- \\ I_{L_n} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -(V_{n-1}^+ - V_{n-1}^-) - \frac{2L}{h_n} I_{L_{n-1}} \end{bmatrix} \quad (202)$$

Josephson Junction

Below the transition region ($V_n < V_{gap} - \frac{1}{2}\Delta V$)

$$\begin{bmatrix} \frac{1}{R_{sg}} + \frac{2C}{h_n} & -\frac{1}{R_{sg}} - \frac{2C}{h_n} & 0 \\ -\frac{1}{R_{sg}} - \frac{2C}{h_n} & \frac{1}{R_{sg}} + \frac{2C}{h_n} & 0 \\ -\frac{h_n}{2} \frac{2e}{\hbar} & \frac{h_n}{2} \frac{2e}{\hbar} & 1 \end{bmatrix} \begin{bmatrix} V_n^+ \\ V_n^- \\ \phi_n \end{bmatrix} = \begin{bmatrix} I_s \\ -I_s \\ \phi_{n-1} + \frac{h_n}{2} \frac{2e}{\hbar} (V_{n-1}^+ - V_{n-1}^-) \end{bmatrix} \quad (203)$$

In the transition region ($V_{gap} - \frac{1}{2}\Delta V \leq V_n < V_{gap} + \frac{1}{2}\Delta V$)

$$\begin{bmatrix} G_T + \frac{2C}{h_n} & -G_T - \frac{2C}{h_n} & 0 \\ -G_T - \frac{2C}{h_n} & G_T + \frac{2C}{h_n} & 0 \\ -\frac{h_n}{2} \frac{2e}{\hbar} & \frac{h_n}{2} \frac{2e}{\hbar} & 1 \end{bmatrix} \begin{bmatrix} V_n^+ \\ V_n^- \\ \phi_n \end{bmatrix} = \begin{bmatrix} I_s - I_T \\ -I_s + I_T \\ \phi_{n-1} + \frac{h_n}{2} \frac{2e}{\hbar} (V_{n-1}^+ - V_{n-1}^-) \end{bmatrix} \quad (204)$$

where

$$G_T = \frac{I_c}{I_{c\text{fact}} \cdot \Delta V} \quad (205)$$

$$I_T = (V_{\text{gap}} - \frac{1}{2}\Delta V) \left(\frac{1}{R_{\text{sg}}} - G_T \right) \quad (206)$$

$$I_{c\text{fact}} = \frac{\pi}{4} \quad (207)$$

Above the transition region ($V_n \geq V_{\text{gap}} + \frac{1}{2}\Delta V$)

$$\begin{bmatrix} \frac{1}{R_n} + \frac{2C}{h_n} & -\frac{1}{R_n} - \frac{2C}{h_n} & 0 \\ -\frac{1}{R_n} - \frac{2C}{h_n} & \frac{1}{R_n} + \frac{2C}{h_n} & 0 \\ -\frac{h_n}{2} \frac{2e}{h} & \frac{h_n}{2} \frac{2e}{h} & 1 \end{bmatrix} \begin{bmatrix} V_n^+ \\ V_n^- \\ \phi_n \end{bmatrix} = \begin{bmatrix} I_s - I_T \\ -I_s + I_T \\ \phi_{n-1} + \frac{h_n}{2} \frac{2e}{h} (V_{n-1}^+ - V_{n-1}^-) \end{bmatrix} \quad (208)$$

where

$$I_T = \frac{I_c}{I_{c\text{fact}}} + \left[\frac{1}{R_{\text{sg}}} \times \left(V_{\text{gap}} - \frac{1}{2}\Delta V \right) \right] - \left[\frac{1}{R_n} \times \left(V_{\text{gap}} + \frac{1}{2}\Delta V \right) \right] \quad (209)$$

I_s is defined as

$$I_s = -I_c \sin \phi_n^g + \frac{2C}{h_n} (V_{n-1}^+ - V_{n-1}^-) + C(\dot{V}_{n-1}^+ - \dot{V}_{n-1}^-) \quad (210)$$

where

$$\phi_n^g = \phi_{n-1} + \frac{h_n}{2} \frac{2e}{h} (V_{n-1}^+ - V_{n-1}^- + v_n^g) \quad (211)$$

$$v_n^g = V_{n-1}^+ - V_{n-1}^- + h_n(\dot{V}_{n-1}^+ - \dot{V}_{n-1}^-) \quad (212)$$

Passive Transmission Line

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & -Z_0 & 0 \\ 0 & 0 & 1 & -1 & 0 & -Z_0 \end{bmatrix} \begin{bmatrix} (V_1)_n^+ \\ (V_1)_n^- \\ (V_2)_n^+ \\ (V_2)_n^- \\ (I_1)_n \\ (I_2)_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ (V_2)_{n-k}^+ - (V_2)_{n-k}^- + Z_0(I_2)_{n-k} \\ (V_1)_{n-k}^+ - (V_1)_{n-k}^- + Z_0(I_1)_{n-k} \end{bmatrix} \quad (213)$$

Mutual Inductance

$$\begin{bmatrix} 0 & -\frac{2M}{h_n} \\ -\frac{2M}{h_n} & 0 \end{bmatrix} \begin{bmatrix} (I_1)_n \\ (I_2)_n \end{bmatrix} = \begin{bmatrix} -\frac{2M}{h_n}(I_2)_{n-1} \\ -\frac{2M}{h_n}(I_1)_{n-1} \end{bmatrix} \quad (214)$$

5 A Brief List of The Phase-based Stamps

The stamps below should NOT be taken literally. That is, we “sum” these stamps into our final matrix that represents the circuit equations. For example, looking at the resistor stamp below, performing the dot product of the first row and the column vector yields $I_G = 0$. This is NOT true in general. This is an incomplete story. All this says is that I_G is being added to the KCL of node V_n^+ .

For simplicity of constructing our system of equations, we include the branch current for all types of circuit elements. Therefore, the currents going through each of the elements are added as variables in our system of equations (as seen below).

Voltage Source

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} \phi_n^+ \\ \phi_n^- \\ I_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \phi_{n-1}^+ - \phi_{n-1}^- + \frac{h_n}{2}(\dot{\phi}_{n-1}^+ - \dot{\phi}_{n-1}^-) + \frac{\pi h_n}{\Phi_0} V_n \end{bmatrix} \quad (215)$$

Phase Source

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} \phi_n^+ \\ \phi_n^- \\ I_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \phi_n \end{bmatrix} \quad (216)$$

Current Source

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \phi_n^+ \\ \phi_n^- \\ I_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ A \end{bmatrix} \quad (217)$$

Resistor

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & -1 & -\frac{\pi h_n R}{\Phi_0} \end{bmatrix} \begin{bmatrix} \phi_n^+ \\ \phi_n^- \\ I_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \phi_{n-1}^+ - \phi_{n-1}^- + \frac{\pi h_n R}{\Phi_0} I_{n-1} \end{bmatrix} \quad (218)$$

Capacitor

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & -1 & -\frac{\pi h_n^2}{2C\Phi_0} \end{bmatrix} \begin{bmatrix} \phi_n^+ \\ \phi_n^- \\ I_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \phi_{n-1}^+ - \phi_{n-1}^- + h_n(\dot{\phi}_{n-1}^+ - \dot{\phi}_{n-1}^-) + \frac{\pi h_n^2}{2C\Phi_0} I_{n-1} \end{bmatrix} \quad (219)$$

Inductor

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & -1 & -\frac{2\pi L}{\Phi_0} \end{bmatrix} \begin{bmatrix} \phi_n^+ \\ \phi_n^- \\ I_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (220)$$

Josephson Junction

Below the transition region ($V_n < V_{gap} - \frac{1}{2}\Delta V$)

$$\begin{bmatrix} 0 & 0 & \frac{2C}{h_n} + \frac{1}{R_{sg}} \\ 0 & 0 & -\frac{2C}{h_n} - \frac{1}{R_{sg}} \\ 1 & -1 & -\frac{h_n}{2} \frac{2e}{\hbar} \end{bmatrix} \begin{bmatrix} \phi_n^+ \\ \phi_n^- \\ V_n \end{bmatrix} = \begin{bmatrix} I_s \\ -I_s \\ \phi_{n-1}^+ - \phi_{n-1}^- + \frac{h_n}{2} \frac{2e}{\hbar} V_{n-1} \end{bmatrix} \quad (221)$$

In the transition region ($V_{gap} - \frac{1}{2}\Delta V \leq V_n < V_{gap} + \frac{1}{2}\Delta V$)

$$\begin{bmatrix} 0 & 0 & \frac{2C}{h_n} + G_T \\ 0 & 0 & -\frac{2C}{h_n} - G_T \\ 1 & -1 & -\frac{h_n}{2} \frac{2e}{\hbar} \end{bmatrix} \begin{bmatrix} \phi_n^+ \\ \phi_n^- \\ V_n \end{bmatrix} = \begin{bmatrix} I_s - I_T \\ -I_s + I_T \\ \phi_{n-1}^+ - \phi_{n-1}^- + \frac{h_n}{2} \frac{2e}{\hbar} V_{n-1} \end{bmatrix} \quad (222)$$

where

$$G_T = \frac{I_c}{I_{c\text{fact}} \cdot \Delta V} \quad (223)$$

$$I_T = \left(V_{gap} - \frac{1}{2}\Delta V \right) \left(\frac{1}{R_{sg}} - G_T \right) \quad (224)$$

$$I_{c fact} = \frac{\pi}{4} \quad (225)$$

Above the transition region ($V_n \geq V_{gap} + \frac{1}{2}\Delta V$)

$$\begin{bmatrix} 0 & 0 & \frac{2C}{h_n} + \frac{1}{R_n} \\ 0 & 0 & -\frac{2C}{h_n} - \frac{1}{R_n} \\ 1 & -1 & -\frac{h_n}{2} \frac{2e}{\hbar} \end{bmatrix} \begin{bmatrix} \phi_n^+ \\ \phi_n^- \\ V_n \end{bmatrix} = \begin{bmatrix} I_s - I_T \\ -I_s + I_T \\ \phi_{n-1}^+ - \phi_{n-1}^- + \frac{h_n}{2} \frac{2e}{\hbar} V_{n-1} \end{bmatrix} \quad (226)$$

where

$$I_T = \frac{I_c}{I_{c fact}} + \left[\frac{1}{R_{sg}} \times \left(V_{gap} - \frac{1}{2}\Delta V \right) \right] - \left[\frac{1}{R_n} \times \left(V_{gap} + \frac{1}{2}\Delta V \right) \right] \quad (227)$$

$$I_{c fact} = \frac{\pi}{4} \quad (228)$$

I_s is defined as

$$I_s = -I_c \sin \phi_n^g + \frac{2C}{h_n} V_{n-1} + C \dot{V}_{n-1} \quad (229)$$

where

$$\phi_n^g = \phi_{n-1} + \frac{h_n}{2} \frac{2e}{\hbar} (V_{n-1} + v_n^g) \quad (230)$$

$$v_n^g = V_{n-1} + h_n \dot{V}_{n-1} \quad (231)$$

Passive Transmission Line

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & -\frac{\pi h_n Z_0}{\Phi_0} & 0 \\ 0 & 0 & 1 & -1 & 0 & -\frac{\pi h_n Z_0}{\Phi_0} \end{bmatrix} \begin{bmatrix} (\phi_1)_n^+ \\ (\phi_1)_n^- \\ (\phi_2)_n^+ \\ (\phi_2)_n^- \\ (I_1)_n \\ (I_2)_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ TL_1 \\ TL_2 \end{bmatrix} \quad (232)$$

where

$$TL_1 = (\phi_1)_{n-1}^+ - (\phi_1)_{n-1}^- + \frac{h_n}{2} \left[(\dot{\phi}_1)_{n-1}^+ - (\dot{\phi}_1)_{n-1}^- + (\dot{\phi}_2)_{n-k}^+ - (\dot{\phi}_2)_{n-k}^- \right] + \frac{\pi h_n Z_0}{\Phi_0} (I_2)_{n-k} \quad (233)$$

$$TL_2 = (\phi_2)_{n-1}^+ - (\phi_2)_{n-1}^- + \frac{h_n}{2} \left[(\dot{\phi}_2)_{n-1}^+ - (\dot{\phi}_2)_{n-1}^- + (\dot{\phi}_1)_{n-k}^+ - (\dot{\phi}_1)_{n-k}^- \right] + \frac{\pi h_n Z_0}{\Phi_0} (I_1)_{n-k} \quad (234)$$

Mutual Inductance

$$\begin{bmatrix} 0 & -\frac{2\pi M}{\Phi_0} \\ -\frac{2\pi M}{\Phi_0} & 0 \end{bmatrix} \begin{bmatrix} (I_1)_n \\ (I_2)_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (235)$$

6 Input Source Waveforms

This section presents the equations of the input sources that are implemented in the VWSIM simulator. The equations are primarily provided by the WRSpice manual [9].

6.1 Piecewise Linear

Parameters

parameter	description	units
$t1$	initial time	seconds
$v1$	initial value	volts, amps, or phase
...	...	...
tn	n^{th} time	seconds
vn	value at tn	volts, amps, or phase

A piecewise linear (PWL) can be specified as

time equation

$t1$	$v1$
$t2$	$v2$
...	...
tn	vn

Each pair of values (ti, vi) specifies that the value of the source is vi at time = ti . The value of the source at intermediate values of time is determined by using linear interpolation on the input values. For times after the final time value, the value of the source is the final value [9].

6.2 Exponential

Parameters

parameter	description	default	units
$v1$	initial value		volts, amps, or phase
$v2$	pulsed value		volts, amps, or phase
$td1$	rise delay time	0.0	seconds
$\tau1$	rise time constant	tstep	seconds
$td2$	fall delay time	$td1 + tstep$	seconds
$\tau2$	fall time constant	tstep	seconds

An exponential waveform can be described as

time	equation
0	$v1$
$td1$	$v1 + (v2 - v1)(1 - e^{-(time - td1)/tau1})$
$td2$	$v1 + (v2 - v1)(1 - e^{-(time - td1)/tau1}) + (v1 - v2)(1 - e^{-(time - td2)/tau2})$

6.3 Sinusoidal

Unlike JoSIM, the WRSpice model allows users to specify a phase delay. Therefore, we decided to implement the sinusoidal input source detailed in the WRSpice manual.

Parameters

parameter	description	default	units
vo	offset		volts, amps, or phase
va	amplitude		volts, amps, or phase
$freq$	frequency	1/tstop	hz
td	delay	0.0	seconds
$theta$	damping factor	0.0	hz
phi	phase delay	0.0	degrees

time	equation
0 to td	$vo + va \cdot \sin(\pi \cdot phi/180)$
td to tstop	$vo + va \cdot e^{-(time - td) \cdot theta} \cdot \sin(2\pi \cdot (freq \cdot (time - td)) + \pi \cdot phi/360)$

6.4 Pulse

parameter	description	default	units
$v1$	initial value		volts, amps, or phase
$v2$	pulsed value		volts, amps, or phase
td	time delay	0.0	seconds
tr	rise time	tstep	seconds
tf	fall time	tstep	seconds
pw	pulse width	tstep	seconds
per	period	tstep	seconds

A single pulse with no period can be specified as

time	equation
0	$v1$
td	$v1$
$td + tr$	$v2$
$td + tr + pw$	$v2$
$td + tr + pw + tf$	$v1$
tstop	$v1$

If a period is specified, a train of pulses is generated. Each pulse in the train will start at $td + (k * per)$ until tstop is reached.

Note: the minimum value for per is $tr + tf + pw$.

7 Simplified 2x2 Stamps

This section explains how the stamps of some primitive devices, such as capacitors and inductors, can be reduced in size from 3x3 to 2x2 matrices. Instead of solving for the current through the device in the matrix equation, we instead calculate the current after the voltage/phase of nodes and their corresponding derivatives have been calculated. Since the time to solve a matrix equation cubes with the dimension of the matrix, we expect the reduction in stamp size to improve the speed and performance of the simulator.

Recall that the matrix equation represents a system of equations. Each of these equations either contribute to the KCL or specify constraints on each node. Now, consider the following stamp:

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} V_n^+ \\ V_n^- \\ I_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ expr \end{bmatrix} \quad (236)$$

In the stamp above, as demanded by KCL, I_n is the contribution of current into or out of the device's two terminals. Notice that the last row, with $expr$ in the “b” vector, defines a constraint on the nodes V_n^+ and V_n^- . Meanwhile, the current I_n is treated as an unknown variable. We will show that keeping I_n in the stamp is unnecessary and redundant, given a history of node values, currents, and derivatives if necessary.

7.1 2x2 Capacitor

7.1.1 Voltage-based

The current through a capacitor in voltage-mode can be written as follows

$$I_n = C \left[\frac{2}{h_n} (V_n - V_{n-1}) - \dot{V}_{n-1} \right] \quad (237)$$

and simplifying

$$I_n = \frac{2C}{h_n} V_n - \frac{2C}{h_n} V_{n-1} - C \dot{V}_{n-1} \quad (238)$$

Finally, writing the voltage as the difference between the voltage of the positive and negative terminals

$$I_n = \frac{2C}{h_n} (V_n^+ - V_n^-) - \frac{2C}{h_n} (V_{n-1}^+ - V_{n-1}^-) - C (\dot{V}_{n-1}^+ - \dot{V}_{n-1}^-) \quad (239)$$

Thus, the capacitor stamp can be written as

$$\begin{bmatrix} \frac{2C}{h_n} & -\frac{2C}{h_n} \\ -\frac{2C}{h_n} & \frac{2C}{h_n} \end{bmatrix} \begin{bmatrix} V_n^+ \\ V_n^- \end{bmatrix} = \begin{bmatrix} \frac{2C}{h_n} (V_{n-1}^+ - V_{n-1}^-) + C (\dot{V}_{n-1}^+ - \dot{V}_{n-1}^-) \\ -(\frac{2C}{h_n} (V_{n-1}^+ - V_{n-1}^-) + C (\dot{V}_{n-1}^+ - \dot{V}_{n-1}^-)) \end{bmatrix} \quad (240)$$

7.1.2 Phase-based

In phase-based simulation, the current through a capacitor can be written as

$$I_n = \frac{2C\Phi_0}{\pi h_n^2} \phi_n - \frac{2C\Phi_0}{\pi h_n^2} \phi_{n-1} - \frac{2C\Phi_0}{\pi h_n} \dot{\phi}_{n-1} - I_{n-1} \quad (241)$$

Writing the phase as the difference between the phase of the positive and negative terminals

$$I_n = \frac{2C\Phi_0}{\pi h_n^2} (\phi_n^+ - \phi_n^-) - \frac{2C\Phi_0}{\pi h_n^2} (\phi_{n-1}^+ - \phi_{n-1}^-) - \frac{2C\Phi_0}{\pi h_n} (\dot{\phi}_{n-1}^+ - \dot{\phi}_{n-1}^-) - I_{n-1} \quad (242)$$

Thus, the capacitor stamp can be written as

$$\begin{bmatrix} \frac{2C\Phi_0}{\pi h_n^2} & -\frac{2C\Phi_0}{\pi h_n^2} \\ -\frac{2C\Phi_0}{\pi h_n^2} & \frac{2C\Phi_0}{\pi h_n^2} \end{bmatrix} \begin{bmatrix} \phi_n^+ \\ \phi_n^- \end{bmatrix} = \begin{bmatrix} \frac{2C\Phi_0}{\pi h_n^2}(\phi_{n-1}^+ - \phi_{n-1}^-) + \frac{2C\Phi_0}{\pi h_n}(\dot{\phi}_{n-1}^+ - \dot{\phi}_{n-1}^-) + I_{n-1} \\ -(\frac{2C\Phi_0}{\pi h_n^2}(\phi_{n-1}^+ - \phi_{n-1}^-) + \frac{2C\Phi_0}{\pi h_n}(\dot{\phi}_{n-1}^+ - \dot{\phi}_{n-1}^-) + I_{n-1}) \end{bmatrix} \quad (243)$$

7.2 2x2 Inductor

7.2.1 Voltage-based

From equation (14d), the voltage across an inductor is

$$V_n = \frac{2L}{h_n} I_n - \frac{2L}{h_n} I_{n-1} - V_{n-1} \quad (244)$$

Rewriting for the current through an inductor,

$$I_n = \frac{h_n}{2L} V_n + \frac{h_n}{2L} V_{n-1} + I_{n-1} \quad (245)$$

Writing the voltage as the difference between the voltage of the positive and negative terminals

$$I_n = \frac{h_n}{2L} (V_n^+ - V_n^-) + \frac{h_n}{2L} (V_{n-1}^+ - V_{n-1}^-) + I_{n-1} \quad (246)$$

Thus, the inductor stamp can be written as

$$\begin{bmatrix} \frac{h_n}{2L} & -\frac{h_n}{2L} \\ -\frac{h_n}{2L} & \frac{h_n}{2L} \end{bmatrix} \begin{bmatrix} V_n^+ \\ V_n^- \end{bmatrix} = \begin{bmatrix} -(\frac{h_n}{2L} (V_{n-1}^+ - V_{n-1}^-) + I_{n-1}) \\ \frac{h_n}{2L} (V_{n-1}^+ - V_{n-1}^-) + I_{n-1} \end{bmatrix} \quad (247)$$

7.2.2 Phase-based

From equation (130),

$$\frac{\Phi_0}{2\pi} \phi_n = L I_n \quad (248)$$

Rewriting for the current through an inductor,

$$I_n = \frac{\Phi_0}{2\pi L} \phi_n \quad (249)$$

Writing the phase as the difference between the phase of the positive and negative terminals

$$I_n = \frac{\Phi_0}{2\pi L} (\phi_n^+ - \phi_n^-) \quad (250)$$

Thus, the inductor stamp can be written as

$$\begin{bmatrix} \frac{\Phi_0}{2\pi L} & -\frac{\Phi_0}{2\pi L} \\ -\frac{\Phi_0}{2\pi L} & \frac{\Phi_0}{2\pi L} \end{bmatrix} \begin{bmatrix} \phi_n^+ \\ \phi_n^- \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (251)$$

7.3 2x2 Mutual Inductance

7.3.1 Voltage-based

From equations (85) and (90),

$$(V_1)_n^+ - (V_1)_n^- - \frac{2L_1}{h_n} (I_1)_n - \frac{2M}{h_n} (I_2)_n = -\frac{2L_1}{h_n} (I_1)_{n-1} - \frac{2M}{h_n} (I_2)_{n-1} - (V_1)_{n-1}^+ + (V_1)_{n-1}^- \quad (252)$$

$$(V_2)_n^+ - (V_2)_n^- - \frac{2M}{h_n} (I_1)_n - \frac{2L_2}{h_n} (I_2)_n = -\frac{2M}{h_n} (I_1)_{n-1} - \frac{2L_2}{h_n} (I_2)_{n-1} - (V_2)_{n-1}^+ + (V_2)_{n-1}^- \quad (253)$$

Multiplying equation (252) by $-\frac{L_2}{M}$,

$$-\frac{L_2}{M} (V_1)_n^+ + \frac{L_2}{M} (V_1)_n^- + \frac{2L_1 L_2}{M h_n} (I_1)_n + \frac{2L_2}{h_n} (I_2)_n = \frac{2L_1 L_2}{M h_n} (I_1)_{n-1} + \frac{2L_2}{h_n} (I_2)_{n-1} + \frac{L_2}{M} (V_1)_{n-1}^+ - \frac{L_2}{M} (V_1)_{n-1}^- \quad (254)$$

Adding equations (253) and (254),

$$\begin{aligned}
& -\frac{L_2}{M}(V_1)_n^+ + \frac{L_2}{M}(V_1)_n^- + \frac{2L_1L_2}{Mh_n}(I_1)_n + (V_2)_n^+ - (V_2)_n^- - \frac{2M}{h_n}(I_1)_n = \\
& \frac{2L_1L_2}{Mh_n}(I_1)_{n-1} + \frac{L_2}{M}(V_1)_{n-1}^+ - \frac{L_2}{M}(V_1)_{n-1}^- + -\frac{2M}{h_n}(I_1)_{n-1} - (V_2)_{n-1}^+ + (V_2)_{n-1}^-
\end{aligned} \tag{255}$$

$$\begin{aligned}
& \left(\frac{2L_1L_2}{Mh_n} - \frac{2M}{h_n}\right)(I_1)_n = \\
& \frac{L_2}{M}(V_1)_n^+ - \frac{L_2}{M}(V_1)_n^- - (V_2)_n^+ + (V_2)_n^- + \\
& \frac{2L_1L_2}{Mh_n}(I_1)_{n-1} + \frac{L_2}{M}(V_1)_{n-1}^+ - \frac{L_2}{M}(V_1)_{n-1}^- + -\frac{2M}{h_n}(I_1)_{n-1} - (V_2)_{n-1}^+ + (V_2)_{n-1}^-
\end{aligned} \tag{256}$$

$$\begin{aligned}
& (I_1)_n = \\
& \left(\frac{Mh_n}{2L_1L_2 - 2M^2}\right)\left(\frac{L_2}{M}(V_1)_n^+ - \frac{L_2}{M}(V_1)_n^- - (V_2)_n^+ + (V_2)_n^- + \right. \\
& \left. \frac{2L_1L_2}{Mh_n}(I_1)_{n-1} + \frac{L_2}{M}(V_1)_{n-1}^+ - \frac{L_2}{M}(V_1)_{n-1}^- + -\frac{2M}{h_n}(I_1)_{n-1} - (V_2)_{n-1}^+ + (V_2)_{n-1}^- \right)
\end{aligned} \tag{257}$$

$$\begin{aligned}
& (I_1)_n = \\
& \frac{1}{1-k^2} \frac{h_n k^2}{2M} \left(\frac{L_2}{M}(V_1)_n^+ - \frac{L_2}{M}(V_1)_n^- - (V_2)_n^+ + (V_2)_n^- + \right. \\
& \left. \frac{2L_1L_2}{Mh_n}(I_1)_{n-1} + \frac{L_2}{M}(V_1)_{n-1}^+ - \frac{L_2}{M}(V_1)_{n-1}^- + -\frac{2M}{h_n}(I_1)_{n-1} - (V_2)_{n-1}^+ + (V_2)_{n-1}^- \right)
\end{aligned} \tag{258}$$

$$\begin{aligned}
(I_1)_n = & \frac{1}{1-k^2} \left(\frac{h_n k^2}{2M} \frac{L_2}{M}(V_1)_n^+ - \frac{h_n k^2}{2M} \frac{L_2}{M}(V_1)_n^- - \frac{h_n k^2}{2M} (V_2)_n^+ + \right. \\
& \frac{h_n k^2}{2M} (V_2)_n^- + \frac{h_n k^2}{2M} \frac{2L_1L_2}{Mh_n}(I_1)_{n-1} + \frac{h_n k^2}{2M} \frac{L_2}{M}(V_1)_{n-1}^+ - \\
& \frac{h_n k^2}{2M} \frac{L_2}{M}(V_1)_{n-1}^- - \frac{h_n k^2}{2M} \frac{2M}{h_n}(I_1)_{n-1} - \\
& \left. \frac{h_n k^2}{2M} (V_2)_{n-1}^+ + \frac{h_n k^2}{2M} (V_2)_{n-1}^- \right)
\end{aligned} \tag{259}$$

$$\begin{aligned}
(I_1)_n = & \frac{1}{1-k^2} \left(\frac{h_n k^2 L_2}{2M^2} (V_1)_n^+ - \frac{h_n k^2 L_2}{2M^2} (V_1)_n^- - \frac{h_n k^2}{2M} (V_2)_n^+ + \right. \\
& \frac{h_n k^2}{2M} (V_2)_n^- + \frac{h_n k^2 2L_1 L_2}{2M^2 h_n} (I_1)_{n-1} + \frac{h_n k^2 L_2}{2M^2} (V_1)_{n-1}^+ - \\
& \frac{h_n k^2 L_2}{2M^2} (V_1)_{n-1}^- - \frac{h_n k^2 2M}{2M h_n} (I_1)_{n-1} - \\
& \left. \frac{h_n k^2}{2M} (V_2)_{n-1}^+ + \frac{h_n k^2}{2M} (V_2)_{n-1}^- \right)
\end{aligned} \tag{260}$$

Using $M^2 = k^2 L_1 L_2$,

$$\begin{aligned}
(I_1)_n = & \frac{1}{1-k^2} \left(\frac{h_n}{2L_1} (V_1)_n^+ - \frac{h_n}{2L_1} (V_1)_n^- - \frac{h_n}{2L_1 L_2} (V_2)_n^+ + \right. \\
& \frac{h_n}{2L_1 L_2} (V_2)_n^- + (I_1)_{n-1} + \frac{h_n}{2L_1} (V_1)_{n-1}^+ - \\
& \frac{h_n}{2L_1} (V_1)_{n-1}^- - k^2 (I_1)_{n-1} - \\
& \left. \frac{h_n}{2L_1 L_2} (V_2)_{n-1}^+ + \frac{h_n}{2L_1 L_2} (V_2)_{n-1}^- \right)
\end{aligned} \tag{261}$$

$$\begin{aligned}
(I_1)_n = & \frac{1}{1-k^2} \left(\frac{h_n}{2L_1} ((V_1)_n^+ - (V_1)_n^-) - \frac{h_n}{2L_1 L_2} (V_2)_n^+ + \right. \\
& \frac{h_n}{2L_1 L_2} (V_2)_n^- + (1-k^2)(I_1)_{n-1} + \frac{h_n}{2L_1} ((V_1)_{n-1}^+ - (V_1)_{n-1}^-) - \\
& \left. \frac{h_n}{2L_1 L_2} (V_2)_{n-1}^+ + \frac{h_n}{2L_1 L_2} (V_2)_{n-1}^- \right)
\end{aligned} \tag{262}$$

$$\begin{aligned}
(I_1)_n = & \frac{1}{1-k^2} \left(\frac{h_n}{2L_1} ((V_1)_n^+ - (V_1)_n^-) + \frac{h_n}{2L_1} ((V_1)_{n-1}^+ - (V_1)_{n-1}^-) + (1-k^2)(I_1)_{n-1} \right. \\
& \left. - \frac{h_n}{2L_1 L_2} ((V_2)_n^+ - (V_2)_n^- + (V_2)_{n-1}^+ - (V_2)_{n-1}^-) \right)
\end{aligned} \tag{263}$$

7.4 4x4 Lossless Passive Transmission Line

Employing the same techniques used on the inductor and capacitor, the transmission line's stamp can be reduced from 6x6 to 4x4.

7.4.1 Voltage-based

The transmission line equations are

$$(V_1)_n - Z_0(I_1)_n = (V_2)_{n-k} + Z_0(I_2)_{n-k} \quad (264)$$

$$(V_2)_n - Z_0(I_2)_n = (V_1)_{n-k} + Z_0(I_1)_{n-k} \quad (265)$$

Rewriting in terms of the two transmission line currents,

$$(I_1)_n = \frac{1}{Z_0}(V_1)_n - \frac{1}{Z_0}(V_2)_{n-k} - (I_2)_{n-k} \quad (266)$$

$$(I_2)_n = \frac{1}{Z_0}(V_2)_n - \frac{1}{Z_0}(V_1)_{n-k} - (I_1)_{n-k} \quad (267)$$

Writing the voltage as the difference between the voltage of the positive and negative terminals

$$(I_1)_n = \frac{1}{Z_0}((V_1)_n^+ - (V_1)_n^-) - \frac{1}{Z_0}((V_2)_{n-k}^+ - (V_2)_{n-k}^-) - (I_2)_{n-k} \quad (268)$$

$$(I_2)_n = \frac{1}{Z_0}((V_2)_n^+ - (V_2)_n^-) - \frac{1}{Z_0}((V_1)_{n-k}^+ - (V_1)_{n-k}^-) - (I_1)_{n-k} \quad (269)$$

$$\begin{bmatrix} \frac{1}{Z_0} & -\frac{1}{Z_0} & 0 & 0 \\ -\frac{1}{Z_0} & \frac{1}{Z_0} & 0 & 0 \\ 0 & 0 & \frac{1}{Z_0} & -\frac{1}{Z_0} \\ 0 & 0 & -\frac{1}{Z_0} & \frac{1}{Z_0} \end{bmatrix} \begin{bmatrix} (V_1)_n^+ \\ (V_1)_n^- \\ (V_2)_n^+ \\ (V_2)_n^- \end{bmatrix} = \begin{bmatrix} \frac{1}{Z_0}((V_2)_{n-k}^+ - (V_2)_{n-k}^-) + (I_2)_{n-k} \\ -\frac{1}{Z_0}((V_2)_{n-k}^+ - (V_2)_{n-k}^-) + (I_2)_{n-k} \\ \frac{1}{Z_0}((V_1)_{n-k}^+ - (V_1)_{n-k}^-) + (I_1)_{n-k} \\ -\frac{1}{Z_0}((V_1)_{n-k}^+ - (V_1)_{n-k}^-) + (I_1)_{n-k} \end{bmatrix} \quad (270)$$

7.4.2 Phase-based

References

- [1] Delport: “Simulation and Verification Software for Superconducting Electronic Circuits”, Stellenbosch University (2019)
- [2] M. Kaufmann and J S. Moore: ACL2 Home Page. <http://www.cs.utexas.edu/users/moore/acl2> (2020)
- [3] Ho, Ruehli, Brennan: “The Modified Nodal Approach to Network Analysis”, IEEE Transactions on Circuits and Systems, Vol. CAS-22, No. 6 (1975)
- [4] Haberkorn, Knauer, Richter: “A Theoretical Study of the Current-Phase Relation in Josephson Contacts”, Physica Status Solidi (a), Vol. 47, No. 2 (1978)
- [5] Golubov, Kupriyanov, Il’ichev: “The Current-Phase Relation in Josephson Junctions”, Reviews of Modern Physics, Vol. 76 (2004)
- [6] Orlando, Delin: “Foundations of Applied Superconductivity”, Addison-Wesley, First Edition (1991)
- [7] Bunyk, Likharev, and Zinoviev: “RSFQ Technology: Physics and Devices”, International Journal of High Speed Electronics and Systems, Vol. 11, No. 1 (2001)
- [8] MIT Lincoln Laboratory: “MIT LL 100 $\mu A/\mu m^2$ Superconductor Electronics Fabrication Process SFQ5ee: Model Guide”, (2018)
- [9] Whiteley Research Incorporated: “WRspice Reference Manual”, Release 4.3.13 (2021)

Appendices

A Backward Euler Method

This appendix section results from the fact that Modified Nodal Analysis (MNA) uses the Euler method to solve ordinary differential equations (ODE), but JoSIM uses the trapezoidal integration method. Why were these choices made?

For the following methods, we will attempt to solve the equation of a capacitor as an example.

$$i_c(t) = C \frac{dv_c}{dt} \quad (271)$$

A.1 Backward Euler Method (The MNA Technique)

Let h be the time step we use to discretize our ODE. We begin with the finite difference approximation

$$y'(t) \approx \frac{y(t) - y(t-h)}{h} \quad (272)$$

Then, we can approximate

$$\frac{dv_c}{dt} \approx \frac{v_c(t) - v_c(t-h)}{h} \quad (273)$$

Consider time $t-h$ to be the $n-1$ step of applying this process. Then, t would be the n th step. Using equation (271)

$$i_{c_n} = \frac{C}{h} (v_{c_n} - v_{c_{n-1}}) \quad (274)$$

$$i_{c_n} = \frac{C}{h} v_{c_n} - \frac{C}{h} v_{c_{n-1}} \quad (275)$$

Let $v_{cp} = (v_c)_{n-1}$ be the voltage at the previous time step. Splitting the voltage as the difference between the two nodes of the element

$$i_{c_n} = \frac{C}{h} (v_{c_n}^+ - v_{c_n}^-) - \frac{C}{h} v_{cp} \quad (276)$$

Rearranging the equation

$$\frac{C}{h}v_{c_n}^+ - \frac{C}{h}v_{c_n}^- - i_{c_n} = \frac{C}{h}v_{cp} \quad (277)$$

Equation (277) is exactly the equation represented by the last row of the conductor stamp in [2].

A.2 Trapezoidal Integration Method (The JoSIM Technique)

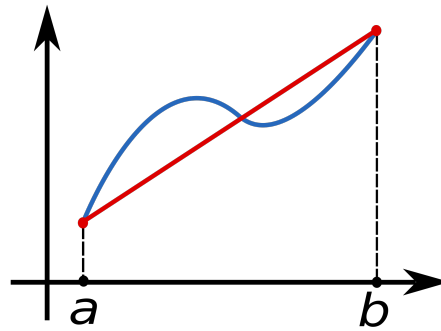


Figure 4: Illustration of Trapezoidal Rule²

Let y be a function of x , where $a, b \in x$. The trapezoidal rule is known to approximate the area under y by

$$\int_a^b y(x)dx \approx (b-a) \cdot \frac{y(a) + y(b)}{2} \quad (278)$$

$$\frac{d}{dx} \left[\int_a^b y(x)dx \right] \approx \frac{d}{dx} \left[(b-a) \cdot \frac{y(a) + y(b)}{2} \right] \quad (279)$$

Differentiating both sides and applying the Fundamental Theorem of Calculus

$$y(b) - y(a) \approx (b-a) \cdot \frac{y'(a) + y'(b)}{2} \quad (280)$$

Rearranging the equation above

$$y'(b) \approx \frac{2}{(b-a)} (y(b) - y(a)) - y'(a) \quad (281)$$

²https://en.wikipedia.org/wiki/Trapezoidal_rule

Let h be the time step we use to discretize our ODE. Using equation (281), the derivative of y at time t can be approximated as

$$y'(t) \approx \frac{2}{h}(y(t) - y(t-h)) - y'(t-h) \quad (282)$$

Writing this in terms of time steps yields

$$y'_n \approx \frac{2}{h}(y_n - y_{n-1}) - y'_{n-1} \quad (283)$$

The rest of the derivation can be seen in the capacitor section under Constructing the Stamps.

A.3 So, Why The Trapezoidal Integration Method?

Compared to the Euler method, the error is much lower for the trapezoidal integration method. Let us explain this more formally.

We define the term local truncation error³ (LTE) as the error induced at every time-step due to the approximation. In general, it is known in numerical analysis that a method with $O(h^{k+1})$ LTE is said to be of k th order.

Therefore, we can see that Euler method's LTE is $O(h^2)$ and trapezoidal integration method's LTE is $O(h^3)$. Essentially, the trapezoidal integration method provides lower LTE for the same step size. This is the major advantage of a higher-order technique.

B Why Euler's Method for the Voltage Guess?

We need an approximation for the voltage guess at time step n (v_n^g).

Suppose we use Euler's method.

Then, we can define the voltage guess to be

$$v_n^g = V_{n-1} + h_n \dot{V}_{n-1} \quad (284)$$

Using this definition, the unknowns (in our simulation) are v_n^g and $\dot{V}_{n-1}$. However, we can eliminate $\dot{V}_{n-1}$ as an unknown by approximating it with the trapezoidal integration method.

$$\dot{V}_{n-1} = \frac{2}{h}(V_{n-1} - V_{n-2}) - \dot{V}_{n-2} \quad (285)$$

³https://web.mit.edu/10.001/Web/Course_Notes/Differential_Equations_Notes/node3.html

Notice how the derivative can be calculated based on values from previous time steps. Therefore, the only unknown is v_n^g .

Suppose, instead of using a finite difference approximation (equation (284)), we use the trapezoidal integration method.

Then, we can define the voltage guess to be

$$v_n^g = V_{n-1} + \frac{h}{2} (\dot{V}_{n-1} + \dot{V}_n) \quad (286)$$

Similarly, using the definition, the unknowns are v_n^g , $\dot{V}_{n-1}$, and $\dot{V}_n$. As shown above, we can eliminate $\dot{V}_{n-1}$ as an unknown. This leaves us with v_n^g and $\dot{V}_n$.

Informally, since these two unknowns are in the same time step, and trapezoidal integration makes us substitute in the same equation, this would lead us to a "circular" process that would not solve the equation. Hence, we would need to use another technique or Euler's method (which rewrites the equation with variables from the $n - 1^{th}$ step).

C Verifying the Continuity of the JJ equations

C.1 Voltage-based Analysis JJ

C.1.1 Below the transition region to the transition region

We verify continuity at $V_n = V_{gap} - \frac{1}{2}\Delta V$

Equation (51) states that

$$\left(\frac{1}{R_{sg}} + \frac{2C}{h_n} \right) V_n^+ - \left(\frac{1}{R_{sg}} + \frac{2C}{h_n} \right) V_n^- - I_n = I_s \quad (287a)$$

$$\left(\frac{1}{R_{sg}} + \frac{2C}{h_n} \right) (V_n^+ - V_n^-) - I_n = I_s \quad (287b)$$

Substituting V_n as $V_{gap} - \frac{1}{2}\Delta V$

$$\left(\frac{1}{R_{sg}} + \frac{2C}{h_n} \right) \left(V_{gap} - \frac{1}{2}\Delta V \right) - I_n = I_s \quad (287c)$$

Rearranging

$$\left(V_{gap} - \frac{1}{2}\Delta V \right) \left(\frac{2C}{h_n} + \frac{1}{R_{sg}} \right) - I_s = I_n \quad (287d)$$

Equation (54) states that

$$(G_T + \frac{2C}{h_n})V_n^+ - (G_T + \frac{2C}{h_n})V_n^- - I_n = I_s - I_T \quad (288a)$$

$$(G_T + \frac{2C}{h_n})(V_n^+ - V_n^-) - I_n = I_s - I_T \quad (288b)$$

$$(G_T + \frac{2C}{h_n})(V_n^+ - V_n^-) - I_n = I_s - \left(V_{gap} - \frac{1}{2}\Delta V\right) \left(\frac{1}{R_{sg}} - G_T\right) \quad (288c)$$

Substituting V_n as $V_{gap} - \frac{1}{2}\Delta V$ and I_T

$$(G_T + \frac{2C}{h_n}) \left(V_{gap} - \frac{1}{2}\Delta V\right) - I_n = I_s - \left(V_{gap} - \frac{1}{2}\Delta V\right) \left(\frac{1}{R_{sg}} - G_T\right) \quad (288d)$$

$$(G_T + \frac{2C}{h_n}) \left(V_{gap} - \frac{1}{2}\Delta V\right) + \left(V_{gap} - \frac{1}{2}\Delta V\right) \left(\frac{1}{R_{sg}} - G_T\right) - I_n = I_s \quad (288e)$$

$$\left(V_{gap} - \frac{1}{2}\Delta V\right) \left(G_T + \frac{2C}{h_n} + \frac{1}{R_{sg}} - G_T\right) - I_n = I_s \quad (288f)$$

$$\left(V_{gap} - \frac{1}{2}\Delta V\right) \left(\frac{2C}{h_n} + \frac{1}{R_{sg}}\right) - I_s = I_n \quad (288g)$$

Since equations (287d) and (288g) are the same, the model is continuous at $V_n = V_{gap} - \frac{1}{2}\Delta V$.

C.1.2 In the transition region to above the transition region

We verify continuity at $V_n = V_{gap} + \frac{1}{2}\Delta V$

Equation (54) states that

$$(G_T + \frac{2C}{h_n})V_n^+ - (G_T + \frac{2C}{h_n})V_n^- - I_n = I_s - I_T \quad (289a)$$

$$(G_T + \frac{2C}{h_n})(V_n^+ - V_n^-) - I_n = I_s - I_T \quad (289b)$$

Substituting V_n as $V_{gap} + \frac{1}{2}\Delta V$ and I_T

$$(G_T + \frac{2C}{h_n}) \left(V_{gap} + \frac{1}{2}\Delta V \right) - I_n = I_s - \left(V_{gap} - \frac{1}{2}\Delta V \right) \left(\frac{1}{R_{sg}} - G_T \right) \quad (289c)$$

$$(G_T + \frac{2C}{h_n}) \left(V_{gap} + \frac{1}{2}\Delta V \right) + \left(V_{gap} - \frac{1}{2}\Delta V \right) \left(\frac{1}{R_{sg}} - G_T \right) - I_n = I_s \quad (289d)$$

$$G_T V_{gap} + \frac{1}{2} G_T \Delta V + \frac{2C}{h_n} V_{gap} + \frac{C}{h_n} \Delta V + \frac{V_{gap}}{R_{sg}} - G_T V_{gap} - \frac{1}{2} \frac{\Delta V}{R_{sg}} + G_T \frac{1}{2} \Delta V - I_n = I_s \quad (289e)$$

$$\frac{2C}{h_n} V_{gap} + \frac{C}{h_n} \Delta V + \frac{V_{gap}}{R_{sg}} - \frac{1}{2} \frac{\Delta V}{R_{sg}} + G_T \Delta V - I_n = I_s \quad (289f)$$

$$\frac{2C}{h_n} \left(V_{gap} + \frac{1}{2}\Delta V \right) + \frac{1}{R_{sg}} \left(V_{gap} - \frac{1}{2}\Delta V \right) + G_T \Delta V - I_n = I_s \quad (289g)$$

Substituting G_T

$$\frac{2C}{h_n} \left(V_{gap} + \frac{1}{2}\Delta V \right) + \frac{1}{R_{sg}} \left(V_{gap} - \frac{1}{2}\Delta V \right) + \frac{I_c}{I_{c fact} \cdot \Delta V} \Delta V - I_n = I_s \quad (289h)$$

Rearranging

$$\frac{2C}{h_n} \left(V_{gap} + \frac{1}{2}\Delta V \right) + \frac{1}{R_{sg}} \left(V_{gap} - \frac{1}{2}\Delta V \right) + \frac{I_c}{I_{c fact}} - I_s = I_n \quad (289i)$$

Equation (60) states that

$$\left(\frac{1}{R_n} + \frac{2C}{h_n} \right) V_n^+ - \left(\frac{1}{R_n} + \frac{2C}{h_n} \right) V_n^- - I_n = I_s - I_T \quad (290a)$$

$$\left(\frac{1}{R_n} + \frac{2C}{h_n} \right) (V_n^+ - V_n^-) - I_n = I_s - I_T \quad (290b)$$

Substituting V_n as $V_{gap} + \frac{1}{2}\Delta V$ and I_T

$$\left(\frac{1}{R_n} + \frac{2C}{h_n} \right) \left(V_{gap} + \frac{1}{2}\Delta V \right) - I_n = I_s - \frac{I_c}{I_{c fact}} - \left[\frac{1}{R_{sg}} \times \left(V_{gap} - \frac{1}{2}\Delta V \right) \right] + \left[\frac{1}{R_n} \times \left(V_{gap} + \frac{1}{2}\Delta V \right) \right] \quad (290c)$$

Rearranging

$$\left(\frac{1}{R_n} + \frac{2C}{h_n} - \frac{1}{R_n} \right) \left(V_{gap} + \frac{1}{2}\Delta V \right) + \frac{1}{R_{sg}} \left(V_{gap} - \frac{1}{2}\Delta V \right) - I_n = I_s - \frac{I_c}{I_{c fact}} \quad (290d)$$

$$\frac{2C}{h_n} \left(V_{gap} + \frac{1}{2}\Delta V \right) + \frac{1}{R_{sg}} \left(V_{gap} - \frac{1}{2}\Delta V \right) + \frac{I_c}{I_{c fact}} - I_s = I_n \quad (290e)$$

Since equations (289i) and (290e) are the same, the model is continuous at $V_n = V_{gap} + \frac{1}{2}\Delta V$

C.2 Phase-based Analysis JJ

C.2.1 Below the transition region to the transition region

We verify continuity at $V_n = V_{gap} - \frac{1}{2}\Delta V$.

Equation (141) states that

$$\left(\frac{2C}{h_n} + \frac{1}{R_{sg}}\right) V_n - I_n = I_s - I_T \quad (291a)$$

Substituting V_n as $V_{gap} - \frac{1}{2}\Delta V$ and $I_T = 0$ below the transition region

$$\left(\frac{2C}{h_n} + \frac{1}{R_{sg}}\right) \left(V_{gap} - \frac{1}{2}\Delta V\right) - I_n = I_s \quad (291b)$$

Equation (144) states that

$$\left(\frac{2C}{h_n} + G_T\right) V_n - I_n = I_s - I_T \quad (292a)$$

Substituting V_n as $V_{gap} - \frac{1}{2}\Delta V$ and I_T in the transition region

$$\left(\frac{2C}{h_n} + G_T\right) \left(V_{gap} - \frac{1}{2}\Delta V\right) - I_n = I_s - \left(V_{gap} - \frac{1}{2}\Delta V\right) \left(\frac{1}{R_{sg}} - G_T\right) \quad (292b)$$

$$\left(\frac{2C}{h_n} + G_T\right) \left(V_{gap} - \frac{1}{2}\Delta V\right) + \left(V_{gap} - \frac{1}{2}\Delta V\right) \left(\frac{1}{R_{sg}} - G_T\right) - I_n = I_s \quad (292c)$$

$$\left(\frac{2C}{h_n} + G_T + \frac{1}{R_{sg}} - G_T\right) \left(V_{gap} - \frac{1}{2}\Delta V\right) - I_n = I_s \quad (292d)$$

Simplifying and rearranging

$$\left(\frac{2C}{h_n} + \frac{1}{R_{sg}}\right) \left(V_{gap} - \frac{1}{2}\Delta V\right) - I_s = I_n \quad (292e)$$

Since equations (291b) and (292e) are the same, the model is continuous at $V_n = V_{gap} - \frac{1}{2}\Delta V$ and I_T .

C.2.2 In the transition region to above the transition region

We verify continuity at $V_n = V_{gap} + \frac{1}{2}\Delta V$.

Equation (144) states that

$$\left(\frac{2C}{h_n} + G_T\right) V_n - I_n = I_s - I_T \quad (293a)$$

Substituting V_n as $V_{gap} + \frac{1}{2}\Delta V$ and I_T in the transition region

$$\left(\frac{2C}{h_n} + G_T\right) \left(V_{gap} + \frac{1}{2}\Delta V\right) - I_n = I_s - \left(V_{gap} - \frac{1}{2}\Delta V\right) \left(\frac{1}{R_{sg}} - G_T\right) \quad (293b)$$

The equation above is the same as the voltage-based analysis case (look at equation (289c)). Thus, we know

$$\frac{2C}{h_n} \left(V_{gap} + \frac{1}{2}\Delta V\right) + \frac{1}{R_{sg}} \left(V_{gap} - \frac{1}{2}\Delta V\right) + \frac{I_c}{I_{cfact}} - I_s = I_n \quad (293c)$$

Equation (150) states that

$$\left(\frac{2C}{h_n} + \frac{1}{R_n}\right) V_n - I_n = I_s - I_T \quad (294a)$$

Substituting V_n as $V_{gap} + \frac{1}{2}\Delta V$ and I_T above the transition region

$$\left(\frac{2C}{h_n} + \frac{1}{R_n}\right) \left(V_{gap} + \frac{1}{2}\Delta V\right) - I_n = I_s - \frac{I_c}{I_{cfact}} - \left[\frac{1}{R_{sg}} \times \left(V_{gap} - \frac{1}{2}\Delta V\right)\right] + \left[\frac{1}{R_n} \times \left(V_{gap} + \frac{1}{2}\Delta V\right)\right] \quad (294b)$$

The equation above is the same as the voltage-based analysis case (look at equation (290c)). Thus, we know

$$\frac{2C}{h_n} \left(V_{gap} + \frac{1}{2}\Delta V\right) + \frac{1}{R_{sg}} \left(V_{gap} - \frac{1}{2}\Delta V\right) + \frac{I_c}{I_{cfact}} - I_s = I_n \quad (294c)$$

Since equations (293c) and (294c) are the same, the model is continuous at $V_n = V_{gap} + \frac{1}{2}\Delta V$

D Errors in the JoSIM Simulator Code

D.1 Transition Current

For the transition region, we know
lower-bound = $V_{gap} - \frac{1}{2}\Delta V$ and

$$\text{upper-bound} = V_{gap} + \frac{1}{2}\Delta V$$

$$G_T = \frac{I_c}{I_{c\text{fact}} \cdot \Delta V}$$

In Delport's dissertation, it is stated that the transition current is

Algorithm 1: Determining Transition Current - Dissertation

```

Result:  $I_T$ 
if  $|v^0| < \text{lower-bound}$  then
  |  $I_T = 0.0$ 
else if  $|v^0| < \text{upper-bound}$  then
  |  $I_T = (V_{gap} - \frac{1}{2}\Delta V) \times \left(\frac{1}{R_{sg}} - G_T\right)$ 
else
  |  $I_T = \frac{I_c}{I_{C\text{fact}}} + \left[\frac{1}{R_{sg}} \times \text{lower-bound}\right] - \left[\frac{1}{R_n} \times \text{upper-bound}\right]$ 
end

```

In the code, we find the transition current as

Algorithm 2: Determining Transition Current - code

```

Result:  $I_T$ 
if  $|v^0| < \text{lower-bound}$  then
  |  $I_T = 0.0$ 
else if  $|v^0| < \text{upper-bound}$  then
  |  $I_T = \text{lower-bound} \times \left(\frac{1}{R_{sg}} - G_T\right)$ 
else
  |  $I_T = \frac{I_c}{I_{C\text{fact}}} + \frac{V_{gap}}{R_{sg}} - \frac{\text{lower-bound}}{R_n}$ 
end

```

We believe the equations in the dissertation to be correct and the code in JoSIM to be in error. Therefore, will use the equations in the dissertation as a guide.

E What is $\bar{D}$ in the Current-Phase Relation for a JJ?

In order to understand the importance of $\bar{D}$, we must first explore a few concepts.

E.1 Andreev Reflection

This section borrows heavily from [5].

In figure 5, we see superconducting materials on the left and right connected by a “weak-

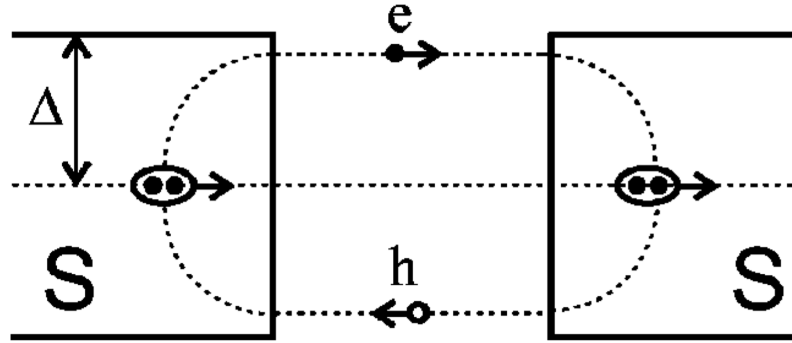


Figure 5: Formation of Andreev levels in a Josephson junction from [5]

link”. A weak-link is usually a non-superconducting material like a normal conductor or an insulator (more are listed in [5]).

A quasiparticle located in the weak-link cannot penetrate directly into a superconductor if its energy is smaller than the superconducting energy gap (Δ). However, there is still a way for the electron to get across! This form of charge transport is known as *Andreev reflection*. An electron with momentum k impinging on one of the interfaces is converted into a hole moving in the opposite direction, thus generating a Cooper pair in a superconductor. This hole is consequently Andreev reflected at the second interface and is converted back to an electron, leading to the destruction of a Cooper pair (see Figure 5). As a result of this cycle, a pair of correlated electrons is transferred from one superconductor to another, creating a supercurrent flow across a junction.

Since Andreev reflection amplitudes depend on the corresponding phases of the superconductors’ wavefunctions, the resulting current depends on the phase difference ϕ , thus leading to the dc Josephson effect ($I_s = I_c \sin \phi$).

What does “converted into a hole” mean?

Assume that electrons are moving from the left to the right superconductor. We think this means that an electron enters the weak-link, leaving a hole in the left superconductor. Since the electron does not have enough energy to jump over the superconducting energy gap, it’s unable to cross the barrier. Meanwhile, another electron jumps to fill up the hole. Then, when the second electron arrives, and since the temperature is low enough, a Cooper pair forms. This causes the pair to have enough energy to jump over the barrier. This process then repeats itself.

E.2 So, What is $\bar{D}$?

$\bar{D}$ is defined as the electron “transparency”. This is not particularly helpful. So, we use the alternate definition: $\bar{D}$ is the electron transmission probability across the barrier.

Thus, this modifies our understanding of electron tunnelling. We would explain it as follows. Normal tunnelling would occur when $\bar{D}$ is 0 (therefore, all electrons tunnel across by Andreev reflection). Meanwhile, ballistic tunnelling occurs when $\bar{D} \rightarrow 1$, which means that there is a non-zero probability that the electron will get across the barrier (into the superconductor) without even forming a cooper pair. Hence the term “ballistic” tunnelling.

F Deriving the Transition Current Equations

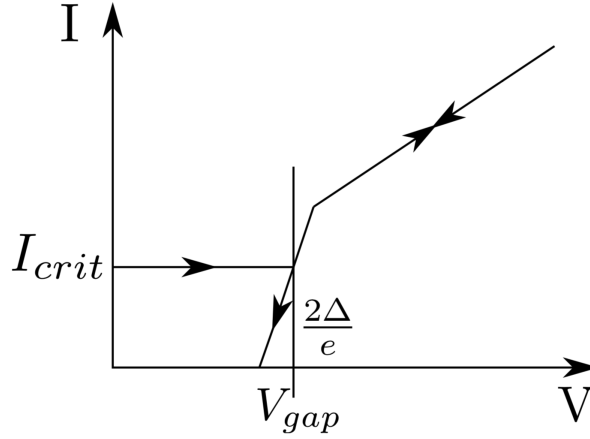


Figure 6: IV Curve of a JJ from Delport's Dissertation

We attempt to derive Delport's equations for the conductances and currents related to the transition region. For convenience, the IV-Curve of a JJ is placed above.

F.1 JSIM Transition Conductance

Restating equation (34),

$$G_T = \frac{\frac{V_{gap} + \Delta V}{R_n} - \frac{V_{gap}}{R_{sg}}}{\Delta V} \quad (295)$$

Notice that in the transition region, the IV relation of the JJ is linear. Hence, treating the JJ as a resistor, we can simply extract its conductance as $\frac{\Delta I}{\Delta V}$. Since we are treating the JJ as a resistor, we know the current at V_{gap} is simply $\frac{V_{gap}}{R_{sg}}$, where R_{sg} is the “subgap resistance”.

Then, we know that at $V_{gap} + \Delta V$, the junction resistance is R_n , where R_n is the “normal resistance”. Summarizing this, we can write

$$I_{V_{gap}} = \frac{V_{gap}}{R_{sg}} \quad (296)$$

$$I_{V_{gap}+\Delta V} = \frac{V_{gap} + \Delta V}{R_n} \quad (297)$$

Finally, we can write the transition conductance as

$$G_T = \frac{\Delta I}{\Delta V} = \frac{(I_{V_{gap}+\Delta V}) - I_{V_{gap}}}{\Delta V} = \frac{\frac{V_{gap}+\Delta V}{R_n} - \frac{V_{gap}}{R_{sg}}}{\Delta V} \quad (298)$$

F.2 JSIM Transition Current

Restating equation (35),

$$I_T = V_{gap} \left(\frac{1}{R_{sg}} - G_T \right) \quad (299)$$

Notice that the subgap current is $I_{subgap} = \frac{V_{gap}}{R_{sg}}$ and $I_{gap} = V_{gap}(G_T)$. We know that we must “transition” from the gap to the transition region. We can easily calculate the difference in current between these two stages as

$$I_T = I_{subgap} - I_{gap} \quad (300)$$

$$= \frac{V_{gap}}{R_{sg}} - V_{gap}(G_T) \quad (301)$$

$$= V_{gap} \left(\frac{1}{R_{sg}} - G_T \right) \quad (302)$$

G Sign of Transition Current

$$\begin{aligned}
I_T &= V_{gap} \left(\frac{1}{R_{sg}} - G_T \right) \\
&= V_{gap} \left(\frac{1}{R_{sg}} - \frac{\frac{V_{gap} + \Delta V}{R_n} - \frac{V_{gap}}{R_{sg}}}{\Delta V} \right) \\
&= V_{gap} \left(\frac{\frac{\Delta V}{R_{sg}}}{\Delta V} - \frac{\frac{V_{gap} + \Delta V}{R_n} - \frac{V_{gap}}{R_{sg}}}{\Delta V} \right) \\
&= V_{gap} \left(\frac{\frac{\Delta V}{R_{sg}} - \frac{V_{gap} + \Delta V}{R_n} + \frac{V_{gap}}{R_{sg}}}{\Delta V} \right) \\
&= V_{gap} \left(\frac{\frac{V_{gap} + \Delta V}{R_{sg}} - \frac{V_{gap} + \Delta V}{R_n}}{\Delta V} \right)
\end{aligned} \tag{303}$$

Note that when $0 < R_n < R_{sg}$, I_T is negative. This is true because, under these conditions, $\frac{V_{gap} + \Delta V}{R_n} < \frac{V_{gap} + \Delta V}{R_{sg}}$ causing the entire term to become negative.